# CURVATURE ESTIMATES FOR MINIMAL HYPERSURFACES IN THE HEISENBERG GROUP

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Abstract. In this paper we solve the Bernstein problem for a broad class of smooth, non-characteristic hypersurfaces in the second sub-Riemannian Heisenberg group $\mathbb{H}^2$.

## 1. Introduction

**The Euclidean Bernstein problem.** The *Bernstein problem*, originally solved by Bernstein in $\mathbb{R}^2$ (cf. [6]), consists in characterizing entire solutions $u : \mathbb{R}^n \to \mathbb{R}$ to the minimal surface equation

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0. \tag{1.1}$$

Thanks to the joint effort of Fleming (cf. [33]), De Giorgi (cf. [27]), Almgren (cf. [2]), Simons (cf. [66]) and Bombieri, De Giorgi and Giusti (cf. [7]), we know that, for $n \leqslant 7$, entire solutions to (1.1) are affine functions, or, equivalently, their graphs are hyperplanes. Moreover, for $n \geqslant 8$, there exist entire analytic solutions to (1.1) which are not affine. Indeed, the well-known monotonicity formula for the perimeter density allows to reduce the solution to the Bernstein problem in $\mathbb{R}^{n+1}$ to the existence of singular minimal cones in $\mathbb{R}^n$ (cf. [33, 27]), which occurs if and only if $n \geqslant 8$ (cf. [2, 66, 7]). The very same approach is suitable for the solution to a way more general formulation of the Bernstein problem, i.e. the characterization of global perimeter minimizers in $\mathbb{R}^n$. In this setting, when $n \leqslant 7$, the unique non-empty global perimeter minimizers in $\mathbb{R}^n$ are half-spaces, while there are counterexamples when $n \geqslant 8$. We refer to [41] for a detailed account on the Bernstein problem in the Euclidean space.

**An alternative approach.** A new approach to the Bernstein problem was proposed by Schoen, Simon and Yau in their seminal paper [63], where the authors solved the latter in the class of complete, stable hypersurfaces satisfying suitable volume growth assumptions and under the additional constraint $n \leqslant 6$. This second approach can be summarized in the following steps.

1. Combining the celebrated *Simons identity* for minimal hypersurfaces $S \subseteq \mathbb{R}^n$ (cf. [66]), namely

$$\Delta^S h = -|h|^2 h, \tag{1.2}$$

with the *Kato-type inequality*

$$\left(1+\frac{2}{n-1}\right)\left|\nabla^S |h|^2\right|^2 \leqslant 4|h|^2 |\nabla^S h|^2 \tag{1.3}$$

(cf. [63]), one provides a lower bound for $\Delta^S |h|^2$ of the form

$$2|h|^2 \Delta^S |h|^2 \geqslant \left(1+\frac{2}{n-1}\right)\left|\nabla^S |h|^2\right|^2 - 4|h|^6. \tag{1.4}$$

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Here, $h$ is the second fundamental form associated to $S$, $\Delta^S$ is the tangential Laplacian and $\nabla^S$ is the tangential gradient.

2. Owing to (1.4), one establishes $L^p$-estimates for stable hypersurfaces such as

$$\int_S |h|^p \varphi^p \, d\sigma \leqslant C \int_S |\nabla^S \varphi|^p \, d\sigma, \tag{1.5}$$

where $C$ is a geometric constant, $p$ lies in a range of exponents which depends on the dimension $n$, $\sigma$ is the surface measure and $\varphi$ is a smooth test function.

3. Assuming that $S$ satisfies volume growth conditions of the form

$$\sigma(S \cap B_r(p)) = O\left(r^{n-1}\right) \tag{1.6}$$

as $r \to \infty$, one exploits (1.5) to show that $S$ is totally geodesic, meaning that $h \equiv 0$.

4. Complete, totally geodesic Euclidean hypersurfaces are hyperplanes.

In particular, since for $n \leqslant 7$ boundaries of perimeter minimizers are smooth, complete, stable hypersurfaces and satisfy (1.6) (cf. [46]), this new approach yields a new solution to the Bernstein problem when $n \leqslant 6$. Although this second approach fails to solve the Bernstein problem in its full generality, as the case $n = 7$ is not covered, it is originally stated in the more general setting of Riemannian manifolds satisfying suitable curvature constraints. Moreover, it has the advantage of being applicable to the solution of the so-called *stable Bernstein problem*, i.e. the characterization of complete, stable hypersurfaces, thus without a priori requiring that they are boundaries of global perimeter minimizers. While the thee-dimensional version of the latter has been solved by do Carmo and Peng (cf. [28]), Fischer-Colbrie and Schoen (cf. [32]) and Pogorelov (cf. [56]) via *ad hoc* techniques, Schoen-Simon-Yau's approach reduces the solution to the higher dimensional case to the establishment of volume growth estimates as in (1.6). Following this approach, Chodosh and Li (cf. [17]), Chodosh, Li, Minter and Stryker (cf. [18]) and Mazet (cf. [48]) recently solved the stable Bernstein problem in $\mathbb{R}^4$, $\mathbb{R}^5$ and $\mathbb{R}^6$ respectively (cf. also [5] for some recent developments).

**The sub-Riemannian Bernstein problem.** Like its Euclidean and Riemannian precursors, the sub-Riemannian Bernstein problem is an intriguing topic within the broader framework of sub-Riemannian geometry, It fits into the more general context of studying minimal hypersurfaces in sub-Riemannian structures (cf. [13, 14, 15, 25, 30, 38, 57, 39, 42, 52, 53, 58, 65] and references therein). This research area is particularly relevant in the sub-Riemannian Heisenberg group $\mathbb{H}^n$, which constitutes a prototypical model in the setting of Carnot groups (cf. [8]), sub-Riemannian manifolds (cf. [1]), CR manifolds (cf. [11]) and Carnot-Carathéodory spaces (cf. [43]). We briefly recall that the $n$-th Heisenberg group $(\mathbb{H}^n, \cdot)$ is $\mathbb{R}^{2n+1}$ endowed with the group law

$$p \cdot p' = (\bar{x}, \bar{y}, t) \cdot (\bar{x}', \bar{y}', t') = \left(\bar{x} + \bar{x}', \bar{y} + \bar{y}', t + t' + \sum_{j=1}^{n} \left(x_j' y_j - x_j y_j'\right)\right),$$

where we denoted points $p \in \mathbb{R}^{2n+1}$ by $p = (\bar{x}, \bar{y}, t) = (x_1, \ldots, x_n, y_1, \ldots, y_n, t)$. With this operation, $\mathbb{H}^n$ is a Carnot group, whose associated *horizontal distribution*, which we denote by $\mathcal{H}$, is generated by the left-invariant vector fields

$$Z_j = X_j = \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial t} \qquad \text{and} \qquad Z_{n+j} = Y_j = \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial t}$$

for $j = 1, \ldots, n$. A vector field which is tangent to $\mathcal{H}$ at every point is called *horizontal*. If we denote by $T$ the left-invariant vector field $\frac{\partial}{\partial t}$, then $X_1, \ldots, X_n, Y_1, \ldots, Y_n, T$ constitutes a global frame of left-invariant vector fields. The only nontrivial commutation relations are

$$[X_j, Y_j] = -[Y_j, X_j] = -2T$$

for any $j = 1, \ldots, n$. $\mathbb{H}^n$ inherits a sub-Riemannian structure by fixing be the unique Riemannian metric $\langle \cdot, \cdot \rangle$ which makes $X_1, \ldots, X_n, Y_1, \ldots, Y_n, T$ orthonormal. Moreover, $\mathbb{H}^n$ can be endowed with an appropriate affine connection $\nabla$, the so-called *pseudohermitian connection*, which is metric but not torsion-free, and in a sense realizes it as a flat sub-Riemannian model. These tools both provide an intrinsic definition of perimeter, the so-called *horizontal perimeter*, and enhance the study of the extrinsic geometry of submanifolds in $\mathbb{H}^n$. Consequently, the sub-Riemannian formulation of the Bernstein problem appears just as natural as its Riemannian counterpart. Nevertheless, its essence is substantially different from the Euclidean one. First of all, a suitable generalization of the Euclidean monotonicity formula is known to hold only for particular classes of hypersurfaces (cf. [24]), whereas its validity for general hypersurfaces remains a major open problem in the field. Moreover, there are several examples of minimal cones, both smooth (cf. [61, 14, 42]) and with low regularity (cf. [58, 50]) which are not flat from the intrinsic viewpoint of $\mathbb{H}^n$. This new phenomenon is caused by the fact that an hypersurface $S \subseteq \mathbb{H}^n$, although smooth from a classical differential standpoint, may be intrinsically irregular due to the presence of the so-called *characteristic points*, i.e. those points $p \in S$ for which the tangent space $T_pS$ coincides with the horizontal distribution $\mathcal{H}_p$. For the above-mentioned reasons, the first approach we have described in the Euclidean setting does not appear to be suitable for this context, neither in $\mathbb{H}^1$ nor in higher dimension. Nevertheless, by means of *ad hoc* techniques, the Bernstein problem in $\mathbb{H}^1$ is now largely solved. In [45], Hurtado, Ritoré and Rosales (cf. also [14, 61, 25, 26]) proved that a complete, two-sided, connected, stable $C^2$-surface in $\mathbb{H}^1$ must be a *vertical plane* (without characteristic points), a *horizontal plane* (with one characteristic point) or the *hyperbolic paraboloid* $t = x_1y_1$ (with a line of characteristic points). In general, we recall that a *vertical hyperplane* in $\mathbb{H}^n$ is an Euclidean hyperplane which is tangent to $T$ at every point. The hyperbolic paraboloid, which is a cone for the intrinsic geometry of $\mathbb{H}^n$, can be easily lifted to provide smooth, non-flat, minimal cones for any $n \geqslant 2$ (cf. [55]). The situation is different when considering surfaces without characteristic points, which we will call *non-characteristic*. In this regard, Barone Adesi, Serra Cassano and Vittone (cf. [4]) showed that entire, stable *intrinsic graphs* associated to a $C^2$-function are vertical planes. The same conclusion was achieved by Galli and Ritoré (cf. [36]) in the class of non-characteristic, complete, two-sided, connected and stable $C^1$-surfaces. The $C^1$-regularity assumption was later improved to Euclidean Lipschitz regularity by Nicolussi Golo and Serra Cassano (cf. [51]) and by the first author and Ritoré (cf. [40]). We point out that the best possible regularity to guarantee the above rigidity in $\mathbb{H}^1$ is still an open problem (cf. [68] for some developments in this direction). On the other hand, although in $\mathbb{H}^1$ there are counterexamples to the regularity of perimeter minimizers, some evidences (cf. e.g. [10]) suggest that it is reasonable to study the higher dimensional Bernstein problem in the smooth category. Therefore, in light of the above results and considerations, the appropriate intrinsic formulation of the Bernstein conjecture in arbitrary dimension reads as follows.

**sub-Riemannian stable Bernstein problem.** Is it true that smooth, complete, two-sided, connected, stable non-characteristic hypersurfaces $S \subseteq \mathbb{H}^n$ are vertical hyperplanes?

While, as we have just noticed, the problem in $\mathbb{H}^1$ is fairly well understood, very little can be said in the higher dimensional case. In [4] the authors provide a negative answer to this question for $n \geqslant 5$, essentially by lifting the Euclidean analytic counterexamples available in $\mathbb{R}^{n+1}$ when $n \geqslant 8$. However, the purely Euclidean character of these counterexamples suggests that the dimensional bound $n \geqslant 5$ might not be optimal. In any case, the validity of this long-standing conjecture in the remaining cases $\mathbb{H}^2$, $\mathbb{H}^3$ and $\mathbb{H}^4$ remains a completely open problem. In this paper we affirmatively solve the Bernstein conjecture in a broad class of hypersurfaces in the second sub-Riemannian Heisenberg group $\mathbb{H}^2$. More precisely, since the classical Euclidean approach does not align well with this setting, we identify some reasonable assumptions whereby to develop and apply an approach in the style of Schoen-Simon-Yau.

**Structural assumptions.** Let us be more precise about our assumptions, that we will call (H1), (H2) and (H3). When a hypersurface $S \subseteq \mathbb{H}^n$ is non-characteristic, we can define its *horizontal unit normal* $\nu$ as the normalization of the projection of the Riemannian unit normal $N$ onto the horizontal distribution $\mathcal{H}$. Then the *horizontal shape operator* $A$ is given by the covariant derivative of $\nu$ with respect to the pseudohermitian connection $\nabla$. Since, differently from the Riemannian setting, $A$ is not necessarily self-adjoint, its symmetrized counterpart $\tilde{A}$ can be considered, and the associated *horizontal second fundamental forms* $h$ and $\tilde{h}$ can be defined (cf. [44, 23, 16, 59, 60]). The first assumption (H1) requires that $J(\nu)$, the ninety-degree rotation of $\nu$ (cf. Section 2.1), is an eigenvector for $\tilde{h}$. This mild assumption, which is automatically satisfied in $\mathbb{H}^1$, emerges naturally in the sub-Riemannian setting, for instance in the study of *umbilic hypersurfaces* as introduced in [12] (cf. Section 3.4). Since $S$ is non-characteristic, the intersection between the horizontal distribution $\mathcal{H}$ and the tangent bundle $TS$ generates a $(2n-1)$-dimensional sub-bundle $\mathcal{H}TS$, the *horizontal tangent bundle*. In turn, the latter admits the orthogonal decomposition $\mathcal{H}TS = \operatorname{span} J(\nu) \oplus \mathcal{H}'TS$, where the $(2n-2)$-dimensional sub-bundle $\mathcal{H}'TS$ is invariant under the complex structure induced by the rotation $J$. The remaining tangent direction of $S$, say $\mathcal{S}$, is non-horizontal and orthogonal to $\mathcal{H}TS$, whence it is given by a linear combination between $T$ and $\nu$. However, as $S$ is non-characteristic, $\mathcal{S}$ cannot coincide with $\nu$, so that there exists a smooth function $\alpha$, the *fundamental function* of $S$, such that $\mathcal{S} = T - \alpha\nu$ belongs to $TS$. The latter appears frequently in the sub-Riemannian theory of hypersurfaces in the Heisenberg group (cf. e.g. [13, 12, 16, 59]) and can be equivalently defined by $\alpha = \frac{\langle N,T\rangle}{|N^{\mathbb{H}}|}$, where $N^{\mathbb{H}}$ is the projection of the Riemannian unit normal $N$ onto the horizontal distribution $\mathcal{H}$. For instance, it is the curvature of a length-minimizing geodesic realizing the distance between a hypersurface and a given point [60]. Moreover, when $S$ is embedded in $\mathbb{H}^n$, the fundamental function can be characterized by the identity $\alpha = Td^S$, where $d^S$ is the signed *Carnot-Carathéodory distance* from $S$ (cf. Section 3.1). Our second assumption (H2) requests that the fundamental function $\alpha$ in constant along the sub-bundle $\mathcal{H}'TS$. Again, since $\mathcal{H}'TS = \{0\}$ when $n = 1$, (H2) is satisfied by every non-characteristic surface in $\mathbb{H}^1$. Moreover, we stress that we are not prescribing any kind of behavior of $\alpha$ along the non-horizontal direction $\mathcal{S}$. In order to describe our last assumption, we recall that a smooth, non-characteristic hypersurface $S$ is *minimal* whether its *horizontal mean curvature* $H$ vanishes, and that it is *stable* if it is minimal and

$$\int_S q\,\xi^2\,d\sigma_{\mathcal{H}} \leqslant \int_S |\nabla^{\mathcal{H},S}\xi|^2\,d\sigma_{\mathcal{H}} \tag{1.7}$$

for any $\xi \in C^1_c(S)$, where $\nabla^{\mathcal{H},S}$ is the *horizontal tangent gradient*, $\sigma_{\mathcal{H}}$ is the sub-Riemannian surface measure and $q$, the *stability function*, is defined by $q = |\tilde{h}|^2 + 4\langle\nabla\alpha, J(\nu)\rangle + 2(n+1)\alpha^2$ (cf. Section 3.8). If compared to the Riemannian stability inequality for minimal hypersurfaces immersed in a Riemannian manifold, the stability function $q$ plays the role of the Riemannian term $|h_R|^2 + \mathrm{Ric}(N,N)$, where $h_R$ is the Riemannian second fundamental form, Ric is the Ricci curvature of the ambient manifold and $N$ is the unit normal. In the Riemannian framework, it is customary to rely on suitable lower bounds for both the Ricci curvature and the sectional curvatures in order to achieve rigidity results (cf. e.g [32, 63, 22] and references therein). Accordingly, we propose with (H3) a lower bound for the stability function $q$ of the form

$$q \geqslant |\tilde{h}|^2 + (2n-2)\alpha^2 - \omega\alpha^2, \tag{1.8}$$

depending on a parameter $\omega \in [0, 2]$. Once more, (1.8) is verified by any complete minimal surface in $\mathbb{H}^1$ with the best possible choice $\omega = 0$, as shown by Galli and Ritoré in [36]. While (H1), (H2) and (H3) are thus satisfied in $\mathbb{H}^1$, the relevance of this set of assumptions in arbitrary dimension is further supported by the fact that they hold, again with $\omega = 0$, for every minimal umbilic hypersurface (cf. Remark 7.7). Furthermore, it would be interesting to understand the consistency of these assumptions in the particular case of entire *intrinsic graphs*, which constitute a relevant class of hypersurfaces to which our approach may apply.

**The sub-Riemannian Schoen-Simon-Yau's approach.** The sub-Riemannian generalization of Schoen-Simon-Yau's approach moves from recent results proved by the last two authors of this paper (cf. [55]). Namely, complete, non-characteristic, embedded hypersurfaces $S \subseteq \mathbb{H}^n$ with vanishing symmetric horizontal second fundamental form $\tilde{h}$ are vertical hyperplanes. Our first step (cf. Theorem 5.1) consists in the establishment of a full sub-Riemannian counterpart of the Simons identity (1.2) for $\Delta^{\mathcal{H},S}h$, where $\Delta^{\mathcal{H},S}$ is the *horizontal tangential Laplacian* of $S$, which relates the latter to the stability function $q$ appearing in (1.7) with the aid of appropriate sub-Riemannian Gauss-Codazzi equations (cf. Proposition 3.12). The previous result, which holds in arbitrary dimension and without requiring (H1), (H2) and (H3), provides a significantly more complex formula than (1.2), and makes clear the influence of the non-commutative structure in which we are operating. In our second step, we exploit the full Simons identity to provide a lower bound for $\hat{\Delta}^{\mathcal{H},S}|\tilde{h}|^2$, where $\hat{\Delta}^{\mathcal{H},S}$ is the self-adjoint counterpart of $\Delta^{\mathcal{H},S}$ introduced by Danielli, Garofalo and Nhieu in [23]. As explained thoroughly in Section 5, it is during this procedure that the importance of assumptions (H1), (H2) and (H3) naturally appears. Moreover, the *a priori* lack of suitable sub-Riemannian geodesic frames requires some delicate *ad hoc* computations (cf. Section 4). Although a result of this kind is actually available in arbitrary dimension (cf. Section 5.2), the specific structure of $\mathbb{H}^2$ (cf. Section 5.3) allows to improve the latter to a more accurate lower bound of the form

$$2|\tilde{h}|^2\hat{\Delta}^{\mathcal{H},S}|\tilde{h}|^2 \geqslant 4|\tilde{h}|^2|\nabla^{\mathcal{H},S}\tilde{h}|^2 - 4q|\tilde{h}|^4 + 4\alpha^2|\tilde{h}|^2\left((4-2\omega)|\tilde{h}|^2 - (6-3\omega)\ell^2\right), \tag{1.9}$$

where $\ell$ is the eigenvalue associated with $J(\nu)$ and $\omega$ is forced by (H3). Next, in order to get rid of the first-order term $|\nabla^{\mathcal{H},S}\tilde{h}|^2$, we exploit (H1) and (H2) to provide a sub-Riemannian parametric Kato-type inequality in arbitrary dimension (cf. Section 5.4), whose version in $\mathbb{H}^2$ reads as

$$\left(1+\frac{k}{3}\right)\left|\nabla^{\mathcal{H},S}|\tilde{h}|^2\right|^2 \leqslant 4|\tilde{h}|^2|\nabla^{\mathcal{H},S}\tilde{h}|^2 + 4\alpha^2|\tilde{h}|^2\left((4k-2)|\tilde{h}|^2 + (2k-6)\ell^2\right), \tag{1.10}$$

where $k \in [0,2]$. The last terms appearing in (1.9) and (1.10) respectively constitute a crucial novelty compared to the Euclidean setting. Indeed, the presence of these two $\alpha^2$-*remainders* is essentially due to the fact that, differently from the Euclidean setting, the lack of torsion-freeness of $\nabla$ prevents $\tilde{h}$ from being a *Codazzi tensor* (cf. Proposition 3.13). As we will see shortly, a key step in our approach consists in managing to control these additional terms. While the $\alpha^2$-reminder in (1.9) is constrained by (H3), the freedom to choose $k \in [0,2]$ in (1.10) allows to balance between the contribution of the gradient term and the $\alpha^2$-reminder. Roughly speaking, for a better control on the latter we need to pay a worse contribution from the former. Combining (1.9) with (1.10) (cf. Section 5.5), we finally establish the sub-Riemannian counterpart of (1.4), namely

$$2|\tilde{h}|^2\hat{\Delta}^{\mathcal{H},S}|\tilde{h}|^2 \geqslant \left(1+\frac{k}{3}\right)|\nabla^S|\tilde{h}|^2|^2 - 4q|\tilde{h}|^4 + 4\alpha^2|\tilde{h}|^2 g_{S,k,\omega}, \tag{1.11}$$

where $g_{S,k,\omega}$ is the appropriate contribution coming from the two $\alpha^2$-reminders of (1.9) and (1.10) (cf. (5.32)). Once (1.11) is achieved, and as soon as it is possible to choose $k \in [0,2]$ small enough to ensure that $g_{S,k,\omega} \geqslant 0$, we can exploit the stability inequality (1.7) to establish sub-Riemannian $L^p$-estimates for $\tilde{h}$ in arbitrary dimension, namely

$$\int_S |\tilde{h}|^{2\beta+2}\varphi^{2\beta+2}\, d\sigma_{\mathcal{H}} \leqslant C\int_S |\nabla^{\mathcal{H},S}\varphi|^{2\beta+2}\, d\sigma_{\mathcal{H}}, \tag{1.12}$$

where $\varphi \in C^1_c(S)$, $C = C(\beta,k)$ is a structural constant and $\beta \in \left[\frac{2n-1-k}{2n-1}, 1+\sqrt{\frac{k}{2n-1}}\right)$. Heuristically, the application of (1.12) to the solution to the Bernstein problem requires $\beta$ to be chosen as close as possible to its upper bound, since the bigger is the latter, the higher is the dimension $n$ in which we can apply this approach. Indeed, under natural sub-Riemannian volume growth assumptions of the form

$$\sigma_{\mathcal{H}}(S \cap B_r(p)) = O\left(r^{2n+1}\right) \tag{1.13}$$

as $r \to \infty$ (cf. Section 7), (1.12) implies that

$$\int_{S\cap B_r(p)} |\tilde{h}|^{2\beta+2}\, d\sigma_{\mathcal{H}} = O\left(r^{2n-1-2\beta}\right)$$

as $r \to \infty$. All in all, then, we would like to choose $k$ large enough to ensure that $2n-1-2\beta \leqslant 0$, but still small enough to ensure that $g_{S,k,\omega} \geqslant 0$. Regarding the first condition, it is easy to verify that even the optimal choice $k = 2$ allows only the case $n = 2$. On the other hand, when $n = 2$, every choice $k \in \left(\frac{3}{4}, 2\right]$ is an admissible candidate (cf. Section 7). Finally, if $\omega$ is sufficiently small (but cf. Section 7 for finer considerations) it is always possible to choose $k \in \left(\frac{3}{4}, 2\right]$ which ensures that $g_{S,k,\omega} \geqslant 0$ (cf Proposition 7.3). In this way $\tilde{h} \equiv 0$, whence, by [55], $S$ is a vertical hyperplane. For instance, when $\omega < \frac{3}{2}$, our main result reads as follows.

**Theorem 1.1.** *Let $S \subseteq \mathbb{H}^2$ be a smooth, complete, connected, embedded, two sided non-characteristic hypersurface. Assume that $S$ is stable. Assume that $S$ verifies (H1), (H2) and (H3) with $\omega < \frac{3}{2}$. Assume in addition that there exists $p \in S$ such that (1.13) holds. Then $S$ is a vertical hyperplane.*

Finally, we point out that the sub-Riemannian Schoen-Simon-Yau's approach would be pointless in $\mathbb{H}^1$. Indeed, in $\mathbb{H}^1$, every minimal surface satisfies $\tilde{h} \equiv 0$, but there are examples of minimal non-characteristic surfaces which are not vertical planes (cf. [25]). This difference between $\mathbb{H}^1$ and the higher dimensional case may be explained by the fact that the horizontal tangent distribution $\mathcal{H}TS$

is bracket-generating if, and only if, $n \geqslant 2$ (cf. [3]). A relevant instance of this phenomenon can be appreciated in the different approaches to regularity employed in $\mathbb{H}^1$ [9] and in higher dimension [10].

**Plan of the paper.** In Section 2 we recall some preliminaries concerning the Heisenberg group. In Section 3 we collect some properties of non-characteristic hypersurfaces. In Section 4 we deduce useful computational features of the symmetric form $\tilde{h}$. In Section 5 we introduce (H1), (H2) and (H3) and we establish (1.11). In Section 6 we deduce (1.12). In Section 7 we prove (a finer version of) Theorem 1.1. In Section 8 we provide the proof of the full Simons identity stated in Theorem 5.1.

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## 2. Preliminaries

2.1. **The Heisenberg group.** In the following, we denote by $\Gamma(T\mathbb{H}^n)$ and $\Gamma(\mathcal{H})$ the families of smooth vector fields and of smooth horizontal vector fields respectively. The *complex structure* $J : \Gamma(T\mathbb{H}^n) \longrightarrow \Gamma(T\mathbb{H}^n)$ is the unique $C^\infty(\mathbb{H}^n)$-linear map which satisfies

$$J(X_i) = Y_i, \qquad J(Y_i) = -X_i \qquad \text{and} \qquad J(T) = 0$$

for any $i = 1, \dots, n$. In particular, observe that

$$J(J(X)) = -X \qquad \text{and} \qquad \langle X, J(X) \rangle = 0 \tag{2.1}$$

for any $X \in \Gamma(\mathcal{H})$, whence the latter, together with the distribution $\mathcal{H}$, realizes $\mathbb{H}^n$ as a *pseudohermitian manifold* (cf. [13, Appendix]). We recall that $\langle \cdot, \cdot \rangle$ is the unique Riemannian metric which makes $X_1, \dots, X_n, Y_1, \dots, Y_n, T$ orthonormal. Restricting the latter to the horizontal distribution $\mathcal{H}$, and still denoting this restriction by $\langle \cdot, \cdot \rangle$, $\mathbb{H}^n$ inherits a *sub-Riemannian structure* which realizes it as a *sub-Riemannian manifold*. We denote by $|\cdot|$ the norm induced by $\langle \cdot, \cdot \rangle$. Moreover, we denote by $\nabla$ the so-called *pseudohermitian connection* (cf. e.g. [60]), i.e. the unique *metric connection* (cf. [29]) whose torsion tensor is

$$\nabla_X Y - \nabla_Y X - [X, Y] = 2\langle J(X), Y \rangle T \tag{2.2}$$

for any $X, Y \in \Gamma(T\mathbb{H}^n)$. Although $\nabla$ is not a torsion-free connection, it has the advantage of vanishing along left-invariant vector fields, meaning that

$$\nabla_{Z_i} Z_j = 0 \tag{2.3}$$

for any $i, j = 1, \dots, 2n + 1$ (cf. [59]). In view of (2.3), it is easy to check that

$$X \in \Gamma(T\mathbb{H}^n),\, Y \in \Gamma(\mathcal{H}) \qquad \Longrightarrow \qquad \nabla_X Y \in \Gamma(\mathcal{H}).$$

Moreover, denoting by $R$ the curvature tensor associated with $\nabla$, i.e.

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for any $X, Y, Z \in \Gamma(T\mathbb{H}^n)$, (2.3) implies that $R \equiv 0$, whence $\mathbb{H}^n$ is flat from the pseudohermitian standpoint. The pseudohermitian connection is related to the complex structure by

$$\nabla_X J(Y) = J(\nabla_X Y). \tag{2.4}$$

for any $X, Y \in \Gamma(T\mathbb{H}^n)$ (cf. e.g. [31]). Given a function $f \in C^\infty(\mathbb{H}^n)$, we denote by

$$\nabla f = \sum_{j=1}^{2n+1} (Z_j f)\, Z_j \qquad \text{and} \qquad \nabla^{\mathcal{H}} f = \sum_{j=1}^{2n} (Z_j f)\, Z_j$$

respectively the gradient and the horizontal gradient associated with the pseudohermitian connection $\nabla$.

