On the geometry of Wasserstein barycenter I

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Abstract

We study the Wasserstein barycenter problem in the setting of infinite-dimensional, non-proper, non-smooth extended metric measure spaces. We introduce a couple of new concepts and study the existence, uniqueness, absolute continuity of the barycenter, and prove Jensen's inequality in an abstract framework. This generalized several results on Euclidean space, Riemannian manifolds and Alexandrov spaces to metric measure spaces satisfying Riemannian Curvature-Dimension condition à la Lott–Sturm–Villani, some extended metric measure spaces including abstract Wiener spaces.

We also introduce a relaxation of the CD condition, we call the Barycenter-Curvature-Dimension condition BCD. We prove its stability under measured-Gromov–Hausdorff convergence and prove the existence of the Wasserstein barycenter under this new condition. In addition, we get some inequalities including a multi-marginal Brunn–Minkowski inequality and a functional Blaschke–Santaló type inequality.

Keywords: Wasserstein barycenter, metric measure space, curvature-dimension condition, Ricci curvature, optimal transport

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Contents

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1 Introduction

This paper has two goals. The first one is to extend results about Wasserstein barycenter problem from the setting of smooth Riemannian manifolds to the setting of non-smooth extended metric measure spaces. The second one is to use Wasserstein barycenter to give a notion for a metric measure space to have Ricci curvature bounded from below. We refer to [BBI01] and [Vil09] for basic material on metric geometry, optimal transport and synthetic theory of curvature bounds. In the introduction, we motivate the questions that we address and we summarize the main results.

Wasserstein barycenter

Let Ω be a Borel probability measure on a complete metric space (Y, d_Y), a barycenter of Ω is defined as a minimizer of

$$x \mapsto \int_Y \mathsf{d}_Y^2(x, y)\,\mathrm{d}\Omega(y)$$

and the variance of Ω is defined as

$$\mathrm{Var}(\Omega) := \inf_{x\in X} \int_Y \mathsf{d}_Y^2(x,y)\,\mathrm{d}\Omega(y).$$

In general, existence or uniqueness is uncertain. In some cases, for example when (Y, d_Y) is NPC (metric spaces of non-positive curvature), we have existence and uniqueness of the barycenter.

A bridge, called Jensen's inequality, between convexity and probability theory was firstly established by Jensen in the seminal paper [Jen06] and had been highly concerned and extensively explored after that. The validity of Jensen's inequality in a general metric space can be formulated in the following way. Let Ω be a probability measure on Y, $K \in \mathbb{R}$. We say that a function $F : Y \to \mathbb{R} \cup \{+\infty\}$ is weakly (strongly) *K-barycentrically convex* (cf. Definition 2.7), if there is (for any) barycenter $\bar{x}$ of Ω, it holds

$$F(\bar{x}) \le \int_Y F(x)\,\mathrm{d}\Omega(x) - \frac{K}{2}\mathrm{Var}(\Omega). \tag{JI}$$

This inequality has been widely studied in the context of Riemannian manifolds [ÉM06, Afs11], CAT spaces [Stu03, Ken90, Kuw97, Yok16], Alexandrov spaces and convex metric spaces [Kuw14].

We say that (X, d) is an extended metric space if $\mathsf{d} : X \times X \to [0, +\infty]$ is a symmetric function satisfying the triangle inequality, with $\mathsf{d}(x, y) = 0$ if and only if $x = y$. Given a Hausdorff topology τ on X and the Borel σ-algebra $\mathcal{B}(\tau)$. Denote by $\mathcal{P}(X)$ the set of Radon probability measures on X, and by $\mathcal{P}_2(X, \mathsf{d})$ the set of probability measures with finite second order moment, i.e. $\mu \in \mathcal{P}_2(X, \mathsf{d})$ if and only if $\mu \in \mathcal{P}(X)$ and $\int \mathsf{d}^2(x, x_0)\,\mathrm{d}\mu(x) < \infty$ for some $x_0 \in X$.

Consider the Wasserstein space $\mathcal{W}_2 := (\mathcal{P}(X), W_2)$ equipped with the so-called L^2-transport distance or 2-Wasserstein distance W_2, defined by

$$W_2^2(\mu, \nu) := \inf_{\Pi} \int \mathsf{d}^2(x, y)\,\mathrm{d}\Pi(x, y)$$

where the infimum is taken among all transport plans Π with marginals μ, ν.

It is known that W_2 is an extended metric on $\mathcal{P}(X)$ (see [AES16, Proposition 5.3]), and W_2 is a metric on $\mathcal{P}_2(X, \mathsf{d})$ if (X, d) is a metric space (see [AG11, Theorem 2.2]). Therefore it makes sense to talk about barycenters in the Wasserstein space. This problem, called Wasserstein barycenter problem, draws particular interests, as it gives a natural but non-linear way to interpolate between a distribution of measures.

Definition 1.1 (Wasserstein barycenter)**.** Let (X, d) be an extended metric space and let $\Omega \in \mathcal{P}_2(\mathcal{P}(X), W_2)$ be a probability measure on the Wasserstein space $(\mathcal{P}(X), W_2)$ with finite variance. We call $\bar{\nu} \in \mathcal{P}(X)$ a Wasserstein barycenter of Ω if

$$\int_{\mathcal{P}(X)} W_2^2(\mu, \bar{\nu})\,\mathrm{d}\Omega(\mu) = \min_{\nu \in \mathcal{P}(X)} \int_{\mathcal{P}(X)} W_2^2(\mu, \nu)\,\mathrm{d}\Omega(\mu).$$

It is worth to mention that, if (X, d) is a geodesic space, any Wasserstein barycenter of $\Omega = \frac{1}{2}\delta_{\mu_0} + \frac{1}{2}\delta_{\mu_1}$ is exactly a mid-point of μ_0, μ_1.

The study of the Wasserstein barycenter problem was initiated by Agueh and Carlier [AC11] in 2011, and got a lot of attention from experts in various fields in the last decade. In mathematical economic, Carlier–Ekeland [CE10] studied team matching problems by considering the interpolation between multiple probability measures, which can be seen as a generalization of Wasserstein barycenter with square of Wasserstein distance replaced by the general cost. From the perspective of statistic, Wasserstein barycenter can be understood as the mean, which is a central topic when dealing with a large number of data, of a data sample composed of probability measures and thus shows its application value in data science [ABA22], image processing [RPDB12] and statistics [BK13].

In metric measure geometry, Agueh–Carlier [AC11] in the Euclidean setting, Kim–Pass [KP17] and Ma [Ma23] in the Riemannian setting, Jiang [Jia17] in the Alexandrov spaces, established the existence and the uniqueness of Wasserstein barycenter, and the absolutely continuity of the Wasserstein barycenter with respect to the canonical reference measures in these cases. **As a corollary**, they prove the following Wasserstein Jensen's inequality, which is a generalized convexity, called barycenter convexity (cf. Definition 2.7): for any K-displacement convex functional $\mathcal{F} : \mathcal{P}_2(X, \mathsf{d}) \to \mathbb{R} \cup \{+\infty\}$ in the sense of McCann [McC97], a probability measure on $\mathcal{P}_2(X, \mathsf{d})$ and its unique barycenter $\bar{\mu}$, it holds

$$\mathcal{F}(\bar{\mu}) \leq \int_{\mathcal{P}_2(X,\mathsf{d})} \mathcal{F}(\mu)\, \mathrm{d}\Omega(\mu) - \frac{K}{2} \int_{\mathcal{P}_2(X,\mathsf{d})} W_2^2(\bar{\mu}, \mu)\, \mathrm{d}\Omega(\mu). \tag{WJI}$$

The first goal of this paper is to extend results [AC11, KP17, Ma23, Jia17] about Wasserstein barycenter, including the key properties **existence, uniqueness, and regularity**, to the setting of metric measure spaces. The main difficulties to achieve this aim, comparing with the previous results, lies in the fact that

- all the known existence results depend on local compactness, which is not available for non-compact spaces such as infinite-dimensional RCD spaces or abstract Wiener spaces;
- in the Euclidean setting, regularity relies on the special geometric and algebraic structure of the space;
- in the case of Riemannian manifolds and Alexandrov spaces, sectional curvature bounds play important roles.

Therefore the known methods are difficult to be realized, in the general metric measure space. To overcome these difficulties, we will show that

> **regularity of the Wasserstein barycenter is a consequence, other than a necessary condition of the Jensen's inequality**.

Furthermore, as one of our main innovations in this paper, we will show

Jensen's inequality plays the role of "a priori estimate" in the theory of partial differential equations.

Our idea is to take advantage of gradient flow theory to study Jensen's inequality (JI), and Wasserstein barycenter problem. Motivated by a work of Daneri and Savaré [DS08], we realize that a special formulation of gradient flows, called **Evolution Variational Inequality**, implies the existence of the barycenter and Jensen's inequality. Consequently, we give a direct, simple proof of Wasserstein Jensen's inequality using the theory of gradient flows on Wasserstein spaces. In particular, our proof is synthetic, dimension-free and does not rely on the properness of (X, d).

More precisely, let F be a lower semi-continuous function on a general extended metric space (X, d), such that any point x_0 with finite distance to the domain of F is the starting point of an EVI_K-type gradient flow (x_t) of F satisfying

$$\varliminf_{t\to 0} F(x_t) \geq F(x_0), \quad \lim_{t\to 0} \mathsf{d}(x_t, y) \to \mathsf{d}(x_0, y), \quad \forall y \in \overline{\mathrm{D}(F)}$$

and

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\frac{1}{2}\mathsf{d}^2(x_t, y) + \frac{K}{2}\mathsf{d}^2(x_t, y) \leq F(y) - F(x_t), \quad \forall t > 0$$

for all $y \in \mathrm{D}(F)$ satisfying $\mathsf{d}(y, x_t) < \infty$ for some (and then all) $t \in (0, \infty)$, where $\frac{\mathrm{d}^+}{\mathrm{d}t}$ denotes the upper right derivative. Then for barycenter $\bar{x}$ of Ω, by considering the gradient flow from $\bar{x}$, we can establish Jensen's type inequality (JI) for this function F.

Examples: The property that the gradient flow of a K-convex function E is EVI_K type, is valid on Euclidean spaces, Hilbert spaces, Alexandrov spaces, CAT spaces. For the Wasserstien space $(\mathcal{P}(X), W_2)$, this property is valid, when X is an Euclidean space, a smooth Riemannian manifold with uniform Ricci lower bound, an Alexandrov space with lower curvature bound, a Wiener spaces or an RCD space (see §3.2). We refer the readers to [AGS05, MS20, AES16] for more discussions on this topic.

In particular, for Wasserstein barycenter problem, we prove the following existence theorems without local compactness or any other fine local structure of the underling space (X, d). Note that if (X, d) is an extended metric space, the definition of $\mathcal{P}_2(X, \mathsf{d})$ may depends on the choice of base point, which is less meaningful. Thus we will write the hypothesis for metric space and extended metric space separately.

Theorem 1.2 (Existence of barycenter, Theorem 5.7)**.** *Let $K \in \mathbb{R}$, $(X, \mathsf{d}, \mathfrak{m})$ be an extended metric measure space, Ω be a probability measure over $\mathcal{P}(X)$ with finite variance. Then Ω has a barycenter if one of the following conditions holds.*

- **A.** *$(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, \infty)$ metric measure space and Ω is concentrated on $\mathcal{P}_2(X, \mathsf{d})$;*
- **B.** *$\mathfrak{m}$ is a probability measure, any $\mu \in \mathcal{P}(X)$ which has finite distance from $\mathrm{D}(\mathrm{Ent}_{\mathfrak{m}})$ is the starting point of an EVI_K gradient flow of $\mathrm{Ent}_{\mathfrak{m}}$ in the Wasserstein space.*

Furthermore, we prove Wasserstein Jensen's inequality. As a corollary, we get the absolute continuity of the Wasserstein barycenter.

Theorem 1.3 (Wasserstein Jensen's inequality, Theorem 5.3)**.** *Assume $(X, \mathsf{d}, \mathfrak{m})$ and Ω satisfy the hypothesis in Theorem 1.2.*

Let $\bar{\mu}$ be a barycenter of Ω. Then Wasserstein Jensen's inequality follows:

$$\mathrm{Ent}_{\mathfrak{m}}(\bar{\mu}) \leq \int_{\mathcal{P}(X)} \mathrm{Ent}_{\mathfrak{m}}(\mu)\,\mathrm{d}\Omega(\mu) - \frac{K}{2}\int_{\mathcal{P}(X)} W_2^2(\bar{\mu}, \mu)\,\mathrm{d}\Omega(\mu). \tag{1.1}$$

Furthermore, if

$$\int_{\mathcal{P}(X)} \mathrm{Ent}_{\mathfrak{m}}(\mu)\,\mathrm{d}\Omega(\mu) < \infty,$$

then the entropy of the barycenter of Ω is finite. In particular, the barycenter is absolutely continuous with respect to $\mathfrak{m}$.

Applying our theorem with Wasserstein spaces over RCD spaces, we get the following existence, uniqueness and regularity theorem.

Theorem 1.4 (Theorem 5.8)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, \infty)$ space with $K \in \mathbb{R}$. Let Ω be a probability measure on $\mathcal{P}_2(X, \mathsf{d})$ with finite variance and*

$$\int_{\mathcal{P}_2(X,\mathsf{d})} \mathrm{Ent}_{\mathfrak{m}}(\mu)\,\mathrm{d}\Omega(\mu) < \infty.$$

Then Ω has a unique barycenter $\bar{\mu}$ and it holds the Wasserstein Jensen's inequality

$$\mathrm{Ent}_{\mathfrak{m}}(\bar{\mu}) \leq \int_{\mathcal{P}_2(X,\mathsf{d})} \mathrm{Ent}_{\mathfrak{m}}(\mu)\,\mathrm{d}\Omega(\mu) - \frac{K}{2}\mathrm{Var}(\Omega).$$

Similarly, using a finite dimensional formulation of the gradient flow called $\mathrm{EVI}_{K,N}$ gradient flow, we prove a finite dimensional version of Jensen's inequality, which seems new in the literature even on $\mathbb{R}^n$.

Theorem 1.5 (Jensen's inequality with dimension parameter, Theorem 5.10, Corollary 5.11 and Corollary 5.12)**.** *Let $N \in [1, \infty)$. Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ metric measure space and Ω be a Borel probability measure over $\mathcal{P}_2(X, \mathsf{d})$. Then the following dimensional Wasserstein Jensen's inequality follows:*

$$\int \frac{W_2(\bar{\mu}, \mu)}{s_{K/N}(W_2(\bar{\mu}, \mu))} U_N(\mu)\,\mathrm{d}\Omega(\mu) \leq U_N(\bar{\mu}) \int \frac{W_2(\bar{\mu}, \mu)}{t_{K/N}(W_2(\bar{\mu}, \mu))}\,\mathrm{d}\Omega(\mu), \tag{1.2}$$

where $\bar{\mu}$ is the unique barycenter of Ω, $s_{K/N}$ and $t_{K/N}$ are distortion coefficients, $U_N(\mu) = e^{-\frac{\mathrm{Ent}_{\mathfrak{m}}(\mu)}{N}}$.

