ABP estimate on metric measure spaces via optimal transport

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Abstract

By using optimal transport theory, we establish a sharp Alexandroff–Bakelman–Pucci (ABP) type estimate on metric measure spaces with synthetic Riemannian Ricci curvature lower bounds, and prove some geometric and functional inequalities including a functional ABP estimate. Our result not only extends the border of ABP estimate, but also provides an effective substitution of Jacobi fields computation in the non-smooth framework, which has potential applications to many problems in non-smooth geometric analysis.

Keywords: ABP estimate, curvature-dimension condition, maximal principle, metric measure space, Ricci curvature, optimal transport

Contents

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1 Introduction

During the sixties, Alexandroff, Bakelman, and Pucci introduced a method, which we call ABP method today, to prove ABP estimate. It plays a key role in the proof of the Krylov–Safonov Harnack inequality and the regularity theory for fully non-linear elliptic equations, see [Cab08, CC95] for an introduction to this important theory and a beautiful proof for the isoperimetric inequality for smooth domains in the Euclidean space.

Formally speaking, the goal of the Alexandroff–Bakelman–Pucci (ABP) estimate is to *estimate the size of the contact sets*, which naturally has a non-linear version. The study of the ABP estimate on Riemannian manifolds was initiated by Cabré in [Cab97] where he considered the square of distance functions (or concave paraboloid) instead of affine functions as the touching functions, and obtained the Harnack inequalities for non-divergent elliptic equations on Riemannian manifolds with non-negative sectional curvature. Based on Cabré's idea and a work of Savin [Sav07], Wang and Zhang [WZ13] introduced a notion of contact set instead of the convex envelope, and established an explicit ABP type estimate on Riemannian manifolds with Ricci curvature bounded from below. More recently, Xia and Zhang [XZ17] established an anisotropic version of the ABP estimate, and proved several geometric inequalities using this estimate.

In the recent achievement of Gigli [Gig23] and Mondino–Semola [MS22] on the regularity theory for harmonic maps from $\mathrm{RCD}(K,N)$ to $\mathrm{CAT}(0)$ spaces, a non-sharp version of this type of estimate plays important role. Given also its importance in both elliptic equations and geometry, we therefore believe that a sharp version of ABP estimate on more general metric measure spaces has its own interest.

In the present work, we continue the study of ABP type estimate on non-smooth metric measure spaces. In particular, we shall establish a sharp version of the ABP estimate on metric measure spaces with Ricci curvature bounded from below.

As pointed by Mondino and Semola in [MS22, §4], there are

> *'a couple of deep difficulties to repeat strategy of Wang–Zhang [WZ13] in the non-smooth setting'.*

Precisely, the difficulties are:

- In the non-smooth setting, Jacobi fields computations are not available and typically one works with Wasserstein geodesics in order to take advantage of optimal transport tools.
- An initial value problem (an ODE which play a key role in the argument) has no clear counterpart in the non-smooth setting.

To overcome these difficulties, we will make full use of the powerful optimal transport techniques developed in the last decade. Optimal transport, or called optimal mass transportation, aims to evaluate the difference between two probability measures. For $p \geq 1$, $\mathcal{P}_p(X)$ denotes the set of probability measures on a metric space (X, d) with finite p-moment, i.e. $\mu \in \mathcal{P}_p(X)$ if $\mu(X) = 1$ and $\int \mathsf{d}^p(x, x_0)\,\mathrm{d}\mu(x) <$

∞ for some (and thus every) $x_0 \in X$. The L^p-transport distance, or called p-Wasserstein distance W_p, is defined by

$$W_p^p(\mu,\nu) := \inf_{\Pi} \int \mathsf{d}^p(x,y)\,\mathrm{d}\Pi(x,y)$$

where the infimum is taken among all transport plans Π with marginals $\mu, \nu \in \mathcal{P}_p(X)$.

Take $p=2$ for example, it is known that W_2 can be computed by duality

$$\frac{1}{2}W_2^2(\mu,\nu) = \sup_{(\varphi,\phi)} \left\{ \int \varphi(x)\,\mathrm{d}\mu(x) + \int \phi(y)\,\mathrm{d}\nu(y) \right\}$$

where the supremum is taken over all pairs of integrable functions (φ, ϕ) satisfying

$$\varphi(x) + \phi(y) \le \frac{\mathsf{d}^2(x,y)}{2} \qquad \forall x, y \in X.$$

Equivalently, we can consider all pairs of functions (φ,ϕ) with $\varphi \in \mathrm{Lip}(\operatorname{supp}\mu, \mathsf{d})$ and

$$\phi(y) = \varphi^c(y) := \inf_{x\in\operatorname{supp}\mu} \left(\frac{\mathsf{d}^2(x,y)}{2} - \varphi(x) \right) \qquad \forall y \in \operatorname{supp}\nu.$$

Note that ϕ defined as above is locally Lipschitz on $\operatorname{supp}\nu$. By optimal transport theory, there is a locally Lipschitz function φ, called Kantorovich potential, such that

$$\frac{1}{2}W_2^2(\mu,\nu) = \int \varphi(x)\,\mathrm{d}\mu(x) + \int \varphi^c(y)\,\mathrm{d}\nu(y).$$

Consider the following problem, which can be seen as an **inverse problem** of the optimal transport problem:

> *Given a function φ, can we find a pair of probability measures, with maximal supports, so that φ is a Kantorovich potential associated with the corresponding optimal transport problem?*

We will see that this inverse problem, together with the curvature-dimension condition, is the essence of the ABP estimate. In particular, we will see that *different optimal transport problems correspond to different contact sets.*

For L^2-optimal transport problem, following Cabré [Cab97] and Wang–Zhang [WZ13], we need to consider the following 2-contact set.

Definition 1.1 (Contact set $\mathbf{R}_2$)**.** Let Ω be a bounded open subset of X and u be a continuous function on X. For a given $t > 0$ and a compact set $\mathbf{D} \subset X$, we define the 2-contact set $\mathbf{R}_2(\mathbf{D}, \Omega, u, t)$ associated to u of opening t with vertex set $\mathbf{D}$ by

$$\mathbf{R}_2(\mathbf{D},\Omega,u,t) := \left\{ x \in \overline{\Omega} : \exists y \in \mathbf{D} \text{ s.t. } \inf_{\overline{\Omega}} \left(u + \frac{\mathsf{d}_y^2}{2t} \right) = u(x) + \frac{\mathsf{d}^2(x,y)}{2t} \right\}$$

where $\mathsf{d}_y(\cdot) := \mathsf{d}(\cdot, y)$ the distance function to a point $y \in X$.

In L^1-optimal transport, the mass will be transported along the trajectories of the gradient of a Kantorovich potential ϕ, each pair of points x, y on the same trajectory satisfies $\mathsf{d}(x,y) = |\phi(x) - \phi(y)|$. So we need to fix the distance between x, y and consider the 1-contact set $\mathbf{R}_1$ in a different way.

Definition 1.2 (Contact sets $\mathbf{R}_1^*$ and $\mathbf{R}_1$)**.** Let Ω be a bounded open subset of X. For a given continuous function u, a compact set $\mathbf{D} \subset X$ and $t \geq 0$, the contact set $\mathbf{R}_1(\mathbf{D}, \Omega, u, t)$ associated with u of opening t, with vertex set $\mathbf{D}$, is defined by

$$\mathbf{R}_1(\mathbf{D}, \Omega, u, t) := \left\{ x \in \overline{\Omega} : \exists y \in \mathbf{D} \text{ s.t. } \mathsf{d}(x,y) = t, \inf_{\overline{\Omega}} \left(u + \mathsf{d}_y\right) = u(x) + \mathsf{d}(x,y) \right\}.$$

We also denote

$$\mathbf{R}_1^*(\mathbf{D}, \Omega, u) = \left\{ x \in \overline{\Omega} : \exists y \in \mathbf{D} \quad \text{s.t.} \quad \inf_{\overline{\Omega}} \left(u + \mathsf{d}_y\right) = u(x) + \mathsf{d}(x,y) \right\}.$$

Remark 1.3. The bold letters **R** and **D** come from French words *remblais* and *déblais* respectively, which appear in the title of the well-known article [Mon81, Mémoire sur la théorie des **D**éblais et de **R**emblais] published in 1781. This article is the starting point of the optimal transport theory, written by French mathematician Gaspard Monge (1746-1818).

Using synthetic curvature-dimension theory initiated by Lott–Villani [LV09] and Sturm [Stu06a, Stu06b], and non-smooth calculus tools developed by Gigli [Gig15], in Subsection 3.1 we extend the classical ABP estimate, and Wang–Zhang's estimate [WZ13] to metric measure spaces with Riemannian Ricci curvatures bounded from below. This improves the recent estimates obtained by Gigli [Gig23, Theorem 5.9] and Mondino–Semola [MS22, Theorem 4.3], to a sharp version with explicit constants.

Theorem 1.4 (ABP estimate)**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *metric measure space with* $K \in \mathbb{R}$ *and* $N \in (1, +\infty)$*. Let* $\Omega \subset X$ *be a bounded open set with* $\mathfrak{m}(\partial\Omega) = 0$*,* $\mathbf{D} \subset X$ *be a compact set. Assume there is a continuous function* $u \in \mathrm{D}(\Delta, \Omega)$ *and* $t > 0$ *such that*

$$\mathbf{R}_2(\mathbf{D}, \Omega, u, t) \subset \Omega$$

or

$$\mathbf{R}_1(\mathbf{D}, \Omega, u, t) \subset \Omega,$$
$$\forall y \in \mathbf{D},\ \exists x \in \mathbf{R}_1(\mathbf{D}, \Omega, u, t),\ \mathsf{d}(x,y) = t,\ \inf_{\overline{\Omega}} \left(u + \mathsf{d}_y\right) = u(x) + \mathsf{d}(x,y)$$

Then for $i = 1, 2$*, we have*

$$\mathfrak{m}(\mathbf{D}) \leq \begin{cases} \mathfrak{m}(\mathbf{R}_i) \left(c_{K/N}(\Theta) + \dfrac{t s_{K/N}(\Theta)}{N\Theta} \|(\Delta u)^+\|_{L^\infty(\overline{\Omega})} \right)^N & \text{if } K < 0, \\ \mathfrak{m}(\mathbf{R}_i) \left(1 + \dfrac{t}{N} \|(\Delta u)^+\|_{L^\infty(\overline{\Omega})} \right)^N & \text{if } K = 0, \\ \mathfrak{m}(\mathbf{R}_i) \left(c_{K/N}(\Phi) + \dfrac{t s_{K/N}(\Phi)}{N\Phi} \|(\Delta u)^+\|_{L^\infty(\overline{\Omega})} \right)^N & \text{if } K > 0. \end{cases}$$

where $(\Delta u)^+$ *denotes the positive part of* Δu, $\Theta := \sup_{(x,y)\in \mathbf{D}\times\Omega} \mathsf{d}(x,y)$ *and* $\Phi := \inf_{(x,y)\in \mathbf{D}\times\Omega} \mathsf{d}(x,y)$, $c_{K/N}$ *and* $s_{K/N}$ *are distortion coefficients.*

In particular, if $K=0$, *we have*

$$\mathfrak{m}(\mathbf{D}) \le \mathfrak{m}(\mathbf{R}_i)\exp(t\|(\Delta u)^+\|_{L^\infty}), \quad i=1,2.$$

This theorem will be proved in general (possibly non-smooth) metric measure spaces $(X,\mathsf{d},\mathfrak{m})$, satisfying the synthetic condition $\mathrm{RCD}(K,N)$ of Lott–Sturm–Villani [LV09, Stu06a, Stu06b]. Here $K\in\mathbb{R}$ denotes Ricci curvature lower bound and $N\in(1,+\infty)$ denotes dimension upper bound.