2.2. **Carnot-Carathéodory structure.** An absolutely continuous curve $\Gamma : [a,b] \longrightarrow \mathbb{H}^n$ is called *horizontal* whenever

$$\dot{\Gamma}(t) \in \mathcal{H}_{\Gamma(t)} \tag{2.5}$$

for a.e. $t \in [a,b]$, and it is called *sub-unit* whenever it is horizontal and $|\dot{\Gamma}(t)| \leqslant 1$ for a.e. $t \in [a,b]$. If we define

$$d(p,q) := \inf\{b \,:\, \Gamma : [0,b] \longrightarrow \mathbb{H}^n \text{ is sub-unit, } \Gamma(0) = p \text{ and } \Gamma(b) = q\},$$

then Chow-Rashevskii theorem (cf. [19]) implies that $d$ is a distance on $\mathbb{H}^n$, the so-called *Carnot-Carathéodory distance*. The metric space $(\mathbb{H}^n, d)$ is a prototype of Carnot-Carathéodory space. For any $r > 0$ and any $p \in \mathbb{H}^n$, we denote by $B_r(p)$ the open metric ball of radius $r$ centered at $p$ induced by $d$.

2.3. **Perimeter and perimeter minimizers.** Let $\Omega \subseteq \mathbb{H}^n$ be open and $E \subseteq \mathbb{H}^n$ measurable. The *horizontal perimeter* (or $\mathbb{H}$-*perimeter*) of $E$ in $\Omega$ (cf. e.g. [34, 35, 37]) is defined by

$$P_{\mathbb{H}}(E,\Omega) := \sup\left\{ \int_E \operatorname{div}_{\mathcal{H}}(\bar{\varphi})\, d\mathcal{L}^{2n+1} \,:\, \bar{\varphi} \in C^1_c(\Omega,\mathcal{H}),\, |\bar{\varphi}|_p \leqslant 1 \text{ for any } p \in \Omega \right\},$$

where by $C^1_c(\Omega,\mathcal{H})$ we denote the class of compactly supported horizontal vector fields defined on $\Omega$, and where $\operatorname{div}_{\mathcal{H}}$ is the so-called *horizontal divergence*, defined by

$$\operatorname{div}_{\mathcal{H}}\left(\sum_{j=1}^n (\varphi_j X_j + \varphi_{n+j} Y_j)\right) := \sum_{j=1}^n (X_j \varphi_j + Y_j \varphi_{n+j})$$

for any $\sum_{j=1}^n (\varphi_j X_j + \varphi_{n+j} Y_j) \in C^1(\Omega,\mathcal{H})$. We say that $E$ is an $\mathbb{H}$-*Caccioppoli set* whenever $P_{\mathbb{H}}(E,\Omega) < +\infty$ for any bounded open set $\Omega \subseteq \mathbb{H}^n$. Finally, we recall (cf. e.g. [64]) that an $\mathbb{H}$-Caccioppoli set $E$ is an $\mathbb{H}$-*perimeter minimizer* in $\Omega$ whenever

$$P_{\mathbb{H}}(E,\Omega) \leqslant P_{\mathbb{H}}(F,\Omega)$$

for any $\Omega \Subset \mathbb{H}^n$ and for any $\mathbb{H}$-Caccioppoli set $F$ such that $E \Delta F \Subset \Omega$. When $E$ is an $\mathbb{H}$-perimeter minimizer in $\Omega = \mathbb{H}^n$, we refer to it as *global* $\mathbb{H}$-*perimeter minimizer*.

## 3. Geometric properties of non-characteristic hypersurfaces

3.1. **Non-characteristic hypersurfaces.** Let $S \subseteq \mathbb{H}^n$ be a smooth hypersurface without boundary. We recall (cf. e.g. [64]) that a point $p \in S$ is called *characteristic* when

$$\mathcal{H}_p = T_p S,$$

and is called *non-characteristic* otherwise. In the latter case, the *horizontal tangent space*

$$\mathcal{H}T_p S = \mathcal{H}_p \cap T_p S$$

is a $(2n-1)$-dimensional vector space. The set of characteristic points of $S$ is denoted by $S_0$ and is called the *characteristic set* of $S$. When $S_0 = \emptyset$, $S$ is called *non-characteristic*, and the *horizontal tangent distribution* $\mathcal{H}TS$ is actually a constant-rank sub-bundle of $TS$. According to the previous notation, we denote by $\Gamma(TS)$ and by $\Gamma(\mathcal{H}TS)$ the families of smooth vector fields which are tangent to $S$ and which are horizontal and tangent to $S$ respectively. In the following, unless otherwise specified, we assume that $S$ is a smooth, embedded, non-characteristic hypersurface without boundary. When our statements are of local nature, we assume without loss of generality that $S$ is two-sided. We denote by $N$ its Riemannian unit normal, and by $N^{\mathbb{H}}$ its projection onto $\mathcal{H}$, that is

$$N^{\mathbb{H}} = N - \langle N, T\rangle T.$$

Being $S$ non-characteristic, then $N^{\mathbb{H}}(p) \neq 0$ for any $p \in S$, so that the *horizontal unit normal*

$$\nu = \frac{N^{\mathbb{H}}}{|N^{\mathbb{H}}|}$$

is well-defined on the whole $S$. Notice that the horizontal unit normal can be characterized to be the unique unitary horizontal vector field which is orthogonal to any horizontal tangent vector field. When $E$ is an $\mathbb{H}$-Caccioppoli set in $\mathbb{H}^n$ with boundary of class $C^1$, it is known (cf. e.g. [23]) that

$P_{\mathbb{H}}(E,\cdot) = |N^{\mathbb{H}}|\mathcal{H}^{2n} \llcorner \partial E$, being $\mathcal{H}^{2n}$ the standard $(2n)$-dimensional Hausdorff measure. Therefore, if $S$ is a two-sided hypersurface as above, in the following we adopt the notation

$$\sigma_{\mathcal{H}} = |N^{\mathbb{H}}|\mathcal{H}^{2n} \llcorner S$$

to denote the relevant sub-Riemannian hypersurface measure as introduced e.g. in [23, 47]. Let $d^S$ be the signed Carnot-Carathéodory distance from $S$. Since $S$ is smooth and non-characteristic, then $d^S$ is smooth and satisfies the eikonal equation $|\nabla^{\mathcal{H}} d^S| = 1$ in a neighborhood of $S$ (cf. [60]). Therefore, in the following, we assume that $\nu$ is defined in a neighborhood of $S$ by

$$\nu = \nabla^{\mathcal{H}} d^S. \tag{3.1}$$

When $\nu$ is locally extended as in (3.1), it follows that

$$Z_k(\nu_h) = Z_h(\nu_k) \tag{3.2}$$

for any $h, k = 1, \dots, 2n$ such that $|h-k| \neq n$, and

$$X_k(\nu_{n+k}) = Y_k(\nu_k) - 2\alpha \qquad \text{and} \qquad Y_k(\nu_k) = X_k(\nu_{n+k}) + 2\alpha \tag{3.3}$$

for any $k = 1, \dots, n$, where here and in the following, according to [16, 12], we adopt the notation $\alpha = T d^S$. In particular, an easy computation (cf. [55]) reveals that

$$\nabla_\nu \nu = -2\alpha J(\nu), \tag{3.4}$$

Moreover, the Riemannian normal $N$ can be locally extended by letting

$$N = \frac{1}{\sqrt{1+\alpha^2}}\nu + \frac{\alpha}{\sqrt{1+\alpha^2}}T. \tag{3.5}$$

Let us provide a more precise description of the tangent space to $S$. First, (2.1) implies that $J(\nu) \in \Gamma(\mathcal{H}TS)$. Moreover, denoting by $\mathcal{H}'TS$ the distribution defined by

$$\mathcal{H}'T_pS = \mathcal{H}T_pS \cap J\left(\mathcal{H}T_pS\right)$$

for any $p \in S$, it is easy to check that it is a $(2n-2)$-dimensional sub-bundle of $\mathcal{H}TS$, and that the latter can be orthogonally decomposed as

$$\mathcal{H}TS = \mathcal{H}'TS \oplus \operatorname{span} J(\nu).$$

Finally, (3.5) implies that the vector field $\mathcal{S}$ defined by

$$\mathcal{S} = T - \alpha\nu$$

belongs to $\Gamma(TS)$ and satisfies $\langle \mathcal{S}, X\rangle = 0$ for any $X \in \Gamma(\mathcal{H}TS)$. Therefore, the tangent space to $S$ admits the orthogonal decomposition

$$TS = \mathcal{H}'TS \oplus \operatorname{span} J(\nu) \oplus \operatorname{span} \mathcal{S}.$$

In the following, we denote by $\pi : \Gamma(\mathcal{H}) \longrightarrow \Gamma(\mathcal{H}TS)$ the projection map

$$\pi(X) = X - \langle X, \nu\rangle\nu = \sum_{i=1}^{2n-1} \langle X, E_i\rangle E_i \tag{3.6}$$

for any $X \in \Gamma(\mathcal{H})$ and any local orthonormal frame $\mathrm{E}_1, \dots, \mathrm{E}_{2n-1}$ of $\mathcal{H}TS$. A simple computation shows that

$$\pi(J(X)) \in \Gamma(\mathcal{H}'TS) \qquad \text{and} \qquad J(\pi(J(X))) = -X + \langle X, J(\nu)\rangle J(\nu). \tag{3.7}$$

The horizontal unit normal $\nu$ evolves along $\mathcal{S}$ as follows.

**Proposition 3.1.** *It holds that*

$$\nabla_{\mathcal{S}}\nu = \nabla^{\mathcal{H}}\alpha + 2\alpha^2 J(\nu). \tag{3.8}$$

*In particular, if* $X \in \Gamma(\mathcal{H}TS)$*, then*

$$\langle \nabla_{\mathcal{S}}\nu, X\rangle = X\alpha + 2\alpha^2\langle J(\nu), X\rangle. \tag{3.9}$$

*Proof.* Let $\nu$ be extended as in (3.1). Then, recalling (3.4),

$$\nabla_{\mathcal{S}}\nu = \nabla_T\nu - \alpha\nabla_\nu\nu = \sum_{j=1}^{2n} T(Z_j d)Z_j + 2\alpha^2 J(\nu) = \sum_{j=1}^{2n} Z_j\alpha Z_j + 2\alpha^2 J(\nu) = \nabla^{\mathcal{H}}\alpha + 2\alpha^2 J(\nu).$$

Finally, (3.9) easily follows. □

3.2. **Tangent pseudohermitian connection.** The *tangent pseudohermitian connection*, denoted by $\nabla^S : \Gamma(TS)\times\Gamma(\mathcal{H}TS)\longrightarrow\Gamma(\mathcal{H}TS)$, is defined by

$$\nabla^S_X Y = \nabla_X Y - \langle\nabla_X Y, \nu\rangle\nu$$

for any $X\in\Gamma(TS)$ and any $Y\in\Gamma(\mathcal{H}TS)$. An easy computation reveals that $\nabla^S$ is a well-defined affine connection, and that it is metric in the sense that

$$X\langle Y, Z\rangle = \langle\nabla^S_X Y, Z\rangle + \langle Y, \nabla^S_X Z\rangle \tag{3.10}$$

for any $X\in\Gamma(TS)$ and any $Y, Z\in\Gamma(\mathcal{H}TS)$. Accordingly, the torsion tensor $\mathrm{Tor}_{\nabla^S}(X,Y):\Gamma(\mathcal{H}TS)\times\Gamma(\mathcal{H}TS)\longrightarrow\Gamma(TS)$ is defined by

$$\mathrm{Tor}_{\nabla^S}(X,Y) = \nabla^S_X Y - \nabla^S_Y X - [X,Y].$$

We stress that we are not requiring $\mathrm{Tor}_{\nabla^S}(X,Y)$ to be horizontal, so that, by Frobenious theorem, $\mathrm{Tor}_{\nabla^S}$ is well-defined. The latter admits the following explicit expression.

**Proposition 3.2.** *Let $X, Y\in\Gamma(\mathcal{H}TS)$. Then*

$$\mathrm{Tor}_{\nabla^S}(X,Y) = 2\langle J(X), Y\rangle\mathcal{S}. \tag{3.11}$$

*Proof.* Let $X, Y\in\Gamma(\mathcal{H}TS)$. If $X=\sum_{j=1}^{2n}X^jZ_j$ and $Y=\sum_{j=1}^{2n}Y^jZ_j$, then

$$\begin{aligned}
-\langle[X,Y],\nu\rangle &= \langle\mathrm{Tor}_\nabla(X,Y),\nu\rangle + \langle\nabla_Y X - \nabla_X Y,\nu\rangle\\
&\overset{(2.2)}{=} 2\langle J(X),Y\rangle\langle\nu,T\rangle + \langle\nabla_X\nu, Y\rangle - \langle\nabla_Y\nu, X\rangle\\
&\overset{(2.3)}{=} \sum_{i,j=1}^{2n} X^iY^j(Z_i\nu_j - Z_j\nu_i)\\
&\overset{(3.2),(3.3)}{=} -2\alpha\sum_{i=1}^{n} X^iY^{n+i} + 2\alpha\sum_{i=1}^{n} X^{n+i}Y^i\\
&= -2\alpha\langle J(X),Y)\rangle.
\end{aligned}$$

In particular,

$$\mathrm{Tor}_{\nabla^S}(X,Y) = \mathrm{Tor}_\nabla(X,Y) - \langle\mathrm{Tor}_\nabla(X,Y),\nu\rangle\nu - \langle[X,Y],\nu\rangle\nu = 2\langle J(X),Y\rangle\mathcal{S}.$$

This concludes the proof. □

The following corollary of Proposition 3.2 will be crucial in the following developments.

**Corollary 3.3.** *Let $X\in\Gamma(\mathcal{H}'TS)$. Then*

$$[J(\nu), X] = \nabla^S_{J(\nu)}X - \nabla^S_X J(\nu)\in\Gamma(\mathcal{H}TS). \tag{3.12}$$

*In particular*

$$\langle[J(\nu),X],X\rangle = \langle\nabla_X\nu, J(X)\rangle. \tag{3.13}$$

If $f\in C^\infty(S)$, we denote by

$$\nabla^{\mathcal{H},S}f = \sum_{j=1}^{2n-1}(\mathrm{E}_j f)\,\mathrm{E}_j$$

the horizontal tangential gradient associated with the connection $\nabla^S$, where $\mathrm{E}_1,\dots,\mathrm{E}_{2n-1}$ is any local orthonormal frame of $\mathcal{H}TS$. More generally, if $p\in\mathbb{N}$ and $T$ is a horizontal $(p,0)$-tensor field (cf. [29]), meaning that $T:\Gamma(\mathcal{H}TS)^p\longrightarrow C^\infty(S)$ is a $C^\infty(S)$-multilinear map, the $(p+1,0)$ tensor field $\nabla^S T:\Gamma(TS)\times\Gamma(\mathcal{H}TS)^p\longrightarrow C^\infty(S)$ is defined by

$$\nabla^S_X T(X_1,\dots,X_p) = X\left(T(X_1,\dots,X_p)\right) - T\left(\nabla^S_X X_1,\dots,X_p\right) - \ldots - T\left(X_1,\dots,\nabla^S_X X_p\right)$$

for any $X \in \Gamma(TS)$ and any $X_1, \ldots, X_p \in \Gamma(\mathcal{H}TS)$. According to the above notation, we denote by $\nabla^{\mathcal{H},S}T$ the restriction of $\nabla^S T$ to $\Gamma(\mathcal{H}TS)^{p+1}$. As a general fact, $\nabla^S$ verifies the Leibniz-type rule

$$\nabla^S_X(TU) = U\nabla^S_X T + T\nabla^S_X U \tag{3.14}$$

for any $X \in \Gamma(TS)$ and any couple of tensor fields $T, U$. If $T$ is as above, we denote its squared norm by

$$|T|^2 = \sum_{i_1,\ldots,i_p=1}^{2n-1} T\left(\mathrm{E}_{i_1}, \ldots, \mathrm{E}_{i_p}\right)^2$$

for any local orthonormal frame $\mathrm{E}_1, \ldots, \mathrm{E}_{2n-1}$ of $\mathcal{H}TS$. The *horizontal tangential Hessian* of $T$ is the $(p+2,0)$-tensor field $\operatorname{Hess}^{\mathcal{H},S} T : \Gamma(\mathcal{H}TS)^{p+2} \longrightarrow C^\infty(S)$ defined by

$$\operatorname{Hess}^{\mathcal{H},S} T(X, Y, X_1, \ldots, X_p) = \nabla^S_X \nabla^S_Y T(X_1, \ldots, X_p)$$

for any $X, Y, X_1, \ldots, X_p \in \Gamma(\mathcal{H}TS)$. Finally, the *horizontal tangential Laplacian* of $T$ is the $(p,0)$-tensor field $\Delta^{\mathcal{H},S}T : \Gamma(\mathcal{H}TS)^p \longrightarrow C^\infty(S)$ defined by

$$\Delta^{\mathcal{H},S}T(X_1, \ldots, X_p) = \operatorname{trace}\operatorname{Hess}^{\mathcal{H},S} T(\cdot,\cdot, X_1, \ldots, X_p) = \sum_{j=1}^{2n-1} \operatorname{Hess}^{\mathcal{H},S} T(\mathrm{E}_j, \mathrm{E}_j, X_1, \ldots, X_p)$$

for any $X_1, \ldots, X_p \in \Gamma(\mathcal{H}TS)$ and any local orthonormal frame $\mathrm{E}_1, \ldots, \mathrm{E}_{2n-1}$ of $\mathcal{H}TS$. We denote by $R^S : \Gamma(\mathcal{H}TS) \times \Gamma(\mathcal{H}TS) \times \Gamma(\mathcal{H}TS) \longrightarrow \Gamma(\mathcal{H}TS)$ the horizontal curvature tensor associated with $\nabla^S$, that is

$$R^S(X,Y)Z = \nabla^S_X \nabla^S_Y Z - \nabla^S_Y \nabla^S_X Z - \nabla^S_{[X,Y]} Z$$

for any $X, Y, Z \in \Gamma(\mathcal{H}TS)$, and with some abuse of notation we let

$$R^S(X,Y,Z,W) = \langle \nabla^S_X \nabla^S_Y Z - \nabla^S_Y \nabla^S_X Z - \nabla^S_{[X,Y]} Z, W\rangle$$

for any $X, Y, Z, W \in \Gamma(\mathcal{H}TS)$. The horizontal Hessian of horizontal tensor fields is affected by $R^S$ as follows.

**Proposition 3.4.** *Let $T : \Gamma(\mathcal{H}TS) \times \Gamma(\mathcal{H}TS) \longrightarrow C^\infty(S)$ be a $(2,0)$-tensor field. Then*

$$\begin{aligned} \operatorname{Hess}^{\mathcal{H},S} T(Y,X,Z,W) &= \operatorname{Hess}^{\mathcal{H},S} T(X,Y,Z,W) \\ &\quad + T(R^S(X,Y)Z, W) + T(Z, R^S(X,Y)W) \\ &\quad + 2\langle J(X), Y\rangle \left(\nabla^S_S T\right)(Z,W) \end{aligned} \tag{3.15}$$

*for any $X, Y, Z, W \in \Gamma(\mathcal{H}TS)$.*

*Proof.* Let

$$S(A,B,C) = \nabla^S_A T(B,C) = A(T(B,C)) - T(\nabla^S_A B, C) - T(B, \nabla^S_A C).$$

Then

$$\begin{aligned} \operatorname{Hess}^{\mathcal{H},S} T(X,Y,Z,W) &= X(S(Y,Z,W)) - S(\nabla^S_X Y, Z, W) - S(Y, \nabla^S_X Z, W) - S(Y, Z, \nabla^S_X W) \\ &= XY(T(Z,W)) - X(T(\nabla^S_Y Z, W)) - X(T(Z, \nabla^S_Y W)) \\ &\quad - \nabla^S_X Y(T(Z,W)) + T(\nabla^S_{\nabla^S_X Y} Z, W) + T(Z, \nabla^S_{\nabla^S_X Y} W) \\ &\quad - Y(T(\nabla^S_X Z, W)) + T(\nabla^S_Y \nabla^S_X Z, W) + T(\nabla^S_X Z, \nabla^S_Y W) \\ &\quad - Y(T(Z, \nabla^S_X W)) + T(\nabla^S_Y Z, \nabla^S_X W) + T(Z, \nabla^S_Y \nabla^S_X W) \end{aligned}$$

and, in the same way,

$$\begin{aligned} \operatorname{Hess}^{\mathcal{H},S} T(Y,X,Z,W) &= YX(T(Z,W)) - Y(T(\nabla^S_X Z, W)) - Y(T(Z, \nabla^S_X W)) \\ &\quad - \nabla^S_Y X(T(Z,W)) + T(\nabla^S_{\nabla^S_Y X} Z, W) + T(Z, \nabla^S_{\nabla^S_Y X} W) \\ &\quad - X(T(\nabla^S_Y Z, W)) + T(\nabla^S_X \nabla^S_Y Z, W) + T(\nabla^S_Y Z, \nabla^S_X W) \\ &\quad - X(T(Z, \nabla^S_Y W)) + T(\nabla^S_X Z, \nabla^S_Y W) + T(Z, \nabla^S_X \nabla^S_Y W). \end{aligned}$$

Hence

$$\begin{aligned}\mathrm{Hess}^{\mathcal{H},S}T(Y,X,Z,W)-\mathrm{Hess}^{\mathcal{H},S}T(X,Y,W,Z)&=\mathrm{Tor}^S_\nabla(X,Y)(T(Z,W))\\&\quad+T(\nabla^S_X\nabla^S_YZ-\nabla^S_Y\nabla^S_XZ-\nabla^S_{\nabla^S_XY-\nabla^S_YX}Z,W)\\&\quad+T(Z,\nabla^S_X\nabla^S_YW-\nabla^S_Y\nabla^S_XW-\nabla^S_{\nabla^S_XY-\nabla^S_YX}W)\\&=\mathrm{Tor}_{\nabla^S}(X,Y)(T(Z,W))-T(\nabla^S_{\mathrm{Tor}_{\nabla^S}(X,Y)}Z,W)-T(Z,\nabla^S_{\mathrm{Tor}_{\nabla^S}(X,Y)}W)\\&\quad+T(R^S(X,Y)Z,W)+T(Z,R^S(X,Y)W)\\&=\nabla^S_{\mathrm{Tor}_{\nabla^S}(X,Y)}T(Z,W)+T(R^S(X,Y)Z,W)+T(Z,R^S(X,Y)W)\\&\overset{(3.11)}{=}2\langle J(X),Y\rangle\nabla^S_ST(Z,W)+T(R^S(X,Y)Z,W)+T(Z,R^S(X,Y)W).\end{aligned}$$

□

3.3. **Second fundamental forms and mean curvature.** In the current literature, different types of second fundamental form are available in the sub-Riemannian Heisenberg group (cf. e.g. [44, 23, 16, 59, 60]). The *horizontal shape operator* $A:\Gamma(TS)\longrightarrow\Gamma(\mathcal{H}TS)$ and the *symmetric horizontal shape operator* $\tilde{A}:\Gamma(\mathcal{H}TS)\longrightarrow\Gamma(\mathcal{H}TS)$ are defined respectively by

$$A(X)=\nabla_X\nu\qquad\text{and}\qquad\tilde{A}(X)=\nabla_X\nu+\alpha J'(X)$$

for any $X\in\Gamma(\mathcal{H}TS)$, where

$$J'=J\text{ on }\mathcal{H}'TS\qquad\text{and}\qquad J'(J(\nu))=0.$$

It is easy to check that $A$ and $\tilde{A}$ are well-defined. Accordingly, the *horizontal second fundamental form* $h$ and the *symmetric horizontal second fundamental form* $\tilde{h}$ are the horizontal $(2,0)$-tensor fields defined by

$$h(X,Y)=\langle A(X),Y\rangle\qquad\text{and}\qquad\tilde{h}(X,Y)=\langle\tilde{A}(X),Y\rangle$$

for any $X,Y\in\Gamma(\mathcal{H}TS)$. As in the Riemannian setting, the horizontal second fundamental form $h$ relates the connections $\nabla$ and $\nabla^S$ by the identity

$$\nabla_XY=\nabla^S_XY-h(X,Y)\nu$$

for any $X,Y\in\Gamma(\mathcal{H}TS)$. It is well known that $\tilde{h}$ is symmetric, while, when $n\geqslant 2$, $h$ may not be symmetric (cf. e.g. [23, 59]). More precisely, $h$ and $\tilde{h}$ are related in the following way.

**Proposition 3.5.** *Let $X,Y\in\Gamma(\mathcal{H}TS)$. Then*

$$\tilde{h}(X,Y)=h(X,Y)+\alpha C(X,Y)=\frac{h(X,Y)+h(Y,X)}{2},\tag{3.16}$$

*where $C:\Gamma(\mathcal{H}TS)\times\Gamma(\mathcal{H}TS)\longrightarrow C^\infty(S)$, which we will refer to as* commutation tensor*, is the skew-symmetric horizontal $(2,0)$-tensor field defined by*

$$C(X,Y)=\langle J(X),Y\rangle.$$

*In particular*

$$h(Y,X)=h(X,Y)+2\alpha C(X,Y).\tag{3.17}$$

*Finally,*

$$|h|^2=|\tilde{h}|^2+2(n-1)\alpha^2.\tag{3.18}$$

*Proof.* Let $X,Y\in\Gamma(\mathcal{H}TS)$. By definition of $h$ and $\tilde{h}$ we have that

$$\tilde{h}(X,Y)=h(X,Y)+\alpha C(X,Y).$$

Moreover, since $\tilde{h}$ is symmetric and $C$ is skew-symmetric in view of (2.1), then

$$\tilde{h}(X,Y)=\tilde{h}(Y,X)=h(Y,X)+\alpha C(Y,X)=h(Y,X)-\alpha C(X,Y),$$

whence (3.16) and (3.17) follow. Finally, (3.18) follows from [55, Proposition 5.4]. □

According to its Riemannian counterpart, the *horizontal mean curvature* $H$ is then defined by

$$H = \operatorname{trace} h = \operatorname{trace} \tilde{h} = \operatorname{div}_{\mathcal{H}} \nu, \tag{3.19}$$

the last identity following from [23]. In the following, we say that $S$ is *minimal* whenever $H \equiv 0$.

**Remark 3.6.** Although we have defined the horizontal shape operator $A$ only for horizontal tangent vector fields, we stress that $A(\mathcal{S})$ is nevertheless well-defined, and admits an explicit expression in view of (3.8). Therefore, with a slight abuse of notation, in the following we shall use the notation $h(\mathcal{S}, X)$ for a given $X \in \Gamma(\mathcal{H}TS)$.

3.4. **Eigenvectors and umbilicity.** Fix $p \in S$. Since $\tilde{h}_p$ is symmetric, then it is diagonalizable. Therefore, in the following, we denote by $\mathrm{F}_1, \ldots, \mathrm{F}_{2n-1}$ any local orthonormal frame of $\mathcal{H}TS$ around $p$ such that

$$\tilde{h}_p(\mathrm{F}_i|_p, \mathrm{F}_j|_p) = \lambda_i \delta_{i,j} \tag{3.20}$$

for any $i, j = 1, \ldots, 2n-1$, where $\lambda_1, \ldots, \lambda_{2n-1}$ are the eigenvalues of $\tilde{h}_p$. When $S$ satisfies milder assumptions, more can be said about $\mathrm{F}_1, \ldots, \mathrm{F}_{2n-1}$. According to [16, 12], in the following we adopt the notation $\ell = h(J(\nu), J(\nu))$, and we let $\mathfrak{X} = \nabla_{J(\nu)}\nu - \ell J(\nu)$. As we know from [12], if $p \in S$, then $\mathfrak{X}|_p = 0$ if and only if $\tilde{A}|_p(\mathcal{H}'T_pS) \subseteq \mathcal{H}'T_pS$. In particular, when

$$\mathfrak{X} \equiv 0, \tag{H1}$$

$J(\nu)|_p$ is an eigenvector of $\tilde{h}_p$ for any $p \in S$. Therefore, when (H1) holds and $\mathrm{F}_1, \ldots, \mathrm{F}_{2n-1}$ is as in (3.20), we will always assume that

$$\mathrm{F}_1, \ldots, \mathrm{F}_{n-1}, \ldots, \mathrm{F}_{n+1}, \ldots, \mathrm{F}_{2n-1} \in \Gamma(\mathcal{H}'TS) \qquad \text{and} \qquad \mathrm{F}_n = J(\nu). \tag{3.21}$$

A relevant class of hypersurfaces which satisfy (H1) is that of *umbilic hypersurfaces* introduced in [12]. We recall that $S$ is called umbilic when (H1) holds and

$$\lambda_1 = \ldots = \lambda_{n-1} = \lambda_{n+1} = \ldots = \lambda_{2n-1} =: \lambda$$

for any $p \in S$, where $\lambda = \lambda(p)$. We collect some basic properties of umbilic hypersurfaces which will be useful in the sequel.

