Calc. Var. (2024) 63:104
https://doi.org/10.1007/s00526-024-02705-9

Calculus of Variations

Sharp uncertainty principles on metric measure spaces

Bang-Xian Han[1] · Zhe-Feng Xu[2]

Received: 9 September 2023 / Accepted: 6 March 2024

Abstract
We prove the rigidity of the Heisenberg–Pauli–Weyl uncertainty principle and the Caffarelli–Kohn–Nirenberg interpolation inequality, on metric measure spaces satisfying measure contraction property. Non-trivial examples fitting our setting include Finsler manifolds with non-negative Ricci curvature and many ideal sub-Riemannian manifolds, such as Heisenberg groups, the Grushin plane and Sasakian structures.

Mathematics Subject Classification 35R06 · 28A50 · 49Q22

Contents

1 Introduction

A fundamental concept in quantum mechanics, called Heisenberg uncertainty principle, named after Heisenberg [9], states that *the position and the momentum of particles cannot be both determined explicitly but only in a probabilistic sense with a certain uncertainty*. A few years later, Pauli and Weyl [24] described it by rigorous mathematical formulation, which states that a function itself and its Fourier transform cannot be well localized simultaneously. The Heisenberg–Pauli–Weyl uncertainty principle on the Euclidean space is described by the

Communicated by A. Mondino.

✉ Zhe-Feng Xu
xzf1998@mail.ustc.edu.cn

Bang-Xian Han
hanbx@sdu.edu.cn

1 School of Mathematics, Shandong University, Jinan 250100, China

2 School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, China

Published online: 13 April 2024

following inequality: for any $u \in C_0^\infty(\mathbb{R}^n)$,

$$\left(\int_{\mathbb{R}^n} |\nabla u(x)|^2 \,\mathrm{d}x\right)\left(\int_{\mathbb{R}^n} |x|^2 u^2(x)\,\mathrm{d}x\right) \geq \frac{n^2}{4}\left(\int_{\mathbb{R}^n} u^2(x)\,\mathrm{d}x\right)^2, \tag{1.1}$$

where $\frac{n^2}{4}$ is sharp and the extremals are $u_\lambda(x) = e^{-\lambda|x|^2}$, $\lambda > 0$.

The Heisenberg–Pauli–Weyl uncertainty principle only had sporadic developments in the fifty years after the initial work in the 1930's, followed by a steady stream of results in the last forty years. We refer to a survey written by Folland and Sitaram [8], where they gave an overview of the history and the relevance of (1.1) in the last century. At the beginning of this century, Ciatti, Ricci and Sundari [5] extended this principle to positive self-adjoint operators on measure spaces, and in the following years, Erb [6, 7], Kombe and Özaydin [12, 13] proved a sharp uncertainty principle on Riemannian manifolds by operator theoretic approach, Huang, Kristály and Zhao [10] got a sharp uncertainty principle on Finsler manifolds. In the context of metric measure spaces, Okoudjou, Saloff-Coste and Teplyaev [21] proved a weak uncertainty principle, Martín and Milman [19] obtained an L^1-uncertainty principle with isoperimetric weights.

Inspired by a recent work of Kristály [15], where he revealed the rigidity of the Heisenberg–Pauli–Weyl uncertainty principle on Riemannian manifolds with non-negative Ricci curvature, we realize that similar rigidity results hold on a larger family of metric measure spaces, called essentially non-branching MCP$(0, N)$ spaces. Examples satisfying MCP$(0, N)$ include *Riemannian manifolds with non-negative Ricci curvature and their Gromov–Hausdorff limits, Finsler manifolds with non-negative Ricci curvature,* RCD$(0, N)$ *spaces and many ideal sub-Riemannian manifolds including generalized H-type groups, the Grushin plane and Sasakian structures.*

We say that a metric measure space $(X, \mathrm{d}, \mathfrak{m})$ admits the Heisenberg–Pauli–Weyl uncertainty principle if there is $x_0 \in X$, such that for any $u \in \mathrm{Lip}_c(X, \mathrm{d})$,

$$\left(\int_X |\,\mathrm{lip}\, u\,|^2 \,\mathrm{d}\mathfrak{m}\right)\left(\int_X \mathrm{d}_{x_0}^2 u^2 \,\mathrm{d}\mathfrak{m}\right) \geq \frac{N^2}{4}\left(\int_X u^2 \,\mathrm{d}\mathfrak{m}\right)^2, \tag{HPW$_{x_0}$}$$

where $\mathrm{d}_{x_0}(x) := \mathrm{d}(x_0, x)$ is the distance function from x_0 and $\mathrm{Lip}_c(X, \mathrm{d})$ denotes the space of Lipschitz functions with compact support, and

$$\mathrm{lip}\, u(x) := \limsup_{y \to x} \frac{|u(y) - u(x)|}{\mathrm{d}(x, y)}$$

is the local Lipschitz constant of u at $x \in X$.

In our first theorem, we generalize Kristály's result [15] to metric measure spaces.

Theorem 1.1 *Let* $(X, \mathrm{d}, \mathfrak{m})$ *be an essentially non-branching metric measure space satisfying* MCP$(0, N)$ *for some* $N \in (1, \infty)$. *Then the following statements are equivalent:*

(a) (HPW)$_{x_0}$ *holds for some* $x_0 \in X$ *and the constant* $\frac{N^2}{4}$ *is sharp.*
(b) $(X, \mathrm{d}, \mathfrak{m})$ *is an* N*-volume cone.*

In Theorem 1.1, the sharpness is understood in the sense that the (HPW)$_{x_0}$ holds on a metric measure space $(X, \mathrm{d}, \mathfrak{m})$ with the same constant $\frac{N^2}{4}$ as in the Euclidean space $\mathbb{R}^N$. The parameters N appearing in the MCP condition, (HPW)$_{x_0}$ and the volume cone are the same. If we allow them to be different, we have the following non-rigid result.

Theorem 1.2 *Let* $(X, \mathrm{d}, \mathfrak{m})$ *be a metric measure space satisfying the Generalized Bishop–Gromov inequality*

$$\frac{\mathfrak{m}(B_r(x))}{r^n} \geq \frac{\mathfrak{m}(B_R(x))}{R^n}, \quad \forall x \in X,\ 0 < r < R,$$

for some $n > 1$. *Then the optimal constant in* $(\mathrm{HPW})_{x_0}$ *is at most* $\frac{n^2}{4}$.

Example. It was shown by Juillet [11] that the n-dimensional Heisenberg group $\mathbb{H}^n$, equipped with the Carnot–Carathéodory metric and the Lebesgue measure, is a metric measure space satisfying $\mathrm{MCP}(0, 2n+3)$, and $2n+3$ is optimal. However, it is a $(2n+2)$-volume cone. By Theorem 1.2 we know the optimal constant in $(\mathrm{HPW})_{x_0}$ is no bigger than $\frac{(2n+2)^2}{4}$.

Next we investigate the Caffarelli–Kohn–Nirenberg interpolation inequality (CKN for short) in the setting of non-smooth metric measure spaces. The classical CKN in the Euclidean setting was first proposed in [1], then Lin [18] generalized it to include derivatives of any order. It is known that the CKN contains the Sobolev inequality and the Hardy inequality as special cases.