Example 1.5 (Notable examples of spaces fitting our framework)**.** *The class of* $\mathrm{RCD}(K,N)$ *spaces includes the following remarkable subclasses:*

- *Measured Gromov–Hausdorff limits of* N*-dimensional Riemannian manifolds with* Ricci $\ge K$, *see [AGS14b].*
- N*-dimensional Alexandrov spaces with curvature bounded from below by* K, *see [ZZ10, Pet11].*

We refer the readers to Villani's Bourbaki seminar [Vil16] and Ambrosio's ICM-Proceeding [Amb18] for more examples and bibliography.

Final remarks:

- With the help of a non-smooth version of Otto's calculus developed by Gigli in [Gig15], we get a non-smooth version of ABP type estimate without any 'Jacobi fields computation'. This improves an estimate obtained by Gigli [Gig23] and Mondino–Semola [MS22].
- In Proposition 3.7, we prove a functional version of ABP estimate, even without the 'essentially non-branching' condition, which seems new even on $\mathbb{R}^n$.
- Our results are essentially dimension-dependent, see [Gig23] for a non-sharp, but dimension-free version.

Organization of the paper: The paper is organized as follows: In Section 2, we collect some preliminaries about the theory of metric measure space, optimal transport and curvature-dimension condition. Section 3 is devoted to proving the main theorems and their applications.

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2 Preliminaries

In this paper, (X, d) represents a complete, proper and separable geodesic space endowed with a positive Radon measure $\mathfrak{m}$ with full support. The triple $(X, \mathsf{d}, \mathfrak{m})$ is called a metric measure space.

2.1 Optimal transport and curvature-dimension condition

Metric space and Wasserstein space

We denote by

$$\mathrm{Geo(X)} := \Big\{\gamma \in \mathrm{C([0,1],X)} : \mathsf{d}(\gamma_{\mathrm{s}}, \gamma_{\mathrm{t}}) = |\mathrm{s}-\mathrm{t}|\mathsf{d}(\gamma_0, \gamma_1),\ \text{for every } \mathrm{s}, \mathrm{t} \in [0,1]\Big\}$$

the space of constant speed geodesics. The metric space (X, d) is assumed to be geodesic, this means, for each $x, y \in X$ there is $\gamma \in \mathrm{Geo(X)}$ so that $\gamma_0 = x, \gamma_1 = y$.

We denote with $\mathcal{P}(X)$ the space of all Borel probability measures over X and with $\mathcal{P}_2(X)$ the space of probability measures with finite second moment. The 2-Wasserstein distance W_2 is defined as follows: for $\mu_0, \mu_1 \in \mathcal{P}_2(X)$, set

$$W_2^2(\mu_0, \mu_1) := \inf_{\Pi} \int_{X \times X} \mathsf{d}^2(x, y)\, \mathrm{d}\Pi(x, y), \tag{2.1}$$

where the infimum is taken over all $\Pi \in \mathcal{P}(X \times X)$ with μ_0 and μ_1 as the first and the second marginal. The space of all measures achieving the minimum in (2.1) will be denoted by $\mathrm{Opt}(\mu_0, \mu_1)$ and any $\Pi \in \mathrm{Opt}(\mu_0, \mu_1)$ will be called *optimal transport plan*.

For any $t \in [0,1]$, let e_t denote the evaluation map:

$$\mathrm{e}_t : \mathrm{Geo(X)} \to \mathrm{X}, \qquad \mathrm{e_t}(\gamma) := \gamma_{\mathrm{t}}.$$

The space $(\mathcal{P}_2(X), W_2)$ is geodesic and any geodesic $(\mu_t)_{t\in[0,1]}$ in $(\mathcal{P}_2(X), W_2)$ can be lifted to a measure $\pi \in \mathcal{P}(\mathrm{Geo(X)})$, so that $(\mathrm{e}_t)_\sharp \pi = \mu_t$ for all $t \in [0,1]$. Given $\mu_0, \mu_1 \in \mathcal{P}_2(X)$, we denote by $\mathrm{OptGeo}(\mu_0, \mu_1)$ the space of all $\pi \in \mathcal{P}(\mathrm{Geo(X)})$ for which $(\mathrm{e}_0, \mathrm{e}_1)_\sharp \pi \in \mathrm{Opt}(\mu_0, \mu_1)$. Such a π will be called *dynamical optimal transport plan*. The set $\mathrm{OptGeo}(\mu_0, \mu_1)$ is non-empty for any $\mu_0, \mu_1 \in \mathcal{P}_2(X)$.

Fundamental theorem of optimal transport

It is known that W_2 can be computed with the following Kantorovich duality formula

$$\frac{1}{2} W_2^2(\mu, \nu) = \sup_{(\varphi, \varphi^c)} \left\{ \int \varphi(x)\, \mathrm{d}\mu(x) + \int \varphi^c(y)\, \mathrm{d}\nu(y) \right\}$$

where the supremum is taken over all pairs of Lipschitz functions φ and its c-transform

$$\varphi^c(y) := \inf_{x \in X} \frac{\mathsf{d}^2(x, y)}{2} - \varphi(x) \quad \forall y \in X.$$

A function $\phi : X \mapsto \mathbb{R} \cup \{-\infty\}$ is called c-concave provided it is not identically $-\infty$ and it holds $\phi = \psi^c$ for some ψ. By optimal transport theory, there is a c-concave function φ, called *Kantorovich potential*, such that

$$\frac{1}{2} W_2^2(\mu, \nu) = \int \varphi(x) \, \mathrm{d}\mu(x) + \int \varphi^c(y) \, \mathrm{d}\nu(y).$$

Definition 2.1 (c-superdifferential)**.** Let φ be a continuous function. The c-superdifferential $\partial^c \varphi \subset X \times X$ is defined as

$$\partial^c \varphi := \left\{ (x, y) \in X \times X : \varphi(x) + \varphi^c(y) = \frac{\mathsf{d}^2(x, y)}{2} \right\}.$$

The c-superdifferential $\partial^c \varphi(x)$ at $x \in X$ is the set of $y \in X$ such that $(x, y) \in \partial^c \varphi$.

We have the following important theorem about the optimality of the transport plan and c-concave functions, see [AG11, Theorem 1.13] for a proof.

Theorem 2.2 (Fundamental theorem of optimal transport)**.** *Let $\Pi \in \mathcal{P}(X \times X)$ be a probability measure with μ and ν as the first and the second marginal, such that $\int \mathsf{d}^2(x, y) \, \mathrm{d}\Pi < +\infty$. Then the following are equivalent:*

(a) The plan Π is optimal, i.e. it realizes the minimum in the Kantorovich problem (2.1).

(b) There exists a c-concave function φ such that $\max\{\varphi, 0\} \in L^1(\mu)$ and $\mathrm{supp}(\Pi) \subset \partial^c \varphi$.

Curvature-dimension condition on metric measure spaces

In order to formulate curvature-dimension conditions, we recall the definition of the distortion coefficients. For $\kappa \in \mathbb{R}$, define the functions $s_\kappa, c_\kappa : [0, +\infty) \mapsto \mathbb{R}$ (on $[0, \pi/\sqrt{\kappa})$ if $\kappa > 0$) as:

$$s_\kappa(\theta) := \begin{cases} (1/\sqrt{\kappa}) \sin(\sqrt{\kappa}\theta), & \text{if } \kappa > 0, \\ \theta, & \text{if } \kappa = 0, \\ (1/\sqrt{-\kappa}) \sinh(\sqrt{-\kappa}\theta), & \text{if } \kappa < 0 \end{cases} \tag{2.2}$$

and

$$c_\kappa(\theta) := \begin{cases} \cos(\sqrt{\kappa}\theta), & \text{if } \kappa > 0, \\ 1, & \text{if } \kappa = 0, \\ \cosh(\sqrt{-\kappa}\theta), & \text{if } \kappa < 0. \end{cases} \tag{2.3}$$

It can be seen that $s'_\kappa = c_\kappa$, and both functions s_κ, c_κ are solutions to the following (Riccati-type) ordinary differential equation

$$u'' + \kappa u = 0. \tag{2.4}$$

For $K \in \mathbb{R}, N \in [1, \infty), \theta \in (0, \infty), t \in [0, 1]$, we define the distortion coefficients $\sigma_{K,N}^{(t)}$ and $\tau_{K,N}^{(t)}(\theta)$ as

$$\sigma_{K,N}^{(t)}(\theta) := \begin{cases} \infty, & \text{if } K\theta^2 \geq N\pi^2, \\ t & \text{if } K\theta^2 = 0, \\ \dfrac{s_{\frac{K}{N}}(t\theta)}{s_{\frac{K}{N}}(\theta)} & \text{otherwise} \end{cases} \tag{2.5}$$

and

$$\tau_{K,N}^{(t)}(\theta) := t^{1/N}\sigma_{K,N-1}^{(t)}(\theta)^{(N-1)/N}. \tag{2.6}$$

The following curvature-dimension conditions were introduced independently by Lott–Villani [LV09] and Sturm [Stu06a, Stu06b] (with some differences).

