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Sharp and rigid isoperimetric inequalities in metric measure spaces with non-negative Ricci curvature

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Abstract Using optimal transport, we prove a sharp, dimension-free isoperimetric inequality involving volume entropy for metric measure spaces with non-negative Ricci curvature in the sense of Lott–Sturm–Villani. Furthermore, we prove rigidity for $\mathrm{RCD}(0,\infty)$ spaces via Bakry–Émery's Γ_2-calculus. These results are new even for Euclidean spaces equipped with log-concave densities, and are of interest in both probability theory and geometry.

Keywords isoperimetric inequality, Cheeger constant, curvature-dimension condition, metric measure space, non-negative Ricci curvature, optimal transport, volume entropy

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1 Introduction

The aim of this paper is to present a *sharp dimension-free isoperimetric inequality*, in metric measure spaces with non-negative Ricci curvature in the sense of Lott–Sturm–Villani, and study its rigidity.

Let $(X,\mathrm{d},\mathfrak{m})$ be a metric measure space, where (X,d) is a complete and separable metric space and $\mathfrak{m}$ is a locally finite, non-negative Radon measure with full support. The *Minkowski content* of a Borel set $\Omega\subset X$ with $\mathfrak{m}(\Omega)<+\infty$ is defined by

$$\mathfrak{m}^+(\Omega):=\liminf_{\epsilon\to 0^+}\frac{\mathfrak{m}(\Omega^\epsilon)-\mathfrak{m}(\Omega)}{\epsilon}$$

where $\Omega^\epsilon\subset X$ is the ϵ-neighbourhood of Ω defined as $\Omega^\epsilon:=\{x:\mathrm{d}(x,\Omega)<\epsilon\}$. An isoperimetric inequality relates the size of the boundary of a set to its measure. Let $\mathcal{M}$ be a family of metric measure spaces. We say that a function $I_{\mathcal{M}}(\cdot):[0,+\infty)\to[0,+\infty)$ is *the isoperimetric profile* associated with $\mathcal{M}$, if

$$\mathfrak{m}^+(\Omega)\geqslant I_{\mathcal{M}}(v)$$

for all $(X,\mathrm{d},\mathfrak{m})\in\mathcal{M}$ and any measurable set $\Omega\subset X$ with $\mathfrak{m}(\Omega)=v$.

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Recently, isoperimetric inequalities in non-compact metric measure spaces with non-negative synthetic Ricci curvature, have been studied in various settings, for example by Agostiniani–Fogagnolo–Mazzieri[1], Brendle[17], Balogh and Kristály[13], Antonelli–Pasqualetto–Pozzetta–Semola[11, 12], Cavalletti and Manini[19, 20]. As discovered by E. Milman[34], the isoperimetric profile for this family of spaces is trivial if there is no restriction on the diameter of the sets. In the above-mentioned papers, a key component in the isoperimetric profile is a parameter called *the asymptotic volume ratio*.

However, the asymptotic volume ratio depends on the dimension parameter, so those isoperimetric inequalities are all *dimension-dependent*. So it is natural to ask for a *dimension-free* isoperimetric inequality in metric measure spaces with non-negative Ricci curvature, in the sense of Lott–Sturm–Villani[32, 37]. Examples satisfying this condition include weighted Riemannian manifolds with non-negative Bakry–Émery curvature, Banach spaces, measured Gromov–Hausdorff limits of Riemannian manifolds with non-negative Ricci curvature, Alexandrov spaces with non-negative curvature, and Finsler manifolds with non-negative weighted Ricci curvature. See Ambrosio's ICM Proceeding[2] and Villani's book[38] for an overview of this field.

Definition 1.1 (Lott–Sturm–Villani[32, 37])**.** We say that a metric measure space $(X, \mathrm{d}, \mathfrak{m})$ has non-negative Ricci curvature, or satisfies $\mathrm{CD}(0, \infty)$ condition, if the relative entropy $\mathrm{Ent}_{\mathfrak{m}}$ defined as

$$\mathrm{Ent}_{\mathfrak{m}}(\mu) := \begin{cases} \int \ln \rho \, \mathrm{d}\mu & \text{if } \mu = \rho\, \mathfrak{m} \\ +\infty & \text{otherwise} \end{cases}$$

is displacement convex. This is to say, for any two probability measures μ_0, μ_1 in the L^2-Wasserstein space $(\mathcal{P}_2(X), W_2)$, there is a geodesic $(\mu_t)_{t\in[0,1]}$ satisfying

$$\mathrm{Ent}_{\mathfrak{m}}(\mu_t) \leqslant t\mathrm{Ent}_{\mathfrak{m}}(\mu_1) + (1-t)\mathrm{Ent}_{\mathfrak{m}}(\mu_0) \quad \forall t \in [0, 1].$$

Remark 1.2. Let (X, d) be a geodesic space and fix $\mu_0, \mu_1 \in \mathcal{P}_2(X)$. A curve $(\mu_t)_{t\in[0,1]} \subset (\mathcal{P}_2(X), W_2)$ is a geodesic, i.e.,

$$W_2(\mu_s, \mu_t) = |s - t| W_2(\mu_0, \mu_1), \quad \forall s, t \in [0, 1],$$

if and only if there exists $\Pi \in \mathcal{P}(\mathrm{Geo(X)}) \subseteq \mathcal{P}(\mathrm{C([0, 1], X)})$, called the optimal dynamical plan, such that

$$\mu_t = (\mathrm{e}_t)_\sharp \Pi \ \ \forall t \in [0, 1] \quad \text{and} \quad (\mathrm{e}_0, \mathrm{e}_1)_\sharp \Pi \in \mathrm{Opt}(\mu_0, \mu_1),$$

where $\mathrm{Opt}(\mu_0, \mu_1)$ denotes the set of optimal transport plans from μ_0 to μ_1.

In order to evaluate the growth of the volume without the dimension parameter, we will use *volume entropy*. This is an important concept in both Riemannian geometry (cf.[14]) and dynamical systems (cf.[33]). For example, it is related to Gromov's simplicial volume, the bottom of the spectrum of the Laplacian, the Cheeger isoperimetric constant, the growth of fundamental groups, and topological entropy of geodesic flows.

Definition 1.3 (Volume entropy)**.** We say that a metric measure space $(X, \mathrm{d}, \mathfrak{m})$ has volume entropy $h_{(X,\mathrm{d},\mathfrak{m})}$, if there is $x_0 \in X$ so that the following limit exists

$$h_{(X,\mathrm{d},\mathfrak{m})} := \lim_{r\to+\infty} \frac{\ln \mathfrak{m}\big(B_r(x_0)\big)}{r} \in [0, \infty].$$

Remark 1.4. While the existence of volume entropy is well-known for the universal cover of a compact Riemannian manifold [14,33], to the best of our knowledge, this is the first result establishing the existence of volume entropy for general non-compact $\mathrm{CD}(0, \infty)$ metric measure spaces without assuming a finite upper dimension bound. See Proposition 2.2 below for the proof.

The first main result of this paper is the following sharp isoperimetric inequality involving volume entropy.

Theorem 1.5 (Sharp isoperimetric inequality, Theorem 2.1 and Theorem 2.3). *Let $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space satisfying the $\mathrm{CD}(0, \infty)$ condition. Then for any $\Omega \subset X$ with $\mathfrak{m}(\Omega) < \infty$, we have*

$$\mathfrak{m}^+(\Omega) \geqslant h_{(X,\mathrm{d},\mathfrak{m})}\mathfrak{m}(\Omega). \tag{1.1}$$

In other words, the Cheeger constant $\mu_{(X,\mathrm{d},\mathfrak{m})} := \inf_\Omega \frac{\mathfrak{m}^+(\Omega)}{\mathfrak{m}(\Omega)}$ is no less than the volume entropy $h_{(X,\mathrm{d},\mathfrak{m})}$. Moreover, the constant $h_{(X,\mathrm{d},\mathfrak{m})}$ in (1.1) cannot be improved.