**Proposition 3.7.** *Let $S$ be umbilic. Then*

$$\nabla^{\mathcal{H},S}\alpha = \langle \nabla\alpha, J(\nu)\rangle J(\nu), \tag{3.22}$$

$$\langle \nabla\alpha, J(\nu)\rangle = \lambda^2 - \alpha^2 - \lambda\ell. \tag{3.23}$$

$$|\tilde{h}|^2 = \ell^2 - \lambda\ell + H\lambda, \tag{3.24}$$

*and*

$$\ell^2 = \frac{2n-2}{2n-1}|\tilde{h}|^2 - \frac{2n-3}{2n-1}H\ell. \tag{3.25}$$

*Assume in addition that $S$ is minimal. Then*

$$\langle \nabla^S\alpha, J(\nu)\rangle = \frac{1}{2n-1}|\tilde{h}|^2 - \alpha^2 \geqslant -\alpha^2. \tag{3.26}$$

*Proof.* First, (3.22) and (3.23) follow from [12, Proposition 4.2]. Recalling that

$$H = (2n-2)\lambda + \ell \qquad \text{and} \qquad |\tilde{h}|^2 = (2n-2)\lambda^2 + \ell^2,$$

(3.24) and (3.25) follow. Finally, if $H \equiv 0$, (3.26) follows from (3.23), (3.24) and (3.25). $\square$

3.5. **Tangential Laplace-Beltrami operators.** The general approach described in Section 3.2 allows to associate to a function $f \in C^\infty(S)$ a natural notion of tangential Laplacian, namely

$$\Delta^{\mathcal{H},S} f = \operatorname{trace}\operatorname{Hess}^{\mathcal{H},S} f. \tag{3.27}$$

On the other hand, the authors of [23] considered a Laplace-Beltrami operator on $S$ of the form

$$\sum_{i=1}^{2n} \nabla_i^{\mathbb{H},S}\nabla_i^{\mathbb{H},S} f, \tag{3.28}$$

where the *horizontal tangential derivatives*

$$\nabla_i^{\mathbb{H},S} f = Z_i f - \langle \nabla^{\mathcal{H}} f, \nu\rangle \nu_i$$

for any $i = 1, \dots, 2n$ do not depend on the chosen smooth extension of $f$ (cf. [23]). As pointed out in [23], the operator defined in (3.28), differently from the Riemannian framework, is not in general self-adjoint. To this aim, the authors of [23] introduced a *modified* version of (3.28), the so-called *modified horizontal tangential Laplacian*

$$\hat{\Delta}^{\mathcal{H},S} f = \sum_{i=1}^{2n} \nabla_i^{\mathbb{H},S}\nabla_i^{\mathbb{H},S} f + 2\alpha \left\langle \nabla^{\mathcal{H}} f, J(\nu)\right\rangle . \tag{3.29}$$

The most relevant feature of $\hat{\Delta}^{\mathcal{H},S}$ is that it is indeed self-adjoint (cf. [23, Corollary 11.4]), so that the following integration-by-parts formula holds.

**Proposition 3.8.** *Let $\varphi \in C^1_c(S)$ and $\psi \in C^2(S)$. Then*

$$\int_S \varphi\, \hat{\Delta}^{\mathcal{H},S}\psi \, d\sigma_{\mathcal{H}} = -\int_S \langle \nabla^{\mathcal{H},S}\varphi, \nabla^{\mathcal{H},S}\psi\rangle \, d\sigma_{\mathcal{H}} \tag{3.30}$$

In order to exploit (3.30), in the next proposition we show that (3.27) and (3.28) agree.

**Proposition 3.9.** *Let $f \in C^2(S)$. Set $g_{\mathbb{H}}^{i,j} = \delta_{i,j} - \nu_i\nu_j$ for any $i, j = 1, \dots, 2n$. Then*

$$\Delta^{\mathcal{H},S} f = \sum_{i=1}^{2n} \nabla_i^{\mathbb{H},S}\nabla_i^{\mathbb{H},S} f = \sum_{i,j=1}^{2n} g_{\mathbb{H}}^{i,j} Z_i Z_j f - H\langle \nabla^{\mathcal{H}} f, \nu\rangle,$$

*Proof.* Recalling that $|\nu| = 1$,

$$\begin{aligned}
\sum_{i=1}^{2n} \nabla_i^{\mathbb{H},S}\nabla_i^{\mathbb{H},S} f &= \sum_{i=1}^{2n} Z_i Z_i f - \langle \nabla^{\mathcal{H}} f, \nu\rangle \sum_{i=1}^{2n} Z_i\nu_i - \sum_{i,j=1}^{2n} Z_i Z_j f \nu_i\nu_j - \sum_{i,j=1}^{2n} Z_j f Z_i\nu_j\nu_i \\
&\quad - \sum_{i}^{2n} \langle \nabla^{\mathcal{H}} Z_i f, \nu\rangle\nu_i + \sum_{i=1}^{2n} \langle \nabla^{\mathcal{H}}(\langle \nabla^{\mathcal{H}} f, \nu\rangle \nu_i), \nu\rangle \nu_i \\
&\overset{(3.19)}{=} \sum_{i,j=1}^{2n} g_{\mathbb{H}}^{i,j} Z_i Z_j f - H\langle \nabla^{\mathcal{H}} f, \nu\rangle - \sum_{i,j=1}^{2n} Z_j f Z_i \nu_j\nu_i - \sum_{i,j=1}^{2n} Z_j Z_i f \nu_i\nu_j \\
&\quad + \sum_{i,j,k=1}^{2n} Z_j Z_k f (\nu_i)^2 \nu_j\nu_k + \sum_{i,j,k=1}^{2n} Z_k f Z_j \nu_k (\nu_i)^2\nu_j + \sum_{i,j,k=1}^{2n} Z_k f \nu_k Z_j \nu_i\nu_i\nu_j \\
&= \sum_{i,j=1}^{2n} g_{\mathbb{H}}^{i,j} Z_i Z_j f - H\langle \nabla^{\mathcal{H}} f, \nu\rangle - \sum_{i,j=1}^{2n} Z_j f Z_i \nu_j\nu_i - \sum_{i,j=1}^{2n} Z_j Z_i f \nu_i\nu_j \\
&\quad + \sum_{j,k=1}^{2n} Z_j Z_k f \nu_j\nu_k + \sum_{j,k=1}^{2n} Z_k f Z_j \nu_k\nu_j \\
&= \sum_{i,j=1}^{2n} g_{\mathbb{H}}^{i,j} Z_i Z_j f - H\langle \nabla^{\mathcal{H}} f, \nu\rangle.
\end{aligned}$$

Let now $\mathrm{E}_1, \ldots, \mathrm{E}_{2n-1}$ be a local orthonormal frame of $\mathcal{H}TS$. Set $a_i^j = \langle \mathrm{E}_i, Z_j\rangle$ for any $i = 1, \ldots, 2n-1$ and any $j = 1, \ldots, 2n$. Then

$$\sum_{k=1}^{2n} a_i^k a_j^k = \delta_{ij}, \qquad \sum_{k=1}^{2n} a_i^k \nu_k = 0 \qquad \text{and} \qquad \sum_{k=1}^{2n-1} a_k^l a_k^m = g_{\mathbb{H}}^{l,m} \tag{3.31}$$

for any $i, j = 1, \ldots, 2n-1$ and any $l, m = 1, \ldots, 2n$. Denoting by $df(\cdot) = \langle \nabla f, \cdot\rangle$, we conclude that

$$\begin{aligned}
\Delta^{\mathcal{H},S} f &= \sum_{j=1}^{2n-1} \nabla^S_{E_j} \nabla^S_{E_j} f \\
&= \sum_{j=1}^{2n-1} \nabla^S_{E_j} df(E_j) \\
&= \sum_{j=1}^{2n-1} E_j(E_j f) - \sum_{j=1}^{2n-1} \nabla^S_{E_j} E_j f \\
&= \sum_{j=1}^{2n-1} \sum_{h,k=1}^{2n} a_j^h Z_h(a_j^k Z_k f) - \sum_{j=1}^{2n-1} \langle \nabla^S_{E_j} E_j, \nabla f\rangle \\
&= \sum_{j=1}^{2n-1} \sum_{h,k=1}^{2n} a_j^h a_j^k Z_h Z_k f + \sum_{j=1}^{2n-1} \sum_{h,k=1}^{2n} a_j^h Z_h(a_j^k) Z_k f \\
&\quad - \sum_{j=1}^{2n-1} \langle \nabla_{E_j} E_j, \nabla f\rangle + \sum_{j=1}^{2n-1} \langle \nabla_{E_j} E_j, \nu\rangle \langle \nabla f, \nu\rangle \\
&\overset{(3.31)}{=} \sum_{h,k=1}^{2n} g_{\mathbb{H}}^{h,k} Z_h Z_k f + \sum_{j=1}^{2n-1} \sum_{h,k=1}^{2n} a_j^h Z_h(a_j^k) Z_k f \\
&\quad - \sum_{j=1}^{2n-1} \sum_{h,k=1}^{2n} a_j^h \langle \nabla_{Z_h} a_j^k Z_k, \nabla f\rangle - H\langle \nabla f, \nu\rangle \\
&= \sum_{h,k=1}^{2n} g_{\mathbb{H}}^{h,k} Z_h Z_k f - H\langle \nabla f, \nu\rangle,
\end{aligned}$$

whence the thesis follows. □

3.6. **The commutation tensor.** We know from Proposition 3.5 how the commutation tensor $C$ intervenes in the lack of commutativity of $h$. Next we discuss how it affects the commutation of the covariant derivative of $h$. First, $C$ evolves along tangent vector fields as follows.

**Proposition 3.10.** *Let $X \in \Gamma(TS)$ and let $Y, Z \in \Gamma(\mathcal{H}TS)$. Then*

$$\left(\nabla^S_X C\right)(Y, Z) = C(Z, \nu) h(X, Y) - C(Y, \nu) h(X, Z). \tag{3.32}$$

*Proof.* Let $X, Y, Z$ be as in the statement. Then, by (2.4),

$$\begin{aligned}
\nabla^S_X C(Y, Z) &= X(C(Y, Z)) - C(\nabla^S_X Y, Z) - C(Y, \nabla^S_X Z) \\
&= X\langle J(Y), Z\rangle + \langle \nabla^S_X Y, J(Z)\rangle - \langle J(Y), \nabla^S_X Z\rangle \\
&= \langle \nabla_X J(Y), Z\rangle + \langle J(Y), \nabla_X Z\rangle + \langle \nabla_X Y, J(Z)\rangle - \langle J(Y), \nabla_X Z\rangle \\
&\quad - \langle \nabla_X Y, \nu\rangle \langle J(Z), \nu\rangle + \langle \nabla_X Z, \nu\rangle \langle J(Y), \nu\rangle \\
&= C(Z, \nu) h(X, Y) - C(Y, \nu) h(X, Z).
\end{aligned}$$

□

With the aid of Proposition 3.5 and Proposition 3.10, we describe the lack of commutativity of $\nabla^S h$ in its last two entries.

**Proposition 3.11.** *Let $X, Y, Z \in \Gamma(\mathcal{H}TS)$. Then*

$$\nabla^S_X h(Y,Z) = \nabla^S_X h(Z,Y) + 2(X\alpha)C(Z,Y) + 2\alpha C(Y,\nu)h(X,Z) - 2\alpha C(Z,\nu)h(X,Y). \tag{3.33}$$

*Proof.* Fix $X, Y, Z$ as in the statement. Then, by Proposition 3.5 and Proposition 3.10,

$$\begin{aligned}\nabla^S_X h(Y,Z) &= X(h(Y,Z)) - h(\nabla^S_X Y, Z) - h(Y, \nabla^S_X Z)\\ &= X(h(Z,Y)) + 2X\alpha C(Z,Y) + 2\alpha X(C(Z,Y))\\ &\quad - h(Z,\nabla^S_X Y) - 2\alpha C(Z,\nabla^S_X Y) - h(\nabla^S_X Z, Y) - 2\alpha C(\nabla^S_X Z, Y)\\ &= \nabla^S_X h(Z,Y) + 2X\alpha C(Z,Y) + 2\alpha \nabla^S_X C(Z,Y)\\ &= \nabla^S_X h(Z,Y) + 2X\alpha C(Z,Y) + 2\alpha C(Y,\nu)h(X,Z) - 2\alpha C(Z,\nu)h(X,Y).\end{aligned}$$

□

3.7. **Gauss-Codazzi equations.** With the next result we derive the sub-Riemannian counterpart of the classical Gauss-Codazzi equations. We refer to [62] for a proof, which we include anyway for the sake of completeness.

**Proposition 3.12** (Gauss-Codazzi equations)**.** *Let $X, Y, Z, W \in \Gamma(\mathcal{H}TS)$. Then the* Gauss equation

$$R^S(X,Y,Z,W) = h(Y,Z)h(X,W) - h(X,Z)h(Y,W) \tag{3.34}$$

*and the* Codazzi equation

$$\left(\nabla^S_Y h\right)(X,Z) = \left(\nabla^S_X h\right)(Y,Z) + 2C(X,Y)h(\mathcal{S},Z) \tag{3.35}$$

*hold.*

*Proof.* We know that

$$\nabla_X Y = \nabla^S_X Y - h(X,Y)\nu.$$

Hence

$$\begin{aligned}\nabla_X\nabla_Y Z &= \nabla_X \nabla^S_Y Z - \nabla_X(h(Y,Z)\nu)\\ &= \nabla^S_X\nabla^S_Y Z - h(X,\nabla^S_Y Z)\nu - X(h(Y,Z))\nu - h(Y,Z)A(X).\end{aligned}$$

Similarly,

$$-\nabla_Y\nabla_X Z = -\nabla^S_Y\nabla^S_X Z + h(Y,\nabla^S_X Z)\nu + Y(h(X,Z))\nu + h(X,Z)A(Y)$$

Moreover,

$$-\nabla_{[X,Y]}Z = -\nabla^S_{[X,Y]}Z + h([X,Y],Z)\nu$$

Summing the three equations term by term we get that

$$\begin{aligned}0 &\overset{R\equiv 0}{=} R^S(X,Y)Z + h(X,Z)A(Y) - h(Y,Z)A(X)\\ &\quad + \left(Y(h(X,Z)) - h(\nabla^S_Y X, Z) - h(X,\nabla^S_Y Z)\right)\nu\\ &\quad \left(-X(h(Y,Z)) + h(\nabla^S_X Y, Z) + h(Y,\nabla^S_X Z)\right)\nu\\ &\quad + h(\nabla^S_Y X - \nabla^S_X Y + [X,Y], Z)\nu\\ &= R^S(X,Y)Z + h(X,Z)A(Y) - h(Y,Z)A(X)\\ &\quad + \nabla^S_Y h(X,Z)\nu - \nabla^S_X h(Y,Z)\nu - h(\mathrm{Tor}_{\nabla^S}(X,Y),Z)\nu\\ &= R^S(X,Y)Z + h(X,Z)A(Y) - h(Y,Z)A(X)\\ &\quad + \nabla^S_Y h(X,Z)\nu - \nabla^S_X h(Y,Z)\nu - 2\langle J(X),Y\rangle h(S,Z)\nu.\end{aligned}$$

The thesis follows projecting the previous identity either on $W$ or on $\nu$. □

In the following, we shall also need the following Codazzi equation for the symmetric form $\tilde{h}$.

**Proposition 3.13** (Codazzi equation for $\tilde{h}$)**.** *Let $X, Y, Z \in \Gamma(\mathcal{H}TS)$. Then*

$$\begin{aligned}\nabla^S_Y\tilde{h}(X,Z) - \nabla^S_X\tilde{h}(Y,Z) &= 2(Z\alpha)C(X,Y) + (Y\alpha)C(X,Z) - (X\alpha)C(Y,Z)\\ &\quad + 2\alpha^2 C(\nu,Z)C(X,Y) + \alpha C(\nu,X)h(Y,Z) - \alpha C(\nu,Y)h(X,Z).\end{aligned}$$

*Proof.* In view of Proposition 3.5, Proposition 3.10 and Proposition 3.12,

$$\begin{aligned}
\nabla^S_Y \tilde{h}(X,Z) - \nabla^S_X \tilde{h}(Y,Z) &= \nabla^S_Y h(X,Z) - \nabla^S_X h(Y,Z) + \nabla^S_Y(\alpha C(X,Z)) - \nabla^S_X(\alpha C(Y,Z)) \\
&= 2C(X,Y)h(\mathcal{S},Z) + (Y\alpha)C(X,Z) - (X\alpha)C(Y,Z) \\
&\quad + \alpha\nabla^S_Y C(X,Z) - \alpha\nabla^S_X C(Y,Z) \\
&\overset{(3.9)}{=} 2(Z\alpha)C(X,Y) + 4\alpha^2 C(X,Y)C(\nu,Z) + (Y\alpha)C(X,Z) - (X\alpha)C(Y,Z) \\
&\quad + \alpha C(Z,\nu)h(Y,X) - \alpha C(X,\nu)h(Y,Z) \\
&\quad - \alpha C(Z,\nu)h(X,Y) + \alpha C(Y,\nu)h(X,Z) \\
&= 2(Z\alpha)C(X,Y) + 4\alpha^2 C(X,Y)C(\nu,Z) + (Y\alpha)C(X,Z) - (X\alpha)C(Y,Z) \\
&\quad + 2\alpha^2 C(Z,\nu)C(X,Y) - \alpha C(X,\nu)h(Y,Z) + \alpha C(Y,\nu)h(X,Z) \\
&= 2(Z\alpha)C(X,Y) + (Y\alpha)C(X,Z) - (X\alpha)C(Y,Z) \\
&\quad + 2\alpha^2 C(\nu,Z)C(X,Y) + \alpha C(\nu,X)h(Y,Z) - \alpha C(\nu,Y)h(X,Z).
\end{aligned}$$

□

**Corollary 3.14.** *Let $X, Y \in \Gamma(\mathcal{H}TS)$. Then*

$$\begin{aligned}
\nabla^S_X \tilde{h}(Y,Y) &= \nabla^S_Y \tilde{h}(X,Y) + 3Y\alpha C(Y,X) + 3\alpha^2\langle J(\nu), Y\rangle C(Y,X) \\
&\quad + \alpha\langle J(\nu), Y\rangle \tilde{h}(Y,X) - \alpha\langle J(\nu), X\rangle \tilde{h}(Y,Y).
\end{aligned} \tag{3.36}$$

3.8. **Variation formulas.** Let $S \subseteq \mathbb{H}^n$ be a smooth, embedded, non-characteristic hypersurface without boundary. Assume in addition that $S$ is two-sided. Let $\Omega \subseteq \mathbb{H}^n$ be an open bounded set such that $\Omega \cap S \neq \emptyset$, and let $\xi \in C^1_c(\Omega)$. Then it is known (cf. [49, 21, 62, 67]) that

$$\frac{d}{dt}\sigma_{\mathcal{H},t}(\Omega)\Big|_{t=0} = \int_S H\xi \, d\sigma_{\mathcal{H}} \tag{3.37}$$

and

$$\frac{d^2}{dt^2}\sigma_{\mathcal{H},t}(\Omega)\Big|_{t=0} = \int_S \left(|\nabla^{\mathcal{H},S}\xi|^2 - \xi^2\left(q - H^2\right)\right) d\sigma_{\mathcal{H}}, \tag{3.38}$$

where

$$q = \sum_{h,k=1}^{2n} Z_h(\nu_k)Z_k(\nu_h) + 4\langle\nabla\alpha, J(\nu)\rangle + 4n\alpha^2 \tag{3.39}$$

and where by $\sigma_{\mathcal{H},t}$ we denote the horizontal surface measure associated with the smooth variation $E_t$ along the vector field $\xi\nu$. Observe that $q$ does not depend on the chosen unitary extension of $\nu$ (cf. [55]). Moreover, in view of [55, Proposition 5.1] and (3.18),

$$q = |h|^2 + 4\langle\nabla\alpha, J(\nu)\rangle + 4\alpha^2 = |\tilde{h}|^2 + 4\langle\nabla\alpha, J(\nu)\rangle + 2(n+1)\alpha^2. \tag{3.40}$$

As customary (cf. e.g. [36]) we say that $S$ is *area stationary* whenever the quantity in (3.37) vanishes for any $\Omega$ and $\xi$ as above, and that $S$ is *stable* if it is area stationary and the quantity in (3.38) is non-negative for any $\Omega$ and $\xi$ as above. Notice that $S$ is minimal if and only if it is area stationary. In particular, when $S$ is stable, the *stability inequality*

$$\int_S q\xi^2 \, d\sigma_{\mathcal{H}} \leqslant \int_S |\nabla^{\mathcal{H},S}\xi|^2 \, d\sigma_{\mathcal{H}} \tag{3.41}$$

holds for any $\xi \in C^1_c(S)$.

## 4. Further properties of the second fundamental forms

In this section we establish some additional properties of $h$ and $\tilde{h}$ which will be useful in the next section. Although we do believe that many of them may have an independent interest, in order to facilitate a more conscious reading we would recommend the reader to skip directly to Section 5, and if necessary to go back to this section in accordance with the references to the latter.

**Proposition 4.1.** *Let* $\mathrm{F}_1, \dots, \mathrm{F}_{2n+1}$ *be as in* (3.20). *Then*

$$\tilde{h}_p(\mathrm{F}_j, J(\mathrm{F}_j)) = 0 \tag{4.1}$$

*for any* $j = 1, \dots, 2n-1$ *such that* $J(\mathrm{F}_j)|_p \in \mathcal{H}' T_p S$, *so that*

$$h_p(\mathrm{F}_j, J(\mathrm{F}_j)) = -\alpha \qquad \text{and} \qquad h_p(J(\mathrm{F}_j), \mathrm{F}_j) = \alpha \tag{4.2}$$

*for any* $j = 1, \dots, 2n-1$ *such that* $J(\mathrm{F}_j)|_p \in \mathcal{H}' T_p S$. *In addition*

$$\tilde{h}_p\left(\nabla^S_X \mathrm{F}_i, \mathrm{F}_i\right) = 0 \tag{4.3}$$

*for any* $i = 1, \dots, 2n-1$ *and any* $X \in \Gamma(TS)$. *Moreover,*

$$\sum_{i=1}^{2n-1} Y\Big(\tilde{h}\left(\nabla^S_X \mathrm{F}_i, \mathrm{F}_i\right)\Big)(p) = 0 \tag{4.4}$$

*for any* $X, Y \in \Gamma(TS)$. *Finally,*

$$\sum_{i=1}^{2n-1} \Big(\tilde{h}_p(\nabla^S_X \nabla^S_Y \mathrm{F}_i, \mathrm{F}_i) + \tilde{h}_p(\nabla^S_Y \mathrm{F}_i, \nabla^S_X \mathrm{F}_i)\Big) = 0 \tag{4.5}$$

*for any* $X, Y \in \Gamma(TS)$.

*Proof.* To prove (4.1), notice that

$$\tilde{h}(\mathrm{F}_j, J(\mathrm{F}_j)) = \sum_{k=1}^{2n-1} \langle J(\mathrm{F}_j), \mathrm{F}_k \rangle \tilde{h}(\mathrm{F}_j, \mathrm{F}_k) = \langle J(\mathrm{F}_j), \mathrm{F}_j \rangle \tilde{h}(\mathrm{F}_j, \mathrm{F}_j) = 0,$$

while (4.2) follows from (3.16). Fix $i = 1, \dots, 2n-1$. Then

$$\tilde{h}\left(\nabla^S_X \mathrm{F}_i, \mathrm{F}_i\right) = \sum_{k=1}^{2n-1} \left\langle \nabla^S_X \mathrm{F}_i, \mathrm{F}_k \right\rangle \tilde{h}\left(\mathrm{F}_k, \mathrm{F}_i\right) = \lambda_i \left\langle \nabla^S_X \mathrm{F}_i, \mathrm{F}_i \right\rangle = 0,$$

whence (4.3) follows. Moreover, (4.4) follows since

$$\begin{aligned}
\sum_{i=1}^{2n-1} Y\Big(\tilde{h}\Big(\nabla^S_X \mathrm{F}_i, \mathrm{F}_i\Big)\Big)(p) &= \sum_{i,k=1}^{2n-1} Y\Big(\left\langle \nabla^S_X \mathrm{F}_i, \mathrm{F}_k \right\rangle \tilde{h}\left(\mathrm{F}_k, \mathrm{F}_i\right)\Big)(p) \\
&= \sum_{i,k=1}^{2n-1} \tilde{h}_p(\mathrm{F}_i, \mathrm{F}_k) Y\left(\left\langle \nabla^S_X \mathrm{F}_i, \mathrm{F}_k \right\rangle\right)(p) + \sum_{i,k=1}^{2n-1} \left\langle \nabla^S_X \mathrm{F}_i, \mathrm{F}_k \right\rangle Y(\tilde{h}(\mathrm{F}_i, \mathrm{F}_k))(p) \\
&\overset{(3.20)}{=} \sum_{i=1}^{2n-1} \lambda_i Y\left(\left\langle \nabla^S_X \mathrm{F}_i, \mathrm{F}_i \right\rangle\right)(p) + \sum_{i,k=1}^{2n-1} \left\langle \nabla^S_X \mathrm{F}_i, \mathrm{F}_k \right\rangle Y(\tilde{h}(\mathrm{F}_i, \mathrm{F}_k))(p) \\
&\overset{(3.10)}{=} \sum_{i,k=1}^{2n-1} \left\langle \nabla^S_X \mathrm{F}_i, \mathrm{F}_k \right\rangle Y(\tilde{h}(\mathrm{F}_i, \mathrm{F}_k))(p) \\
&\overset{(3.10)}{=} -\sum_{i,k=1}^{2n-1} \left\langle \nabla^S_X \mathrm{F}_i, \mathrm{F}_k \right\rangle Y(\tilde{h}(\mathrm{F}_i, \mathrm{F}_k))(p) \\
&= 0,
\end{aligned}$$

the semi-last equality following by the symmetry of $\tilde{h}$. Finally,

$$\begin{aligned}
&\sum_{i=1}^{2n-1}\Big(\tilde{h}(\nabla^S_X\nabla^S_Y \mathrm{F}_i,\mathrm{F}_i)+\tilde{h}(\nabla^S_Y\mathrm{F}_i,\nabla^S_X\mathrm{F}_i)\Big)\\
&\qquad=\sum_{i,j=1}^{2n-1}\langle\nabla^S_X\nabla^S_Y\mathrm{F}_i,\mathrm{F}_j\rangle\tilde{h}(\mathrm{F}_i,\mathrm{F}_j)+\sum_{i,j,k=1}^{2n-1}\langle\nabla^S_Y\mathrm{F}_i,\mathrm{F}_j\rangle\langle\nabla^S_X\mathrm{F}_i,\mathrm{F}_k\rangle\tilde{h}(\mathrm{F}_j,\mathrm{F}_k)\\
&\qquad\overset{(3.20)}{=}\sum_{i=1}^{2n-1}\lambda_i\langle\nabla^S_X\nabla^S_Y\mathrm{F}_i,\mathrm{F}_i\rangle+\sum_{i,j=1}^{2n-1}\lambda_j\langle\nabla^S_Y\mathrm{F}_i,\mathrm{F}_j\rangle\langle\nabla^S_X\mathrm{F}_i,\mathrm{F}_j\rangle\\
&\qquad\overset{(3.10)}{=}-\sum_{i=1}^{2n-1}\lambda_i\langle\nabla^S_Y\mathrm{F}_i,\nabla^S_X\mathrm{F}_i\rangle+\sum_{i,j=1}^{2n-1}\lambda_j\langle\nabla^S_Y\mathrm{F}_i,\mathrm{F}_j\rangle\langle\nabla^S_X\mathrm{F}_i,\mathrm{F}_j\rangle\\
&\qquad=-\sum_{i,j,k=1}^{2n-1}\lambda_i\langle\nabla^S_Y\mathrm{F}_i,\mathrm{F}_j\rangle\langle\nabla^S_X\mathrm{F}_i,\mathrm{F}_k\rangle\langle\mathrm{F}_j,\mathrm{F}_k\rangle+\sum_{i,j=1}^{2n-1}\lambda_j\langle\nabla^S_Y\mathrm{F}_i,\mathrm{F}_j\rangle\langle\nabla^S_X\mathrm{F}_i,\mathrm{F}_j\rangle\\
&\qquad=\sum_{i,j=1}^{2n-1}\langle\nabla^S_Y\mathrm{F}_i,\mathrm{F}_j\rangle\langle\nabla^S_X\mathrm{F}_i,\mathrm{F}_j\rangle(\lambda_j-\lambda_i).
\end{aligned}$$

From one hand, exchanging the indices in the previous equation, we get that

$$\sum_{i,j=1}^{2n-1}\langle\nabla^S_Y\mathrm{F}_i,\mathrm{F}_j\rangle\langle\nabla^S_X\mathrm{F}_i,\mathrm{F}_j\rangle(\lambda_j-\lambda_i)=\sum_{i,j=1}^{2n-1}\langle\nabla^S_Y\mathrm{F}_j,\mathrm{F}_i\rangle\langle\nabla^S_X\mathrm{F}_j,\mathrm{F}_i\rangle(\lambda_i-\lambda_j). \tag{4.6}$$

From the other hand, recalling (3.10), we infer that

$$\sum_{i,j=1}^{2n-1}\langle\nabla^S_Y\mathrm{F}_i,\mathrm{F}_j\rangle\langle\nabla^S_X\mathrm{F}_i,\mathrm{F}_j\rangle(\lambda_j-\lambda_i)=\sum_{i,j=1}^{2n-1}\langle\nabla^S_Y\mathrm{F}_j,\mathrm{F}_i\rangle\langle\nabla^S_X\mathrm{F}_j,\mathrm{F}_i\rangle(\lambda_j-\lambda_i). \tag{4.7}$$

Therefore, combining (4.6) and (4.7), (4.5) follows. □

**Proposition 4.2.** *Let* $X, Y \in \Gamma(\mathcal{H}TS)$. *Then*

$$\operatorname{trace}\nabla^S_X h(\cdot,\cdot)=XH \tag{4.8}$$

*and*

$$\operatorname{trace}\operatorname{Hess}^{\mathcal{H},S}h(X,Y,\cdot,\cdot)=\operatorname{Hess}^{\mathcal{H},S}H(X,Y). \tag{4.9}$$