Definition 2.3 (CD(K,N) condition)**.** Let $K \in \mathbb{R}$ and $N \in [1,\infty)$. A metric measure space $(X, \mathsf{d}, \mathfrak{m})$ verifies CD(K,N) if for any two $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ with bounded support there exist a dynamical optimal plan $\pi \in \mathrm{OptGeo}(\mu_0,\mu_1)$ and an optimal transport plan $\Pi \in \mathrm{Opt}(\mu_0,\mu_1)$, such that $\mu_t := (e_t)_\sharp \pi \ll \mathfrak{m}$ and for any $N' \geq N, t \in [0,1]$:

$$\mathcal{E}_{N'}(\mu_t) \geq \int_{X\times X} \tau_{K,N'}^{(1-t)}(\mathsf{d}(x,y))\rho_0^{-1/N'}(x) + \tau_{K,N'}^{(t)}(\mathsf{d}(x,y))\rho_1^{-1/N'}(y)\,\mathrm{d}\Pi(x,y) \tag{2.7}$$

where the Rényi entropy $\mathcal{E}_N$ is defined as

$$\mathcal{E}_N(\mu) := \begin{cases} \int \rho^{1-1/N}\,\mathrm{d}\mathfrak{m} & \text{if } \mu = \rho\,\mathfrak{m} \\ +\infty & \text{otherwise.} \end{cases}$$

Remark 2.4. It is worth recalling that if (M,g) is a Riemannian manifold of dimension n and $h \in C^2(M)$ with $h > 0$, then the metric measure space $(M, \mathsf{d}_g, h\,\mathrm{Vol}_g)$ (where d_g and Vol_g denote the Riemannian distance and volume induced by g) verifies CD(K,N) with $N \geq n$ if and only if (see [Stu06b, Theorem 1.7])

$$\mathrm{Ricci}_{g,h,N} := \mathrm{Ricci}_g - (N-n)\frac{\nabla_g^2 h^{\frac{1}{N-n}}}{h^{\frac{1}{N-n}}} \geq Kg.$$

In particular if $N = n$ the generalized Ricci tensor $\mathrm{Ricci}_{g,h,N} = \mathrm{Ricci}_g$ makes sense only if h is constant.

Remark 2.5. A variant of the CD(K,N) condition, called reduced curvature dimension condition and denoted by CD$^*(K,N)$ [BS10], asks for the same inequality (2.7) of CD(K,N) but the coefficients $\tau_{K,N}^{(t)}(\mathsf{d}(\gamma_0,\gamma_1))$ and $\tau_{K,N}^{(1-t)}(\mathsf{d}(\gamma_0,\gamma_1))$ are replaced by $\sigma_{K,N}^{(t)}(\mathsf{d}(\gamma_0,\gamma_1))$ and $\sigma_{K,N}^{(1-t)}(\mathsf{d}(\gamma_0,\gamma_1))$, respectively. For both definitions there is a local version and it was recently proved in [CM21] that on an essentially non-branching metric measure spaces with $\mathfrak{m}(X) < \infty$ (and in [Li24] for general σ-finite $\mathfrak{m}$), the CD$^*_{loc}(K,N)$, CD$^*(K,N)$, CD$_{loc}(K,N)$, CD(K,N) conditions are all equivalent.

2.2 Differential structure of metric measure spaces

We recall some facts about calculus in metric measure spaces following the approach of [AGS14a, AGS14b, Gig15].

A function $f : X \to \mathbb{R}$ is called Lipschitz (or more precisely L-Lipschitz) if there exists a constant $L \geq 0$ such that

$$|f(x) - f(y)| \leq L\,\mathsf{d}(x,y), \quad \forall x,y \in X.$$

We denote by $\mathrm{Lip}(X,\mathsf{d})$ the space of real valued Lipschitz functions on (X,d) and with $\mathrm{Lip}_c(\Omega,\mathsf{d})\subset\mathrm{Lip}(X,\mathsf{d})$ the sub-space of Lipschitz functions on X with compact support contained in an open subset $\Omega\subset X$.

Given $f\in\mathrm{Lip}(X,\mathsf{d})$, the *local Lipschitz constant* $\mathrm{lip}(f)(x_0)$ of f at $x_0\in X$ is defined as

$$\mathrm{lip}(f)(x_0):=\limsup_{x\to x_0}\frac{|f(x)-f(x_0)|}{\mathsf{d}(x,x_0)}\ \text{ if } x_0 \text{ is not isolated,}\quad \mathrm{lip}(f)(x_0)=0\ \text{ otherwise.}$$

We say that $f\in L^2(X,\mathfrak{m})$ is a Sobolev function in $W^{1,2}(X,\mathsf{d},\mathfrak{m})$ if

$$\inf\left\{\liminf_{n\to\infty}\int_X \mathrm{lip}(f_n)^2\mathrm{d}\mathfrak{m} : f_n\in\mathrm{Lip}_c(X,\mathsf{d}),\ f_n\to f \text{ in } L^2(X,\mathfrak{m})\right\}<+\infty.$$

For any $f\in W^{1,2}(X,\mathsf{d},\mathfrak{m})$, there exists a sequence of Lipschitz functions $(f_n)\subset L^2(X,\mathfrak{m})$, such that $f_n\to f$ and $\mathrm{lip}(f_n)\to G$ in L^2 for some $G\in L^2(X,\mathfrak{m})$. There exists a minimal function G in $\mathfrak{m}$-a.e. sense, called minimal weak upper gradient (or weak gradient for simplicity) of f, and we denote it by $|\mathrm{D}f|$.

If $W^{1,2}(X,\mathsf{d},\mathfrak{m})$ is a Hilbert space, $(X,\mathsf{d},\mathfrak{m})$ is called infinitesimally Hilbertian (cf. [AGS14b, Gig15]). In this case, for $f,u\in W^{1,2}(X,\mathsf{d},\mathfrak{m})$, we define

$$\mathrm{D}f(\nabla u):=\inf_{\epsilon>0}\frac{|\mathrm{D}(u+\epsilon f)|^2-|\mathrm{D}u|^2}{2\epsilon},$$

and we have $\mathrm{D}f(\nabla u)=\mathrm{D}u(\nabla f)$.

For infinitesimally Hilbertian metric measure spaces, Cavalletti–E.Milman [CM21] and Li [Li24] prove the following equivalence.

Proposition 2.6 (cf. Remark 2.5). *For infinitesimally Hilbertian metric measure spaces, the* $\mathrm{CD}^*_{loc}(K,N)$*,* $\mathrm{CD}^*(K,N)$*,* $\mathrm{CD}_{loc}(K,N)$*,* $\mathrm{CD}(K,N)$ *conditions are all equivalent, and we denote them by* $\mathrm{RCD}(K,N)$.

Definition 2.7 (Measure valued Laplacian, cf. [Gig15]). Let $\Omega\subset X$ be an open subset and let $u\in W^{1,2}(X,\mathsf{d},\mathfrak{m})$. We say that u is in the domain of the Laplacian of Ω, and write $u\in\mathrm{D}(\mathbf{\Delta},\Omega)$, provided there exists a signed measure μ on Ω such that for any $f\in\mathrm{Lip}_c(\Omega,\mathsf{d})$ it holds

$$\int \mathrm{D}f(\nabla u)\,\mathrm{d}\mathfrak{m}=-\int f\,\mathrm{d}\mu. \tag{2.8}$$

If μ is unique, we denote it by $\mathbf{\Delta}u$. If $\mathbf{\Delta}u\ll\mathfrak{m}$ and its density Δu is locally finite, we write $u\in\mathrm{D}(\Delta,\Omega)$.

3 Alexandroff–Bakelman–Pucci estimate

3.1 Contact sets

2-contact set

Let $\Omega\subset X$ be a bounded set, $\mathbf{D}$ be a compact set, u be a continuous function and $t>0$. Recall that the 2-contact set is defined as

$$\mathbf{R}_2(\mathbf{D},\Omega,u,t):=\left\{x\in\overline{\Omega}:\ \exists y\in\mathbf{D}\ \text{s.t.}\ \inf_{\overline{\Omega}}\left(u+\frac{\mathsf{d}_y^2}{2t}\right)=u(x)+\frac{\mathsf{d}^2(x,y)}{2t}\right\}.$$

Lemma 3.1. *Let $u \in C(\Omega)$. Then $-tu$ has a c-concave, (upper) representative on $\mathbf{R}_2$, i.e. there is a c-concave function φ so that $\varphi = -tu$ on $\mathbf{R}_2$ and $\varphi \geq -tu$ on Ω.*

Proof. Define

$$v(y) := \begin{cases} \inf_{z\in\overline{\Omega}} \left(u(z) + \frac{\mathsf{d}^2(z,y)}{2t}\right) & \text{if } y \in \mathbf{D} \\ -\infty & \text{otherwise.} \end{cases}$$

By definition,

$$v(y) - u(z) \leq \frac{\mathsf{d}^2(z,y)}{2t} \qquad \forall (y,z) \in \mathbf{D} \times \overline{\Omega} \tag{3.1}$$

and for any $x \in \mathbf{R}_2$ there is $y \in \mathbf{D}$ so that

$$v(y) - u(x) = \frac{\mathsf{d}^2(x,y)}{2t}. \tag{3.2}$$

So for any $x \in \mathbf{R}_2$, it holds

$$-tu(x) = \inf_{y\in\mathbf{D}} \left(-tv(y) + \frac{\mathsf{d}^2(x,y)}{2}\right) = \inf_{y\in X} \left(-tv(y) + \frac{\mathsf{d}^2(x,y)}{2}\right) = (tv)^c(x). \tag{3.3}$$

For $x \in \Omega \setminus \mathbf{R}_2$, by (3.1) it holds

$$(tv)^c(x) = \inf_{y\in\mathbf{D}} \left(-tv(y) + \frac{\mathsf{d}^2(x,y)}{2}\right) \geq -tu(x).$$

Then we define φ by

$$\varphi(x) := (tv)^c(x)$$

which fulfils our request. □

Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space. By the Laplacian comparison theorem [Gig15, Theorem 5.14], we know that the c-concave function φ obtained in the last lemma is in $\mathrm{D}(\mathbf{\Delta}, \Omega)$, and the positive part of $\mathbf{\Delta}\varphi$, denoted by $(\mathbf{\Delta}\varphi)^+$, is absolutely continuous and has bounded density. In addition, we have the following estimate concerning the negative part of $\mathbf{\Delta}\varphi$.

Lemma 3.2. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space. Let $\mathbf{\Delta}\varphi = \mathbf{\Delta}\varphi\llcorner_{\mathbf{R}_2} + \mathbf{\Delta}\varphi\llcorner_{\Omega\setminus\mathbf{R}_2}$ be a decomposition of $\mathbf{\Delta}\varphi$. Assume $u \in \mathrm{D}(\Delta, \Omega)$ with $\Delta u \in L^\infty$. We have*

$$\mathbf{\Delta}\varphi\llcorner_{\mathbf{R}_2} \geq -\Delta u\, \mathfrak{m}\llcorner_{\mathbf{R}_2}.$$

Proof. Let $(P_t\varphi)_{t\geq 0}$ be the heat flow from φ. We claim that $\frac{P_t\varphi - \varphi}{t}\,\mathfrak{m}$ converge weakly to $\mathbf{\Delta}\varphi$ as $t \downarrow 0$. Given a non-negative function $g \in \mathrm{TestF} := \mathrm{D}(\Delta, \Omega) \cap C_c(\Omega)$ with $\Delta g \in L^\infty$. By [MS23, Lemma 2.55] we know

$$\lim_{t\to 0} \frac{P_t g(x) - g(x)}{t} = \Delta g(x) \quad \text{for } \mathfrak{m}\text{-a.e. } x \in \Omega.$$

By dominated convergence theorem we have

$$\begin{aligned}\lim_{t\to 0}\int g\frac{P_t\varphi-\varphi}{t}\,\mathrm{d}\mathfrak{m} &= \lim_{t\to 0}\int \varphi\frac{P_tg-g}{t}\,\mathrm{d}\mathfrak{m}\\ &= \int \varphi\Delta g\,\mathrm{d}\mathfrak{m} = \int g\,\mathrm{d}\boldsymbol{\Delta}\varphi.\end{aligned}$$

By Riesz–Markov–Kakutani representation theorem and the density of TestF in $\mathrm{Lip}_c(\Omega,\mathsf{d})$, we know $\frac{P_t\varphi-\varphi}{t}$ converge to $\boldsymbol{\Delta}\varphi$ as $t\downarrow 0$.