In [18, Theorem 1], R. Brooks proved that the bottom of the essential spectrum λ_0^{ess} is bounded from above by $\frac{1}{4}h^2_{(X,\mathrm{d},\mathfrak{m})}$ if $\mathfrak{m}(X) = +\infty$. Combining this with Cheeger's inequality [22] we get the following inequality (cf. [18, Corollary 2])

$$\frac{1}{4}h^2_{(X,\mathrm{d},\mathfrak{m})} \geqslant \lambda_0^{\mathrm{ess}} \geqslant \frac{1}{4}\mu^2_{(X,\mathrm{d},\mathfrak{m})}.$$

Then we obtain the following corollary.

Corollary 1.6. *Let $(X, \mathrm{d}, \mathfrak{m})$ be a $\mathrm{CD}(0, \infty)$ metric measure space with infinite volume. The following equality holds:*

$$\frac{1}{4}h^2_{(X,\mathrm{d},\mathfrak{m})} = \lambda_0^{\mathrm{ess}} = \frac{1}{4}\mu^2_{(X,\mathrm{d},\mathfrak{m})}.$$

It has been observed by De Ponti–Mondino–Semola [25] (see also [24]) that the equality in Cheeger's isoperimetric inequality can never be attained in the family of spaces with finite diameter or positive Ricci curvature (Theorem 1.5 provides another interpretation of this fact). In the next theorem we show that, in metric measure spaces satisfying the Riemannian curvature-dimension condition $\mathrm{RCD}(0, \infty)$, the isoperimetric inequality is rigid. Here "Riemannian" means that $(X, \mathrm{d}, \mathfrak{m})$ is infinitesimally Hilbertian (cf. [6, 26]). In this case, we can use non-smooth Bakry–Émery theory and Γ_2-calculus to prove the splitting theorem.

Theorem 1.7 (Rigidity theorem, Theorem 3.3). *Let $(X, \mathrm{d}, \mathfrak{m})$ be an $\mathrm{RCD}(0, \infty)$ metric measure space having positive volume entropy $h_{(X,\mathrm{d},\mathfrak{m})}$.*

If there is a measurable set $\Omega \subset X$ with $\mathfrak{m}(\Omega) < \infty$ such that the equality in the isoperimetric inequality (1.1) is attained

$$\mathfrak{m}^+(\Omega) = h_{(X,\mathrm{d},\mathfrak{m})}\mathfrak{m}(\Omega)$$

or the Cheeger constant is achieved

$$\mu_{(X,\mathrm{d},\mathfrak{m})} = \frac{\mathfrak{m}^+(\Omega)}{\mathfrak{m}(\Omega)},$$

then

$$(X, \mathrm{d}, \mathfrak{m}) \cong \left(\mathbb{R}, |\cdot|, \mathrm{e}^{h_{(X,\mathrm{d},\mathfrak{m})}t}\mathrm{d}t\right) \times (Y, \mathrm{d}_Y, \mathfrak{m}_Y)$$

for some $\mathrm{RCD}(0, \infty)$ metric measure space $(Y, \mathrm{d}_Y, \mathfrak{m}_Y)$ with $\mathfrak{m}_Y(Y) < +\infty$. In a suitable choice of coordinates, Ω can be identified as

$$\Omega = (-\infty, c] \times Y \subset \mathbb{R} \times Y$$

with $c \in \mathbb{R}$ satisfying $\mathfrak{m}_Y(Y)\mathrm{e}^{h_{(X,\mathrm{d},\mathfrak{m})}c} = h_{(X,\mathrm{d},\mathfrak{m})}\mathfrak{m}(\Omega)$.

The rest of this paper is organized as follows. In Section 2 we prove the sharp isoperimetric inequality and its corollaries. In Section 3 we study its rigidity.

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2 Sharp Isoperimetric Inequalities

In this section we will prove a sharp isoperimetric inequality in $\mathrm{CD}(0,\infty)$ metric measure spaces.

Theorem 2.1 (Sharp isoperimetric inequality)**.** *Let $(X,\mathrm{d},\mathfrak{m})$ be a $\mathrm{CD}(0,\infty)$ metric measure space having volume entropy $h_{(X,\mathrm{d},\mathfrak{m})}$. Then for any measurable set $\Omega\subset X$ with $\mathfrak{m}(\Omega)<\infty$, the following isoperimetric inequality holds:*

$$\mathfrak{m}^+(\Omega)\geqslant h_{(X,\mathrm{d},\mathfrak{m})}\mathfrak{m}(\Omega). \tag{2.1}$$

Proof. **Step 1.** Assume Ω to be bounded.

Given $x_0\in\Omega$, let $R>0$ be such that $\Omega\subset B_R(x_0):=\{x:\mathrm{d}(x,x_0)<R\}$. Define $\mu_0=\frac{1}{\mathfrak{m}(\Omega)}\mathfrak{m}\llcorner_\Omega$ and $\mu_1=\frac{1}{\mathfrak{m}(B_R(x_0))}\mathfrak{m}\llcorner_{B_R(x_0)}$. According to Definition 1.1, there exists an L^2-Wasserstein geodesic $(\mu_t)_{t\in[0,1]}$ connecting μ_0,μ_1 such that

$$\mathrm{Ent}_\mathfrak{m}(\mu_t)\leqslant t\mathrm{Ent}_\mathfrak{m}(\mu_1)+(1-t)\mathrm{Ent}_\mathfrak{m}(\mu_0). \tag{2.2}$$

Denote the set of t-intermediate points by

$$Z_t:=\left\{z:\exists\ x\in\Omega,y\in B_R(x_0),\text{such that }\frac{\mathrm{d}(z,x)}{t}=\frac{\mathrm{d}(z,y)}{1-t}=\mathrm{d}(x,y)\right\}.$$

It can be seen from the super-position theorem (cf. [4, Theorem 2.10] and Remark 1.2) that μ_t is concentrated on Z_t. Then by (2.2), Jensen's inequality and monotonicity of the function $t\to\ln(t)$ we have

$$-\ln\big(\mathfrak{m}(Z_t)\big)\leqslant -t\ln\big(\mathfrak{m}(B_R(x_0))\big)-(1-t)\ln\big(\mathfrak{m}(\Omega)\big). \tag{2.3}$$

Let $\epsilon:=t\big(\mathrm{diam}(\Omega)+R\big)$. For any $z\in Z_t$, there are $x\in\Omega$ and $y\in B_R(x_0)$ so that $\mathrm{d}(z,x)=t\mathrm{d}(x,y)$. By the triangle inequality,

$$\mathrm{d}(x,y)\leqslant\mathrm{d}(x,x_0)+\mathrm{d}(y,x_0)<\mathrm{diam}(\Omega)+R.$$

So $\mathrm{d}(z,x)<\epsilon$, $z\in\Omega^\epsilon$ and $Z_t\subset\Omega^\epsilon$.