In particular, if Ω gives mass to the set $\{\mu, \mathrm{Ent}_{\mathfrak{m}}(\mu) < \infty\}$. Then the entropy of the barycenter of Ω is finite and the barycenter $\bar{\mu}$ is absolutely continuous with respect to $\mathfrak{m}$.

Barycenter-Curvature-Dimension condition

The second goal of this paper is to provide a new notion, in therms of Wasserstein barycenters, for an extended metric measure spaces to have synthetic Ricci curvature bounded below and dimension bounded above.

There are various approaches to extend notions of curvature from smooth Riemannian manifolds to more general spaces. A good notion of a length space having "sectional curvature bounded below" is Alexandrov space, which is defined in terms of Toponogov's comparison theorem concerning geodesic triangles. More recently, a notion of a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ having "Ricci curvature bounded below" was introduced by Sturm [Stu06a, Stu06b] and Lott–Villani [LV09] independently. This new synthetic curvature-dimension condition is defined in terms of a notion of geodesic convexity for functionals on the Wasserstein space $(\mathcal{P}_2(X, \mathsf{d}), W_2)$, called displacement convexity, introduced by McCann in his celebrated paper [McC97]. This theory, called Lott–Sturm–Villani theory today, has been widely used in the study of Ricci limit spaces, geometric and functional inequalities and many areas in applied mathematics.

From [KP17], we know that the Wasserstein Jensen's inequality of certain functional is equivalent to lower Ricci curvature bounds in the setting of Riemannian manifolds. In Section 5 we find more examples satisfying the Wasserstein Jensen's inequality, including non-compact spaces such as RCD spaces and extended metric spaces such as abstract Wiener spaces. Based on these results, it is natural to provide a notion for general extended metric measure spaces to have Ricci curvature bounded from below, via the Wasserstein Jensen's inequality. This approach is surely compatible with Lott–Sturm–Villani theory and has its own highlights and interests.

Definition 1.6 (BCD(K, ∞) condition, Definition 6.1)**.** Let $K \in \mathbb{R}$. We say that an **extended** metric measure space $(X, \mathsf{d}, \mathfrak{m})$ verifies BCD(K, ∞) condition, if for any probability measure $\Omega \in \mathcal{P}_2(\mathcal{P}(X), W_2)$, concentrated on **finitely many** measures, there exists a barycenter $\bar{\mu}$ of Ω such that the following Jensen's inequality holds:

$$\mathrm{Ent}_{\mathfrak{m}}(\bar{\mu}) \leq \int_{\mathcal{P}(X)} \mathrm{Ent}_{\mathfrak{m}}(\mu)\, \mathrm{d}\Omega(\mu) - \frac{K}{2}\mathrm{Var}(\Omega). \tag{1.3}$$

The next theorem, says that the moduli space of compact metric measure spaces satisfying barycenter curvature-dimension condition is closed in measured Gromov–Hausdorff convergence.

Theorem 1.7 (Stability in measured Gromov–Hausdorff topology, Theorem 6.6)**.** *Let* $\{(X_i, \mathsf{d}_i, \nu_i)\}_{i=1}^{\infty}$ *be a sequence of compact* BCD(K, ∞) *metric measure spaces with* $K \in \mathbb{R}$*. If* $\{(X_i, \mathsf{d}_i, \nu_i)\}$ *converges to* (X, d, ν) *in the measured Gromov–Hausdorff sense as* $n \to \infty$*, then* (X, d, ν) *is also a* BCD(K, ∞) *space.*

Our theory has various applications, one of the most important and surprising ones is the resolvability of the Wasserstein barycenter problem on BCD spaces. Note that a BCD space does not necessarily have a Riemannian structure, the scope of our theorem is far beyond Ricci-limit and RCD spaces.

Theorem 1.8 (Existence of Wasserstein barycenter, Theorem 6.7)**.** *Let* $(X, \mathrm{d}, \mathfrak{m})$ *be an extended metric measure space satisfying* $\mathrm{BCD}(K, \infty)$ *curvature-dimension condition,* Ω *be a probability measure on* $\mathcal{P}(X)$ *satisfying*

$$\mathrm{Var}(\Omega) < \infty \qquad \textit{and} \qquad \int_{\mathcal{P}(X)} \mathrm{Ent}_{\mathfrak{m}}(\mu) \, \mathrm{d}\Omega(\mu) < \infty.$$

Then Ω *has a barycenter if one of the following conditions holds.*

A. $(X, \mathsf{d}, \mathfrak{m})$ *satisfies the exponential volume growth condition and* Ω *is concentrated on* $\mathcal{P}_2(X, \mathsf{d})$*;*

B. $\mathfrak{m}$ *is a probability measure.*

In the definition of $\mathrm{BCD}(K, \infty)$ condition, there is no parameter for the dimension. This indicates that we need to specify a dimension parameter in order to define a "finite dimensional" BCD space. Following Theorem 1.5, it is natural to define:

Definition 1.9 ($\mathrm{BCD}(K, N)$ condition, Definition 6.3)**.** Let $K \in \mathbb{R}, N > 0$. We say that a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ verifies $\mathrm{BCD}(K, N)$ condition, if for any probability measure $\Omega \in \mathcal{P}_2(\mathcal{P}(X), W_2)$, concentrated on **finitely many** measures, there exists a barycenter $\bar{\mu}$ of Ω such that the following Jensen-type inequality holds:

$$\int \frac{W_2(\bar{\mu}, \mu)}{s_{K/N}(W_2(\bar{\mu}, \mu))} U_N(\mu) \, \mathrm{d}\Omega(\mu) \leq U_N(\bar{\mu}) \int \frac{W_2(\bar{\mu}, \mu)}{t_{K/N}(W_2(\bar{\mu}, \mu))} \, \mathrm{d}\Omega(\mu), \tag{1.4}$$

where $U_N(\mu) = e^{-\frac{\mathrm{Ent}_{\mathfrak{m}}(\mu)}{N}}$.

We will see in § 6.4 that BCD condition implies several geometric and functional inequalities. In addition, by letting one (or more) marginal measure be Dirac mass, we can propose a variant of "Measure Contraction Property" (cf. [Oht07b, Stu06b]) in the setting of $\mathrm{BCD}(K, N)$. We will study the geometric and analysis consequence of this property in a forthcoming paper.

Multi-marginal optimal transport problem

Given $\mu_1, ..., \mu_n \in \mathcal{P}(X)$ and a lower semi-continuous cost function $c : X^n \to \mathbb{R}$. The multi-marginal optimal transport problem of Monge type is to minimize

$$\inf_{T_2,\ldots,T_n} \int_X c(x_1, T_2(x_1), \ldots, T_n(x_1)) \, \mathrm{d}\mu_1(x_1), \tag{MP}$$

among $(n-1)$-tuples of map $(T_2, \ldots, T_n)$, such that for each $i = 2, \ldots, n$, the map $T_i : X \to X$ pushes the measure μ_1 forward to μ_i; that means, for any Borel $A \subseteq X$, $\mu_1(T_i^{-1}(A)) = \mu_i(A)$.

The multi-marginal optimal transport problem of Kantorovich type is to solve

$$\inf_{\pi \in \Pi} \int_{X^n} c(x_1, \ldots, x_n) \, \mathrm{d}\pi(x_1, \ldots, x_n), \tag{KP}$$

where the infimum is taken over all probability measures π on X^n whose marginals are $\mu_1, \ldots, \mu_n$, denote by $\pi \in \Pi(\mu_1, \ldots, \mu_n)$.

When $n = 2$ and $c(x_1, x_2) = \mathsf{d}(x_1, x_2)$ or $\mathsf{d}^2(x_1, x_2)$, (MP) and (KP) correspond to the Monge and Kantorovich problems in classical optimal transport theory respectively. This problem has been studied extensively over the past 25 years, which is one of the fundamental problems in the study of spaces satisfying synthetic curvature-dimension condition à la Lott–Sturm–Villani, see [Gig12a, RS14, CM17, GRS15]. In particular,during the study of this problem, many important by-products have been produced, including the notion of essentially non-branching.

We will prove uniqueness and resolvability of Monge problem (MP) for the multi-marginal optimal transport problem in the setting of metric measure space, with cost function

$$c(x_1, \ldots, x_n) := \inf_{y \in X} \sum_{i=1}^{n} \frac{1}{2} \mathsf{d}^2(x_i, y). \tag{1.5}$$

Alternatively, we will show that (MP) = (KP). It is worth mentioning that this problem was solved by Gangbo and Swiech [GŚ98] in the Euclidean setting, and generalized by Kim and Pass [KP15] in the Riemannian setting and Jiang [Jia17] in the Alexandrov spaces. The link between multi-marginal optimal transport with cost function (1.5) and the Wasserstein barycenter problem associated with the measures $\mu_1, \mu_2, \ldots, \mu_n$ was discovered by Agueh and Carlier [AC11] in the Euclidean case.

Our theorem is the following.

Theorem 1.10 (Existence and uniqueness of multi-marginal optimal transport map, Theorem 5.15)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space. Then the multi-marginal optimal transport problem of Monge type, associated with the cost function $c(x_1, \ldots, x_n)$, has a unique solution, if one of the following conditions holds:*

(i) *$(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, N)$ space, and $\mu_1 \ll \mathfrak{m}$.*

(ii) *$(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, \infty)$ space, and $\mu_i \ll \mathfrak{m}, i = 1, \ldots, n$.*

This theorem is, to the best of our knowledge, the first of this kind for multi-marginal optimal transport problems without using any of the local structure of the underling space; and the first result concerning infinite-dimensional spaces. Furthermore, we prove the existence, uniqueness and absolute continuity of the Wasserstein barycenter of measure with finite support, without the finite entropy condition. This generalized some results concerning Wasserstein geodesics, by T. Rajala and his co-authors [Raj12, Raj13, RS14, GRS15], to Wasserstein barycenters.

Theorem 1.11 (Theorem 5.16)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space. Assume $\mu_1, \ldots, \mu_n \in \mathcal{P}_2(X, \mathsf{d})$, then there exists a unique Wasserstein barycenter $\bar{\mu}$ and it is absolutely continuous with respect to $\mathfrak{m}$ if one of the following conditions holds:*

(i) *$(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, N)$ space, and $\mu_1 \ll \mathfrak{m}$.*

(ii) *$(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, \infty)$ space, and $\mu_i \ll \mathfrak{m}, i = 1, \ldots, n$.*

Organization of the paper: In Section 3, we collect some preliminaries in the theory of metric measure spaces, optimal transport and curvature-dimension condition. In Section 4 we introduce some basic results, concerning existence and uniqueness of the Wasserstein barycenter. Section 5 is devoted to proving the main theorems. In Section 6, we introduce the concept of barycenter curvature-dimension condition, and apply our theory to prove some geometric inequalities. More detailed descriptions appear at the beginning of each section.

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2 Geodesic and barycenter

Let us summarize some definitions and basic results about (extended) metric spaces. For proofs and further details we refer to the text books [AT04] and [BBI01]. Throughout this section, we denote by $X = (X, \tau)$ a Hausdorff topological space.

2.1 Geodesic spaces

Let d be a metric on X, this means $\mathsf{d} : X \times X \to [0, +\infty)$ is a symmetric function satisfying the triangle inequality, with $\mathsf{d}(x, y) = 0$ if and only if $x = y$. Let γ be a curve in X, i.e. a continuous map from $[0, 1]$ to X, its length is defined by

$$L(\gamma) := \sup_{J\in\mathbb{N}} \sup_{0=t_0\leq t_1\leq \ldots \leq t_J=1} \sum_{j=1}^{J} \mathsf{d}(\gamma_{t_{j-1}}, \gamma_{t_j}).$$

Clearly $L(\gamma) \geq \mathsf{d}(\gamma_0, \gamma_1)$. We say that X is a length space, if the distance between two points $x, y \in X$ is the infimum of the lengths of curves from x to y. Such a space is path connected.

We denote by

$$\mathrm{Geo(X)} := \Big\{\gamma \in \mathrm{C([0, 1], X)} : \mathsf{d}(\gamma_{\mathrm{s}}, \gamma_{\mathrm{t}}) = |\mathrm{s} - \mathrm{t}|\mathsf{d}(\gamma_0, \gamma_1),\ \text{for every } \mathrm{s}, \mathrm{t} \in [0, 1]\Big\}$$

the space of constant speed geodesics, equipped with the canonical supremum norm. The metric space (X, d) is called a geodesic space if for each $x, y \in X$ there exists $\gamma \in \mathrm{Geo(X)}$ so that $\gamma_0 = x, \gamma_1 = y$. This curve is **not** required to be unique.

The following lemma is a characterization of geodesic space using midpoints.

Lemma 2.1. *A complete metric space (X, d) is a geodesic space if and only if for any $x, y \in X$, there is $z \in X$ such that*

$$\mathsf{d}(x, z) = \mathsf{d}(x, z) = \frac{1}{2}\mathsf{d}(x, y).$$

Any point $z \in X$ with the above properties will be called midpoint of x and y.

In addition, we have the following characterization of the completeness of a length space.

Lemma 2.2 (Hopf–Rinow)**.** *Let (X, d) be a complete length space, then:*

(1) The closure of $B_r(x)$, the open ball of radius r around $x \in X$, is the closed ball $\{y \in X : \mathsf{d}(x, y) \leq r\}$;

(2) X is locally compact if and only if each closed ball is compact;

(3) If X is locally compact, then it is a geodesic space.

2.2 Extended metric spaces

In this subsection we introduce the notion of extended metric space. Abstract Wiener spaces and configuration spaces over Riemannian manifolds are particular examples of extended metric spaces.

It can be seen from [AES16] that most of the analytic tools in metric spaces can be extend in a natural way to extended metric spaces.

Definition 2.3 (Extended metric spaces)**.** We say that (X, d) is an extended metric space if $\mathsf{d} : X \times X \to [0, +\infty]$ is a symmetric function satisfying the triangle inequality, with $\mathsf{d}(x, y) = 0$ if and only if $x = y$.

Since an extended metric space can be seen as the disjoint union of the equivalence classes induced by the equivalence relation

$$x \backsim y \iff \mathsf{d}(x, y) < +\infty.$$

and since any equivalence class is indeed a metric space, many results and definitions extend with no effort to extended metric spaces. For example, we say that an extended metric space (X, d) is complete (resp. geodesic, length,...) if all metric spaces $X[x] = \{y : y \backsim x\}$ are complete (resp. geodesic, length,...).

2.3 Barycenter spaces

Let (X, d) be an extended metric space and τ be a Hausdorff topology on X. We assume that τ and d are compatible, in the sense of [AES16, Definition 4.1]. In this case, we say that (X, τ, d) is an extended metric-topological space.