Let $N, p, q \in \mathbb{R}$ be such that

$$0 < q < 2 < p \text{ and } 2 < N < \frac{2(p-q)}{p-2}. \tag{1.2}$$

Fix $x_0 \in X$. We say that CKN holds if for all $u \in \mathrm{Lip}_c(X, \mathrm{d})$,

$$\left(\int_X |\operatorname{lip} u|^2 \,\mathrm{d}\mathfrak{m}\right)\left(\int_X \frac{|u|^{2p-2}}{\mathrm{d}_{x_0}^{2q-2}}\,\mathrm{d}\mathfrak{m}\right) \geq \frac{(N-q)^2}{p^2}\left(\int_X \frac{|u|^p}{\mathrm{d}_{x_0}^q}\,\mathrm{d}\mathfrak{m}\right)^2. \tag{CKN$_{x_0}$}$$

An endpoint of $(\mathrm{CKN})_{x_0}$ is exactly $(\mathrm{HPW})_{x_0}$ as $p \to 2$ and $q \to 0$. In the Euclidean setting, Xia [25] proved the sharpness of $\frac{(N-q)^2}{p^2}$ and the existence of a class of extremals

$$u_\lambda(x) = (\lambda + |x - x_0|^{2-q})^{\frac{1}{2-p}},\ \lambda > 0. \tag{1.3}$$

Similar to Theorem 1.1, we have the rigidity of $(\mathrm{CKN})_{x_0}$.

Theorem 1.3 *Let* N, p, q *be real numbers satisfying (1.2) and* $(X, \mathrm{d}, \mathfrak{m})$ *be an essentially non-branching metric measure space satisfying* $\mathrm{MCP}(0, N)$. *Then the following statements are equivalent:*

(a) $(\mathrm{CKN})_{x_0}$ *holds for some* $x_0 \in X$ *and* $\frac{(N-q)^2}{p^2}$ *is sharp.*
(b) $(X, \mathrm{d}, \mathfrak{m})$ *is an* N*-volume cone.*

Plan of the paper. In Sect. 2, we introduce some basic concepts and results about MCP metric measure spaces. In Sect. 3, we prove the rigidity of $(\mathrm{HPW})_{x_0}$ on metric measure spaces, with the help of the needle decomposition and the Generalized Bishop–Gromov inequality. In Sect. 4, we prove the rigidity of $(\mathrm{CKN})_{x_0}$.

2 Preliminaries

In this paper, (X, d) is a Polish space (i.e. a complete and separable metric space), and $\mathfrak{m}$ is a Radon measure on X such that $0 < \mathfrak{m}(U) < \infty$ for any non-empty bounded open set $U \in X$ (i.e. $\operatorname{supp} \mathfrak{m} = X$). The triple $(X, \mathrm{d}, \mathfrak{m})$ is said to be a metric measure space.

Denote by

$$\mathrm{Geo}(X) := \left\{ \gamma \in C([0,1], X) : \mathrm{d}(\gamma_s, \gamma_t) = |s-t| \mathrm{d}(\gamma_0, \gamma_1), \ \forall s, t \in [0,1] \right\}$$

the space of constant-speed geodesics. We assume that (X, d) is a geodesic space, this means, for any $x, y \in X$ there exists $\gamma \in \mathrm{Geo}(X)$ so that $\gamma_0 = x, \gamma_1 = y$.

Denote by $\mathscr{P}(X)$ the space of all Borel probability measures on X and by $\mathscr{P}_2(X)$ the space of probability measures with finite second moment. The L^2-Kantorovich–Wasserstein distance W_2 is defined as follows: for any $\mu_0, \mu_1 \in \mathscr{P}_2(X)$, set

$$W_2^2(\mu_0, \mu_1) := \inf_{\pi} \int_{X \times X} \mathrm{d}^2(x, y) \,\mathrm{d}\pi(x, y), \tag{2.1}$$

where the infimum is taken over all $\pi \in \mathscr{P}(X \times X)$ with marginals μ_0 and μ_1.

For any geodesic $(\mu_t)_{t \in [0,1]}$ in $(\mathscr{P}_2(X), W_2)$, there is $\nu \in \mathscr{P}(\mathrm{Geo}(X))$, so that

$$(e_t)_\sharp \nu = \mu_t \ \text{ for all } t \in [0,1],$$

where e_t is the evaluation map

$$e_t : \mathrm{Geo}(X) \to X, \quad e_t(\gamma) := \gamma_t.$$

We denote by $\mathrm{OptGeo}(\mu_0, \mu_1)$ the space of all $\nu \in \mathscr{P}(\mathrm{Geo}(X))$ for which $(e_0, e_1)_\sharp \nu$ realizes the minimum in (2.1), such a ν will be called dynamical optimal plan. If (X, d) is geodesic, $\mathrm{OptGeo}(\mu_0, \mu_1)$ is non-empty for any $\mu_0, \mu_1 \in \mathscr{P}_2(X)$.

Recall the following notion of essentially non-branching [22].

Definition 2.1 A set $G \in \mathrm{Geo}(X)$ is called a set of non-branching geodesics if for any $\gamma^1, \gamma^2 \in G$, it holds:

$$\exists t \in (0,1) \ \text{ s.t. } \ \forall s \in [0,t] \ \gamma_s^1 = \gamma_s^2 \Rightarrow \forall s \in [0,1] \ \ \gamma_s^1 = \gamma_s^2.$$

Definition 2.2 A metric measure space $(X, \mathrm{d}, \mathfrak{m})$ is called essentially non-branching if for any $\mu_0, \mu_1 \in \mathscr{P}_2(X)$ with $\mu_0, \mu_1 \ll \mathfrak{m}$, any element of $\mathrm{OptGeo}(\mu_0, \mu_1)$ is concentrated on a set of non-branching geodesics.

If (X, d) is a smooth Riemannian manifold, then any subset $G \subset \mathrm{Geo}(X)$ is a set of non-branching geodesics. More generally, it is known that RCD spaces are essentially non-branching.

Given $K \in \mathbb{R}$ and $N \geq 0$, for $(t, \theta) \in [0,1] \times \mathbb{R}_+$, we define the distortion coefficients as

$$\sigma_{K,N}^{(t)}(\theta) := \begin{cases} \infty & \text{if } K\theta^2 \geq N\pi^2, \\ \frac{\sin(t\theta\sqrt{K/N})}{\sin(\theta\sqrt{K/N})} & \text{if } 0 < K\theta^2 < N\pi^2, \\ t & \text{if } K\theta^2 < 0 \text{ and } N = 0, \text{ or if } K\theta^2 = 0, \\ \frac{\sinh(t\theta\sqrt{-K/N})}{\sinh(\theta\sqrt{-K/N})} & \text{if } K\theta^2 \leq 0 \text{ and } N > 0. \end{cases}$$

We also set, for $K \in \mathbb{R}$, $N \in [1, \infty)$ and $(t, \theta) \in [0,1] \times \mathbb{R}_+$,

$$\tau_{K,N}^{(t)}(\theta) := t^{\frac{1}{N}} \sigma_{K,N-1}^{(t)}(\theta)^{\frac{N-1}{N}}.$$

The notion of *measure contraction property* $\mathrm{MCP}(K, N)$, was proposed independently by Ohta and Sturm in [20] and [23], as a synthetic notion of lower Ricci curvature bounds. Generally, these two definitions are slightly different, but on essentially non-branching spaces they coincide (see for instance Appendix A in [3] or Proposition 9.1 in [2]). We adopt the one in [20].