*Proof.* Fix $X, Y$ as in the statement. Let $p \in S$. Being the trace operator independent of the choice of the orthonormal basis, we let Let $\mathrm{F}_1,\dots,\mathrm{F}_{2n+1}$ be as in (3.20). To prove (4.8), we observe that

$$\operatorname{trace}\nabla^S_X h(\cdot,\cdot)=\sum_{i=1}^{2n-1}\nabla^S_X h(\mathrm{F}_i,\mathrm{F}_i)=XH-2\sum_{i=1}^{2n-1}\tilde{h}(\nabla^S_X\mathrm{F}_i,\mathrm{F}_i)\overset{(4.3)}{=}XH.$$

Notice that

$$\operatorname{Hess}^{\mathcal{H},S}H(X,Y)=\nabla^S_X\nabla^S_Y H=\nabla^S_X(YH)=XYH-\nabla^S_X YH. \tag{4.10}$$

On the other hand, exploiting Proposition 4.1,

$$
\begin{aligned}
\operatorname{trace}\operatorname{Hess}^{\mathcal{H},S} h(X,Y,\cdot,\cdot) &= \sum_{i=1}^{2n-1} \operatorname{Hess}^{\mathcal{H},S} h(X,Y,\mathrm{F}_i,\mathrm{F}_i) \\
&= \sum_{i=1}^{2n-1} \nabla^S_X \nabla^S_Y h(\mathrm{F}_i,\mathrm{F}_i) \\
&= \sum_{i=1}^{2n-1} \nabla^S_X \left( Yh(\mathrm{F}_i,\mathrm{F}_i) - 2\tilde{h}(\nabla^S_Y \mathrm{F}_i,\mathrm{F}_i)\right) \\
&= \sum_{i=1}^{2n-1} \Big( XYh(\mathrm{F}_i,\mathrm{F}_i) - \nabla^S_X Y h(\mathrm{F}_i,\mathrm{F}_i) - 2Y\tilde{h}(\nabla^S_X \mathrm{F}_i,\mathrm{F}_i) \\
&\quad - 2X\tilde{h}(\nabla^S_Y \mathrm{F}_i,\mathrm{F}_i) + 2\tilde{h}(\nabla^S_{\nabla^S_X Y}\mathrm{F}_i,\mathrm{F}_i) + 2\tilde{h}(\nabla^S_Y\nabla^S_X \mathrm{F}_i,\mathrm{F}_i) + 2\tilde{h}(\nabla^S_Y \mathrm{F}_i,\nabla^S_X \mathrm{F}_i)\Big) \\
&\overset{(4.3)}{=} \sum_{i=1}^{2n-1} \Big( XYh(\mathrm{F}_i,\mathrm{F}_i) - \nabla^S_X Y h(\mathrm{F}_i,\mathrm{F}_i) - 2Y\tilde{h}(\nabla^S_X \mathrm{F}_i,\mathrm{F}_i) \\
&\quad - 2X\tilde{h}(\nabla^S_Y \mathrm{F}_i,\mathrm{F}_i) + 2\tilde{h}(\nabla^S_Y\nabla^S_X \mathrm{F}_i,\mathrm{F}_i) + 2\tilde{h}(\nabla^S_Y \mathrm{F}_i,\nabla^S_X \mathrm{F}_i)\Big) \\
&\overset{(4.4)}{=} \sum_{i=1}^{2n-1} \Big( XYh(\mathrm{F}_i,\mathrm{F}_i) - \nabla^S_X Y h(\mathrm{F}_i,\mathrm{F}_i) + 2\tilde{h}(\nabla^S_Y\nabla^S_X \mathrm{F}_i,\mathrm{F}_i) + 2\tilde{h}(\nabla^S_Y \mathrm{F}_i,\nabla^S_X \mathrm{F}_i)\Big) \\
&\overset{(4.5)}{=} \sum_{i=1}^{2n-1} \Big( XYh(\mathrm{F}_i,\mathrm{F}_i) - \nabla^S_X Y h(\mathrm{F}_i,\mathrm{F}_i)\Big) \\
&\overset{(4.10)}{=} \operatorname{Hess}^{\mathcal{H},S} H(X,Y).
\end{aligned}
$$

□

**Proposition 4.3.** *Let* $\mathrm{F}_1,\dots,\mathrm{F}_{2n-1}$ *be as in* (3.20). *Then*

$$
\frac{1}{2}\Delta^{\mathcal{H},S}|\tilde{h}|^2 = |\nabla^{\mathcal{H},S}\tilde{h}|^2 + \sum_{j=1}^{2n-1} \tilde{h}(\mathrm{F}_j,\mathrm{F}_j)\left(\Delta^{\mathcal{H},S}\tilde{h}\right)(\mathrm{F}_j,\mathrm{F}_j) \tag{4.11}
$$

*at* $p$.

*Proof.* Notice that

$$
\begin{aligned}
|\nabla^{\mathcal{H},S}\tilde{h}|^2 &= \sum_{i,j,k=1}^{2n-1} \left(\nabla^S_{\mathrm{F}_i}\tilde{h}(\mathrm{F}_j,\mathrm{F}_k)\right)^2 \\
&= \sum_{i,j,k=1}^{2n-1} \left(\mathrm{F}_i\tilde{h}(\mathrm{F}_j,\mathrm{F}_k) - \tilde{h}(\nabla^S_{\mathrm{F}_i}\mathrm{F}_j,\mathrm{F}_k) - \tilde{h}(\mathrm{F}_j,\nabla^S_{\mathrm{F}_i}\mathrm{F}_k)\right)^2 \\
&= \sum_{i,j,k=1}^{2n-1} \left(\mathrm{F}_i\tilde{h}(\mathrm{F}_j,\mathrm{F}_k)\right)^2 + 2\sum_{i,j,k=1}^{2n-1} \left(\tilde{h}(\nabla^S_{\mathrm{F}_i}\mathrm{F}_j,\mathrm{F}_k)\right)^2 \\
&\quad - 4\sum_{i,j,k=1}^{2n-1} \mathrm{F}_i\tilde{h}(\mathrm{F}_j,\mathrm{F}_k)\tilde{h}(\nabla^S_{\mathrm{F}_i}\mathrm{F}_j,\mathrm{F}_k) + 2\sum_{i,j,k=1}^{2n-1} \tilde{h}(\nabla^S_{\mathrm{F}_i}\mathrm{F}_j,\mathrm{F}_k)\tilde{h}(\mathrm{F}_j,\nabla^S_{\mathrm{F}_i}\mathrm{F}_k) \\
&= \sum_{i,j,k=1}^{2n-1} \left(\mathrm{F}_i\tilde{h}(\mathrm{F}_j,\mathrm{F}_k)\right)^2 + 2\sum_{i,j,k=1}^{2n-1} \langle\nabla^S_{\mathrm{F}_i}\mathrm{F}_j,\mathrm{F}_k\rangle^2\tilde{h}(\mathrm{F}_k,\mathrm{F}_k)^2 \\
&\quad - 4\sum_{i,j,k=1}^{2n-1} \tilde{h}(\mathrm{F}_k,\mathrm{F}_k)\mathrm{F}_i\tilde{h}(\mathrm{F}_j,\mathrm{F}_k)\langle\nabla^S_{\mathrm{F}_i}\mathrm{F}_j,\mathrm{F}_k\rangle \\
&\quad + 2\sum_{i,j,k=1}^{2n-1} \tilde{h}(\mathrm{F}_j,\mathrm{F}_j)\tilde{h}(\mathrm{F}_k,\mathrm{F}_k)\langle\nabla^S_{\mathrm{F}_i}\mathrm{F}_j,\mathrm{F}_k\rangle\langle\nabla^S_{\mathrm{F}_i}\mathrm{F}_k,\mathrm{F}_j\rangle \\
&\overset{(3.10)}{=} \sum_{i,j,k=1}^{2n-1} \left(\mathrm{F}_i\tilde{h}(\mathrm{F}_j,\mathrm{F}_k)\right)^2 + 2\sum_{i,j,k=1}^{2n-1} \langle\nabla^S_{\mathrm{F}_i}\mathrm{F}_j,\mathrm{F}_k\rangle^2\tilde{h}(\mathrm{F}_k,\mathrm{F}_k)^2 \\
&\quad + 4\sum_{i,j,k=1}^{2n-1} \tilde{h}(\mathrm{F}_j,\mathrm{F}_j)\mathrm{F}_i\tilde{h}(\mathrm{F}_j,\mathrm{F}_k)\langle\nabla^S_{\mathrm{F}_i}\mathrm{F}_j,\mathrm{F}_k\rangle - 2\sum_{i,j,k=1}^{2n-1} \tilde{h}(\mathrm{F}_j,\mathrm{F}_j)\tilde{h}(\mathrm{F}_k,\mathrm{F}_k)\langle\nabla^S_{\mathrm{F}_i}\mathrm{F}_j,\mathrm{F}_k\rangle^2
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\sum_{j=1}^{2n-1}\tilde{h}(\mathrm{F}_j,\mathrm{F}_j)\left(\Delta^{\mathcal{H},S}\tilde{h}\right)(\mathrm{F}_j,\mathrm{F}_j) &= \sum_{i,j=1}^{2n-1}\tilde{h}(\mathrm{F}_j,\mathrm{F}_j)\nabla^S_{\mathrm{F}_i}\nabla^S_{\mathrm{F}_i}\tilde{h}(\mathrm{F}_j,\mathrm{F}_j) \\
&= \sum_{i,j=1}^{2n-1}\tilde{h}(\mathrm{F}_j,\mathrm{F}_j)\nabla^S_{\mathrm{F}_i}\left(\mathrm{F}_i\tilde{h}(\mathrm{F}_j,\mathrm{F}_j) - 2\tilde{h}(\nabla^S_{\mathrm{F}_i}\mathrm{F}_j,\mathrm{F}_j)\right) \\
&= \sum_{i,j=1}^{2n-1}\tilde{h}(\mathrm{F}_j,\mathrm{F}_j)\mathrm{F}_i\mathrm{F}_i\tilde{h}(\mathrm{F}_j,\mathrm{F}_j) - 2\sum_{i,j=1}^{2n-1}\tilde{h}(\mathrm{F}_j,\mathrm{F}_j)\mathrm{F}_i\tilde{h}(\nabla^S_{\mathrm{F}_i}\mathrm{F}_j,\mathrm{F}_j) \\
&\quad - \sum_{i,j=1}^{2n-1}\tilde{h}(\mathrm{F}_j,\mathrm{F}_j)\nabla^S_{\mathrm{F}_i}\mathrm{F}_i\tilde{h}(\mathrm{F}_j,\mathrm{F}_j) + 2\sum_{i,j=1}^{2n-1}\tilde{h}(\mathrm{F}_j,\mathrm{F}_j)\tilde{h}\left(\nabla^S_{\nabla^S_{\mathrm{F}_i}\mathrm{F}_i}\mathrm{F}_j,\mathrm{F}_j\right) \\
&\quad - \sum_{i,j=1}^{2n-1}\tilde{h}(\mathrm{F}_j,\mathrm{F}_j)\mathrm{F}_i\tilde{h}(\nabla^S_{\mathrm{F}_i}\mathrm{F}_j,\mathrm{F}_j) + 2\sum_{i,j=1}^{2n-1}\tilde{h}(\mathrm{F}_j,\mathrm{F}_j)\tilde{h}(\nabla^S_{\mathrm{F}_i}\nabla^S_{\mathrm{F}_i}\mathrm{F}_j,\mathrm{F}_j) \\
&\quad - \sum_{i,j=1}^{2n-1}\tilde{h}(\mathrm{F}_j,\mathrm{F}_j)\mathrm{F}_i\tilde{h}(\mathrm{F}_j,\nabla^S_{\mathrm{F}_i}\mathrm{F}_j) + 2\sum_{i,j=1}^{2n-1}\tilde{h}(\mathrm{F}_j,\mathrm{F}_j)\tilde{h}(\nabla^S_{\mathrm{F}_i}\mathrm{F}_j,\nabla^S_{\mathrm{F}_i}\mathrm{F}_j) \\
&\overset{(4.3)}{=} \sum_{j=1}^{2n-1}\tilde{h}(\mathrm{F}_j,\mathrm{F}_j)\Delta^{\mathcal{H},S}(\tilde{h}(\mathrm{F}_j,\mathrm{F}_j)) - 4\sum_{i,j=1}^{2n-1}\tilde{h}(\mathrm{F}_j,\mathrm{F}_j)\mathrm{F}_i\tilde{h}(\nabla^S_{\mathrm{F}_i}\mathrm{F}_j,\mathrm{F}_j) \\
&\quad + 2\sum_{i,j=1}^{2n-1}\tilde{h}(\mathrm{F}_j,\mathrm{F}_j)\tilde{h}(\nabla^S_{\mathrm{F}_i}\nabla^S_{\mathrm{F}_i}\mathrm{F}_j,\mathrm{F}_j) + 2\sum_{i,j=1}^{2n-1}\tilde{h}(\mathrm{F}_j,\mathrm{F}_j)\tilde{h}(\nabla^S_{\mathrm{F}_i}\mathrm{F}_j,\nabla^S_{\mathrm{F}_i}\mathrm{F}_j).
\end{aligned}
$$

Notice that

$$\begin{aligned}
-4\sum_{i,j=1}^{2n-1}\tilde h(\mathrm{F}_j,\mathrm{F}_j)\mathrm{F}_i\tilde h(\nabla^S_{\mathrm{F}_i}\mathrm{F}_j,\mathrm{F}_j) &= -4\sum_{i,j,k=1}^{2n-1}\tilde h(\mathrm{F}_j,\mathrm{F}_j)\mathrm{F}_i\left(\langle\nabla^S_{\mathrm{F}_i}\mathrm{F}_j,\mathrm{F}_k\rangle\tilde h(\mathrm{F}_k,\mathrm{F}_j)\right)\\
&\overset{(3.20)}{=} -4\sum_{i,j=1}^{2n-1}\tilde h(\mathrm{F}_j,\mathrm{F}_j)^2\mathrm{F}_i\left(\langle\nabla^S_{\mathrm{F}_i}\mathrm{F}_j,\mathrm{F}_j\rangle\right) - 4\sum_{i,j,k=1}^{2n-1}\tilde h(\mathrm{F}_j,\mathrm{F}_j)\mathrm{F}_i\tilde h(\mathrm{F}_k,\mathrm{F}_j)\langle\nabla^S_{\mathrm{F}_i}\mathrm{F}_j,\mathrm{F}_k\rangle\\
&\overset{(3.10)}{=} -4\sum_{i,j,k=1}^{2n-1}\tilde h(\mathrm{F}_j,\mathrm{F}_j)\mathrm{F}_i\tilde h(\mathrm{F}_j,\mathrm{F}_k)\langle\nabla^S_{\mathrm{F}_i}\mathrm{F}_j,\mathrm{F}_k\rangle.
\end{aligned}$$

Moreover,

$$\begin{aligned}
2\sum_{i,j=1}^{2n-1}\tilde h(\mathrm{F}_j,\mathrm{F}_j)\tilde h(\nabla^S_{\mathrm{F}_i}\nabla^S_{\mathrm{F}_i}\mathrm{F}_j,\mathrm{F}_j) &\overset{(3.20)}{=} 2\sum_{i,j=1}^{2n-1}\tilde h(\mathrm{F}_j,\mathrm{F}_j)^2\langle\nabla^S_{\mathrm{F}_i}\nabla^S_{\mathrm{F}_i}\mathrm{F}_j,\mathrm{F}_j\rangle\\
&\overset{(3.10)}{=} -2\sum_{i,j=1}^{2n-1}\tilde h(\mathrm{F}_j,\mathrm{F}_j)^2\langle\nabla^S_{\mathrm{F}_i}\mathrm{F}_j,\nabla^S_{\mathrm{F}_i}\mathrm{F}_j\rangle\\
&= -2\sum_{i,j,k=1}^{2n-1}\tilde h(\mathrm{F}_j,\mathrm{F}_j)^2\langle\nabla^S_{\mathrm{F}_i}\mathrm{F}_j,\mathrm{F}_k\rangle^2\\
&\overset{(3.10)}{=} -2\sum_{i,j,k=1}^{2n-1}\tilde h(\mathrm{F}_k,\mathrm{F}_k)^2\langle\nabla^S_{\mathrm{F}_i}\mathrm{F}_j,\mathrm{F}_k\rangle^2.
\end{aligned}$$

Finally,

$$2\sum_{i,j=1}^{2n-1}\tilde h(\mathrm{F}_j,\mathrm{F}_j)\tilde h(\nabla^S_{\mathrm{F}_i}\mathrm{F}_j,\nabla^S_{\mathrm{F}_i}\mathrm{F}_j) = 2\sum_{i,j,k=1}^{2n-1}\tilde h(\mathrm{F}_j,\mathrm{F}_j)\tilde h(\mathrm{F}_k,\mathrm{F}_k)\langle\nabla^S_{\mathrm{F}_i}\mathrm{F}_j,\mathrm{F}_k\rangle^2.$$

Therefore we infer that

(4.12)
$$|\nabla^{\mathcal{H},S}\tilde h|^2 + \sum_{j=1}^{2n-1}\tilde h(\mathrm{F}_j,\mathrm{F}_j)\left(\Delta^{\mathcal{H},S}\tilde h\right)(\mathrm{F}_j,\mathrm{F}_j) = \sum_{i,j,k=1}^{2n-1}\left(\mathrm{F}_i\tilde h(\mathrm{F}_j,\mathrm{F}_k)\right)^2 + \sum_{j=1}^{2n-1}\tilde h(\mathrm{F}_j,\mathrm{F}_j)\Delta^{\mathcal{H},S}(\tilde h(\mathrm{F}_j,\mathrm{F}_j)).$$

We conclude noticing that

$$\begin{aligned}
\frac12\Delta^{\mathcal{H},S}|\tilde h|^2 &= \frac12\sum_{i,j,k=1}^{2n-1}\nabla^S_{\mathrm{F}_i}\nabla^S_{\mathrm{F}_i}\left(\tilde h(\mathrm{F}_j,\mathrm{F}_k)^2\right)\\
&= \sum_{i,j,k=1}^{2n-1}\nabla^S_{\mathrm{F}_i}\left(\tilde h(\mathrm{F}_j,\mathrm{F}_k)\mathrm{F}_i\tilde h(\mathrm{F}_j,\mathrm{F}_k)\right)\\
&= \sum_{i,j,k=1}^{2n-1}\left(\mathrm{F}_i\tilde h(\mathrm{F}_j,\mathrm{F}_k)\right)^2 + \sum_{j=1}^{2n-1}\tilde h(\mathrm{F}_j,\mathrm{F}_j)\Delta^{\mathcal{H},S}(\tilde h(\mathrm{F}_j,\mathrm{F}_j))\\
&\overset{(4.12)}{=} |\nabla^{\mathcal{H},S}\tilde h|^2 + \sum_{j=1}^{2n-1}\tilde h(\mathrm{F}_j,\mathrm{F}_j)\left(\Delta^{\mathcal{H},S}\tilde h\right)(\mathrm{F}_j,\mathrm{F}_j).
\end{aligned}$$

□

**Proposition 4.4.** *Assume that* (H1) *holds. Then*

$$\langle\nabla^S|\tilde h|^2, J(\nu)\rangle = 2\ell\langle\nabla\ell, J(\nu)\rangle - 4\alpha|\tilde h|^2 + 2\alpha\ell^2 + 2\alpha H\ell.$$

*In particular,*

$$\frac12\hat\Delta^{\mathcal{H},S}|\tilde h|^2 = \frac12\Delta^{\mathcal{H},S}|\tilde h|^2 + 2\alpha\ell\langle\nabla^S\ell, J(\nu)\rangle - 4\alpha^2|\tilde h|^2 + 2\alpha^2\ell^2 + 2\alpha^2 H\ell.$$

*Proof.* Let $\mathrm{F}_1, \ldots, \mathrm{F}_{2n-1}$ be as in (3.20). Since $\mathfrak{X} \equiv 0$, we can assume that $\mathrm{F}_n = J(\nu)$. Therefore

$$\begin{aligned}
\langle \nabla^S |\tilde{h}|^2, J(\nu) \rangle &= J(\nu) \left( \sum_{j,k=1}^{2n-1} \tilde{h}(\mathrm{F}_j, \mathrm{F}_k)^2 \right) \\
&= 2 \sum_{j,k=1}^{2n-1} \tilde{h}(\mathrm{F}_j, \mathrm{F}_k) J(\nu) \left( \tilde{h}(\mathrm{F}_j, \mathrm{F}_k) \right) \\
&\overset{(3.20)}{=} 2 \sum_{j=1}^{2n-1} \tilde{h}(\mathrm{F}_j, \mathrm{F}_j) J(\nu) \left( \tilde{h}(\mathrm{F}_j, \mathrm{F}_j) \right) \\
&\overset{(3.17)}{=} 2 \sum_{j=1}^{2n-1} \tilde{h}(\mathrm{F}_j, \mathrm{F}_j) J(\nu) \left( h(\mathrm{F}_j, \mathrm{F}_j) \right) \\
&= 2\ell \langle \nabla \ell, J(\nu) \rangle + 2 \sum_{\substack{j=1 \\ j \neq n}}^{2n-1} \tilde{h}(\mathrm{F}_j, \mathrm{F}_j) J(\nu) \left( h(\mathrm{F}_j, \mathrm{F}_j) \right).
\end{aligned}$$

Notice that

$$2 \sum_{\substack{j=1 \\ j \neq n}}^{2n-1} \tilde{h}(\mathrm{F}_j, \mathrm{F}_j) J(\nu) \left( h(\mathrm{F}_j, \mathrm{F}_j) \right) = 2 \sum_{\substack{j=1 \\ j \neq n}}^{2n-1} \tilde{h}(\mathrm{F}_j, \mathrm{F}_j) \langle \nabla^S_{J(\nu)} \nabla_{\mathrm{F}_j} \nu, \mathrm{F}_j \rangle + 2 \sum_{\substack{j=1 \\ j \neq n}}^{2n-1} \tilde{h}(\mathrm{F}_j, \mathrm{F}_j) \langle \nabla_{\mathrm{F}_j} \nu, \nabla^S_{J(\nu)} \mathrm{F}_j \rangle.$$

First, we have that

$$\begin{aligned}
2 \sum_{\substack{j=1 \\ j \neq n}}^{2n-1} \tilde{h}(\mathrm{F}_j, \mathrm{F}_j) \langle \nabla^S_{J(\nu)} \nabla_{\mathrm{F}_j} \nu, \mathrm{F}_j \rangle &= 2 \sum_{\substack{j=1 \\ j \neq n}}^{2n-1} \tilde{h}(\mathrm{F}_j, \mathrm{F}_j) \langle \nabla_{J(\nu)} \nabla_{\mathrm{F}_j} \nu, \mathrm{F}_j \rangle \\
&= 2 \sum_{\substack{j=1 \\ j \neq n}}^{2n-1} \tilde{h}(\mathrm{F}_j, \mathrm{F}_j) \langle R(J(\nu), \mathrm{F}_j) \nu, \mathrm{F}_j \rangle \\
&\quad + 2 \sum_{\substack{j=1 \\ j \neq n}}^{2n-1} \tilde{h}(\mathrm{F}_j, \mathrm{F}_j) \langle \nabla_{\mathrm{F}_j} \nabla_{J(\nu)} \nu, \mathrm{F}_j \rangle + 2 \sum_{\substack{j=1 \\ j \neq n}}^{2n-1} \tilde{h}(\mathrm{F}_j, \mathrm{F}_j) \langle \nabla_{[J(\nu), \mathrm{F}_j]} \nu, \mathrm{F}_j \rangle \\
&\overset{R \equiv 0}{=} 2 \sum_{\substack{j=1 \\ j \neq n}}^{2n-1} \tilde{h}(\mathrm{F}_j, \mathrm{F}_j) \langle \nabla_{\mathrm{F}_j} \nabla_{J(\nu)} \nu, \mathrm{F}_j \rangle + 2 \sum_{\substack{j=1 \\ j \neq n}}^{2n-1} \tilde{h}(\mathrm{F}_j, \mathrm{F}_j) \langle \nabla_{[J(\nu), \mathrm{F}_j]} \nu, \mathrm{F}_j \rangle \\
&\overset{(H1)+(3.12)}{=} 2 \sum_{\substack{j=1 \\ j \neq n}}^{2n-1} \tilde{h}(\mathrm{F}_j, \mathrm{F}_j) \langle \nabla_{\mathrm{F}_j} \left( \ell J(\nu) \right), \mathrm{F}_j \rangle + 2 \sum_{\substack{j=1 \\ j \neq n}}^{2n-1} \tilde{h}(\mathrm{F}_j, \mathrm{F}_j) h([J(\nu), \mathrm{F}_j], \mathrm{F}_j) \\
&\overset{(3.17)}{=} -2\ell \sum_{\substack{j=1 \\ j \neq n}}^{2n-1} \tilde{h}(\mathrm{F}_j, \mathrm{F}_j) h(\mathrm{F}_j, J(\mathrm{F}_j)) + 2 \sum_{\substack{j=1 \\ j \neq n}}^{2n-1} \tilde{h}(\mathrm{F}_j, \mathrm{F}_j) h([J(\nu), \mathrm{F}_j], \mathrm{F}_j) \\
&\overset{(4.2)}{=} 2\alpha\ell \sum_{\substack{j=1 \\ j \neq n}}^{2n-1} \tilde{h}(\mathrm{F}_j, \mathrm{F}_j) + 2 \sum_{\substack{j=1 \\ j \neq n}}^{2n-1} \tilde{h}(\mathrm{F}_j, \mathrm{F}_j) h([J(\nu), \mathrm{F}_j], \mathrm{F}_j) \\
&\overset{(4.2)}{=} 2\alpha H \ell - 2\alpha \ell^2 + \underbrace{2 \sum_{\substack{j=1 \\ j \neq n}}^{2n-1} \tilde{h}(\mathrm{F}_j, \mathrm{F}_j) h([J(\nu), \mathrm{F}_j], \mathrm{F}_j)}_{\mathrm{I}}.
\end{aligned}$$

Observe that

$$
\begin{aligned}
\mathrm{I} &\overset{(3.17)}{=} 2\sum_{\substack{j=1\\ j\neq n}}^{2n-1} \tilde h(\mathrm{F}_j,\mathrm{F}_j)\tilde h([J(\nu),\mathrm{F}_j],\mathrm{F}_j) + 2\alpha \sum_{\substack{j=1\\ j\neq n}}^{2n-1} \tilde h(\mathrm{F}_j,\mathrm{F}_j)\langle [J(\nu),\mathrm{F}_j], J(\mathrm{F}_j)\rangle \\
&\overset{(3.20)}{=} 2\sum_{\substack{j=1\\ j\neq n}}^{2n-1} \tilde h(\mathrm{F}_j,\mathrm{F}_j)^2\langle [J(\nu),\mathrm{F}_j],\mathrm{F}_j\rangle + 2\alpha \sum_{\substack{j=1\\ j\neq n}}^{2n-1} \tilde h(\mathrm{F}_j,\mathrm{F}_j)\langle [J(\nu),\mathrm{F}_j], J(\mathrm{F}_j)\rangle \\
&\overset{(3.13)}{=} 2\sum_{\substack{j=1\\ j\neq n}}^{2n-1} \tilde h(\mathrm{F}_j,\mathrm{F}_j)^2 h(\mathrm{F}_j, J(\mathrm{F}_j)) + 2\alpha \sum_{\substack{j=1\\ j\neq n}}^{2n-1} \tilde h(\mathrm{F}_j,\mathrm{F}_j)\langle [J(\nu),\mathrm{F}_j], J(\mathrm{F}_j)\rangle \\
&\overset{(4.2)}{=} -2\alpha\sum_{\substack{j=1\\ j\neq n}}^{2n-1} \tilde h(\mathrm{F}_j,\mathrm{F}_j)^2 + 2\alpha \sum_{\substack{j=1\\ j\neq n}}^{2n-1} \tilde h(\mathrm{F}_j,\mathrm{F}_j)\langle [J(\nu),\mathrm{F}_j], J(\mathrm{F}_j)\rangle \\
&= -2\alpha|\tilde h|^2 + 2\alpha\ell^2 + 2\alpha \sum_{\substack{j=1\\ j\neq n}}^{2n-1} \tilde h(\mathrm{F}_j,\mathrm{F}_j)\langle [J(\nu),\mathrm{F}_j], J(\mathrm{F}_j)\rangle .
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
2\sum_{\substack{j=1\\ j\neq n}}^{2n-1} \tilde h(\mathrm{F}_j,\mathrm{F}_j)\langle \nabla_{\mathrm{F}_j}\nu, \nabla^S_{J(\nu)}\mathrm{F}_j\rangle &= 2\sum_{\substack{j=1\\ j\neq n}}^{2n-1} \tilde h(\mathrm{F}_j,\mathrm{F}_j) h(\mathrm{F}_j, \nabla^S_{J(\nu)}\mathrm{F}_j) \\
&\overset{(3.17)}{=} 2\sum_{\substack{j=1\\ j\neq n}}^{2n-1} \tilde h(\mathrm{F}_j,\mathrm{F}_j)\tilde h(\mathrm{F}_j, \nabla^S_{J(\nu)}\mathrm{F}_j) - 2\alpha \sum_{\substack{j=1\\ j\neq n}}^{2n-1} \tilde h(\mathrm{F}_j,\mathrm{F}_j)\langle \nabla^S_{J(\nu)}\mathrm{F}_j, J(\mathrm{F}_j)\rangle \\
&\overset{(4.3)}{=} -2\alpha\sum_{\substack{j=1\\ j\neq n}}^{2n-1} \tilde h(\mathrm{F}_j,\mathrm{F}_j)\langle \nabla^S_{J(\nu)}\mathrm{F}_j, J(\mathrm{F}_j)\rangle \\
&= 2\alpha\sum_{\substack{j=1\\ j\neq n}}^{2n-1} \tilde h(\mathrm{F}_j,\mathrm{F}_j)\langle \mathrm{Tor}_\nabla(\mathrm{F}_j, J(\nu)), J(\mathrm{F}_j)\rangle \\
&\quad - 2\alpha\sum_{\substack{j=1\\ j\neq n}}^{2n-1} \tilde h(\mathrm{F}_j,\mathrm{F}_j)\langle \nabla_{\mathrm{F}_j} J(\nu), J(\mathrm{F}_j)\rangle - 2\alpha \sum_{\substack{j=1\\ j\neq n}}^{2n-1} \tilde h(\mathrm{F}_j,\mathrm{F}_j)\langle [J(\nu),\mathrm{F}_j], J(\mathrm{F}_j)\rangle \\
&\overset{(2.2)}{=} -2\alpha\sum_{\substack{j=1\\ j\neq n}}^{2n-1} \tilde h(\mathrm{F}_j,\mathrm{F}_j)\langle \nabla_{\mathrm{F}_j} J(\nu), J(\mathrm{F}_j)\rangle - 2\alpha \sum_{\substack{j=1\\ j\neq n}}^{2n-1} \tilde h(\mathrm{F}_j,\mathrm{F}_j)\langle [J(\nu),\mathrm{F}_j], J(\mathrm{F}_j)\rangle \\
&= -2\alpha\sum_{\substack{j=1\\ j\neq n}}^{2n-1} \tilde h(\mathrm{F}_j,\mathrm{F}_j)^2 - 2\alpha \sum_{\substack{j=1\\ j\neq n}}^{2n-1} \tilde h(\mathrm{F}_j,\mathrm{F}_j)\langle [J(\nu),\mathrm{F}_j], J(\mathrm{F}_j)\rangle \\
&= -2\alpha|\tilde h|^2 + 2\alpha\ell^2 - 2\alpha \sum_{\substack{j=1\\ j\neq n}}^{2n-1} \tilde h(\mathrm{F}_j,\mathrm{F}_j)\langle [J(\nu),\mathrm{F}_j], J(\mathrm{F}_j)\rangle .
\end{aligned}
$$

Putting all the pieces together, the thesis follows. □

## 5. Simons and Kato inequalities

The aim of this section is to provide a lower bound for $\hat{\Delta}^{\mathcal{H},S}|\tilde{h}|^2$. To this aim we establish and properly combine suitable sub-Riemannian Simons and Kato inequalities.