If $\mathfrak{m}^+(\Omega)=+\infty$, there is nothing to prove. Otherwise, $\lim_{\epsilon\to 0}\mathfrak{m}(\Omega^\epsilon)=\mathfrak{m}(\Omega)$. So we have

$$\begin{aligned}
\frac{\mathfrak{m}^+(\Omega)}{\mathfrak{m}(\Omega)} &= \liminf_{\epsilon\to 0}\frac{1}{\mathfrak{m}(\Omega)}\frac{\mathfrak{m}(\Omega^\epsilon)-\mathfrak{m}(\Omega)}{\epsilon}\\
&\overset{\text{L'Hôpital}}{=} \liminf_{\epsilon\to 0}\frac{\ln\big(\mathfrak{m}(\Omega^\epsilon)\big)-\ln\big(\mathfrak{m}(\Omega)\big)}{\mathfrak{m}(\Omega^\epsilon)-\mathfrak{m}(\Omega)}\frac{\mathfrak{m}(\Omega^\epsilon)-\mathfrak{m}(\Omega)}{\epsilon}\\
&\geqslant \liminf_{t\to 0}\frac{\ln\big(\mathfrak{m}(Z_t)\big)-\ln\big(\mathfrak{m}(\Omega)\big)}{t(\mathrm{diam}(\Omega)+R)}\\
&\overset{(2.3)}{\geqslant} \liminf_{t\to 0}\frac{t\ln\big(\mathfrak{m}(B_R(x_0))\big)+(1-t)\ln\big(\mathfrak{m}(\Omega)\big)-\ln\big(\mathfrak{m}(\Omega)\big)}{t(\mathrm{diam}(\Omega)+R)}\\
&= \frac{\ln\big(\mathfrak{m}(B_R(x_0))\big)-\ln\big(\mathfrak{m}(\Omega)\big)}{\mathrm{diam}(\Omega)+R}.
\end{aligned}$$

By Proposition 2.2, the volume entropy $h_{(X,\mathrm{d},\mathfrak{m})}$ exists. Letting $R\to\infty$, we get

$$\frac{\mathfrak{m}^+(\Omega)}{\mathfrak{m}(\Omega)}\geqslant h_{(X,\mathrm{d},\mathfrak{m})} \tag{2.4}$$

which proves the claim.

Step 2. For $\Omega\subset X$ with $\mathfrak{m}(\Omega)<\infty$, we adopt an argument used by Cavalletti–Manini [19, Theorem 3.2], based on a relaxation principle investigated in [3, Theorem 3.6]. For any $\Omega\subset X$ with $\mathfrak{m}(\Omega)<\infty$, we have

$$\mathrm{Per}(\Omega)=\inf\left\{\liminf_{n\to\infty}\int\mathrm{lip}(f_n)\,\mathrm{d}\mathfrak{m}:f_n\in\mathrm{Lip}(X,\mathrm{d}),\ \lim_{n\to\infty}\int|f_n-\chi_\Omega|\,\mathrm{d}\mathfrak{m}=0\right\}$$

$$= \inf \left\{ \liminf_{n\to\infty} \mathfrak{m}^+(\Omega_n) : \mathfrak{m}(\Omega \Delta \Omega_n) \to 0 \right\}$$

where we require Ω_n to be bounded. Applying (2.4) with Ω_n and letting $n \to \infty$ we get

$$\mathrm{Per}(\Omega) \geqslant h_{(X,\mathrm{d},\mathfrak{m})} \mathfrak{m}(\Omega). \tag{2.5}$$

By [3, Theorem 3.6], $\mathrm{Per}(\Omega) \leqslant \mathfrak{m}^+(\Omega)$, we complete the proof. □

In the same spirit we can prove the existence of the volume entropy on $\mathrm{CD}(0,\infty)$ spaces. Note that on Riemannian manifolds having lower Ricci curvature bounds, this result can be proved using the Bishop–Gromov volume comparison inequality (see [14] and [33]).

Proposition 2.2. *Let $(X, \mathrm{d}, \mathfrak{m})$ be a $\mathrm{CD}(0,\infty)$ metric measure space. Then the volume entropy $h_{(X,\mathrm{d},\mathfrak{m})} \in [0,+\infty]$ exists in the sense of Definition 1.3.*

Proof. If $\mathfrak{m}(X) < +\infty$, we have $h_{(X,\mathrm{d},\mathfrak{m})} = 0$, otherwise there is $\epsilon > 0$ such that $\mathfrak{m}(B_\epsilon(x_0)) > 1$. Applying (2.3) with $\Omega = B_{r+\delta}(x_0)$ and $R = \epsilon$ for some $r > \epsilon$ and $\delta, > 0$, we get

$$\ln\big(\mathfrak{m}(Z_t)\big) \geqslant t \ln\big(\mathfrak{m}(B_\epsilon(x_0))\big) + (1-t)\ln\big(\mathfrak{m}(B_{r+\delta}(x_0))\big) \quad \forall t \in [0,1]. \tag{2.6}$$

For any $z \in Z_t$, by the triangle inequality

$$\mathrm{d}(z, x_0) < (1-t)[(r+\delta)+\epsilon] + \epsilon.$$

So for $t = \frac{\delta+\epsilon}{r+\delta}$, we have $Z_t \subset B_{r+\epsilon}(x_0)$. Thus (2.6) implies

$$\ln\big(\mathfrak{m}(B_{r+\epsilon}(x_0))\big) \geqslant \frac{\delta+\epsilon}{r+\delta}\ln\big(\mathfrak{m}(B_\epsilon(x_0))\big) + \frac{r-\epsilon}{r+\delta}\ln\big(\mathfrak{m}(B_{r+\delta}(x_0))\big). \tag{2.7}$$

Dividing both sides of (2.7) by r, we get

$$\frac{\ln\big(\mathfrak{m}(B_{r+\epsilon}(x_0))\big)}{r} \geqslant \left(1 - \frac{\epsilon}{r}\right)\frac{\ln\big(\mathfrak{m}(B_{r+\delta}(x_0))\big)}{r+\delta} \qquad \forall r, \delta > 0.$$

Then

$$\liminf_{r\to+\infty} \frac{\ln\big(\mathfrak{m}(B_r(x_0))\big)}{r} \geqslant \lim_{r\to+\infty}\left(1-\frac{\epsilon}{r}\right)\limsup_{\delta\to+\infty}\frac{\ln\big(\mathfrak{m}(B_\delta(x_0))\big)}{\delta},$$

which is the thesis. □

Next we will show that the inequality (2.1) is sharp. This can be proved by combining [18, Corollary 2] where Brooks showed that the Cheeger constant is no larger than the volume entropy, and our Theorem 2.1. We will give a different proof which has its own interest.

Theorem 2.3 (Sharpness)**.** *The inequality* (2.1) *in Theorem 2.1 is sharp. This means that, for any* $\mathrm{CD}(0,\infty)$ *space* $(X,\mathrm{d},\mathfrak{m})$ *and* $C > h_{(X,\mathrm{d},\mathfrak{m})}$, *the inequality* $\mathfrak{m}^+(\Omega) \geqslant C\mathfrak{m}(\Omega)$ *does not always hold.*

Proof. We will prove the theorem by contradiction. Assume there is a constant $C > h_{(X,\mathrm{d},\mathfrak{m})}$, such that

$$\mathfrak{m}^+(\Omega) \geqslant C\mathfrak{m}(\Omega) > 0 \tag{2.8}$$

for any bounded set $\Omega \subset X$.

