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On the asymptotic behaviour of the fractional Sobolev seminorms: A geometric approach

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A R T I C L E   I N F O

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A B S T R A C T

We study the well-known asymptotic formulas for fractional Sobolev functions à la Bourgain–Brezis–Mironescu and Maz'ya–Shaposhnikova, in a geometric approach. We show that the key to these asymptotic formulas are Rademacher's theorem and volume growth at infinity respectively. Examples fitting our framework includes Euclidean spaces, Riemannian manifolds, Alexandrov spaces, finite dimensional Banach spaces, and some ideal sub-Riemannian manifolds.

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1. Introduction

Since the pioneer works of Bourgain, Brezis, Mironescu [5], and Maz'ya, Shaposhnikova [22], the study of fractional seminorms got new interest. In [5,22], the authors revealed that the fractional s-seminorms can be seen as intermediary functionals between the L^p-norm and the $W^{1,p}$-seminorms. Precisely, for any $s \in (0,1)$, $N \in \mathbb{N}$ and $p \geq 1$, the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is defined as the union of $f \in L^p(\mathbb{R}^N)$ with

$$\|f\|_{W^{s,p}} := \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x)-f(y)|^p}{|x-y|^{N+sp}} \, \mathrm{d}x \, \mathrm{d}y \right)^{\frac{1}{p}} < +\infty.$$

The following well-known asymptotic formulas were proved in [5] and [22]:

$$\lim_{s \uparrow 1} (1-s)\|f\|^p_{W^{s,p}} = K\|\nabla f\|^p_{L^p}, \quad \forall f \in W^{1,p}(\mathbb{R}^N), \tag{BBM}$$

$$\lim_{s \downarrow 0} s\|f\|^p_{W^{s,p}} = L\|f\|^p_{L^p}, \quad \forall f \in \bigcup_{0<s<1} W^{s,p}(\mathbb{R}^N) \tag{MS}$$

where K, L are constants depending only on p and N.

Both formulas (BBM) and (MS) have been widely studied in *hundreds of papers* in the view of analysis, probability theory and geometry, and generalized to many different settings such as Carnot groups [11], Riemannian manifolds [19], anisotropic spaces [20], RCD metric measure spaces [16,17]), heat semi-group mollifiers [25], ball Banach function spaces [12], and many new approaches to these formulas such as [9,13].

Until today, it is still an interesting and challenging problem to find more examples satisfying such asymptotic formulas. Motivated by a recent seminal work of Górny [15] about a Bourgain–Brezis–Mironescu type formula in metric spaces with Euclidean tangents, we realized that (BBM) and (MS) hold in great generality.

In this paper, we will give a **geometric understanding** to the asymptotic formulas, and focus on three basic models: *Euclidean space, finite-dimensional Banach space and Carnot group*. We will show that

the key of (BBM) is the infinitesimal structure (small scale),

the key of (MS) is the volume growth at infinity (large scale).

In our three models, the tangent cone at a point and the tangent cone at infinity are isometric to the underlying spaces, so these properties and their differences are often overlooked.

As an application, we get *a unified proof* to several already known results, including [5] and [22] in $\mathbb{R}^n$, [20] in finite dimensional Banach spaces and [19] in weighted Riemannian manifolds, and we give *a full characterization of the constants* K *and* L in (BBM) and (MS) respectively. We also show that the asymptotic formulas are valid for more mollifiers.

Structure of the paper: In Section 2 we recall some preliminary notions, including Sobolev spaces and Rademacher's theorem, in the setting of metric measure spaces. In Section 3, we start by posing the basic assumptions and then prove the main results Theorem 3.9 and Theorem 3.11. In order to focus on the 'geometric approach', we will not pose the most general assumptions, but the readers can find some weaker assumptions without modifying our proof a lot. In Section 4, we provide several non-trivial and relevant examples satisfying our assumptions.

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2. Preliminaries

Sobolev spaces

In this paper, a metric measure space $(X, \mathrm{d}, \mathfrak{m})$ is a triple, where (X, d) is a complete separable metric space, $\mathfrak{m}$ is the N-dimensional Hausdorff measure w.r.t. d for some $N \in \mathbb{N}$.

Given $f : X \to \mathbb{R}$, the local Lipschitz constant $\mathrm{lip}(f) : X \to [0, \infty]$ is defined as

$$\mathrm{lip}(f)(x) := \overline{\lim_{y \to x}} \frac{|f(y) - f(x)|}{\mathrm{d}(x, y)} \quad \text{if } x \text{ is not isolated, 0 otherwise.} \tag{2.1}$$

The Lipschitz constant is defined as

$$\mathrm{Lip}(f) := \sup_{x \neq y} \frac{|f(y) - f(x)|}{\mathrm{d}(x, y)}.$$

If $\mathrm{Lip}(f) < \infty$, we call f Lipschitz and write $f \in \mathrm{Lip}(X, \mathrm{d})$. We denote by $\mathrm{Lip}_b(X, \mathrm{d})$ the collection of Lipschitz functions with bounded support.

Let $1 < p < \infty$. We say that a function $f \in L^p(X, \mathfrak{m})$ is in the Sobolev space $W^{1,p}(X, \mathrm{d}, \mathfrak{m})$ if there is a sequence of Lipschitz functions $(f_n)_{n \in \mathbb{N}}$ converging to f in $L^p(X, \mathfrak{m})$, such that

$$\liminf_{n \to \infty} \int_X \mathrm{lip}(f_n)^p \, \mathrm{d}\mathfrak{m} < \infty.$$

It is known (cf. [2]) that for any $f \in W^{1,p}(X, \mathrm{d}, \mathfrak{m})$, there is a unique function $|\mathrm{D}f|_p \in L^p(X, \mathfrak{m})$, called minimal p-weak upper gradient, such that

$$\int_X |\mathrm{D}f|_p^p \,\mathrm{d}\mathfrak{m} = \inf\Big\{ \liminf_{n\to\infty} \int_X \mathrm{lip}(f_n)^p \mathrm{d}\mathfrak{m} : f_n \in \mathrm{Lip}_c(X), f_n \to f \text{ in } L^p(X,\mathfrak{m})\Big\}.$$

If $(X, \mathrm{d}, \mathfrak{m})$ is the Euclidean space, $|\mathrm{D}f|_p$ coincides $\mathfrak{m}$-a.e. with the modulus of the distributional differential of f. In many situations, like PI spaces (i.e. it is doubling and it satisfies a p-Poincaré inequality, cf. [10]) or $\mathrm{RCD}(K,\infty)$ spaces (cf. [14]), $|\mathrm{D}f|_p$ is independent on p. In this paper, we will neglect the parameter p and denote $|\mathrm{D}f|_p$ by $|\mathrm{D}f|$.

The Sobolev space $W^{1,p}(X, \mathrm{d}, \mathfrak{m})$ endowed with the norm

$$\|f\|^p_{W^{1,p}(X,\mathrm{d},\mathfrak{m})} := \|f\|^p_{L^p(X,\mathfrak{m})} + \||\mathrm{D}f|\|^p_{L^p(X,\mathfrak{m})}$$

is a Banach space. For any $f \in W^{1,p}(X, \mathrm{d}, \mathfrak{m})$, by [2] there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathrm{Lip}_b(X, \mathrm{d})$ converging to f in $L^p(X,\mathfrak{m})$ such that

$$\lim_{n\to\infty} \int \big|\mathrm{lip}(f_n) - |\mathrm{D}f|\big|^p \,\mathrm{d}\mathfrak{m} = 0.$$

Furthermore, if $(X, \mathrm{d}, \mathfrak{m})$ is a PI space, by [1, Corollary 7.5, Proposition 7.6], there is $(f_n)_{n\in\mathbb{N}} \subset \mathrm{Lip}_b(X, \mathrm{d})$ converging to f strongly in $W^{1,p}$.

The Hajlasz–Sobolev space $M^{1,p}(X, \mathrm{d}, \mathfrak{m})$ is the space consisting of all $u \in W^{1,p}(X, \mathrm{d}, \mathfrak{m})$ satisfying

$$|u(x) - u(y)| \le \mathrm{d}(x, y)\big(g(x) + g(y)\big) \quad \text{a.e. } x, y \in X \tag{2.2}$$

for some $g \in L^p(X,\mathfrak{m})$ with $\|g\|_{L^p(X,\mathfrak{m})} \le M\|u\|_{W^{1,p}}$ for some universal constant M. If $(X, \mathrm{d}, \mathfrak{m})$ is a PI space (cf. [15, Lemma 2.3]), $W^{1,p}(X, \mathrm{d}, \mathfrak{m})$ coincides with $M^{1,p}(X, \mathrm{d}, \mathfrak{m})$.

Rademacher's theorem

Given $f \in \mathrm{Lip}(X, \mathrm{d})$, $x \in X$ and $r > 0$. The rescaling function $f_{r,x}$ is defined as

$$f_{r,x}(y) := \frac{f(y) - f(x)}{r}, \qquad y \in X. \tag{2.3}$$

It can be seen that $f_{r,x}$ is Lipschitz on $(X, r^{-1}\mathrm{d})$, with Lipschitz constant bounded from above by $\mathrm{Lip}(f)$.

Fix $x \in X$, assume that the pointed metric spaces $(X, r^{-1}\mathrm{d}, x)$ converge to a pointed metric space (Y, d_Y, y) (e.g. in the Gromov–Hausdorff sense, see [4] for details). This space (Y, d_Y, y) is called a tangent cone at x, and in general it depends on x and is not unique.

In [10, §10, page 487], Cheeger introduced the following abstract characterization of uniform convergence of rescaling functions $f_{r,x}$, along with the convergence of $(X, r^{-1}\mathrm{d}, x)$.