Let $\mathrm{K}\subset\mathbf{R}_2$ be a compact set and $h\in C(\mathrm{K})$ be such that $h>0$ on K and $h=0$ on $\Omega\setminus\mathrm{K}$. We can find a sequence $(h_n)_{n\in\mathbb{N}}\subset C_c(\Omega)$ such that $h_n\geq h$ and $h_n\downarrow h$ pointwisely. Then

$$\begin{aligned}\lim_{t\to 0}\int h\frac{P_t\varphi-\varphi}{t}\,\mathrm{d}\mathfrak{m} &\leq \lim_{t\to 0}\int h_n\frac{P_t\varphi-\varphi}{t}\,\mathrm{d}\mathfrak{m}\\ &= \int h_n\,\mathrm{d}\boldsymbol{\Delta}\varphi = \int (h_n-h)\,\mathrm{d}\boldsymbol{\Delta}\varphi+\int h\,\mathrm{d}\boldsymbol{\Delta}\varphi\\ &\leq \int (h_n-h)\,\mathrm{d}(\boldsymbol{\Delta}\varphi)^+ + \int h\,\mathrm{d}\boldsymbol{\Delta}\varphi.\end{aligned}$$

Letting $n\to\infty$ we get

$$\lim_{t\to 0}\int h\frac{P_t\varphi-\varphi}{t}\,\mathrm{d}\mathfrak{m}\leq\int h\,\mathrm{d}\boldsymbol{\Delta}\varphi. \tag{3.4}$$

For any $x\in\mathbf{R}_2$, by Lemma 3.1 we know

$$P_t\varphi(x)-\varphi(x)\geq -t\big(P_tu(x)-u(x)\big).$$

Thus

$$\lim_{t\to 0}\int h\frac{P_t\varphi-\varphi}{t}\,\mathrm{d}\mathfrak{m}\geq -\lim_{t\to 0}\int_{\mathrm{K}} h(x)\frac{P_tu(x)-u(x)}{t}\,\mathrm{d}\mathfrak{m}(x) = -\int h\Delta u\,\mathrm{d}\mathfrak{m}.$$

Combining with (3.4) and the arbitrariness of h,K, we know

$$\boldsymbol{\Delta}\varphi\llcorner_{\mathbf{R}_2}\geq -\Delta u\,\mathfrak{m}\llcorner_{\mathbf{R}_2}.$$

□

1-contact set

Recall that the 1-contact set $\mathbf{R}_1(\mathbf{D},\Omega,u,t)$ is defined as

$$\mathbf{R}_1(\mathbf{D},\Omega,u,t) := \left\{x\in\overline{\Omega}:\ \exists y\in\mathbf{D}\text{ s.t. }\mathsf{d}(x,y)=t,\inf_{\overline{\Omega}}\big(u+\mathsf{d}_y\big)=u(x)+\mathsf{d}(x,y)\right\}$$

and

$$\begin{aligned}\mathbf{R}_1^*(\mathbf{D},\Omega,u) &:= \bigcup_{t\geq 0}\mathbf{R}_1(\mathbf{D},\Omega,u,t)\\ &= \left\{x\in\overline{\Omega}:\exists y\in\mathbf{D}\ \text{ s.t. }\inf_{\overline{\Omega}}\big(u+\mathsf{d}_y\big)=u(x)+\mathsf{d}(x,y)\right\}.\end{aligned}$$

We have the following lemma concerning the Lipschitz regularity of u on the contact set $\mathbf{R}_1^*$. This lemma has its own interest in the viewpoint of optimal transport theory.

Lemma 3.3. *Let $\Omega \subset X$ be a bounded open set and u be a continuous function defined on Ω. Define a 1-Lipschitz function u^{d} on $\mathbf{D}$ by*

$$u^{\mathsf{d}}(y) := \inf_{x \in \overline{\Omega}} \big(u(x) + \mathsf{d}_y(x)\big) \qquad y \in \mathbf{D}.$$

Then for any $t > 0$ we have

(1) *u is 1-Lipschitz on $\mathbf{R}_1^*(\mathbf{D}, \Omega, u)$.*

(2) *$u^{\mathsf{d}} = u$ on $\mathbf{D} \cap \mathbf{R}_1^*(\mathbf{D}, \Omega, u)$.*

(3) *For any $(x, z) \in \mathbf{R}_1^*(\mathbf{D}, \Omega, u) \times \mathbf{D}$, it holds*

$$-\mathsf{d}(x, z) \le u^{\mathsf{d}}(z) - u(x) \le \mathsf{d}(x, z).$$

Furthermore, for any $x \in \mathbf{R}_1^*(\mathbf{D}, \Omega, u)$, $y \in \mathbf{D}$ with $u(x) + \mathsf{d}(x, y) = u^{\mathsf{d}}(y)$, and any geodesic $\gamma \subset X$ connecting x and y, we have

$$u(x') - u(x) = \mathsf{d}(x', x) \qquad \forall x' \in \mathbf{R}_1^*(\mathbf{D}, \Omega, u) \cap \gamma.$$

Proof. (1) By definition, for any $x \in \mathbf{R}_1^*(\mathbf{D}, \Omega, u)$ there is $y \in \mathbf{D}$ so that

$$u^{\mathsf{d}}(y) = \inf_{\overline{\Omega}} \big(u + \mathsf{d}_y\big) = u(x) + \mathsf{d}_y(x). \tag{3.5}$$

Then for any $x' \in \mathbf{R}_1^*(\mathbf{D}, \Omega, u)$,

$$u(x') + \mathsf{d}_y(x') \ge u(x) + \mathsf{d}_y(x). \tag{3.6}$$

By triangle inequality

$$u(x) - u(x') \le \mathsf{d}_y(x') - \mathsf{d}_y(x) \le \mathsf{d}(x', x).$$

By symmetry we also have

$$u(x') - u(x) \le \mathsf{d}(x', x)$$

so u is 1-Lipschitz on $\mathbf{R}_1^*(\mathbf{D}, \Omega, u)$.

(2) For $y \in \mathbf{D} \cap \mathbf{R}_1^*(\mathbf{D}, \Omega, u)$, on one hand, since u is 1-Lipschitz on $\mathbf{R}_1^*(\mathbf{D}, \Omega, u)$, we have

$$u(y) - u(x) \le \mathsf{d}(x, y) \qquad \forall x \in \mathbf{R}_1^*(\mathbf{D}, \Omega, u)$$

so that

$$u^{\mathsf{d}}(y) = \inf_{\overline{\Omega}} \big(u + \mathsf{d}_y\big) = \inf_{x \in \mathbf{R}_1^*} \big(u(x) + \mathsf{d}(x, y)\big) \ge u(y).$$

On the other hand, we have the following trivial inequality

$$u^{\mathsf{d}}(y) = \inf_{\overline{\Omega}} \big(u + \mathsf{d}_y\big) \le u(y) + \mathsf{d}_y(y) = u(y).$$

In conclusion, $u^{\mathsf{d}} = u$ on $\mathbf{D} \cap \mathbf{R}_1^*(\mathbf{D}, \Omega, u)$.

(3) By definition of u^{d}, we have

$$u^{\mathsf{d}}(z) - u(x) \leq \mathsf{d}(x, z) \qquad (x, z) \in \overline{\Omega} \times \mathbf{D}, \tag{3.7}$$

and for any $x \in \mathbf{R}_1^*(\mathbf{D}, \Omega, u)$, there is $y \in \mathbf{D}$ such that

$$u^{\mathsf{d}}(y) - u(x) = \mathsf{d}(x, y).$$

So for any $(x, z) \in \mathbf{R}_1^*(\mathbf{D}, \Omega, u) \times \mathbf{D}$, we have

$$-\mathsf{d}(x, z) \leq u^{\mathsf{d}}(z) + \mathsf{d}(x, y) - u^{\mathsf{d}}(y) = u^{\mathsf{d}}(z) - u(x) \leq \mathsf{d}(x, z)$$

where in the first inequality we use the fact that u^{d} is 1-Lipschitz on $\mathbf{D}$.

(4) At last, let γ be a geodesic connecting $x \in \mathbf{R}_1^*(\mathbf{D}, \Omega, u)$ and $y \in \mathbf{D}$. By (3.6), for any $x' \in \mathbf{R}_1^*(\mathbf{D}, \Omega, u) \cap \gamma$ it holds

$$u(x') - u(x) \geq \mathsf{d}_y(x) - \mathsf{d}_y(x') = \mathsf{d}(x, x').$$

By (1), u is 1-Lipschitz on $\mathbf{R}_1^*(\mathbf{D}, \Omega, u)$, so we have $u(x') - u(x) = \mathsf{d}(x, x')$.

□

Lemma 3.4. *The function $-tu$ has a c-concave (upper) representative φ on $\mathbf{R}_1(\mathbf{D}, \Omega, u, t)$. If $u \in \mathrm{D}(\Delta, \Omega)$ with $\Delta u \in L^\infty$, then it holds a Laplacian estimate*

$$\mathbf{\Delta} \varphi_{\llcorner \mathbf{R}_1} \geq -\Delta u \, \mathfrak{m}_{\llcorner \mathbf{R}_1}.$$

Proof. Define

$$v(y) := \begin{cases} \inf_{z \in \overline{\Omega}} \left(u(z) + \frac{\mathsf{d}^2(z,y)}{2t} \right) & \text{if } y \in \mathbf{D} \\ -\infty & \text{otherwise.} \end{cases}$$

By definition,

$$v(y) - u(z) \leq \frac{\mathsf{d}^2(z, y)}{2t} \qquad \forall (y, z) \in \mathbf{D} \times \overline{\Omega}. \tag{3.8}$$

Given $x \in \mathbf{R}_1(\mathbf{D}, \Omega, u, t)$, there is $y_x \in \mathbf{D}$ so that

$$u^{\mathsf{d}}(y_x) = u(x) + \mathsf{d}(x, y_x) = u(x) + t. \tag{3.9}$$

Then

$$u(x) + \frac{\mathsf{d}^2(x, y_x)}{2t} = u^{\mathsf{d}}(y_x) - t + \frac{t}{2} = u^{\mathsf{d}}(y_x) - \frac{t}{2}.$$

For any other $z \in \overline{\Omega}$, by Cauchy inequality and (3.7), it holds

$$\begin{aligned} & u(z) + \frac{\mathsf{d}^2(z, y_x)}{2t} \\ \overset{\text{Cauchy}}{\geq} \; & u(z) - \frac{t}{2} + \mathsf{d}(z, y_x) \overset{(3.7)}{\geq} u^{\mathsf{d}}(y_x) - \frac{t}{2} = u(x) + \frac{\mathsf{d}^2(x, y_x)}{2t}, \end{aligned}$$

so

$$u(x)+\frac{\mathsf{d}^2(x,y_x)}{2t}=\inf_{z\in\overline{\Omega}}\left(u(z)+\frac{\mathsf{d}^2(z,y_x)}{2t}\right)=v(y_x). \tag{3.10}$$

Combining (3.8) and (3.10) we get

$$-tu(x)=\inf_{y\in\mathbf{D}}\left(-tv(y)+\frac{\mathsf{d}^2(x,y)}{2}\right)=\inf_{y\in X}\left(-tv(y)+\frac{\mathsf{d}^2(x,y)}{2}\right)=(tv)^c(x) \tag{3.11}$$

for $x\in\mathbf{R}_1$ and $-tu(x)\le(tv)^c(x)$ for $x\in\overline{\Omega}$.

