Sharp and rigid isoperimetric inequality in metric measure spaces with non-negative Ricci curvature

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Abstract

By using optimal transport theory, we prove a sharp dimension-free isoperimetric inequality involving the volume entropy, in metric measure spaces with non-negative Ricci curvature in the sense of Lott–Sturm–Villani. We show that this isoperimetric inequality is attained by a non-trivial open set, if and only if the space satisfies a certain foliation property. For metric measure spaces with non-negative Riemannian Ricci curvature, we show that the sharp Cheeger constant is achieved by a non-trivial measurable set, if and only if a one-dimensional space is split off. Our isoperimetric inequality and the rigidity theorems are proved in non-smooth framework, totally dimension-free, new even in the smooth setting. In particular, our results provide some new understanding of logarithmically concave measures.

Keywords: isoperimetric inequality, Cheeger constant, curvature-dimension condition, metric measure space, non-negative Ricci curvature, optimal transport, relative entropy, volume entropy.

Contents

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1 Introduction

In the study of functional and geometric inequalities, such as isoperimetric inequality, log-Sobolev and Talagrand inequality, *strictly positive curvature* in the sense of Ricci, Bakry–Émery, Alexandrov or Lott-Sturm-Villani, often play critical roles. In lots of situations, we only have *non-negative curvature*, and many problems are even *dimension-free*.

The aim of this paper is to present a *sharp, dimension-free isoperimetric inequality*, in metric measure spaces with non-negative Ricci curvature in the sense of Lott–Sturm–Villani, and prove its rigidity.

Let $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space, where (X, d) is a complete and separable metric space and $\mathfrak{m}$ is a locally finite, non-negative Radon measure with full support. The *Minkowski content* of a Borel set $\Omega \subset X$ with $\mathfrak{m}(\Omega) < +\infty$ is defined by

$$\mathfrak{m}^+(\Omega) := \liminf_{\epsilon \to 0^+} \frac{\mathfrak{m}(\Omega^\epsilon) - \mathfrak{m}(\Omega)}{\epsilon}$$

where $\Omega^\epsilon \subset X$ is the ϵ-neighbourhood of Ω defined as $\Omega^\epsilon := \{x : \mathrm{d}(x, \Omega) < \epsilon\}$. An isoperimetric inequality relates the size of the boundary of a set to its measure. Precisely, let $\mathcal{M}$ be a family of metric measure spaces, there is a function $I_{\mathcal{M}}(\cdot)$, called *isoperimetric profile*, such that

$$\mathfrak{m}^+(\Omega) \geq I_{\mathcal{M}}(v)$$

for all $(X, \mathrm{d}, \mathfrak{m}) \in \mathcal{M}$ and any measurable set $\Omega \subset X$ with $\mathfrak{m}(\Omega) = v$.

Recently, isoperimetric inequalities in non-compact metric measure spaces with non-negative synthetic Ricci curvature, are studied in various settings, for example by Agostiniani–Fogagnolo–Mazzieri [AFM20], Brendle [Bre20], Balogh–Kristály [BK22], Antonelli–Pasqualetto–Pozzetta–Semola [APPS22a, APPS22b] and Cavalletti–Manini [CM22a, CM22b]. As discovered by E. Milman [Mil15], the isoperimetric profile for this family of spaces is trivial if there is no restriction on the diameter of the sets. In the above mentioned papers, a key component in the isoperimetric profile is a parameter called *asymptotic volume ratio*.

However, the asymptotic volume ratio depends on the dimension parameter, so those isoperimetric inequalities are all **dimension-dependent**. So it is natural to ask for a **dimension-free** isoperimetric inequality in metric measure spaces with non-negative Ricci curvature, in the sense of Lott–Sturm–Villani [LV09, Stu06]. Examples satisfying this condition includes weighted Riemannian manifolds with non-negative Bakry–Émery curvature, measured-Gromov Hausdorff limits of Riemannian manifolds with non-negative Ricci curvature, Alexandrov spaces with non-negative curvature, and reversible Finsler manifolds with non-negative Ricci curvature. See Ambrosio's ICM-Proceeding [Amb18] for an overview of this fast-growing field and bibliography.

Definition 1.1 (Lott–Sturm–Villani [LV09, Stu06])**.** We say that a metric measure space $(X, \mathrm{d}, \mathfrak{m})$ has Ricci curvature lower bound $K \in \mathbb{R}$, or satisfies $\mathrm{CD}(K, \infty)$ condition, if the relative entropy $\mathrm{Ent}_{\mathfrak{m}}$ defined as

$$\mathrm{Ent}_{\mathfrak{m}}(\mu) := \begin{cases} \int \ln \rho \, \mathrm{d}\mu & \text{if } \mu = \rho\, \mathfrak{m} \\ +\infty & \text{otherwise} \end{cases}$$

is K-displacement convex. This is to say, for any two probability measures μ_0, μ_1 in the L^2-Wasserstein space $(\mathcal{P}_2(X), W_2)$, there is a geodesic $(\mu_t)_{t\in[0,1]}$ satisfying

$$\mathrm{Ent}_{\mathfrak{m}}(\mu_t) \le t\mathrm{Ent}_{\mathfrak{m}}(\mu_1) + (1-t)\mathrm{Ent}_{\mathfrak{m}}(\mu_0) - \frac{t(1-t)K}{2} W_2^2(\mu_0,\mu_1) \quad \forall t \in [0,1].$$

In order to evaluate the growth of the volume without the dimension parameter, we will use **volume entropy**. This is an important concept in both Riemannian geometry (cf. [BCG95]) and dynamical system (cf. [Man79]). For example, it is related with Gromov's simplicial volume, the bottom of the spectrum of Laplacian, the Cheeger isoperimetric constant, the growth of fundamental groups, topological entropy of geodesic flows, etc.

Definition 1.2 (Volume entropy)**.** We say that a metric measure space $(X, \mathrm{d}, \mathfrak{m})$ admits the volume entropy $h_{(X,\mathrm{d},\mathfrak{m})}$, if there is $x_0 \in X$ so that the following limit exists

$$h_{(X,\mathrm{d},\mathfrak{m})} := \lim_{r\to+\infty} \frac{\ln \mathfrak{m}\big(B_r(x_0)\big)}{r} \in [0,\infty].$$

It can be seen that a metric measure space with non-negative Ricci curvature in the sense of Definition 1.1, surely admits the volume entropy which is independent on the choice of $x_0 \in X$ (see Proposition 2.2).

The first main result of this paper is the following sharp isoperimetric inequality involving volume entropy.

Theorem 1.3 (Sharp isoperimetric inequality, Theorem 2.1 and Theorem 2.3)**.** *Let $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space with non-negative synthetic Ricci curvature in the sense of Lott–Sturm–Villani. Then for any $\Omega \subset X$ with finite measure, we have*

$$\mathfrak{m}^+(\Omega) \ge h_{(X,\mathrm{d},\mathfrak{m})}\mathfrak{m}(\Omega). \tag{1.1}$$

In other words, the Cheeger constant $\mu_{(X,\mathrm{d},\mathfrak{m})} := \inf_\Omega \frac{\mathfrak{m}^+(\Omega)}{\mathfrak{m}(\Omega)}$ is no less than the volume entropy $h_{(X,\mathrm{d},\mathfrak{m})}$.

Moreover, the constant $h_{(X,\mathrm{d},\mathfrak{m})}$ in (1.1) can not be improved.

As a direct consequence of this theorem, we have the following corollary. Particular examples fitting the hypothesis includes $\mathrm{CD}(0,N)$ spaces with $N < +\infty$, and $\mathrm{CD}(K,\infty)$ spaces with $K > 0$.

Corollary 1.4. *Let $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space with non-negative Ricci curvature and $h_{(X,\mathrm{d},\mathfrak{m})} = 0$. Then there is no isoperimetric inequality in the form of*

$$\mathfrak{m}^+(\Omega) \ge C\mathfrak{m}(\Omega), \quad \forall \Omega \subset X$$

from some $C > 0$.

In [Bro81, Theorem 1], R. Brooks proved that the bottom of the essential spectrum λ_0^{ess} is bounded from above by $\frac{1}{4}h^2_{(X,\mathrm{d},\mathfrak{m})}$ if $\mathfrak{m}(X) = +\infty$. Combining with Cheeger's inequality [Che70] we get the following inequality (cf. [Bro81, Corollary 2])

$$\frac{1}{4}h^2_{(X,\mathrm{d},\mathfrak{m})} \ge \lambda_0^{\mathrm{ess}} \ge \frac{1}{4}\mu^2_{(X,\mathrm{d},\mathfrak{m})}.$$

Then we obtain the following corollary.