Definition 2.1. Given a family of maps $\{\phi_r\}_{r>0}$ from (X, d) to (Y, d_Y) satisfying $\phi_r(x) = y \in Y$. We say that the rescaling functions $\{f_{r,x}\}_{r>0}$ converge to a function $f_{0,x}$ on (Y, d_Y) (associated with $\{\phi_r\}_{r>0}$) if there is a function $\alpha(r)$ satisfying $\alpha(r) \downarrow 0$ as $r \to 0$, such that

$$\left\| f_{0,x} \circ \phi_r - f_{r,x} \right\|_{L^\infty(B_r(x),\mathfrak{m})} \le \alpha(r), \qquad \forall r > 0 \tag{2.4}$$

where $B_r(x) = \{y \in X : \mathrm{d}(y, x) < r\}$ is the geodesic ball.

3. Main results

3.1. Assumptions on the spaces

Model Spaces: the triple $\mathfrak{C} := (C, \mathrm{d}_C, \mathfrak{m}_C)$ denotes one of the following spaces:

$$N\text{-dimensional Euclidean space} \quad (\mathbb{R}^N, |\cdot|, \mathcal{L}^N = \mathcal{H}^N_{|\cdot|}),$$
$$N\text{-dimensional Banach space} \quad (\mathbb{R}^N, \|\cdot\|, \mathcal{L}^N = \mathcal{H}^N_{\|\cdot\|}),$$
$$\text{Carnot group with homogeneous dimension } N \quad (\mathbb{R}^m, \mathrm{d}_{CC}, \mathcal{L}^m = \mathcal{H}^N_{\mathrm{d}_{CC}}).$$

This space $\mathfrak{C}$ plays the role as the unique tangent space to X at $\mathfrak{m}$-a.e. point.

Assumption 3.1 *(Small scale: infinitesimal structure).* We assume that $(X, \mathrm{d}, \mathfrak{m})$ is PI, so properties about the Sobolev spaces stated in Section 2 are valid. In addition, given a model space $(C, \mathrm{d}_C, \mathfrak{m}_C)$, we also assume:

A) (**Unique tangent space:**) For $\mathfrak{m}$-a.e. $x \in X$, there is a family of maps $\{\phi_\delta\}_{\delta>0}$ from X to C satisfying $\phi_\delta(x) = 0 \in C$ and

$$\left| \frac{\frac{1}{\delta}\mathrm{d}(y, z)}{\mathrm{d}_C(\phi_\delta(y), \phi_\delta(z))} - 1 \right| < \eta(\delta), \quad \forall y, z \in B_\delta(x), \delta \in (0, 1) \tag{3.1}$$

where $\eta : (0, 1) \to (0, 1)$ is an increasing function with $\lim_{\delta\downarrow 0} \eta(\delta) = 0$.

B) (**Rademacher's theorem**) For any $u \in \mathrm{Lip}(X, \mathrm{d})$, for $\mathfrak{m}$-a.e. $x \in X$, there is a function $u_{0,x}$ on C, such that the rescaling functions $\{u_{r,x}\}_{r>0}$ converge to $u_{0,x}$ associated with the maps $\{\phi_\delta\}_{\delta>0}$, in the sense of Definition 2.1.

Remark 3.2. Here are some remarks on the Assumption 3.1.

(1) For any $r > 0$, there is a dilation map D_r, which is an isometry between $(C, \mathrm{d}_C, \mathfrak{m}_C)$ and $(C, r^{-1}\mathrm{d}_C, r^{-N}\mathfrak{m}_C)$, such that $D_r(0) = 0$ and $D_r \circ D_{r'} = D_{rr'}$ for any $r, r' > 0$. In particular,

$$(D_r)_\sharp \mathfrak{m}_C = r^{-N}\mathfrak{m}_C. \tag{3.2}$$

(2) For any $r > 0$ and any Borel set $\Omega \subset S_r^C := \{y : \mathrm{d}_C(y, 0) = r\}$, we define the 'boundary measure' of Ω by

$$\mathfrak{m}_C^+|_{S_r^C}(\Omega) := \lim_{\epsilon \downarrow 0} \frac{\mathfrak{m}_C\big(\cup_{s\in[1,1+\epsilon]} D_s(\Omega)\big)}{\epsilon r}.$$

We can see that

$$\mathfrak{m}_C^+|_{S_r^C} = r^{N-1}(D_r)_\sharp\big(\mathfrak{m}_C^+|_{S_1^C}\big). \tag{3.3}$$

(3) Since both $\mathfrak{m}$ and $\mathfrak{m}_C$ are N-dimensional Hausdorff measures, the condition (3.1) implies that

$$(\phi_\delta)_\sharp\big(\mathfrak{m}|_{B_\delta(x)}\big) = \big(1 + o(1)\big)\delta^N \mathfrak{m}_C|_{\phi_\delta(B_\delta(x))} \quad \text{as } \delta \to 0. \tag{3.4}$$

Equivalently, for any $\epsilon > 0$ there is $\delta_0 > 0$ such that

$$(1-\epsilon)\delta^N \mathfrak{m}_C|_{\phi_\delta(B_\delta(x))} < (\phi_\delta)_\sharp\big(\mathfrak{m}|_{B_\delta(x)}\big) < (1+\epsilon)\delta^N \mathfrak{m}_C|_{\phi_\delta(B_\delta(x))} \quad \forall\, \delta < \delta_0.$$

It can be seen from our proofs, the assumption that $\mathfrak{m}$ is the N-dimensional Hausdorff measure can be replaced by (3.4).

Remark 3.3. In general, neither $\{\phi_\delta\}_{\delta>0}$ nor $u_{0,x}$ is unique. However, in many situations, the value $⨍_{S_1^C} |u_{0,x}(v)|^p \,\mathrm{d}\mathfrak{m}_C^+(v)$ is independent on the choice of ϕ_δ and $u_{0,x}$. For example, by a deep result of Cheeger [10, Theorem 10.2], any limit function $u_{0,x}$ is *generalized linear* and

$$-\mathrm{lip}(u)(x) b_\gamma \le u_{0,x} \le \mathrm{lip}(u)(x) b_{-\gamma}$$

where b_γ denotes the *Busemann function* associated to a ray γ. Therefore, if $\mathfrak{C}$ is an N-dimensional Euclidean space (cf. [10, Theorem 8.11.]) or an N-dimensional Banach space equipped with a smooth norm (cf. [18, §2.3]), then there is γ so that $-\mathrm{lip}(u)(x)b_\gamma = u_{0,x} = \mathrm{lip}(u)(x)b_{-\gamma}$, $u_{0,x}$ is linear and unique.

Assumption 3.4 *(Large scale: volume growth condition).* For any point $o \in X$ and $R > 0$, the following generalized Bishop–Gromov volume growth inequality holds

$$\frac{\mathfrak{m}\big(B_R(o)\big)}{\mathfrak{m}\big(B_r(o)\big)} \le \left(\frac{R}{r}\right)^N \quad \forall r \in (0, R] \tag{3.5}$$

where $B_R(o), B_r(o)$ are geodesic balls. In this case, the limit $\lim_{r\to+\infty} \mathfrak{m}\big(B_r(o)\big)/r^N$ exists and it is independent on o, we will denote it by $\mathrm{AVR}_{(X,\mathrm{d},\mathfrak{m})}$.

3.2. Assumptions on the mollifiers

Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of mollifiers:

$$\rho_n : \Big\{(x,y) \in X \times X :\ x \neq y\Big\} \to (0,\infty) \text{ is measurable.}$$

We assume that $(\rho_n)_{n\in\mathbb{N}}$ satisfies the following *approximation of the identity of radial type*. Examples of mollifiers fulfilling these assumptions can be found in [12, §2] and [8].

Assumption 3.5 *(Approximation of the identity: small scale).*

A) (**Polynomial decay at infinity**) There is a constant c_1 such that

$$\left\| \int_X \rho_n(x,y)\,\mathrm{d}\mathfrak{m}(y) \right\|_{L^\infty(X,\mathfrak{m})} < c_1 \quad \forall n \in \mathbb{N}, x \in X. \tag{3.6}$$

For any $\delta \in (0,1)$ and $x \in X$, denote $E_\delta(x) = \{y \in X : \mathrm{d}(y,x) \geq \delta\}$. It holds

$$\lim_{n\to\infty} \int_{E_\delta(x)} \rho_n(x,y)\,\mathrm{d}\mathfrak{m}(y) = 0.$$

B) (**Radial distribution**) There is a sequence of non-increasing functions $(\tilde\rho_n)_{n\in\mathbb{N}}$ such that

$$\rho_n(x,y) = \tilde\rho_n\big(\mathrm{d}(x,y)\big) \quad \text{for all } x,y \in X,\ x \neq y.$$

C) (**Approximation of the identity**) For any $\delta > 0$,

$$\lim_{n\to\infty} \mathfrak{m}_C^+\big(S_1^C\big) \int_0^\delta r^{N-1}\tilde\rho_n(r)\,\mathrm{d}r = 1$$

where $N \in \mathbb{N}$ is the same constant as before.

Assumption 3.6 *(Approximation of the identity: large scale).*

A) (**Radial distribution**) There are strictly decreasing functions $(\tilde\rho_n)_{n\in\mathbb{N}}$ with

$$\lim_{n\to\infty} \tilde\rho_n(r) = 0, \quad \forall r \in (0,+\infty), \tag{3.7}$$

such that

$$\rho_n(x,y) = \tilde\rho_n\big(\mathrm{d}(x,y)\big) \quad \text{for all } x,y \in X,\ x \neq y.$$

B) For any $n, m \in \mathbb{N}$ with $n > m$, it holds

$$(0, +\infty) \ni r \to \frac{\tilde{\rho}_n(r)}{\tilde{\rho}_m(r)} \text{ is non-decreasing.}$$

C) (**Approximation of the identity**) For any $\delta > 0$ and $x \in X$,

$$\lim_{n\to\infty} \mathrm{AVR}_{(X,\mathrm{d},\mathfrak{m})} \int_\delta^{+\infty} N r^{N-1} \tilde{\rho}_n(r)\, \mathrm{d}r = 1 \tag{3.8}$$

where $\mathrm{AVR}_{(X,\mathrm{d},\mathfrak{m})}$ is well-defined under Assumption 3.4.