By (2.7) we have

$$\begin{aligned}
&\frac{r-\epsilon}{r+\delta}\left(\frac{\ln\big(\mathfrak{m}(B_{r+\delta}(x_0))\big) - \ln\big(\mathfrak{m}(B_r(x_0))\big)}{\delta}\right)\\
&\leqslant \frac{\delta+\epsilon}{\delta(r+\delta)}\Big(\ln\big(\mathfrak{m}(B_r(x_0))\big) - \ln\big(\mathfrak{m}(B_\epsilon(x_0))\big)\Big).
\end{aligned}$$

Applying (2.8) with geodesic balls, we get

$$\ln\Big(\mathfrak{m}(B_{r+\delta}(x_0))\Big) - \ln\Big(\mathfrak{m}(B_r(x_0))\Big) \geqslant \delta C,$$

so

$$\frac{r-\epsilon}{r+\delta}C \leqslant \frac{\delta+\epsilon}{\delta(r+\delta)}\Big(\ln\Big(\mathfrak{m}(B_r(x_0))\Big) - \ln\Big(\mathfrak{m}(B_\epsilon(x_0))\Big)\Big).$$

Letting $r\to\infty$, we get

$$C \leqslant \frac{\delta+\epsilon}{\delta}h_{(X,\mathrm{d},\mathfrak{m})}.$$

Letting $\epsilon\to 0$ we get the contradiction. □

3 Rigidity

In this section we will study the rigidity of the isoperimetric inequality (2.1). We first deal with the rigidity for 1-dimensional spaces. The idea behind its proof is essential, which will be used directly or indirectly later. Then we will study the equality case of the isoperimetric inequality in the general $\mathrm{RCD}(0,\infty)$ setting.

3.1 Rigidity for log-concave densities

By a well-known result of Bobkov [15], for any log-concave density on $\mathbb{R}$, the infimum in the corresponding isoperimetric problem is attained by a half line. Among all log-concave densities, the log-linear densities e^{ht} play a special role.

Proposition 3.1 (Rigidity for log-concave densities). *Let $(X,\mathrm{d},\mathfrak{m}) = \Big(\mathbb{R}, |\cdot|, \mathrm{e}^f\mathcal{L}^1\Big)$ be a 1-dimensional metric measure space, where f is concave and*

$$\lim_{t\to+\infty} f'(t) = h > 0.$$

Then the volume entropy $h_{(X,\mathrm{d},\mathfrak{m})} = h$. If there is $\Omega\subset\mathbb{R}$ such that

$$\mu_{(X,\mathrm{d},\mathfrak{m})} = \frac{\mathfrak{m}^+(\Omega)}{\mathfrak{m}(\Omega)} = h,$$

then $f' = h$ and $\Omega = (-\infty, b)$ for some $b\in\mathbb{R}$.

Proof. Since f is concave, f' is well-defined almost everywhere, and the limits $\lim_{t\to-\infty} f'(t)$, $\lim_{t\to+\infty} f'(t)$ exist. Assume $\lim_{t\to+\infty} f'(t) = h > 0$. We can see that the volume entropy of $(X,\mathrm{d},\mathfrak{m})$ is the same as the volume entropy of $\Big(\mathbb{R}, |\cdot|, \mathrm{e}^{ht}\mathrm{d}t\Big)$, which is exactly h. By Theorem 2.1 and Theorem 2.3 we know the Cheeger constant $\mu_{(X,\mathrm{d},\mathfrak{m})}$ is h.

Assume there is $\Omega\subset\mathbb{R}$ attaining $\mu_{(X,\mathrm{d},\mathfrak{m})}$, and by Bobkov's result [15], Ω must be $(-\infty,\mathrm{C})$ for some $C\in\mathbb{R}$. Assume by contradiction that $f'(\mathrm{C}) > h$. We replace f by

$$\tilde f(t) := \begin{cases} f(t) & t\leqslant \mathrm{C},\\ f(\mathrm{C}) + f'(\mathrm{C})(t-\mathrm{C}) & t > \mathrm{C}.\end{cases}$$

Similarly, we can see that the volume entropy of $\Big(\mathbb{R}, |\cdot|, \mathrm{e}^{\tilde f(t)}\mathrm{d}t\Big)$ is $f'(\mathrm{C})$ and its corresponding Cheeger constant $\mu_{\mathrm{e}^{\tilde f(t)}} \geqslant f'(\mathrm{C}) > h$. However, by the definition of the Cheeger constant,

$$\mu_{\mathrm{e}^{\tilde f(t)}} \leqslant \frac{e^{\tilde f(\mathrm{C})}}{\int_{-\infty}^{\mathrm{C}} \mathrm{e}^{\tilde f(t)}\,\mathrm{d}t} = \frac{\mathrm{e}^{f(\mathrm{C})}}{\int_{-\infty}^{\mathrm{C}} \mathrm{e}^{f(t)}\,\mathrm{d}t} = \mu_{(X,\mathrm{d},\mathfrak{m})} = h \tag{3.1}$$

which leads to a contradiction. Therefore $f'(\mathrm{C}) = h$ and by concavity of f, $f' = h$ on right-hand. Notice that the inequalities in (3.1) must be equalities. So

$$\int_{-\infty}^{\mathrm{C}} e^{\tilde{f}(t)}\,\mathrm{d}t = \int_{-\infty}^{\mathrm{C}} e^{f(\mathrm{C})+h(t-\mathrm{C})}\,\mathrm{d}t$$

which proves the proposition.

□

3.2 Rigidity for RCD spaces

In the rest of this section, we study rigidity of the isoperimetric inequality in $\mathrm{RCD}(0,\infty)$ spaces. Recall that the Sobolev space $W^{1,2}(X,\mathrm{d},\mathfrak{m})$ is a Hilbert space, as a part of the definition of the RCD condition (cf. [5,6]). In this case, for $u, v \in W^{1,2}(X,\mathrm{d},\mathfrak{m})$, we define

$$\nabla u \cdot \nabla v := \inf_{\epsilon>0} \frac{|\mathrm{D}(v+\epsilon u)|^2 - |\mathrm{D}v|^2}{2\epsilon},$$

and we have $\nabla u \cdot \nabla v = \nabla v \cdot \nabla u$. Here $|\mathrm{D}u|$ denotes the weak upper gradient of u satisfying

$$\int |\mathrm{D}u|^2\,\mathrm{d}\mathfrak{m} = \inf\left\{\liminf_{n\to\infty} \int \mathrm{lip}(u_n)^2 \mathrm{d}\mathfrak{m} : u_n \in \mathrm{Lip}_c(X,\mathrm{d}),\ u_n \to u \text{ in } L^2\right\}$$

where

$$\mathrm{lip}(f)(x) := \limsup_{y\to x} \frac{|f(y)-f(x)|}{\mathrm{d}(x,y)} \ \text{ if } x \text{ is not isolated}, \quad \mathrm{lip}(f)(x) = 0 \ \text{ otherwise.}$$

Definition 3.2 (Measure valued Laplacian, cf. [26])**.** Let $\Omega \subset X$ be an open subset and let $u \in W^{1,2}_{loc}(X,\mathrm{d},\mathfrak{m})$. We say that u is in the domain of the Laplacian, and write $u \in \mathrm{D}(\mathbf{\Delta},\Omega)$, provided there exists a signed measure μ on Ω such that for any $f \in \mathrm{Lip}_c(\Omega)$ it holds that

$$\int \nabla f \cdot \nabla u\,\mathrm{d}\mathfrak{m} = -\int f\,\mathrm{d}\mu. \tag{3.2}$$

If μ is unique, we denote it by $\mathbf{\Delta}u$. If $\mathbf{\Delta}u \ll \mathfrak{m}$, we write $u \in \mathrm{D}(\Delta,\Omega)$ and denote its density by Δu.

In the next theorem we will prove the rigidity of the isoperimetric inequality. The strategy for proving the theorem is as follows. Supposing there is $\Omega \subset X$ with positive measure attaining the equality, we first show the existence of a curve $(\mu_\sigma)_{\sigma>0}$ in the Wasserstein space, such that $\mathrm{supp}\,\mu_\sigma \subset \Omega^\sigma$ and $\sigma \mapsto \mathrm{Ent}_{\mathfrak{m}}(\mu_\sigma)$ is linear. Then by showing that the Kantorovich potential ϕ_σ associated with μ_0 and μ_σ is an affine function, we prove X splits off a one-dimensional space. This can be seen as a dimension-free version of De Philippis–Gigli's theorem [23] in the $\mathrm{RCD}(0,N)$ setting. At last, by showing the "normal vector" of Ω is parallel to $\nabla\phi_\sigma$ in the non-smooth sense, we prove that Ω is a half space.