Definition 2.3 Let $K \in \mathbb{R}$ and $N \in [1, \infty)$. We say that a metric measure space $(X, \mathrm{d}, \mathfrak{m})$ satisfies $\mathrm{MCP}(K, N)$ if for any $\mu_0 \in \mathscr{P}_2(X)$ of the form

$$\mu_0 = \frac{1}{\mathfrak{m}(A)}\mathfrak{m}\llcorner_A, \quad A \subset X \text{ is Borel, and } \mathfrak{m}(A) \in (0, \infty),$$

and any $o \in X$, there exists $\nu \in \mathrm{OptGeo}(\mu_0, \delta_o)$ such that:

$$\frac{1}{\mathfrak{m}(A)}\mathfrak{m} \geq (e_t)_\sharp \left(\tau_{K,N}^{(1-t)}(\mathrm{d}(\gamma_0, \gamma_1))^N \nu(\gamma)\right) \quad \forall t \in [0, 1].$$

From [2], we know that in the setting of essentially non-branching spaces, Definition 2.3 is equivalent to the following: for all $\mu_0, \mu_1 \in \mathscr{P}_2(X)$ with $\mu_0 \ll \mathfrak{m}$, there exists a unique $\nu \in \mathrm{OptGeo}(\mu_0, \mu_1)$ such that for all $t \in [0, 1)$, $\mu_t = (e_t)_\sharp \nu \ll \mathfrak{m}$ and

$$\rho_t^{-\frac{1}{N}}(\gamma_t) \geq \tau_{K,N}^{(1-t)}(\mathrm{d}(\gamma_0, \gamma_1))\rho_0^{-\frac{1}{N}}(\gamma_0), \quad \text{for } \nu\text{-}a.e.\, \gamma \in \mathrm{Geo}(X), \tag{2.2}$$

where $\mu_t = \rho_t \mathfrak{m}$.

In order to prove our main results, we need a corollary of the powerful needle decomposition theorem. Here we only consider the case $K = 0$ and the simplest 1-Lipschitz function $\mathrm{d}_{x_0} := \mathrm{d}(x, x_0)$. We refer to Cavalletti–Mondino [4, Theorem 3.6] for more general cases.

Theorem 2.4 *Let $(X, \mathrm{d}, \mathfrak{m})$ be an essentially non-branching $\mathrm{MCP}(0, N)$ metric measure space for some $N \in (1, \infty)$. Then for any $x_0 \in X$ and $R > 0$, there exists an $\mathfrak{m}$-measurable transport subset $\mathcal{T} \subset B_R(x_0)$ and a family $\{X_\alpha\}_{\alpha \in Q}$ of subsets of $B_R(x_0)$, such that*

1. *there exists a disintegration of $\mathfrak{m}$*

$$\mathfrak{m}\llcorner_{\mathcal{T}} = \int_Q \mathfrak{m}_\alpha \, \mathrm{d}\mathfrak{q}(\alpha), \qquad \mathfrak{q}(Q) = 1,$$

2. *$\mathfrak{m}(B_R(x_0) \setminus \mathcal{T}) = 0$,*
3. *for $\mathfrak{q}$-a.e. $\alpha \in Q$, X_α is a closed geodesic with an extremal point x_0,*
4. *for $\mathfrak{q}$-a.e. $\alpha \in Q$, $\mathfrak{m}_\alpha$ is a Radon measure supported on X_α with $\mathfrak{m}_\alpha = h_\alpha \mathcal{H}^1\llcorner_{X_\alpha} \ll \mathcal{H}^1\llcorner_{X_\alpha}$,*
5. *for $\mathfrak{q}$-a.e. $\alpha \in Q$, the metric measure space $(X_\alpha, \mathrm{d}, \mathfrak{m}_\alpha)$ verifies $\mathrm{MCP}(0, N)$.*

Here $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure, $\{X_\alpha\}_{\alpha \in Q}$ are called transport rays and two distinct transport rays can only meet at x_0.

It is worth recalling that, if h_α is an $\mathrm{MCP}(0, N)$ density on $I \subset \mathbb{R}$, then for all $x_0, x_1 \in I$ and $t \in [0, 1]$,

$$h_\alpha(tx_1 + (1-t)x_0) \geq (1-t)^{N-1} h(x_0). \tag{2.3}$$

At the end of this part, we recall the *Generalized Bishop–Gromov volume growth inequality* (cf. [23, Remark 5.3]) and the definition of the volume cone.

Theorem 2.5 *(Generalized Bishop–Gromov inequality) Assume that $(X, \mathrm{d}, \mathfrak{m})$ satisfies $\mathrm{MCP}(0, N)$ for some $N > 1$. Then for any $x \in X$,*

$$\frac{\mathfrak{m}(B_r(x))}{r^N} \geq \frac{\mathfrak{m}(B_R(x))}{R^N}, \quad \forall 0 < r < R, \tag{GBGI}$$

where $B_r(x) := \{y \in X : \mathrm{d}(y, x) < r\}$.

Definition 2.6 Given $N \in [1, \infty)$, we say that a metric measure space is an N-volume cone if there exists $O \in X$, such that

$$\frac{\mathfrak{m}(B_R(O))}{\mathfrak{m}(B_r(O))} = \left(\frac{R}{r}\right)^N, \quad \forall 0 < r < R.$$

3 Heisenberg–Pauli–Weyl uncertainty principle

In this section, we will prove the Heisenberg–Pauli–Weyl uncertainty principle on the volume cones using needle decomposition, which reduces the problem into one-dimensional metric measure spaces. Conversely, we deduce the rigidity from the Generalized Bishop–Gromov inequality.

Proof of Theorem 1.1 **Part1** : (**b**) $\Rightarrow$ (**a**).

Step 1. Assume that the vertex of the N-volume cone is O and denote $\mathrm{d}_O(x) := \mathrm{d}(x, O)$. By Theorem 2.4, for any $R > 0$ we have a measure disintegration:

1. $$\mathfrak{m}\llcorner_{\mathcal{T}} = \int_Q \mathfrak{m}_\alpha \, \mathrm{d}\mathfrak{q}(\alpha), \qquad \mathfrak{q}(Q) = 1,$$
2. $\mathfrak{m}(B_R(O) \setminus \mathcal{T}) = 0$,
3. for $\mathfrak{q}$-a.e. $\alpha \in Q$, X_α is a closed geodesic with an extremal point O,
4. for $\mathfrak{q}$-a.e. $\alpha \in Q$, $\mathfrak{m}_\alpha \ll \mathcal{H}^1\llcorner_{X_\alpha}$,
5. for $\mathfrak{q}$-a.e. $\alpha \in Q$, the metric measure space $(X_\alpha, \mathrm{d}, \mathfrak{m}_\alpha)$ verifies MCP$(0, N)$.

Consider the optimal transport problem between the probability measures

$$\mu_0 = \frac{1}{\mathfrak{m}(B_R(O))}\mathfrak{m}\llcorner_{B_R(O)}, \qquad \text{and} \qquad \mu_1 = \delta_O.$$

By Definition 2.3, there exists a unique geodesic $(\mu_t)_{t\in[0,1]}$ in $(\mathscr{P}_2(X), W_2)$ connecting μ_0 and μ_1, and there is a measure $\nu \in \mathscr{P}(\mathrm{Geo}(X))$, so that $(e_t)_\sharp \nu = \mu_t$ for all $t \in [0, 1]$. We can see that μ_t is concentrated on a subset $\Omega_t \subset B_{(1-t)R}(O)$. By measure contraction property (2.2), we have $\mathfrak{m}(\Omega_t) \geq (1-t)^N \mathfrak{m}(B_R(O))$. Since $(X, \mathrm{d}, \mathfrak{m})$ is an N-volume cone, we know $\mathfrak{m}(B_{(1-t)R}(O) \setminus \Omega_t) = 0$ so that almost every point in $B_{(1-t)R}(O)$ is a t-intermediate point of some X_α. Combining (GBGI) and (2.3), we can also see that for $\mathfrak{q}$-a.e.$\alpha \in Q$, $(X_\alpha, \mathrm{d}, \mathfrak{m}_\alpha)$ can be identified with $([0, R], |\cdot|, c_\alpha x^{N-1}\mathrm{d}x)$ for some positive constant c_α.