Thus $(tv)^c$ is the desired upper representative, and the Laplacian estimate can be proved using the same argument as Lemma 3.2. □

3.2 Functional ABP estimate

To study the curvature-dimension condition with finite dimension parameter (cf. Definition 2.3), Erbar–Kuwada–Sturm [EKS15] introduced a notion called *entropic curvature-dimension condition* $\mathrm{CD}^e(K,N)$. For infinitesimally Hilbertian metric measure spaces, they show that $\mathrm{CD}^e(K,N)$ is equivalent to the reduced curvature-dimension condition $\mathrm{CD}^*(K,N)$ introduced by Bacher–Sturm [BS10] (cf. Remark 2.5 and Proposition 2.6). Following their footprints, we can prove the following differential inequality (see [EKS15, Lemma 2.2]).

Lemma 3.5 ((K,N)-convexity). *Let* $\mathrm{Ent}_{\mathfrak{m}}$ *be the relative entropy defined as*

$$\mathrm{Ent}_{\mathfrak{m}}(\mu):=\begin{cases}\int\rho\ln\rho\,\mathrm{d}\mathfrak{m} & \text{if } \mu=\rho\,\mathfrak{m}\\ +\infty & \text{otherwise.}\end{cases}$$

and $U_N:=\exp\left(-\frac{1}{N}\mathrm{Ent}_{\mathfrak{m}}\right)$. *Assume that* $(X,\mathsf{d},\mathfrak{m})$ *is an* $\mathrm{RCD}(K,N)$ *space. Then for each pair* $\mu_0,\mu_1\in\mathcal{P}_2(X)$ *with* $\mu_0,\mu_1\ll\mathfrak{m}$, *there is a geodesic* $(\mu_t)_{t\in[0,1]}$ *in* $\left(\mathcal{P}_2(X),W_2\right)$ *from* μ_0 *to* μ_1 *so that for all* $t\in[0,1]$ *we have*

$$U_N(\mu_1)\le c_{K/N}\big(W_2(\mu_0,\mu_1)\big)U_N(\mu_0)+\frac{s_{K/N}\big(W_2(\mu_0,\mu_1)\big)}{W_2(\mu_0,\mu_1)}\frac{\mathrm{d}^-}{\mathrm{d}t}\Big|_{t=0}U_N(\mu_t) \tag{3.12}$$

where

$$\frac{\mathrm{d}^-}{\mathrm{d}t}\Big|_{t=0}U_N(\mu_t):=\liminf_{h\downarrow0}\frac{U_N(\mu_h)-U_N(\mu_0)}{h}.$$

Proof. By [EKS15, Definition 3.1, Definition 2.7], there is a constant speed geodesic $(\mu_t)_{t\in[0,1]}$ in the Wasserstein space $\left(\mathcal{P}_2(X),W_2\right)$ connecting μ_0 and μ_1, so that for all $t\in[0,1]$ it holds

$$U_N(\mu_t)\ge\sigma^{(1-t)}_{K/N}\big(W_2(\mu_0,\mu_1)\big)U_N(\mu_0)+\sigma^{(t)}_{K/N}\big(W_2(\mu_0,\mu_1)\big)U_N(\mu_1). \tag{3.13}$$

Subtracting $U_N(\mu_0)$ on both sides of (3.13), dividing by t and letting $t\downarrow0$, we get (3.12). □

Next, we need to find an upper bound of $\frac{\mathrm{d}^-}{\mathrm{d}t}|_{t=0}U_N(\mu_t)$. To do this, we make use of a strategy of Gigli [Gig15, Proposition 5.10].

Lemma 3.6 (Bound from above on the derivative of the entropy)**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K,N)$ *metric measure space. Let* $\mu_0 \in \mathcal{P}_2(\overline{\Omega})$ *be such that* $\mu_0 \ll \mathfrak{m}\llcorner_\Omega$ *with* $\mu_0 = \rho\mathfrak{m}\llcorner_\Omega$*. Assume that* $\mathfrak{m}(\partial\Omega) = 0$ *and assume also that the restriction of* ρ *to* Ω *is Lipschitz and bounded from below by a positive constant. Let* $(\mu_t)_{t\in[0,1]}$ *be a geodesic in* $\big(\mathcal{P}_2(X), W_2\big)$*. Then it holds*

$$\liminf_{t\downarrow 0} \frac{U_N(\mu_t) - U_N(\mu_0)}{t} \leq \frac{1}{N} U_N(\mu_0) \int_\Omega \mathrm{D}\rho(\nabla\varphi)\,\mathrm{d}\mathfrak{m} \tag{3.14}$$

where φ *is any Kantorovich potential from* μ_0 *to* μ_1*.*

Proof. We work on the space $(\overline{\Omega}, \mathsf{d}, \mathfrak{m}\llcorner_\Omega)$. For any $\nu \in \mathcal{P}(\overline{\Omega})$ with $\nu \ll \mathfrak{m}\llcorner_\Omega$, the concavity of the function e^{-x} gives

$$U_N(\nu) - U_N(\mu_0) \leq -U_N(\mu_0)\left(\frac{1}{N}\mathrm{Ent}_{\mathfrak{m}}(\nu) - \frac{1}{N}\mathrm{Ent}_{\mathfrak{m}}(\mu_0)\right).$$

Since ρ, ρ^{-1} are positive and bounded, the function $\ln\rho : \overline{\Omega} \to \mathbb{R}$ is bounded. Thus the convexity of $x\ln x$ gives

$$\mathrm{Ent}_{\mathfrak{m}}(\nu) - \mathrm{Ent}_{\mathfrak{m}}(\mu_0) \geq \int_\Omega \ln(\rho)\Big(\frac{\mathrm{d}\nu}{\mathrm{d}\mathfrak{m}} - \rho\Big)\,\mathrm{d}\mathfrak{m}, \qquad \forall \nu \in \mathcal{P}(\overline{\Omega}),\ \nu \ll \mathfrak{m}.$$

Then

$$U_N(\nu) - U_N(\mu_0) \leq -\frac{1}{N} U_N(\mu_0) \int_\Omega \ln(\rho)\,\mathrm{d}\Big(\nu - \mu_0\Big).$$

Plugging $\nu := \mu_t$, dividing by t and letting $t \downarrow 0$ we get

$$\varliminf_{t\to 0} \frac{U_N(\mu_t) - U_N(\mu_0)}{t} \leq -\frac{1}{N} U_N(\mu_0) \varliminf_{t\to 0} \frac{\int_\Omega \ln\rho\,\mathrm{d}\mu_t - \int_\Omega \ln\rho\,\mathrm{d}\mu_0}{t}.$$

Applying [Gig15, Proposition 5.9], we get

$$\varliminf_{t\to 0} \frac{\int_\Omega \ln\rho\,\mathrm{d}\mu_t - \int_\Omega \ln\rho\,\mathrm{d}\mu_0}{t} \geq -\int_\Omega \mathrm{D}\rho(\nabla\varphi)\,\mathrm{d}\mathfrak{m}.$$

Combining the estimates above we complete the proof.

□

Next we will prove a functional ABP estimate.

Theorem 3.7 (A functional ABP estimate)**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K,N)$ *metric measure space. Assume that* $\mathfrak{m}(\partial\Omega) = 0$*. Let* $\mu_0 \in \mathcal{P}_2(\overline{\Omega})$ *be with non-negative density* $\rho \in \mathrm{Lip}(\Omega, \mathsf{d})$*,* $\mu_1 \in \mathcal{P}_2(X)$ *be with bounded support. Then it holds*

$$U_N(\mu_1) \leq \left(c_{K/N}\big(W_2(\mu_0,\mu_1)\big) - \frac{s_{K/N}\big(W_2(\mu_0,\mu_1)\big)}{NW_2(\mu_0,\mu_1)}\left(\int_\Omega \rho\,\mathrm{d}\mathbf{\Delta}\varphi\right)\right) U_N(\mu_0) \tag{3.15}$$

where φ *is any Kantorovich potential from* μ_0 *to* μ_1*.*

Proof. **Step 1.** Exponential map:

Let φ be a Kantorovich potential from μ_0 to μ_1, which is a c-concave function. Without loss of generality, we may assume that φ is a real-valued function on Ω. Consider the multivalued map $\Omega \ni x \mapsto T_\varphi(x) \subset \mathcal{P}(X)$ defined by:

$$\nu \in T_\varphi(x) \ \text{ if and only if } \ \operatorname{supp} \nu \subset \partial^c \varphi(x).$$

By measurable selection theorem (see e.g. [Bog07, Theorem 6.9.3] and [Gig15, proof of THEOREM 5.14, page 71]), there exists a measurable map $x \mapsto \eta_x$ such that $\eta_x \in T_\varphi(x)$ for any $x \in \Omega$.