5.1. **The full Simons identity.** Our first step consists in providing a full Simons-type identity for the $(2,0)$-horizontal tensor field $\Delta^{\mathcal{H},S}h$ associated to a minimal hypersurface. Due to its computational complexity, in order to facilitate the reader's overall understanding of our approach, we have postponed its proof until Section 8. The sub-Riemannian Simons identity reads as follows.

**Theorem 5.1.** *Let $S$ be a smooth, embedded, non-characteristic hypersurface without boundary. Assume that $S$ is minimal. Then*

$$\begin{aligned}\Delta^{\mathcal{H},S}h(X,Y) = &-qh(X,Y) + 8\alpha^2 h(X,Y)\\ &+ 4\,\mathrm{Hess}^{\mathcal{H},S}\,\alpha(\pi(J(X)),Y) + 4\,\mathrm{Hess}^{\mathcal{H},S}\,\alpha(X,\pi(J(Y)))\\ &+ \Big(16\alpha\pi(J(X))\alpha - 8\alpha^2 h(X,J(\nu)) + 4\left(\nabla_X\nu\right)\alpha\Big)\langle Y, J(\nu)\rangle\\ &- 2X\alpha h(Y,J(\nu)) - 2Y\alpha h(X,J(\nu))\\ &+ 2\alpha h(Y,\nabla_{\pi(J(X))}\nu) - 2\alpha\langle\nabla_X\nabla_{J(\nu)}\nu, Y\rangle - 4\alpha^2 h(\pi(J(X)),\pi(J(Y)))\\ &+ 2\alpha\left\langle J\left(\nabla_X\nu\right),\nabla_Y\nu\right\rangle\end{aligned}$$

*for any $X,Y\in\Gamma(\mathcal{H}TS)$.*

5.2. **Simons inequality in arbitrary dimension.** We are going to combine Proposition 4.3, Proposition 4.4 and Theorem 5.1 to provide a lower bound for $\hat{\Delta}^{\mathcal{H},S}|\tilde{h}|^2$. To this aim, we assume that (H1) holds, so that, here and in the rest of this section, we can fix a local orthonormal frame $\mathrm{F}_1,\dots,\mathrm{F}_{2n-1}$ which satisfies (3.20) and (3.21) at a given $p\in S$. Under these assumptions, we deduce the following properties.

**Proposition 5.2.** *Let $j=1,\dots,2n-1$, $j\neq n$. Then*

$$2\alpha h(\mathrm{F}_j,\nabla_{J(\mathrm{F}_j)}\nu) = 2\alpha^2 h(\mathrm{F}_j,\mathrm{F}_j) - 2\alpha^2 h(J(\mathrm{F}_j),J(\mathrm{F}_j)) \tag{5.1}$$

*and*

$$-2\alpha\langle\nabla_{\mathrm{F}_j}\nabla_{J(\nu)}\nu,\mathrm{F}_j\rangle = -2\alpha^2\ell. \tag{5.2}$$

*Proof.* Fix $j=1,\dots,2n-1$, $j\neq n$. Then

$$\begin{aligned}2\alpha h(\mathrm{F}_j,\nabla_{J(\mathrm{F}_j)}\nu) &\overset{(3.16)}{=} 2\alpha\tilde{h}(\mathrm{F}_j,\nabla_{J(\mathrm{F}_j)}\nu) - 2\alpha^2 C(\mathrm{F}_j,\nabla_{J(\mathrm{F}_j)}\nu)\\ &= \sum_{k=1}^{2n-1} 2\alpha\langle\nabla_{J(\mathrm{F}_j)}\nu,\mathrm{F}_k\rangle\tilde{h}(\mathrm{F}_j,\mathrm{F}_k) - 2\alpha^2 h(J(\mathrm{F}_j),J(\mathrm{F}_j))\\ &= 2\alpha\langle\nabla_{J(\mathrm{F}_j)}\nu,\mathrm{F}_j\rangle\tilde{h}(\mathrm{F}_j,\mathrm{F}_j) - 2\alpha^2 h(J(\mathrm{F}_j),J(\mathrm{F}_j))\\ &= 2\alpha h(J(\mathrm{F}_j),\mathrm{F}_j)\tilde{h}(\mathrm{F}_j,\mathrm{F}_j) - 2\alpha^2 h(J(\mathrm{F}_j),J(\mathrm{F}_j))\\ &\overset{(4.2)}{=} 2\alpha^2 h(\mathrm{F}_j,\mathrm{F}_j) - 2\alpha^2 h(J(\mathrm{F}_j),J(\mathrm{F}_j)).\end{aligned}$$

Moreover,

$$\begin{aligned}-2\alpha\langle\nabla_{\mathrm{F}_j}\nabla_{J(\nu)}\nu,\mathrm{F}_j\rangle &= -2\alpha\mathrm{F}_j h(J(\nu),\mathrm{F}_j) + 2\alpha\langle\nabla_{J(\nu)}\nu,\nabla_{\mathrm{F}_j}\mathrm{F}_j\rangle\\ &\overset{(H1)}{=} 2\alpha\ell\langle J(\nu),\nabla_{\mathrm{F}_j}\mathrm{F}_j\rangle\\ &= -2\alpha\ell\langle\nabla_{\mathrm{F}_j}J(\nu),\mathrm{F}_j\rangle\\ &= 2\alpha\ell h(\mathrm{F}_j,J(\mathrm{F}_j))\\ &\overset{(4.2)}{=} -2\alpha^2\ell.\end{aligned}$$

□

Owing to Theorem 5.1, we deduce that

$$\begin{aligned}\Delta^{\mathcal{H},S}h(J(\nu),J(\nu)) &= -q\ell + 4(\nabla_{J(\nu)}\nu)\alpha - 4\ell\langle\nabla\alpha, J(\nu)\rangle - 2\alpha\langle\nabla_{J(\nu)}\nabla_{J(\nu)}\nu, J(\nu)\rangle\\ &\overset{(H1)}{=} -q\ell - 2\alpha\langle\nabla_{J(\nu)}\nabla_{J(\nu)}\nu, J(\nu)\rangle\\ &= -q\ell - 2\alpha\langle\nabla\ell, J(\nu)\rangle.\end{aligned}$$

On the other hand, if $j = 1,\dots,2n-1$, $j \neq n$,

$$\begin{aligned}\Delta^{\mathcal{H},S}h(\mathrm{F}_j,\mathrm{F}_j) &= -qh(\mathrm{F}_j,\mathrm{F}_j) + 8\alpha^2 h(\mathrm{F}_j,\mathrm{F}_j) + B_\alpha(\mathrm{F}_j) - 4\mathrm{F}_j\alpha h(\mathrm{F}_j, J(\nu))\\ &\quad + 2\alpha h(\mathrm{F}_j, \nabla_{\pi(J(\mathrm{F}_j))}\nu) - 2\alpha\langle\nabla_{\mathrm{F}_j}\nabla_{J(\nu)}\nu, \mathrm{F}_j\rangle - 4\alpha^2 h(\pi(J(\mathrm{F}_j)), \pi(J(\mathrm{F}_j)))\\ &\overset{(H1)}{=} -qh(\mathrm{F}_j,\mathrm{F}_j) + 8\alpha^2 h(\mathrm{F}_j,\mathrm{F}_j) + B_\alpha(\mathrm{F}_j)\\ &\quad + 2\alpha h(\mathrm{F}_j, \nabla_{J(\mathrm{F}_j)}\nu) - 2\alpha\langle\nabla_{\mathrm{F}_j}\nabla_{J(\nu)}\nu, \mathrm{F}_j\rangle - 4\alpha^2 h(J(\mathrm{F}_j), J(\mathrm{F}_j))\\ &\overset{(5.1),(5.2)}{=} -qh(\mathrm{F}_j,\mathrm{F}_j) + 10\alpha^2 h(\mathrm{F}_j,\mathrm{F}_j) + B_\alpha(\mathrm{F}_j) - 2\alpha^2\ell - 6\alpha^2 h(J(\mathrm{F}_j), J(\mathrm{F}_j))\end{aligned}$$

where $B_\alpha$ is defined by

$$B_\alpha(X) = 4\,\mathrm{Hess}^{\mathcal{H},S}\alpha(X, J(X)) + 4\,\mathrm{Hess}^{\mathcal{H},S}\alpha(J(X), X) \tag{5.3}$$

for any $X \in \Gamma(\mathcal{H}'TS)$. Notice that

$$B_\alpha(J(X)) = -B_\alpha(X) \tag{5.4}$$

for any $X \in \Gamma(\mathcal{H}'TS)$. In this way, recalling (4.11),

$$\begin{aligned}\frac{1}{2}\Delta^{\mathcal{H},S}|\tilde h|^2 &= |\nabla^{\mathcal{H},S}\tilde h|^2 + \sum_{j=1}^{2n-1}\tilde h(\mathrm{F}_j,\mathrm{F}_j)\left(\Delta^{\mathcal{H},S}\tilde h\right)(\mathrm{F}_j,\mathrm{F}_j)\\ &= |\nabla^{\mathcal{H},S}\tilde h|^2 + \ell\left(\Delta^{\mathcal{H},S}\tilde h\right)(J(\nu),J(\nu)) + \sum_{\substack{j=1\\ j\neq n}}^{2n-1}\tilde h(\mathrm{F}_j,\mathrm{F}_j)\left(\Delta^{\mathcal{H},S}\tilde h\right)(\mathrm{F}_j,\mathrm{F}_j)\\ &= |\nabla^{\mathcal{H},S}\tilde h|^2 - q\ell^2 - 2\alpha\ell\langle\nabla^S\ell, J(\nu)\rangle\\ &\quad - q\sum_{\substack{j=1\\ j\neq n}}^{2n-1}h(\mathrm{F}_j,\mathrm{F}_j)^2 + 10\alpha^2\sum_{\substack{j=1\\ j\neq n}}^{2n-1}h(\mathrm{F}_j,\mathrm{F}_j)^2 - 6\alpha^2\sum_{\substack{j=1\\ j\neq n}}^{2n-1}h(\mathrm{F}_j,\mathrm{F}_j)h(J(\mathrm{F}_j),J(\mathrm{F}_j))\\ &\quad + \sum_{\substack{j=1\\ j\neq n}}^{2n-1}h(\mathrm{F}_j,\mathrm{F}_j)B_\alpha(\mathrm{F}_j) - 2\alpha^2\ell\sum_{\substack{j=1\\ j\neq n}}^{2n-1}h(\mathrm{F}_j,\mathrm{F}_j)\\ &\overset{H\equiv 0}{=} |\nabla^{\mathcal{H},S}\tilde h|^2 - q\ell^2 - 2\alpha\ell\langle\nabla^S\ell, J(\nu)\rangle\\ &\quad - q\sum_{\substack{j=1\\ j\neq n}}^{2n-1}h(\mathrm{F}_j,\mathrm{F}_j)^2 + 10\alpha^2\sum_{\substack{j=1\\ j\neq n}}^{2n-1}h(\mathrm{F}_j,\mathrm{F}_j)^2 - 6\alpha^2\sum_{\substack{j=1\\ j\neq n}}^{2n-1}h(\mathrm{F}_j,\mathrm{F}_j)h(J(\mathrm{F}_j),J(\mathrm{F}_j))\\ &\quad + \sum_{\substack{j=1\\ j\neq n}}^{2n-1}h(\mathrm{F}_j,\mathrm{F}_j)B_\alpha(\mathrm{F}_j) + 2\alpha^2\ell^2\\ &= |\nabla^{\mathcal{H},S}\tilde h|^2 - q|\tilde h|^2 + 10\alpha^2|\tilde h|^2 - 8\alpha^2\ell^2 - 2\alpha\ell\langle\nabla^S\ell, J(\nu)\rangle\\ &\quad - 6\alpha^2\sum_{\substack{j=1\\ j\neq n}}^{2n-1}h(\mathrm{F}_j,\mathrm{F}_j)h(J(\mathrm{F}_j),J(\mathrm{F}_j)) + \sum_{\substack{j=1\\ j\neq n}}^{2n-1}h(\mathrm{F}_j,\mathrm{F}_j)B_\alpha(\mathrm{F}_j).\end{aligned}$$

Recalling Proposition 4.4,

$$\begin{aligned}\frac{1}{2}\hat{\Delta}^{\mathcal{H},S}|\tilde{h}|^2 &= |\nabla^{\mathcal{H},S}\tilde{h}|^2 - q|\tilde{h}|^2 + 6\alpha^2|\tilde{h}|^2 - 6\alpha^2\ell^2 \\ &\quad - 6\alpha^2 \sum_{\substack{j=1\\ j\neq n}}^{2n-1} h(\mathrm{F}_j,\mathrm{F}_j)h(J(\mathrm{F}_j),J(\mathrm{F}_j)) + \sum_{\substack{j=1\\ j\neq n}}^{2n-1} h(\mathrm{F}_j,\mathrm{F}_j)B_\alpha(\mathrm{F}_j).\end{aligned} \tag{5.5}$$

Let us consider the family

$$(\mathrm{K}_1,\dots,\mathrm{K}_{n-1},\mathrm{K}_n,\mathrm{K}_{n+1},\dots,\mathrm{K}_{2n-1}) = (J(\mathrm{F}_1),\dots,J(\mathrm{F}_{n-1}),J(\nu),J(\mathrm{F}_{n+1}),\dots,J(\mathrm{F}_{2n-1})).$$

Notice that $(\mathrm{K}_1,\dots,\mathrm{K}_{2n-1})$ is still a local orthonormal frame for $\mathcal{H}TS$ near $p$. Exploiting the above notation, we can rewrite (5.5) as

$$\begin{aligned}\frac{1}{2}\hat{\Delta}^{\mathcal{H},S}|\tilde{h}|^2 &= |\nabla^{\mathcal{H},S}\tilde{h}|^2 - q|\tilde{h}|^2 + 6\alpha^2|\tilde{h}|^2 \\ &\quad - 6\alpha^2 \sum_{j=1}^{2n-1} h(\mathrm{F}_j,\mathrm{F}_j)h(\mathrm{K}_j,\mathrm{K}_j) + \sum_{\substack{j=1\\ j\neq n}}^{2n-1} h(\mathrm{F}_j,\mathrm{F}_j)B_\alpha(\mathrm{F}_j).\end{aligned} \tag{5.6}$$

In order to proceed, we further assume that

$$\nabla^{\mathcal{H},S}\alpha \equiv \langle\nabla\alpha, J(\nu)\rangle J(\nu). \tag{H2}$$

**Proposition 5.3.** *Assume that both* (H1) *and* (H2) *hold. Let* $\mathrm{F}_1,\dots,\mathrm{F}_{2n-1}$ *be as in* (3.20), *and assume moreover that* $\mathrm{F}_n = J(\nu)$. *Then*

$$\sum_{\substack{j=1\\ j\neq n}}^{2n-1} h(\mathrm{F}_j,\mathrm{F}_j)B_\alpha(\mathrm{F}_j) = 4\langle\nabla\alpha,J(\nu)\rangle|\tilde{h}|^2 - 4\langle\nabla\alpha,J(\nu)\rangle\sum_{j=1}^{2n-1} h(\mathrm{F}_j,\mathrm{F}_j)h(\mathrm{K}_j,\mathrm{K}_j).$$

*Proof.* Fix $j \neq n$, $j = 1,\dots,2n-1$. By our choice of $\mathrm{F}_1,\dots,\mathrm{F}_{2n-1}$ and by (H2), we deduce that

$$\mathrm{F}_j\alpha = 0 \qquad \text{and} \qquad J(\mathrm{F}_j)\alpha = 0. \tag{5.7}$$

Therefore

$$\begin{aligned}B_\alpha(\mathrm{F}_j) &\overset{(5.3)}{=} 4\,\mathrm{Hess}^{\mathcal{H},S}\alpha(\mathrm{F}_j,J(\mathrm{F}_j)) + 4\,\mathrm{Hess}^{\mathcal{H},S}\alpha(J(\mathrm{F}_j),\mathrm{F}_j) \\ &= 4\nabla^S_{\mathrm{F}_j}(J(\mathrm{F}_j)\alpha) + 4\nabla^S_{J(\mathrm{F}_j)}(\mathrm{F}_j\alpha) \\ &= 4\mathrm{F}_j(J(\mathrm{F}_j)\alpha) - 4\langle\nabla^S_{\mathrm{F}_j}J(\mathrm{F}_j),\nabla^S\alpha\rangle + 4J(\mathrm{F}_j)(\mathrm{F}_j\alpha)) - 4\langle\nabla^S_{J(\mathrm{F}_j)}\mathrm{F}_j,\nabla^S\alpha\rangle \\ &\overset{(5.7)}{=} -4\langle\nabla^S_{\mathrm{F}_j}J(\mathrm{F}_j),\nabla^S\alpha\rangle - 4\langle\nabla^S_{J(\mathrm{F}_j)}\mathrm{F}_j,\nabla^S\alpha\rangle \\ &\overset{(5.7)}{=} 4\langle\nabla\alpha,J(\nu)\rangle\left(-\langle\nabla^S_{\mathrm{F}_j}J(\mathrm{F}_j),J(\nu)\rangle - \langle\nabla^S_{J(\mathrm{F}_j)}\mathrm{F}_j,J(\nu)\rangle\right) \\ &= 4\langle\nabla\alpha,J(\nu)\rangle\left(-\langle\nabla_{\mathrm{F}_j}J(\mathrm{F}_j),J(\nu)\rangle - \langle\nabla_{J(\mathrm{F}_j)}\mathrm{F}_j,J(\nu)\rangle\right) \\ &= 4\langle\nabla\alpha,J(\nu)\rangle\left(-\langle\nabla_{\mathrm{F}_j}\mathrm{F}_j,\nu\rangle + \langle\nabla_{J(\mathrm{F}_j)}J(\mathrm{F}_j),\nu\rangle\right) \\ &= 4\langle\nabla\alpha,J(\nu)\rangle\left(h(\mathrm{F}_j,\mathrm{F}_j) - h(J(\mathrm{F}_j),J(\mathrm{F}_j))\right).\end{aligned}$$

In this way,

$$\begin{aligned}\sum_{\substack{j=1\\ j\neq n}}^{2n-1} h(\mathrm{F}_j,\mathrm{F}_j)B_\alpha(\mathrm{F}_j) &= 4\langle\nabla\alpha, J(\nu)\rangle \sum_{\substack{j=1\\ j\neq n}}^{2n-1} h(\mathrm{F}_j,\mathrm{F}_j)^2 \\ &\quad - 4\langle\nabla\alpha, J(\nu)\rangle \sum_{\substack{j=1\\ j\neq n}}^{2n-1} h(\mathrm{F}_j,\mathrm{F}_j)h(J(\mathrm{F}_j),J(\mathrm{F}_j)) \\ &= 4\langle\nabla\alpha, J(\nu)\rangle|\tilde{h}|^2 - 4\langle\nabla\alpha, J(\nu)\rangle \sum_{j=1}^{2n-1} h(\mathrm{F}_j,\mathrm{F}_j)h(\mathrm{K}_j,\mathrm{K}_j).\end{aligned}$$

□

Since (H2) holds, we can exploit Proposition 5.3 to write (5.6) as

$$\begin{aligned}\frac{1}{2}\hat{\Delta}^{\mathcal{H},S}|\tilde{h}|^2 &= |\nabla^{\mathcal{H},S}\tilde{h}|^2 - q|\tilde{h}|^2 + 6\alpha^2|\tilde{h}|^2 \\ &\quad + 4\langle\nabla\alpha, J(\nu)\rangle|\tilde{h}|^2 + \left(-6\alpha^2 - 4\langle\nabla\alpha, J(\nu)\rangle\right) \sum_{j=1}^{2n-1} h(\mathrm{F}_j,\mathrm{F}_j)h(\mathrm{K}_j,\mathrm{K}_j).\end{aligned} \tag{5.8}$$

Let us further assume that

$$4\langle\nabla\alpha, J(\nu)\rangle \geqslant -6\alpha^2. \tag{5.9}$$

In this way, observing that

$$\left|\sum_{j=1}^{2n-1} h(\mathrm{F}_j,\mathrm{F}_j)h(\mathrm{K}_j,\mathrm{K}_j)\right| \leqslant \sqrt{\sum_{j=1}^{2n-1} \tilde{h}(\mathrm{F}_j,\mathrm{F}_j)^2}\sqrt{\sum_{j=1}^{2n-1} \tilde{h}(\mathrm{K}_j,\mathrm{K}_j)^2} \leqslant |\tilde{h}|^2,$$

we exploit (5.9) to infer that

$$\left(-6\alpha^2 - 4\langle\nabla\alpha, J(\nu)\rangle\right) \sum_{j=!}^{2n-1} h(\mathrm{F}_j,\mathrm{F}_j)h(\mathrm{K}_j,\mathrm{K}_j) \geqslant \left(-6\alpha^2 - 4\langle\nabla\alpha, J(\nu)\rangle\right)|\tilde{h}|^2,$$

so that

$$\frac{1}{2}\hat{\Delta}^{\mathcal{H},S}|\tilde{h}|^2 \geqslant |\nabla^{\mathcal{H},S}\tilde{h}|^2 - q|\tilde{h}|^2. \tag{5.10}$$

**Remark 5.4.** We point out that the inequality provided by (5.10) is sharp in the class of minimal hypersurfaces which satisfy (H1), (H2) and (5.9). First, observe that, owing to Proposition 3.7 every minimal umbilic hypersurface satisfies (H1), (H2) and (5.9). We claim that, when $S$ is umbilic, equality holds in (5.10). Indeed, notice that

$$\sum_{j=1}^{2n-1} h(\mathrm{F}_j,\mathrm{F}_j)h(\mathrm{K}_j,\mathrm{K}_j) = \ell^2 + \lambda \sum_{\substack{j=1\\ j\neq n}}^{2n-1} h(\mathrm{K}_j,\mathrm{K}_j) \overset{H\equiv 0}{=} \ell^2 - \lambda\ell \overset{(3.24)}{=} |\tilde{h}|^2. \tag{5.11}$$

In particular, plugging this identity into (5.8) implies the claim.

**Remark 5.5.** Let us recall (cf. (3.40)) that

$$q = |\tilde{h}|^2 + 4\langle\nabla\alpha, J(\nu)\rangle + (2n+2)\alpha^2.$$

Therefore, if we assume (5.9), then

$$q \geqslant |\tilde{h}|^2 + (2n-4)\alpha^2 \geqslant |\tilde{h}|^2 \geqslant 0. \tag{5.12}$$

5.3. **A refined Simons inequality in $\mathbb{H}^2$.** In the forthcoming sections, we shall need a more refined version of (5.10) in the second Heisenberg group $\mathbb{H}^2$. Therefore, in the rest of this section we assume unless otherwised specified that $n = 2$. The advantage of working in $\mathbb{H}^2$ consists in the fact that, as in the umbilic case, we can provide an explicit expression of the term appearing in the left hand side of (5.11). More precisely, the following holds.

**Proposition 5.6.** *Assume that* (H1) *holds. Assume that $H \equiv 0$. Then*

$$\sum_{j=1}^{2n-1} h(\mathrm{F}_j, \mathrm{F}_j) h(\mathrm{K}_j, \mathrm{K}_j) = 3\ell^2 - |\tilde{h}|^2. \tag{5.13}$$

*Proof.* Let $\mathrm{F}_1, \mathrm{F}_2, \mathrm{F}_3$ be as in (3.20). Since (H1) holds, we can assume that $\mathrm{F}_2 = J(\nu)$, so that $\mathrm{F}_1, \mathrm{F}_3 \in \Gamma(\mathcal{H}'TS)$. But then either $\mathrm{F}_3 = J(\mathrm{F}_1)$ or $\mathrm{F}_3 = -J(\mathrm{F}_1)$. Therefore, without loss of generality, we let

$$\mathrm{F}_2 = J(\nu) \qquad \text{and} \qquad \mathrm{F}_3 = J(\mathrm{F}_1). \tag{5.14}$$

Let us set $\lambda_1 = h(\mathrm{F}_1, \mathrm{F}_1)$ and $\lambda_3 = h(\mathrm{F}_3, \mathrm{F}_3)$. Exploiting (5.14), we infer that

$$\sum_{j=1}^{2n-1} h(\mathrm{F}_j, \mathrm{F}_j) h(\mathrm{K}_j, \mathrm{K}_j) = \ell^2 + 2\lambda_1\lambda_3.$$

Moreover, as $H \equiv 0$, then $\lambda_1 + \lambda_3 = -\ell$, so that

$$0 = H^2 = (\ell + \lambda_1 + \lambda_3)^2 = |\tilde{h}|^2 + 2\ell(\lambda_1 + \lambda_3) + 2\lambda_1\lambda_3 = |\tilde{h}|^2 - 2\ell^2 + 2\lambda_1\lambda_3,$$

whence (5.13) follows. □

Exploiting (5.13), we rely on (5.8) to infer that

$$\begin{aligned} \frac{1}{2}\hat{\Delta}^{\mathcal{H},S}|\tilde{h}|^2 &= |\nabla^{\mathcal{H},S}\tilde{h}|^2 - q|\tilde{h}|^2 + 6\alpha^2|\tilde{h}|^2 + 4\langle \nabla\alpha, J(\nu)\rangle|\tilde{h}|^2 + \left(-6\alpha^2 - 4\langle \nabla\alpha, J(\nu)\rangle\right)(3\ell^2 - |\tilde{h}|^2) \\ &= |\nabla^{\mathcal{H},S}\tilde{h}|^2 - q|\tilde{h}|^2 + \left(6\alpha^2 + 4\langle \nabla\alpha, J(\nu)\rangle\right)(2|\tilde{h}|^2 - 3\ell^2). \end{aligned} \tag{5.15}$$

We claim that the second factor in the last term of the previous line is non-negative. To this aim, let us recall the following straightforward lemma.

**Lemma 5.7.** *Let $K \in \mathbb{N}_{>0}$, and let $\alpha_1, \ldots, \alpha_K \in \mathbb{R}$. Then*

$$\left(\sum_{j=1}^{K} \alpha_j\right)^2 \leqslant K \sum_{j=1}^{K} \alpha_j^2. \tag{5.16}$$

*Moreover, equality in* (5.16) *holds if and only if $\alpha_1 = \ldots = \alpha_K$.*

Minimal hypersurfaces share the following relation between $\ell$ and $|\tilde{h}|$.