Since R is arbitrary and transport rays can only meet at O, we can rewrite the measure disintegration as

$$\mathfrak{m} = \int_Q \mathfrak{m}_\alpha \, \mathrm{d}\mathfrak{q}(\alpha), \qquad (X_\alpha, \mathrm{d}, \mathfrak{m}_\alpha) \cong ([0, +\infty), |\cdot|, c_\alpha x^{N-1}\mathrm{d}x), \tag{3.1}$$

where for $\mathfrak{q}$-a.e.$\alpha \in Q$, X_α has O as an extremal point.

Step 2. Fix $\alpha \in Q$, denote by $\tilde{\Delta}_\alpha$ the weighted Laplacian

$$\tilde{\Delta}_\alpha := \Delta - \langle \nabla V_\alpha, \nabla \cdot \rangle,$$

where $V_\alpha(x)$ is given by $e^{-V_\alpha(x)} = c_\alpha x^{N-1}$, and Δ, ∇ are understood as directional derivatives on $[0, +\infty)$ in the usual sense.

Fix $u \in \mathrm{Lip}_c(X, \mathrm{d}) \setminus \{0\}$. On one hand, by (3.1) we have $\tilde{\Delta}_\alpha(\mathrm{d}_O^2) = 2N$ and

$$\left(\int_{X_\alpha} \tilde{\Delta}_\alpha(\mathrm{d}_O^2) u^2 \,\mathrm{d}\mathfrak{m}_\alpha\right)^2 = 4N^2 \left(\int_{X_\alpha} u^2 \,\mathrm{d}\mathfrak{m}_\alpha\right)^2. \tag{3.2}$$

On the other hand, by integration by parts

$$\begin{aligned}\left(\int_{X_\alpha} \tilde{\Delta}_\alpha(\mathrm{d}_O^2) u^2 \,\mathrm{d}\mathfrak{m}_\alpha\right)^2 &= \left(-\int_0^\infty \left\langle \nabla(u^2), \nabla(\mathrm{d}_O^2)\right\rangle \,\mathrm{d}\mathfrak{m}_\alpha\right)^2 \\ &= \left(-\int_0^\infty 4u\mathrm{d}_O \left\langle \nabla u, \nabla \mathrm{d}_O\right\rangle \,\mathrm{d}\mathfrak{m}_\alpha\right)^2 \\ \text{Cauchy–Schwarz} \quad &\leq 16 \left(\int_{X_\alpha} \mathrm{d}_O^2 u^2 \,\mathrm{d}\mathfrak{m}_\alpha\right)\left(\int_{X_\alpha} |\nabla u|^2 \,\mathrm{d}\mathfrak{m}_\alpha\right).\end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3), we obtain

$$\left(\int_{X_\alpha} |\nabla u|^2 \,\mathrm{d}\mathfrak{m}_\alpha\right)\left(\int_{X_\alpha} \mathrm{d}_O^2 u^2 \,\mathrm{d}\mathfrak{m}_\alpha\right) \geq \frac{N^2}{4}\left(\int_{X_\alpha} u^2 \,\mathrm{d}\mathfrak{m}_\alpha\right)^2. \tag{3.4}$$

Step 3. Denote $C_\alpha := \int_{X_\alpha} \mathrm{d}_O^2 u^2 \,\mathrm{d}\mathfrak{m}_\alpha > 0$, $\widetilde{\mathfrak{m}}_\alpha := \frac{\mathfrak{m}_\alpha}{C_\alpha}$, $\widetilde{\mathfrak{q}} := C_\alpha \mathfrak{q}$. We can see that $\{\widetilde{\mathfrak{m}}_\alpha\}_{\alpha\in Q}$ is also a disintegration of $\mathfrak{m}$ since

$$\int_Q \widetilde{\mathfrak{m}}_\alpha \,\mathrm{d}\widetilde{\mathfrak{q}}(\alpha) = \int_Q \frac{\mathfrak{m}_\alpha}{C_\alpha} \cdot C_\alpha \mathrm{d}\mathfrak{q}(\alpha) = \int_Q \mathfrak{m}_\alpha \,\mathrm{d}\mathfrak{q}(\alpha) = \mathfrak{m}.$$

Fix $\alpha \in Q$, multiplying $1/C_\alpha^2$ on both sides of (3.4), then we have

$$\left(\int_{X_\alpha} |\nabla u|^2 \,\mathrm{d}\widetilde{\mathfrak{m}}_\alpha\right)\left(\int_{X_\alpha} \mathrm{d}_O^2 u^2 \,\mathrm{d}\widetilde{\mathfrak{m}}_\alpha\right) \geq \frac{N^2}{4}\left(\int_{X_\alpha} u^2 \,\mathrm{d}\widetilde{\mathfrak{m}}_\alpha\right)^2. \tag{3.5}$$

Note that

$$\begin{aligned}\int_{X_\alpha} \mathrm{d}_O^2 u^2 \,\mathrm{d}\widetilde{\mathfrak{m}}_\alpha &\equiv 1, \quad \forall \alpha \in Q, \\ \int_X \mathrm{d}_O^2 u^2 \mathrm{d}\mathfrak{m} = \int_Q \int_{X_\alpha} \mathrm{d}_O^2 u^2 \,\mathrm{d}\widetilde{\mathfrak{m}}_\alpha \mathrm{d}\widetilde{\mathfrak{q}} &= \widetilde{\mathfrak{q}}(Q).\end{aligned} \tag{3.6}$$

Combining the above identities and integrating α on both sides of (3.5), we conclude that

$$\int_Q \left(\int_{X_\alpha} |\nabla u|^2 \,\mathrm{d}\widetilde{\mathfrak{m}}_\alpha\right) \mathrm{d}\widetilde{\mathfrak{q}}(\alpha) \geq \frac{N^2}{4} \int_Q \left(\int_{X_\alpha} u^2 \,\mathrm{d}\widetilde{\mathfrak{m}}_\alpha\right)^2 \mathrm{d}\widetilde{\mathfrak{q}}(\alpha). \tag{3.7}$$

Multiplying $\widetilde{\mathfrak{q}}(Q)$ on both sides of (3.7) and combining

$$|\nabla u(x)| \leq |\mathrm{lip}\, u(x)| \qquad a.e.\, x \in X_\alpha, \ \forall \alpha \in Q,$$

by Cauchy–Schwarz inequality, we get

$$\begin{aligned}
&\left(\int_X |\operatorname{lip} u|^2 \,\mathrm{d}\mathfrak{m}\right)\left(\int_X \mathrm{d}_O^2 u^2 \,\mathrm{d}\mathfrak{m}\right)\\
&\geq \frac{N^2}{4}\left(\int_Q \left(\int_{X_\alpha} u^2 \,\mathrm{d}\widetilde{\mathfrak{m}}_\alpha\right)^2 \mathrm{d}\widetilde{\mathfrak{q}}(\alpha)\right)\left(\int_Q 1^2 \,\mathrm{d}\widetilde{\mathfrak{q}}(\alpha)\right)\\
&\geq \frac{N^2}{4}\left(\int_Q \int_{X_\alpha} u^2 \,\mathrm{d}\widetilde{\mathfrak{m}}_\alpha \mathrm{d}\widetilde{\mathfrak{q}}\right)^2\\
&= \frac{N^2}{4}\left(\int_X u^2 \,\mathrm{d}\mathfrak{m}\right)^2,
\end{aligned} \tag{3.8}$$

which is $(\mathrm{HPW})_{x_0}$ with $x_0 = O$.