Let $\mathcal{B}(\tau)$ be the Borel σ-algebra of τ and let $\mathcal{P}(X)$ be the set of Radon probability measures on X. Denote by $\mathcal{P}_2(X, \mathsf{d})$ the set of $\mu \in \mathcal{P}(X)$ such that

$$\int \mathsf{d}^2(x_0, y)\, \mathrm{d}\mu(y) < \infty \quad \text{for some} \quad x_0 \in X.$$

In general, the choice of such x_0 is not arbitrary. If (X, d) is a metric space, $\int \mathsf{d}^2(x_0, y)\, \mathrm{d}\mu(y) < \infty$ for some $x_0 \in X$ if and only if $\int \mathsf{d}^2(x, y)\, \mathrm{d}\mu(y) < \infty$ for any $x \in X$. For $\mu \in \mathcal{P}(X)$, the value $\mathrm{Var}(\mu) := \inf_{x \in X} \int \mathsf{d}^2(x, y)\, \mathrm{d}\mu(y) \in [0, +\infty]$ is called the *variance* of μ. We can see that $\mu \in \mathcal{P}_2(X, \mathsf{d})$ if and only if $\mathrm{Var}(\mu) < \infty$.

We also denote by $\mathcal{P}_0(X)$ the set of all $\mu \in \mathcal{P}(X)$ of the form $\mu = \frac{1}{n} \sum_{i=1}^n \delta_{x_i}$ with finite points $x_i \in X$. Here and henceforth, δ_x denotes the Dirac measure on the point $x \in X$.

Definition 2.4 (Barycenter)**.** Let (X, d) be an extended metric space and let $\mu \in \mathcal{P}_2(X, \mathsf{d})$ be a probability measure with finite variance. We call $\bar{x} \in X$ a barycenter of μ if

$$\int_X \mathsf{d}^2(\bar{x}, z)\, \mathrm{d}\mu(z) = \min_{x \in X} \int \mathsf{d}^2(x, y)\, \mathrm{d}\mu(y) < \infty.$$

Now we can introduce the notion of barycenter space. We remark that there is a different notion of "barycenter metric space", as a generalization of Hadamard space, introduced by Lee and Naor in [LN05].

Definition 2.5 (Barycenter space)**.** We say that an extended metric space is a *barycenter extended metric space*, or *barycenter space* for simplicity, if any $\mu \in \mathcal{P}_0(X)$ with finite variance has a barycenter.

Here we list a couple of examples which are barycenter spaces. More discussions can be found in [Stu03], [NS11], [Kuw14] and [Oht12, Oht07a].

Example 2.6. *The following spaces are barycenter spaces:*

- *proper spaces, or locally compact geodesic spaces;*
- *metric spaces with non-positive curvature (NPC spaces), this includes complete, simply connected Riemannian manifold with non-positive (sectional) curvature and Hilbert spaces;*
- *uniform convex metric spaces, introduced by James A. Clarkson [Cla36] in 1936, this includes uniformly convex Banach spaces such as L^p spaces with $p > 1$;*
- *abstract Wiener space (X, H, μ) where X is a Banach space that contains a Hilbert space H as a dense subspace, equipped with the Cameron–Martin distance*
$$\mathsf{d}_H(x, y) := \begin{cases} \|x - y\|_H & \text{if } x - y \in H \\ +\infty & \text{otherwise.} \end{cases}$$
- *Wasserstein spaces over Riemannian manifolds [KP17] and Alexandrov spaces [Jia17].*

2.4 Convex functions

By definition, a subset $\Omega \subset X$ is convex if for any $x, y \in \Omega$, there **is** a geodesic from x to y that lies entirely in Ω. It is totally convex if for any $x, y \in \Omega$, **any** geodesic in X from x to y lies in Ω. Given $K \in \mathbb{R}$, a function $F : X \to \mathbb{R} \cup \{+\infty\}$ is said to be weakly (strongly) *K-geodesically convex* if there is (for any) geodesic $(\gamma_t)_{t \in [0,1]}$ and any $t \in [0, 1]$,

$$F(\gamma_t) \le tF(\gamma_1) + (1 - t)F(\gamma_0) - Kt(1 - t)\mathsf{d}^2(\gamma_0, \gamma_1).$$

In the case when (X, d) is the Euclidean space $\mathbb{R}^n$ equipped with the Euclidean norm, and $F \in C^2(X)$, this is equivalent to say that $\mathrm{Hess}_F \ge K$.

With the notion of barycenter, we introduce a notion called barycenter convexity. It can be seen that this convexity is equivalent to the geodesic convexity, if (X, d) is the Euclidean space.

Definition 2.7 (Barycenter convexity)**.** Let (X, d) be a barycenter space. Given $K \in \mathbb{R}$, a function $F : X \to \mathbb{R} \cup \{+\infty\}$ is said to be weakly (strongly) K-*barycentrically convex* if for any $\mu \in \mathcal{P}(X)$ with finite variance, there is a (for any) barycenter $\bar{x}$ of μ, such that

$$F(\bar{x}) \leq \int F(x) \,\mathrm{d}\mu(x) - \frac{K}{2}\mathrm{Var}(\mu).$$

3 Geometry of the Wasserstein space

In this section, we add a reference measure in an extended metric-topological space as follows.

Definition 3.1 (Extended metric measure space, cf. [AES16])**.** We say that $(X, \mathsf{d}, \mathfrak{m})$ is an extended metric measure space if:

(a) (X, τ, d) is an extended metric-topological space and (X, d) is complete;

(b) $\mathfrak{m}$ is a non-negative Radon probability measure on $(X, \mathcal{B}(\tau))$ with full support.

3.1 Wasserstein barycenter and mult-marginal optimal transport

The L^2-Kantorovich-Wasserstein or L^2-optimal transport distance $W_2(\mu, \nu)$ between $\mu, \nu \in \mathcal{P}(X)$ is given by

$$W_2^2(\mu, \nu) := \inf_{\pi \in \Pi(\mu,\nu)} \int_{X \times X} \mathsf{d}^2(x, y) \,\mathrm{d}\pi(x, y), \tag{3.1}$$

where d is the extended metric and the infimum is taken over all probability Radon measures π on $X \times X$ whose marginals are μ and ν. It is known that W_2 is an extended metric on $\mathcal{P}(X)$ (in fact, $(\mathcal{P}_2(X, \mathsf{d}), W_2)$ is a metric space if (X, d) is a metric space, $(\mathcal{P}_2(X, \mathsf{d}), W_2)$ is geodesic if (X, d) is geodesic) and therefore it makes sense to talk about barycenters in the Wasserstein space $\mathcal{W}_2 := (\mathcal{P}(X), W_2)$.

Definition 3.2 (Wasserstein barycenter)**.** Let (X, d) be an extended metric space and let $\Omega \in \mathcal{P}_2(\mathcal{W}_2) = \mathcal{P}_2(\mathcal{P}(X), W_2)$ be a probability Radon measure on $\mathcal{P}(X)$ with finite variance, compatible with the extended metric W_2. We call $\bar{\nu} \in \mathcal{P}(X)$ a *Wasserstein barycenter* of Ω if

$$\int_{\mathcal{P}(X)} W_2^2(\mu, \bar{\nu}) \,\mathrm{d}\Omega(\mu) = \min_{\nu \in \mathcal{P}(X)} \int_{\mathcal{P}(X)} W_2^2(\mu, \nu) \,\mathrm{d}\Omega(\mu). \tag{3.2}$$

We will also study a multi-marginal version of optimal transport, which is a nature generalization of two-marginal optimal transport. This theory has attracted considerable attention, and has got wide applications in different subjects such as economics and statistics. In particular, Gangbo and Swiech [GŚ98] studied the multi-marginal optimal transport map in the Euclidean setting, Agueh–Carlier [AC11] studied Wasserstein barycenter problem via multi-marginal optimal transport, we refer to [Pas15] for a survey on this fast-developing topic.

Definition 3.3 (Multi-marginal optimal transport)**.** Let $\mu_1, ..., \mu_n \in \mathcal{P}(X)$ and let $c : X^n \to \mathbb{R}$ be a lower semi-continuous cost function. The multi-marginal optimal transport problem of Monge type is to minimize

$$\inf_{T_2,\dots,T_n} \int_X c(x_1, T_2(x_1), \dots, T_n(x_1))\, \mathrm{d}\mu_1(x_1), \tag{3.3}$$

where for each $i = 2, \dots, n$, the map $T_i : X \to X$ pushes the measure μ_1 forward to μ_i, that means, for any Borel $A \subseteq X$, $\mu_1(T_i^{-1}(A)) = \mu_i(A)$.

The multi-marginal optimal transport problem of Kantorovich type is to minimize

$$\inf_{\pi\in\Pi} \int_{X^n} c(x_1, \dots, x_n)\, \mathrm{d}\pi(x_1, \dots, x_n), \tag{3.4}$$

where the infimum is taken over all probability measures π on X^n whose marginals are $\mu_1, \dots, \mu_n$, denoted by $\Pi(\mu_1, \dots, \mu_n)$.

From [Vil09, Theorem 4.1, Lemma 4.4], we can see that the set $\Pi(\mu_1, \dots, \mu_n)$ is tight, so by Prohkhorov's theorem, the multi-marginal optimal transport plan always exists, this implies there exists $\theta \in \Pi(\mu_1, \dots, \mu_n)$, such that

$$\int_{X^n} c(x_1, \dots, x_n)\, \mathrm{d}\theta = \min_\pi \int_{X^n} c(x_1, \dots, x_n)\, \mathrm{d}\pi.$$

It is worth to mention that if we choose

$$c(x_1, \dots, x_n) = \inf_{y\in X} \sum_{i=1}^n \frac{1}{2}\mathsf{d}^2(x_i, y),$$

there is an intimate link between multi-marginal optimal transport and Wasserstein barycenter problem. This connection was studied by Gangbo–Swiech [GŚ98], Agueh–Carlier [AC11] in the Euclidean setting, by Kim–Pass [KP15, KP17] in Riemannian manifolds and by Jiang [Jia17] in Alexandrov spaces.

3.2 Curvature-dimension condition of metric measure spaces

Throughout this subsection, $(X, \mathsf{d}, \mathfrak{m})$ is a metric measure space, where (X, d) is a separable complete geodesic space and $\mathfrak{m}$ is a non-negative Borel measure with full support.

For any $t \in [0, 1]$, let e_t denote the evaluation map:

$$\mathrm{e}_t : \mathrm{Geo(X)} \to \mathrm{X}, \qquad \gamma \mapsto \gamma_\mathrm{t}.$$

By super-position theorem [AG11, Theorem 2.10], any geodesic $(\mu_t)_{t\in[0,1]}$ in the Wasserstein space $(\mathcal{P}_2(X,\mathsf{d}),W_2)$ can be lifted to a measure $\nu\in\mathcal{P}(\mathrm{Geo(X)})$, so that $(\mathrm{e}_t)_\sharp\,\nu=\mu_t$ for all $t\in[0,1]$. Given $\mu_0,\mu_1\in\mathcal{P}_2(X,\mathsf{d})$, we denote by $\mathrm{OptGeo}(\mu_0,\mu_1)$ the space of all $\nu\in\mathcal{P}(\mathrm{Geo(X)})$ for which $(\mathrm{e}_0,\mathrm{e}_1)_\sharp\,\nu$ realizes the minimum in the Kantorovich problem (3.1). Such a ν will be called dynamical optimal plan. Since (X,d) is geodesic, the set $\mathrm{OptGeo}(\mu_0,\mu_1)$ is non-empty for any $\mu_0,\mu_1\in\mathcal{P}_2(X,\mathsf{d})$.

Next we introduce the curvature-dimension condition of non-smooth metric measure space, which was introduced independently by Lott–Villani [LV09] and Sturm [Stu06a, Stu06b]. Recall that the relative entropy of $\mu\in\mathcal{P}_2(X,\mathsf{d})$ with respect to $\mathfrak{m}$ is defined by

$$\mathrm{Ent}_{\mathfrak{m}}(\mu):=\begin{cases}\int\rho\ln\rho\,\mathrm{d}\mathfrak{m} & \text{if } \mu=\rho\,\mathfrak{m}\\ +\infty & \text{otherwise.}\end{cases}$$

Definition 3.4 (CD(K,∞) condition)**.** Let $K\in\mathbb{R}$. A metric measure space $(X,\mathsf{d},\mathfrak{m})$ verifies CD(K,∞), if for any two $\mu_0,\mu_1\in\mathcal{P}_2(X,\mathsf{d})$ there exists $\nu\in\mathrm{OptGeo}(\mu_0,\mu_1)$, such that for all $t\in(0,1)$:

$$\mathrm{Ent}_{\mathfrak{m}}(\mu_t)\le(1-t)\mathrm{Ent}_{\mathfrak{m}}(\mu_0)+t\mathrm{Ent}_{\mathfrak{m}}(\mu_1)-\frac{K}{2}(1-t)tW_2^2(\mu_0,\mu_1),\tag{3.5}$$

where $\mu_t:=(e_t)_\sharp\nu$.