Corollary 1.5. *Let $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space with non-negative synthetic Ricci curvature and infinite volume. It holds the equality*

$$\frac{1}{4}h^2_{(X,\mathrm{d},\mathfrak{m})} = \lambda_0^{\mathrm{ess}} = \frac{1}{4}\mu^2_{(X,\mathrm{d},\mathfrak{m})}.$$

It has been noticed by De Ponti–Mondino–Semola [DPMS21] (see also [DPM21]) that the equality in Cheeger's isoperimetric inequality can never be attained in the family of spaces with finite diameter or positive Ricci curvature (Corollary 1.4 provides a different interpretation of this fact). In the next theorem we show that, in metric measure spaces with non-negative Riemannian Ricci curvature, the isoperimetric inequality is rigid. Here 'Riemannian' means that $(X, \mathrm{d}, \mathfrak{m})$ is infinitesimally Hilbertian (cf. [AGS14, Gig15])

Theorem 1.6 (Rigidity theorem, Theorem 3.6)**.** *Let $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space with non-negative Riemannian Ricci curvature, and with positive volume entropy $h_{(X,\mathrm{d},\mathfrak{m})}$.*

If there is a measurable set $\Omega \subset X$ with finite measure such that the equality in the isoperimetric inequality (1.1) is attained

$$\mathfrak{m}^+(\Omega) = h_{(X,\mathrm{d},\mathfrak{m})}\mathfrak{m}(\Omega)$$

or the Cheeger constant is achieved

$$\mu_{(X,\mathrm{d},\mathfrak{m})} = \frac{\mathfrak{m}^+(\Omega)}{\mathfrak{m}(\Omega)},$$

then

$$(X, \mathrm{d}, \mathfrak{m}) \cong \left(\mathbb{R}, |\cdot|, e^{h_{(X,\mathrm{d},\mathfrak{m})}t}\mathrm{d}t\right) \times (Y, \mathrm{d}_Y, \mathfrak{m}_Y)$$

for some $\mathrm{RCD}(0,\infty)$ metric measure space $(Y, \mathrm{d}_Y, \mathfrak{m}_Y)$ with $\mathfrak{m}_Y(Y) < +\infty$. In a suitable choice of coordinates, Ω can be identified as

$$\Omega = (-\infty, c] \times Y \subset \mathbb{R} \times Y$$

with $c \in \mathbb{R}$ satisfying $\mathfrak{m}_Y(Y)c^{h_{(X,\mathrm{d},\mathfrak{m})}e} = h_{(X,\mathrm{d},\mathfrak{m})}\mathfrak{m}(\Omega)$.

The rest of this paper is organized as follows. In Section 2 we prove the sharp isoperimetric inequality and in Section 3 we study its rigidity.

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2 Sharp Isoperimetric Inequality

In this section we will prove a sharp isoperimetric inequality in metric measure spaces with non-negative Ricci curvature.

Theorem 2.1 (A dimension-free isoperimetric inequality)**.** *Let* $(X, \mathrm{d}, \mathfrak{m})$ *be a metric measure space with non-negative Ricci curvature admitting the volume entropy* $h_{(X,\mathrm{d},\mathfrak{m})}$. *Then for any measurable set* $\Omega \subset X$ *with finite measure, it holds the following isoperimetric inequality*

$$\mathfrak{m}^+(\Omega) \geq h_{(X,\mathrm{d},\mathfrak{m})}\mathfrak{m}(\Omega). \tag{2.1}$$

Proof. **Step 1.** Assume Ω to be bounded.

Given $x_0 \in \Omega$. Let $R > 0$ be such that $\Omega \subset B_R(x_0) := \{x : \mathrm{d}(x, x_0) < R\}$. Define $\mu_0 = \frac{1}{\mathfrak{m}(\Omega)}\mathfrak{m}|_\Omega$ and $\mu_1 = \frac{1}{\mathfrak{m}(B_R(x_0))}\mathfrak{m}|_{B_R(x_0)}$. According to Definition 1.1, there exists an L^2-Wasserstein geodesic $(\mu_t)_{t\in[0,1]}$ connecting μ_0, μ_1 such that

$$\mathrm{Ent}_\mathfrak{m}(\mu_t) \leq t\mathrm{Ent}_\mathfrak{m}(\mu_1) + (1-t)\mathrm{Ent}_\mathfrak{m}(\mu_0). \tag{2.2}$$

Denote the set of t-intermediate points by

$$Z_t := \left\{ z : \exists\, x \in \Omega, y \in B_R(x_0), \text{such that } \frac{\mathrm{d}(z,x)}{t} = \frac{\mathrm{d}(z,y)}{1-t} = \mathrm{d}(x,y) \right\}.$$

It can be seen from the super-position theorem (cf. [AG11, Theorem 2.10]) that μ_t is concentrated on Z_t. Then by (2.2), Jensen's inequality and monotonicity of the function $t \to \ln(t)$ we have

$$-\ln\big(\mathfrak{m}(Z_t)\big) \leq -t\ln\big(\mathfrak{m}(B_R(x_0))\big) - (1-t)\ln\big(\mathfrak{m}(\Omega)\big). \tag{2.3}$$

Let $\epsilon := t\big(\operatorname{diam}(\Omega) + R\big)$. For any $z \in Z_t$, there is $x \in \Omega$ and $y \in B_R(x_0)$ so that $\mathrm{d}(z,x) = t\mathrm{d}(x,y)$. By triangle inequality,

$$\mathrm{d}(x,y) \leq \mathrm{d}(x,x_0) + \mathrm{d}(y,x_0) < \operatorname{diam}(\Omega) + R.$$

So $\mathrm{d}(z,x) < \epsilon$, $z \in \Omega^\epsilon$ and $Z_t \subset \Omega^\epsilon$.

If $\mathfrak{m}^+(\Omega) = +\infty$, there is nothing to prove. Otherwise, $\lim_{\epsilon\to 0}\mathfrak{m}(\Omega^\epsilon) = \mathfrak{m}(\Omega)$. So we have

$$\begin{aligned}
\frac{\mathfrak{m}^+(\Omega)}{\mathfrak{m}(\Omega)} &= \liminf_{\epsilon\to 0}\frac{1}{\mathfrak{m}(\Omega)}\frac{\mathfrak{m}(\Omega^\epsilon)-\mathfrak{m}(\Omega)}{\epsilon}\\
\text{By L'Hôpital's rule} &= \liminf_{\epsilon\to 0}\frac{\ln\big(\mathfrak{m}(\Omega^\epsilon)\big)-\ln\big(\mathfrak{m}(\Omega)\big)}{\mathfrak{m}(\Omega^\epsilon)-\mathfrak{m}(\Omega)}\frac{\mathfrak{m}(\Omega^\epsilon)-\mathfrak{m}(\Omega)}{\epsilon}\\
&\geq \liminf_{t\to 0}\frac{\ln\big(\mathfrak{m}(Z_t)\big)-\ln\big(\mathfrak{m}(\Omega)\big)}{t(\operatorname{diam}(\Omega)+R)}\\
\text{By (2.3)} &\geq \liminf_{t\to 0}\frac{t\ln\big(\mathfrak{m}(B_R(x_0))\big)+(1-t)\ln\big(\mathfrak{m}(\Omega)\big)-\ln\big(\mathfrak{m}(\Omega)\big)}{t(\operatorname{diam}(\Omega)+R)}\\
&= \frac{\ln\big(\mathfrak{m}(B_R(x_0))\big)-\ln\big(\mathfrak{m}(\Omega)\big)}{\operatorname{diam}(\Omega)+R}.
\end{aligned}$$

By Proposition 2.2, the volume entropy $h_{(X,\mathrm{d},\mathfrak{m})}$ exists. Letting $R \to \infty$, we get

$$\frac{\mathfrak{m}^+(\Omega)}{\mathfrak{m}(\Omega)} \geq h_{(X,\mathrm{d},\mathfrak{m})} \tag{2.4}$$

which is the thesis.

Step 2. Any $\Omega \subset X$ with finite measure.

We adopt an argument used by Cavalletti–Manini [CM22a, Theorem 3.2], based on a relaxation principle investigated in [ADMG17, Theorem 3.6]. For any $\Omega \subset X$ with finite measure, we have

$$\begin{aligned} \mathrm{Per}(\Omega) &= \inf\left\{\liminf_{n\to\infty} \int \mathrm{lip}(f_n)\,\mathrm{d}\mathfrak{m} : f_n \in \mathrm{Lip}(X,\mathrm{d}),\ \lim_{n\to\infty}\int |f_n - \chi_\Omega|\,\mathrm{d}\mathfrak{m} = 0\right\} \\ &= \inf\left\{\liminf_{n\to\infty} \mathfrak{m}^+(\Omega_n) : \mathfrak{m}(\Omega\Delta\Omega_n) \to 0\right\} \end{aligned}$$

where we impose Ω_n to be bounded. Applying (2.4) with Ω_n and letting $n \to \infty$ we get

$$\mathrm{Per}(\Omega) \geq h_{(X,\mathrm{d},\mathfrak{m})}\mathfrak{m}(\Omega). \tag{2.5}$$

By [ADMG17, Theorem 3.6], $\mathrm{Per}(\Omega) \leq \mathfrak{m}^+(\Omega)$, we complete the proof. □

In the same spirit we can prove the existence of the volume entropy under non-negative curvature condition.