Step 4. Following Kristály and Ohta [14–16] we can prove the sharpness of $\frac{N^2}{4}$. Since $(X, \mathrm{d}, \mathfrak{m})$ is an N-volume cone with vertex O, we have

$$\frac{\mathfrak{m}(B_R(O))}{\mathfrak{m}(B_r(O))} = \left(\frac{R}{r}\right)^N, \quad \forall 0 < r < R.$$

Without loss of generality, we can assume that

$$\mathfrak{m}(B_\rho(O)) = A\omega_N \rho^N, \quad \forall \rho > 0, \tag{3.9}$$

where A is a positive constant and $\omega_N := \pi^{\frac{N}{2}}/\Gamma(\frac{N}{2}+1)$ is the volume of the unit ball in $\mathbb{R}^N$ (cf. [17]).

For each $\lambda > 0$, consider the sequence of functions $u_{\lambda,k} : X \to \mathbb{R}$, $k \in \mathbb{N}$ defined as

$$u_{\lambda,k}(x) := \max\Big\{0, \min\{0, k - \mathrm{d}_O(x)\} + 1\Big\} e^{-\lambda \mathrm{d}_O^2(x)}.$$

Notice that supp $u_{\lambda,k}=\{x \in X : \mathrm{d}_O(x) \leq k+1\}$ and MCP spaces are locally compact. For any $\lambda > 0$ and $k \in \mathbb{N}$, we have $u_{\lambda,k} \in \mathrm{Lip}_c(X, \mathrm{d})$ so it satisfies (3.8). Set

$$u_\lambda(x) := \lim_{k\to\infty} u_{\lambda,k}(x) = e^{-\lambda \mathrm{d}_O^2(x)}.$$

A simple approximation procedure based on (3.9) shows that u_λ verifies (3.8) as well. Next, we will prove that u_λ attains the equalities in (3.8).

Define a function $T : (0, \infty) \to \mathbb{R}$ by

$$T(\lambda) := \int_X u_\lambda^2 \,\mathrm{d}\mathfrak{m} = \int_X e^{-2\lambda \mathrm{d}_O^2} \,\mathrm{d}\mathfrak{m}.$$

It is well-defined and differentiable, and we have

$$T'(\lambda) = \int_X (-2\mathrm{d}_O^2) e^{-2\lambda \mathrm{d}_O^2} \,\mathrm{d}\mathfrak{m}.$$

Note that

$$|\operatorname{lip} u_\lambda|^2 = |-2\lambda \mathrm{d}_O \operatorname{lip}(\mathrm{d}_O)\, e^{-\lambda \mathrm{d}_O^2}|^2 \leq 4\lambda^2 \mathrm{d}_O^2 e^{-2\lambda \mathrm{d}_O^2}.$$

To prove (3.8), it is sufficient to check the following equation:

$$-\lambda T'(\lambda) = \frac{N}{2} T(\lambda), \qquad \forall \lambda > 0, \tag{3.10}$$

equivalently,

$$T(\lambda) = C\lambda^{-\frac{N}{2}} \quad \text{for some } C > 0. \tag{3.11}$$

In fact, by the layer cake representation (or Cavalieri's formula) and changing a variable, we have

$$\begin{aligned} T(\lambda) &= \int_0^\infty \mathfrak{m}(\{x \in X : e^{-2\lambda \mathrm{d}_O^2} > t\})\,\mathrm{d}t \\ &= 4\lambda \int_0^\infty \mathfrak{m}\big(B_\rho(O)\big)\rho e^{-2\lambda\rho^2}\,\mathrm{d}\rho \\ &= 4\lambda A\omega_N \int_0^\infty \rho^{N+1} e^{-2\lambda\rho^2}\,\mathrm{d}\rho \\ \text{set } 2\lambda\rho^2 = y \quad &= \frac{1}{(2\lambda)^{\frac{N}{2}}} A\omega_N \int_0^\infty y^{\frac{N}{2}} e^{-y}\,\mathrm{d}y \\ &= \frac{1}{(2\lambda)^{\frac{N}{2}}} A\omega_N \Gamma(\frac{N}{2}+1), \end{aligned}$$

which is the thesis.

Part2 : (**a**) $\Rightarrow$ (**b**).

By Theorem 2.5 we know $\rho \mapsto \frac{\mathfrak{m}(B_\rho(x_0))}{\rho^N}$ is non-increasing on $(0,\infty)$. Without loss of generality, we assume

$$\lim_{\rho\to 0^+} \frac{\mathfrak{m}(B_\rho(x_0))}{\omega_N \rho^N} = A, \quad A > 0 \text{ is a finite constant}, \tag{3.12}$$

so that

$$\mathfrak{m}(B_\rho(x_0)) \le A\omega_N\rho^N, \qquad \forall \rho > 0. \tag{3.13}$$

Consider the function

$$T(\lambda) = 4\lambda\omega_N \int_0^\infty \rho^{N+1} e^{-2\lambda\rho^2}\,\mathrm{d}\rho = \frac{1}{(2\lambda)^{\frac{N}{2}}}\omega_N\Gamma(\frac{N}{2}+1), \tag{3.14}$$

which certainly satisfies (3.10).

Fix $x_0 \in X$, we consider the class of functions

$$\widetilde{u}_\lambda(x) = e^{-\lambda \mathrm{d}_{x_0}^2(x)}, \quad \lambda > 0. \tag{3.15}$$

Similarly, we can approximate $\widetilde{u}_\lambda$ by elements in $\mathrm{Lip}_c(X, \mathrm{d})$. Inserting $\widetilde{u}_\lambda$ into $(\mathrm{HPW})_{x_0}$, we obtain

$$2\lambda \int_X \mathrm{d}_{x_0}^2 e^{-2\lambda \mathrm{d}_{x_0}^2}\,\mathrm{d}\mathfrak{m} \ge \frac{N}{2}\int_X e^{-2\lambda \mathrm{d}_{x_0}^2}\,\mathrm{d}\mathfrak{m}, \quad \lambda > 0. \tag{3.16}$$

We introduce a function $P : (0,\infty) \to \mathbb{R}$ defined by

$$P(\lambda) = \int_X e^{-2\lambda \mathrm{d}_{x_0}^2}\,\mathrm{d}\mathfrak{m}.$$

It is well-defined and differentiable. By the layer cake representation, the function can be equivalently rewritten as

$$P(\lambda) = \int_0^\infty \mathfrak{m}\big(\{x \in X : e^{-2\lambda \mathrm{d}_{x_0}^2} > t\}\big)\,\mathrm{d}t = 4\lambda \int_0^\infty \mathfrak{m}(B_\rho(x_0))\rho e^{-2\lambda\rho^2}\,\mathrm{d}\rho.$$

Thus relation (3.16) is equivalent to

$$-\lambda P'(\lambda) \geq \frac{N}{2} P(\lambda), \quad \lambda > 0. \tag{3.17}$$

By (3.10) and (3.17), it turns out that

$$\frac{P'(\lambda)}{P(\lambda)} \leq \frac{T'(\lambda)}{T(\lambda)}, \quad \lambda > 0.$$

Integrating this inequality, it yields that the function $\lambda \mapsto \frac{P(\lambda)}{T(\lambda)}$ is non-increasing; in particular, for every $\lambda > 0$,

$$\frac{P(\lambda)}{T(\lambda)} \geq \liminf_{\lambda \to \infty} \frac{P(\lambda)}{T(\lambda)}.$$