In order to formulate the curvature-dimension condition with finite dimensional parameter, we recall the definition of the distortion coefficients. For $\kappa\in\mathbb{R}$, define the functions $s_\kappa,c_\kappa:[0,+\infty)\to\mathbb{R}$ (on $[0,\pi/\sqrt{\kappa})$ if $\kappa>0$) as:

$$s_\kappa(\theta):=\begin{cases}(1/\sqrt{\kappa})\sin(\sqrt{\kappa}\theta), & \text{if } \kappa>0,\\ \theta, & \text{if } \kappa=0,\\ (1/\sqrt{-\kappa})\sinh(\sqrt{-\kappa}\theta), & \text{if } \kappa<0\end{cases}\tag{3.6}$$

and

$$c_\kappa(\theta):=\begin{cases}\cos(\sqrt{\kappa}\theta), & \text{if } \kappa>0,\\ 1, & \text{if } \kappa=0,\\ \cosh(\sqrt{-\kappa}\theta), & \text{if } \kappa<0.\end{cases}\tag{3.7}$$

It can be seen that s_κ is a solution to the following ordinary differential equation

$$s_\kappa''+\kappa s_\kappa=0.\tag{3.8}$$

For $K\in\mathbb{R},N\in[1,\infty),\theta\in(0,\infty),t\in[0,1]$, set we define the distortion coefficients $\sigma_{K,N}^{(t)}$ and $\tau_{K,N}^{(t)}(\theta)$ as

$$\sigma_{K,N}^{(t)}(\theta):=\begin{cases}\infty, & \text{if } K\theta^2\ge N\pi^2,\\ t & \text{if } K\theta^2=0,\\ \dfrac{s_{\frac{K}{N}}(t\theta)}{s_{\frac{K}{N}}(\theta)} & \text{otherwise}\end{cases}\tag{3.9}$$

and

$$\tau_{K,N}^{(t)}(\theta):=t^{1/N}\sigma_{K,N-1}^{(t)}(\theta)^{(N-1)/N}.\tag{3.10}$$

Recall that the Rényi Entropy functional $\mathcal{E}_N:\mathcal{P}_2(X,\mathsf{d})\to[0,\infty]$, if defined by

$$\mathcal{E}_N(\mu):=\begin{cases}\int\rho^{1-1/N}\,\mathrm{d}\mathfrak{m} & \text{if } \mu=\rho\,\mathfrak{m}\\ +\infty & \text{otherwise.}\end{cases}$$

Definition 3.5 (CD(K,N) condition)**.** Let $K \in \mathbb{R}$ and $N \in [1,\infty)$. A metric measure space $(X,\mathsf{d},\mathfrak{m})$ verifies CD(K,N) if for any two $\mu_0,\mu_1 \in \mathcal{P}_2(X,\mathsf{d})$ with bounded support there exist $\nu \in \mathrm{OptGeo}(\mu_0,\mu_1)$ and $\pi := (\mathrm{e}_0,\mathrm{e}_1)_\sharp \nu$, such that $\mu_t := (e_t)_\sharp \nu \ll \mathfrak{m}$ and for any $N' \geq N, t \in [0,1]$:

$$\mathcal{E}_{N'}(\mu_t) \geq \int \tau_{K,N'}^{(1-t)}(\mathsf{d}(x,y))\rho_0^{-1/N'} + \tau_{K,N'}^{(t)}(\mathsf{d}(x,y))\rho_1^{-1/N'}\,\mathrm{d}\pi(x,y). \tag{3.11}$$

Remark 3.6. It is worth recalling that if (M,g) is a Riemannian manifold of dimension n and $h \in C^2(M)$ with $h > 0$, then the m.m.s. $(M,\mathsf{d}_g,h\,\mathrm{Vol}_g)$ (where d_g and Vol_g denote the Riemannian distance and volume induced by g) verifies CD(K,N) with $N \geq n$ if and only if (see [Stu06b, Theorem 1.7])

$$\mathrm{Ricci}_{g,h,N} \geq Kg, \qquad \mathrm{Ricci}_{g,h,N} := \mathrm{Ricci}_g - (N-n)\frac{\nabla_g^2 h^{\frac{1}{N-n}}}{h^{\frac{1}{N-n}}}.$$

In particular if $N = n$ the generalized Ricci tensor $\mathrm{Ricci}_{g,h,N} = \mathrm{Ricci}_g$ makes sense only if h is constant.

Riemannian curvature-dimension condition

In [AGS14], Ambrosio, Gigli and Savaré introduced a Riemannian version of the curvature-dimension condition, called **R**iemannian **c**urvature-**d**imension condition, ruling out Finsler manifolds.

Let us first recall the definition of the Cheeger energy [Che99]. The Cheeger energy is the functional defined in $L^2(X,\mathfrak{m})$ by

$$\mathrm{Ch}(f) := \frac{1}{2}\inf\left\{\liminf_{i\to\infty}\int_X |\mathrm{lip}(f_i)|^2\,\mathrm{d}\mathfrak{m} : f_i \in \mathrm{Lip}(X,\mathrm{d}), \|f_i - f\|_{L^2} \to 0\right\},$$

where

$$|\mathrm{lip}(h)|(x) := \limsup_{y\to x}\frac{|h(y)-h(x)|}{\mathsf{d}(x,y)}$$

for $h \in \mathrm{Lip}(X,\mathsf{d})$. We define the Sobolev space $W^{1,2}(X,\mathsf{d},\mathfrak{m})$ by

$$W^{1,2}(X,\mathsf{d},\mathfrak{m}) := \left\{f \in L^2(X,\mathfrak{m}) : \mathrm{Ch}(f) < \infty\right\}.$$

Definition 3.7 (RCD(K,∞) condition)**.** Let $K \in \mathbb{R}$. A metric measure space $(X,\mathsf{d},\mathfrak{m})$ verifies Riemannian curvature bounded from below by K (or $(X,\mathsf{d},\mathfrak{m})$ is an RCD(K,∞) space) if it satisfies CD(K,∞) and the Cheeger energy Ch is a quadratic form in the sense that

$$\mathrm{Ch}(f+g) + \mathrm{Ch}(f-g) = 2\mathrm{Ch}(f) + 2\mathrm{Ch}(g) \qquad \text{for all} f,g \in W^{1,2}(X). \tag{3.12}$$

It is known that a smooth Riemannian manifold (M^n,g) equipped with a reference measure $\mathfrak{m} := e^{-V}\mathrm{Vol}_g$ is an RCD(K,∞) if the modified Ricci tensor

$$\mathrm{Ricci}_V := \mathrm{Ricci}_g + \mathrm{Hess}_g V$$

is bounded from below by Kg.

For the finite dimensional case, there also holds a Riemannian version of the curvature-dimension condition [AMS15, EKS15]. It is known that an n-dimensional Riemannian manifold (M^n, g) equipped with a reference measure $\mathfrak{m} := e^{-V}\mathrm{Vol}_g$ is $\mathrm{RCD}(K, N)$ if the tensor $\mathrm{Ricci}_g + \mathrm{Hess}_g V$ is bounded from below by $Kg + \frac{1}{N-n}\nabla V \otimes \nabla V$ [EKS15].

Definition 3.8 (RCD(K, N) condition)**.** Let $K \in \mathbb{R}$ and $N \in (0, \infty)$. A metric measure space $(X, \mathsf{d}, \mathfrak{m})$ verifies Riemannian curvature bounded from below by K and dimension bounded from above by N (or $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, N)$ if it satisfies $\mathrm{CD}(K, N)$ and the Cheeger energy Ch is a quadratic form.

Remark 3.9. A variant of the $\mathrm{CD}(K, N)$ condition, called reduced curvature dimension condition and denoted by $\mathrm{CD}^*(K, N)$ [BS10], asks for the same inequality (3.11) of $\mathrm{CD}(K, N)$ but the coefficients $\tau^{(t)}_{K,N}(\mathsf{d}(\gamma_0, \gamma_1))$ and $\tau^{(1-t)}_{K,N}(\mathsf{d}(\gamma_0, \gamma_1))$ are replaced by $\sigma^{(t)}_{K,N}(\mathsf{d}(\gamma_0, \gamma_1))$ and $\sigma^{(1-t)}_{K,N}(\mathsf{d}(\gamma_0, \gamma_1))$, respectively. For both definitions there is a local version and it was recently proved in [CM21] that on an essentially non-branching metric measure spaces with $\mathfrak{m}(X) < \infty$ (and in [Li24] for general $\mathfrak{m}$), the $\mathrm{CD}^*_{loc}(K, N)$, $\mathrm{CD}^*(K, N)$, $\mathrm{CD}_{loc}(K, N)$, $\mathrm{CD}(K, N)$ conditions are all equivalent, for all $K \in \mathbb{R}, N \in (1, \infty)$, via the needle decomposition method. In particular, $\mathrm{RCD}^*(K, N)$ and $\mathrm{RCD}(K, N)$ are equivalent.

Example 3.10 (Notable examples of RCD spaces)**.** *The class of* $\mathrm{RCD}(K, N)$ *spaces includes the following remarkable subclasses:*

- *Measured Gromov–Hausdorff limits of N-dimensional Riemannian manifolds with* $\mathrm{Ricci} \geq K$*, see [AGS14, GMS15];*
- *N-dimensional Alexandrov spaces with curvature bounded from below by K, see [ZZ10, Pet11];*
- *Cones, spherical suspensions, Warped products over* RCD *space, see [Ket13, Ket12].*

We refer the readers to Villani's Bourbaki seminar [Vil16], Ambrosio's ICM lecture [Amb18] and Sturm's ECM lecture [Stu23] for an overview of this fast-growing field and bibliography.

3.3 Gradient Flows: EVI type

We recall some basic results about gradient flow theory on an extended metric space (X, d), which play key roles in our proof of Jensen's inequality. We refer the readers to [AGS05, AES16] for comprehensive discussion and references about this topic.

Definition 3.11 (EVI formulation of gradient flows)**.** Let (X, d) be an extended metric space, $K \in \mathbb{R}$, $E : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous functional, $\mathrm{D}(E) := \{E < +\infty\}$. For any $y_0 \in \mathrm{D}(E)$, we say that $(0, \infty) \ni t \to y_t \in X$ is an EVI_K gradient flow of E, starting from y_0, if it is a locally absolutely continuous curve, such that $y_t \xrightarrow{\mathsf{d}} y_0$ as $t \to 0$ and the following inequality is satisfied for any $z \in \mathrm{D}(E)$ satisfying $\mathsf{d}(z, y_t) < \infty$ for some (and then all) $t \in (0, \infty)$:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{d}^2(y_t, z) + \frac{K}{2}\mathsf{d}^2(y_t, z) \le E(z) - E(y_t), \qquad \text{for a.e. } t \in (0, \infty). \tag{3.13}$$

If $x_0 \notin \mathrm{D}(E)$ we ask for

$$\liminf_{t \downarrow 0} F(x_t) \ge F(x_0), \quad \lim_{t \downarrow 0} \mathsf{d}(x_t, y) \to \mathsf{d}(x_0, y), \qquad \forall y \in \overline{\mathrm{D}(F)}.$$

In brief, the fact that gradient flow of a functional E satisfies EVI_K formulation relies on the properties of both E and (X, d). The existence of EVI_K gradient flow of K-convex functionals has been studied in many cases, e.g. smooth Riemannian manifolds, Hilbert spaces, $\mathrm{CAT}(k)$ spaces, Wasserstein spaces over Riemannian manifolds. We refer to [MS20, Vil09, Stu03] for more discussions.

In particular, Ambrosio–Gigli–Savaré [AGS14] showed that Riemannian curvature-dimension conditions $\mathrm{RCD}(K, \infty)$ can be characterized via EVI_K gradient flows of the relative entropy.

Theorem 3.12 ([AGS14], Theorem 5.1; [AGMR15], Theorem 6.1)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space and $K \in \mathbb{R}$. Then $(X, \mathsf{d}, \mathfrak{m})$ satisfies $\mathrm{RCD}(K, \infty)$ if only and if $(X, \mathsf{d}, \mathfrak{m})$ is a length space satisfying the exponential growth condition*

$$\int_X e^{-c\mathsf{d}(x_0,x)^2}\,\mathrm{d}\mathfrak{m}(x) < \infty, \qquad \textit{for all } x_0 \in X \textit{ and } c > 0,$$

and for all $\mu \in \mathcal{P}_2(X, \mathsf{d})$ there exists an EVI_K gradient flow, in $(\mathcal{P}_2(X, \mathsf{d}), W_2)$, of $\mathrm{Ent}_{\mathfrak{m}}$ starting from μ.

Concerning abstract Wiener spaces, which are extended metric measure spaces, Ambrosio–Erbar–Savaré [AES16] proved the following theorem.

Theorem 3.13 ([AES16], Theorem 11.1, §11 and §13)**.** *Let (X, H, γ) be an abstract Wiener space equipped with the canonical Cameron–Martin distance d_H and the Gaussian measure γ. Then for all $\mu \in \mathcal{P}(X)$ with $\mu \ll \gamma$, there exists an EVI_1 gradient flow of Ent_γ starting from μ.*

For the finite dimensional case, similarly, metric measure spaces with Riemannian Ricci curvature bounded from below and with dimension bounded from above can also be characterized via EVI-type gradient flows (see [EKS15]). Firstly, we recall the definition of finite dimensional $\mathrm{EVI}_{K,N}$ gradient flows.

Definition 3.14 ($\mathrm{EVI}_{K,N}$ formulation of gradient flows)**.** Let (X, d) be a metric space, $E : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous functional, $y_0 \in \overline{\{E < \infty\}}$, and $K \in \mathbb{R}$, $N \in (0, \infty)$. We say that $(0, \infty) \ni t \to y_t \in X$ is an $\mathrm{EVI}_{K,N}$ gradient flow of E, starting from y_0, if it is a locally absolutely continuous curve, such that $y_t \xrightarrow{\mathsf{d}} y_0$ as $t \to 0$ and the following inequality is satisfied for any $z \in X$

$$\frac{\mathrm{d}}{\mathrm{d}t} s^2_{K/N}\left(\frac{1}{2}\mathsf{d}(y_t, z)\right) + K s^2_{K/N}\left(\frac{1}{2}\mathsf{d}(y_t, z)\right) \le \frac{N}{2}\left(1 - \frac{U_N(z)}{U_N(y_t)}\right), \tag{3.14}$$

for a.e. $t \in (0, \infty)$, where $U_N(x) := e^{-E(x)/N}$.

Theorem 3.15 ([EKS15], Theorem 3.17). *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space. Let $K \in \mathbb{R}$ and $N \in (0, \infty)$. Then $(X, \mathsf{d}, \mathfrak{m})$ is an* $\mathrm{RCD}(K, N)$ *space if only and if $(X, \mathsf{d}, \mathfrak{m})$ is a length space satisfying the exponential growth condition*

$$\int_X e^{-c\mathsf{d}(x_0,x)^2} \, \mathrm{d}\mathfrak{m}(x) < \infty, \qquad \text{for all } x_0 \in X \text{ and } c > 0,$$

and for all $\mu \in \mathcal{P}_2(X, \mathsf{d})$ there exists an $\mathrm{EVI}_{K,N}$ *gradient flow, in $(\mathcal{P}_2(X, \mathsf{d}), W_2)$, of* $\mathrm{Ent}_{\mathfrak{m}}$ *starting from μ.*

4 Wasserstein barycenter: general setting

4.1 Existence

The study of the Wasserstein barycenter problem was initiated by Agueh and Carlier [AC11], who established the existence and uniqueness results for finitely supported measure $\Omega \in \mathcal{P}_2(\mathcal{P}_2(\mathbb{R}^n, |\cdot|), W_2)$ when the underlying space is Euclidean. Later on, there are lots of works about the existence, uniqueness and absolute continuity, such as Kim–Pass [KP17] in the Riemannian setting and Jiang [Jia17] in the Alexandrov spaces.

For the existence of the Wasserstein barycenter, we have the following existence result proved by Le Gouic and Loubes [LGL17]. Note that, in general, the Wasserstein space is neither locally compact nor NPC, even if the underling space is locally compact and NPC, so we can not deduce the existence directly with the known examples (cf. Example 2.6).

Proposition 4.1 ([LGL17],Theorem 2). *Let (X, d) be a separable locally compact geodesic space, then any $\Omega \in \mathcal{P}_2(\mathcal{P}_2(X, \mathsf{d}), W_2)$ has a Wasserstein barycenter.*

Up to our knowledge, the proposition above is the best known result concerning the existence of the Wasserstein barycenter. However, the local compactness condition is not valid in infinite-dimensional spaces, such as infinite-dimensional Hilbert space and many other $\mathrm{RCD}(K, \infty)$ spaces. By adapting the proof of [AC11, Proposition 4.2] (or [LGL17, Theorem 8]) we can prove an existence theorem for barycenter extended metric spaces, without any compactness condition.

We begin with a lemma concerning the existence of measurable barycenter-selection map.