Proposition 2.2. *Let $(X,\mathrm{d},\mathfrak{m})$ be a metric measure space with non-negative Ricci curvature. Then $h_{(X,\mathrm{d},\mathfrak{m})} \in [0,+\infty]$ exists in the sense of Definition 1.2.*

Proof. If $\mathfrak{m}(X) < +\infty$, we have $h_{(X,\mathrm{d},\mathfrak{m})} = 0$, otherwise there is $\epsilon > 0$ such that $\mathfrak{m}(B_\epsilon(x_0)) > 1$. Applying (2.3) with $\Omega = B_{r+\delta}(x_0)$ and $R = \epsilon$ for some $r > \epsilon, \delta, > 0$, we get

$$\ln\big(\mathfrak{m}(Z_t)\big) \geq t\ln\big(\mathfrak{m}(B_\epsilon(x_0))\big) + (1-t)\ln\big(\mathfrak{m}(B_{r+\delta}(x_0))\big) \quad \forall t \in [0,1]. \tag{2.6}$$

For any $z \in Z_t$, by triangle inequality

$$\mathrm{d}(z,x_0) < (1-t)[(r+\delta)+\epsilon] + \epsilon.$$

So for $t = \frac{\delta+\epsilon}{r+\delta}$, we have $Z_t \subset B_{r+\epsilon}(x_0)$. Thus (2.6) implies

$$\ln\big(\mathfrak{m}(B_{r+\epsilon}(x_0))\big) \geq \frac{\delta+\epsilon}{r+\delta}\ln\big(\mathfrak{m}(B_\epsilon(x_0))\big) + \frac{r-\epsilon}{r+\delta}\ln\big(\mathfrak{m}(B_{r+\delta}(x_0))\big). \tag{2.7}$$

Dividing r on both sides of (2.7), we get

$$\frac{\ln\big(\mathfrak{m}(B_{r+\epsilon}(x_0))\big)}{r} \geq \left(1-\frac{\epsilon}{r}\right)\frac{\ln\big(\mathfrak{m}(B_{r+\delta}(x_0))\big)}{r+\delta} \qquad \forall r,\delta > 0.$$

Then

$$\liminf_{r\to+\infty}\frac{\ln\big(\mathfrak{m}(B_r(x_0))\big)}{r} \geq \lim_{r\to+\infty}\left(1-\frac{\epsilon}{r}\right)\limsup_{\delta\to+\infty}\frac{\ln\big(\mathfrak{m}(B_\delta(x_0))\big)}{\delta}$$

which is the thesis. □

Next we will show that the inequality (2.1) is sharp. This can be proved by combining [Bro81, Corollary 2] where Brooks showed that the Cheeger constant is no larger than the volume entropy, and our Theorem 2.1. We will give a different proof which has its own interest.

Theorem 2.3 (Sharpness)**.** *The inequality* (2.1) *in Theorem 2.1 is sharp. This means, for any* $(X, \mathrm{d}, \mathfrak{m})$ *with non-negative curvature and any* $C > h_{(X,\mathrm{d},\mathfrak{m})}$*, the inequality*

$$\mathfrak{m}^+(\Omega) \geq C\mathfrak{m}(\Omega) \quad \textit{for all bounded } \Omega$$

does not hold.

Proof. We will prove the theorem by contradiction. Assume there is a constant $C > h_{(X,\mathrm{d},\mathfrak{m})}$, such that

$$\mathfrak{m}^+(\Omega) \geq C\mathfrak{m}(\Omega) > 0 \tag{2.8}$$

for any bounded set $\Omega \subset X$.

By (2.7) we have

$$\begin{aligned} &\frac{r-\epsilon}{r+\delta}\left(\frac{\ln\big(\mathfrak{m}(B_{r+\delta}(x_0))\big)-\ln\big(\mathfrak{m}(B_r(x_0))\big)}{\delta}\right) \\ \leq\ &\frac{\delta+\epsilon}{\delta(r+\delta)}\Big(\ln\big(\mathfrak{m}(B_r(x_0))\big)-\ln\big(\mathfrak{m}(B_\epsilon(x_0))\big)\Big). \end{aligned}$$

Applying (2.8) with geodesic balls, we get

$$\ln\big(\mathfrak{m}(B_{r+\delta}(x_0))\big)-\ln\big(\mathfrak{m}(B_r(x_0))\big) \geq \delta C,$$

so

$$\frac{r-\epsilon}{r+\delta}C \leq \frac{\delta+\epsilon}{\delta(r+\delta)}\Big(\ln\big(\mathfrak{m}(B_r(x_0))\big)-\ln\big(\mathfrak{m}(B_\epsilon(x_0))\big)\Big).$$

Letting $r \to \infty$, we get

$$C \leq \frac{\delta+\epsilon}{\delta} h_{(X,\mathrm{d},\mathfrak{m})}.$$

Letting $\epsilon \to 0$ we get the contradiction. □

3 Cheeger Constant, Volume Entropy and Rigidity

In this section we will prove the rigidity of the isoperimetric inequality (2.1). As the needle decomposition theorem has not been established for $\mathrm{CD}(K, \infty)$ spaces, we can not adopt the powerful localization method used by Cavalletti and Manini in [CM22a, CM22b]. We will use a more direct method to study the rigidity.

We first deal with the rigidity for 1-dimensional spaces, in Subsection 3.1. The idea behind its proof is essential, which will be used directly or indirectly later. In Subsection 3.2 we will study the equality case of the isoperimetric inequality in general $\mathrm{CD}(0, \infty)$ setting. Then in Subsection 3.3 we will prove the rigidity of the isoperimetric inequality in $\mathrm{RCD}(0, \infty)$ setting.

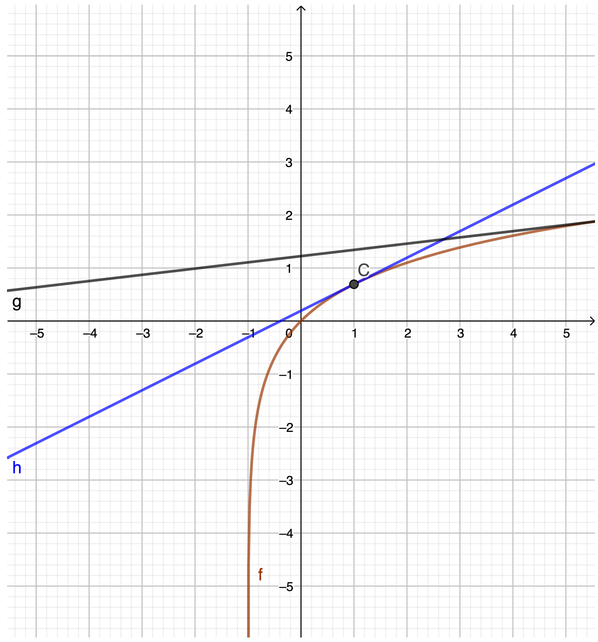

Figure 1: A concave function

3.1 Rigidity for log-concave densities

By a well-known result of Bobkov [Bob96], for any log-concave density on $\mathbb{R}$, the infimum in the corresponding isoperimetric problem is attained by a half line. Among all log-concave densities, the log-linear densities e^{ht} play special roles. For example, the isoperimetric profile is precisely given by E. Milman [Mil15, Corollary 1.4, Case 7]

$$\inf_{h\geq 0} I_{([0,D],e^{ht}\mathrm{d}t)}(v) = \frac{1}{D}\inf_{w>0}(v+w)\ln(1+1/w)$$

where $v \in (0, \frac{1}{2})$ is the volume and D is the upper bound of the diameter.

Firstly we prove the rigidity of the isoperimetric inequality in 1-dimensional spaces.