For any $\mu \in \mathcal{P}(X)$ with $\mu\llcorner\Omega \ll \mathfrak{m}\llcorner\Omega$, define a probability measure $T_\varphi(\mu)$ by

$$T_\varphi(\mu) := \int \eta_x \,\mathrm{d}\mu(x). \tag{3.16}$$

By the fundamental theorem of optimal transport (cf. Theorem 2.2), φ is a Kantorovich potential from μ to $T_\varphi(\mu)$. By [RS14], the Monge's problem is uniquely solvable on $\mathrm{RCD}(K,N)$ spaces, and up to an additive constant, φ is the unique Kantorovich potential from μ to $T_\varphi(\mu)$. In this case, for $\mathfrak{m}$-a.e. $x \in \Omega$, there is a $y_x \in \partial^c\varphi(x)$. Then we can define a map $\nabla\varphi : \Omega \to \partial^c\varphi$ by $\nabla\varphi(x) = y_x$ such that

$$T_\varphi(\mu) = (\nabla\varphi)_\sharp \mu \quad \forall \mu \ll \mathfrak{m} \text{ on } \Omega. \tag{3.17}$$

Step 2. Entropy estimate:

Firstly, let $(\zeta_m)_{m\in\mathbb{N}} \subset \mathrm{Lip}_c(\Omega, \mathsf{d})$ be a sequence of non-negative functions, such that $\zeta_m \uparrow \rho$ pointwisely. Denote

$$\nu_{m,n,0} = C_{m,n}\left(\zeta_m + \frac{1}{n}\right)\mathfrak{m}\llcorner\Omega, \quad \zeta_{m,n,0} = C_{m,n}\left(\zeta_m + \frac{1}{n}\right)$$

where $C_{m,n}$ are normalizing constants such that $\nu_{m,n,0} \in \mathcal{P}(\Omega)$. By Step 1, there is a (unique) probability measure $\nu_{m,n,1} = T_\varphi(\nu_{m,n,0})$ so that φ is a Kantorovich potential from $\nu_{m,n,0}$ to $\nu_{m,n,1}$. By Lemma 3.5 and Lemma 3.6 we get

$$\frac{U_N(\nu_{m,n,1})}{U_N(\nu_{m,n,0})} \le c_{K/N}\big(W_2(\nu_{m,n,0},\nu_{m,n,1})\big) + \frac{s_{K/N}\big(W_2(\nu_{m,n,0},\nu_{m,n,1})\big)}{N W_2(\nu_{m,n,0},\nu_{m,n,1})}\left(\int_\Omega \mathrm{D}\zeta_{m,n,0}(\nabla\varphi)\,\mathrm{d}\mathfrak{m}\right). \tag{3.18}$$

Denote $\nu_{m,0} = C_m\zeta_m\mathfrak{m}$, $\nu_{m,1} = T_\varphi(\nu_{m,0}) \in \mathcal{P}(\Omega)$. By (3.16), (3.17) and monotone convergence theorem, we can see that $\nu_{m,n,i} \overset{W_2}{\to} \nu_{m,i}$ and $U_N(\nu_{m,n,i}) \to U_N(\nu_{m,i})$ for $i = 0,1$, as $n \to \infty$. By locality of the weak gradient, we also have

$$\lim_{n\to\infty}\int_\Omega \mathrm{D}\zeta_{m,n,0}(\nabla\varphi)\,\mathrm{d}\mathfrak{m} = C_m\int_\Omega \mathrm{D}\zeta_m(\nabla\varphi)\,\mathrm{d}\mathfrak{m} = -C_m\int_\Omega \zeta_m\,\mathrm{d}\mathbf{\Delta}\varphi. \tag{3.19}$$

Letting $n \to \infty$ in (3.18) and combining with (3.19) we get

$$\frac{U_N(\nu_{m,1})}{U_N(\nu_{m,0})} \le c_{K/N}\big(W_2(\nu_{m,0},\nu_{m,1})\big) - C_m\frac{s_{K/N}\big(W_2(\nu_{m,0},\nu_{m,1})\big)}{N W_2(\nu_{m,0},\nu_{m,1})}\left(\int_\Omega \zeta_m\,\mathrm{d}\mathbf{\Delta}\varphi\right) \tag{3.20}$$

By (3.16) and (3.17) again, we can see that $\nu_{m,i} \overset{W_2}{\to} \mu_i$ and $U_N(\nu_{m,i}) \to U_N(\mu_i)$ for $i = 0,1$, as $m \to \infty$. Letting $m \to \infty$ in (3.20) and noticing that $\lim_{m\to\infty} C_m = 1$, we get (3.15). $\square$

3.3 Main results

Proof of the main theorem

Proof of Theorem 1.4. The proof is divided into four steps. Without loss of generality, we assume that $\mathfrak{m}(\mathbf{D}) > 0$.

Step 1.

Given a c-concave function ϕ. From the proof of Theorem 3.7, for almost every x in the support of ϕ, there is a unique geodesic γ^x satisfying $\gamma^x_0 = x$, $\gamma^x_1 \in \partial^c\phi(x)$. Then we can define a map $\nabla\phi : \Omega \to X$ by $\nabla\phi(x) = \gamma^x_1$ such that for any $\mu \ll \mathfrak{m}$, ϕ is a Kantorovich potential from μ to $T_\phi(\mu) := (\nabla\phi)_\sharp\mu$. In addition, we have a family of maps $\nabla\phi_t(x) := \gamma^x_t, t \in (0,1)$ such that $((\nabla\phi_t)_\sharp\mu)_{t\in[0,1]}$ is the unique geodesic from μ to $T_\phi(\mu)$ in the Wasserstein space.

Step 2.

Let φ be a c-concave representative of $-tu$ given in Lemma 3.1 (or Lemma 3.4 respectively). By Lemma 3.1 and the compactness of $\mathbf{D}$, we have

$$\mathbf{D} \subset \Big\{y \in \partial^c\varphi(x) : x \in \mathbf{R}_2\Big\} \text{ and } \mathbf{R}_2 \subset \Big\{x \in \partial^c\varphi^c(y) : y \in \mathbf{D}\Big\}. \tag{3.21}$$

By Lemma 3.3, Lemma 3.4 (and the assumption of the theorem), we also have

$$\mathbf{D} \subset \Big\{y \in \partial^c\varphi(x) : x \in \mathbf{R}_1\Big\} \text{ and } \mathbf{R}_1 \subset \Big\{x \in \partial^c\varphi^c(y) : y \in \mathbf{D}\Big\}. \tag{3.22}$$

Let $\nu := \frac{1}{\mathfrak{m}(\mathbf{D})}\mathfrak{m}\llcorner_{\mathbf{D}} \in \mathcal{P}_2(X)$. By Step 1, there is a (unique) measure $\mu \in \mathcal{P}(\mathbf{R}_i)$ such that: φ^c is a Kantorovich potential from ν to μ, and φ is a Kantorovich potential from μ to ν. By [MS22, Theorem 4.3, Step 4], $\mu \ll \mathfrak{m}$ and has bounded density ρ. So we have $\nu = T_\varphi(\mu)$.

Denote by $\mathrm{M}_\Omega \subset \mathcal{P}(\Omega)$ the space of all probability measures on Ω with Lipschitz density bounded from below by a positive constant.

Claim: Let $\mathrm{K} \subset \mathbf{R}_i \subset \Omega$ be a compact set such that ρ is bounded from below by a positive constant on K. Let c_K be the normalizing constant so that $\mu_\mathrm{K} := c_\mathrm{K}\mu\llcorner_\mathrm{K} \in \mathcal{P}(\Omega)$. We can find a sequence $(\mu_n)_{n\in\mathbb{N}} \subset \mathrm{M}_\Omega$ such that

(1) $\mu_n = \rho_n\,\mathfrak{m}$ with uniformly bounded $\rho_n \in \mathrm{Lip}_c(\Omega, \mathsf{d})$;

(2) $\lim_{n\to\infty}\mu_n(\mathrm{K}) = 1$ and

$$\lim_{n\to\infty} W_2(\mu_\mathrm{K}, \mu_n) = \lim_{n\to\infty} \mathrm{Ent}_{\mu_\mathrm{K}}(\mu_n\llcorner_\mathrm{K}) = 0.$$

(3) for $\nu = \mathbf{\Delta}\varphi\llcorner_{\Omega\setminus\mathbf{R}_i}$ it holds

$$\lim_{n\to\infty}\int \rho_n\,\mathrm{d}\nu = 0 \tag{3.23}$$

Proof of the claim:

Let $(H_t\mu_{\mathrm{K}})_{t>0}$ be the gradient flow of the relative entropy $\mathrm{Ent}_{\mathfrak{m}}$, in the 2-Wasserstein space, from μ_{K}. Denote $\mu_{\mathrm{K}} = \rho_{\mathrm{K}}\,\mathfrak{m} = c_{\mathrm{K}}\rho\chi_{\mathrm{K}}\,\mathfrak{m}$. From [AGS14a] we know that the heat flow $P_t\rho_{\mathrm{K}}$ from ρ_{K} coincides with the density of $H_t\mu_{\mathrm{K}}$, so we can write $H_t\mu_{\mathrm{K}} = P_t\rho_{\mathrm{K}}\,\mathfrak{m}$. By [AGS14b, THEOREM 6.1, PROPOSITION 6.4] we also know $P_t\rho_{\mathrm{K}}$ is Lipschitz and uniformly bounded in $t > 0$.

Furthermore, it is known that $\lim_{t\to 0} W_2(H_t\mu_{\mathrm{K}}, \mu_{\mathrm{K}}) = 0$, $\lim_{t\to 0}\mathrm{Ent}_{\mathfrak{m}}(H_t\mu_{\mathrm{K}}) = \mathrm{Ent}_{\mathfrak{m}}(\mu_{\mathrm{K}})$, and $P_t\chi_{\mathrm{K}}$ converge to χ_{K} in L^2. So for $\tilde{\mathfrak{m}} := \frac{1}{\mathfrak{m}(\mathrm{K})}\mathfrak{m}\llcorner_{\mathrm{K}}$, we have

$$\lim_{t\to 0}\int \chi_{\mathrm{K}}\,\mathrm{d}H_t\tilde{\mathfrak{m}} = \lim_{t\to 0}\int P_t\chi_{\mathrm{K}}\,\mathrm{d}\tilde{\mathfrak{m}} = \int \chi_{\mathrm{K}}\,\mathrm{d}\tilde{\mathfrak{m}} = 1.$$

Thus $\lim_{t\to 0} H_t\tilde{\mathfrak{m}}(\mathrm{K}) = 1$ and $\lim_{t\to 0} H_t\tilde{\mathfrak{m}}(X\setminus\mathrm{K}) = 0$. Recall that ρ_{K} is bounded, by maximal principle of the heat flow we have

$$\lim_{t\to 0} H_t\mu_{\mathrm{K}}(X\setminus\mathrm{K}) = 0 \quad\text{and}\quad \lim_{t\to 0} H_t\mu_{\mathrm{K}}(\mathrm{K}) = 1. \tag{3.24}$$

By direct computation we have

$$\begin{aligned}
&\mathrm{Ent}_{\mu_{\mathrm{K}}}\big((H_t\mu_{\mathrm{K}})\llcorner_{\mathrm{K}}\big)\\
=&\int_{\mathrm{K}} \ln(P_t\rho_{\mathrm{K}}/\rho_{\mathrm{K}})\,\mathrm{d}H_t\mu_{\mathrm{K}}\\
=&\int_{\mathrm{K}} \ln(P_t\rho_{\mathrm{K}})\,\mathrm{d}H_t\mu_{\mathrm{K}} - \int_{\mathrm{K}} \ln\rho_{\mathrm{K}}\,\mathrm{d}H_t\mu_{\mathrm{K}}\\
=&\underbrace{\int_X \ln(P_t\rho_{\mathrm{K}})\,\mathrm{d}H_t\mu_{\mathrm{K}}}_{\mathrm{Ent}_{\mathfrak{m}}(H_t\mu_{\mathrm{K}})} - \int_{X\setminus\mathrm{K}} \ln(P_t\rho_{\mathrm{K}})\,\mathrm{d}H_t\mu_{\mathrm{K}} - \int_{\mathrm{K}} \ln\rho_{\mathrm{K}}\,\mathrm{d}H_t\mu_{\mathrm{K}}.
\end{aligned}$$