We claim that

$$P(\lambda) \geq AT(\lambda), \qquad \forall \lambda > 0.$$

To prove the claim, we only need to show

$$\liminf_{\lambda \to \infty} \frac{P(\lambda)}{T(\lambda)} \geq A. \tag{3.18}$$

Due to (3.12), for any $\varepsilon > 0$ small enough, we can find $\rho_\varepsilon > 0$ such that

$$\mathfrak{m}(B_\rho(x_0)) \geq (A - \varepsilon)\omega_N \rho^N, \quad \forall \rho \in [0, \rho_\varepsilon].$$

Consequently, we have

$$\begin{aligned}
P(\lambda) &= 4\lambda \int_0^\infty \mathfrak{m}(B_\rho(x_0))\rho e^{-2\lambda\rho^2}\,\mathrm{d}\rho \\
&\geq 4\lambda(A - \varepsilon)\omega_N \int_0^{\rho_\varepsilon} \rho^{N+1} e^{-2\lambda\rho^2}\,\mathrm{d}\rho \\
\text{set } 2\lambda\rho^2 = y \quad &= \frac{1}{(2\lambda)^{\frac{N}{2}}}(A - \varepsilon)\omega_N \int_0^{2\lambda\rho_\varepsilon^2} y^{\frac{N}{2}} e^{-y}\,\mathrm{d}y.
\end{aligned}$$

Combining (3.14) we get

$$\liminf_{\lambda \to \infty} \frac{P(\lambda)}{T(\lambda)} \geq A - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, relation (3.18) holds and we complete the proof of the claim.

From the claim we know

$$4\lambda \int_0^\infty \left[\mathfrak{m}(B_\rho(x_0)) - A\omega_N \rho^N\right] \rho e^{-2\lambda\rho^2}\mathrm{d}\rho \geq 0, \qquad \forall \lambda > 0.$$

By (3.13) we have

$$\mathfrak{m}(B_\rho(x_0)) = A\omega_N \rho^N, \qquad \forall \rho > 0,$$

so $(X, \mathrm{d}, \mathfrak{m})$ is an N-volume cone. □

Proof of Theorem 1.2 Without loss of generality, we can assume

$$0 < \lim_{\rho \to 0^+} \frac{\mathfrak{m}(B_\rho(x_0))}{\omega_n \rho^n} = A < +\infty \tag{3.19}$$

and

$$\mathfrak{m}(B_\rho(x_0)) \leq A\omega_n \rho^n, \qquad \forall \rho > 0. \tag{3.20}$$

Consider the function

$$T_n(\lambda) = 4\lambda\omega_n \int_0^\infty \rho^{n+1} e^{-2\lambda\rho^2}\, \mathrm{d}\rho = \frac{1}{(2\lambda)^{\frac{n}{2}}} \omega_n \Gamma(\frac{n}{2}+1). \tag{3.21}$$

By (3.17) we have

$$\frac{P'(\lambda)}{P(\lambda)} \leq \frac{N}{n} \frac{T_n'(\lambda)}{T_n(\lambda)}, \quad \lambda > 0.$$

Integrating this inequality, it yields that the function $\lambda \mapsto \frac{P(\lambda)}{[T_n(\lambda)]^{N/n}}$ is non-increasing. Using the same arguments as in the proof of Theorem 1.1 and combining (3.20), we can prove $\lim_{\lambda \to \infty} \frac{P(\lambda)}{T_n(\lambda)} = A$. Since $T_n(\lambda) \downarrow 0$ as $\lambda \to \infty$, it holds the inequality $N/n \leq 1$ which is the thesis. □

4 Caffarelli–Kohn–Nirenberg inequality

In this section, we will prove the rigidity of $(\mathrm{CKN})_{x_0}$. Its proof is similar to Theorem 1.1, so we adopt the same notations as in the proof of Theorem 1.1 and omit some details.

Proof of Theorem 1.3 **Part1** : (**b**) $\Rightarrow$ (**a**).

Step 1. Assume that the vertex of the N-volume cone is O and denote $\mathrm{d}_O(x) := \mathrm{d}(x, O)$. Similarly, for $\mathfrak{q}$-a.e.$\alpha \in Q$, $(X_\alpha, \mathrm{d}, \mathfrak{m}_\alpha)$ can be identified with the 1-dimensional space $([0,\infty), |\cdot|, c_\alpha x^{N-1}\mathrm{d}x)$ and $\mathrm{d}_O \tilde{\Delta}_\alpha \mathrm{d}_O = N-1$ on $(X_\alpha, \mathrm{d}, \mathfrak{m}_\alpha)$.

Fix $u \in \mathrm{Lip}_c(X, \mathrm{d}) \setminus \{0\}$. We have

$$\begin{aligned}\int_{X_\alpha} \frac{|u|^p}{\mathrm{d}_O^q}\, \mathrm{d}\mathfrak{m}_\alpha =& \frac{1}{N-1} \int_{X_\alpha} \frac{|u|^p}{\mathrm{d}_O^{q-1}} \tilde{\Delta}_\alpha \mathrm{d}_O\, \mathrm{d}\mathfrak{m}_\alpha \\ =& -\frac{p}{N-1} \int_{X_\alpha} \frac{|u|^{p-1}}{\mathrm{d}_O^{q-1}} \langle \nabla |u|, \nabla \mathrm{d}_O \rangle\, \mathrm{d}\mathfrak{m}_\alpha \\ &+ \frac{q-1}{N-1} \int_{X_\alpha} \frac{|u|^p}{\mathrm{d}_O^q} |\nabla \mathrm{d}_O|^2\, \mathrm{d}\mathfrak{m}_\alpha.\end{aligned} \tag{4.1}$$

Note that $|\nabla \mathrm{d}_O| = 1$ $\mathfrak{m}_\alpha$-a.e. on X_α, a reorganization of the above estimate implies that

$$\begin{aligned}\frac{N-q}{p} \int_{X_\alpha} \frac{|u|^p}{\mathrm{d}_O^q}\, \mathrm{d}\mathfrak{m}_\alpha &\leq \int_{X_\alpha} \frac{|u|^{p-1}}{\mathrm{d}_O^{q-1}} |\nabla u|\, \mathrm{d}\mathfrak{m}_\alpha \\ \text{Cauchy–Schwarz} \quad &\leq \left(\int_{X_\alpha} |\nabla u|^2\, \mathrm{d}\mathfrak{m}_\alpha \right)^{\frac{1}{2}} \left(\int_{X_\alpha} \frac{|u|^{2p-2}}{\mathrm{d}_O^{2q-2}}\, \mathrm{d}\mathfrak{m}_\alpha \right)^{\frac{1}{2}}.\end{aligned} \tag{4.2}$$

Similar to the proof of Theorem 1.1, we can adjust the decomposition and obtain

$$\left(\int_X |\operatorname{lip} u|^2 \, \mathrm{d}\mathfrak{m}\right)\left(\int_X \frac{|u|^{2p-2}}{\mathrm{d}_O^{2q-2}} \, \mathrm{d}\mathfrak{m}\right) \geq \frac{(N-q)^2}{p^2}\left(\int_X \frac{|u|^p}{\mathrm{d}_O^q} \, \mathrm{d}\mathfrak{m}\right)^2 \tag{4.3}$$

which is the thesis.

Step 2. Similarly, to prove the sharpness of $\frac{(N-q)^2}{p^2}$, we consider functions

$$u_{\lambda,k}(x) := \max\left\{0, \min\{0, k - \mathrm{d}_O(x)\} + 1\right\}\left(\lambda + \max\left\{\mathrm{d}_O(x), \frac{1}{k}\right\}^{2-q}\right)^{\frac{1}{2-p}}.$$

Define

$$u_\lambda(x) := \lim_{k\to\infty} u_{\lambda,k}(x) = (\lambda + \mathrm{d}_O^{2-q})^{\frac{1}{2-p}}.$$

By an approximation argument we can prove that u_λ verifies (4.3) as well. Next we will show that u_λ attains the equality in (4.3).