Proposition 3.1 (Rigidity for log-concave densities)**.** *Let* $(X, \mathrm{d}, \mathfrak{m}) = \left(\mathbb{R}, |\cdot|, e^V\mathcal{L}^1\right)$ *be a 1-dimensional metric measure space, where* V *is concave and*

$$\lim_{t\to+\infty} V'(t) = h > 0.$$

Then the volume entropy $h_{(X,\mathrm{d},\mathfrak{m})} = h$*. If there is* $\Omega \subset \mathbb{R}$ *such that*

$$\mu_{(X,\mathrm{d},\mathfrak{m})} = \frac{\mathfrak{m}^+(\Omega)}{\mathfrak{m}(\Omega)} = h,$$

then $V' = h$ *and* $\Omega = (-\infty, b)$ *for some* $b \in \mathbb{R}$*.*

Proof. Since V is concave, V' is well-defined almost everywhere, and the limits $\lim_{t\to-\infty} V'(t)$, $\lim_{t\to+\infty} V'(t)$ exist. Assume $\lim_{t\to+\infty} V'(t) = h > 0$. In Figure 1, the graph of V is represented by a red curve f, line g is the tangent line of V at infinity whose slope is h. We can see that the volume entropy of $(X, \mathrm{d}, \mathfrak{m})$ is the same as the volume entropy of $\left(\mathbb{R}, |\cdot|, e^{ht}\mathrm{d}t\right)$, which is exactly h. By Theorem 2.1 and Theorem 2.3 we know the Cheeger constant $\mu_{(X,\mathrm{d},\mathfrak{m})}$ is h.

Assume there is $\Omega \subset \mathbb{R}$ attaining $\mu_{(X,\mathrm{d},\mathfrak{m})}$, by Bobokov's result [Bob96], Ω must be a half-line $(-\infty, C)$. Assume by contradiction that $V'(C) > h$. In Figure 1, draw a blue line h, which is tangent to f at C. Now, we replace V by

$$\tilde{V}(t) := \begin{cases} V(t) & t \le C, \\ V(C) + V'(C)(t - C) & t > C. \end{cases}$$

Similarly, we can see that the volume entropy of $\left(\mathbb{R}, |\cdot|, e^{\tilde{V}(t)}\mathrm{d}t\right)$ is $V'(C)$ and its corresponding Cheeger constant $\mu_{e^{\tilde{V}(t)}} \ge V'(C) > h$. However, by definition of Cheeger constant,

$$\mu_{e^{\tilde{V}(t)}} \le e^{\tilde{V}(C)} / \int_{-\infty}^{C} e^{\tilde{V}(t)}\,\mathrm{d}t = e^{V(C)} / \int_{-\infty}^{C} e^{V(t)}\,\mathrm{d}t = \mu_{(X,\mathrm{d},\mathfrak{m})} = h, \tag{3.1}$$

which is a contradiction. Therefore $V'(C) = h$ and by concavity of V, $V' = h$ on $[C, +\infty)$. In Figure 1, the blue line and the curve f coincide on the right hand side of C. Notice that the inequalities in (3.1) must be equalities. So

$$\int_{-\infty}^{C} e^{\tilde{V}(t)}\,\mathrm{d}t = \int_{-\infty}^{C} e^{V(C)+h(t-C)}\mathrm{d}t.$$

In other words, the blue line h coincides with f on the left hand side of C.

□

3.2 Rigidity in CD setting

Lemma 3.2. *Let $(X, \mathrm{d}, \mathfrak{m})$ be a $\mathrm{CD}(0, \infty)$ metric measure space admitting positive volume entropy $h_{(X,\mathrm{d},\mathfrak{m})}$. Assume there is an open set Ω with positive volume, such that $\mathfrak{m}^+(\Omega) = h_{(X,\mathrm{d},\mathfrak{m})}\mathfrak{m}(\Omega)$. Then for any $\sigma > 0$, the σ-neighbourhood Ω^σ of Ω satisfies*

$$\mathfrak{m}(\Omega^\sigma) = \mathfrak{m}(\Omega)e^{\sigma h_{(X,\mathrm{d},\mathfrak{m})}}, \quad \mathfrak{m}^+(\Omega^\sigma) = h_{(X,\mathrm{d},\mathfrak{m})}\mathfrak{m}(\Omega^\sigma).$$

Proof. Since $\mathfrak{m}^+(\Omega) < +\infty$, we have $\mathfrak{m}(\Omega^\sigma) < \infty$ for some $\sigma > 0$. By $\mathrm{CD}(0, \infty)$ condition, we can see that the function $\sigma \mapsto \mathfrak{m}(\Omega^\sigma)$ is log-concave (cf. (2.3) in the proof of Theorem 2.1), so $\mathfrak{m}(\Omega^\sigma) < \infty$ for all $\sigma > 0$. Moreover, $\sigma \mapsto \mathfrak{m}(\Omega^\sigma)$ is almost everywhere differentiable, and for almost all $\sigma > 0$ we have $\mathfrak{m}^+(\Omega^\sigma) < \infty$ and

$$\mathfrak{m}(\Omega^{\sigma+\epsilon}) = \mathfrak{m}(\Omega^\sigma) + \epsilon\mathfrak{m}^+(\Omega^\sigma) + o(\epsilon) \quad \text{as } \epsilon \to 0. \tag{3.2}$$

By log-concavity and (3.2), we have

$$\begin{aligned} \left(\mathfrak{m}(\Omega^\sigma)\right)^n &\ge \mathfrak{m}(\Omega)\left(\mathfrak{m}(\Omega^{\frac{n}{n-1}\sigma})\right)^{n-1} \\ &= \mathfrak{m}(\Omega)\left(\mathfrak{m}(\Omega^\sigma) + \frac{\sigma}{n-1}\mathfrak{m}^+(\Omega^\sigma) + o(1/n)\right)^{n-1}. \end{aligned}$$

Combining with the isoperimetric inequality (2.1) we get

$$\big(\mathfrak{m}(\Omega^\sigma)\big)^n \geq \mathfrak{m}(\Omega)\big(\mathfrak{m}(\Omega^\sigma)\big)^{n-1}\Big(1+\frac{\sigma}{n-1}h_{(X,\mathrm{d},\mathfrak{m})}+o(1/n)\Big)^{n-1}$$

Dividing $\big(\mathfrak{m}(\Omega^\sigma)\big)^{n-1}$ on both sides of the inequality and letting $n \to \infty$ we get

$$\mathfrak{m}(\Omega^\sigma) \geq \mathfrak{m}(\Omega)e^{\sigma h_{(X,\mathrm{d},\mathfrak{m})}}. \tag{3.3}$$

Similarly, by log-concavity and the hypothesis $\mathfrak{m}^+(\Omega) = h_{(X,\mathrm{d},\mathfrak{m})}\mathfrak{m}(\Omega)$, we have

$$\begin{aligned}\mathfrak{m}(\Omega^\sigma)\big(\mathfrak{m}(\Omega)\big)^{n-1} &\leq \big(\mathfrak{m}(\Omega^{\frac{\sigma}{n}})\big)^n\\ &= \Big(\mathfrak{m}(\Omega)+\frac{\sigma}{n}\mathfrak{m}^+(\Omega)+o(\frac{1}{n})\Big)^n\\ &= \big(\mathfrak{m}(\Omega)\big)^n\Big(1+\frac{\sigma}{n}h_{(X,\mathrm{d},\mathfrak{m})}+o(\frac{1}{n})\Big)^n.\end{aligned}$$

Dividing $\big(\mathfrak{m}(\Omega)\big)^{n-1}$ on both sides of the inequality and letting $n \to \infty$ we get

$$\mathfrak{m}(\Omega^\sigma) \leq \mathfrak{m}(\Omega)e^{\sigma h_{(X,\mathrm{d},\mathfrak{m})}}.$$

Combining with (3.3) we get

$$\mathfrak{m}(\Omega^\sigma) = \mathfrak{m}(\Omega)e^{\sigma h_{(X,\mathrm{d},\mathfrak{m})}}$$

for almost all $\sigma > 0$. Hence $\mathfrak{m}(\Omega^\sigma) = \mathfrak{m}(\Omega)e^{\sigma h_{(X,\mathrm{d},\mathfrak{m})}}$ and $\mathfrak{m}^+(\Omega^\sigma) = h_{(X,\mathrm{d},\mathfrak{m})}\mathfrak{m}(\Omega^\sigma)$ for all $\sigma \geq 0$. □

Proposition 3.3 (Rigidity in CD setting)**.** *Denote* Rad $:= \{\sigma \in \mathbb{R} : \mathfrak{m}(\Omega^\sigma) > 0\}$ *where* $\Omega^\sigma := \{x \in \Omega : \mathrm{d}(x, \Omega^c) > -\sigma\}$ *for* $\sigma < 0$*. Under the same assumption as Lemma 3.3, we have* Rad $= \mathbb{R}$ *and*

$$\mathfrak{m}(\Omega^\sigma) = e^{\sigma h_{(X,\mathrm{d},\mathfrak{m})}}\mathfrak{m}(\Omega) \qquad \forall \sigma \in \mathbb{R}.$$

Proof. By log-concavity, for any negative $-\sigma \in$ Rad we have

$$\big(\mathfrak{m}(\Omega)\big)^2 \geq \mathfrak{m}(\Omega^{-\sigma})\mathfrak{m}(\Omega^\sigma) = \mathfrak{m}(\Omega^{-\sigma})\mathfrak{m}(\Omega)e^{\sigma h_{(X,\mathrm{d},\mathfrak{m})}}.$$