Theorem 3.3 (Rigidity theorem)**.** *Let $(X,\mathrm{d},\mathfrak{m})$ be an $\mathrm{RCD}(0,\infty)$ metric measure space with positive volume entropy $h_{(X,\mathrm{d},\mathfrak{m})}$.*

If there is a measurable set $\Omega \subset X$ with positive measure such that

$$\mathfrak{m}^+(\Omega) = h_{(X,\mathrm{d},\mathfrak{m})}\mathfrak{m}(\Omega), \tag{3.3}$$

then

$$(X,\mathrm{d},\mathfrak{m}) \cong \left(\mathbb{R}, |\cdot|, e^{h_{(X,\mathrm{d},\mathfrak{m})}t}\mathrm{d}t\right) \times (Y,\mathrm{d}_Y,\mathfrak{m}_Y)$$

for some $\mathrm{RCD}(0,\infty)$ space $(Y,\mathrm{d}_Y,\mathfrak{m}_Y)$ with $\mathfrak{m}_Y(Y) < +\infty$, where the product space on the right-hand side is a metric measure space with the canonical L^2-product metric and the product measure. In a suitable choice of coordinates, up to a negligible set, Ω can be identified as

$$\Omega = (-\infty, c) \times Y \subset \mathbb{R} \times Y$$

with $\mathfrak{m}_Y(Y)\int_{-\infty}^{c} e^{h_{(X,\mathrm{d},\mathfrak{m})}t}\,\mathrm{d}t = \mathfrak{m}(\Omega)$.

Proof. The proof is divided into five steps.

Step 1. We can assume that Ω is open:

By Bakry–Émery's gradient estimate for the heat flow $f_t := H_t(\chi_\Omega), t > 0$, we can see (cf. [30, Remark 3.5])

$$\int |\mathrm{D} f_t|\,\mathrm{d}\mathfrak{m} \leqslant \mathrm{Per}(\Omega) \leqslant \mathfrak{m}^+(\Omega) = h_{(X,\mathrm{d},\mathfrak{m})} \int f_t\,\mathrm{d}\mathfrak{m}.$$

By Cavalieri's formula (cf. [8, Chapter 6]) and the inequality (2.5) in Theorem 2.1

$$h_{(X,\mathrm{d},\mathfrak{m})} \int f_t\,\mathrm{d}\mathfrak{m} = h_{(X,\mathrm{d},\mathfrak{m})} \int_0^1 \mathfrak{m}\big(\{f_t > s\}\big)\,\mathrm{d}s \leqslant \int_0^1 \mathrm{Per}\big(\{f_t > s\}\big)\,\mathrm{d}s.$$

By the coarea formula of Fleming–Rishel (see [35] and [3, §4])

$$\int |\mathrm{D} f_t|\,\mathrm{d}\mathfrak{m} = \int_0^1 \mathrm{Per}\big(\{f_t > s\}\big)\,\mathrm{d}s.$$

Combining the inequalities above, for $\mathcal{L}^1$-a.e. $s \in [0,1]$, we have

$$h_{(X,\mathrm{d},\mathfrak{m})}\mathfrak{m}\big(\{f_t > s\}\big) = \mathfrak{m}^+\big(\{f_t > s\}\big)$$

By regularization of the heat flow, f_t is Lipschitz (cf. [6, THEOREM 6.5]), so $\{f_t > s\}$ is a non-trivial open set for some s. Since the isoperimetric profile is linear, without loss of generality, we may assume that Ω is a connected open set.

Step 2. The potential function ϕ_σ and the optimal transport map T_{ϕ_σ}:

For $r > 0$, let $\mu_{r,0} := \frac{1}{\mathfrak{m}(\Omega\cap B_r)}\mathfrak{m}_{\llcorner \Omega\cap B_r}$ and $\mu_{r,1} := \frac{1}{\mathfrak{m}\left((\Omega\cap B_r)^R\right)}\mathfrak{m}_{\llcorner (\Omega\cap B_r)^R}$. Consider the Wasserstein geodesic $(\mu_{r,t})_{t\in[0,1]}$ from $\mu_{r,0}$ to $\mu_{r,1}$. For $\sigma > 0$, by the super-position theorem (cf. [4, Theorem 2.10] and Remark 1.2), we can see that $\mu_{r,\sigma/R}$ is concentrated on $\Omega^{\sigma(r+R)/R}$.

By the $\mathrm{CD}(0,\infty)$ condition we have

$$\mathrm{Ent}_\mathfrak{m}(\mu_{r,\sigma/R}) \leqslant -\left(1 - \frac{\sigma}{R}\right) \ln\left(\mathfrak{m}(\Omega\cap B_r)\right) - \frac{\sigma}{R} \ln\left(\mathfrak{m}((\Omega\cap B_r)^R)\right). \tag{3.4}$$

By (3.4) we know the family of measures $\{\mu_{r,\sigma/R}\}_{r,R>0}$ is tight (cf. [5, Lemma 4.4]), so it converges narrowly, up to taking a subsequence, to a measure μ_σ as $R \to \infty$ and $r \to \infty$. By the positive volume entropy condition and the super-position theorem again, we can see that $(\mu_\sigma)_{\sigma>0}$ is a Wasserstein geodesic ray and μ_σ is concentrated on $\overline{\Omega^\sigma}$. From the construction, we can see that $\mu_0 = \frac{1}{\mathfrak{m}(\Omega)}\mathfrak{m}_{\llcorner\Omega}$ and we may assume that $\mathfrak{m}(\partial\Omega^\sigma) = 0$ for all σ without loss of generality. By Jensen's inequality, we can also see

$$\mathrm{Ent}_\mathfrak{m}(\mu_\sigma) \geqslant -\ln\left(\mathfrak{m}(\Omega^\sigma)\right). \tag{3.5}$$

Letting $R \to \infty, r \to \infty$ in (3.4), and combining (3.5) and the lower semi-continuity of the entropy, we get

$$-\ln\left(\mathfrak{m}(\Omega^\sigma)\right) \leqslant \mathrm{Ent}_\mathfrak{m}(\mu_\sigma) \leqslant \underbrace{-\ln\left(\mathfrak{m}(\Omega)\right)}_{=\mathrm{Ent}_\mathfrak{m}(\mu_0)} - \sigma h_{(X,\mathrm{d},\mathfrak{m})}. \tag{3.6}$$

By (3.6) and the assumption (3.3), we can see that

$$\varliminf_{\sigma\to 0} \frac{\mathrm{Ent}_\mathfrak{m}(\mu_\sigma) - \mathrm{Ent}_\mathfrak{m}(\mu_0)}{\sigma} \geqslant \varliminf_{\sigma\to 0} \frac{-\ln\left(\mathfrak{m}(\Omega^\sigma)\right) + \ln\left(\mathfrak{m}(\Omega)\right)}{\sigma} = -\frac{\mathfrak{m}^+(\Omega)}{\mathfrak{m}(\Omega)} = -h_{(X,\mathrm{d},\mathfrak{m})}.$$

By the $\mathrm{CD}(0,\infty)$ condition, $\sigma \mapsto \mathrm{Ent}_\mathfrak{m}(\mu_\sigma)$ is convex. Combining with (3.6) we get

$$-h_{(X,\mathrm{d},\mathfrak{m})} \leqslant \frac{\mathrm{d}^+}{\mathrm{d}\sigma}_{\llcorner\sigma=0} \mathrm{Ent}_\mathfrak{m}(\mu_\sigma) \leqslant \frac{\mathrm{Ent}_\mathfrak{m}(\mu_\sigma) - \mathrm{Ent}_\mathfrak{m}(\mu_0)}{\sigma} \leqslant -h_{(X,\mathrm{d},\mathfrak{m})}.$$

Thus $\sigma \mapsto \mathrm{Ent}_\mathfrak{m}(\mu_\sigma)$ is linear and $\mathrm{Ent}_\mathfrak{m}(\mu_\sigma) = -\ln\left(\mathfrak{m}(\Omega)\right) - h_{(X,\mathrm{d},\mathfrak{m})}\sigma$.