By (3.24) and Jensen's inequality we know

$$\lim_{t\to 0}\int_{X\setminus\mathrm{K}} \ln(P_t\rho_{\mathrm{K}})\,\mathrm{d}H_t\mu_{\mathrm{K}} = 0.$$

By choice of K, $\ln\rho_{\mathrm{K}}$ is bounded. Then we have

$$\lim_{t\to 0}\int_{\mathrm{K}} \ln\rho_{\mathrm{K}}\,\mathrm{d}H_t\mu_{\mathrm{K}} = \lim_{t\to 0}\int_{\mathrm{K}} P_t\rho_{\mathrm{K}}\ln\rho_{\mathrm{K}}\,\mathrm{d}\mathfrak{m} = \mathrm{Ent}_{\mathfrak{m}}(\mu_{\mathrm{K}}).$$

Combining with the continuity of the entropy along the heat flow, we get

$$\lim_{t\to 0}\mathrm{Ent}_{\mu_{\mathrm{K}}}\big((H_t\mu_{\mathrm{K}})\llcorner_{\mathrm{K}}\big) = 0.$$

At last, for any $n\in\mathbb{N}$, there is a compact set $E_n\subset\Omega\setminus\mathbf{R}_i$ so that

$$\|\rho_{\mathrm{K}}\|_{L^\infty}\nu(\Omega\setminus E_n) < \frac{1}{2n}. \tag{3.25}$$

Denote $\nu_n := \nu\llcorner_{E_n}/\nu(E_n)$. Note that $\lim_{t\to 0} W_2(H_t\nu_n, \nu_n) = 0$ and $\mathsf{d}(\mathrm{K}, E_n) > 0$, it holds

$$\lim_{t\to 0}\int_{E_n} P_t\rho_{\mathrm{K}}\,\mathrm{d}\nu = \lim_{t\to 0}\nu(E_n)\int \rho_{\mathrm{K}}\,\mathrm{d}H_t\nu_n = 0.$$

So there is $t_n > 0$ so that
$$\int_{E_n} P_{t_n}\rho_{\mathrm{K}}\,\mathrm{d}\nu < \frac{1}{2n}. \tag{3.26}$$
Combining (3.25) and (3.26) we get
$$\begin{aligned}\left|\int P_{t_n}\rho_{\mathrm{K}}\,\mathrm{d}\nu\right| &= \left|\int_{E_n} P_{t_n}\rho_{\mathrm{K}}\,\mathrm{d}\nu + \int_{\Omega\setminus E_n} P_{t_n}\rho_{\mathrm{K}}\,\mathrm{d}\nu\right| \\ &\le \frac{1}{2n}+\frac{1}{2n}=\frac{1}{n}.\end{aligned}$$

For any $n\in\mathbb{N}$, we define
$$\mu_n = a_n\left((H_{t_n}\mu_{\mathrm{K}})\llcorner\Omega + \frac{1}{n}\right)$$
where a_n is the normalizing constant. From the construction above we can see that μ_n fulfils our request.

Step 3. Given a compact set $\mathrm{K}\subset \mathbf{R}_i$ and a sequence $(\mu_n)_{n\in\mathbb{N}}\subset \mathrm{M}_\Omega$ constructed in the last step. For any $t\in(0,1)$, denote $\mu_{n,t}=(\nabla\varphi_t)_\sharp\mu_n$. By local compactness of $(X,\mathsf{d},\mathfrak{m})$, we may assume that $\mu_{n,t}\rightharpoonup\mu_t$ for some $\mu_t\in\mathcal{P}(\Omega)$ as $n\to\infty$. By stability of optimal transport (cf. [Vil09, Theorem 5.20]) and the uniqueness of optimal transport map again, we know $\mu_t=(\nabla\varphi_t)_\sharp\mu_{\mathrm{K}}$ is the unique t-intermediate point between μ_{K} and a uniform distribution $T_\varphi(\mu_{\mathrm{K}})=\frac{1}{\mathfrak{m}(\mathbf{D}_{\mathrm{K}})}\mathfrak{m}\llcorner_{\mathbf{D}_{\mathrm{K}}}$ on some measurable set $\mathbf{D}_{\mathrm{K}}\subset\mathbf{D}$. By $\mathrm{RCD}(K,N)$ condition [Raj12] and [MS22, Theorem 4.3], we also know $\mu_{n,t},\mu_t\ll\mathfrak{m}$, and for any $t>0$, $(\mu_{n,t})_{n\in\mathbb{N}}$ have uniformly bounded densities $\rho_{n,t}$.

Furthermore, the density of μ_t satisfies
$$\|\rho_t\|_{L^\infty}\le\frac{1}{\mathfrak{m}(\mathbf{D}_{\mathrm{K}})}+o(1),\quad \text{as } t\to 1. \tag{3.27}$$
Given $\epsilon>0$, by (3.27) we have
$$\|\rho_t\|_{L^\infty}\le\frac{1}{\mathfrak{m}(\mathbf{D}_{\mathrm{K}})}+\epsilon \tag{3.28}$$
for $t\in(0,1)$ close enough to 1.

Next we will estimate $\mathrm{Ent}_{\mathfrak{m}}(\mu_{n,t})$. Denote $\mu_t=\rho_t\,\mathfrak{m}$, $\mu_{n,t}=\rho_{n,t}\,\mathfrak{m}$, and $\tilde\mu_n=c_n\mu_n\llcorner\mathrm{K}\in\mathcal{P}(\Omega)$ for some normalizing constants c_n, $n\in\mathbb{N}$. We have
$$\begin{aligned}&\mathrm{Ent}_{\mathfrak{m}}(\mu_{n,t})\\ =&\int_{\nabla\varphi_t(\Omega)}\ln\rho_{n,t}\,\mathrm{d}\mu_{n,t}\\ =&\int_{\nabla\varphi_t(\mathrm{K})}\ln\rho_{n,t}\,\mathrm{d}\mu_{n,t}+\int_{\nabla\varphi_t(\Omega\setminus\mathrm{K})}\ln\rho_{n,t}\,\mathrm{d}\mu_{n,t}\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Ent}_{(\nabla\varphi_t)_\sharp(\mu_{\mathrm{K}})}\big((\nabla\varphi_t)_\sharp(\tilde\mu_n)\big)\\
=&\int_{\nabla\varphi_t(\mathrm{K})}\ln(c_n\rho_{n,t})\,\mathrm{d}(c_n\mu_{n,t})-\int_{\nabla\varphi_t(\mathrm{K})}\ln\rho_t\,\mathrm{d}(c_n\mu_{n,t})\\
=&\ c_n\int_{\nabla\varphi_t(\mathrm{K})}\ln\rho_{n,t}\,\mathrm{d}\mu_{n,t}+\underbrace{\int_{\nabla\varphi_t(\mathrm{K})}c_n\ln c_n\,\mathrm{d}\mu_{n,t}}_{=\ln c_n}-\int_{\nabla\varphi_t(\mathrm{K})}\ln\rho_t\,\mathrm{d}(c_n\mu_{n,t}).
\end{aligned}$$

Combining with (3.28), we get

$$\begin{aligned}
&\mathrm{Ent}_{\mathfrak{m}}(\mu_{n,t})\\
=&\ \frac{1}{c_n}\mathrm{Ent}_{(\nabla\varphi_t)_\sharp(\mu_{\mathrm{K}})}\big((\nabla\varphi_t)_\sharp(\tilde\mu_n)\big)-\frac{\ln c_n}{c_n}+\frac{1}{c_n}\int_{\nabla\varphi_t(\mathrm{K})}\ln\rho_t\,\mathrm{d}(c_n\mu_{n,t})\\
&+\int_{\nabla\varphi_t(\Omega\setminus\mathrm{K})}\ln\rho_{n,t}\,\mathrm{d}\mu_{n,t}\\
\le&\ \frac{1}{c_n}\mathrm{Ent}_{(\nabla\varphi_t)_\sharp(\mu)}\big((\nabla\varphi_t)_\sharp(\tilde\mu_n)\big)-\frac{\ln c_n}{c_n}+\frac{1}{c_n}\ln\big(1/\mathfrak{m}(\mathbf{D}_{\mathrm{K}})+\epsilon\big)\\
&+\ln\|\rho_{n,t}\|_{L^\infty}\mu_n(\Omega\setminus\mathbf{R}_i).
\end{aligned}$$

By [AGS08, Lemma 9.4.5] we have

$$0\le\mathrm{Ent}_{(\nabla\varphi_t)_\sharp(\mu_{\mathrm{K}})}((\nabla\varphi_t)_\sharp(\tilde\mu_n))\le\mathrm{Ent}_{\mu_{\mathrm{K}}}(\tilde\mu_n).\tag{3.29}$$

Combining these estimates above, the properties (1) and (2) in Step 2, and $\lim_{n\to\infty}c_n=1$, we obtain

$$\varlimsup_{n\to\infty}\mathrm{Ent}_{\mathfrak{m}}(\mu_{n,t})\le\ln\big(1/\mathfrak{m}(\mathbf{D}_{\mathrm{K}})+\epsilon\big)\tag{3.30}$$

for $t\in(0,1)$ close enough to 1.

Step 4.