So

$$\mathfrak{m}(\Omega^{-\sigma}) \leq \mathfrak{m}(\Omega)e^{-\sigma h_{(X,\mathrm{d},\mathfrak{m})}} \quad \forall \sigma \in \mathrm{Rad} \cap (-\infty, 0).$$

Since the function Rad $\ni \sigma \mapsto \mathfrak{m}(\Omega^\sigma)$ is log-concave. For the same reason as Proposition 3.1, we know $(-\infty, 0) \subset$ Rad and $\sigma \mapsto \ln\big(\mathfrak{m}(\Omega^\sigma)\big)$ is linear on $(-\infty, 0)$. Combining with Lemma 3.2, we have $\mathfrak{m}(\Omega^\sigma) = \mathfrak{m}(\Omega)e^{\sigma h_{(X,\mathrm{d},\mathfrak{m})}}$ for all $\sigma \in \mathbb{R}$. □

Corollary 3.4. *Given* $\sigma \in \mathbb{R}$ *and* $r > 0$*. Let* $\mu_0, \mu_1 \in \mathcal{P}(X)$ *be uniform distributions on* Ω^σ *and* $\Omega^{\sigma+r}$ *respectively. Under the same assumption as Lemma 3.3, we have that the* t*-intermediate point between* μ_0, μ_1 *is unique and is the uniform distribution on* $\Omega^{\sigma+tr}$*.*

Proof. By displacement convexity of $\mathrm{Ent}_{\mathfrak{m}}$, there is a t-intermediate point μ of μ_0, μ_1 so that

$$\mathrm{Ent}_{\mathfrak{m}}(\mu) \leq -(1-t)\ln(\Omega^{\sigma}) - t\ln(\Omega^{\sigma+r}) \overset{\text{Proposition 3.3}}{=} -(\sigma+tr)h_{(X,\mathrm{d},\mathfrak{m})} - \ln\big(\mathfrak{m}(\Omega)\big).$$

Note that μ is concentrated on $\Omega^{\sigma+tr}$, by Jensen's inequality we have

$$\mathrm{Ent}_{\mathfrak{m}}(\mu) \geq -\ln(\Omega^{\sigma+tr}) = -(\sigma+tr)h_{(X,\mathrm{d},\mathfrak{m})} - \ln\big(\mathfrak{m}(\Omega)\big).$$

So μ is the uniform distribution on $\Omega^{\sigma+tr}$. □

3.3 Rigidity in RCD setting

In this part we will prove the rigidity of the isoperimetric inequality in $\mathrm{RCD}(0,\infty)$ spaces. Recall that the Sobolev space $W^{1,2}(X,\mathrm{d},\mathfrak{m})$ is a Hilbert space, as a part of the definition of RCD condition (cf. [AGS14, AGMR15]). In this case, for $u, v \in W^{1,2}(X,\mathrm{d},\mathfrak{m})$, we define

$$\nabla u \cdot \nabla v := \inf_{\epsilon>0} \frac{|\mathrm{D}(v+\epsilon u)|^2 - |\mathrm{D}v|^2}{2\epsilon},$$

and we have $\nabla u \cdot \nabla v = \nabla v \cdot \nabla u$. Here $|\mathrm{D}u|$ denotes the weak upper gradient of u satisfying

$$\int |\mathrm{D}u|^2 \,\mathrm{d}\mathfrak{m} = \inf\left\{\liminf_{n\to\infty} \int \mathrm{lip}(u_n)^2 \mathrm{d}\mathfrak{m} : u_n \in \mathrm{Lip}_c(X,\mathrm{d}),\ u_n \to u \text{ in } L^2\right\}$$

where

$$\mathrm{lip}(f)(x) := \limsup_{y\to x} \frac{|f(y)-f(x)|}{\mathrm{d}(x,y)} \ \text{ if } x \text{ is not isolated}, \quad \mathrm{lip}(f)(x) = 0 \ \text{ otherwise.}$$

Definition 3.5 (Measure valued Laplacian, cf. [Gig15])**.** Let $\Omega \subset X$ be an open subset and let $u \in W^{1,2}_{loc}(X,\mathrm{d},\mathfrak{m})$. We say that u is in the domain of the Laplacian, and write $u \in \mathrm{D}(\mathbf{\Delta},\Omega)$, provided there exists a signed measure μ on Ω such that for any $f \in \mathrm{Lip}_c(\Omega)$ it holds

$$\int \nabla f \cdot \nabla u \,\mathrm{d}\mathfrak{m} = -\int f \,\mathrm{d}\mu. \tag{3.4}$$

If μ is unique, we denote it by $\mathbf{\Delta}u$. If $\mathbf{\Delta}u \ll \mathfrak{m}$, we write $u \in \mathrm{D}(\Delta,\Omega)$ and denote its density by Δu.

Theorem 3.6 (Rigidity theorem)**.** *Let $(X,\mathrm{d},\mathfrak{m})$ be an $\mathrm{RCD}(0,\infty)$ metric measure space with positive volume entropy $h_{(X,\mathrm{d},\mathfrak{m})}$.*

If there is a measurable set $\Omega \subset X$ with positive volume such that

$$\mathfrak{m}^+(\Omega) = h_{(X,\mathrm{d},\mathfrak{m})}\mathfrak{m}(\Omega). \tag{3.5}$$

Then

$$(X,\mathrm{d},\mathfrak{m}) \cong \Big(\mathbb{R}, |\cdot|, e^{h_{(X,\mathrm{d},\mathfrak{m})}t}\mathrm{d}t\Big) \times (Y,\mathrm{d}_Y,\mathfrak{m}_Y)$$

for some $\mathrm{RCD}(0,\infty)$ *space* $(Y, \mathrm{d}_Y, \mathfrak{m}_Y)$ *with* $\mathfrak{m}_Y(Y) < +\infty$*, where the product space on the right hand side is a metric measure space with the canonical* L^2*-product metric and the product measure. In a suitable choice of coordinates,* Ω *can be identified as*

$$\Omega = (-\infty, c] \times Y \subset \mathbb{R} \times Y$$

with $c \in \mathbb{R}$ *satisfying* $\mathfrak{m}_Y(Y) \int_{-\infty}^{c} e^{h_{(X,\mathrm{d},\mathfrak{m})} t}\,\mathrm{d}t = \mathfrak{m}(\Omega)$.

Proof. The proof is divided into six steps.

Step 1. We can assume that Ω is open:

By Bakry–Émery's gradient estimate for the heat flow $f_t := H_t(\chi_\Omega)$, we can see (cf. [GH16, Remark 3.5])

$$\int |\mathrm{D} f_t|\,\mathrm{d}\mathfrak{m} \le \mathrm{Per}(\Omega) \le \mathfrak{m}^+(\Omega) = h_{(X,\mathrm{d},\mathfrak{m})} \int f_t\,\mathrm{d}\mathfrak{m}.$$

By Cavalieri's formula (cf. [AT04, Chapter 6]) and the inequality (2.5) in Theorem 2.1

$$h_{(X,\mathrm{d},\mathfrak{m})} \int f_t\,\mathrm{d}\mathfrak{m} = h_{(X,\mathrm{d},\mathfrak{m})} \int_0^1 \mathfrak{m}\big(\{f_t \ge t\}\big)\,\mathrm{d}t \le \int_0^1 \mathrm{Per}\big(\{f_t \ge t\}\big)\,\mathrm{d}t.$$

By coarea formula of Fleming–Rishel (see [Mir03] and [ADMG17, §4])

$$\int |\mathrm{D} f_t|\,\mathrm{d}\mathfrak{m} = \int_0^1 \mathrm{Per}\big(\{f_t \ge t\}\big)\,\mathrm{d}t.$$

Combining the inequalities above, for $\mathcal{L}^1$-a.e. $t \in [0,1]$, we have

$$h_{(X,\mathrm{d},\mathfrak{m})}\mathfrak{m}\big(\{f_t \ge t\}\big) = \mathfrak{m}^+\big(\{f_t \ge t\}\big)$$

By regularization of the heat flow, f_t is Lipschitz (cf. [AGS14, THEOREM 6.5]), so $\{f_t \ge t\}$ has non-empty interior for some t. Since the isoperimetric profile is linear, without loss of generality, we may assume that Ω is a connected open set.

Step 2. Potential function ϕ and optimal transport map $\nabla\phi$:

Given $\sigma \in \mathbb{R}$, $R > 0$. Let $\mu_0, \mu_1 \in \mathcal{P}(X)$ be uniform distributions on Ω^σ and $\Omega^{\sigma+R}$ respectively. Consider the optimal transport from μ_0 to μ_1. By [RS14], up to an additive constant, there is a unique Kantorovich potential ϕ, and the Monge problem has a unique solution $\nabla\phi : \Omega^\sigma \to \Omega^{\sigma+R}$, so that $\mu_1 = (\nabla\phi)_\sharp \mu_0$.