3.3. Bourgain–Brezis–Mironescu's formula

Firstly we study Lipschitz functions with bounded support.

Proposition 3.7. *Let $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space satisfying Assumption 3.1. Let $p > 1$, $(\rho_n)_{n\in\mathbb{N}}$ be mollifiers satisfying Assumption 3.5, and $u \in \mathrm{Lip}_b(X, \mathrm{d})$ be a Lipschitz functions with bounded support. For any $n \in \mathbb{N}$, define*

$$\mathcal{E}_n(u) := \int_X \int_X \frac{|u(x) - u(y)|^p}{\mathrm{d}^p(x, y)} \rho_n(x, y)\, \mathrm{d}\mathfrak{m}(x)\mathrm{d}\mathfrak{m}(y).$$

It holds

$$\lim_{n\to\infty} \mathcal{E}_n(u) = \|\nabla u\|^p_{K_{p,\mathfrak{C}}} \le \int_X |\mathrm{lip}(u)|^p \,\mathrm{d}\mathfrak{m}, \quad \forall u \in \mathrm{Lip}_b(X, \mathrm{d}) \tag{3.9}$$

where

$$\|\nabla u\|^p_{K_{p,\mathfrak{C}}} := \int_X ⨍_{S_1^C} |u_{0,x}(v)|^p \,\mathrm{d}\mathfrak{m}^+_C(v)\, \mathrm{d}\mathfrak{m}(x) \tag{3.10}$$

and the function $u_{0,x}$ is given in Assumption 3.1-B).

Proof. Let $x \in X$ be a point for which the statements A) and B) in Assumption 3.1 hold. There exist maps $\{\phi_\delta\}_{\delta>0}$ satisfying (3.1) and $\phi_\delta(x) = 0 \in C$, and there is a function $\alpha(\delta)$ satisfying $\alpha(\delta) \downarrow 0$ as $\delta \downarrow 0$, such that the rescaling functions $u_{\delta,x}(y) = \frac{u(y)-u(x)}{\delta}$ converge to $u_{0,x}$ as $\delta \to 0$:

$$\big|u_{0,x}(\phi_\delta(y)) - u_{\delta,x}(y)\big| \le \alpha(\delta) \quad \text{for almost every } y \in B_\delta(x). \tag{3.11}$$

By the Lagrange mean value theorem for $t \mapsto t^p$, (3.11) and the fact that $|u_{\delta,x}| \le \mathrm{Lip}(u)$ on $B_\delta(x)$, there is a constant $K = K(u, x) > 0$ such that

$$\Big| \big|u_{\delta,x}(y)\big|^p - \big|u_{0,x}(\phi_\delta(y))\big|^p \Big| \le K\alpha(\delta) \quad \text{as } \delta \to 0.$$

For any $\delta > 0$ and $i \in \mathbb{N}$, set $\delta_i := 2^{-i}\delta$. It holds the identity

$$\int_{B_\delta(x)} \frac{|u(x)-u(y)|^p}{\mathrm{d}^p(x,y)} \rho_n(x,y)\,\mathrm{d}\mathfrak{m}(y) = \sum_{i=0}^{\infty} \delta_i^p \int_{B_{\delta_i}(x)\setminus B_{\delta_{i+1}}(x)} \big|u_{\delta,x}(y)\big|^p \frac{\rho_n(x,y)}{\mathrm{d}^p(x,y)}\,\mathrm{d}\mathfrak{m}(y).$$

Denote $B_{i,\delta} := B_{\delta_i}(x) \setminus B_{\delta_{i+1}}(x)$. We have

$$\left| \int_{B_\delta(x)} \frac{|u(x)-u(y)|^p}{\mathrm{d}^p(x,y)} \rho_n(x,y)\,\mathrm{d}\mathfrak{m}(y) - \sum_{i=0}^{\infty} \underbrace{\delta_i^p \int_{B_{i,\delta}} \big|u_{0,x}(\phi_{\delta_i}(y))\big|^p \frac{\rho_n(x,y)}{\mathrm{d}^p(x,y)}\,\mathrm{d}\mathfrak{m}(y)}_{\mathrm{I}(i,\delta,n)} \right|$$
$$\le \underbrace{\sum_{i=0}^{\infty} \delta_i^p \int_{B_{i,\delta}} K\alpha(\delta_i) \frac{\rho_n(x,y)}{\mathrm{d}^p(x,y)}\,\mathrm{d}\mathfrak{m}(y)}_{\mathrm{II}(\delta,n)}.$$

Estimate of $\mathrm{I}(i,\delta,n)$: Given $\epsilon > 0$. By Assumption 3.1-A) and symmetry of mollifiers in Assumption 3.5-B), for $\delta > 0$ small enough, it holds

$$\frac{\tilde\rho_n\big(\delta \mathrm{d}_C(\phi_\delta(y),0)(1+\epsilon)\big)}{\big(\delta \mathrm{d}_C(\phi_\delta(y),0)(1+\epsilon)\big)^p} \le \frac{\rho_n(x,y)}{\mathrm{d}^p(x,y)} \le \frac{\tilde\rho_n\big(\delta \mathrm{d}_C(\phi_\delta(y),0)(1-\epsilon)\big)}{\big(\delta \mathrm{d}_C(\phi_\delta(y),0)(1-\epsilon)\big)^p}. \tag{3.12}$$

By change of variable, for δ small enough, we have

$$\begin{aligned}
&\mathrm{I}(i,\delta,n) \\
\overset{(3.12)}{\le}\ & \delta_i^p \int_{B_{i,\delta}} \big|u_{0,x}(\phi_{\delta_i}(y))\big|^p \frac{\tilde\rho_n\big(\delta_i \mathrm{d}_C(\phi_{\delta_i}(y),0)(1-\epsilon)\big)}{\big(\delta_i \mathrm{d}_C(\phi_{\delta_i}(y),0)(1-\epsilon)\big)^p}\,\mathrm{d}\mathfrak{m}(y) \\
=\ & \delta_i^p \int_{\phi_{\delta_i}(B_{i,\delta})} \big|u_{0,x}(v)\big|^p \frac{\tilde\rho_n\big(\delta_i \mathrm{d}_C(v,0)(1-\epsilon)\big)}{\big(\delta_i \mathrm{d}_C(v,0)(1-\epsilon)\big)^p}\,\mathrm{d}(\phi_{\delta_i})_\sharp\mathfrak{m}(v) \\
\overset{(3.4)}{\le}\ & (1+\epsilon)\delta_i^{N+p} \int_{\phi_{\delta_i}(B_{i,\delta})} \big|u_{0,x}(v)\big|^p \frac{\tilde\rho_n\big(\delta_i \mathrm{d}_C(v,0)(1-\epsilon)\big)}{\big(\delta_i \mathrm{d}_C(v,0)(1-\epsilon)\big)^p}\,\mathrm{d}\mathfrak{m}_C(v) \\
=\ & \underbrace{(1+\epsilon)\,\delta_i^{N+p} \int_{B_1^C\setminus B_{1/2}^C} \big|u_{0,x}(v)\big|^p \frac{\tilde\rho_n\big(\delta_i \mathrm{d}_C(v,0)(1-\epsilon)\big)}{\big(\delta_i \mathrm{d}_C(v,0)(1-\epsilon)\big)^p}\,\mathrm{d}\mathfrak{m}_C(v)}_{\mathrm{I}_a(i,\delta,n)}
\end{aligned}$$

$$+(1+\epsilon)\,\delta_i^{N+p} \underbrace{\int\limits_{\phi_{\delta_i}(B_{i,\delta})\setminus(B_1^C\setminus B_{1/2}^C)} \big|u_{0,x}(v)\big|^p \frac{\tilde{\rho}_n\big(\delta_i \mathrm{d}_C(v,0)(1-\epsilon)\big)}{\big(\delta_i \mathrm{d}_C(v,0)(1-\epsilon)\big)^p}\,\mathrm{d}\mathfrak{m}_C(v)}_{\mathrm{I}_b(i,\delta,n)}.$$

We can see that

$$\begin{aligned}
&\mathrm{I}_a(i,\delta,n)\\
&\le \delta_i^p \frac{1}{(1-\epsilon)^N} \int\limits_{B^C_{(1-\epsilon)\delta_i}\setminus B^C_{(1-\epsilon)\delta_{i+1}}} \big|u_{0,x}\big(D_{\delta_i^{-1}(1-\epsilon)^{-1}}(v)\big)\big|^p \frac{\tilde{\rho}_n\big(\mathrm{d}_C(v,0)\big)}{\big(\mathrm{d}_C(v,0)\big)^p}\,\mathrm{d}\mathfrak{m}_C(v)\\
&= \delta_i^p \frac{1}{(1-\epsilon)^N} \int\limits_{(1-\epsilon)\delta_{i+1}}^{(1-\epsilon)\delta_i} \int\limits_{S_r^C} \big|u_{0,x}\big(D_{\delta_i^{-1}(1-\epsilon)^{-1}}(v)\big)\big|^p \frac{\tilde{\rho}_n(r)}{r^p}\,\mathrm{d}\mathfrak{m}_C^+(v)\,\mathrm{d}r
\end{aligned}$$

where $\mathfrak{m}_C^+$ is the boundary measure and D_r is the dilation on $\mathfrak{C}$ (see Remark 3.2).