Consider a function $T : (0, \infty) \to \mathbb{R}$ defined by

$$T(\lambda) = \frac{p-2}{p}\int_X \frac{(\lambda + \mathrm{d}_O^{2-q})^{\frac{p}{2-p}}}{\mathrm{d}_O^q} \, \mathrm{d}\mathfrak{m}.$$

It is well-defined and differentiable, and we have

$$T'(\lambda) = -\int_X \frac{(\lambda + \mathrm{d}_O^{2-q})^{\frac{2p-2}{2-p}}}{\mathrm{d}_O^q} \, \mathrm{d}\mathfrak{m}.$$

Notice that

$$|\operatorname{lip} u_\lambda|^2 \leq \left(\frac{2-q}{2-p}\right)^2 (\lambda + \mathrm{d}_O^{2-q})^{\frac{2p-2}{2-p}} \mathrm{d}_O^{2-2q}$$

and

$$\frac{p}{p-2}T(\lambda) + \lambda T'(\lambda) = \int_X \frac{(\lambda + \mathrm{d}_O^{2-q})^{\frac{2p-2}{2-p}}}{\mathrm{d}_O^{2q-2}} \, \mathrm{d}\mathfrak{m}.$$

It is sufficient to prove

$$\lambda T'(\lambda) = \left(\frac{N-q}{2-q} - \frac{p}{p-2}\right) T(\lambda), \qquad \forall \lambda > 0. \tag{4.4}$$

Note that $\alpha := \frac{N-q}{2-q} - \frac{p}{p-2} < 0$. It is equivalent to prove

$$T(\lambda) = C\lambda^\alpha, \quad \forall \lambda > 0 \text{ and for some } C > 0. \tag{4.5}$$

By the layer cake representation and changing a variable, we have

$$\begin{aligned} T(\lambda) &= \frac{p-2}{p}\int_X \frac{(\lambda+\mathrm{d}_O^{2-q})^{\frac{p}{2-p}}}{\mathrm{d}_O^q}\,\mathrm{d}\mathfrak{m} \\ &= \frac{p-2}{p}\int_0^\infty \mathfrak{m}\Big(\{x\in X : \frac{(\lambda+\mathrm{d}_O^{2-q})^{\frac{p}{2-p}}}{\mathrm{d}_O^q} > t\}\Big)\,\mathrm{d}t \\ &= \frac{p-2}{p}A\omega_N\int_0^\infty \frac{(\lambda+\rho^{2-q})^{\frac{2p-2}{2-p}}}{\rho^{2q-1-N}}\left(\frac{(2-q)p}{p-2}+q(\lambda\rho^{q-2}+1)\right)\mathrm{d}\rho \\ \text{set } \rho=\lambda^{\frac{1}{2-q}}y \quad &= \frac{p-2}{p}A\omega_N\int_0^\infty \frac{(\lambda+\lambda y^{2-q})^{\frac{2p-2}{2-p}}}{\lambda^{\frac{2q-1-N}{2-q}}y^{2q-1-N}}\left(\frac{(2-q)p}{p-2}+q(y^{q-2}+1)\right)\lambda^{\frac{1}{2-q}}\,\mathrm{d}y \\ &= \lambda^\alpha\frac{p-2}{p}A\omega_N\int_0^\infty \frac{(1+y^{2-q})^{\frac{2p-2}{2-p}}}{y^{2q-1-N}}\left(\frac{(2-q)p}{p-2}+q(y^{q-2}+1)\right)\mathrm{d}y, \end{aligned}$$

which is (4.5) and we complete the proof of (**b**) ⇒ (**a**).

Part2 : (**a**) ⇒ (**b**).

Assume that

$$\lim_{\rho\to 0^+}\frac{\mathfrak{m}(B_\rho(x_0))}{\omega_N\rho^N}=A,\quad A>0 \text{ is a finite constant,} \tag{4.6}$$

and

$$\mathfrak{m}(B_\rho(x_0))\le A\omega_N\rho^N,\qquad \forall\rho>0. \tag{4.7}$$

Similarly, we consider the function

$$\begin{aligned} T(\lambda) &= \frac{p-2}{p}\omega_N\int_0^\infty \frac{(\lambda+\rho^{2-q})^{\frac{2p-2}{2-p}}}{\rho^{2q-1-N}}\left(\frac{(2-q)p}{p-2}+q(\lambda\rho^{q-2}+1)\right)\mathrm{d}\rho \\ &= \lambda^\alpha\frac{p-2}{p}\omega_N\int_0^\infty \frac{(1+y^{2-q})^{\frac{2p-2}{2-p}}}{y^{2q-1-N}}\left(\frac{(2-q)p}{p-2}+q(y^{q-2}+1)\right)\mathrm{d}y, \end{aligned} \tag{4.8}$$

and the class of functions

$$\widetilde{u}_\lambda=(\lambda+\mathrm{d}_{x_0}^{2-q})^{\frac{1}{2-p}},\quad \lambda>0. \tag{4.9}$$

Similarly, the functions $\widetilde{u}_\lambda$ can be approximated by elements in $\mathrm{Lip}_c(X,\mathrm{d})$ for every $\lambda>0$. By inserting $\widetilde{u}_\lambda$ into $(\mathrm{CKN})_{x_0}$, we obtain that

$$\frac{2-q}{p-2}\int_X\frac{(\lambda+\mathrm{d}_{x_0}^{2-q})^{\frac{2p-2}{2-p}}}{\mathrm{d}_{x_0}^{2q-2}}\,\mathrm{d}\mathfrak{m}\ge\frac{N-q}{p}\int_X\frac{(\lambda+\mathrm{d}_{x_0}^{2-q})^{\frac{p}{2-p}}}{\mathrm{d}_{x_0}^q}\,\mathrm{d}\mathfrak{m}. \tag{4.10}$$

Define a function $P:(0,\infty)\to\mathbb{R}$ by

$$P(\lambda)=\frac{p-2}{p}\int_X\frac{(\lambda+\mathrm{d}_{x_0}^{2-q})^{\frac{p}{2-p}}}{\mathrm{d}_{x_0}^q}\,\mathrm{d}\mathfrak{m}.$$

It is well-defined and differentiable. Through similar arguments, (4.10) is equivalent to

$$\lambda P'(\lambda)\ge\left(\frac{N-q}{2-q}-\frac{p}{p-2}\right)P(\lambda),\qquad\forall\lambda>0. \tag{4.11}$$

Due to (4.6), for any $\varepsilon > 0$ small enough, we can find $\rho_\varepsilon > 0$ such that

$$\mathfrak{m}(B_\rho(x_0)) \geq (A-\varepsilon)\omega_N \rho^N, \quad \forall \rho \in [0, \rho_\varepsilon].$$