Lemma 4.2. *Let (X, d) be a barycenter space. Given an integer $n \geq 1$, let $\lambda_i > 0, 1 \leq i \leq n$, be n positive real numbers such that $\sum_{i=1}^n \lambda_i = 1$, then there exists a measurable barycenter selection map $T : X^n \to X$, such that $T(x_1, \ldots, x_n)$ is a barycenter of $\sum_{i=1}^n \lambda_i \delta_{x_i} \in \mathcal{P}_0(X) \cap \mathcal{P}_2(X, \mathsf{d})$.*

Proof. The proof is similar to the geodesic case, see [AG11, Theorem 2.10 and Lemma 2.11]. Since (X, d) is a barycenter space, the barycenters of $\sum_{i=1}^n \lambda_i \delta_{x_i}$ exist if $\sum_{i=1}^n \lambda_i \delta_{x_i} \in \mathcal{P}_2(X, \mathsf{d})$. So we can define a multivalued map from $G : X^n \to X$, which associates to each pair $(x_1, \ldots, x_n)$ the set $G(x_1, \ldots, x_n)$ of all barycenters of $\sum_{i=1}^n \lambda_i \delta_{x_i}$ if $\sum_{i=1}^n \lambda_i \delta_{x_i} \in \mathcal{P}_2(X, \mathsf{d})$; otherwise, we define $G(x_1, \ldots, x_n) = y$ for a fixed point $y \in X$. We can see that G has closed graph. Then using Kuratowski–Ryll–Nardzewski measurable selection theorem [BR07, Theorem 6.9.3], we obtain the existence of measurable barycenter selection map T. □

Next we prove an existence theorem concerning the Wasserstein barycenter problem in barycenter spaces. From this theorem, we know the Wasserstein spaces over abstract Wiener spaces are barycentric.

Theorem 4.3 (Wasserstein space over a barycenter space is barycentric)**.** *Let* (X, d) *be a barycenter space, then* $\mathcal{W}_2 = (\mathcal{P}(X), W_2)$ *is barycentric as well.*

Proof. Note that (X, d) is a barycenter space, by Lemma 4.2, there exists a measurable barycenter selection map $T : X^n \to X$, for any $x_1, \ldots, x_n \in X$, $\mathbf{x} := (x_1, \ldots, x_n)$, $T(\mathbf{x}) = T(x_1, \ldots, x_n) \in X$, such that

$$c(x_1, \ldots, x_n) := \inf_{y \in X} \sum_{i=1}^n \lambda_i \mathsf{d}^2(x_i, y) = \sum_{i=1}^n \lambda_i \mathsf{d}^2(x_i, T(\mathbf{x})) \in [0, +\infty]. \tag{4.1}$$

Let $\mu_1, \ldots, \mu_n \in \mathcal{P}(X)$, $\lambda_i, i = 1, ..., n$ be positive real numbers with $\sum_{i=1}^n \lambda_i = 1$. Assume $\Omega := \sum_{i=1}^n \lambda_i \delta_{\mu_i} \in \mathcal{P}_2(\mathcal{P}(X), W_2)$. We claim $\nu = T_\sharp \pi$ is a Wasserstein barycenter of Ω, where $\pi \in \Pi(\mu_1, \ldots, \mu_n)$ is an optimal plan of the multi-marginal optimal transport problem of Kantorovich type (3.4).

On one hand: by the disintegration theorem [AGS05, Theorem 5.3.1], for $\eta_i \in \Pi(\mu_i, \mu)$, $i = 1, \ldots, n$, we can write $\eta_i = \eta_i^y \otimes \mu$ for a Borel family of probability measures $\{\eta_i^y\}_{y \in X} \subseteq \mathcal{P}(X)$. This means, for any Borel-measurable function $f : X^2 \to [0, +\infty)$,

$$\int_{X \times X} f \, \mathrm{d}\eta_i = \int_X \int_X f \, \mathrm{d}\eta_i^y \mathrm{d}\mu(y).$$

Denote $\eta_y = \eta_1^y \ldots \eta_n^y$ and $\eta := \eta_y \mu \in \mathcal{P}(X^{n+1})$. It can be seen that $\eta \in \Pi(\mu_1, \ldots, \mu_n, \mu)$ and for any Borel-measurable function $f : X^{n+1} \to [0, +\infty)$, we have

$$\int_{X^{n+1}} f(x_1, ..., x_n, y) \, \mathrm{d}\eta(x_1, ..., x_n, y) = \int_X \int_{X^n} f(x_1, ..., x_n, y) \, \mathrm{d}\eta_y(x_1, ..., x_n) \mathrm{d}\mu(y). \tag{4.2}$$

As $\sum \lambda_i \delta_{\mu_i}$ has finite variance, there is $\mu \in \mathcal{P}(X)$ so that

$$\sum_{i=1}^n \lambda_i W_2^2(\mu_i, \mu) < +\infty.$$

Therefore,

$$\begin{aligned}
&+\infty > \sum_{i=1}^n \lambda_i W_2^2(\mu_i,\mu)\\
&= \sum_{i=1}^n \lambda_i \int_{X\times X} \mathsf{d}^2(x_i,y)\,\mathrm{d}\eta_i = \int_{X^{n+1}} \sum_{i=1}^n \lambda_i \mathsf{d}^2(x_i,y)\,\mathrm{d}\eta\\
&\overset{(4.1)}{\geq} \int_{X^{n+1}} \sum_{i=1}^n \lambda_i \mathsf{d}^2(x_i,T(\mathbf{x}))\,\mathrm{d}\eta\\
&\overset{(4.2)}{=} \int_X \int_{X^n} \sum_{i=1}^n \lambda_i \mathsf{d}^2(x_i,T(\mathbf{x}))\,\mathrm{d}\eta_y(\mathbf{x})\,\mathrm{d}\mu(y)\\
&\overset{\text{Fubini}}{=} \int_{X^n} \sum_{i=1}^n \lambda_i \mathsf{d}^2(x_i,T(\mathbf{x}))\,\mathrm{d}\tilde\eta\\
&\geq \int_{X^n} \sum_{i=1}^n \lambda_i \mathsf{d}^2(x_i,T(\mathbf{x}))\,\mathrm{d}\pi \overset{(4.1)}{=} \int_{X^n} c(x_1,\dots,x_n)\,\mathrm{d}\pi,
\end{aligned} \tag{4.3}$$

where we have used the facts $\eta_i \in \Pi(\mu_i,\mu), i = 1,\dots,n$, $\eta \in \Pi(\mu_1,\dots,\mu_n,\mu)$, and $\tilde\eta := \int_X \eta_y\,\mathrm{d}\mu(y) \in \Pi(\mu_1,\dots,\mu_n)$.

On the other hand: for every $i = 1,\dots,n$, denote by θ_i the i-th canonical projection from X^n to X, and denote $\eta_i = (\theta_i, T)_\sharp \pi$. By construction, $\eta_i \in \Pi(\mu_i,\nu)$. Then by definition of L^2-Kantorovich–Wasserstein distance, we have

$$W_2^2(\mu_i,\nu) \leq \int_{X\times X} \mathsf{d}^2(x_i,y)\,\mathrm{d}\eta_i = \int_{X^n} \mathsf{d}^2(x_i,T(\mathbf{x}))\,\mathrm{d}\pi. \tag{4.4}$$

Combining with (4.1) and (4.3) we get

$$\sum_{i=1}^n \lambda_i W_2^2(\mu_i,\nu) \leq \sum_{i=1}^n \lambda_i \int_{X^n} \mathsf{d}^2(x_i,T(\mathbf{x}))\,\mathrm{d}\pi = \int_{X^n} c(x_1,\dots,x_n)\,\mathrm{d}\pi < +\infty. \tag{4.5}$$

Applying (4.3) with $\mu=\nu$ and combining with (4.5), we have

$$\inf_\nu \sum_{i=1}^n \lambda_i W_2^2(\mu_i,\nu) = \int_{X^n} c(x_1,\dots,x_n)\,\mathrm{d}\pi = \int_{X^n} \sum_{i=1}^n \lambda_i \mathsf{d}^2(x_i,T(\mathbf{x}))\,\mathrm{d}\pi. \tag{4.6}$$

Therefore, $\nu = T_\sharp \pi$ is a Wasserstein barycenter of Ω and we prove the claim. □

4.2 Uniqueness

By [KP17, Theorem3.1] and [Pas13, Lemma3.2.1], the uniqueness of Wasserstein barycenter can be deduced from the strict convexity of L^2-Kantorovich-Wasserstein distance with respect to the linear interpolation. It is worth to note that this strict convexity can be obtained from the existence of optimal transport map. Thus Wasserstein barycenter's uniqueness can be obtained in more general spaces, such as non-branching CD(K,N) spaces [Gig12b, Theorem 3.3], essentially non-branching MCP(K,N) spaces [CM17, Theorem 1.1] and abstract Wiener spaces [FU04, Theorem 6.1]. In this section, to make our paper complete and self-contained, we give a sketched proof to the uniqueness of the Wasserstein barycenter, under the existence of optimal transport map.

Given an extended metric measure space $(X, \mathsf{d}, \mathfrak{m})$. For $\Omega \in \mathcal{P}_2(\mathcal{W}_2)$, we always assume that the set of $\mathfrak{m}$-absolutely continuous probability measures $\mathcal{P}_{ac}(X, \mathsf{d}, \mathfrak{m}) \subset \mathcal{P}(X)$ is Ω measurable.

Proposition 4.4. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an extended metric measure space, such that the optimal transport problem of Monge type is solvable if one of the marginal measures is in $\mathcal{P}_{ac}(X, \mathsf{d}, \mathfrak{m})$ (in this case we call $(X, \mathsf{d}, \mathfrak{m})$ a Monge Space, see E. Milman's paper [Mil20, §3.3] for more discussions). Let $\Omega \in \mathcal{P}_2(\mathcal{W}_2)$ be such that $\Omega(\mathcal{P}_{ac}(X, \mathsf{d}, \mathfrak{m})) > 0$, then the barycenter of Ω is unique.*

Proof. Note that by the definition of L^2-Kantorovich–Wasserstein distance, we know that $\mathcal{P}(X) \ni \nu \mapsto W_2^2(\mu, \nu)$ is convex (w.r.t. linear interpolation) for any $\mu \in \mathcal{P}(X)$, i.e.

$$W_2^2(\mu, \lambda\nu_1 + (1-\lambda)\nu_2) \le \lambda W_2^2(\mu, \nu_1) + (1-\lambda) W_2^2(\mu, \nu_2). \tag{4.7}$$

We claim that this inequality is strict if $\mu \in \mathcal{P}_{ac}(X, \mathsf{d}, \mathfrak{m})$.

In order to prove this claim, it is sufficiently to show that for any $\nu_1, \nu_2 \in \mathcal{P}(X)$, $\nu_1 \neq \nu_2$, $0 < \lambda < 1$, it holds

$$W_2^2(\mu, \lambda\nu_1 + (1-\lambda)\nu_2) < \lambda W_2^2(\mu, \nu_1) + (1-\lambda) W_2^2(\mu, \nu_2)$$

for any $\mu \in \mathcal{P}_{ac}(X, \mathsf{d}, \mathfrak{m})$ with $W_2^2(\mu, \nu_1), W_2^2(\mu, \nu_2) < \infty$.

Indeed, if (4.7) is an equality for some $0 < \lambda < 1$, denote by $\nu = \lambda\nu_1 + (1-\lambda)\nu_2 \in \mathcal{P}(X)$. Since $\mu \in \mathcal{P}_{ac}(X, \mathsf{d}, \mathfrak{m})$, by assumption, there exists a unique optimal transport map, push forward μ to ν_1, ν_2, ν, i.e. $T_i : \mathrm{supp}(\mu) \to \mathrm{supp}(\nu_i), i = 1, 2$, $T : \mathrm{supp}(\mu) \to \mathrm{supp}(\nu)$. Let $\gamma = \lambda T_1 + (1-\lambda) T_2$, and note that (4.7) is an equality implies γ is an optimal plan between μ and ν, then γ must induced by a map T, this contradicts with $\nu_1 \neq \nu_2$.

Therefore, integrating $\nu \to W_2^2(\mu, \nu)$ with respect to Ω yields the strict convexity of the functional $\nu \to \int_{\mathcal{W}_2} W_2^2(\mu, \nu) \, \mathrm{d}\Omega(\mu)$ under the assumption $\Omega(\mathcal{P}_{ac}(X, \mathsf{d}, \mathfrak{m})) > 0$. This then implies the uniqueness of its minimizer, the Wasserstein barycenter of Ω. □

5 Functionals on the Wasserstein space

5.1 EVI$_K$ gradient flows

5.1.1 EVI implies Jensen's inequality

Motivated by a result of Daneri–Savaré [DS08, Theorem 3.2], which states that a functional E is K-convex along **any** geodesic contained in $\overline{\{E < \infty\}}$ if EVI$_K$ gradient flow of E exists for any starting point, we realize that the existence of EVI$_K$ gradient flows of E is closely related to Jensen's inequality .

The following proposition, proved in [DS08, Proposition 3.1] (see also [AES16]), provides an integral version of EVI type gradient flows. We will use this formulation to prove Jensen's inequality.

Proposition 5.1 (Integral version of EVI)**.** *Let E, K and $(y_t)_{t>0}$ be as in Definition 3.11, then $(y_t)_{t>0}$ is an EVI_K gradient flow if and only if it satisfies*

$$\frac{e^{K(t-s)}}{2}\mathsf{d}^2(y_t,z)-\frac{1}{2}\mathsf{d}^2(y_s,z)\le I_K(t-s)\big(E(z)-E(y_t)\big),\quad \forall 0\le s\le t, \tag{5.1}$$

where $I_K(t):=\int_0^t e^{Kr}\,\mathrm{d}r$.