**Proposition 5.8.** *Let $n \geqslant 2$. Assume that* (H1) *holds. Assume that $H \equiv 0$. Then*

$$(2n-1)\ell^2 \leqslant (2n-2)|\tilde{h}|^2. \tag{5.17}$$

*Moreover, equality holds in* (5.17) *if and only if $S$ is umbilic.*

*Proof.* Fix $\beta \in [0,1]$. Then

$$\begin{aligned} \ell^2 &= \beta\ell^2 + (1-\beta)\ell^2 \\ &\overset{H\equiv 0}{=} \beta \left(\sum_{\substack{j=1 \\ j\neq n}}^{2n-1} h(\mathrm{F}_j, \mathrm{F}_j)\right)^2 + (1-\beta)\ell^2 \\ &\overset{(5.16)}{\leqslant} (2n-2)\beta \sum_{\substack{j=1 \\ j\neq n}}^{2n-1} h(\mathrm{F}_j, \mathrm{F}_j)^2 + (1-\beta)\ell^2 \\ &\leqslant \max\{(2n-2)\beta, 1-\beta\}|\tilde{h}|^2. \end{aligned}$$

Noticing that

$$\min_{\beta\in[0,1]}\{\max\{(2n-2)\beta, 1-\beta\}\} = \frac{2n-2}{2n-1},$$

(5.17) follows. If $S$ is umbilic, we already know from (3.25) that equality holds in (5.17). Finally, assume that equality holds in (5.17). Then

$$\ell^2 = \frac{2n-2}{2n-1}\ell^2 + \frac{2n-2}{2n-1}\sum_{\substack{j=1\\ j\neq n}}^{2n-1} h(\mathrm{F}_j,\mathrm{F}_j)^2,$$

so that

$$\ell^2 = (2n-2)\sum_{\substack{j=1\\ j\neq n}}^{2n-1} h(\mathrm{F}_j,\mathrm{F}_j)^2.$$

On the other hand, being $S$ minimal, then

$$\ell = -\sum_{\substack{j=1\\ j\neq n}}^{2n-1} h(\mathrm{F}_j,\mathrm{F}_j),$$

so that

$$\left(\sum_{\substack{j=1\\ j\neq n}}^{2n-1} h(\mathrm{F}_j,\mathrm{F}_j)\right)^2 = (2n-2)\sum_{\substack{j=1\\ j\neq n}}^{2n-1} h(\mathrm{F}_j,\mathrm{F}_j)^2.$$

In light of Lemma 5.7, we conclude that $S$ is umbilic. □

In view of (5.17), and recalling (5.9), it is clear that (5.15) implies (5.10). Therefore, to improve our estimate, we refine (5.9) by requiring that

$$4\langle\nabla\alpha, J(\nu)\rangle \geqslant -(4+\omega)\alpha^2 \tag{H3}$$

for a fixed constant $\omega\in[0,2]$.

**Remark 5.9.** The constraint $\omega\in[0,2]$ in (H3) is motivated by two reasons. On the one hand, since $\omega\geqslant 0$, in view of Proposition 3.7 every minimal umbilic hypersurface satisfies (H1), (H2) and (H3). On the other hand, since $\omega\leqslant 2$, (H3) implies (5.9), so that all our previous considerations still hold.

Nevertheless, coupling (5.15) with (5.17) and (H3), we can improve (5.10) to

$$\begin{aligned}\frac{1}{2}\hat\Delta^{\mathcal{H},S}|\tilde h|^2 &\overset{(5.15)}{=} |\nabla^{\mathcal{H},S}\tilde h|^2 - q|\tilde h|^2 + \left(6\alpha^2 + 4\langle\nabla\alpha, J(\nu)\rangle\right)(2|\tilde h|^2 - 3\ell^2)\\ &\overset{(5.17),(H3)}{\geqslant} |\nabla^{\mathcal{H},S}\tilde h|^2 - q|\tilde h|^2 + (2-\omega)\alpha^2(2|\tilde h|^2 - 3\ell^2).\end{aligned}$$

In the end, assuming (H1), (H2) and (H3), we conclude that

$$\frac{1}{2}\hat\Delta^{\mathcal{H},S}|\tilde h|^2 \geqslant |\nabla^{\mathcal{H},S}\tilde h|^2 - q|\tilde h|^2 + (4-2\omega)\alpha^2|\tilde h|^2 - (6-3\omega)\alpha^2\ell^2. \tag{5.18}$$

5.4. **Kato inequalities.** Our goal here is to provide a lower bound for $|\nabla^{\mathcal{H},S}\tilde h|^2$ in terms of $|\nabla^{\mathcal{H},S}|\tilde h|^2|^2$, basically under assumptions (H1) and (H2).

**Theorem 5.10.** *Assume* (H1). *Then*

$$|\nabla^{\mathcal{H},S}|\tilde h|^2|^2 \leqslant 4|\tilde h|^2|\nabla^{\mathcal{H},S}\tilde h|^2. \tag{5.19}$$

*If in addition $H$ is constant and* (H2) *holds, then*

$$\left(1+\frac{k}{2n-1}\right)|\nabla^{\mathcal{H},S}|\tilde h|^2|^2 \leqslant 4|\tilde h|^2|\nabla^{\mathcal{H},S}\tilde h|^2 + 4\alpha^2|\tilde h|^2\left((4k-2)|\tilde h|^2 + (2+2kn-2k-4n)\ell^2\right) \tag{5.20}$$

*for any $k\in[0,2]$.*

*Proof.* Let $\mathrm{F}_1, \ldots, \mathrm{F}_{2n-1}$ be as in (3.20). Since we are assuming (H1), we can choose $\mathrm{F}_n = J(\nu)$. Notice that

$$
\begin{aligned}
\mathrm{F}_i|\tilde{h}|^2 &= \sum_{j,k=1}^{2n-1} \mathrm{F}_i(\tilde{h}(\mathrm{F}_j, \mathrm{F}_k)^2) \\
&= 2\sum_{j,k=1}^{2n-1} \tilde{h}(\mathrm{F}_j, \mathrm{F}_k)\mathrm{F}_i(\tilde{h}(\mathrm{F}_j, \mathrm{F}_k)) \\
&\overset{(3.20)}{=} 2\sum_{j=1}^{2n-1} \tilde{h}(\mathrm{F}_j, \mathrm{F}_j)\mathrm{F}_i(\tilde{h}(\mathrm{F}_j, \mathrm{F}_j)) \qquad (5.21)\\
&= 2\sum_{j=1}^{2n-1} \tilde{h}(\mathrm{F}_j, \mathrm{F}_j)\nabla^S_{\mathrm{F}_i}\tilde{h}(\mathrm{F}_j, \mathrm{F}_j) + 4\sum_{j=1}^{2n-1} \tilde{h}(\mathrm{F}_j, \mathrm{F}_j)\tilde{h}(\nabla^S_{\mathrm{F}_i}\mathrm{F}_j, \mathrm{F}_j) \\
&\overset{(4.3)}{=} 2\sum_{j=1}^{2n-1} \tilde{h}(\mathrm{F}_j, \mathrm{F}_j)\nabla^S_{\mathrm{F}_i}\tilde{h}(\mathrm{F}_j, \mathrm{F}_j)
\end{aligned}
$$

for any $i = 1, \ldots, 2n-1$, so that

$$
\begin{aligned}
|\nabla^{\mathcal{H},S}|\tilde{h}|^2|^2 &= \sum_{i=1}^{2n-1} (\mathrm{F}_i|\tilde{h}|^2)^2 \\
&\overset{(5.21)}{=} 4\sum_{i=1}^{2n-1}\left(\sum_{j=1}^{2n-1} \tilde{h}(\mathrm{F}_j, \mathrm{F}_j)\nabla^S_{\mathrm{F}_i}\tilde{h}(\mathrm{F}_j, \mathrm{F}_j)\right)^2 \\
&\leqslant 4|\tilde{h}|^2 \sum_{i,j=1}^{2n-1} \nabla^S_{\mathrm{F}_i}\tilde{h}(\mathrm{F}_j, \mathrm{F}_j)^2,
\end{aligned}
$$

where in the last passage we used the Cauchy-Schwarz inequality. Therefore

$$
|\nabla^{\mathcal{H},S}|\tilde{h}|^2|^2 \leqslant 4|\tilde{h}|^2 \sum_{i,j=1}^{2n-1} \nabla^S_{\mathrm{F}_i}\tilde{h}(\mathrm{F}_j, \mathrm{F}_j)^2. \qquad (5.22)
$$

In particular, (5.19) follows from (5.22). Assume now that $H$ is constant. Then

$$
\begin{aligned}
|\nabla^{\mathcal{H},S}|\tilde{h}|^2|^2 &\overset{(5.22)}{\leqslant} 4|\tilde{h}|^2 \sum_{i,j=1}^{2n-1} \nabla^S_{\mathrm{F}_i}\tilde{h}(\mathrm{F}_j, \mathrm{F}_j)^2 \\
&= 4|\tilde{h}|^2 \sum_{\substack{i,j=1\\ i\neq j}}^{2n-1} \nabla^S_{\mathrm{F}_i}\tilde{h}(\mathrm{F}_j, \mathrm{F}_j)^2 + 4|\tilde{h}|^2 \sum_{i=1}^{2n-1} \nabla^S_{\mathrm{F}_i}\tilde{h}(\mathrm{F}_i, \mathrm{F}_i)^2 \\
&\overset{(4.8)}{=} 4|\tilde{h}|^2 \sum_{\substack{i,j=1\\ i\neq j}}^{2n-1} \nabla^S_{\mathrm{F}_i}\tilde{h}(\mathrm{F}_j, \mathrm{F}_j)^2 + 4|\tilde{h}|^2 \sum_{i=1}^{2n-1}\left(\sum_{\substack{j=1\\ j\neq i}}^{2n-1} \nabla^S_{\mathrm{F}_i}\tilde{h}(\mathrm{F}_j, \mathrm{F}_j)\right)^2 \\
&\overset{(5.16)}{\leqslant} 4|\tilde{h}|^2 \sum_{\substack{i,j=1\\ i\neq j}}^{2n-1} \nabla^S_{\mathrm{F}_i}\tilde{h}(\mathrm{F}_j, \mathrm{F}_j)^2 + 4(2n-2)|\tilde{h}|^2 \sum_{\substack{i,j=1\\ i\neq j}}^{2n-1} \nabla^S_{\mathrm{F}_i}\tilde{h}(\mathrm{F}_j, \mathrm{F}_j)^2 \\
&= 4(2n-1)|\tilde{h}|^2 \sum_{\substack{i,j=1\\ i\neq j}}^{2n-1} \nabla^S_{\mathrm{F}_i}\tilde{h}(\mathrm{F}_j, \mathrm{F}_j)^2,
\end{aligned}
$$

whence

$$|\nabla^{\mathcal{H},S}|\tilde{h}|^2|^2 \leqslant 4(2n-1)|\tilde{h}|^2 \sum_{\substack{i,j=1\\ i\neq j}}^{2n-1} \nabla^S_{\mathrm{F}_i}\tilde{h}(\mathrm{F}_j,\mathrm{F}_j)^2. \tag{5.23}$$

Exploiting (5.23), we see that

$$\begin{aligned}
\left(1+\frac{k}{2n-1}\right)|\nabla^{\mathcal{H},S}|\tilde{h}|^2|^2 &= |\nabla^{\mathcal{H},S}|\tilde{h}|^2|^2 + \frac{k}{2n-1}|\nabla^{\mathcal{H},S}|\tilde{h}|^2|^2 \\
&\overset{(5.22),(5.23)}{\leqslant} 4|\tilde{h}|^2 \sum_{i,j=1}^{2n-1} \nabla^S_{\mathrm{F}_i}\tilde{h}(\mathrm{F}_j,\mathrm{F}_j)^2 + 4k|\tilde{h}|^2 \sum_{\substack{i,j=1\\ i\neq j}}^{2n-1} \nabla^S_{\mathrm{F}_i}\tilde{h}(\mathrm{F}_j,\mathrm{F}_j)^2 \\
&= 4|\tilde{h}|^2 \sum_{\substack{i,j=1\\ i\neq j}}^{2n-1} \nabla^S_{\mathrm{F}_i}\tilde{h}(\mathrm{F}_j,\mathrm{F}_j)^2 + 4|\tilde{h}|^2 \sum_{i=1}^{2n-1} \nabla^S_{\mathrm{F}_i}\tilde{h}(\mathrm{F}_i,\mathrm{F}_i)^2 + 4k|\tilde{h}|^2 \sum_{\substack{i,j=1\\ i\neq j}}^{2n-1} \nabla^S_{\mathrm{F}_i}\tilde{h}(\mathrm{F}_j,\mathrm{F}_j)^2 \\
&= 4|\tilde{h}|^2 \sum_{\substack{i,j=1\\ i\neq j}}^{2n-1} \nabla^S_{\mathrm{F}_i}\tilde{h}(\mathrm{F}_j,\mathrm{F}_j)^2 + 4|\tilde{h}|^2 \sum_{i=1}^{2n-1} \nabla^S_{\mathrm{F}_i}\tilde{h}(\mathrm{F}_i,\mathrm{F}_i)^2 \\
&\quad + 4|\tilde{h}|^2 \left( \underbrace{k\sum_{\substack{j=1\\ j\neq n}}^{2n-1} \nabla^S_{J(\nu)}\tilde{h}(\mathrm{F}_j,\mathrm{F}_j)^2}_{\mathrm{I}} + \underbrace{k\sum_{\substack{i=1\\ i\neq n}}^{2n-1} \nabla^S_{\mathrm{F}_i}\tilde{h}(J(\nu),J(\nu))^2}_{\mathrm{II}} + \underbrace{k\sum_{\substack{i,j=1\\ i\neq j\\ i,j\neq n}}^{2n-1} \nabla^S_{\mathrm{F}_i}\tilde{h}(\mathrm{F}_j,\mathrm{F}_j)^2}_{\mathrm{III}} \right).
\end{aligned}$$

At this stage we need to apply the Codazzi equation (3.36) to the terms I, II and III. First, notice that

$$\begin{aligned}
\mathrm{I} &\overset{(3.36)}{=} k\sum_{\substack{j=1\\ j\neq n}}^{2n-1} \Big(\nabla^S_{\mathrm{F}_j}\tilde{h}(J(\nu),\mathrm{F}_j) - \alpha h(\mathrm{F}_j,\mathrm{F}_j)\Big)^2 \\
&= k\sum_{\substack{j=1\\ j\neq n}}^{2n-1} \nabla^S_{\mathrm{F}_j}\tilde{h}(J(\nu),\mathrm{F}_j)^2 - 2k\alpha \sum_{\substack{j=1\\ j\neq n}}^{2n-1} h(\mathrm{F}_j,\mathrm{F}_j)\nabla^S_{\mathrm{F}_j}\tilde{h}(J(\nu),\mathrm{F}_j) + k\alpha^2|\tilde{h}|^2 - k\alpha^2\ell^2.
\end{aligned} \tag{5.24}$$

Moreover,

$$
\begin{aligned}
-2k\alpha \sum_{\substack{j=1\\ j\neq n}}^{2n-1} h(\mathrm{F}_j,\mathrm{F}_j)\nabla^S_{\mathrm{F}_j}\tilde{h}(J(\nu),\mathrm{F}_j)
&\overset{(\mathrm{H1})}{=} 2k\alpha \sum_{\substack{j=1\\ j\neq n}}^{2n-1} h(\mathrm{F}_j,\mathrm{F}_j)\tilde{h}(\nabla^S_{\mathrm{F}_j}J(\nu),\mathrm{F}_j) + 2k\alpha \sum_{\substack{j=1\\ j\neq n}}^{2n-1} h(\mathrm{F}_j,\mathrm{F}_j)\tilde{h}(J(\nu),\nabla^S_{\mathrm{F}_j}\mathrm{F}_j)\\
&\overset{(3.20)}{=} 2k\alpha \sum_{\substack{j=1\\ j\neq n}}^{2n-1} h(\mathrm{F}_j,\mathrm{F}_j)^2\langle\nabla^S_{\mathrm{F}_j}J(\nu),\mathrm{F}_j\rangle + 2k\alpha\ell \sum_{\substack{j=1\\ j\neq n}}^{2n-1} h(\mathrm{F}_j,\mathrm{F}_j)\langle\nabla^S_{\mathrm{F}_j}\mathrm{F}_j,J(\nu)\rangle\\
&= 2k\alpha \sum_{\substack{j=1\\ j\neq n}}^{2n-1} h(\mathrm{F}_j,\mathrm{F}_j)^2\langle\nabla_{\mathrm{F}_j}J(\nu),\mathrm{F}_j\rangle + 2k\alpha\ell \sum_{\substack{j=1\\ j\neq n}}^{2n-1} h(\mathrm{F}_j,\mathrm{F}_j)\langle\nabla_{\mathrm{F}_j}\mathrm{F}_j,J(\nu)\rangle\\
&= -2k\alpha \sum_{\substack{j=1\\ j\neq n}}^{2n-1} h(\mathrm{F}_j,\mathrm{F}_j)^2 h(\mathrm{F}_j,J(\mathrm{F}_j)) + 2k\alpha\ell \sum_{\substack{j=1\\ j\neq n}}^{2n-1} h(\mathrm{F}_j,\mathrm{F}_j)h(\mathrm{F}_j,J(\mathrm{F}_j))\\
&\overset{(4.2)}{=} 2k\alpha^2 \sum_{\substack{j=1\\ j\neq n}}^{2n-1} h(\mathrm{F}_j,\mathrm{F}_j)^2 - 2k\alpha^2\ell \sum_{\substack{j=1\\ j\neq n}}^{2n-1} h(\mathrm{F}_j,\mathrm{F}_j)\\
&\overset{H\equiv 0}{=} 2k\alpha^2 \sum_{\substack{j=1\\ j\neq n}}^{2n-1} h(\mathrm{F}_j,\mathrm{F}_j)^2 + 2k\alpha^2\ell^2\\
&= 2k\alpha^2|\tilde{h}|^2.
\end{aligned} \tag{5.25}
$$

Finally,

$$
\begin{aligned}
\sum_{\substack{j=1\\ j\neq n}}^{2n-1} \nabla^S_{\mathrm{F}_j}\tilde{h}(J(\nu),\mathrm{F}_j)^2 &= \sum_{\substack{j=1\\ j\neq n}}^{2n-1} \Big(\mathrm{F}_j\Big(\tilde{h}(J(\nu),\mathrm{F}_j)\Big) - \tilde{h}(\nabla^S_{\mathrm{F}_j}J(\nu),\mathrm{F}_j) - \tilde{h}(J(\nu),\nabla^S_{\mathrm{F}_j}\mathrm{F}_j)\Big)^2\\
&\overset{(\mathrm{H1})}{=} \sum_{\substack{j=1\\ j\neq n}}^{2n-1} \Big(\tilde{h}(\nabla^S_{\mathrm{F}_j}J(\nu),\mathrm{F}_j) + \tilde{h}(J(\nu),\nabla^S_{\mathrm{F}_j}\mathrm{F}_j)\Big)^2\\
&\overset{(3.20)}{=} \sum_{\substack{j=1\\ j\neq n}}^{2n-1} \Big(\langle\nabla^S_{\mathrm{F}_j}J(\nu),\mathrm{F}_j\rangle\tilde{h}(\mathrm{F}_j,\mathrm{F}_j) + \langle\nabla^S_{\mathrm{F}_j}\mathrm{F}_j,J(\nu)\rangle\ell\Big)^2\\
&= \sum_{\substack{j=1\\ j\neq n}}^{2n-1} \Big(-h(\mathrm{F}_j,J(\mathrm{F}_j))\tilde{h}(\mathrm{F}_j,\mathrm{F}_j) + h(\mathrm{F}_j,J(\mathrm{F}_j))\ell\Big)^2\\
&\overset{(4.2)}{=} \sum_{\substack{j=1\\ j\neq n}}^{2n-1} \Big(\alpha\tilde{h}(\mathrm{F}_j,\mathrm{F}_j) - \alpha\ell\Big)^2\\
&= \alpha^2 \sum_{\substack{j=1\\ j\neq n}}^{2n-1} \tilde{h}(\mathrm{F}_j,\mathrm{F}_j)^2 - 2\alpha^2\ell \sum_{\substack{j=1\\ j\neq n}}^{2n-1} \tilde{h}(\mathrm{F}_j,\mathrm{F}_j) + (2n-2)\alpha^2\ell^2\\
&\overset{H\equiv 0}{=} \alpha^2|\tilde{h}|^2 - \alpha^2\ell^2 + 2\alpha^2\ell^2 + (2n-2)\alpha^2\ell^2\\
&= \alpha^2|\tilde{h}|^2 + (2n-1)\alpha^2\ell^2.
\end{aligned} \tag{5.26}
$$

In this way we can conclude that

$$\begin{aligned}
\mathrm{I} &\overset{(5.24)}{=} k\sum_{\substack{j=1\\ j\neq n}}^{2n-1}\nabla^S_{\mathrm{F}_j}\tilde h(J(\nu),\mathrm{F}_j)^2-2k\alpha\sum_{\substack{j=1\\ j\neq n}}^{2n-1}h(\mathrm{F}_j,\mathrm{F}_j)\nabla^S_{\mathrm{F}_j}\tilde h(J(\nu),\mathrm{F}_j)+k\alpha^2|\tilde h|^2-k\alpha^2\ell^2\\
&\overset{(5.25)}{=} k\sum_{\substack{j=1\\ j\neq n}}^{2n-1}\nabla^S_{\mathrm{F}_j}\tilde h(J(\nu),\mathrm{F}_j)^2+3k\alpha^2|\tilde h|^2-k\alpha^2\ell^2\\
&= 2\sum_{\substack{j=1\\ j\neq n}}^{2n-1}\nabla^S_{\mathrm{F}_j}\tilde h(J(\nu),\mathrm{F}_j)^2-(2-k)\sum_{\substack{j=1\\ j\neq n}}^{2n-1}\nabla^S_{\mathrm{F}_j}\tilde h(J(\nu),\mathrm{F}_j)^2+3k\alpha^2|\tilde h|^2-k\alpha^2\ell^2\\
&\overset{(5.26)}{=} \sum_{\substack{j=1\\ j\neq n}}^{2n-1}\nabla^S_{\mathrm{F}_j}\tilde h(J(\nu),\mathrm{F}_j)^2+\sum_{\substack{j=1\\ j\neq n}}^{2n-1}\nabla^S_{\mathrm{F}_j}\tilde h(\mathrm{F}_j,J(\nu))^2-(2-k)\alpha^2|\tilde h|^2-(2-k)(2n-1)\alpha^2\ell^2\\
&\quad+3k\alpha^2|\tilde h|^2-k\alpha^2\ell^2\\
&= \sum_{\substack{j=1\\ j\neq n}}^{2n-1}\nabla^S_{\mathrm{F}_j}\tilde h(J(\nu),\mathrm{F}_j)^2+\sum_{\substack{j=1\\ j\neq n}}^{2n-1}\nabla^S_{\mathrm{F}_j}\tilde h(\mathrm{F}_j,J(\nu))^2+(4k-2)\alpha^2|\tilde h|^2+2(1+kn-k-2n)\alpha^2\ell^2.
\end{aligned}$$

On the other hand,

$$\mathrm{II}\overset{(3.36)}{=} k\sum_{\substack{i=1\\ i\neq n}}^{2n-1}\nabla^S_{J(\nu)}\tilde h(\mathrm{F}_i,J(\nu))^2\leqslant\sum_{\substack{i=1\\ i\neq n}}^{2n-1}\nabla^S_{J(\nu)}\tilde h(\mathrm{F}_i,J(\nu))^2+\sum_{\substack{i=1\\ i\neq n}}^{2n-1}\nabla^S_{J(\nu)}\tilde h(J(\nu),\mathrm{F}_i)^2.$$

Finally,

$$\begin{aligned}
\mathrm{III} &\overset{(3.36)}{=} k\sum_{\substack{i,j=1\\ i\neq j\\ i,j\neq n}}^{2n-1}\Big(\nabla^S_{\mathrm{F}_j}\tilde h(\mathrm{F}_i,\mathrm{F}_j)+3\mathrm{F}_j\alpha C(\mathrm{F}_j,\mathrm{F}_i)\Big)^2\\
&= k\sum_{\substack{i,j=1\\ i\neq j\\ i,j\neq n}}^{2n-1}\nabla^S_{\mathrm{F}_j}\tilde h(\mathrm{F}_i,\mathrm{F}_j)^2+6k\sum_{\substack{i,j=1\\ i\neq j\\ i,j\neq n}}^{2n-1}\mathrm{F}_j\alpha\nabla^S_{\mathrm{F}_j}\tilde h(\mathrm{F}_i,\mathrm{F}_j)C(\mathrm{F}_j,\mathrm{F}_i)+9k\sum_{\substack{i,j=1\\ i\neq j\\ i,j\neq n}}^{2n-1}(\mathrm{F}_j\alpha)^2C(\mathrm{F}_j,\mathrm{F}_i)^2\\
&= \frac{k}{2}\sum_{\substack{i,j=1\\ i\neq j\\ i,j\neq n}}^{2n-1}\nabla^S_{\mathrm{F}_j}\tilde h(\mathrm{F}_i,\mathrm{F}_j)^2+\frac{k}{2}\sum_{\substack{i,j=1\\ i\neq j\\ i,j\neq n}}^{2n-1}\nabla^S_{\mathrm{F}_j}\tilde h(\mathrm{F}_j,\mathrm{F}_i)^2+6k\sum_{\substack{j=1\\ j\neq n}}^{2n-1}\mathrm{F}_j\alpha\nabla^S_{\mathrm{F}_j}\tilde h(J(\mathrm{F}_j),\mathrm{F}_j)+9k\sum_{\substack{j=1\\ j\neq n}}^{2n-1}(\mathrm{F}_j\alpha)^2\\
&= \frac{k}{2}\sum_{\substack{i,j=1\\ i\neq j\\ i,j\neq n}}^{2n-1}\nabla^S_{\mathrm{F}_j}\tilde h(\mathrm{F}_i,\mathrm{F}_j)^2+\frac{k}{2}\sum_{\substack{i,j=1\\ i\neq j\\ i,j\neq n}}^{2n-1}\nabla^S_{\mathrm{F}_j}\tilde h(\mathrm{F}_j,\mathrm{F}_i)^2+6k\sum_{\substack{j=1\\ j\neq n}}^{2n-1}\mathrm{F}_j\alpha\nabla^S_{\mathrm{F}_j}\tilde h(J(\mathrm{F}_j),\mathrm{F}_j)\\
&\quad+9k|\nabla^{\mathcal{H},S}\alpha|^2-9k\langle\nabla^S\alpha,J(\nu)\rangle^2.
\end{aligned}$$

In this way, exploiting (H2), we conclude that

$$\mathrm{III}\leqslant\sum_{\substack{i,j=1\\ i\neq j\\ i,j\neq n}}^{2n-1}\nabla^S_{\mathrm{F}_j}\tilde h(\mathrm{F}_i,\mathrm{F}_j)^2+\sum_{\substack{i,j=1\\ i\neq j\\ i,j\neq n}}^{2n-1}\nabla^S_{\mathrm{F}_j}\tilde h(\mathrm{F}_j,\mathrm{F}_i)^2,$$

so that (5.20) follows. □

5.5. **Simons-Kato inequalities.** Combining the estimates of Section 5.2, Section 5.3 and Section 5.4, we can derive several consequences. For future convenience, we focus on the case $n = 2$, since the higher dimensional estimate, owing to (5.10), follows in the same way. To this aim, we fix a minimal hypersurface $S \subseteq \mathbb{H}^2$ such that (H1), (H2) and (H3) hold. Moreover, for any $\delta \geqslant 0$ we define the function $A(\delta)$ by

$$A(\delta)(p) = \sqrt{|\tilde{h}_p|^2 + \delta} \tag{5.27}$$

for any $p \in S$. Notice that the function $A(\delta)$ belongs to $C^\infty(S)$, for any $\delta > 0$. This desingularization will be crucial in the forthcoming Section 6. Observe that $A_\delta \geqslant |\tilde{h}|$ and that $A_\delta \to |\tilde{h}|$ uniformly on $S$ as $\delta \to 0^+$. Since $A(\delta) \geqslant 0$, we can multiply (5.18) by $4A(\delta)^2$, so that

$$2A(\delta)^2 \hat{\Delta}^{\mathcal{H},S}|\tilde{h}|^2 \geqslant 4A(\delta)^2|\nabla^{\mathcal{H},S}\tilde{h}|^2 - 4qA(\delta)^2|\tilde{h}|^2 + 4\alpha^2 A(\delta)^2 \left((4-2\omega)|\tilde{h}|^2 - (6-3\omega)\ell^2\right) \tag{5.28}$$

for any $\delta \geqslant 0$ and any $\omega \in [0,2]$. In particular, since $\omega \leqslant 2$, then $6 - 3\omega \geqslant 0$, so that we can apply Proposition 5.8 to infer that

$$(4-2\omega)|\tilde{h}|^2 - (6-3\omega)\ell^2 \overset{(5.17)}{\geqslant} (4-2\omega)|\tilde{h}|^2 - \frac{2}{3}(6-3\omega)|\tilde{h}|^2 = 0.$$

Therefore, we deduce from (5.28) that

$$2A(\delta)^2 \hat{\Delta}^{\mathcal{H},S}|\tilde{h}|^2 \geqslant 4A(\delta)^2|\nabla^{\mathcal{H},S}\tilde{h}|^2 - 4qA(\delta)^2|\tilde{h}|^2 + 4\alpha^2 |\tilde{h}|^2 \left((4-2\omega)|\tilde{h}|^2 - (6-3\omega)\ell^2\right) \tag{5.29}$$

for any $\delta \geqslant 0$ and any $\omega \in [0,2]$. Moreover, (5.20) implies that

$$\left(1 + \frac{k}{2n-1}\right)|\nabla^S|\tilde{h}|^2|^2 \leqslant 4A(\delta)^2|\nabla^{\mathcal{H},S}\tilde{h}|^2 + 4\alpha^2|\tilde{h}|^2\left((4k-2)|\tilde{h}|^2 + (2+2kn-2k-4n)\ell^2\right) \tag{5.30}$$

for any $\delta \geqslant 0$ and any $k \in [0,2]$. Therefore, combining (5.29) and (5.30) and recalling (5.12), we conclude that

$$2A(\delta)^2 \hat{\Delta}^{\mathcal{H},S}|\tilde{h}|^2 \geqslant \left(1 + \frac{k}{2n-1}\right)|\nabla^S|\tilde{h}|^2|^2 - 4qA(\delta)^4 + 4\alpha^2|\tilde{h}|^2 g_{S,k,\omega} \tag{5.31}$$

for any $\delta \geqslant 0$ and any $k, \omega \in [0,2]$, where

$$g_{S,k,\omega}(p) := (6 - 2\omega - 4k)|\tilde{h}_p|^2 + (3\omega - 2k)\,\ell(p)^2 \tag{5.32}$$

for any $p \in S$. On the other hand, when $n \geqslant 2$ the very same argument allows to combine (5.10) with (5.20) to deduce that

$$2A(\delta)^2 \hat{\Delta}^{\mathcal{H},S}|\tilde{h}|^2 \geqslant \left(1 + \frac{k}{2n-1}\right)|\nabla^S|\tilde{h}|^2|^2 - 4qA(\delta)^4 + 4\alpha^2|\tilde{h}|^2 \tilde{g}_{S,n,k} \tag{5.33}$$

for any $\delta \geqslant 0$ and any $k \in [0,2]$, where

$$\tilde{g}_{S,n,k}(p) = (2-4k)\alpha^2|\tilde{h}_p|^2 - (2+2kn-2k-4n)\alpha^2\ell(p)^2$$

for any $p \in S$.