By linearity of $u_{0,x}$ and homogeneity of $\mathfrak{C}$

$$\begin{aligned}
&\int\limits_{S_r^C} \Big(\delta_i \Big|u_{0,x}\big(D_{\delta_i^{-1}(1-\epsilon)^{-1}}(v)\big)\Big|\Big)^p \,\mathrm{d}\mathfrak{m}_C^+(v)\\
&= \big((1-\epsilon)^{-1}r\big)^p \int\limits_{S_r^C} \big|u_{0,x}\big(D_{r^{-1}}(v)\big)\big|^p \,\mathrm{d}\mathfrak{m}_C^+(v)\\
&= \big((1-\epsilon)^{-1}r\big)^p r^{N-1} \int\limits_{S_1^C} |u_{0,x}(v)|^p \,\mathrm{d}\mathfrak{m}_C^+(v).
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{i=0}^{\infty} \int\limits_{(1-\epsilon)\delta_{i+1}}^{(1-\epsilon)\delta_i} \int\limits_{S_r^C} \Big(\frac{\delta_i \Big|u_{0,x}\big(D_{\delta_i^{-1}(1-\epsilon)^{-1}}(v)\big)\Big|}{r}\Big)^p \tilde{\rho}_n(r)\,\mathrm{d}\mathfrak{m}_C^+(v)\,\mathrm{d}r\\
&= (1-\epsilon)^{-p} \left(\sum_{i=0}^{\infty} \int\limits_{(1-\epsilon)\delta_{i+1}}^{(1-\epsilon)\delta_i} r^{N-1}\tilde{\rho}_n(r)\,\mathrm{d}r\right) \left(\int\limits_{S_1^C} |u_{0,x}(v)|^p \,\mathrm{d}\mathfrak{m}_C^+(v)\right)\\
&= \big(1-p\epsilon+o(\epsilon)\big) \left(\int\limits_0^{(1-\epsilon)\delta} r^{N-1}\tilde{\rho}_n(r)\,\mathrm{d}r\right) \left(\int\limits_{S_1^C} |u_{0,x}(v)|^p \,\mathrm{d}\mathfrak{m}_C^+(v)\right).
\end{aligned}$$

In conclusion

$$\sum_{i=0}^{\infty} \mathrm{I}_a(i,\delta,n) \le \big(1+O(\epsilon)\big) \left(\int_0^{(1-\epsilon)\delta} r^{N-1}\tilde{\rho}_n(r)\,\mathrm{d}r \right) \left(\int_{S_1^C} |u_{0,x}(v)|^p \,\mathrm{d}\mathfrak{m}_C^+(v) \right). \tag{3.13}$$

Estimate of $\mathrm{I}_b(i,\delta,n)$: As $\delta \to 0$, by (3.1) we have

$$\phi_{\delta_i}(B_{i,\delta}) \subset B^C_{1+\eta(\delta_i)} \setminus B^C_{(1-\eta(\delta_i))/2}. \tag{3.14}$$

So

$$\mathfrak{m}_C\Big(\phi_{\delta_i}(B_{i,\delta}) \setminus \big(B_1^C \setminus B_{1/2}^C\big)\Big) = O\big(\eta(\delta_i)\big). \tag{3.15}$$

Assume $\big(1-\eta(\delta)\big)(1-\epsilon) > \frac{1}{2}$. Note that $\mathrm{Lip}(u_{0,x}) = \mathrm{lip}(u)(x)$, we have

$$\begin{aligned}
&\mathrm{I}_b(i,\delta,n)\\
&= \delta_i^{N+p} \int_{\phi_{\delta_i}(B_{i,\delta})\setminus(B_1^C\setminus B_{1/2}^C)} \big|u_{0,x}(v)\big|^p \frac{\tilde{\rho}_n\big(\delta_i \mathrm{d}_C(v,0)(1-\epsilon)\big)}{\big(\delta_i \mathrm{d}_C(v,0)(1-\epsilon)\big)^p}\,\mathrm{d}\mathfrak{m}_C(v)\\
&\le \delta_i^{N+p} \int_{\phi_{\delta_i}(B_{i,\delta})\setminus(B_1^C\setminus B_{1/2}^C)} \Big[\big(1+\eta(\delta)\big)|\mathrm{lip}(u)(x)|\Big]^p \frac{\tilde{\rho}_n\Big(\delta_{i+1}\big(1-\eta(\delta)\big)(1-\epsilon)\Big)}{\Big(\delta_{i+1}\big(1-\eta(\delta)\big)(1-\epsilon)\Big)^p}\,\mathrm{d}\mathfrak{m}_C(v)\\
&\lesssim \eta(\delta_i)\frac{\delta_i^N}{\delta_{i+1}^N - \delta_{i+2}^N}\Big((\delta_{i+1}^N - \delta_{i+2}^N)\tilde{\rho}_n(\delta_{i+2})\Big)\\
&\lesssim \eta(\delta) \int_{(B_{\delta_{i+1}}(x)\setminus B_{\delta_{i+2}}(x))} \rho_n(x,y)\,\mathrm{d}\mathfrak{m}(y)
\end{aligned}$$

so that by Assumption 3.5-A)

$$\sum_{i=0}^{\infty} \mathrm{I}_b(i,\delta,n) \lesssim \eta(\delta) \tag{3.16}$$

Estimate of $\mathrm{II}(\delta,n)$: By monotonicity of $\alpha(\delta)$,

$$\begin{aligned}
&\mathrm{II}(\delta,n)\\
&\le C\alpha(\delta)\sum_{i=0}^{\infty}\delta_i^p \int_{B_{\delta_i}(x)\setminus B_{\delta_{i+1}}(x)} \frac{\rho_n(x,y)}{\delta_{i+1}^p}\,\mathrm{d}\mathfrak{m}(y)\\
&= 2^p C\alpha(\delta)\int_{B_\delta(x)} \rho_n(x,y)\,\mathrm{d}\mathfrak{m}(y)\\
&\lesssim \alpha(\delta).
\end{aligned}$$

Conclusion: By Assumption 3.5-A),

$$\lim_{n\to\infty}\int\limits_{B^c_\delta(x)}\frac{|u(x)-u(y)|^p}{\mathrm{d}^p(x,y)}\rho_n(x,y)\,\mathrm{d}\mathfrak{m}(y)\le\lim_{n\to\infty}[\mathrm{Lip}(u)]^p\int\limits_{B^c_\delta(x)}\rho_n(x,y)\,\mathrm{d}\mathfrak{m}(y)=0.$$

Combining the estimates obtained above

$$\begin{aligned}
&\overline{\lim_{n\to\infty}}\int\limits_X\frac{|u(x)-u(y)|^p}{\mathrm{d}^p(x,y)}\rho_n(x,y)\,\mathrm{d}\mathfrak{m}(y)\\
&=\overline{\lim_{n\to\infty}}\int\limits_{B_\delta(x)}\frac{|u(x)-u(y)|^p}{\mathrm{d}^p(x,y)}\rho_n(x,y)\,\mathrm{d}\mathfrak{m}(y)\\
&\le\overline{\lim_{n\to\infty}}\Big(\mathrm{I}(\delta,n)+\mathrm{II}(\delta,n)\Big)\\
&\le\overline{\lim_{n\to\infty}}\,(1+O(\epsilon))\Big(\int\limits_0^\delta r^{N-1}\tilde\rho_n(r)\,\mathrm{d}r\Big)\Big(\int\limits_{S^C_1}|u_{0,x}(v)|^p\;\mathrm{d}\mathfrak{m}^+_C(v)\Big)+O\big(\eta(\delta)+\alpha(\delta)\big)\\
&=\big(1+O(\epsilon)\big)\Big(\mathfrak{m}^+_C(S^C_1(x_0))\Big)^{-1}\Big(\int\limits_{S^C_1(x_0)}|u_{0,x}(v)|^p\;\mathrm{d}\mathfrak{m}^+_C(v)\Big)+O\big(\eta(\delta)+\alpha(\delta)\big)
\end{aligned}$$

where in the last equality we use Assumption 3.5-C).

Letting $\delta\to0$ and $\epsilon\to0$ we get

$$\overline{\lim_{n\to\infty}}\int\limits_X\frac{|u(x)-u(y)|^p}{\mathrm{d}^p(x,y)}\rho_n(x,y)\,\mathrm{d}\mathfrak{m}(y)\le\underbrace{\fint\limits_{S^C_1(x_0)}|u_{0,x}(v)|^p\;\mathrm{d}\mathfrak{m}^+_C(v)}_{|\nabla u|_{K_{p,\mathfrak{C}}}}.$$

Note that $u_{0,x}$ is $|\mathrm{lip}(u)(x)|$-Lipschitz, we have

$$\begin{aligned}
\overline{\lim_{n\to\infty}}\int\limits_X\frac{|u(x)-u(y)|^p}{\mathrm{d}^p(x,y)}\rho_n(x,y)\,\mathrm{d}\mathfrak{m}(y)&\le|\nabla u|_{K_{p,\mathfrak{C}}}\\
&\le\fint\limits_{S^C_1(x_0)}|\mathrm{lip}(u)(x)|^p\,\mathrm{d}\mathfrak{m}^+_C(v)=|\mathrm{lip}(u)(x)|^p.
\end{aligned}$$

Similarly, from the first inequality in (3.12) we can deduce

$$\underline{\lim_{n\to\infty}}\int\limits_X\frac{|u(x)-u(y)|^p}{\mathrm{d}^p(x,y)}\rho_n(x,y)\,\mathrm{d}\mathfrak{m}(y)\ge\fint\limits_{S^C_1}|u_{0,x}(v)|^p\;\mathrm{d}\mathfrak{m}^+_C(v).$$

Integrating the inequalities above and using Fatou's lemma, we get

$$\||\nabla u|_{K_{p,\mathfrak{c}}}\|_{L^1} \leq \varliminf_{n\to\infty} \mathcal{E}_n(u) \leq \varlimsup_{n\to\infty} \mathcal{E}_n(u) \leq \||\nabla u|_{K_{p,\mathfrak{c}}}\|_{L^1} \leq \int |\mathrm{lip}(u)|^p \,\mathrm{d}\mathfrak{m}$$

which is the thesis. □

Definition 3.8. For any $u \in W^{1,p}(X, \mathrm{d}, \mathfrak{m})$, $\|\nabla u\|^p_{K_{p,\mathfrak{c}}}$ is defined as

$$\|\nabla u\|^p_{K_{p,\mathfrak{c}}} := \lim_{k\to\infty} \|\nabla u_k\|^p_{K_{p,\mathfrak{c}}}$$

where $(u_k)_{k\in\mathbb{N}}$ is a sequence of Lipschitz functions converging to u in $W^{1,p}(X, \mathrm{d}, \mathfrak{m})$.