Therefore, by layer cake representation and changing the variable $\rho = \lambda^{\frac{1}{2-q}} y$, it turns out that

$$\begin{aligned}
P(\lambda) &= \frac{p-2}{p} \int_0^\infty \mathfrak{m}\left(\left\{x \in X : \frac{(\lambda + \mathrm{d}_{x_0}^{2-q})^{\frac{p}{2-p}}}{\mathrm{d}_{x_0}^q} > t\right\}\right) \mathrm{d}t \\
&= \frac{p-2}{p} \int_0^\infty \mathfrak{m}(B_\rho(x_0)) \frac{(\lambda + \rho^{2-q})^{\frac{2p-2}{2-p}}}{\rho^{2q-1}} \left(\frac{(2-q)p}{p-2} + q(\lambda\rho^{q-2}+1)\right) \mathrm{d}\rho \\
&\geq \frac{p-2}{p}(A-\varepsilon)\omega_N \int_0^{\rho_\varepsilon} \frac{(\lambda + \rho^{2-q})^{\frac{2p-2}{2-p}}}{\rho^{2q-1-N}} \left(\frac{(2-q)p}{p-2} + q(\lambda\rho^{q-2}+1)\right) \mathrm{d}\rho \\
&= \lambda^\alpha \frac{p-2}{p}(A-\varepsilon)\omega_N \int_0^{\lambda^{\frac{1}{q-2}}\rho_\varepsilon} \frac{(1 + y^{2-q})^{\frac{2p-2}{2-p}}}{y^{2q-1-N}} \left(\frac{(2-q)p}{p-2} + q(y^{q-2}+1)\right) \mathrm{d}y.
\end{aligned}$$

Combining (4.8) and the fact that $\frac{1}{q-2} < 0$, we have

$$\liminf_{\lambda \to 0^+} \frac{P(\lambda)}{T(\lambda)} \geq A - \varepsilon.$$

Letting $\varepsilon \to 0$, we obtain

$$\liminf_{\lambda \to 0^+} \frac{P(\lambda)}{T(\lambda)} \geq A. \tag{4.12}$$

By (4.4) and (4.11), we have

$$\frac{P'(\lambda)}{P(\lambda)} \geq \frac{T'(\lambda)}{T(\lambda)}, \quad \forall \lambda > 0. \tag{4.13}$$

Integrating this inequality, it yields that the function $\lambda \mapsto \frac{P(\lambda)}{T(\lambda)}$ is non-decreasing. Combining (4.12) we get

$$\frac{P(\lambda)}{T(\lambda)} \geq A, \quad \forall \lambda > 0,$$

which means, for any $\lambda > 0$,

$$\frac{p-2}{p} \int_0^\infty \left(\mathfrak{m}(B_\rho(x_0)) - A\omega_N\rho^N\right) \frac{(\lambda + \rho^{2-q})^{\frac{2p-2}{2-p}}}{\rho^{2q-1}} \left(\frac{(2-q)p}{p-2} + q(\lambda\rho^{q-2}+1)\right) \mathrm{d}\rho \geq 0.$$

By (4.7), we have

$$\mathfrak{m}(B_\rho(x_0)) = A\omega_N\rho^N, \qquad \forall \rho > 0,$$

so $(X, \mathrm{d}, \mathfrak{m})$ is an N-volume cone and we complete the proof of Theorem 1.3. □

Remark 4.1 By the same methods, we can prove the rigidity of more general CKN inequalities in [25]. To achieve this, we just need to replace Cauchy–Schwarz inequality by Hölder inequality in (4.2).

Concerning the dimension parameter N and the optimal constant in $(\mathrm{CKN})_{x_0}$, we can prove a similar result in the same way as Theorem 1.2, we leave the proof to the readers.

Acknowledgements This project is supported in part by the Ministry of Science and Technology of China, through the Young Scientist Programs (No. 2021YFA1000900 and 2021YFA1002200), and NSFC grant (No.12201596). The authors want to thank Alexandru Kristály for his suggestions on the first draft of this paper.

Declarations

Conflict of interest The authors declare that there is no Conflict of interest and the manuscript has no associated data.

References

1. Caffarelli, L., Kohn, R., Nirenberg, L.: First order interpolation inequalities with weights. Compositio Math. **53**, 259–275 (1984)
2. Cavalletti, F., Milman, E.: The globalization theorem for the curvature-dimension condition. Invent. Math. **226**, 1–137 (2021)
3. Cavalletti, F., Mondino, A.: Optimal maps in essentially non-branching spaces. Commun. Contemp. Math. **19**, 1750007 (2017)
4. Cavalletti, F., Mondino, A.: New formulas for the Laplacian of distance functions and applications. Anal. PDE **13**, 2091–2147 (2020)
5. Ciatti, P., Ricci, F., Sundari, M.: Heisenberg-Pauli-Weyl uncertainty inequalities and polynomial volume growth. Adv. Math. **215**, 616–625 (2007)
6. Erb, W.: *Uncertainty Principles on Riemannian Manifolds*, PhD thesis, Technical University Munchen, (2009)
7. Erb, W.: Uncertainty principles on compact Riemannian manifolds. Appl. Comput. Harmon. Anal. **29**, 182–197 (2010)
8. Folland, G., Sitaram, A.: The uncertainty principle: a mathematical survey. J. Fourier Anal. Appl. **3**, 207–238 (1997)
9. Heisenberg, W.: Über den anschaulichen inhalt der quantentheoretischen kinematik und mechanik. Z. Phys. **43**, 172–198 (1927)
10. Huang, L., Kristály, A., Zhao, W.: Sharp uncertainty principles on general Finsler manifolds. Trans. Amer. Math. Soc. **373**, 8127–8161 (2020)
11. Juillet, N.: Geometric inequalities and generalized Ricci bounds in the Heisenberg group. Int. Math. Res. Not. IMRN **13**, 2347–2373 (2009)
12. Kombe, I., Özaydin, M.: Improved Hardy and Rellich inequalities on Riemannian manifolds. Trans. Amer. Math. Soc. **361**, 6191–6203 (2009)
13. Kombe, I., Özaydin, M.: Hardy-Poincaré, Rellich and uncertainty principle inequalities on Riemannian manifolds. Trans. Amer. Math. Soc. **365**, 5035–5050 (2013)
14. Kristály, A.: Metric measure spaces supporting Gagliardo-Nirenberg inequalities: volume non-collapsing and rigidities. Calc. Var. Partial Differ. Equ. **55**, 27 (2016)
15. Kristály, A.: Sharp uncertainty principles on Riemannian manifolds: the influence of curvature. J. Math. Pures Appl. **119**(9), 326–346 (2018)
16. Kristály, A., Ohta, S.: Caffarelli-Kohn-Nirenberg inequality on metric measure spaces with applications. Math. Ann. **357**, 711–726 (2013)
17. Lieb, E.: Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities. Ann. of Math. **118**(2), 349–374 (1983)
18. Lin, C.: Interpolation inequalities with weights. Comm. Partial Differ. Equ. **11**, 1515–1538 (1986)
19. Martín, J., Milman, M.: Isoperimetric weights and generalized uncertainty inequalities in metric measure spaces. J. Funct. Anal. **270**, 3307–3343 (2016)
20. Ohta, S.: On the measure contraction property of metric measure spaces. Comment. Math. Helv. **82**, 805–828 (2007)
21. Okoudjou, K., Saloff-Coste, L., Teplyaev, A.: Weak uncertainty principle for fractals, graphs and metric measure spaces. Trans. Amer. Math. Soc. **360**, 3857–3873 (2008)

22. Rajala, T., Sturm, K.-T.: Non-branching geodesics and optimal maps in strong $CD(K, \infty)$-spaces. Calc. Var. Partial Differ. Equ. **50**, 831–846 (2014)
23. Sturm, K.-T.: On the geometry of metric measure spaces II. Acta Math. **196**, 133–177 (2006)
24. Weyl, H.: The Theory of Groups and Quantum Mechanics. Dover Publications, New York (1931)
25. Xia, C.: The Caffarelli-Kohn-Nirenberg inequalities on complete manifolds. Math. Res. Lett. **14**, 875–885 (2007)