Step 3. $|\mathrm{D}\phi|(x) = R$ and $\mathrm{d}(\nabla\phi(x), x) = R$ $\mathfrak{m}$-a.e. on Ω^σ:

For $\epsilon > 0$, denote $E_0 = \Omega^{\sigma,\sigma-\epsilon} := \Omega^\sigma \setminus \Omega^{\sigma-\epsilon}, E_1 = \Omega^{\sigma+R,\sigma+R-\epsilon}$ and $F_0 = \Omega^{\sigma-\epsilon}, F_1 = \Omega^{\sigma+R-\epsilon}$. Let $\mu_0, \nu_0, \mu_1, \nu_1$ be uniform distributions on E_0, F_0, E_1, F_1 respectively. Let $(\mu_t)_{t\in[0,1]}$ and $(\nu_t)_{t\in[0,1]}$ be geodesics in the Wasserstein space. By Corollary 3.4, for any $t \in [0,1]$, μ_t is the uniform distribution on $E_t := \Omega^{\sigma+tR,\sigma+tR-\epsilon}$ and ν_t is the uniform distribution on $F_t := \Omega^{\sigma+tR-\epsilon}$. By Lemma 3.7, ϕ is a Kantorovich potential relative to both (μ_0, μ_1) and (ν_0, ν_1). In particular, the optimal transport map $\nabla\phi$ transport mass from E_0 to E_1. By metric Brenier's theorem [AGS14, PROPOSITION 3.5],

$$R - \epsilon \le |\mathrm{D}\phi| \le R + \epsilon \quad \text{on} \ \ \Omega^{\sigma,\sigma-\epsilon}.$$

Similarly, by induction, we can prove that

$$R-\epsilon \le |\mathrm{D}\phi| \le R+\epsilon \quad \text{on} \quad \Omega^{\sigma,\sigma-m\epsilon}, \qquad \forall m\in\mathbb{N}.$$

Letting $\epsilon\to 0$ we have

$$|\mathrm{D}\phi| = R \quad \text{on} \quad \Omega^{\sigma}.$$

Then by [AGS14, PROPOSITION 3.5] again, we know $\mathrm{d}(\nabla\phi(x),x)=R$ for all $x\in\Omega^\sigma$.

Step 4. $\phi\in \mathrm{D}(\boldsymbol{\Delta},\Omega^\sigma)$ and $\Delta\phi = h_{(X,\mathrm{d},\mathfrak{m})}$.

Let ρ be a Lipschitz probability density with compact support in Ω^σ. For any $\epsilon>0$, denote $\rho_\epsilon := c_\epsilon(\rho+\epsilon)\chi_{\Omega^\sigma}$ where c_ϵ is the normalizing constant. Let τ be the density of $(\nabla\phi)_\sharp(\rho\,\mathfrak{m})$ and τ_ϵ be the density of $(\nabla\phi)_\sharp(\rho_\epsilon\,\mathfrak{m})$. By the derivative of the entropy formula [AGS14, THEOREM 4.8-(b)] (see also [Gig15, Proposition 5.10]), we have

$$\mathrm{Ent}_{\mathfrak{m}}(\tau_\epsilon)-\mathrm{Ent}_{\mathfrak{m}}(\rho_\epsilon)\ge -\int_{\Omega^\sigma}\nabla\phi\cdot\nabla\rho_\epsilon\,\mathrm{d}\mathfrak{m}.$$

Letting $\epsilon\downarrow 0$, by monotone convergence theorem and the locality of the weak upper gradient, we get

$$\mathrm{Ent}_{\mathfrak{m}}(\tau)-\mathrm{Ent}_{\mathfrak{m}}(\rho)\ge -\int_{\Omega^\sigma}\nabla\phi\cdot\nabla\rho\,\mathrm{d}\mathfrak{m}.$$

By Step 3 and Proposition 3.3, we can see that $\mathrm{Ent}_{\mathfrak{m}}(\tau)-\mathrm{Ent}_{\mathfrak{m}}(\rho) = -h_{(X,\mathrm{d},\mathfrak{m})}R$. So

$$-h_{(X,\mathrm{d},\mathfrak{m})}R\ge -\int_{\Omega^\sigma}\nabla\phi\cdot\nabla\rho\,\mathrm{d}\mathfrak{m}.$$

Similarly, by considering the optimal transport induced by $-\phi$ (cf. [GH15, Proposition 5.3]), we can prove

$$h_{(X,\mathrm{d},\mathfrak{m})}R\ge \int_{\Omega^\sigma}\nabla\phi\cdot\nabla\rho\,\mathrm{d}\mathfrak{m}.$$

Then by Riesz–Markov–Kakutani representation theorem we know $\phi\in\mathrm{D}(\boldsymbol{\Delta},\Omega^\sigma)$ and

$$\Delta\phi = h_{(X,\mathrm{d},\mathfrak{m})} \quad \text{on } \Omega^\sigma.$$

Step 5. The gradient flow of ϕ induces an isometric splitting.

The existence of the isometric splitting map has been well-studied in the framework of non-smooth metric measure spaces. This argument was used by Gigli and his co-authors in [Gig13, GKKO20]. For convenience, we will omit some details here.

Indeed, note that $|\nabla\phi|=R$, we have $\int|\nabla\phi|^2\Delta\varphi\,\mathrm{d}\mathfrak{m} = \int\Delta\varphi\,\mathrm{d}\mathfrak{m}=0$ for any $\varphi\in\mathrm{Lip}_c(\Omega^\sigma)\cap\mathrm{D}(\Delta,\Omega^\sigma)$. By Step 4, $\Delta\phi=h$, so $\nabla\phi\cdot\nabla\Delta\phi=0$, by Bochner formula [Gig18, Theorem 3.3.8] we know ϕ is an affine function (in the sense of [GKKO20, Proposition 3.2], $\mathrm{D}^{\mathrm{sym}}(\nabla\phi)=0$ and $|\mathrm{D}\phi|$ is constant). In particular, since the choice of $\sigma>0$ is arbitrary, by [GKKO20, Theorem 4.4], there is a globally defined map $F:\mathbb{R}\times X\to X$, called the *Regular Lagrangian Flow*, studied by Ambrosio–Trevisan [AT14] in the metric measure setting, such that

(i) $F_t(\cdot) := F(t,\cdot)$ is an isometry on X for each $t \in \mathbb{R}$;

(ii) $(F(t,x))_{t\in\mathbb{R}}$ is a geodesic (line) in X for every $x \in X$.

Following the same argument as [Gig13, Section 6] and [GKKO20, Section 5], F induces an isometry between (X,d) and the product space $(Y,\mathrm{d}_Y)\times\big(\mathbb{R},|\cdot|\big)$ equipped with the L^2-product distance, where Y can be identified as $\phi^{-1}(0)$. Precisely, there are isometries Φ,Ψ defined by

$$\Phi : X \ni x \mapsto (y,t) \in Y\times\mathbb{R} \ \text{ s.t. } F_t(y) = x$$

and

$$\Psi : Y\times\mathbb{R} \ni (y,t) \mapsto x = F_t(y) \in X.$$

By disintegration of measure, $\mathfrak{m}$ has a decomposition

$$\mathfrak{m} = \int_Y \mathfrak{m}_y \,\mathrm{d}\mathfrak{q}(y), \quad \mathfrak{m}_y \in \mathrm{Meas}(X_y), \quad X_y = \big\{F_t(y) : t\in\mathbb{R}\big\}. \tag{3.6}$$

Following Cavalletti–Mondino [CM20, 4b], with the help of (3.6), we can represent the measure-valued Laplacian in the following way

$$\mathbf{\Delta}\phi = \int_Y h_{(X,\mathrm{d},\mathfrak{m})}\phi \,\mathrm{d}\mathfrak{m}_y \mathrm{d}\mathfrak{q}(y)$$

By integration by parts formula on $\mathbb{R}$ (cf. [CM20, Theorem 4.8]), this implies that $\mathfrak{m}_y = e^{V_y}\,\mathrm{d}t$ with $V_y' = h_{(X,\mathrm{d},\mathfrak{m})}$ on X_y. In particular,

$$(F_t)_\sharp \mathfrak{m} = e^{h_{(X,\mathrm{d},\mathfrak{m})}t}\mathfrak{m} \qquad \forall t\in\mathbb{R}.$$

Then we have

$$\mathfrak{m}_Y(A) := \lim_{\epsilon\to 0}\frac{\mathfrak{m}\big(\Psi(A\times[0,\epsilon])\big)}{\epsilon} = \mathfrak{q}(A), \qquad \forall A\subset Y \ \text{ is measurable,}$$

and

$$\Phi_\sharp\mathfrak{m} = \mathfrak{m}_Y \times e^{h_{(X,\mathrm{d},\mathfrak{m})}t}\mathrm{d}t.$$

Following the same argument as [Gig13, Section 6] and [GKKO20, Section 5], we can prove that $(Y,\mathrm{d}_Y,\mathfrak{m}_Y)$ is $\mathrm{RCD}(0,\infty)$ and

$$(X,\mathrm{d},\mathfrak{m}) \underset{\Phi,\Psi}{\cong} (Y,\mathrm{d}_Y,\mathfrak{m}_Y)\times\Big(\mathbb{R},|\cdot|, e^{h_{(X,\mathrm{d},\mathfrak{m})}t}\,\mathrm{d}t\Big).$$

Step 6. Characterization of Ω.