Theorem 5.2 (EVI implies Jensen's inequality)**.** *Let (X,d) be an extended metric space, $K\in\mathbb{R}$, E be as in Definition 3.11 and μ be a probability measure over X with finite variance. Let $\epsilon\ge 0$. Assume that EVI_K gradient flow of E exists for some initial data $y\in X$ satisfying*

$$\int_X \mathsf{d}^2(y,z)\,\mathrm{d}\mu(z)\le \mathrm{Var}(\mu)+\epsilon. \tag{5.2}$$

Then the following inequality holds:

$$E(y_t)\le \int_X E(z)\,\mathrm{d}\mu(z)-\frac{K}{2}\mathrm{Var}(\mu)+\frac{\epsilon}{2I_K(t)}, \tag{5.3}$$

where y_t is the EVI_K gradient flow of E starting from y. In particular, if μ has a barycenter $\bar{y}$ and there is an EVI_K gradient flow of E starting from $\bar{y}$, then we have Jensen's inequality:

$$E(\bar{y})\le \int_X E(z)\,\mathrm{d}\mu(z)-\frac{K}{2}\int_X \mathsf{d}^2(\bar{y},z)\,\mathrm{d}\mu(z). \tag{5.4}$$

Proof. Integrating (5.1) in z with μ and choosing $s=0$, we get

$$\frac{e^{Kt}}{2}\int_X \mathsf{d}^2(y_t,z)\,\mathrm{d}\mu(z)-\frac{1}{2}\int_X \mathsf{d}^2(y,z)\,\mathrm{d}\mu(z)\le I_K(t)\left(\int_X E(z)\,\mathrm{d}\mu(z)-E(y_t)\right). \tag{5.5}$$

By Definition 3.2, we know $\int_X \mathsf{d}^2(y_t,z)\,\mathrm{d}\mu(z)\ge \mathrm{Var}(\mu)$. Combining with (5.2) and (5.5), we get

$$\frac{e^{Kt}-1}{2}\mathrm{Var}(\mu)-\frac{\epsilon}{2}\le I_K(t)\left(\int_X E(z)\,\mathrm{d}\mu(z)-E(y_t)\right). \tag{5.6}$$

Dividing both sides of (5.6) by $I_K(t)$, we get (5.3). In particular, if y is a barycenter of μ, we can take $\epsilon=0$ in (5.3), so that

$$E(y_t)\le \int_X E(z)\,\mathrm{d}\mu(z)-\frac{K}{2}\mathrm{Var}(\mu).$$

Letting $t\to 0$, and recalling the lower semicontinuity of E and $y_t\overset{\mathsf{d}}{\to} y_0$, we get Jensen's inequality (5.4). □

5.1.2 Existence and uniqueness

Using Theorem 5.2, we can extend Wasserstein Jensen's inequality (for the Wasserstein barycenter) on Riemannian manifolds (cf. [KP17]) to non-smooth extended metric measure spaces, including $\mathrm{RCD}(K,\infty)$ spaces and abstract Wiener spaces. Unlike the known approaches, this approach (using EVI gradient flow) does not rely on absolute continuity of the barycenter or local compactness of the space. In fact, we can deduce the absolute continuity of barycenter(s), as a corollary of this Wasserstein Jensen's inequality.

Theorem 5.3 (Wasserstein Jensen's inequality)**.** *Let $K \in \mathbb{R}$ and let $(X,\mathsf{d},\mathfrak{m})$ be an extended metric measure space, Ω be a probability measure over $\mathcal{P}(X)$ with finite variance.*

Assume any $\mu \in \mathcal{P}(X)$ which has finite distance from $\mathrm{D}(\mathrm{Ent}_\mathfrak{m})$ is the starting point of an EVI_K gradient flow of $\mathrm{Ent}_\mathfrak{m}$ in the Wasserstein space. Then the following Wasserstein Jensen's inequality is valid for any barycenter $\bar\mu$ of Ω:

$$\mathrm{Ent}_\mathfrak{m}(\bar\mu) \le \int_{\mathcal{P}(X)} \mathrm{Ent}_\mathfrak{m}(\mu)\,\mathrm{d}\Omega(\mu) - \frac{K}{2}\int_{\mathcal{P}(X)} W_2^2(\bar\mu,\mu)\,\mathrm{d}\Omega(\mu). \tag{5.7}$$

As a corollary , if

$$\int_{\mathcal{P}(X)} \mathrm{Ent}_\mathfrak{m}(\mu)\,\mathrm{d}\Omega(\mu) < \infty,$$

then the entropy of the barycenter of Ω is finite. In particular, the barycenter is absolutely continuous with respect to $\mathfrak{m}$.

Proof. If $\int_{\mathcal{P}(X)} \mathrm{Ent}_\mathfrak{m}(\mu)\,\mathrm{d}\Omega(\mu) = +\infty$, there is nothing to prove. Otherwise, Ω is concentrated on $\mathrm{D}(\mathrm{Ent}_\mathfrak{m})$. Since Ω has finite variance, $\int_{\mathcal{P}(X)} W_2^2(\bar\mu,\mu)\,\mathrm{d}\Omega(\mu) < +\infty$, we can see that $\bar\mu$ has finite distance from $\mathrm{D}(\mathrm{Ent}_\mathfrak{m})$. By hypothesis, $\bar\mu$ is the starting point of an EVI_K gradient flow of $\mathrm{Ent}_\mathfrak{m}$. Then (5.7) follows from Theorem 5.2. □

Remark 5.4. Note that the condition $\int_{\mathcal{P}(X)} \mathrm{Ent}_\mathfrak{m}(\mu)\,\mathrm{d}\Omega(\mu) < \infty$ implies that Ω is concentrated on the absolutely continuous measures. If $(X,\mathsf{d},\mathfrak{m})$ is RCD, Ω is concentrated on finitely many absolutely continuous measures, in Theorem 5.16 we will prove the absolute continuity of the Wasserstein barycenter, without the condition $\int_{\mathcal{P}(X)} \mathrm{Ent}_\mathfrak{m}(\mu)\,\mathrm{d}\Omega(\mu) < \infty$.

Corollary 5.5. *Let (X,H,γ) be an abstract Wiener space equipped with the Cameron–Martin distance d_H and the Gaussian measure γ. Let Ω be a probability measure over $\mathcal{P}(X)$ with finite variance.*

Then for any barycenter $\bar\mu$ of Ω it holds

$$\mathrm{Ent}_\gamma(\bar\mu) \le \int \mathrm{Ent}_\gamma(\mu)\,\mathrm{d}\Omega(\mu) - \frac{1}{2}\int W_2^2(\bar\mu,\mu)\,\mathrm{d}\Omega(\mu). \tag{5.8}$$

Proof. Assume μ has finite distance from $\mathrm{D}(\mathrm{Ent}_\gamma)$, then by [FU04, Theorem 6.1] (and its proof), we know there is a unique geodesic (μ_t) connecting μ and a measure in $\mathrm{D}(\mathrm{Ent}_\gamma)$, satisfying $\mu_t \ll \gamma$ for any $t \in (0,1)$. By [AES16, §11 and §13], any measure in $\mathcal{P}_{ac}(X,\mathsf{d}_H,\gamma)$ is the starting point of an EVI_1 gradient flow of Ent_γ.

Therefore for any $t \in (0,1)$, μ_t is the starting point of an EVI_1 gradient flow. By completeness of the Wasserstein space (cf. [FU04, Proposition 5.4]) and the Wasserstein contraction of EVI_1 gradient flows, we know μ is the starting point of an EVI_1 gradient flow. Then by Theorem 5.3 we get (5.8). □

Corollary 5.6. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, \infty)$ space. Let Ω be a probability measure over $\mathcal{P}_2(X, \mathsf{d})$ with finite variance. Then for any barycenter $\bar{\mu}$ of Ω it holds*

$$\mathrm{Ent}_{\mathfrak{m}}(\bar{\mu}) \leq \int \mathrm{Ent}_{\mathfrak{m}}(\mu)\, \mathrm{d}\Omega(\mu) - \frac{K}{2} \int W_2^2(\bar{\mu}, \mu)\, \mathrm{d}\Omega(\mu). \tag{5.9}$$

Proof. By [AGS14, Theorem 5.1], any $\mu \in \mathcal{P}_2(X, \mathsf{d})$ is the starting point of an EVI_K gradient flow. Then the assertion follows from Theorem 5.3. □

Similar to [AGS14, Lemma 5.2], we can use (5.3) to show the existence of Wasserstein barycenter, without local compactness of the underlying space.

Theorem 5.7 (Existence of barycenter)**.** *Let $K \in \mathbb{R}$ and let $(X, \mathsf{d}, \mathfrak{m})$ be an extended metric measure space, Ω be a probability measure over $\mathcal{P}(X)$ with finite variance and*

$$\int_{\mathcal{P}(X)} \mathrm{Ent}_{\mathfrak{m}}(\mu)\, \mathrm{d}\Omega(\mu) < \infty.$$

Assume that one of the following conditions holds:

- **A.** *$(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, \infty)$ metric measure space and Ω is concentrated on $\mathcal{P}_2(X, \mathsf{d})$;*
- **B.** *$\mathfrak{m}$ is a probability measure, any $\mu \in \mathcal{P}(X)$ which has finite distance from $\mathrm{D}(\mathrm{Ent}_{\mathfrak{m}})$ is the starting point of an EVI_K gradient flow of $\mathrm{Ent}_{\mathfrak{m}}$ in the Wasserstein space.*

Then Ω has a barycenter.

Proof. Without loss of generality, we assume that Ω is concentrated on $\mathrm{D}(\mathrm{Ent}_{\mathfrak{m}})$. Let $0 < \epsilon < 1$. By the definition of the variance, there exists $\bar{\mu}^\epsilon \in \mathcal{P}(X)$ such that

$$\int_{\mathcal{P}(X)} W_2^2(\mu, \bar{\mu}^\epsilon)\, \mathrm{d}\Omega(\mu) \leq \mathrm{Var}(\Omega) + \epsilon. \tag{5.10}$$

By hypothesis and Theorem 5.2, (5.3), we get

$$\mathrm{Ent}_{\mathfrak{m}}(\bar{\mu}^\epsilon_t) \leq \int_{\mathcal{P}(X)} \mathrm{Ent}_{\mathfrak{m}}(\mu)\, \mathrm{d}\Omega(\mu) - \frac{K}{2}\mathrm{Var}(\Omega) + \frac{\epsilon}{2I_K(t)}, \tag{5.11}$$

where $\bar{\mu}^\epsilon_t$ is the EVI_K gradient flow of $\mathrm{Ent}_{\mathfrak{m}}$ starting from $\bar{\mu}^\epsilon$. Set $\bar{\nu}_\epsilon := \bar{\mu}^\epsilon_\epsilon$. It can be checked that $\epsilon / I_K(\epsilon)$ is bounded for $0 < \epsilon < 1$. Combining this with (5.11), we know that the family $\{\bar{\nu}_\epsilon\}$ has uniformly bounded entropy. Thus, it is tight (cf. [AGMR15, Lemma 4.4] and [FSS10, Theorem 1.2]). Without loss of generality, we assume that $\bar{\nu}_\epsilon$ converges to $\bar{\nu} \in \mathcal{P}(X)$ in the weak sense as $\epsilon \to 0$.

We claim that $\bar{\nu}$ is a barycenter of Ω. It is known that (cf. [AGS08, Chapter 4], [AGS14, Proposition 2.22] and [AES16, §10])

$$W_2(x_t, y_t) \le e^{-Kt} W_2(x_0, y_0) \quad \forall t > 0 \tag{5.12}$$

for any EVI_K gradient flows of $\mathrm{Ent}_{\mathfrak{m}}$ starting from x_0, y_0. Thus

$$W_2(\mu, \bar{\nu}_\epsilon) - W_2(\mu_\epsilon, \mu) \le W_2(\mu_\epsilon, \bar{\nu}_\epsilon) \le e^{-K\epsilon} W_2(\mu, \bar{\mu}^\epsilon), \tag{5.13}$$

where μ_ϵ is the EVI_K gradient flow of $\mathrm{Ent}_{\mathfrak{m}}$ starting from μ.

Integrating (5.13) in μ with respect to Ω, we get

$$\int_{\mathcal{P}(X)} W_2^2(\mu, \bar{\nu}_\epsilon)\, \mathrm{d}\Omega(\mu) \le \int_{\mathcal{P}(X)} \left(e^{-K\epsilon} W_2(\mu, \bar{\mu}^\epsilon) + W_2(\mu_\epsilon, \mu)\right)^2 \mathrm{d}\Omega(\mu). \tag{5.14}$$

Note that $W_2(\mu, \mu_\epsilon)$ converges to 0 as $\epsilon \to 0$ and

$$W_2(\bar{\nu}, \mu) \le \varliminf_{\epsilon \to 0} W_2(\bar{\nu}_\epsilon, \mu), \qquad \text{for any } \mu \in \mathcal{P}_2(X),$$

we deduce from (5.14) that

$$\begin{aligned}
\mathrm{Var}(\Omega) &\le \int_{\mathcal{P}_2(X,\mathsf{d})} W_2^2(\mu, \bar{\nu})\, \mathrm{d}\Omega(\mu) \\
&\le \int_{\mathcal{P}_2(X,\mathsf{d})} \varliminf_{\epsilon \to 0} W_2^2(\mu, \bar{\nu}_\epsilon)\, \mathrm{d}\Omega(\mu) \\
&\le \varliminf_{\epsilon \to 0} \int_{\mathcal{P}_2(X,\mathsf{d})} W_2^2(\mu, \bar{\mu}^\epsilon)\, \mathrm{d}\Omega(\mu) \\
&\le \varliminf_{\epsilon \to 0} \int_{\mathcal{P}_2(X,\mathsf{d})} \left(e^{-K\epsilon} W_2(\mu, \bar{\mu}^\epsilon) + W_2(\mu_\epsilon, \mu)\right)^2 \mathrm{d}\Omega(\mu) \\
&\le \varliminf_{\epsilon \to 0} \int_{\mathcal{P}_2(X,\mathsf{d})} (e^{-2K\epsilon} + C) W_2^2(\mu, \bar{\mu}^\epsilon) + (1 + \frac{1}{C}) W_2^2(\mu_\epsilon, \mu)\, \mathrm{d}\Omega(\mu) \\
&\le \varliminf_{\epsilon \to 0} \int_{\mathcal{P}_2(X,\mathsf{d})} (1 + C) W_2^2(\mu, \bar{\mu}^\epsilon)\, \mathrm{d}\Omega(\mu),
\end{aligned}$$

for any $C > 0$. Combining this with (5.10), we prove the claim. □

Using similar argument as Proposition 4.4 in Subsection 4.2, we can prove the uniqueness of Wasserstein barycenter, under a slightly weaker resolvability of the Monge's problem. Important examples satisfying our assumption includes $\mathrm{RCD}(K, \infty)$ spaces [GRS16] and abstract Wiener spaces [FSS10, AES16].

Theorem 5.8. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space satisfying one of the two conditions in Theorem 5.7 and satisfying the weak Monge property: for any $\mu, \nu \in \mathcal{P}_{ac}(X, \mathsf{d}, \mathfrak{m})$ with $W_2(\mu, \nu) < \infty$, there exists a unique optimal transport map between μ and ν.*

Then for any $\Omega \in \mathcal{P}_2(\mathcal{P}(X), W_2)$ with

$$\int_{\mathcal{P}(X)} \mathrm{Ent}_{\mathfrak{m}}(\mu)\, \mathrm{d}\Omega(\mu) < \infty,$$

there exists a unique Wasserstein barycenter of Ω.