## 6. The improved stability inequality

In this section we prove the sub-Riemannian analogue of [63, Theorem 1] in the Heisenberg group. We aim to rely on the Simons-Kato inequalities which we achieved in Section 5.5. For future purposes, we specialize our exposition in $\mathbb{H}^2$, but the reader will not encounter difficulties in adapting the same strategy to higher dimensional statements (cf. Remark 6.2).

**Theorem 6.1.** *Let $S \subseteq \mathbb{H}^2$ be a smooth, connected, two-sided, embedded, non-characteristic hypersurface. Assume that $S$ is stable. Assume that (H1), (H2) and (H3) hold for some $\omega \in [0,2]$. Assume that*

$$g_{S,k,\omega}(p) \geqslant 0 \tag{6.1}$$

*for a given* $k \in [0,2]$ *and for any* $p \in S$. *Let* $\beta \in \left[\frac{2n-1-k}{2n-1}, 1+\sqrt{\frac{k}{2n-1}}\right)$. *There exists a constant* $C = C(\beta,k) > 0$, *thus independent on* $S$, *such that*

$$\int_S |\tilde{h}|^{2\beta+2}\varphi^{2\beta+2}\, d\sigma_{\mathcal{H}} \leqslant C \int_S |\nabla^{\mathcal{H},S}\varphi|^{2\beta+2}\, d\sigma_{\mathcal{H}} \tag{6.2}$$

*for any* $\varphi \in C^1_c(S)$.

*Proof.* Let us recall (cf. [24, Lemma 11.6]) that

$$\hat{\Delta}^{\mathcal{H},S}(F\circ u) = \left(\ddot{F}\circ u\right)|\nabla^{\mathcal{H},S}u|^2 + \left(\dot{F}\circ u\right)\hat{\Delta}^{\mathcal{H},S}u \tag{6.3}$$

for any $F \in C^2(\mathbb{R})$ and any $u \in C^2(S)$. Fix $\beta, \delta > 0$, $k \in [0,2]$ and a test function $\varphi \in C^1_c(S)$. Choosing $\xi = A(\delta)^\beta\varphi$ in the stability inequality (3.41), where $A(\delta)$ is as in (5.27), we infer that

$$\begin{aligned}
\int_S q\left(A(\delta)^\beta\varphi\right)^2 d\sigma_{\mathcal{H}} &\overset{(3.41)}{\leqslant} \int_S \left|\nabla^{\mathcal{H},S}\left(A(\delta)^\beta\varphi\right)\right|^2 d\sigma_{\mathcal{H}}\\
&= \int_S \left|\frac{\beta}{2}\varphi A(\delta)^{\beta-2}\nabla^{\mathcal{H},S}|\tilde{h}|^2 + A(\delta)^\beta\nabla^{\mathcal{H},S}\varphi\right|^2 d\sigma_{\mathcal{H}}\\
&= \int_S \frac{\beta^2}{4}\varphi^2 A(\delta)^{2\beta-4}|\nabla^{\mathcal{H},S}|\tilde{h}|^2|^2 + \beta\varphi A(\delta)^{2\beta-2}\langle\nabla^{\mathcal{H},S}|\tilde{h}|^2, \nabla^{\mathcal{H},S}\varphi\rangle + A(\delta)^{2\beta}|\nabla^{\mathcal{H},S}\varphi|^2\, d\sigma_{\mathcal{H}}\\
&= \int_S \frac{\beta^2}{4}\varphi^2 A(\delta)^{2\beta-4}|\nabla^{\mathcal{H},S}|\tilde{h}|^2|^2 + \frac{1}{2}\langle\nabla^{\mathcal{H},S}A(\delta)^{2\beta}, \nabla^{\mathcal{H},S}\varphi^2\rangle + A(\delta)^{2\beta}|\nabla^{\mathcal{H},S}\varphi|^2\, d\sigma_{\mathcal{H}}\\
&\overset{(3.30)}{=} \int_S \frac{\beta^2}{4}\varphi^2 A(\delta)^{2\beta-4}|\nabla^{\mathcal{H},S}|\tilde{h}|^2|^2 - \frac{1}{2}\varphi^2\hat{\Delta}^{\mathcal{H},S}A(\delta)^{2\beta} + A(\delta)^{2\beta}|\nabla^{\mathcal{H},S}\varphi|^2\, d\sigma_{\mathcal{H}}\\
&\overset{(6.3)}{=} \int_S \frac{\beta}{2}\left(1-\frac{\beta}{2}\right)\varphi^2 A(\delta)^{2\beta-4}|\nabla^{\mathcal{H},S}|\tilde{h}|^2|^2 - \frac{\beta}{4}\varphi^2 A(\delta)^{2\beta-4}\left(2A(\delta)^2\hat{\Delta}^{\mathcal{H},S}|\tilde{h}|^2\right) + A(\delta)^{2\beta}|\nabla^{\mathcal{H},S}\varphi|^2\, d\sigma_{\mathcal{H}}\\
&\overset{(5.31)}{\leqslant} \int_S \frac{\beta}{2}\left(1-\frac{\beta}{2}\right)\varphi^2 A(\delta)^{2\beta-4}|\nabla^{\mathcal{H},S}|\tilde{h}|^2|^2 + A(\delta)^{2\beta}|\nabla^{\mathcal{H},S}\varphi|^2 - \beta\varphi^2\alpha^2|\tilde{h}|^2 A(\delta)^{2\beta-4}g_{S,k,\omega}\, d\sigma_{\mathcal{H}}\\
&\quad + \int_S -\frac{\beta}{4}\left(1+\frac{k}{2n-1}\right)\varphi^2 A(\delta)^{2\beta-4}|\nabla^{\mathcal{H},S}|\tilde{h}|^2|^2 + \beta q\left(A(\delta)^\beta\varphi\right)^2 d\sigma_{\mathcal{H}}\\
&\overset{(6.1)}{\leqslant} \int_S \frac{\beta}{4}\left(1-\beta-\frac{k}{2n-1}\right)\varphi^2 A(\delta)^{2\beta-4}|\nabla^{\mathcal{H},S}|\tilde{h}|^2|^2 + \beta q\left(A(\delta)^\beta\varphi\right)^2 + A(\delta)^{2\beta}|\nabla^{\mathcal{H},S}\varphi|^2\, d\sigma_{\mathcal{H}}.
\end{aligned}$$

Therefore, factoring out, we deduce that

$$\frac{\beta}{4}\left(\frac{k}{2n-1}+\beta-1\right)\int_S \varphi^2 A(\delta)^{2\beta-4}|\nabla^{\mathcal{H},S}|\tilde{h}|^2|^2\, d\sigma_{\mathcal{H}} \leqslant \int_S (\beta-1)q\left(A(\delta)^\beta\varphi\right)^2 + A(\delta)^{2\beta}|\nabla^{\mathcal{H},S}\varphi|^2\, d\sigma_{\mathcal{H}}. \tag{6.4}$$

Assume first that $\beta \in \left[1, 1+\sqrt{\frac{k}{2n-1}}\right)$. In particular, $\beta - 1 \geqslant 0$. Therefore, exploiting again (3.41) and combining Cauchy-Schwarz and Young inequalities,

$$\begin{aligned}
\int_S q\left(A(\delta)^\beta\varphi\right)^2 d\sigma_{\mathcal{H}} &\overset{(3.41)}{\leqslant} \int_S \left|\nabla^{\mathcal{H},S}\left(A(\delta)^\beta\varphi\right)\right|^2 d\sigma_{\mathcal{H}}\\
&= \int_S \frac{\beta^2}{4}\varphi^2 A(\delta)^{2\beta-4}|\nabla^{\mathcal{H},S}|\tilde{h}|^2|^2 + \beta\varphi A(\delta)^{2\beta-2}\langle\nabla^{\mathcal{H},S}|\tilde{h}|^2, \nabla^{\mathcal{H},S}\varphi\rangle + A(\delta)^{2\beta}|\nabla^{\mathcal{H},S}\varphi|^2\, d\sigma_{\mathcal{H}}\\
&\leqslant \int_S \frac{\beta^2}{4}\varphi^2 A(\delta)^{2\beta-4}|\nabla^{\mathcal{H},S}|\tilde{h}|^2|^2 + \beta\left(\varphi A(\delta)^{\beta-2}\left|\nabla^{\mathcal{H},S}|\tilde{h}|^2\right|\right)\left(A(\delta)^\beta\left|\nabla^{\mathcal{H},S}\varphi\right|\right) + A(\delta)^{2\beta}|\nabla^{\mathcal{H},S}\varphi|^2\, d\sigma_{\mathcal{H}}\\
&\leqslant \int_S \frac{\beta^2}{4}\varphi^2 A(\delta)^{2\beta-4}|\nabla^{\mathcal{H},S}|\tilde{h}|^2|^2 + A(\delta)^{2\beta}|\nabla^{\mathcal{H},S}\varphi|^2\, d\sigma_{\mathcal{H}}\\
&\quad + \int_S \frac{\varepsilon\beta}{4}\varphi^2 A(\delta)^{2\beta-4}\left|\nabla^{\mathcal{H},S}|\tilde{h}|^2\right|^2 + \frac{\beta}{\varepsilon}A(\delta)^{2\beta}\left|\nabla^{\mathcal{H},S}\varphi\right|^2 d\sigma_{\mathcal{H}}\\
&= \int_S \frac{\beta}{4}(\beta+\varepsilon)\varphi^2 A(\delta)^{2\beta-4}\left|\nabla^{\mathcal{H},S}|\tilde{h}|^2\right|^2 + \left(1+\frac{\beta}{\varepsilon}\right)A(\delta)^{2\beta}\left|\nabla^{\mathcal{H},S}\varphi\right|^2 d\sigma_{\mathcal{H}}
\end{aligned}$$

for any given $\varepsilon > 0$, so that

$$(\beta-1)\int_S q\left(A(\delta)^\beta\varphi\right)^2 d\sigma_{\mathcal{H}} \leqslant \frac{\beta}{4}(\beta-1)(\beta+\varepsilon)\int_S \varphi^2 A(\delta)^{2\beta-4}\left|\nabla^{\mathcal{H},S}|\tilde{h}|^2\right|^2 d\sigma_{\mathcal{H}} + (\beta-1)\left(1+\frac{\beta}{\varepsilon}\right)\int_S A(\delta)^{2\beta}\left|\nabla^{\mathcal{H},S}\varphi\right|^2 d\sigma_{\mathcal{H}}. \tag{6.5}$$

Combining (6.4) and (6.5),

$$\frac{\beta}{4}\left(\frac{k}{2n-1}+\beta-1\right)\int_S \varphi^2 A(\delta)^{2\beta-4}|\nabla^{\mathcal{H},S}|\tilde{h}|^2|^2\,d\sigma_{\mathcal{H}} \leqslant \frac{\beta}{4}(\beta-1)(\beta+\varepsilon)\int_S \varphi^2 A(\delta)^{2\beta-4}\left|\nabla^{\mathcal{H},S}|\tilde{h}|^2\right|^2 d\sigma_{\mathcal{H}}$$
$$+(\beta-1)\left(1+\frac{\beta}{\varepsilon}\right)\int_S A(\delta)^{2\beta}\left|\nabla^{\mathcal{H},S}\varphi\right|^2 d\sigma_{\mathcal{H}} + \int_S A(\delta)^{2\beta}|\nabla^{\mathcal{H},S}\varphi|^2\,d\sigma_{\mathcal{H}},$$

so that, factoring out and dividing by $\frac{\beta}{4}$,

$$P(\varepsilon,\beta,k)\int_S \varphi^2 A(\delta)^{2\beta-4}|\nabla^{\mathcal{H},S}|\tilde{h}|^2|^2\,d\sigma_{\mathcal{H}} \leqslant Q(\varepsilon,\beta)\int_S A(\delta)^{2\beta}\left|\nabla^{\mathcal{H},S}\varphi\right|^2 d\sigma_{\mathcal{H}}, \tag{6.6}$$

where

$$P(\varepsilon,\beta,k) = -\beta^2+(2-\varepsilon)\beta+\frac{k}{2n-1}-1+\varepsilon \qquad \text{and} \qquad Q(\varepsilon,\beta) = 4+\frac{4\beta-4}{\varepsilon}.$$

Notice that

$$Q(\varepsilon,\beta) \geqslant 4$$

for any $\varepsilon > 0$ and any $\beta \geqslant 1$. Moreover,

$$P(0,\beta,k)\begin{cases} >0 & \text{if } \beta \in \left[1, 1+\sqrt{\frac{k}{2n-1}}\right) \\ =0 & \text{if } \beta = 1+\sqrt{\frac{k}{2n-1}} \\ <0 & \text{if } \beta > 1+\sqrt{\frac{k}{2n-1}}. \end{cases}$$

Since the map $\varepsilon \mapsto P(\varepsilon,\beta,k)$ is continuous and $\beta < 1+\sqrt{\frac{k}{2n-1}}$, there exists $\hat{\varepsilon} = \hat{\varepsilon}(k,\beta)$ such that

$$P(\hat{\varepsilon},\beta,k) > 0. \tag{6.7}$$

Choosing $\varepsilon = \hat{\varepsilon}$ in (6.6), we exploit (6.7) to infer that

$$\int_S \varphi^2 A(\delta)^{2\beta-4}|\nabla^{\mathcal{H},S}|\tilde{h}|^2|^2\,d\sigma_{\mathcal{H}} \leqslant \frac{Q(\hat{\varepsilon},\beta)}{P(\hat{\varepsilon},\beta,k)}\int_S A(\delta)^{2\beta}\left|\nabla^{\mathcal{H},S}\varphi\right|^2 d\sigma_{\mathcal{H}}. \tag{6.8}$$

Finally, recalling (H3), we combine (6.5) and (6.6) to conclude that

$$\begin{aligned}\int_S \varphi^2|\tilde{h}|^{2\beta+2}\,d\sigma_{\mathcal{H}} &\overset{(5.12)}{\leqslant} \int_S q\left(A(\delta)^\beta\varphi\right)^2 d\sigma_{\mathcal{H}} \\ &\overset{(6.5)}{\leqslant} \frac{\beta}{4}(\beta+\hat{\varepsilon})\int_S \varphi^2 A(\delta)^{2\beta-4}\left|\nabla^{\mathcal{H},S}|\tilde{h}|^2\right|^2 d\sigma_{\mathcal{H}} + \left(1+\frac{\beta}{\hat{\varepsilon}}\right)\int_S A(\delta)^{2\beta}\left|\nabla^{\mathcal{H},S}\varphi\right|^2 d\sigma_{\mathcal{H}} \\ &\overset{(6.8)}{\leqslant} \left(\frac{\beta(\beta+\hat{\varepsilon})Q(\hat{\varepsilon},\beta)}{4P(\hat{\varepsilon},\beta,k)}+1+\frac{\beta}{\hat{\varepsilon}}\right)\int_S A(\delta)^{2\beta}\left|\nabla^{\mathcal{H},S}\varphi\right|^2 d\sigma_{\mathcal{H}}.\end{aligned}$$

Since the constant

$$C(\beta,k) = \left(\frac{\beta(\beta+\hat{\varepsilon})Q(\hat{\varepsilon},\beta)}{4P(\hat{\varepsilon},\beta,k)}+1+\frac{\beta}{\hat{\varepsilon}}\right)$$

is independent on $\delta > 0$, the dominated convergence theorem allows to let $\delta \to 0^+$ in the previous inequality to deduce that

$$\int_S \varphi^2|\tilde{h}|^{2\beta+2}\,d\sigma_{\mathcal{H}} \leqslant C(\beta,k)\int_S |\tilde{h}|^{2\beta}|\nabla^{\mathcal{H},S}\varphi|^2\,d\sigma_{\mathcal{H}}. \tag{6.9}$$

In order to conclude, we exploit (6.9) replacing $\varphi$ with $\varphi^{\beta+1}$. In this way Hölder's inequality implies that

$$\begin{aligned}\int_S |\tilde{h}|^{2\beta+2}\varphi^{2\beta+2}\, d\sigma_{\mathcal{H}} &= \int_S |\tilde{h}|^{2\beta+2}\left(\varphi^{\beta+1}\right)^2\, d\sigma_{\mathcal{H}}\\ &\overset{(6.9)}{\leqslant} C(\beta,k)\int_S |\tilde{h}|^{2\beta}|\nabla^{\mathcal{H},S}(\varphi^{\beta+1})|^2\, d\sigma_{\mathcal{H}}\\ &= (\beta+1)^2 C(\beta,k)\int_S \left(|\tilde{h}|\varphi\right)^{2\beta}|\nabla^{\mathcal{H},S}\varphi|^2\, d\sigma_{\mathcal{H}}\\ &\leqslant (\beta+1)^2 C(\beta,k)\left(\int_S |\tilde{h}|^{2\beta+2}\varphi^{2\beta+2}\, d\sigma_{\mathcal{H}}\right)^{\frac{\beta}{\beta+1}}\left(\int_S |\nabla^{\mathcal{H},S}\varphi|^{2\beta+2}\, d\sigma_{\mathcal{H}}\right)^{\frac{1}{\beta+1}},\end{aligned}$$

whence

$$\int_S |\tilde{h}|^{2\beta+2}\varphi^{2\beta+2}\, d\sigma_{\mathcal{H}} \leqslant (\beta+1)^{2\beta+2}C(\beta,k)^{\beta+1}\int_S |\nabla^{\mathcal{H},S}\varphi|^{2\beta+2}\, d\sigma_{\mathcal{H}}.$$

The thesis follows when $\beta \geqslant 1$. Finally, assume that $\beta \in \left[\frac{2n-1-k}{2n-1}, 1\right)$. In this case, (6.4) implies that

$$\int_S |\tilde{h}|^{2\beta+2}\varphi^2\, d\sigma_{\mathcal{H}} \leqslant \int_S q\left(A(\delta)^\beta\varphi\right)^2\, d\sigma_{\mathcal{H}} \leqslant \frac{1}{1-\beta}\int_S A(\delta)^{2\beta}|\nabla^S\varphi|^2\, dS,$$

and the thesis follows as in the previous case. □

**Remark 6.2.** As already pointed out, the vary same proof of Theorem 6.1 applies for an arbitrary $n \geqslant 2$. In this case, it suffices to assume (5.9) instead of (H3), and to rely on (5.33) rather than on (5.31). Accordingly, (6.1) has to be replaced by requiring that $\tilde{g}_{S,n,k}(p) \geqslant 0$ for any $p \in S$.

## 7. The Bernstein problem

We are ready to solve the Bernstein problem in the class of hypersurfaces $S \subseteq \mathbb{H}^2$ which satisfy (H1), (H2) and (H3), providing the sub-Riemannian analogue of [63, Theorem 2]. To this aim, we are going to assume the validity of suitable sub-Riemannian volume growth conditions, inspired by the behavior of perimeter minimizers in the Heisenberg group (cf. [54, Theorem 2.2]).

**Proposition 7.1.** *Let $E \subseteq \mathbb{H}^n$ be a global perimeter minimizer with smooth, non-characteristic boundary. Then there exists a constant $c > 0$ such that*

$$P_{\mathbb{H}}(E, B_r(p)) \leqslant cr^{2n+1}$$

*for any $r > 0$ and any $p \in \partial E$.*

**Remark 7.2.** With regard to the above volume growth condition, notice that $2n+1 = Q-1$, where $Q := 2n+2$ is both the *homogeneous* and the *metric* dimension of $\mathbb{H}^n$ (cf. [64]).

Since we wish to apply Theorem 6.1, we need to ensure the validity of (6.1) for suitable values of $k$. Let us describe our approach as follows. Let $m_\ell \in [0, \frac{2}{3}]$ be such that

$$\ell(p)^2 \geqslant m_\ell|\tilde{h}_p|^2 \tag{7.1}$$

for any $p \in S$. Notice that the upper bound for $m_\ell$ follows from (5.17), while $m_\ell = 0$ can be chosen whether no further information is available.

**Proposition 7.3.** *Assume that*

$$m_\ell \leqslant \frac{2}{9} \quad\Longrightarrow\quad \omega < u(m_\ell) := \frac{3m_\ell - 6}{6m_\ell - 4}. \tag{7.2}$$

*Then there exists $k \in (\frac{3}{4}, 2]$ such that (6.1) is satisfied.*

**Remark 7.4.** The relevance of having $k > \frac{3}{4}$ in Proposition 7.3 will be evident in a few lines. Notice that the function $s \mapsto u(s)$ is continuous and increasing on $\left[0, \frac{2}{9}\right]$, and moreover $u(0) = \frac{3}{2}$ and $u\left(\frac{2}{9}\right) = 2$. In particular, (7.2) holds for any $\omega < \frac{3}{2}$ without any further information on $m_\ell$, while the best choice $\omega = 2$ can be made as soon as $m_\ell > \frac{2}{9}$.

*Proof of Proposition 7.3.* Let $k \in (\frac{3}{4}, 2]$, and assume first that $0 \leqslant \omega \leqslant \frac{1}{2}$. In this way $3\omega - 2k < 0$, so that

$$g_{S,k,\omega}(p) = (6 - 2\omega - 4k)|\tilde{h}_p|^2 + (3\omega - 2k)\,\ell(p)^2 \overset{(5.17)}{\geqslant} \left(6 - \frac{16}{3}k\right)|\tilde{h}_p|^2$$

for any $p \in S$, whence (6.1) holds for any $k \in \left(\frac{3}{4}, \frac{9}{8}\right]$. On the other hand, assume that $\omega > \frac{1}{2}$. Since $\omega > \frac{1}{2}$, we can chose $k > \frac{3}{4}$ small enough to ensure that $3\omega - 2k > 0$. Assume first that $m_\ell \in \left[0, \frac{2}{9}\right]$. In this way, by (7.2), $\omega < u(m_\ell)$, so that

$$\begin{aligned} g_{S,k,\omega}(p) &= (6 - 2\omega - 4k)|\tilde{h}_p|^2 + (3\omega - 2k)\,\ell(p)^2 \\ &\overset{(7.1)}{\geqslant} (6 - \omega(2 - 3m_\ell) - 4k - 2km_\ell)|\tilde{h}_p|^2 \\ &\overset{(7.2)}{>} \left(4\left(\frac{3}{4} - k\right) + 2m_\ell\left(\frac{3}{4} - k\right)\right)|\tilde{h}_p|^2 \end{aligned}$$

for any $p \in S$. As the last term in the above inequality vanishes when $k = \frac{3}{4}$, and due to the last strict inequality, the thesis follows when $m_\ell \in \left[0, \frac{2}{9}\right]$. Finally, assume that $m_\ell > \frac{2}{9}$. In this case, recalling that $w \leqslant 2$,

$$\begin{aligned} g_{S,k,\omega}(p) &= (6 - 2\omega - 4k)|\tilde{h}_p|^2 + (3\omega - 2k)\,\ell(p)^2 \\ &\overset{(7.1)}{\geqslant} (6 - 2\omega - 4k + m_\ell(3\omega - 2k))\,|\tilde{h}_p|^2 \\ &> \left(6 - \frac{4}{3}\omega - \frac{40}{9}k\right)|\tilde{h}_p|^2 \\ &\geqslant \frac{40}{9}\left(\frac{3}{4} - k\right)|\tilde{h}_p|^2 \end{aligned}$$

for any $p \in S$, whence the thesis follows as in the previous case. □

Before stating our main result, we point out that, in view of [3], when $n \geqslant 2$ the notion of completeness for an embedded hypersurface $S \subseteq \mathbb{H}^n$ can be equivalently given by the restriction of the ambient metric on $S$ or by the intrinsic metric of $S$, provided that the latter are induced by the Euclidean, Riemannian or sub-Riemannian structure of $\mathbb{H}^n$. Therefore, in the following we will talk without ambiguity of complete hypersurfaces.

**Theorem 7.5.** *Let $S \subseteq \mathbb{H}^2$ be a smooth, complete, connected, embedded, two sided non-characteristic hypersurface. Assume that $S$ is stable. Assume that $S$ verifies (H1), (H2) and (H3), where $\omega$ is as in (7.2). Assume in addition that there exists $p \in S$ and a constant $c > 0$ such that*

$$\lim_{r \to +\infty} \frac{\sigma_{\mathcal{H}}(S \cap B_r(p))}{r^{2n+1}} \leqslant c. \tag{7.3}$$

*Then $S$ is a vertical hyperplane.*

*Proof.* Fix $p \in S$ as in the statement. Let $(R_j)_j$ be a sequence of positive numbers such that $\lim_{j\to\infty} R_j = +\infty$. In this way, up to a subsequence, we deduce from (7.3) that

$$\sigma_{\mathcal{H}}(S \cap B_{2R_j}(p)) \leqslant \hat{c} R_j^{2n+1} \tag{7.4}$$

for any $j \in \mathbb{N}$ and a suitable constant $\hat{c} > 0$. In view of [20, Lemma 3.6], it is possible to find a positive constant $\tilde{C} > 0$ and a sequence of non-negative functions $(\varphi_j)_j \subseteq C^1_c(\mathbb{H}^2)$ such that

$$\varphi_j \equiv 1 \text{ in } B_{R_j}(p), \qquad \varphi_j \equiv 0 \text{ in } \mathbb{H}^2 \setminus B_{2R_j}(p) \qquad \text{and} \qquad |\nabla^{\mathcal{H}}\varphi_j| \leqslant \frac{\tilde{C}}{R_j}. \tag{7.5}$$

We wish to apply Theorem 6.1 to the sequence $(\varphi_j)_j$. To this aim, since $S$ is complete, then $S \cap \operatorname{supp} \varphi_j$ is compact in $S$ for any $j \in \mathbb{N}$, whence $(\varphi_j)_j \subseteq C^1_c(S)$. Since $\omega$ satisfies (7.2), Proposition 7.3 implies the existence of $k \in \left(\frac{3}{4}, 2\right]$ such that (6.1) holds. Therefore, for any fixed $j \in \mathbb{N}$, we can apply (6.2), so that

$$\begin{aligned}\int_{S\cap B_{R_j}(p)} |\tilde{h}|^{2\beta+2}\, d\sigma_{\mathcal{H}} &\overset{(7.5)}{\leqslant} \int_{S\cap B_{2R_j}(p)} |\tilde{h}|^{2\beta+2}\varphi^{2\beta+2}\, d\sigma_{\mathcal{H}} \\ &\overset{(6.2)}{\leqslant} \int_{S\cap B_{2R_j}(p)} |\nabla^{\mathcal{H},S}\varphi|^{2\beta+2}\, d\sigma_{\mathcal{H}} \\ &\leqslant \int_{S\cap B_{2R_j}(p)} |\nabla^{\mathcal{H}}\varphi|^{2\beta+2}\, d\sigma_{\mathcal{H}} \\ &\overset{(7.5)}{\leqslant} \left(\frac{\tilde{C}}{R_j}\right)^{2\beta+2} \sigma_{\mathcal{H}}(S\cap B_{2R_j}(p)) \\ &\overset{(7.4)}{\leqslant} \tilde{c}R_j^{3-2\beta}\end{aligned}$$

for a suitable positive constant $\tilde{c}$ independent of $j \in \mathbb{N}$. Observe that, if we could choose $\beta > \frac{3}{2}$ in the previous inequality, we could pass to the limit as $j \to \infty$ to infer that

$$\int_S |\tilde{h}|^{2\beta+2}\, d\sigma_{\mathcal{H}} = 0,$$

whence $\tilde{h} \equiv 0$. To this aim, it suffices to notice that

$$\frac{3}{2} < 1 + \sqrt{\frac{k}{3}} \qquad \Longleftrightarrow \qquad k > \frac{3}{4}.$$

Therefore, we can conclude that $\tilde{h} \equiv 0$. Finally, being $S$ non-characteristic, we can apply [55, Theorem 1.1] to conclude that $S$ is a vertical hyperplane. □

As a consequence of Theorem 7.5 and Proposition 7.1, we get the following corollary.