By decomposition (3.6) and Theorem 2.1, it holds

$$\mathfrak{m}^+(\Omega) \overset{\text{Fatou}}{\geq} \int_Y \mathfrak{m}_y^+(\Omega)\,\mathrm{d}\mathfrak{q}(y) \geq h_{(X,\mathrm{d},\mathfrak{m})}\int_Y \mathfrak{m}_y(\Omega)\,\mathrm{d}\mathfrak{q}(y) = h_{(X,\mathrm{d},\mathfrak{m})}\mathfrak{m}(\Omega).$$

Thus

$$\mathfrak{m}_y^+(\Omega) = h_{(X,\mathrm{d},\mathfrak{m})}\mathfrak{m}_y(\Omega) \quad \mathfrak{q}\text{-a.e. } \ y\in Y.$$

By 1-dimensional rigidity in Proposition 3.1, for almost every $y \in Y$, $\Omega \cap X_y$ is a half line, and we denote it by $(-\infty, e(y)]$. So we can identify Ω as

$$\Omega \cong \Big\{(y,r) : r \in \big(-\infty, e(y)\big), y \in Y, e(y) \in \mathbb{R}\Big\}$$

and $\partial\Omega$ is the graph of a measurable function $e(\cdot)$ on Y.

Next we will show that e is a constant function. By Step 2 in the proof of Theorem 2.1, there is a sequence of Lipschitz functions $(f_n)_{n\in\mathbb{N}}$ such that $f_n \to \chi_\Omega$ in L^1 and

$$\mathrm{Per}(\Omega) = \mathfrak{m}^+(\Omega) = \lim_{n\to+\infty} \int |\mathrm{D} f_n| \,\mathrm{d}\mathfrak{m}.$$

For simplicity, we write $f_n = f_n(y,r)$ as a function on $Y \times \mathbb{R}$, and $\mathrm{d}\mathfrak{m} = \mathrm{d}\mathfrak{m}_Y \times \mathrm{d}\mathfrak{m}_\mathbb{R}$ where $\mathrm{d}\mathfrak{m}_\mathbb{R} = e^{h_{(X,\mathrm{d},\mathfrak{m})}t}\mathrm{d}t$. Denote $f_n^r = f_n(\cdot, r)$, $f_n^y = f_n(y,\cdot)$ and $\chi_\Omega^y = \chi_{\Omega^y} = \chi_{\Omega\cap\{(y,r):r\in\mathbb{R}\}}$. By Fubini's theorem, $f_n \to \chi_\Omega$ in L^1 implies that

$$\int_Y \left(\int_\mathbb{R} |f_n^y(t) - \chi_\Omega^y| \,\mathrm{d}\mathfrak{m}_\mathbb{R}\right) \mathrm{d}\mathfrak{m}_Y \to 0 \quad \text{as } n \to \infty.$$

So there is a subsequence of (f_n), still denoted by (f_n), such that

$$\lim_{n\to\infty} \int_\mathbb{R} |f_n^y(t) - \chi_\Omega^y| \,\mathrm{d}\mathfrak{m}_\mathbb{R} = 0, \quad \mathfrak{m}_Y\text{-a.e. } y \in Y,$$

and

$$\lim_{n\to\infty} \int_\mathbb{R} f_n^y(t) \,\mathrm{d}\mathfrak{m}_\mathbb{R} = \int_\mathbb{R} \chi_\Omega^y \,\mathrm{d}\mathfrak{m}_\mathbb{R} = \frac{1}{h_{(X,\mathrm{d},\mathfrak{m})}} e^{h_{(X,\mathrm{d},\mathfrak{m})}e(y)}, \quad \mathfrak{m}_Y\text{-a.e. } y \in Y. \tag{3.7}$$

So by lower semi-continuity,

$$\varliminf_{n\to\infty} \int_\mathbb{R} |\mathrm{D} f_n^y(t)| \,\mathrm{d}\mathfrak{m}_\mathbb{R} \geq \mathfrak{m}_\mathbb{R}^+(\Omega^y) = e^{h_{(X,\mathrm{d},\mathfrak{m})}e(y)}. \tag{3.8}$$

By [AGS15, Theorem 5.2] (see also [Gig13, Theorem 6.1]), $|\mathrm{D} f_n|^2 = |\mathrm{D} f_n^r|^2 + |\mathrm{D} f_n^y|^2$, where $|\mathrm{D} f_n^r| = |\mathrm{D} f_n^r|_Y$ is the weak gradient of f_n^r in Y, and $|\mathrm{D} f_n^y| = |\mathrm{D} f_n^y|_\mathbb{R}$ is the weak gradient of f_n^y in $\mathbb{R}$ which can be see as the norm of partial derivatives in smooth setting. So for any $\epsilon > 0$ we have

$$\begin{aligned}
&\int |\mathrm{D} f_n| \,\mathrm{d}\mathfrak{m} \\
=\ &\int \sqrt{|\mathrm{D} f_n^r|^2 + |\mathrm{D} f_n^y|^2} \,\mathrm{d}\mathfrak{m}_\mathbb{R}\mathrm{d}\mathfrak{m}_Y \\
=\ &\int_{\{|\mathrm{D} f_n^r| > \epsilon |\mathrm{D} f_n^y|\}} \left(\frac{|\mathrm{D} f_n^r|^2}{\sqrt{|\mathrm{D} f_n^r|^2 + |\mathrm{D} f_n^y|^2} + |\mathrm{D} f_n^y|} + |\mathrm{D} f_n^y|\right) \mathrm{d}\mathfrak{m}_\mathbb{R}\mathrm{d}\mathfrak{m}_Y \\
&+ \int_{|\mathrm{D} f_n^r| \leq \epsilon |\mathrm{D} f_n^y|} \sqrt{|\mathrm{D} f_n^r|^2 + |\mathrm{D} f_n^y|^2} \,\mathrm{d}\mathfrak{m}_\mathbb{R}\mathrm{d}\mathfrak{m}_Y \\
\geq\ &\int_{\{|\mathrm{D} f_n^r| > \epsilon |\mathrm{D} f_n^y|\}} \left(\frac{|\mathrm{D} f_n^r|}{2\sqrt{1+\epsilon^{-2}}} + |\mathrm{D} f_n^y|\right) \mathrm{d}\mathfrak{m}_\mathbb{R}\mathrm{d}\mathfrak{m}_Y \\
&+ \int_{|\mathrm{D} f_n^r| \leq \epsilon |\mathrm{D} f_n^y|} |\mathrm{D} f_n^y| \,\mathrm{d}\mathfrak{m}_\mathbb{R}\mathrm{d}\mathfrak{m}_Y.
\end{aligned}$$

Then

$$\begin{aligned}\int |\mathrm{D}f_n|\,\mathrm{d}\mathfrak{m} &\geq \int_{\{|\mathrm{D}f_n^r|>\epsilon|\mathrm{D}f_n^y|\}} \frac{|\mathrm{D}f_n^r|}{2\sqrt{1+\epsilon^{-2}}}\,\mathrm{d}\mathfrak{m} + \int |\mathrm{D}f_n^y|\,\mathrm{d}\mathfrak{m}\\ &\geq \int \frac{|\mathrm{D}f_n^r|}{2\sqrt{1+\epsilon^{-2}}}\,\mathrm{d}\mathfrak{m} + \left(1-\frac{\epsilon}{2\sqrt{1+\epsilon^{-2}}}\right)\int |\mathrm{D}f_n^y|\,\mathrm{d}\mathfrak{m}.\end{aligned}$$