By density of Lipschitz functions (with bounded support) in $W^{1,p}$, we know $\|\nabla u\|^p_{K_{p,\mathfrak{c}}}$ is well-defined. In other words, the value of $\lim_{k\to\infty} \|\nabla u_k\|^p_{K_{p,\mathfrak{c}}}$ is independent of the choice of $(u_k)_{k\in\mathbb{N}}$. In general, such value can not be written as $K\|\mathrm{D}u\|^p$ for some universal constant K, since the space can be anisotropic and the function $u_{0,x}$ is not linear (See Example 4.1 and [20]).

Theorem 3.9 *(Generalized Bourgain–Brezis–Mironescu's formula). Let $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space satisfying Assumption 3.1, let $(\rho_n)_{n\in\mathbb{N}}$ be mollifiers satisfying Assumption 3.5 and $1 < p < \infty$. Then for any $u \in W^{1,p}(X, \mathrm{d}, \mathfrak{m})$*

$$\lim_{n\to\infty} \underbrace{\int_X \int_X \frac{|u(x) - u(y)|^p}{\mathrm{d}^p(x, y)} \rho_n(x, y) \,\mathrm{d}\mathfrak{m}(x)\mathrm{d}\mathfrak{m}(y)}_{=\mathcal{E}_n(u)} = \|\nabla u\|^p_{K_{p,\mathfrak{c}}}. \tag{3.17}$$

Proof. Let $(u_k) \subset \mathrm{Lip}_c(X, \mathrm{d})$ be such that $u_k \to u$ strongly in $W^{1,p}(X, \mathrm{d}, \mathfrak{m})$. For any $\epsilon \in (0, 1)$, there is $k_0 \in \mathbb{N}$ such that

$$\|u - u_{k_0}\|_{W^{1,p}(X,\mathrm{d},\mathfrak{m})} < \epsilon, \quad \left| \|\nabla u_{k_0}\|_{K_{p,\mathfrak{c}}} - \|\nabla u\|_{K_{p,\mathfrak{c}}} \right| < \epsilon. \tag{3.18}$$

By (2.2) and Assumption 3.5-A), there exists $g \in L^p$ with $\|g\|_{L^p(X,\mathfrak{m})} \leq M\|u\|_{W^{1,p}}$ such that

$$\begin{aligned} \left| \mathcal{E}_n^{\frac{1}{p}}(u_{k_0}) - \mathcal{E}_n^{\frac{1}{p}}(u) \right|^p &\overset{\text{Minkowski inequality}}{\leq} \mathcal{E}_n(u_{k_0} - u) \\ \leq^{(2.2)} &\int_X \int_X \big(g(x) + g(y)\big)^p \rho_n(x, y) \,\mathrm{d}\mathfrak{m}(x)\mathrm{d}\mathfrak{m}(y) \\ \leq^{(3.6)} &\ 2^p c_1 M^p \|u - u_{k_0}\|^p_{W^{1,p}(X,\mathrm{d},\mathfrak{m})} < 2^p c_1 M^p \epsilon^p. \end{aligned}$$

By Proposition 3.7 and (3.18), there is $n_0 \in \mathbb{N}$ such that for any $n > n_0$, it holds

$$\left| \mathcal{E}_n^{\frac{1}{p}}(u_{k_0}) - \|\nabla u_{k_0}\|_{K_{p,\mathfrak{c}}} \right| < \epsilon. \tag{3.19}$$

Combining the estimates above, we obtain

$$\begin{aligned}
&\left|\mathcal{E}_n^{\frac{1}{p}}(u) - \|\nabla u\|_{K_{p,\mathfrak{C}}}\right| \\
&\le \left|\mathcal{E}_n^{\frac{1}{p}}(u) - \mathcal{E}_n^{\frac{1}{p}}(u_{k_0})\right| + \left|\mathcal{E}_n^{\frac{1}{p}}(u_{k_0}) - \|\nabla u_{k_0}\|_{K_{p,\mathfrak{C}}}\right| + \left|\|\nabla u_{k_0}\|_{K_{p,\mathfrak{C}}} - \|\nabla u\|_{K_{p,\mathfrak{C}}}\right| \\
&\le 2^p c_1 M^p \epsilon^p + 2\epsilon
\end{aligned}$$

for any $n > n_0$, which is the thesis. □

Remark 3.10. The value $\|\nabla u\|^p_{K_{p,\mathfrak{C}}}$ depends on the choice of ϕ_δ. So our results depend on a **given** family of maps ϕ_δ. So Theorem 3.9 should be understood in this way: if there are ϕ_δ and $u_{0,x}$ fulfils our assumption, then the limit on the left hand side of (3.17) **exists and it is** $\|\nabla u\|^p_{K_{p,\mathfrak{C}}}$. This is irrelevant to the uniqueness of ϕ_δ nor $u_{0,x}$.

In case $\mathfrak{C}$ is an N-dimensional Euclidean space (cf. [10, Theorem 8.11.]), or an N-dimensional Banach space equipped with a smooth norm (cf. [18, §2.3]), or an Heisenberg group (cf. [15, §4.2]), the limit function $u_{0,x}$ is unique up to composing a rotation. In these cases, the value $\|\nabla u\|^p_{K_{p,\mathfrak{C}}}$ is independent on the choice of ϕ_δ.

3.4. Maz'ya–Shaposhnikova's formula

Next we will prove Maz'ya–Shaposhnikova's formula in a geometric way. In the formula (3.20), the constant on the right-hand side is equal to 2 by the assumption on the mollifiers, and up to rescaling the mollifiers and the measure we could recover the original constant in the Maz'ya–Shaposhnikova's formula (MS).

Theorem 3.11 *(Generalized Maz'ya–Shaposhnikova's formula). Let $(X, \mathrm{d}, \mathfrak{m})$ be a non-compact metric measure space satisfying Assumption 3.4, $(\rho_n)_{n\in\mathbb{N}}$ be mollifiers satisfying Assumption 3.6. For any $u \in L^p$ with $\mathcal{E}_{n_0}(u) < +\infty$ for some $n_0 \in \mathbb{N}$, we have*

$$\lim_{n\to\infty} \underbrace{\int_X \int_X |u(x) - u(y)|^p \rho_n(x, y)\, \mathrm{d}\mathfrak{m}(x)\mathrm{d}\mathfrak{m}(y)}_{=\mathcal{E}_n(u)} = 2\|u\|^p_{L^p}. \tag{3.20}$$

Proof. For $x_0 \in X$, $\delta > 0$, we have a decomposition of $X \times X$

$$\begin{cases} A := \{(x, y) : \mathrm{d}(x, y) \le \delta\} \\ B := \{(x, y) : \mathrm{d}(x, y) > \delta\} \cap \{(x, y) : \mathrm{d}(y, x_0) > 2\mathrm{d}(x, x_0) \text{ or } \mathrm{d}(y, x_0) < \frac{1}{2}\mathrm{d}(x, x_0)\} \\ C := \{(x, y) : \mathrm{d}(x, y) > \delta\} \cap \{(x, y) : \frac{1}{2}\mathrm{d}(x, x_0) \le \mathrm{d}(y, x_0) \le 2\mathrm{d}(x, x_0)\} \end{cases}$$

and we divide $\mathcal{E}_n(u)$ into the following three parts

$$\mathcal{E}_n(u) = \underbrace{\int_A |u(x)-u(y)|^p \rho_n(x,y)\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y)}_{\mathrm{I}(\delta,n)}$$
$$+ \underbrace{\int_B |u(x)-u(y)|^p \rho_n(x,y)\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y)}_{\mathrm{II}(\delta,n)}$$
$$+ \underbrace{\int_C |u(x)-u(y)|^p \rho_n(x,y)\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y)}_{\mathrm{III}(\delta,n)}.$$

Estimate of $\mathrm{I}(\delta,n)$**:** For any $n > n_0$, by Assumption 3.6-B), it holds

$$\begin{aligned}
\mathrm{I}(\delta,n) &= \int_X \left(\int_{B_\delta(y)} |u(x)-u(y)|^p \rho_{n_0}(x,y)\frac{\rho_n(x,y)}{\rho_{n_0}(x,y)}\,\mathrm{d}\mathfrak{m}(x)\right)\mathrm{d}\mathfrak{m}(y) \\
&\le \int_X \left(\int_{B_\delta(y)} |u(x)-u(y)|^p \rho_{n_0}(x,y)\frac{\tilde{\rho}_n(\delta)}{\tilde{\rho}_{n_0}(\delta)}\,\mathrm{d}\mathfrak{m}(x)\right)\mathrm{d}\mathfrak{m}(y) \\
&\le \mathcal{E}^p_{n_0}(u)\frac{\tilde{\rho}_n(\delta)}{\tilde{\rho}_{n_0}(\delta)}.
\end{aligned}$$