**Corollary 7.6.** *Let $E \subseteq \mathbb{H}^2$ be a global perimeter minimizer. Assume that $\partial E$ is smooth, connected and non-characteristic. Assume in addition that $\partial E$ verifies* (H1)*,* (H2) *and* (H3)*. Then $\partial E$ is a vertical hyperplane.*

**Remark 7.7.** When $S$ is a minimal umbilic hypersurface, then $S$ satisfies (H1), (H2) and (H3) with $\omega = 0$. In particular, umbilic hypersurfaces automatically satisfy (7.2).

## 8. Proof of Theorem 5.1

In this final section we provide the proof of the full Simons identity presented in Theorem 5.1.

*Proof of Theorem 5.1.* Let $\mathrm{E}_1, \ldots, \mathrm{E}_{2n-1}$ be a local orthonormal frame of $\mathcal{H}TS$. In order to avoid ambiguities when computing derivatives, we let $h_{\mathcal{S}}$ and $C_\nu$ be the $(1,0)$-tensor fields defined respectively by

$$h_{\mathcal{S}}(Z) = h(\mathcal{S}, Z) \qquad \text{and} \qquad C_\nu(Z) = C(Z,\nu)$$

for any $Z \in \Gamma(\mathcal{H}TS)$. Then

$$
\begin{aligned}
\Delta^{\mathcal{H},S} h(X,Y) &= \sum_{i=1}^{2n-1} \mathrm{Hess}^{\mathcal{H},S} h(\mathrm{E}_i, \mathrm{E}_i, X, Y)\\
&= \sum_{i=1}^{2n-1} \nabla^S_{\mathrm{E}_i}(\nabla^S_{\mathrm{E}_i} h(X,Y))\\
&\overset{(3.35)}{=} \sum_{i=1}^{2n-1} \nabla^S_{\mathrm{E}_i}(\nabla^S_X h(\mathrm{E}_i, Y) + 2C(X,\mathrm{E}_i) h_{\mathcal{S}}(Y))\\
&= \sum_{i=1}^{2n-1} \mathrm{Hess}^{\mathcal{H},S} h(\mathrm{E}_i, X, \mathrm{E}_i, Y) + \underbrace{\sum_{i=1}^{2n-1} \nabla^S_{\mathrm{E}_i}(2C(X,\mathrm{E}_i) h_{\mathcal{S}}(Y))}_{\mathrm{I}}\\
&\overset{(3.15)}{=} \sum_{i=1}^{2n-1} \mathrm{Hess}^{\mathcal{H},S} h(X, \mathrm{E}_i, \mathrm{E}_i, Y) + \underbrace{\sum_{i=1}^{2n-1} h(R^S(X,\mathrm{E}_i)\mathrm{E}_i, Y) + \sum_{i=1}^{2n-1} h(\mathrm{E}_i, R^S(X,\mathrm{E}_i)Y)}_{\mathrm{II}}\\
&\quad + \underbrace{\sum_{i=1}^{2n-1} 2C(X,\mathrm{E}_i)\nabla^S_{\mathcal{S}} h(\mathrm{E}_i, Y)}_{\mathrm{III}} + \mathrm{I}\\
&= \sum_{i=1}^{2n-1} \nabla^S_X(\nabla^S_{\mathrm{E}_i} h(\mathrm{E}_i, Y)) + \mathrm{I} + \mathrm{II} + \mathrm{III}\\
&\overset{(3.33)}{=} \sum_{i=1}^{2n-1} \nabla^S_X(\nabla^S_{\mathrm{E}_i} h(Y, \mathrm{E}_i)) + \underbrace{\sum_{i=1}^{2n-1} \nabla^S_X(2\mathrm{E}_i \alpha C(Y,\mathrm{E}_i))}_{\mathrm{IV}} + \underbrace{\sum_{i=1}^{2n-1} \nabla^S_X(2\alpha C_\nu(\mathrm{E}_i) h(\mathrm{E}_i, Y))}_{\mathrm{V}}\\
&\quad \underbrace{- \sum_{i=1}^{2n-1} \nabla^S_X(2\alpha C_\nu(Y) h(\mathrm{E}_i, \mathrm{E}_i))}_{\mathrm{VI}} + \mathrm{I} + \mathrm{II} + \mathrm{III}\\
&\overset{(3.35)}{=} \underbrace{\sum_{i=1}^{2n-1} \nabla^S_X(\nabla^S_Y h(\mathrm{E}_i, \mathrm{E}_i))}_{\mathrm{VII}} + \underbrace{\sum_{i=1}^{2n-1} \nabla^S_X(2C(Y,\mathrm{E}_i) h_{\mathcal{S}}(\mathrm{E}_i))}_{\mathrm{VIII}} + \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V} + \mathrm{VI}\\
&= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V} + \mathrm{VI} + \mathrm{VII} + \mathrm{VIII}.
\end{aligned}
$$

We need to compute separately the terms I, . . . , VIII. To this aim, we denote by $\mathrm{G}_1, \dots, \mathrm{G}_{2n-1}$ a local orthonormal frame of $\mathcal{H}TS$ such that

$$
\mathrm{G}_n = J(\nu) \qquad \text{and} \qquad J(\mathrm{G}_i) = \mathrm{G}_{n+i} \tag{8.1}
$$

for any $i = 1, \dots, n-1$ (cf. [16]). **Computation of** I.

$$\begin{aligned}
\mathrm{I} &= \operatorname{trace} \nabla^S_{\cdot}(2C(X,\cdot)h_{\mathcal{S}}(Y)) \\
&= \sum_{i=1}^{2n-1} \nabla^S_{\mathrm{G}_i}(2C(X,\mathrm{G}_i)h_{\mathcal{S}}(Y)) \\
&\overset{(3.14)}{=} \sum_{i=1}^{2n-1} 2h(\mathcal{S},Y)\nabla^S_{\mathrm{G}_i}C(X,\mathrm{G}_i) + \sum_{i=1}^{2n-1} 2C(X,\mathrm{G}_i)\nabla^S_{\mathrm{G}_i}h_{\mathcal{S}}(Y) \\
&\overset{(3.32),(3.6)}{=} \sum_{i=1}^{2n-1} 2h(\mathcal{S},Y)C(\mathrm{G}_i,\nu)h(\mathrm{G}_i,X) - \sum_{i=1}^{2n-1} 2h(\mathcal{S},Y)C(X,\nu)h(\mathrm{G}_i,\mathrm{G}_i) + 2\nabla^S_{\pi(J(X))}h_{\mathcal{S}}(Y) \\
&\overset{H\equiv 0}{=} \sum_{i=1}^{2n-1} 2h(\mathcal{S},Y)C(\mathrm{G}_i,\nu)h(\mathrm{G}_i,X) + 2\nabla^S_{\pi(J(X))}h_{\mathcal{S}}(Y) \\
&\overset{(8.1)}{=} -2h(\mathcal{S},Y)h(J(\nu),X) + 2\nabla^S_{\pi(J(X))}h_{\mathcal{S}}(Y).
\end{aligned}$$

Notice that

$$-2h(\mathcal{S},Y)h(J(\nu),X) \overset{(3.17)}{=} -2h(\mathcal{S},Y)h(X,J(\nu)) \overset{(3.9)}{=} -2Y\alpha h(X,J(\nu)) - 4\alpha^2 h(X,J(\nu))\langle Y,J(\nu)\rangle.$$

On the other hand,

$$\begin{aligned}
2\nabla^S_{\pi(J(X))}h_{\mathcal{S}}(Y) &= 2\pi(J(X))(h(\mathcal{S},Y)) - 2h(\mathcal{S},\nabla^S_{\pi(J(X))}Y) \\
&\overset{(3.9)}{=} 2\pi(J(X))(Y\alpha + 2\alpha^2\langle Y,J(\nu)\rangle - 2\nabla^S_{\pi(J(X))}Y\alpha - 4\alpha^2\langle\nabla^S_{\pi(J(X))}Y,J(\nu)\rangle \\
&= 2\operatorname{Hess}^{\mathcal{H},S}\alpha(\pi(J(X)),Y) + 8\alpha\pi(J(X))\alpha\langle Y,J(\nu)\rangle \\
&\quad + 4\alpha^2\pi(J(X))\langle Y,J(\nu)\rangle - 4\alpha^2\langle\nabla_{\pi(J(X))}Y,J(\nu)\rangle \\
&= 2\operatorname{Hess}^{\mathcal{H},S}\alpha(\pi(J(X)),Y) + 8\alpha\pi(J(X))\alpha\langle Y,J(\nu)\rangle + 4\alpha^2\langle Y,\nabla_{\pi(J(X))}J(\nu)\rangle \\
&\overset{(2.4)}{=} 2\operatorname{Hess}^{\mathcal{H},S}\alpha(\pi(J(X)),Y) + 8\alpha\pi(J(X))\alpha\langle Y,J(\nu)\rangle - 4\alpha^2\left\langle J(Y),\nabla_{\pi(J(X))}\nu\right\rangle \\
&= 2\operatorname{Hess}^{\mathcal{H},S}\alpha(\pi(J(X)),Y) + 8\alpha\pi(J(X))\alpha\langle Y,J(\nu)\rangle - 4\alpha^2 h(\pi(J(X)),\pi(J(Y))).
\end{aligned}$$

In conclusion,

$$\begin{aligned}
\mathrm{I} = {} & 2\operatorname{Hess}^{\mathcal{H},S}\alpha(\pi(J(X)),Y) + 8\alpha\pi(J(X))\alpha\langle Y,J(\nu)\rangle - 4\alpha^2 h(\pi(J(X)),\pi(J(Y))) \\
& - 2Y\alpha h(X,J(\nu)) - 4\alpha^2 h(X,J(\nu))\langle Y,J(\nu)\rangle.
\end{aligned}$$

**Computation of** VI.

$$\begin{aligned}
\mathrm{VI} &= -\sum_{i=1}^{2n-1} \nabla^S_X(2\alpha C_\nu(Y)h(\mathrm{E}_i,\mathrm{E}_i)) \\
&= -\sum_{i=1}^{2n-1} \nabla^S_X(2\alpha C_\nu(Y))h(\mathrm{E}_i,\mathrm{E}_i) - 2\alpha C(Y,\nu)\sum_{i=1}^{2n-1} \nabla^S_X h(\mathrm{E}_i,\mathrm{E}_i) \\
&= -\nabla^S_X(2\alpha C_\nu(Y))H - 2\alpha C(Y,\nu)\operatorname{trace}\nabla^S_X h(\cdot,\cdot) \\
&\overset{(4.8)}{=} -\nabla^S_X(2\alpha C_\nu(Y))H - 2\alpha C(Y,\nu)XH \\
&\overset{H\equiv 0}{=} 0.
\end{aligned}$$

**Computation of** VII. Thanks to Proposition 4.2, we infer that

$$\mathrm{VII} = \operatorname{trace}\operatorname{Hess}^{\mathcal{H},S} h(X,Y,\cdot,\cdot) = \operatorname{Hess}^{\mathcal{H},S} H(X,Y) \overset{H\equiv 0}{=} 0.$$

**Computation of VIII.**

$$\begin{aligned}
\mathrm{VIII} &= \operatorname{trace} \nabla^S_X (2C(Y,\cdot)h_{\mathcal{S}}(\cdot)) \\
&= \sum_{i=1}^{2n-1} \nabla^S_X (2C(Y,\mathrm{G}_i)h_{\mathcal{S}}(\mathrm{G}_i)) \\
&\overset{(3.14)}{=} \sum_{i=1}^{2n-1} 2h(\mathcal{S},\mathrm{G}_i)\nabla^S_X C(Y,\mathrm{G}_i) + \sum_{i=1}^{2n-1} 2C(Y,\mathrm{G}_i)\nabla^S_X h_{\mathcal{S}}(\mathrm{G}_i) \\
&\overset{(3.32)}{=} \sum_{i=1}^{2n-1} 2h(\mathcal{S},\mathrm{G}_i)C(\mathrm{G}_i,\nu)h(X,Y) - \sum_{i=1}^{2n-1} 2h(\mathcal{S},\mathrm{G}_i)C(Y,\nu)h(X,\mathrm{G}_i) + \sum_{i=1}^{2n-1} 2C(Y,\mathrm{G}_i)\nabla^S_X h_{\mathcal{S}}(\mathrm{G}_i) \\
&\overset{(8.1),(3.6)}{=} -2h(\mathcal{S},J(\nu))h(X,Y) - 2h(\mathcal{S},\nabla_X\nu)C(Y,\nu) + 2\nabla^S_X h_{\mathcal{S}}(\pi(J(Y))).
\end{aligned}$$

Notice that

$$-2h(\mathcal{S},J(\nu))h(X,Y) \overset{(3.9)}{=} \left(-2\langle\nabla\alpha,J(\nu)\rangle - 4\alpha^2\right)h(X,Y).$$

On the other hand,

$$-2h(\mathcal{S},\nabla_X\nu)C(Y,\nu) = 2\left(\nabla_X\nu\right)\alpha\langle Y,J(\nu)\rangle + 4\alpha^2 h(X,J(\nu))\langle Y,J(\nu)\rangle.$$

Finally,

$$\begin{aligned}
2\nabla^S_X h_{\mathcal{S}}(\pi(J(Y))) &= 2X\left(h(\mathcal{S},\pi(J(Y)))\right) - 2h(\mathcal{S},\nabla^S_X\pi(J(Y))) \\
&\overset{(3.9),(3.7)}{=} 2\operatorname{Hess}^{\mathcal{H},S}\alpha(X,\pi(J(Y))) + 4\alpha^2\langle\pi(J(Y)),\nabla_X J(\nu)\rangle \\
&= 2\operatorname{Hess}^{\mathcal{H},S}\alpha(X,\pi(J(Y))) - 4\alpha^2 h(X,J(\pi(J(Y)))) \\
&\overset{(3.7)}{=} 2\operatorname{Hess}^{\mathcal{H},S}\alpha(X,\pi(J(Y))) + 4\alpha^2 h(X,Y) - 4\alpha^2 h(X,J(\nu))\langle Y,J(\nu)\rangle.
\end{aligned}$$

Putting the previous equations together, we conclude that

$$\mathrm{VIII} = -2\langle\nabla\alpha,J(\nu)\rangle h(X,Y) + 2\left(\nabla_X\nu\right)\alpha\langle Y,J(\nu)\rangle + 2\operatorname{Hess}^{\mathcal{H},S}\alpha(X,\pi(J(Y))).$$

**Computation of II.**

$$\begin{aligned}
\mathrm{II} &= \sum_{i,j=1}^{2n-1} R^S(X,\mathrm{G}_i,\mathrm{G}_i,\mathrm{G}_j)h(\mathrm{G}_j,Y) + \sum_{i,j=1}^{2n-1} R^S(X,\mathrm{G}_i,Y,\mathrm{G}_j)h(\mathrm{G}_i,\mathrm{G}_j) \\
&\overset{(3.34)}{=} \sum_{i,j=1}^{2n-1} h(\mathrm{G}_i,\mathrm{G}_i)h(X,\mathrm{G}_j)h(\mathrm{G}_j,Y) - \sum_{i,j=1}^{2n-1} h(X,\mathrm{G}_i)h(\mathrm{G}_i,\mathrm{G}_j)h(\mathrm{G}_j,Y) \\
&\quad + \sum_{i,j=1}^{2n-1} h(\mathrm{G}_i,Y)h(X,\mathrm{G}_j)h(\mathrm{G}_i,\mathrm{G}_j) - \sum_{i,j=1}^{2n-1} h(X,Y)h(\mathrm{G}_i,\mathrm{G}_j)^2 \\
&\overset{H\equiv 0}{=} \sum_{i,j=1}^{2n-1} h(\mathrm{G}_i,Y)h(X,\mathrm{G}_j)(h(\mathrm{G}_i,\mathrm{G}_j) - h(\mathrm{G}_j,\mathrm{G}_i)) - |h|^2 h(X,Y) \\
&\overset{(3.17)}{=} 2\alpha\sum_{i,j=1}^{2n-1} h(\mathrm{G}_i,Y)h(X,\mathrm{G}_j)C(\mathrm{G}_j,\mathrm{G}_i) - |h|^2 h(X,Y) \\
&\overset{(3.17)}{=} 2\alpha\sum_{i,j=1}^{2n-1} h(Y,\mathrm{G}_i)h(X,\mathrm{G}_j)C(\mathrm{G}_j,\mathrm{G}_i) + 4\alpha^2\sum_{i,j=1}^{2n-1} h(X,\mathrm{G}_j)C(Y,\mathrm{G}_i)C(\mathrm{G}_j,\mathrm{G}_i) - |h|^2 h(X,Y).
\end{aligned}$$

Notice that

$$\begin{aligned}\sum_{i,j=1}^{2n-1} h(Y,\mathrm{G}_i)h(X,\mathrm{G}_j)C(\mathrm{G}_j,\mathrm{G}_i) &= \sum_{j=1}^{2n-1} h(X,\mathrm{G}_j)\left\langle \nabla_Y\nu, \sum_{i=1}^{2n-1}\langle J(\mathrm{G}_j),\mathrm{G}_i\rangle \mathrm{G}_i\right\rangle \\ &\overset{(3.10)}{=} \sum_{j=1}^{2n-1} h(X,\mathrm{G}_j)\langle\nabla_Y\nu, J(\mathrm{G}_j)\rangle \\ &= -\left\langle \nabla_X\nu, \sum_{j=1}^{2n-1}\langle J(\nabla_Y\nu),\mathrm{G}_j\rangle\mathrm{G}_j\right\rangle \\ &= -\langle\nabla_X\nu, J(\nabla_Y\nu)\rangle + \langle\nabla_X\nu,\nu\rangle\langle J(\nabla_Y\nu),\nu\rangle \\ &\overset{(3.10)}{=} -\langle\nabla_X\nu, J(\nabla_Y\nu)\rangle.\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{i,j=1}^{2n-1} h(X,\mathrm{G}_j)C(Y,\mathrm{G}_i)C(\mathrm{G}_j,\mathrm{G}_i) &= \sum_{j=1}^{2n-1} h(X,\mathrm{G}_j)\left\langle J(Y), \sum_{i=1}^{2n-1}\langle J(\mathrm{G}_j),\mathrm{G}_i\rangle\mathrm{G}_i\right\rangle \\ &= \sum_{j=1}^{2n-1} h(X,\mathrm{G}_j)\left\langle J(Y),J(\mathrm{G}_j)\right\rangle - \sum_{j=1}^{2n-1} h(X,\mathrm{G}_j)\left\langle J(Y),\nu\right\rangle\left\langle J(\mathrm{G}_j),\nu\right\rangle \\ &= \sum_{j=1}^{2n-1} h(X,\mathrm{G}_j)\left\langle Y,\mathrm{G}_j\right\rangle + h(X,J(\nu))\left\langle J(Y),\nu\right\rangle \\ &= h(X,Y) - h(X,J(\nu))\left\langle Y,J(\nu)\right\rangle.\end{aligned}$$

Therefore we conclude that

$$\mathrm{II} = \left(-|h|^2+4\alpha^2\right)h(X,Y) - 4\alpha^2 h(X,J(\nu))\left\langle Y,J(\nu)\right\rangle - 2\alpha\langle\nabla_X\nu, J(\nabla_Y\nu)\rangle.$$

**Computation of IV.**

$$\begin{aligned}\mathrm{IV} &= \sum_{i=1}^{2n-1} 2\nabla_X^S(\mathrm{G}_i\alpha)C(Y,\mathrm{G}_i) + \sum_{i=1}^{2n-1} 2\mathrm{G}_i\alpha\nabla_X^S C(Y,\mathrm{G}_i) \\ &\overset{(3.32)}{=} 2\nabla_X^S\left(\pi(J(Y))\alpha\right) + \sum_{i=1}^{2n-1} 2G_i\alpha C(\mathrm{G}_i,\nu)h(X,Y) - \sum_{i=1}^{2n-1} 2\mathrm{G}_i\alpha C(Y,\nu)h(X,\mathrm{G}_i) \\ &= 2\,\mathrm{Hess}^{\mathcal{H},S}\alpha(X,\pi(J(Y))) - 2\langle\nabla\alpha, J(\nu)\rangle h(X,Y) + 2\langle Y,J(\nu)\rangle\left(\nabla_X\nu\right)\alpha.\end{aligned}$$

**Computation of** V

$$
\begin{aligned}
\mathrm{V} =& \sum_{i=1}^{2n-1} 2X\alpha C(\mathrm{G}_i,\nu)h(\mathrm{G}_i,Y) + \sum_{i=1}^{2n-1} 2\alpha X(C(\mathrm{G}_i,\nu)h(\mathrm{G}_i,Y)) \\
&- \sum_{i=1}^{2n-1} 2\alpha C(\nabla^S_X \mathrm{G}_i,\nu)h(\mathrm{G}_i,Y) - \sum_{i=1}^{2n-1} 2\alpha C(\mathrm{G}_i,\nu)h(\nabla^S_X \mathrm{G}_i,Y) - \sum_{i=1}^{2n-1} 2\alpha C(\mathrm{G}_i,\nu)h(\mathrm{G}_i,\nabla^S_X Y) \\
\overset{(3.17)}{=}& -2X\alpha h(Y,J(\nu)) - 2\alpha X h(J(\nu),Y) + \sum_{i=1}^{2n-1} 2\alpha\langle \nabla^S_X \mathrm{G}_i, J(\nu)\rangle h(\mathrm{G}_i,Y) \\
&+ 2\alpha h(\nabla^S_X J(\nu),Y) + 2\alpha h(J(\nu),\nabla^S_X Y) \\
=& -2X\alpha h(Y,J(\nu)) - 2\alpha X\langle \nabla_{J(\nu)}\nu, Y\rangle - \sum_{i=1}^{2n-1} 2\alpha\langle \mathrm{G}_i, \nabla^S_X J(\nu)\rangle h(\mathrm{G}_i,Y) \\
&+ 2\alpha h(\nabla^S_X J(\nu),Y) + 2\alpha h(J(\nu),\nabla^S_X Y) \\
=& -2X\alpha h(Y,J(\nu)) - 2\alpha X\langle \nabla_{J(\nu)}\nu, Y\rangle + 2\alpha h(J(\nu),\nabla^S_X Y) \\
=& -2X\alpha h(Y,J(\nu)) - 2\alpha\langle \nabla^S_X \nabla_{J(\nu)}\nu, Y\rangle \\
=& -2X\alpha h(Y,J(\nu)) - 2\alpha\langle \nabla_X \nabla_{J(\nu)}\nu, Y\rangle.
\end{aligned}
$$

**Computation of** III**.**

$$
\begin{aligned}
\mathrm{III} =& 2\nabla^S_{\mathcal{S}} h(\pi(J(X)),Y) \\
=& 2\mathcal{S}\langle \nabla_{\pi(J(X))}\nu, Y\rangle - 2\langle \nabla_{\nabla^S_{\mathcal{S}}\pi(J(X))}\nu, Y\rangle - 2\langle \nabla_{\pi(J(X))}\nu, \nabla^S_{\mathcal{S}} Y\rangle \\
=& 2\langle \nabla^S_{\mathcal{S}} \nabla_{\pi(J(X))}\nu, Y\rangle + 2\langle \nabla_{\pi(J(X))}\nu, \nabla^S_{\mathcal{S}} Y\rangle - 2\langle \nabla_{\nabla^S_{\mathcal{S}}\pi(J(X))}\nu, Y\rangle - 2\langle \nabla_{\pi(J(X))}\nu, \nabla^S_{\mathcal{S}} Y\rangle \\
=& 2\langle \nabla_{\mathcal{S}} \nabla_{\pi(J(X))}\nu, Y\rangle - 2\langle \nabla_{\nabla_{\mathcal{S}}\pi(J(X))}\nu, Y\rangle + 2\langle \nabla_{\mathcal{S}}\pi(J(X)),\nu\rangle\langle \nabla_\nu \nu, Y\rangle \\
=& 2\langle R(\mathcal{S},\pi(J(X)))\nu, Y\rangle + 2\langle \nabla_{\pi(J(X))}\nabla_{\mathcal{S}}\nu, Y\rangle + 2\langle \nabla_{[\mathcal{S},\pi(J(X))]}\nu, Y\rangle \\
&- 2\langle \nabla_{\nabla_{\mathcal{S}}\pi(J(X))}\nu, Y\rangle + 2\langle \nabla_{\mathcal{S}}\pi(J(X)),\nu\rangle\langle \nabla_\nu \nu, Y\rangle \\
\overset{R\equiv 0}{=}& 2\langle \nabla_{\pi(J(X))}\nabla_{\mathcal{S}}\nu, Y\rangle + 2\langle \nabla_{[\mathcal{S},\pi(J(X))]-\nabla_{\mathcal{S}}\pi(J(X))}\nu, Y\rangle + 2\langle \nabla_{\mathcal{S}}\pi(J(X)),\nu\rangle\langle \nabla_\nu \nu, Y\rangle \\
\overset{(3.8),(3.4)}{=}& 2\langle \nabla_{\pi(J(X))}\nabla^{\mathcal{H}}\alpha, Y\rangle + 4\langle \nabla_{\pi(J(X))}\alpha^2 J(\nu), Y\rangle + 2\langle \nabla_{\mathrm{Tor}_\nabla(\pi(J(X)),\mathcal{S})}\nu, Y\rangle \\
&- 2\langle \nabla_{\nabla_{\pi(J(X))}\mathcal{S}}\nu, Y\rangle + 4\alpha\langle Y, J(\nu)\rangle h(\mathcal{S},\pi(J(X))).
\end{aligned}
$$

Notice that

$$
2\langle \nabla_{\pi(J(X))}\nabla^{\mathcal{H}}\alpha, Y\rangle = 2\langle \nabla^S_{\pi(J(X))}\nabla^{\mathcal{H}}\alpha, Y\rangle = 2\,\mathrm{Hess}^{\mathcal{H},S}\,\alpha(\pi(J(X)),Y).
$$

Moreover,

$$
\begin{aligned}
4\langle \nabla_{\pi(J(X))}\alpha^2 J(\nu), Y\rangle &= 8\alpha\langle Y, J(\nu)\rangle \pi(J(X))\alpha + 4\alpha^2\langle \nabla_{\pi(J(X))}J(\nu), Y\rangle \\
&= 8\alpha\langle Y, J(\nu)\rangle \pi(J(X))\alpha - 4\alpha^2 h(\pi(J(X)),\pi(J(Y))).
\end{aligned}
$$

In addition,

$$
\begin{aligned}
2\langle \nabla_{\mathrm{Tor}_\nabla(\pi(J(X)),\mathcal{S})}\nu, Y\rangle &\overset{(2.2)}{=} -4\langle \pi(J(X)), J(\mathcal{S})\rangle\langle \nabla_T \nu, Y\rangle \\
&= -4\langle \pi(J(X)), J(T)\rangle\langle \nabla_T \nu, Y\rangle + 4\alpha\langle \pi(J(X)), J(\nu)\rangle\langle \nabla_T \nu, Y\rangle \\
&\overset{(3.7)}{=} 0.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
-2\langle \nabla_{\nabla_{\pi(J(X))}\mathcal{S}}\nu, Y\rangle &\overset{(2.3)}{=} 2\langle \nabla_{\nabla_{\pi(J(X))}\alpha\nu}\nu, Y\rangle \\
&= 2\pi(J(X))\alpha\langle \nabla_\nu \nu, Y\rangle + 2\alpha\langle \nabla_{\nabla_{\pi(J(X))}\nu}\nu, Y\rangle \\
&\overset{(3.4)}{=} -4\alpha\pi(J(X))\alpha\langle Y, J(\nu)\rangle + 2\alpha h(\nabla_{\pi(J(X))}\nu, Y) \\
&\overset{(3.17)}{=} -4\alpha\pi(J(X))\alpha\langle Y, J(\nu)\rangle + 2\alpha h(Y, \nabla_{\pi(J(X))}\nu) + 4\alpha^2 h(\pi(J(X)),\pi(J(Y))).
\end{aligned}
$$

Finally,

$$4\alpha\langle Y, J(\nu)\rangle h(\mathcal{S}, \pi(J(X))) \overset{(3.9),(3.7)}{=} 4\alpha\pi(J(X))\alpha\langle Y, J(\nu)\rangle.$$

In conclusion, we infer that

$$\mathrm{III} = 2\,\mathrm{Hess}^{\mathcal{H},S}\,\alpha(\pi(J(X)), Y) + 8\alpha\pi(J(X))\alpha\langle Y, J(\nu)\rangle + 2\alpha h(Y, \nabla_{\pi(J(X))}\nu).$$

The thesis follows from adding the terms that we have just computed. □

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