Letting $n \to \infty$ and combining with (3.8), we get

$$\begin{aligned}&\mathfrak{m}^+(\Omega) = \lim_{n\to\infty}\int |\mathrm{D}f_n|\,\mathrm{d}\mathfrak{m}\\ \geq\ &\varliminf_{n\to\infty}\int \frac{|\mathrm{D}f_n^r|}{2\sqrt{1+\epsilon^{-2}}}\,\mathrm{d}\mathfrak{m} + \left(1-\frac{\epsilon}{2\sqrt{1+\epsilon^{-2}}}\right)\int \mathfrak{m}_{\mathbb{R}}^+(\Omega^y)\,\mathrm{d}\mathfrak{m}_Y\\ \geq\ &\varliminf_{n\to\infty}\int \frac{|\mathrm{D}f_n^r|}{2\sqrt{1+\epsilon^{-2}}}\,\mathrm{d}\mathfrak{m} + \left(1-\frac{\epsilon}{2\sqrt{1+\epsilon^{-2}}}\right)h_{(X,\mathrm{d},\mathfrak{m})}\int \mathfrak{m}_{\mathbb{R}}(\Omega^y)\,\mathrm{d}\mathfrak{m}_Y\\ =\ &\varliminf_{n\to\infty}\int \frac{|\mathrm{D}f_n^r|}{2\sqrt{1+\epsilon^{-2}}}\,\mathrm{d}\mathfrak{m} + \left(1-\frac{\epsilon}{2\sqrt{1+\epsilon^{-2}}}\right)h_{(X,\mathrm{d},\mathfrak{m})}\mathfrak{m}(\Omega).\end{aligned}$$

Combining with $\mathfrak{m}^+(\Omega) = h_{(X,\mathrm{d},\mathfrak{m})}\mathfrak{m}(\Omega)$ we get

$$\epsilon\mathfrak{m}^+(\Omega) \geq \varliminf_{n\to\infty}\int |\mathrm{D}f_n^r|\,\mathrm{d}\mathfrak{m}.$$

Letting $\epsilon \to 0$ we obtain

$$\varliminf_{n\to\infty}\int |\mathrm{D}f_n^r|\,\mathrm{d}\mathfrak{m} = 0. \tag{3.9}$$

Define a Lipschitz function g_n on Y by $g_n(y) = \int_{\mathbb{R}} f_n^y\,\mathrm{d}\mathfrak{m}_{\mathbb{R}} = \int_{\mathbb{R}} f_n(r,y)\,\mathrm{d}\mathfrak{m}_{\mathbb{R}}(r)$. We can approximate g_n in L^1 with functions in the form of $\sum_{k\in I,|I|<\infty} c_k f_n^{r_k}(y)$, and approximate $\int_{\mathbb{R}} |\mathrm{D}f_n^r|\,\mathrm{d}\mathfrak{m}_{\mathbb{R}}(r)$ with $\sum_{k\in I,|I|<\infty} c_k|\mathrm{D}f_n^{r_k}|(y)$. Then by a diagonal argument we can approximate g_n in L^1 with Lipschitz functions in the form of $\sum_{k\in I,|I|<\infty} c_k h_k(y)$, and approximate $\int_{\mathbb{R}} |\mathrm{D}f_n^r|\,\mathrm{d}\mathfrak{m}_{\mathbb{R}}(r)$ with $\sum_{k\in I,|I|<\infty} c_k|\mathrm{D}h_k|$. Combining with the lower semi-continuity (or the pointwise minimality of the weak upper gradients), one can prove

$$|\mathrm{D}g_n| \leq \int_{\mathbb{R}} |\mathrm{D}f_n^r|\,\mathrm{d}\mathfrak{m}_{\mathbb{R}}(r).$$

Combining with (3.9) we get

$$\varliminf_{n\to\infty}\int |\mathrm{D}g_n|\,\mathrm{d}\mathfrak{m}_Y \leq \varliminf_{n\to\infty}\int |\mathrm{D}f_n^r|\,\mathrm{d}\mathfrak{m} = 0.$$

By (3.7) and the lower semi-continuity again, we know $\int |\mathrm{D}e^{h_{(X,\mathrm{d},\mathfrak{m})}e(y)}|\,\mathrm{d}\mathfrak{m}_Y(y) = 0$ and $e(\cdot)$ is constant. □

Lemma 3.7. *Let $(X, \mathrm{d}, \mathfrak{m})$ be an $\mathrm{RCD}(0,\infty)$ metric measure space. Let $(\mu_t)_{t\in[0,1]}$ and $(\nu_t)_{t\in[0,1]}$ be two geodesics in the Wasserstein space $(\mathcal{P}_2(X), W_2)$, with $\mu_t, \nu_t \ll \mathfrak{m}$. Assume that $(\lambda\mu_t + (1-\lambda)\nu_t))_{t\in[0,1]}$ is also a geodesic for some $\lambda \in (0,1)$, then $(\mu_t)_{t\in[0,1]}, (\nu_t)_{t\in[0,1]}$ are induced by the same Kantorovich potential, i.e. there is a globally defined function ϕ which is a Kantorovich potential from μ_0 to μ_1, as well as a Kantorovich potential from ν_0 to ν_1.*

Proof. For the convenience of writing, we assume that $\lambda = \frac{1}{2}$. General cases can be proved in the same way. By [RS14], up to an additive constant, there is a unique Kantorovich potential ϕ_t from $\frac{1}{2}(\mu_0+\nu_0)$ to $\frac{1}{2}(\mu_t+\nu_t)$, and there is a measurable map $\nabla\phi_t : X \to X$ so that

$$\frac{\mu_t+\nu_t}{2} = (\nabla\phi_t)_\sharp\left(\frac{\mu_0+\nu_0}{2}\right). \tag{3.10}$$

In particular, $\left(\frac{1}{2}(\mu_t+\nu_t)\right)_{t\in[0,1]}$ is the unique geodesic from $\frac{1}{2}(\mu_0+\nu_0)$ to $\frac{1}{2}(\mu_1+\nu_1)$.

Let $\tilde\mu_t := (\nabla\phi_t)_\sharp\mu_0, \tilde\nu_t := (\nabla\phi_t)_\sharp\nu_0$ be probability measures so that ϕ_t is the Kantorovich potential from μ_0 to $\tilde\mu_t$, and ν_0 to $\tilde\nu_t$ respectively. By (3.10), we have $\frac{1}{2}(\mu_t+\nu_t) = \frac{1}{2}(\tilde\mu_t+\tilde\nu_t)$. In addition,

$$\begin{aligned}
&W_2^2\big((\tilde\mu_t+\tilde\nu_t)/2, (\mu_0+\nu_0)/2\big)\\
=&\int \mathrm{d}^2(x, \nabla\phi_t(x))\,\mathrm{d}(\mu_0+\nu_0)/2\\
=&\frac{1}{2}\Big(W_2^2(\tilde\mu_t,\mu_0) + W_2^2(\tilde\nu_t,\nu_0)\Big)
\end{aligned}$$

and

$$W_2^2\big((\tilde\mu_t+\tilde\nu_t)/2, (\mu_1+\nu_1)/2\big) = \frac{1}{2}\Big(W_2^2(\tilde\mu_t,\mu_1) + W_2^2(\tilde\nu_t,\nu_1)\Big).$$

However, by Kantorovich duality formula, we have

$$\frac{1}{2}\Big(W_2^2(\mu_1,\mu_0) + W_2^2(\nu_1,\nu_0)\Big) \ge W_2^2\big((\mu_1+\nu_1)/2, (\mu_0+\nu_0)/2)\big),$$

so that

$$\begin{aligned}
&W_2^2\big((\mu_1+\nu_1)/2, (\mu_0+\nu_0)/2\big) = W_2^2\big((\tilde\mu_1+\tilde\nu_1)/2, (\mu_0+\nu_0)/2\big)\\
=&\frac{W_2^2\big((\tilde\mu_t+\tilde\nu_t)/2, (\mu_0+\nu_0)/2\big)}{t} + \frac{W_2^2\big((\tilde\mu_t+\tilde\nu_t)/2, (\mu_1+\nu_1)/2\big)}{1-t}\\
=&\frac{1}{2}\left(\frac{W_2^2(\tilde\mu_t,\mu_0)}{t} + \frac{W_2^2(\tilde\mu_t,\mu_1)}{1-t}\right) + \frac{1}{2}\left(\frac{W_2^2(\tilde\nu_t,\nu_0)}{t} + \frac{W_2^2(\tilde\nu_t,\nu_1)}{1-t}\right)\\
\ge&\frac{1}{2}\Big(W_2^2(\mu_1,\mu_0) + W_2^2(\nu_1,\nu_0)\Big) \ge W_2^2\big((\mu_1+\nu_1)/2, (\mu_0+\nu_0)/2)\big)
\end{aligned}$$

where in the first inequality we use the inequality $x^2/t + y^2/(1-t) \ge (x+y)^2$.

In conclusion, $(\tilde\mu_t)_{t\in[0,1]}$ is also a geodesic from μ_0 to μ_1, and $(\tilde\nu_t)_{t\in[0,1]}$ is also a geodesic from ν_0 to ν_1. By uniqueness of the geodesic, we know $\tilde\mu_t = \mu_t$ and $\tilde\nu_t = \nu_t$ and ϕ_1 is the Kantorovich potential from μ_0 to μ_1 and ν_0 to ν_1. □

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