By (3.7) in Assumption 3.6 we get

$$\lim_{n\to+\infty} \mathrm{I}(\delta,n) = 0. \tag{3.21}$$

Estimate of $\mathrm{II}(\delta,n)$**:** For $\delta > 0$, $y \in X$ and

$$x \in \Big\{x : \mathrm{d}(x,y) > \delta,\ \mathrm{d}(y,x_0) > 2\mathrm{d}(x,x_0)\Big\},$$

by triangle inequality,

$$\mathrm{d}(x,y) \ge \mathrm{d}(y,x_0) - \mathrm{d}(x_0,x) > \mathrm{d}(y,x_0) - \frac{1}{2}\mathrm{d}(y,x_0) = \frac{1}{2}\mathrm{d}(y,x_0)$$

and

$$\mathrm{d}(x,y) \le \mathrm{d}(x_0,x) + \mathrm{d}(y,x_0) < \frac{3}{2}\mathrm{d}(y,x_0).$$

Therefore

$$\Big\{x : \mathrm{d}(x,y) > \delta,\ \mathrm{d}(y,x_0) > 2\mathrm{d}(x,x_0)\Big\} \subset \Big\{x : \frac{3}{2}\mathrm{d}(y,x_0) \ge \mathrm{d}(x,y) > \frac{1}{2}\mathrm{d}(y,x_0) \vee \delta\Big\}$$

so that

$$
\begin{aligned}
&\frac{1}{2}\int_B |u(y)|^p \rho_n(x,y)\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y)\\
=&\int_X |u(y)|^p \left(\int_{\{x:\ \mathrm{d}(x,y)>\delta,\ \mathrm{d}(y,x_0)>2\mathrm{d}(x,x_0)\}} \rho_n(x,y)\,\mathrm{d}\mathfrak{m}(x)\right)\mathrm{d}\mathfrak{m}(y)\\
\le&\int_X |u(y)|^p \left(\int_{\{x:\ \frac{3}{2}\mathrm{d}(y,x_0)\ge \mathrm{d}(x,y)>\frac{1}{2}\mathrm{d}(y,x_0)\vee\delta\}} \rho_n(x,y)\,\mathrm{d}\mathfrak{m}(x)\right)\mathrm{d}\mathfrak{m}(y)\\
=&\int_X |u(y)|^p \Big(\int_{\{x:\ \mathrm{d}(x,y)>\frac{1}{2}\mathrm{d}(y,x_0)\vee\delta\}} \rho_n(x,y)\,\mathrm{d}\mathfrak{m}(x)\\
&-\int_{\{x:\ \mathrm{d}(x,y)>\frac{3}{2}\mathrm{d}(y,x_0)\vee\delta\}} \rho_n(x,y)\,\mathrm{d}\mathfrak{m}(x)\Big)\,\mathrm{d}\mathfrak{m}(y).
\end{aligned}
$$

By Lemma 3.12, Fatou's lemma and monotone convergence theorem, we have

$$
\begin{aligned}
&\lim_{\delta\to\infty}\lim_{n\to\infty}\int_X |u(y)|^p \Big(\int_{\{x:\ \mathrm{d}(x,y)>\frac{1}{2}\mathrm{d}(y,x_0)\vee\delta\}} \rho_n(x,y)\,\mathrm{d}\mathfrak{m}(x)\Big)\,\mathrm{d}\mathfrak{m}(y)\\
=&\lim_{\delta\to\infty}\lim_{n\to\infty}\int_X |u(y)|^p \Big(\int_{\{x:\ \mathrm{d}(x,y)>\frac{3}{2}\mathrm{d}(y,x_0)\vee\delta\}} \rho_n(x,y)\,\mathrm{d}\mathfrak{m}(x)\Big)\,\mathrm{d}\mathfrak{m}(y)\\
=&\|u\|^p_{L^p}.
\end{aligned}
$$

Hence

$$
\lim_{\delta\to\infty}\lim_{n\to\infty}\int_B |u(y)|^p \rho_n(x,y)\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y)=0. \tag{3.22}
$$

For $x, x_0 \in X$, by triangle inequality we can also prove

$$
E_{4\mathrm{d}(x,x_0)}(x) \subset \Big\{y \in X : \mathrm{d}(y,x_0) > 2\mathrm{d}(x,x_0)\Big\} \subset E_{\mathrm{d}(x,x_0)}(x),
$$

where $E_r(x)$ denotes the set $\big\{y :\ \mathrm{d}(y,x) > r\big\}$. So

$$
2\int_X |u(x)|^p \left(\int_{B^c_{\mathrm{d}(x,x_0)\vee\delta}(x)} \rho_n(x,y)\,\mathrm{d}\mathfrak{m}(y)\right)\mathrm{d}\mathfrak{m}(x)
$$

$$
\begin{aligned}
&\geq \int_B |u(x)|^p \rho_n(x,y)\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y)\\
&= 2\int_X |u(x)|^p \left(\int_{\{y:\ \mathrm{d}(y,x)>\delta,\ \mathrm{d}(y,x_0)>2\mathrm{d}(x,x_0)\}} \rho_n(x,y)\,\mathrm{d}\mathfrak{m}(y)\right)\mathrm{d}\mathfrak{m}(x)\\
&\geq 2\int_X |u(x)|^p \left(\int_{E_{4\mathrm{d}(x,x_0)\vee\delta}} \rho_n(x,y)\,\mathrm{d}\mathfrak{m}(y)\right)\mathrm{d}\mathfrak{m}(x).
\end{aligned}
$$

By Lemma 3.12, Fatou's lemma and monotone convergence theorem, we obtain

$$
\lim_{\delta\to\infty}\lim_{n\to\infty}\int_B |u(x)|^p \rho_n(x,y)\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y) = 2\|u\|_{L^p}^p. \tag{3.23}
$$

Combining with (3.22), we get

$$
\lim_{\delta\to\infty}\lim_{n\to\infty} \mathrm{II}(\delta,n) = 2\|u\|_{L^p}^p. \tag{3.24}
$$

Estimate of $\mathrm{III}(\delta,n)$**:** By triangle inequality we can also prove

$$
C\subset\left\{(x,y):\ \mathrm{d}(x,y)>\delta,\ \mathrm{d}(y,x_0)>\frac{\delta}{3},\ \mathrm{d}(x,x_0)>\frac{\delta}{3}\right\}. \tag{3.25}
$$

Thus

$$
\begin{aligned}
&\mathrm{III}(\delta,n) = \int_C |u(x)-u(y)|^p \rho_n(x,y)\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y)\\
&\leq 2^{p-1}\left(\int_C |u(x)|^p \rho_n(x,y)\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y) + \int_C |u(y)|^p \rho_n(x,y)\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y)\right)\\
&\leq 2^{p-1}\int_{\mathrm{d}(x,x_0)>\frac{\delta}{3}} |u(x)|^p \Big(\int_{\mathrm{d}(x,y)>\delta} \rho_n(x,y)\,\mathrm{d}\mathfrak{m}(y)\Big)\,\mathrm{d}\mathfrak{m}(x)\\
&+ 2^{p-1}\int_{\mathrm{d}(y,x_0)>\frac{\delta}{3}} |u(y)|^p \Big(\int_{\mathrm{d}(x,y)>\delta} \rho_n(x,y)\,\mathrm{d}\mathfrak{m}(x)\Big)\,\mathrm{d}\mathfrak{m}(y)\\
&\leq 2^{p}\int_{\mathrm{d}(x,x_0)>\frac{\delta}{3}} |u(x)|^p \Big(\int_{\mathrm{d}(x,y)>\delta} \rho_n(x,y)\,\mathrm{d}\mathfrak{m}(y)\Big)\,\mathrm{d}\mathfrak{m}(x).
\end{aligned}
$$

By Fatou's lemma

$$\overline{\lim_{n\to\infty}}\, \mathrm{III}(\delta,n) \le 2^p \int_{\mathrm{d}(x,x_0)>\frac{\delta}{3}} |u(x)|^p \,\overline{\lim_{n\to\infty}} \Big(\int_{\mathrm{d}(x,y)>\delta} \rho_n(x,y)\,\mathrm{d}\mathfrak{m}(y)\Big)\,\mathrm{d}\mathfrak{m}(x).$$

Fix $R > 0$. By monotone convergence theorem

$$\lim_{\delta\to\infty}\overline{\lim_{n\to\infty}}\, \mathrm{III}(\delta,n) \le 2^p \int_{\mathrm{d}(x,x_0)>\frac{R}{3}} |u(x)|^p \lim_{\delta\to\infty}\overline{\lim_{n\to\infty}} \Big(\int_{\mathrm{d}(x,y)>\delta} \rho_n(x,y)\,\mathrm{d}\mathfrak{m}(y)\Big)\,\mathrm{d}\mathfrak{m}(x).$$

Then by Lemma 3.12 below, we get

$$\lim_{\delta\to\infty}\overline{\lim_{n\to\infty}}\, \mathrm{III}(\delta,n) \le 2^p \int_{\mathrm{d}(x,x_0)>\frac{R}{3}} |u(x)|^p \,\mathrm{d}\mathfrak{m}(x) \overset{R\to\infty}{\to} 0.$$

Combining with (3.21) and (3.24) we get the conclusion. □

Lemma 3.12. *Let $(X, \mathrm{d}, \mathfrak{m})$ be a space satisfying Assumption 3.4 and $(\rho_n)_{n\in\mathbb{N}}$ be mollifiers satisfying Assumption 3.6. Then for any $x \in X$,*

$$\lim_{\delta\to+\infty}\lim_{n\to+\infty}\int_{E_\delta(x)} \rho_n(x,y)\,\mathrm{d}\mathfrak{m}(y) = 1.$$