By [36], there exist a Kantorovich potential ϕ_σ, and a map $T_{\phi_\sigma}:\Omega\to\Omega^\sigma$, so that $\mu_\sigma=(T_{\phi_\sigma})_\sharp\mu_0$. By construction of μ_σ, we have $W_2(\mu_0,\mu_\sigma)=\sigma$ and $\mathrm{d}(T_{\phi_\sigma}(x),x)=\sigma$ for $\mathfrak{m}$-a.e. $x\in\Omega$. Thus by the metric Brenier's theorem [6, PROPOSITION 3.5] we have $|\mathrm{D}\phi_\sigma|=\sigma$. Furthermore, by an approximation argument with simple functions, we can also prove that $\mathrm{Ent}_\mathfrak{m}((T_{\phi_\sigma})_\sharp\nu)-\mathrm{Ent}_\mathfrak{m}(\nu)=-h_{(X,\mathrm{d},\mathfrak{m})}\sigma$ for any ν which is concentrated on Ω.

Step 3. $\phi_\sigma\in\mathrm{D}(\mathbf{\Delta},\Omega)$ and $\Delta\phi_\sigma=\sigma h_{(X,\mathrm{d},\mathfrak{m})}$.

Let ρ be a Lipschitz probability density with compact support in Ω. For any $\epsilon>0$, set $\rho_\epsilon:=c_\epsilon(\rho+\epsilon)\chi_{\Omega^\sigma}$ where c_ϵ is the normalizing constant. Let $\tau=(T_{\phi_\sigma})_\sharp(\rho\,\mathfrak{m})$ and $\tau_\epsilon=(T_{\phi_\sigma})_\sharp(\rho_\epsilon\,\mathfrak{m})$. By the derivative of the entropy formula [6, THEOREM 4.8-(b)], we have

$$\mathrm{Ent}_\mathfrak{m}(\tau_\epsilon)-\mathrm{Ent}_\mathfrak{m}(\rho_\epsilon\,\mathfrak{m})\geqslant-\int_\Omega\nabla\phi_\sigma\cdot\nabla\rho_\epsilon\,\mathrm{d}\mathfrak{m}.$$

Letting $\epsilon\downarrow0$, by the monotone convergence theorem and the locality of the weak upper gradient, we get

$$\mathrm{Ent}_\mathfrak{m}(\tau)-\mathrm{Ent}_\mathfrak{m}(\rho\,\mathfrak{m})\geqslant-\int_\Omega\nabla\phi_\sigma\cdot\nabla\rho\,\mathrm{d}\mathfrak{m}.$$

Combining Step 2, we get

$$-h_{(X,\mathrm{d},\mathfrak{m})}\sigma\geqslant-\int_\Omega\nabla\phi_\sigma\cdot\nabla\rho\,\mathrm{d}\mathfrak{m}.$$

By Step 2 we know almost all points in Ω have a pre-image of T_{ϕ_σ}. So $\{r\in\mathbb{R}:\mathfrak{m}(\Omega^r)>0\}=\mathbb{R}$ where $\Omega^r:=\{x\in\Omega:\mathrm{d}(x,\Omega^c)>|r|\}$. Considering the optimal transport induced by $-\phi_\sigma$ (cf. [29, Proposition 5.3]), we can also prove

$$h_{(X,\mathrm{d},\mathfrak{m})}\sigma\geqslant\int_\Omega\nabla\phi_\sigma\cdot\nabla\rho\,\mathrm{d}\mathfrak{m}.$$

Then by the Riesz–Markov–Kakutani representation theorem we know $\phi_\sigma\in\mathrm{D}(\mathbf{\Delta},\Omega)$ and

$$\Delta\phi_\sigma=-h_{(X,\mathrm{d},\mathfrak{m})}\sigma\quad\text{on}\quad\Omega.$$

Step 4. The gradient flow of ϕ_σ induces an isometric splitting.

The existence of the isometric splitting map has been well-studied by Gigli and his co-authors in [28,31] in the framework of non-smooth metric measure spaces. For convenience, we will omit some details here.

Let $\phi=\phi_\sigma/\sigma$. From Step 2 and Step 3, we know that $|\nabla\phi|=1$ and $\int|\nabla\phi|^2\Delta\varphi\,\mathrm{d}\mathfrak{m}=\int\Delta\varphi\,\mathrm{d}\mathfrak{m}=0$ for any $\varphi\in\mathrm{Lip}_c(\Omega)\cap\mathrm{D}(\Delta,\Omega)$. By Step 3, $\Delta\phi=-h_{(X,\mathrm{d},\mathfrak{m})}$, so $\nabla\phi\cdot\nabla\Delta\phi=0$, by Bochner's formula [27, Theorem 3.3.8] we know ϕ is an affine function (in the sense of [31, Proposition 3.2], $\mathrm{D}^{\mathrm{sym}}(\nabla\phi)=0$ and $|\mathrm{D}\phi|$ is constant). By [31, Theorem 4.4], there is a map $F:(-\infty,0)\times\Omega\to\Omega$, called the Regular Lagrangian Flow, and studied by Ambrosio–Trevisan [9] in the metric measure setting, such that

(i) $F_t(\cdot):=F(t,\cdot)$ is an isometry on X for each $t\in(-\infty,0)$;

(ii) $(F(t,x))_{t\in(-\infty,0)}$ is a geodesic (ray) in X for every $x\in\Omega$.

By disintegration, $\mathfrak{m}\llcorner_\Omega$ has a decomposition

$$\mathfrak{m}\llcorner_\Omega=\int_Y\mathfrak{m}_y\,\mathrm{d}\mathfrak{q}(y),\quad \mathfrak{m}_y\in\mathrm{Meas}(X_y),\quad X_y=\big\{F_t(y):t\in(-\infty,0)\big\}.\tag{3.7}$$

Following Cavalletti–Mondino [21, 4b], with the help of (3.7), we can represent the measure-valued Laplacian in the following way

$$\mathbf{\Delta}\phi=\int_Y h_{(X,\mathrm{d},\mathfrak{m})}\phi\,\mathrm{d}\mathfrak{m}_y\mathrm{d}\mathfrak{q}(y)$$

By integration by parts on $\mathbb{R}$ (cf. [21, Theorem 4.8]), this implies that $\mathfrak{m}_y=e^{V_y}\,\mathrm{d}t$ with $V_y'=h_{(X,\mathrm{d},\mathfrak{m})}$ on X_y. Furthermore, using the same argument as in Step 2, we can prove that

$$\ln(\mathfrak{m}(\Omega))\geqslant\ln(\mathfrak{m}(\Omega^\sigma))-h_{(X,\mathrm{d},\mathfrak{m})}\sigma.$$

Combining this inequality with (3.6) we get $\mathfrak{m}^+(\Omega^\sigma) = h_{(X,\mathrm{d},\mathfrak{m})}\mathfrak{m}(\Omega^\sigma)$. Since σ is arbitrary, we can see that F can be defined on the whole product space $\mathbb{R} \times X$. Similar to (3.7) we can write $\mathfrak{m} = \int_Y \mathfrak{m}_y \,\mathrm{d}\mathfrak{q}(y)$ with $\mathfrak{m}_y = e^{V_y}\,\mathrm{d}t$ and $V_y' = h_{(X,\mathrm{d},\mathfrak{m})}$.