Proof. By Theorem 5.7, we know Ω has a Wasserstein barycenter, denote by $\bar{\mu}$, then by Jensen's inequality in Theorem 5.3,

$$\mathrm{Ent}_{\mathfrak{m}}(\bar{\mu}) \leq \int_{\mathcal{P}(X)} \mathrm{Ent}_{\mathfrak{m}}(\mu)\,\mathrm{d}\Omega(\mu) - \frac{K}{2}\int_{\mathcal{P}(X)} W_2^2(\bar{\mu},\mu)\,\mathrm{d}\Omega(\mu) < \infty. \tag{5.15}$$

This implies that the Wasserstein barycenter $\bar{\mu}$ is absolutely continuous with respect to $\mathfrak{m}$. By assumption, for any $\mu,\nu \in \mathcal{P}(X)$ with $W_2(\mu,\nu) < \infty$ and $\mu,\nu \ll m$, there exists a unique optimal transport map from μ to ν. Now, assume there exist two different Wasserstein barycenters $\bar{\mu}_1, \bar{\mu}_2$ of Ω, so that

$$\int_{\mathcal{P}(X)} W_2^2(\mu,\bar{\mu}_i)\,\mathrm{d}\Omega(\mu) = \min_{\nu\in\mathcal{P}(X)} \int_{\mathcal{P}(X)} W_2^2(\mu,\nu)\,\mathrm{d}\Omega(\mu) \quad i = 1,2. \tag{5.16}$$

However, by the same argument as in the proof of Proposition 4.4, we know

$$W_2^2(\mu,(\bar{\mu}_1+\bar{\mu}_2)/2) < \frac{1}{2}\Big(W_2^2(\mu,\bar{\mu}_1) + W_2^2(\mu,\bar{\mu}_2)\Big). \tag{5.17}$$

Thus, denote by $\bar{\mu} = (\bar{\mu}_1+\bar{\mu}_2)/2 \in \mathcal{P}(X)$, integrating (5.17) with respect to Ω, noticing that $\int_{\mathcal{P}(X)} \mathrm{Ent}_{\mathfrak{m}}(\mu)\,\mathrm{d}\Omega(\mu) < \infty$ implies $\Omega(\mathcal{P}_{ac}(X,\mathsf{d},\mathfrak{m})) = 1$, we obtain

$$\begin{aligned}\int_{\mathcal{P}(X)} W_2^2(\mu,\bar{\mu})\,\mathrm{d}\Omega(\mu) &< \frac{1}{2}\int_{\mathcal{P}(X)} W_2^2(\mu,\bar{\mu}_1)\,\mathrm{d}\Omega(\mu) + \frac{1}{2}\int_{\mathcal{P}(X)} W_2^2(\mu,\bar{\mu}_2)\,\mathrm{d}\Omega(\mu)\\ &= \min_{\nu\in\mathcal{P}(X)} \int_{\mathcal{P}(X)} W_2^2(\mu,\nu)\,\mathrm{d}\Omega(\mu),\end{aligned} \tag{5.18}$$

which contradict to (5.16). Thus the Wasserstein barycenter of Ω is unique. □

5.2 $\mathrm{EVI}_{K,N}$ gradient flows

Similar to Theorem 5.2, we exploit a finite dimensional formulation of the gradient flow, $\mathrm{EVI}_{K,N}$ gradient flow, to prove a finite dimensional version of Jensen's inequality, which is new even on $\mathbb{R}^n$. For simplicity, throughout this subsection, (X,d) is a metric space.

Firstly, we have the following estimates.

Lemma 5.9. *Let (X,d) be a metric space. Let $K\in\mathbb{R}, N\in(0,\infty)$ and let U_N, (y_t) be as in Definition 3.14. Then for $L = \pi\sqrt{\frac{N}{|K|}} \vee 2$, there exists $C_1 = C_1(K,N) > 0$, such that for t sufficiently small, it holds*

- *(i) for $z\in X$ with $\mathsf{d}(y,z)\leq L$, $\mathsf{d}(y_t,z)\leq C_1$.*
- *(ii) for $z\in X$ with $\mathsf{d}(y,z) > L$, $\mathsf{d}(y_t,z)\leq L\mathsf{d}(y,z)$.*

Proof. For $K>0$, by Definition 3.14 we can see that (X,d) is bounded, so there is nothing to prove. Then we may assume $K<0$.

We recall that the formulation of $\mathrm{EVI}_{K,N}$ gradient flow is

$$\frac{\mathrm{d}}{\mathrm{d}t} s^2_{K/N}\left(\frac{1}{2}\mathsf{d}(y_t,z)\right) + K s^2_{K/N}\left(\frac{1}{2}\mathsf{d}(y_t,z)\right) \le \frac{N}{2}\left(1-\frac{U_N(z)}{U_N(y_t)}\right) \tag{5.19}$$

holds for a.e. $t>0$. And its integral version [EKS15, Proposition 2.18] (see also Proposition 5.1) is

$$e^{Kt} s^2_{K/N}\left(\frac{1}{2}\mathsf{d}(y_t,z)\right) - s^2_{K/N}\left(\frac{1}{2}\mathsf{d}(y,z)\right) \le I_K(t)\frac{N}{2}\left(1-\frac{U_N(z)}{U_N(y_t)}\right). \tag{5.20}$$

Notice that $U_N \ge 0$, from (5.20), we get

$$c_{K/N}(\mathsf{d}(y_t,z)) \le e^{-Kt} c_{K/N}(\mathsf{d}(y,z)) + \sqrt{-\frac{N}{K}}\big(1-e^{-Kt}-Ke^{-Kt}I_K(t)\big), \tag{5.21}$$

where we use the identity that $-\frac{K}{N}s^2_{K/N}(\frac{x}{2}) = \frac{\sqrt{-K/N}c_{K/N}(x)-1}{2}$. Then the first statement of lemma follows.

Next, we prove the second statement of the lemma by contradiction. We assume that $\mathsf{d}(y_t,z) \ge L\mathsf{d}(y,z)$. By monotonicity of $x \mapsto \sinh x$ and $\mapsto \cosh x$ on $[0,\infty)$, we have

$$\begin{aligned} c_{K/N}(\mathsf{d}(y_t,z)) \ge c_{K/N}(2\mathsf{d}(y,z)) &= \sqrt{-\frac{K}{N}}\big(c^2_{K/N}(\mathsf{d}(y,z)) + s^2_{K/N}(\mathsf{d}(y,z))\big) \\ &\ge \cosh(\pi) c_{K/N}(\mathsf{d}(y,z)) + \sqrt{-\frac{N}{K}}\sinh^2(\pi), \end{aligned} \tag{5.22}$$

When t is sufficiently small, such that

$$e^{-Kt} < \cosh(\pi)$$

and

$$1-e^{-Kt}-Ke^{-Kt}I_K(t) < \sqrt{-\frac{N}{K}}\sinh^2(\pi),$$

we can see (5.22) contradicts to (5.21). □

Then we can prove a Jensen-type inequality with dimension parameter in the next theorem.

Theorem 5.10 (Jensen-type inequality with dimension parameter)**.** *Let (X,d) be a metric space. Let N, K, U_N and (y_t) be as in Lemma 5.9. Let μ be a probability measure over X with finite variance. Assume that $\mathrm{EVI}_{K,N}$ gradient flow of E exists for any initial data $y_0 \in X$, then for any barycenter $\bar{y}$ of μ, the following Jensen-type inequality holds:*

$$\frac{1}{U_N(\bar{y})}\int_X \frac{\mathsf{d}(\bar{y},z)}{s_{K/N}(\mathsf{d}(\bar{y},z))}U_N(z)\,\mathrm{d}\mu(z) \le \int_X \frac{\mathsf{d}(\bar{y},z)}{t_{K/N}(\mathsf{d}(\bar{y},z))}\,\mathrm{d}\mu(z) < +\infty. \tag{5.23}$$

where $t_{K/N} := s_{K/N}/c_{K/N}$.

Proof. Recall that the $\mathrm{EVI}_{K,N}$ gradient flow (y_t) from y satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} s_{K/N}^2\left(\frac{1}{2}\mathsf{d}(y_t,z)\right) + K s_{K/N}^2\left(\frac{1}{2}\mathsf{d}(y_t,z)\right) \le \frac{N}{2}\left(1-\frac{U_N(z)}{U_N(y_t)}\right) \tag{5.24}$$

holds for a.e. $t>0$. It can be seen that (5.24) is equivalent to the following inequality (cf. [EKS15, Lemma 2.15, (2.20)]):

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{d}^2(y_t,z) \le \frac{N\mathsf{d}(y_t,z)}{s_{K/N}(\mathsf{d}(y_t,z))}\left[c_{K/N}(\mathsf{d}(y_t,z)) - \frac{U_N(z)}{U_N(y_t)}\right]. \tag{5.25}$$

And its integral version is

$$\frac{1}{2}\mathsf{d}^2(y_t,z) - \frac{1}{2}\mathsf{d}^2(y,z) \le N\left[\int_0^t \frac{\mathsf{d}(y_s,z)}{t_{K/N}(\mathsf{d}(y_s,z))}\,\mathrm{d}s - \frac{U_N(z)}{U_N(y_t)}\int_0^t \frac{\mathsf{d}(y_s,z)}{s_{K/N}(\mathsf{d}(y_s,z))}\,\mathrm{d}s\right]. \tag{5.26}$$

Integrating (5.26) in z with given probability measure $\mu \in \mathcal{P}(X,\mathsf{d})$, choosing $y=\bar{y}$ and (y_t) as the $\mathrm{EVI}_{K,N}$ gradient flow of U_N starting from the barycenter $\bar{y}$, we get

$$\begin{aligned}&\frac{1}{2}\int_X \mathsf{d}^2(y_t,z)\,\mathrm{d}\mu(z) - \frac{1}{2}\int_X \mathsf{d}^2(\bar{y},z)\,\mathrm{d}\mu(z)\\ \le& N\left[\int_X\int_0^t \frac{\mathsf{d}(y_s,z)}{t_{K/N}(\mathsf{d}(y_s,z))}\,\mathrm{d}s\,\mathrm{d}\mu(z) - \int_X \frac{U_N(z)}{U_N(y_t)}\int_0^t \frac{\mathsf{d}(y_s,z)}{s_{K/N}(\mathsf{d}(y_s,z))}\,\mathrm{d}s\,\mathrm{d}\mu(z)\right].\end{aligned} \tag{5.27}$$

By Definition 3.2, we know that $\int_X \mathsf{d}(y_t,z)^2\,\mathrm{d}\mu(z) \ge \int_X \mathsf{d}(\bar{y},z)^2\,\mathrm{d}\mu(z)$. Combining this with (5.27), we get

$$0 \le N\left[\int_X\int_0^t \frac{\mathsf{d}(y_s,z)}{t_{K/N}(\mathrm{d}(y_s,z))}\,\mathrm{d}s\,\mathrm{d}\mu(z) - \int_X \frac{U_N(z)}{U_N(y_t)}\int_0^t \frac{\mathsf{d}(y_s,z)}{s_{K/N}(\mathsf{d}(y_s,z))}\,\mathrm{d}s\,\mathrm{d}\mu(z)\right]. \tag{5.28}$$

Rearrange (5.28) and divide both sides of it by Nt:

$$\int_X \frac{U_N(z)}{U_N(y_t)}\left[\frac{1}{t}\int_0^t \frac{\mathsf{d}(y_s,z)}{s_{K/N}(\mathsf{d}(y_s,z))}\,\mathrm{d}s\right]\mathrm{d}\mu(z) \le \int_X \left[\frac{1}{t}\int_0^t \frac{\mathsf{d}(y_s,z)}{t_{K/N}(\mathsf{d}(y_s,z))}\,\mathrm{d}s\right]\mathrm{d}\mu(z). \tag{5.29}$$

Note that $\lim_{s\to 0}\mathrm{d}(y_s,z) = \mathrm{d}(\bar{y},z)$ and $x/s_{K/N}(x)$ is continuous and bounded. We have

$$\lim_{t\to 0}\frac{1}{t}\int_0^t \frac{\mathsf{d}(y_s,z)}{s_{K/N}(\mathsf{d}(y_s,z))}\,\mathrm{d}s = \frac{\mathsf{d}(\bar{y},z)}{s_{K/N}(\mathsf{d}(\bar{y},z))}.$$

By Fatou's lemma and upper semicontinuity of the non-negative functional U_N, we get

$$\begin{aligned}&\frac{1}{U_N(\bar{y})}\int_X \frac{\mathsf{d}(\bar{y},z)}{s_{K/N}(\mathsf{d}(\bar{y},z))}U_N(z)\,\mathrm{d}\mu(z)\\ \le& \liminf_{t\to 0}\frac{1}{U_N(y_t)}\int_X U_N(z)\left[\frac{1}{t}\int_0^t \frac{\mathsf{d}(y_s,z)}{s_{K/N}(\mathsf{d}(y_s,z))}\,\mathrm{d}s\right]\mathrm{d}\mu(z).\end{aligned} \tag{5.30}$$

Next, we consider separately the following two cases: $K>0$ and $K<0$.

Case 1: $K > 0$. By definition, the diameter of (X, d) is bounded from above by $\pi\sqrt{\frac{N}{K}}$. Thus, we can see that $\frac{\mathsf{d}(y,z)}{t_{K/N}(\mathsf{d}(y,z))}$ is bounded. Notice that μ is a probability measure. By using Fatou's lemma with respect to the non-positive function, we get

$$\begin{aligned}\limsup_{t\to 0}\int_X\left[\frac{1}{t}\int_0^t\frac{\mathsf{d}(y_s,z)}{t_{K/N}(\mathsf{d}(y_s,z))}\,\mathrm{d}s\right]\mathrm{d}\mu(z) &\le \int_X \limsup_{t\to 0}\left[\frac{1}{t}\int_0^t\frac{\mathsf{d}(y_s,z)}{t_{K/N}(\mathsf{d}(y_s,z))}\,\mathrm{d}s\right]\mathrm{d}\mu(z)\\ &= \int_X\frac{\mathsf{d}(\bar{y},z)}{t_{K/N}(\mathsf{d}(\bar{y},z))}\,\mathrm{d}\mu(z) < +\infty.\end{aligned} \tag{5.31}$$

Combining (5.29), (5.30) and (5.31), we get the desired inequality.