Proof. For $\delta > 0$ and $n \in \mathbb{N}$, by Cavalieri's formula (cf. [3, Chapter 6])

$$\int_{E_\delta(x)} \rho_n(x,y)\,\mathrm{d}\mathfrak{m}(y) = \int_0^{\tilde\rho_n(\delta)} \mathfrak{m}\Big(\big\{y : r < \rho_n(x,y) < \tilde\rho_n(\delta)\big\}\Big)\,\mathrm{d}r$$

$$= \int_0^{\tilde\rho_n(\delta)} \mathfrak{m}\big(B_{\tilde\rho_n^{-1}(r)}(x) \setminus B_\delta(x)\big)\,\mathrm{d}r = \int_0^{\tilde\rho_n(\delta)} \mathfrak{m}\big(B_{\tilde\rho_n^{-1}(r)}(x)\big)\,\mathrm{d}r - \mathfrak{m}\big(B_\delta(x)\big)\tilde\rho_n(\delta).$$

By assumption, $\mathfrak{m}\big(B_\delta(x)\big) = \mathrm{AVR}_{(X,\mathrm{d},\mathfrak{m})}\big(1+o(1)\big)\delta^N$ as $\delta \to +\infty$. So

$$\begin{aligned} &\int_0^{\tilde\rho_n(\delta)} \mathfrak{m}\big(B_{\tilde\rho_n^{-1}(r)}(x)\big)\,\mathrm{d}r \\ &= \int_0^{\tilde\rho_n(\delta)} \mathrm{AVR}_{(X,\mathrm{d},\mathfrak{m})}\big(1+o(1)\big)\big(\tilde\rho_n^{-1}(r)\big)^N\,\mathrm{d}r \\ \text{let } t = \tilde\rho_n^{-1}(r) \quad &= \big(1+o(1)\big)\mathrm{AVR}_{(X,\mathrm{d},\mathfrak{m})}\int_{+\infty}^{\delta} t^N\tilde\rho_n'(t)\,\mathrm{d}t \end{aligned}$$

by integration by parts $= \big(1 + o(1)\big)\mathrm{AVR}_{(X,\mathrm{d},\mathfrak{m})}\left(\tilde{\rho}_n(\delta)\delta^N + \int\limits_{\delta}^{+\infty} Nt^{N-1}\tilde{\rho}_n(t)\,\mathrm{d}t\right).$

Then by (3.7) and (3.8) in Assumption 3.6 we get

$$\lim_{\delta\to+\infty}\lim_{n\to+\infty}\int\limits_{E_\delta(x)} \rho_n(x,y)\,\mathrm{d}\mathfrak{m}(y) = 1, \quad \forall x \in X. \quad \square$$

4. Applications and examples

We present several applications of Theorem 3.9 and Theorem 3.11.

Bourgain–Brezis–Mironescu type formula

The first application extends a result of M. Ludwig [20] concerning finite dimensional Banach spaces with general mollifiers. We remark that the proof in [20] relies on Blaschke–Petkantschin formula, which works only for a very specific class of mollifiers.

Example 4.1 *(Anisotropic spaces).* Let $\mathfrak{C} = (\mathbb{R}^N, \|\cdot\|, \mathcal{L}^N)$ be an N-dimensional Banach space equipped with the Lebesgue measure $\mathcal{L}^N$, and let $(\rho_n)_{n\in\mathbb{N}}$ be mollifiers satisfying Assumption 3.5. Then

$$\lim_{n\to\infty}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{|f(x)-f(y)|^p}{\|x-y\|^p}\rho_n(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int\limits_{\mathbb{R}^N}\fint\limits_{S_1^{\mathfrak{C}}}|\nabla f\cdot v|^p\,\mathrm{d}\mathcal{H}^{N-1}_{\|\cdot\|}(v)\,\mathrm{d}x \tag{4.1}$$

where $B_1^{\mathfrak{C}}$ is the unit ball in $\mathfrak{C}$ centred at 0 and $S_1^{\mathfrak{C}}$ is its boundary, $\mathcal{H}^{N-1}_{\|\cdot\|}$ is the boundary measure.

Proof. Let $\{\varphi_r\}_{r>0}$ be a family of dilations with respect to a fixed point x, i.e. $\varphi_r(y) = x + r(y-x)$. Obviously $\{\varphi_r\}_{r>0}$ satisfy Assumption 3.1-A) with $\eta \equiv 0$.

Let f be a Lipschitz function and $x \in \mathbb{R}^N$ be a differentiable point of f with respect to the Euclidean norm $|\cdot|$. By Rademacher's theorem on Euclidean spaces and [10, Theorem 10.2], the union of such points has full measure, and $\{f_{r,x}\}_{r>0}$ converge uniformly to a linear function $f_{0,x}(v) = \nabla f\cdot v$ as $r \to 0$. Since the norm $\|\cdot\|$ and the Euclidean norm are equivalent, $\{f_{r,x}\}_{r>0}$ also converge to $f_{0,x}$ in $\|\cdot\|$. So Assumption 3.1-B) is fulfilled. $\square$

Remark 4.2. In [20], the author studies

$$\lim_{s\to1^-}(1-s)\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{|f(x)-f(y)|^p}{\|x-y\|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y$$

for $f \in W^{1,p}(\mathbb{R}^N, \|\cdot\|)$. Note that the mollifiers $\frac{1-s}{\|x-y\|^{N+sp-p}}$ are not globally integrable, which do not satisfy Assumption 3.5-A). However, the limit

$$\lim_{s\to 1^-}(1-s)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|f(x)-f(y)|^p}{\|x-y\|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y$$

exists for $f\in L^p(\mathbb{R}^N)$ if and only if the limit

$$\lim_{s\to 1^-}(1-s)\int_{\Omega}\int_{\Omega}\frac{|f(x)-f(y)|^p}{\|x-y\|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y$$

exists for any $f\in L^p(\Omega)$ and any bounded open set Ω containing the origin. In the latter case, mollifiers $\frac{1-s}{\|x-y\|^{N+sp-p}}$ are uniformly integrable on Ω so that we can apply our theorem.

In this case, we obtain

$$\begin{aligned}&\lim_{s\to 1^-}(1-s)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|f(x)-f(y)|^p}{\|x-y\|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y\\ &=C_0\int_{\mathbb{R}^N}\int_{S_1^{\mathfrak{c}}}|\nabla f\cdot v|^p\,\mathrm{d}\mathcal{H}^{N-1}_{\|\cdot\|}(v)\,\mathrm{d}x\end{aligned}$$

where by definition the constant C_0 is given by

$$C_0=\lim_{s\to 1^-}(1-s)\int_0^{\delta}r^{N-1}r^{-N-sp+p}\,\mathrm{d}r=\frac{1}{p}.$$

Note that

$$\begin{aligned}&\int_{B_1^{\mathfrak{c}}}|\nabla f\cdot v|^p\,\mathrm{d}\mathcal{L}^N(v)\\ &=\int_0^1\int_{S_r^{\mathfrak{c}}}|\nabla f\cdot v|^p\,\mathrm{d}\mathcal{H}^{N-1}_{\|\cdot\|}(v)\,\mathrm{d}r\\ \text{By change of variable}&=\int_0^1\int_{S_1^{\mathfrak{c}}}|\nabla f\cdot rv|^p\,\mathrm{d}\mathcal{H}^{N-1}_{\|\cdot\|}(rv)\,\mathrm{d}r\\ &=\int_0^1 r^{p+N-1}\Big(\int_{S_1^{\mathfrak{c}}}|\nabla f\cdot v|^p\,\mathrm{d}\mathcal{H}^{N-1}_{\|\cdot\|}(v)\Big)\,\mathrm{d}r\\ &=\frac{1}{p+N}\int_{S_1^{\mathfrak{c}}}|\nabla f\cdot v|^p\,\mathrm{d}\mathcal{H}^{N-1}_{\|\cdot\|}(v).\end{aligned}$$

Then we reprove the formula obtained in [20]

$$\lim_{s\to 1^-}(1-s)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|f(x)-f(y)|^p}{\|x-y\|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y$$

$$=\frac{p+N}{p}\int_{\mathbb{R}^N}\int_K|\nabla f\cdot v|^p\,\mathrm{d}\mathcal{L}^N(v)\,\mathrm{d}x$$

where $K:=B_1^{\mathfrak{C}}$ is a convex body.

Example 4.3 *(Euclidean spaces).* Let $\mathfrak{C}=(\mathbb{R}^N,|\cdot|,\mathcal{L}^N)$ be the N-dimensional Euclidean space equipped with the Euclidean distance and the Lebesgue measure, and $(\rho_n)_n$ be a family of mollifiers satisfying Assumption 3.5. Then

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_n(x,y)\,\mathrm{d}x\,\mathrm{d}y=K_{p,N}\|\nabla f\|^p_{L^p} \tag{4.2}$$

where

$$K_{p,N}:=\mathcal{L}^N(B_1^N)\fint_{S_1^N}|w\cdot v|^p\,\mathrm{d}\mathcal{H}^{N-1}(v)$$

is a constant independent of the choice of w in an N-dimensional unit sphere S_1^N, and $\mathcal{L}^N(B_1^N)=\frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}+1)}$ is the volume of the N-dimensional unit ball B_1^N.