By Proposition 3.7 and a re-parametrization argument, we obtain

$$\begin{aligned}
U_N(\mu_{n,t})\ \le&\ c_{K/N}\big(W_2(\mu_n,\mu_{n,t})\big)U_N(\mu_n)\\
&-t\frac{s_{K/N}\big(W_2(\mu_n,\mu_{n,t})\big)}{W_2(\mu_n,\mu_{n,t})}\left(\frac{1}{N}U_N(\mu_n)\int\rho_n\,\mathrm{d}\mathbf{\Delta}\varphi\right).
\end{aligned}$$

Letting $n\to\infty$ and combining with (3.30), upper semi-continuity of the functional $U_N(\cdot)$ (cf. [Stu06a, Lemma 4.1]), (3.23) and the Laplacian estimate in Lemma 3.2, Lemma 3.4, we obtain

$$\frac{\big(1/\mathfrak{m}(\mathbf{D}_{\mathrm{K}})+\epsilon\big)^{-\frac{1}{N}}}{U_N(\mu_{\mathrm{K}})}\le c_{K/N}\big(W_2(\mu_{\mathrm{K}},\mu_t)\big)+\frac{ts_{K/N}\big(W_2(\mu_{\mathrm{K}},\mu_t)\big)}{NW_2(\mu_{\mathrm{K}},\mu_t)}\int\rho_{\mathrm{K}}\Delta u\,\mathrm{d}\mathfrak{m}.$$

By Jensen's inequality (cf. [Stu06a, Lemma 4.1]), we get

$$U_N(\mu_{\mathrm{K}})\le\mathfrak{m}(\mathrm{K})^{\frac{1}{N}}\le\mathfrak{m}(\mathbf{R}_i)^{\frac{1}{N}}.$$

By monotonicity of $\frac{s_{K/N}(x)}{x}$ and $c_{K/N}(x)$, letting $t \to 1$, $\epsilon \to 0$ and $\mathfrak{m}(\mathbf{R}_i \setminus \mathrm{K}) \to 0$, we get

$$\mathfrak{m}(\mathbf{D}) \leq \begin{cases} \mathfrak{m}(\mathbf{R}_i)\left(c_{K/N}(\Theta) + \frac{ts_{K/N}(\Theta)}{N\Theta}\|(\Delta u)^+\|_{L^\infty(\overline{\Omega})}\right)^N, & \text{if } K < 0, \\ \mathfrak{m}(\mathbf{R}_i)\left(1 + \frac{t}{N}\|(\Delta u)^+\|_{L^\infty(\overline{\Omega})}\right)^N & \text{if } K = 0, \\ \mathfrak{m}(\mathbf{R}_i)\left(c_{K/N}(\Phi) + \frac{ts_{K/N}(\Phi)}{N\Phi}\|(\Delta u)^+\|_{L^\infty(\overline{\Omega})}\right)^N & \text{if } K > 0, \end{cases}$$

where $(\Delta u)^+$ denotes the positive part of Δu, $\Theta := \sup_{(x,y)\in\mathbf{D}\times\Omega} \mathsf{d}(x,y)$ and $\Phi := \inf_{(x,y)\in\mathbf{D}\times\Omega} \mathsf{d}(x,y)$. □

Applications

In [Gig23, MS22], the authors adopt a perturbation argument of [ZZ18], in the spirit of the classical Jensen's maximum principle and ideas from Petrunin [Pet] and Zhang–Zhu [ZZ12], to study harmonic maps from RCD metric measure spaces to CAT(0) spaces. By Theorem 1.4, we get the following estimate which plays a key role in their perturbation argument.

Corollary 3.8 (cf. [MS22], Theorem 4.3)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ metric measure space for some $K \in \mathbb{R}$ and $N \in (1, \infty)$. Let $\Omega \subset X$ be a bounded open domain with $\mathfrak{m}(\partial\Omega) = 0$. Assume that $u \in \mathrm{D}(\Delta, \Omega) \cap C(\Omega)$ with $\Delta u \leq L$ for some positive constant L. Then, for any compact set $\mathbf{D} \subset X$ and for any $t > 0$ so that*

$$\mathbf{R}_2(\mathbf{D}, \Omega, u, t) \subset \Omega,$$

or

$$\mathbf{R}_1(\mathbf{D}, \Omega, u, t) \subset \Omega,$$
$$\forall y \in \mathbf{D},\ \exists x \in \mathbf{R}_1(\mathbf{D}, \Omega, u, t),\ \mathsf{d}(x,y) = t,\ \inf_{\Omega}\left(u + \mathsf{d}_y\right) = u(x) + \mathsf{d}(x,y).$$

It holds the following estimate:

$$\mathfrak{m}(\mathbf{D}) \leq C(K, N, \mathbf{D}, \Omega, t, L)\, \mathfrak{m}\big(\mathbf{R}_i(\mathbf{D}, \Omega, u, t)\big) \quad i = 1, 2 \tag{3.31}$$

for some explicit constant

$$C(K, N, \mathbf{D}, \Omega, t, L) := \begin{cases} \left(c_{K/N}(\Theta) + \frac{ts_{K/N}(\Theta)}{N\Theta}L\right)^N, & \textit{if } K < 0, \\ \left(1 + \frac{t}{N}L\right)^N & \textit{if } K = 0, \\ \left(c_{K/N}(\Phi) + \frac{ts_{K/N}(\Phi)}{N\Phi}L\right)^N & \textit{if } K > 0, \end{cases}$$

where $\Theta := \sup_{(x,y)\in\mathbf{D}\times\Omega} \mathsf{d}(x,y)$ and $\Phi := \inf_{(x,y)\in\mathbf{D}\times\Omega} \mathsf{d}(x,y)$. In particular, if $\mathfrak{m}(\mathbf{D}) > 0$, then $\mathfrak{m}\big(\mathbf{R}_i(\mathbf{D}, \Omega, u, t)\big) > 0$.

On the basis of our ABP estimate and the ideas of Cabré [Cab97] and Wang–Zhang [WZ13], one can also prove the Harnack inequality and certain geometric inequalities on metric measure spaces with suitable assumptions. In this paper, we will only study an isoperimetric type inequality.

Let $\Omega \subset X$ be an open set. Recall that the *upper Minkowski content* is defined as

$$\mathfrak{m}^+(\Omega) := \limsup_{\epsilon \downarrow 0} \frac{\mathfrak{m}(\Omega^\epsilon) - \mathfrak{m}(\Omega)}{\epsilon}$$

where $\Omega^\epsilon \subset X$ is the ϵ-neighbourhood of Ω defined as $\Omega^\epsilon := \{x : \mathsf{d}(x, \Omega) < \epsilon\}$. In metric-measure setting, this notion plays the role of 'boundary area' (cf. [ADMG17] for more discussions).

Definition 3.9 (Uniform exterior sphere condition)**.** Let $\Omega \subset X$ be an open set. We say that Ω satisfies the *uniform exterior sphere condition* if there exists $r > 0$ such that for all $x \in \partial\Omega$ there is $p_x \in \Omega^c$ such that $\mathsf{d}(x, p_x) = r$ and $B_r(p_x) \subset \Omega^c$.

Definition 3.10 (H-mean convex)**.** Let Ω be an open set in a $\mathrm{CD}(0, N)$ space. Let u be the signed distance function defined by

$$u(x) := \begin{cases} \mathsf{d}(x, \Omega), & \text{if } \ x \in \Omega^c \\ -\mathsf{d}(x, \Omega^c), & \text{if } \ x \in \Omega. \end{cases} \tag{3.32}$$

We say that $\partial\Omega$ is H-mean convex if there is $\sigma > 0$ so that $u \in \mathrm{D}(\Delta, \Omega_\sigma \setminus \overline{\Omega})$ and $\Delta u \llcorner_{\Omega_\sigma \setminus \overline{\Omega}} \leq -H \mathfrak{m} \llcorner_{\Omega_\sigma \setminus \overline{\Omega}}$.

Remark 3.11. By the representation formula for the Laplacian of the signed distance function, proved by Cavalletti–Mondino [CM20, Corollary 4.16] based on needle decomposition, one can prove: if Ω satisfies uniform exterior sphere condition, and the outer mean curvature of $\partial\Omega$ is bounded from below by H in the sense of Ketterer [Ket20], then it is H-mean convex.

In particular, in the smooth setting, $\partial\Omega$ is H-mean convex if and only if its (outer) mean curvature is bounded from below by H.

Then we can prove a generalized Steiner-type formula. We refer the readers to [Ket20] for a Heintze–Karcher type inequality.

Theorem 3.12 (Generalized Steiner-type formula)**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(0, N)$ *metric measure space with* $N \geq 2$*. Let* Ω *be a bounded open set satisfying uniform exterior sphere condition with* $\mathfrak{m}(\partial\Omega) = 0$*. Assume that* $\partial\Omega$ *is* H*-mean convex for some* $H \geq 0$*. Then, as* $\epsilon \to 0$*,*

$$\mathfrak{m}(\Omega_\epsilon) \leq \mathfrak{m}(\Omega) + \left(\epsilon - \frac{1}{2} H \epsilon^2\right) \mathfrak{m}^+(\Omega) + o(\epsilon^2).$$

Proof. For $\epsilon \in (0, r/2)$ and $\delta \in (0, \epsilon/2)$, consider the contact set $\mathbf{R}_1(\overline{\Omega_{2\epsilon-\delta}} \setminus \Omega_{\epsilon+\delta}, \Omega_\epsilon \setminus \overline{\Omega}, u, \epsilon)$. By uniform exterior sphere condition, we can see that

$$\mathbf{R}_1(\overline{\Omega_{2\epsilon-\delta}} \setminus \Omega_{\epsilon+\delta}, \Omega_\epsilon \setminus \overline{\Omega}, u, \epsilon) \subset \Omega_\epsilon \setminus \overline{\Omega}$$

and

$$\forall y \in \overline{\Omega_{2\epsilon-\delta}} \setminus \Omega_{\epsilon+\delta},\ \exists x \in \mathbf{R}_1,\ \mathsf{d}(x,y) = \epsilon,\ \inf_{\Omega_\epsilon \setminus \overline{\Omega}} \big(u + \mathsf{d}_y\big) = u(x) + \mathsf{d}(x,y).$$

Applying Corollary 3.8 to u, we obtain

$$\mathfrak{m}(\overline{\Omega_{2\epsilon-\delta}} \setminus \Omega_{\epsilon+\delta}) \leq \mathfrak{m}(\Omega_\epsilon \setminus \overline{\Omega})\Big(1 - \frac{\epsilon H}{N-1}\Big)^{N-1}.$$

Letting $\delta \to 0$ we get

$$\mathfrak{m}(\Omega_{2\epsilon} \setminus \overline{\Omega_\epsilon}) \leq \mathfrak{m}(\Omega_\epsilon \setminus \overline{\Omega})\Big(1 - \frac{\epsilon H}{N-1}\Big)^{N-1}$$

By Taylor's expansion with respect to ϵ, we know

$$\Big(1 - \frac{\epsilon H}{N-1}\Big)^{N-1} \leq 1 - H\epsilon + \frac{1}{2}H^2\epsilon^2.$$

So

$$\begin{aligned}
\mathfrak{m}(\Omega_{2\epsilon} \setminus \overline{\Omega}) &\leq \left(2 - H\epsilon + \frac{1}{2}H^2\epsilon^2\right)\mathfrak{m}(\Omega_\epsilon \setminus \overline{\Omega}) \\
&\leq \left(2 - H\epsilon + \frac{1}{2}H^2\epsilon^2\right)\left(2 - \frac{1}{2}H\epsilon + \frac{1}{2^3}H^2\epsilon^2\right)\mathfrak{m}(\Omega_{\epsilon/2} \setminus \overline{\Omega}) \\
&\cdots \\
&\leq 2^n\left(1 - H\epsilon + \frac{1}{2^n}H\epsilon + O(\epsilon^2)\right)\mathfrak{m}(\Omega_{\epsilon/2^{n-1}} \setminus \overline{\Omega}).
\end{aligned}$$

Letting $n \to \infty$ we get

$$\mathfrak{m}(\Omega_{2\epsilon} \setminus \overline{\Omega}) \leq (2\epsilon - 2H\epsilon^2)\mathfrak{m}^+(\Omega) + o(\epsilon^2)$$

Replacing ϵ by $\epsilon/2$ we prove the theorem. □

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