Following [28, Section 6] and [31, Section 5], we can prove that F induces an isometry between (X, d) and the product space $(Y, \mathrm{d}_Y) \times \big(\mathbb{R}, |\cdot|\big)$ equipped with the L^2-product distance, where Y can be identified as $\phi^{-1}(0)$. Precisely, there are isometries Φ, Ψ defined by

$$\Phi : X \ni x \mapsto (y, t) \in Y \times \mathbb{R} \;\text{ s.t. } F_t(y) = x$$

and

$$\Psi : Y \times \mathbb{R} \ni (y, t) \mapsto x = F_t(y) \in X.$$

Furthermore, note that

$$(F_t)_\sharp \mathfrak{m} = e^{h_{(X,\mathrm{d},\mathfrak{m})}t}\mathfrak{m} \qquad \forall t \in \mathbb{R}.$$

We can define

$$\mathfrak{m}_Y(A) := \lim_{\epsilon\to 0} \frac{\mathfrak{m}\big(\Psi(A \times [0,\epsilon])\big)}{\epsilon} = \mathfrak{q}(A), \qquad \forall A \subset Y \;\text{ is measurable},$$

so that

$$\Phi_\sharp \mathfrak{m} = \mathfrak{m}_Y \times e^{h_{(X,\mathrm{d},\mathfrak{m})}t}\mathrm{d}t.$$

Using the same argument as in [28, Section 6] and [31, Section 5], we can prove that $(Y, \mathrm{d}_Y, \mathfrak{m}_Y)$ is $\mathrm{RCD}(0, \infty)$ and

$$(X, \mathrm{d}, \mathfrak{m}) \underset{\Phi,\Psi}{\cong} (Y, \mathrm{d}_Y, \mathfrak{m}_Y) \times \Big(\mathbb{R}, |\cdot|, e^{h_{(X,\mathrm{d},\mathfrak{m})}t}\,\mathrm{d}t\Big).$$

Step 5. Characterization of Ω.

By the decomposition (3.7) and Theorem 2.1, it holds that

$$\mathfrak{m}^+(\Omega) \overset{\text{Fatou}}{\geqslant} \int_Y \mathfrak{m}_y^+(\Omega)\,\mathrm{d}\mathfrak{q}(y) \geqslant h_{(X,\mathrm{d},\mathfrak{m})} \int_Y \mathfrak{m}_y(\Omega)\,\mathrm{d}\mathfrak{q}(y) = h_{(X,\mathrm{d},\mathfrak{m})}\mathfrak{m}(\Omega).$$

Thus

$$\mathfrak{m}_y^+(\Omega) = h_{(X,\mathrm{d},\mathfrak{m})}\mathfrak{m}_y(\Omega) \quad \mathfrak{q}\text{-a.e. } \; y \in Y.$$

By 1-dimensional rigidity in Proposition 3.1, for almost every $y \in Y$, $\Omega \cap X_y$ is a half line, and we denote it by $(-\infty, \mathsf{b}(y)]$. So we can identify Ω as

$$\Omega \cong \Big\{(y, r) : r \in \big(-\infty, \mathsf{b}(y)\big),\; y \in Y,\; \mathsf{b}(y) \in \mathbb{R}\Big\}$$

and $\partial\Omega$ is the graph of a measurable function $\mathsf{b}(\cdot)$ on Y.

Claim: b is a constant function. In the smooth case, the optimality of Ω surely implies that the line $t \mapsto F_t(x)$ is "vertical" to the boundary of Ω. If b is not constant, the "normal vector" of $\partial\Omega$ has a non-trivial "horizontal component" which leads to a contradiction. See Remark 3.4 for more explanations and related references.

From the proof of Theorem 2.1, we know there is a sequence of Lipschitz functions $(f_n)_{n\in\mathbb{N}}$ such that $f_n \to \chi_\Omega$ in L^1 and

$$\mathrm{Per}(\Omega) = \mathfrak{m}^+(\Omega) = \lim_{n\to+\infty} \int |\mathrm{D}f_n|\,\mathrm{d}\mathfrak{m}.$$

For simplicity, we write $f_n = f_n(y, r)$ as a function on $Y \times \mathbb{R}$, and $\mathfrak{m} = \mathfrak{m}_Y \times \mathfrak{m}_\mathbb{R}$ where $\mathfrak{m}_\mathbb{R} = e^{h_{(X,\mathrm{d},\mathfrak{m})}t}\mathrm{d}t$. Set $f_n^r = f_n(\cdot, r)$, $f_n^y = f_n(y, \cdot)$ and $\chi_\Omega^y = \chi_{\Omega^y} = \chi_{\Omega\cap\{(y,r):r\in\mathbb{R}\}}$. By Fubini's theorem, $f_n \to \chi_\Omega$ in L^1 implies that

$$\int_Y \left(\int_\mathbb{R} |f_n^y(t) - \chi_\Omega^y|\,\mathrm{d}\mathfrak{m}_\mathbb{R}\right)\mathrm{d}\mathfrak{m}_Y \to 0 \quad \text{as } n \to \infty.$$

So there is a subsequence of (f_n), still denoted by (f_n), such that

$$\lim_{n\to\infty}\int_{\mathbb{R}} |f_n^y(t)-\chi_\Omega^y|\,\mathrm{d}\mathfrak{m}_{\mathbb{R}}=0,\quad \mathfrak{m}_Y\text{-a.e. } y\in Y,$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}} f_n^y(t)\,\mathrm{d}\mathfrak{m}_{\mathbb{R}}=\int_{\mathbb{R}}\chi_\Omega^y\,\mathrm{d}\mathfrak{m}_{\mathbb{R}}=\frac{1}{h_{(X,\mathrm{d},\mathfrak{m})}}e^{h_{(X,\mathrm{d},\mathfrak{m})}\mathsf{b}(y)},\quad \mathfrak{m}_Y\text{-a.e. } y\in Y. \tag{3.8}$$

So by lower semi-continuity,

$$\varliminf_{n\to\infty}\int_{\mathbb{R}} |\mathrm{D}f_n^y(t)|\,\mathrm{d}\mathfrak{m}_{\mathbb{R}}\geqslant \mathfrak{m}_{\mathbb{R}}^+(\Omega^y)=e^{h_{(X,\mathrm{d},\mathfrak{m})}\mathsf{b}(y)}. \tag{3.9}$$

By [7, Theorem 5.2], $|\mathrm{D}f_n|^2=|\mathrm{D}f_n^r|^2+|\mathrm{D}f_n^y|^2$, where $|\mathrm{D}f_n^r|=|\mathrm{D}f_n^r|_Y$ is the weak gradient of f_n^r in Y, and $|\mathrm{D}f_n^y|=|\mathrm{D}f_n^y|_{\mathbb{R}}$ is the weak gradient of f_n^y in $\mathbb{R}$ which is the norm of partial derivatives. So for any $\epsilon>0$ we have