Proof. Notice that

$$\fint_{S_1^N}|\nabla f\cdot v|^p\,\mathrm{d}\mathcal{H}^{N-1}(v)=|\nabla f|^p\fint_{S_1^N}\left|\frac{\nabla f}{|\nabla f|}\cdot v\right|^p\,\mathrm{d}\mathcal{H}^{N-1}(v).$$

By isotropicity of the Euclidean space, we know

$$\fint_{S_1^N}\left|\frac{\nabla f}{|\nabla f|}\cdot v\right|^p\,\mathrm{d}\mathcal{H}^{N-1}(v)=\fint_{S_1^N}|w\cdot v|^p\,\mathrm{d}\mathcal{H}^{N-1}(v)\quad\forall w\in S_1^N.$$

Then the assertion follows from Example 4.1. □

Example 4.4 *(Riemannian manifolds, cf. [19]).* Let $(M,\mathrm{d_g},\mathrm{Vol_g})$ be an N-dimensional compact Riemannian manifold, and $(\rho_n)_n$ be mollifiers satisfying Assumption 3.5. Then

$$\lim_{n\to\infty}\int_M\int_M\frac{|f(x)-f(y)|^p}{\mathrm{d}^p(x,y)}\rho_n(x,y)\,\mathrm{dVol_g}(x)\mathrm{dVol_g}(y)=K_{p,N}\|\nabla f\|^p_{L^p(M,\mathrm{Vol_g})} \tag{4.3}$$

where

$$K_{p,N} := \mathcal{L}^N(B_1^N) \fint_{S_1^N} |w \cdot v|^p \, \mathrm{d}\mathcal{H}^{N-1}(v)$$

is the same constant as the constant appeared in (4.2).

Proof. On an N-dimensional Riemannian manifold, the tangent space is unique and isometric to $\mathbb{R}^N$. We can construct $\{\varphi_\delta\}_{\delta>0}$ by exponential maps. Then the assertion follows from Theorem 3.9 and the constant $K_{p,N}$ is the same as the asymptotic formula for Euclidean space. □

Example 4.5 *(Carnot groups).* Let $(X, \mathrm{d}, \mathfrak{m})$ be an m-dimensional equi-regular sub-Riemannian manifold with homogeneous dimension $N \geq m$, equipped with the Carnot–Carathéodory metric d and the associated Hausdorff measure $\mathfrak{m} = \mathcal{H}^N_{\mathrm{d}}$. Let $(\rho_n)_n$ be mollifiers satisfying Assumption 3.5. We have

$$\lim_{n\to\infty} \int_X \int_X \frac{|f(x) - f(y)|^p}{\mathrm{d}^p(x,y)} \rho_n(x,y) \, \mathrm{d}\mathfrak{m}(x) \mathrm{d}\mathfrak{m}(y) = \|\nabla f\|^p_{K_{p,\mathfrak{C}}} \tag{4.4}$$

where

$$\|\nabla f\|^p_{K_{p,\mathfrak{C}}} = \mathcal{L}^m(B_1^{\mathfrak{C}}) \int_X \fint_{S_1^{\mathfrak{C}}} |\mathrm{D}_v f|^p \, \mathrm{d}\mathcal{H}^{N-1}_{\mathrm{d}_{CC}}(v) \, \mathrm{d}\mathfrak{m}$$

where $B_1^{\mathfrak{C}}$ is the unit ball in the tangent cone $\mathfrak{C} = (\mathbb{R}^m, \mathrm{d}_{CC}, \mathcal{L}^m = \mathcal{H}^N_{\mathrm{d}_{CC}})$ centred at 0, $S_1^{\mathfrak{C}}$ denotes its boundary and $\mathcal{H}^{N-1}_{\mathrm{d}_{CC}}$ is the boundary measure, and $\mathrm{D}_v f$ is Pansu's derivative of f in the direction v.

Proof. Let us check Assumption 3.1.

A) It was proved by Mitchell in [21, Theorem 1] (see also [24]) that the tangent space of a sub-Riemannian manifold equipped with a equi-regular (or called generic) distribution, is isometric to a nilpotent Lie group (Carnot group) with a left-invariant Carnot–Carathéodory metric d_{CC}.

Similar to Riemannian manifolds, at any point x on a sub-Riemannian manifold, there are almost isometries ϕ_δ induced by exponential maps. More precisely, there exist positive constants c and r, such that (see [6, Theorem 6.4]) $\varphi_\delta(x) = 0$ and

$$-c\mathrm{d}(x,y)\Big(\delta \mathrm{d}_{CC}\big(0, \phi_\delta(y)\big)\Big)^{\frac{1}{r}} < \mathrm{d}(x,y) - \delta \mathrm{d}_{CC}\big(0, \phi_\delta(y)\big) < c\mathrm{d}(x,y)\Big(\delta \mathrm{d}_{CC}\big(0, \phi_\delta(y)\big)\Big)^{\frac{1}{r}} \tag{4.5}$$

for some $c > 0$. Denote $\delta \mathrm{d}_{CC}\big(0, \phi_\delta(y)\big)$ by $|w|$. By iteration use of (4.5) we get

$$\begin{aligned}\mathrm{d}(x,y) &< |w| + c\mathrm{d}(x,y)|w|^{\frac{1}{r}} < |w| + c\big(|w| + c\mathrm{d}(x,y)|w|^{\frac{1}{r}}\big)|w|^{\frac{1}{r}} < \ldots \\ &< |w|(1+O(|w|)).\end{aligned}$$

Similarly, we can prove

$$\mathrm{d}(x,y) > |w|(1+O(|w|)).$$

Thus

$$\frac{\mathrm{d}(x,y)}{\delta \mathrm{d}_{CC}\big(x_0,\phi_\delta(y)\big)} = 1 + O(\delta).$$

B) By Pansu's theorem [24, Théorèm 2] concerning Rademacher–Stepanov theorem on sub-Riemannian manifolds, we know Lipschitz functions are almost everywhere differentiable and the limit of the rescaling functions can be written as a linear function $\mathrm{D}_v f(x)$ in the sense of Pansu (cf. [24, A. Différentiabilité]). In particular, this limit function is unique with respect to the almost isometries ϕ_δ, in the sense of Definition 2.1.

Then the formula (4.4) follows from Theorem 3.9. □

Maz'ya–Shaposhnikova type formula

We list some spaces satisfying the volume growth condition in Assumption 3.4. More mollifiers other than $\rho_s(x,y) = \frac{s}{\mathrm{d}(x,y)^{N+sp}}$, and more discussions concerning asymptotic volume ratio, curvature-dimension conditions and rigidity, will be discussed in [17].

Example 4.6 *(Euclidean spaces).* It is known that $\mathrm{AVR}_{(\mathbb{R}^N,|\cdot|,\mathcal{L}^N)} = \omega_N = \frac{|S^{N-1}|}{N}$ where $\omega_N = \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}+1)}$ denotes the volume of an N-dimensional unit ball and $|S^{N-1}|$ denotes its surface area. By Theorem 3.11 and a direct computation

$$\int_\delta^{+\infty} sNr^{N-1}/r^{N+sp}\,\mathrm{d}r = \frac{N}{p}\delta^{-sp}$$

we get Maz'ya–Shaposhnikova's formula [22, Theorem 3]:

$$\lim_{s\downarrow 0} s \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}\mathcal{L}^N(x)\,\mathrm{d}\mathcal{L}^N(y) = \frac{2|S^{N-1}|}{p}\|u\|^p_{L^p(\mathbb{R}^N)}.$$

Example 4.7 *(Finite dimensional Banach spaces).* Let $(\mathbb{R}^N,\|\cdot\|,\mathcal{L}^N)$ be an N-dimensional Banach space. Denote by $|K|$ the volume of a unit ball K. Applying Theorem 3.11, we get Ludwig's result [20, Theorem 2] for anisotropic fractional Sobolev norms:

$$\lim_{s\downarrow 0} s\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{\|x-y\|^{N+sp}}\,\mathrm{d}\mathcal{L}^N(x)\,\mathrm{d}\mathcal{L}^N(y)=\frac{2N}{p}|K|\|u\|^p_{L^p(\mathbb{R}^N)}.$$

Example 4.8 *(*MCP *spaces).* Let $(X,\mathrm{d},\mathfrak{m})$ be a metric measure space satisfying the so-called Measure Contraction Property $\mathrm{MCP}(0,N)$, a synthetic curvature-dimension condition of metric measure spaces introduced independently by Ohta [23] and Sturm [26], as a generalization of N-dimensional Riemannian manifolds with non-negative Ricci curvature. By [26, Theorem 2.3], the generalized Bishop–Gromov volume growth inequality holds.

In this case, for the mollifiers $\rho_s(x,y)=\frac{s}{\mathrm{d}(x,y)^{N+sp}}$, $s>1$, we have

$$\lim_{s\downarrow 0}\int_X\int_X\frac{|u(x)-u(y)|^p}{\mathrm{d}(x,y)^{N+sp}}\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y)=\frac{2N}{p}\mathrm{AVR}_{(X,\mathrm{d},\mathfrak{m})}\|u\|^p_{L^p}.$$

Example 4.9 *(Sub-Riemannian manifolds).* Let $\mathbb{G}=(\mathbb{R}^d,\mathrm{d}_{CC},\mathcal{L}^d)$ be a Carnot group endowed with the Carnot–Carathéodory distance d_{CC} and the Lebesgue measure $\mathcal{L}^d$. It is well known that $\mathcal{L}^d(B(x,r))=r^N\mathcal{L}^d(B(0,1))$ where $N\in\mathbb{N}$ is the homogeneous dimension. It can be seen that $\mathrm{AVR}_{(\mathbb{G},\mathrm{d}_{CC},\mathcal{L}^d)}=\mathcal{L}^d\big(B_1^{\mathbb{G}}(0)\big)$.

Recently, as a consequence of interpolation inequalities proved by Barilari and Rizzi [7] on some ideal sub-Riemannian manifolds, more examples of spaces verifying MCP have been found, such as generalized H-type groups, the Grushin plane and Sasakian structures.

Declaration of competing interest

The author states that there is no conflict of interest and the manuscript has no associated data.

Data availability

No data was used for the research described in the article.

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