$$\begin{aligned}
&\int |\mathrm{D}f_n|\,\mathrm{d}\mathfrak{m}=\int\sqrt{|\mathrm{D}f_n^r|^2+|\mathrm{D}f_n^y|^2}\,\mathrm{d}\mathfrak{m}_{\mathbb{R}}\mathrm{d}\mathfrak{m}_Y\\
=&\int_{\{|\mathrm{D}f_n^r|>\epsilon|\mathrm{D}f_n^y|\}}\left(\frac{|\mathrm{D}f_n^r|^2}{\sqrt{|\mathrm{D}f_n^r|^2+|\mathrm{D}f_n^y|^2}+|\mathrm{D}f_n^y|}+|\mathrm{D}f_n^y|\right)\mathrm{d}\mathfrak{m}_{\mathbb{R}}\mathrm{d}\mathfrak{m}_Y\\
&+\int_{|\mathrm{D}f_n^r|\leqslant\epsilon|\mathrm{D}f_n^y|}\sqrt{|\mathrm{D}f_n^r|^2+|\mathrm{D}f_n^y|^2}\,\mathrm{d}\mathfrak{m}_{\mathbb{R}}\mathrm{d}\mathfrak{m}_Y\\
\geqslant&\int_{\{|\mathrm{D}f_n^r|>\epsilon|\mathrm{D}f_n^y|\}}\left(\frac{|\mathrm{D}f_n^r|}{2\sqrt{1+\epsilon^{-2}}}+|\mathrm{D}f_n^y|\right)\mathrm{d}\mathfrak{m}_{\mathbb{R}}\mathrm{d}\mathfrak{m}_Y+\int_{|\mathrm{D}f_n^r|\leqslant\epsilon|\mathrm{D}f_n^y|}|\mathrm{D}f_n^y|\,\mathrm{d}\mathfrak{m}_{\mathbb{R}}\mathrm{d}\mathfrak{m}_Y.
\end{aligned}$$

Then

$$\begin{aligned}
\int |\mathrm{D}f_n|\,\mathrm{d}\mathfrak{m}&\geqslant\int_{\{|\mathrm{D}f_n^r|>\epsilon|\mathrm{D}f_n^y|\}}\frac{|\mathrm{D}f_n^r|}{2\sqrt{1+\epsilon^{-2}}}\,\mathrm{d}\mathfrak{m}+\int|\mathrm{D}f_n^y|\,\mathrm{d}\mathfrak{m}\\
&\geqslant\int\frac{|\mathrm{D}f_n^r|}{2\sqrt{1+\epsilon^{-2}}}\,\mathrm{d}\mathfrak{m}+\left(1-\frac{\epsilon}{2\sqrt{1+\epsilon^{-2}}}\right)\int|\mathrm{D}f_n^y|\,\mathrm{d}\mathfrak{m}.
\end{aligned}$$

Letting $n\to\infty$ and combining the inequalities above with (3.9), we get

$$\begin{aligned}
&\mathfrak{m}^+(\Omega)=\lim_{n\to\infty}\int|\mathrm{D}f_n|\,\mathrm{d}\mathfrak{m}\\
\geqslant&\varliminf_{n\to\infty}\int\frac{|\mathrm{D}f_n^r|}{2\sqrt{1+\epsilon^{-2}}}\,\mathrm{d}\mathfrak{m}+(1-\frac{\epsilon}{2\sqrt{1+\epsilon^{-2}}})\int\mathfrak{m}_{\mathbb{R}}^+(\Omega^y)\,\mathrm{d}\mathfrak{m}_Y\\
\geqslant&\varliminf_{n\to\infty}\int\frac{|\mathrm{D}f_n^r|}{2\sqrt{1+\epsilon^{-2}}}\,\mathrm{d}\mathfrak{m}+(1-\frac{\epsilon}{2\sqrt{1+\epsilon^{-2}}})h_{(X,\mathrm{d},\mathfrak{m})}\int\mathfrak{m}_{\mathbb{R}}(\Omega^y)\,\mathrm{d}\mathfrak{m}_Y\\
=&\varliminf_{n\to\infty}\int\frac{|\mathrm{D}f_n^r|}{2\sqrt{1+\epsilon^{-2}}}\,\mathrm{d}\mathfrak{m}+(1-\frac{\epsilon}{2\sqrt{1+\epsilon^{-2}}})h_{(X,\mathrm{d},\mathfrak{m})}\mathfrak{m}(\Omega).
\end{aligned}$$

Combining this with $\mathfrak{m}^+(\Omega)=h_{(X,\mathrm{d},\mathfrak{m})}\mathfrak{m}(\Omega)$ we get

$$\epsilon\mathfrak{m}^+(\Omega)\geqslant\varliminf_{n\to\infty}\int|\mathrm{D}f_n^r|\,\mathrm{d}\mathfrak{m}.$$

Letting $\epsilon\to 0$ we obtain

$$\varliminf_{n\to\infty}\int|\mathrm{D}f_n^r|\,\mathrm{d}\mathfrak{m}=0. \tag{3.10}$$

Define $g_n(y)=\int_{\mathbb{R}}f_n(y,r)\,\mathrm{d}\mathfrak{m}_{\mathbb{R}}(r)$. We can approximate g_n in $L^1(\mathfrak{m})$ with functions of the form $\sum_{k\in I,|I|<\infty}c_kf_n^{r_k}(y)$, and approximate $\int_{\mathbb{R}}|\mathrm{D}f_n^r|\,\mathrm{d}\mathfrak{m}_{\mathbb{R}}(r)$ with functions of the form $\sum_{k\in I,|I|<\infty}c_k|\mathrm{D}f_n^{r_k}|(y)$. Then by a diagonal argument we can approximate g_n in $L^1(\mathfrak{m})$ with Lipschitz functions of the form $\sum_{k\in I,|I|<\infty} c_k h_k(y)$, and approximate $\int_{\mathbb{R}} |\mathrm{D} f_n^r| \,\mathrm{d}\mathfrak{m}_{\mathbb{R}}(r)$ with $\sum_{k\in I,|I|<\infty} c_k |\mathrm{D} h_k|$. Combining this approximation with the lower semi-continuity, one can prove

$$|\mathrm{D} g_n| \leqslant \int_{\mathbb{R}} |\mathrm{D} f_n^r| \,\mathrm{d}\mathfrak{m}_{\mathbb{R}}(r).$$

Thus (3.10) implies

$$\varliminf_{n\to\infty} \int |\mathrm{D} g_n| \,\mathrm{d}\mathfrak{m}_Y \leqslant \varliminf_{n\to\infty} \int |\mathrm{D} f_n^r| \,\mathrm{d}\mathfrak{m} = 0.$$

By (3.8) and the lower semi-continuity again, we know $\int |\mathrm{D} e^{h_{(X,\mathsf{d},\mathfrak{m})}\mathsf{b}(y)}| \,\mathrm{d}\mathfrak{m}_Y(y) = 0$ and $\mathsf{b}(\cdot)$ is constant. $\square$

Remark 3.4. From the proof of Theorem 2.1, we can see that χ_Ω is a BV function. By a general integration-by-parts formula of Brena–Gigli [16, Theorem 4.13], we have

$$\int_\Omega \mathrm{div} v \,\mathrm{d}\mathfrak{m} = -\int v \cdot v_\Omega \,\mathrm{d}\mu$$

for any "sufficiently smooth" vector field v, where μ is the total variation of χ_Ω and v_Ω plays the role of the "outward normal vector" of Ω.

Suppose that Ω attains the equality in the sharp isoperimetric inequality. If X is a Riemannian manifold equipped with a product metric structure induced by $\nabla\phi$, from the proof of Theorem 3.3 and integration by parts, we can see that the normal vector $v_\Omega = \nabla\phi$, so Ω is surely a half space (i.e., the function e in the proof of previous theorem is constant). In the setting of $\mathrm{RCD}(0, N)$ spaces, Antonelli–Brena–Pasqualetto [10, Theorem 3] identified the sub-graph of BV functions on Y and sets of finite perimeter on the cylinder $Y \times \mathbb{R}$. Using their result, one can also see that Ω is a half space and the proof can be **significantly shortened**. Unfortunately, it is still unknown whether [10, Theorem 3] is valid for general $\mathrm{RCD}(0, \infty)$ spaces or not.

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