Case 2: $K < 0$. We estimate the right side of (5.29) when t is sufficiently small. To achieve this aim, we exploit two estimates given in Lemma 5.9:

$$\begin{aligned}&\int_X\frac{1}{t}\int_0^t\frac{\mathsf{d}(y_s,z)}{t_{K/N}(\mathsf{d}(y_s,z))}\,\mathrm{d}s\,\mathrm{d}\mu(z)\\ =\ &\underbrace{\int_{\{z\in X,\mathsf{d}(\bar{y},z)\le L\}}\frac{1}{t}\int_0^t\frac{\mathsf{d}(y_s,z)}{t_{K/N}(\mathsf{d}(y_s,z))}\,\mathrm{d}s\,\mathrm{d}\mu(z)}_{A(t)}+\underbrace{\int_{\{z\in X,\mathsf{d}(\bar{y},z)> L\}}\frac{1}{t}\int_0^t\frac{\mathsf{d}(y_s,z)}{t_{K/N}(\mathsf{d}(y_s,z))}\,\mathrm{d}s\,\mathrm{d}\mu(z)}_{B(t)}.\end{aligned}$$

By Lemma 5.9-(i) and the monotonicity of $x \mapsto \frac{x}{\tanh x}$, we know

$$\frac{\mathsf{d}(y_s,z)}{t_{K/N}(\mathsf{d}(y_s,z))} \le \frac{C_1}{t_{K/N}(C_1)}$$

on $\{z \in X, \mathsf{d}(\bar{y},z) \le L\}$. So by dominated convergence theorem, we get

$$\lim_{t\to 0} A(t) = \int_{\{z\in X,\mathsf{d}(\bar{y},z)\le L\}}\frac{\mathsf{d}(\bar{y},z)}{t_{K/N}(\mathsf{d}(\bar{y},z))}\,\mathrm{d}\mu(z). \tag{5.32}$$

By Lemma 5.9-(ii) and the inequality $t_{K/N}(x) \ge \frac{L^2}{x}$ for $x > L$, we have

$$\frac{1}{t}\int_0^t\frac{\mathsf{d}(y_s,z)}{t_{K/N}(\mathsf{d}(y_s,z))} \le \frac{1}{L^2}\mathsf{d}(\bar{y},z)^2, \quad \forall z\in X,\ \mathsf{d}(\bar{y},z) > 2.$$

Note that $z \mapsto \mathsf{d}(\bar{y},z)^2$ is μ-integrable, so by dominated convergence theorem we get

$$\lim_{t\to 0} B(t) = \int_{\{z\in X,\mathsf{d}(\bar{y},z)> L\}}\frac{\mathsf{d}(\bar{y},z)}{t_{K/N}(\mathsf{d}(\bar{y},z))}\,\mathrm{d}\mu(z). \tag{5.33}$$

Combining (5.28), (5.29), (5.32) and (5.33), we get

$$\frac{1}{U_N(\bar{y})}\int_X\frac{\mathsf{d}(\bar{y},z)}{\sinh(\mathsf{d}(\bar{y},z))}U_N(z)\,\mathrm{d}\mu(z) \le \int_X\frac{\mathsf{d}(\bar{y},z)}{\tanh(\mathsf{d}(\bar{y},z))}\,\mathrm{d}\mu(z) \le \frac{1}{L^2}\mathrm{Var}(\mu)+\frac{C_1}{\tanh(C_1)}$$

which is the thesis. □

Combining Theorem 3.15, 5.7 and 5.10 we obtain the following corollaries.

Corollary 5.11. *Let $N \in [1, \infty)$. Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space and Ω be a Borel probability measure over $\mathcal{P}_2(X, \mathsf{d})$. Then Wasserstein Jensen's inequality follows:*

$$\int \frac{W_2(\bar{\mu}, \mu)}{s_{K/N}(W_2(\bar{\mu}, \mu))} U_N(\mu)\, \mathrm{d}\Omega(\mu) \le U_N(\bar{\mu}) \int \frac{W_2(\bar{\mu}, \mu)}{t_{K/N}(W_2(\bar{\mu}, \mu))}\, \mathrm{d}\Omega(\mu), \tag{5.34}$$

where $\bar{\mu}$ is a barycenter of Ω, $U_N(\mu) = e^{-\frac{\mathrm{Ent}_{\mathfrak{m}}(\mu)}{N}}$.

Corollary 5.12. *Let $N \in [1, \infty)$ and $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space. Let Ω be a Borel probability measure over $\mathcal{P}_2(X, \mathsf{d})$, which gives mass to the set $\mathrm{D}(\mathrm{Ent}_{\mathfrak{m}}) = \{\mu : \mathrm{Ent}_{\mathfrak{m}}(\mu) < \infty\}$. Then the entropy of the barycenter of Ω is finite. In particular, the barycenter is absolutely continuous with respect to $\mathfrak{m}$.*

Proof. Denote a barycenter of Ω by $\bar{\mu}$. Since $\Omega(\mathrm{D}(\mathrm{Ent}_{\mathfrak{m}})) > 0$, we have

$$\int_{\mathcal{P}_2(X,\mathsf{d})} \frac{W_2(\bar{\mu}, \mu)}{s_{K/N}(W_2(\bar{\mu}, \mu))} U_N(\mu)\, \mathrm{d}\Omega(\mu) > 0,$$

where $U_N(\mu) = e^{-\frac{\mathrm{Ent}_{\mathfrak{m}}(\mu)}{N}}$. By (5.34), we know that $U_N(\bar{\mu}) = e^{-\frac{\mathrm{Ent}_{\mathfrak{m}}(\bar{\mu})}{N}} > 0$. This implies $\mathrm{Ent}(\bar{\mu}) < \infty$. □

5.3 Multi-marginal optimal transport

Our goal in this subsection is to study multi-marginal optimal transport problem of Monge type in RCD spaces. This extends Gangbo and Swiech's result in Euclidean spaces [GŚ98], Kim and Pass's results in Riemannian manifolds [KP15] and Jiang's result [Jia17] in Alexandrov spaces. It is worth to emphasize the relationship between multi-marginal optimal transport problem and Wasserstein barycenter problem, was observed and investigated by Carlier and Ekeland [CE10], then studied by Agueh–Carlier [AC11], Kim–Pass [KP15] in different settings. In the following discussion, we focus on the cost function

$$c(x_1, \ldots, x_n) = \inf_{y \in X} \sum_{i=1}^{n} \frac{1}{2} \mathsf{d}^2(x_i, y).$$

Before stating our main theorem, we collect some results we have proved in the previous sections.

Proposition 5.13. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space. Assume $\mu_1, \ldots, \mu_n \in \mathcal{P}_2(X, \mathsf{d})$, then there exists a unique Wasserstein barycenter $\bar{\mu}$ of the measure $\sum_{i=1}^n \delta_{\mu_i} \in \mathcal{P}_0(\mathcal{P}_2(X, \mathsf{d}))$ and it is absolutely continuous with respect to $\mathfrak{m}$ if one of the following conditions holds:*

(i) $(X, \mathrm{d}, \mathfrak{m})$ *is an* $\mathrm{RCD}(K, N)$ *space, and* $\mu_1 \ll \mathfrak{m}$ *with* $\mathrm{Ent}_{\mathfrak{m}}(\mu_1) < \infty$.

(ii) $(X, \mathrm{d}, \mathfrak{m})$ *is an* $\mathrm{RCD}(K, \infty)$ *space, and* $\mu_i \ll \mathfrak{m}$ *with* $\mathrm{Ent}_{\mathfrak{m}}(\mu_i) < \infty$, $i = 1, \ldots, n$.

Proof. We give a sketched proof. For existence and uniqueness, by Proposition 4.1 and Theorem 5.7, we obtain the existence of Wasserstein barycenter for $\mathrm{RCD}(K,N)$ and $\mathrm{RCD}(K,\infty)$ spaces respectively. Then by Proposition 4.4 and Theorem 5.8, we have the uniqueness of Wasserstein barycenter, and denote it by $\bar{\mu}$.

The absolute continuity of $\bar{\mu}$ follows from Corollary 5.12 and Theorem 5.3 respectively. □

Using Proposition 5.13, we can prove the existence and uniqueness of multi-marginal optimal transport map, in case the marginal measures have finite entropy.

Proposition 5.14. *Let $(X,\mathsf{d},\mathfrak{m})$ be a metric measure space. Then the multi-marginal optimal transport problem of Monge type, associated with the cost function $c(x_1,\dots,x_n)$, has a unique solution, if one of the following conditions holds:*

(i) $(X,\mathsf{d},\mathfrak{m})$ is an $\mathrm{RCD}(K,N)$ space, and $\mu_1 \ll \mathfrak{m}$ with $\mathrm{Ent}_{\mathfrak{m}}(\mu_1)<\infty$.

(ii) $(X,\mathsf{d},\mathfrak{m})$ is an $\mathrm{RCD}(K,\infty)$ space, and $\mu_i \ll \mathfrak{m}$ with $\mathrm{Ent}_{\mathfrak{m}}(\mu_i)<\infty$, $i=1,\dots,n$.

In particular, any multi-marginal optimal transport plan π is concentrated on the graph of a X^{n-1}-valued map, and for π-a.e. $(x_1,\dots,x_n)\in X^n$, there is a unique barycenter of $\sum_i \delta_{x_i}$.

Proof. First of all, we claim:

$$\inf_{\nu\in\mathcal{P}(X,\mathsf{d})}\sum_{i=1}^n \frac{1}{2}W_2^2(\mu_i,\nu)\le \inf_{\pi\in\Pi}\int_{X^n} c(x_1,\dots,x_n)\,\mathrm{d}\pi. \tag{5.35}$$

Given $\epsilon>0$, by measurable selection theorem, there exists a measurable map $T_\epsilon: X^n\to X$, such that for any $\mathbf{x}:=(x_1,\dots,x_n)$, it holds

$$c(\mathbf{x})=\inf_{y\in X}\sum_{i=1}^n\frac{1}{2}\mathsf{d}^2(x_i,y)>\sum_{i=1}^n\frac{1}{2}\mathsf{d}^2(x_i,T_\epsilon(\mathbf{x}))-\epsilon.$$

On one hand, similar to the proof of Theorem 4.3, for the i-th canonical projection θ_i from X^n to X, and $\pi\in\Pi(\mu_1,\dots,\mu_n)$, we set $\nu_\epsilon:=(T_\epsilon)_\sharp\pi$ and $\eta_i:=(\theta_i,T_\epsilon)_\sharp\pi\in\Pi(\mu_i,\nu)$. Then

$$W_2^2(\mu_i,\nu)\le\int_{X\times X}\mathsf{d}^2(x_i,y)\,\mathrm{d}\eta_i=\int_{X^n}\mathsf{d}^2(x_i,T(\mathbf{x}))\,\mathrm{d}\pi.$$

Thus,

$$\frac{1}{2}\sum_{i=1}^n W_2^2(\mu_i,\nu)\le\sum_{i=1}^n\int_{X^n}\frac{1}{2}\mathrm{d}^2(x_i,T(\mathbf{x}))\,\mathrm{d}\pi=\int_{X^n}c(x_1,\dots,x_n)\,\mathrm{d}\pi+\epsilon. \tag{5.36}$$

Letting $\epsilon\to 0$, we prove the claim.

By Proposition 5.13, we know the unique Wasserstein barycenter $\bar{\mu}$ is absolutely continuous with respect to $\mathfrak{m}$. For $i\in\{1,\dots,n\}$, by [CM17] and [GRS16] respectively, there exists a unique optimal transport map from $\bar{\mu}$ to μ_i, denote by T_i, and there is a unique optimal transport map from μ_1 to $\bar{\mu}$, this map is the inverse map of T_1, denote it by T_1^{-1}. This implies

$$W_2^2(\mu_i, \bar{\mu}) = \int_X \mathsf{d}^2(x, T_i(\mathbf{x}))\, \mathrm{d}\bar{\mu}, \quad i = 1, \dots, n,$$

$$W_2^2(\mu_1, \bar{\mu}) = \int_X \mathsf{d}^2(x_1, T_1^{-1}(x_1))\, \mathrm{d}\mu_1.$$

We claim $(T_2 \circ T_1^{-1}, \dots, T_n \circ T_1^{-1})$ is a multi-marginal optimal transport map. Note that it is surely a multi-marginal transport map, and

$$\begin{aligned}
\inf_{\pi \in \Pi} \int_{X^n} c(x_1, \dots, x_n)\, \mathrm{d}\pi &\leq \int_X c\left(x_1, T_2 \circ T_1^{-1}(x_1), \dots, T_n \circ T_1^{-1}(x_1)\right)\, \mathrm{d}\mu_1 \\
&= \int_X \inf_{y \in X} \sum_{i=1}^n \frac{1}{2} \mathsf{d}^2(T_i \circ T_1^{-1}(x_1), y)\, \mathrm{d}\mu_1 \\
&\leq \int_X \sum_{i=1}^n \frac{1}{2} \mathsf{d}^2(T_i \circ T_1^{-1}(x_1), T_1^{-1}(x_1))\, \mathrm{d}\mu_1 \\
&= \int_X \sum_{i=1}^n \frac{1}{2} \mathsf{d}^2(T_i(z), z)\, \mathrm{d}(T_1^{-1})_\sharp \mu_1(z) \\
&= \int_X \sum_{i=1}^n \frac{1}{2} \mathsf{d}^2(T_i(z), z)\, \mathrm{d}\bar{\mu}(z) \\
&\overset{*}{=} \min_{\nu \in \mathcal{P}(X, \mathsf{d})} \sum_{i=1}^n \frac{1}{2} W_2^2(\mu_i, \nu) \overset{(5.35)}{\leq} \inf_\pi \int_{X^n} c(x_1, \dots, x_n)\, \mathrm{d}\pi,
\end{aligned} \tag{5.37}$$

where $(*)$ holds since

$$\int_X \sum_{i=1}^n \frac{1}{2} \mathrm{d}^2(T_i(z), z) \mathrm{d}\bar{\mu}(z) = \sum_{i=1}^n \frac{1}{2} W_2^2(\mu_i, \bar{\mu}).$$

Therefore, the inequalities are all equalities. In particular,

$$\inf_{\pi \in \Pi} \int_{X^n} c(x_1, \dots, x_n) \mathrm{d}\pi = \int_X c(x_1, T_2 \circ T_1^{-1}(x_1), \dots, T_n \circ T_1^{-1}(x_1)) \mathrm{d}\mu_1. \tag{5.38}$$

This then implies $(T_2 \circ T_1^{-1}, \dots, T_n \circ T_1^{-1})$ is a multi-marginal optimal transport map. Furthermore, from the second inequality in (5.37), we can see that $T_1^{-1}(x_1)$ is a barycenter of $\sum_i \delta_{T_i \circ T_1^{-1}(x_1)}$.

Finally, we show the uniqueness of multi-marginal optimal transport map. This consequence follows from the uniqueness of T_1^{-1} and $T_i, i = 1, \dots, n$. Note that from (5.37),

$$\inf_{\pi \in \Pi} \int_{X^n} c(x_1, \dots, x_n)\, \mathrm{d}\pi = \int_X \sum_{i=1}^n \frac{1}{2} \mathsf{d}^2(T_i(z), z)\, \mathrm{d}\bar{\mu}(z). \tag{5.39}$$

Assume, for sake of contradiction, there exists $(F_2, \dots, F_n)$, which is distinct with $(T_2 \circ T_1^{-1}, \dots, T_n \circ T_1^{-1})$, also a multi-marginal optimal transport map. This means there exists a subset $A \subseteq X$, $\mu_1(A) > 0$, and exists i, $1 \le i \le n$, such that $F_i(x_1) \neq T_i \circ T_1^{-1}(x_1), \forall x_1 \in A$.

Now, define $H : X \to X$, such that $F_i(x_1) = H(T_1^{-1}(x_1)), \forall x_1 \in A$, by uniqueness of T_i,

$$\int_{T_1^{-1}(A)} \frac{1}{2}\mathsf{d}^2(T_i(z), z)\,\mathrm{d}\bar{\mu}(z) < \int_{T_1^{-1}(A)} \frac{1}{2}\mathsf{d}^2(H(z), z)\,\mathrm{d}\bar{\mu}(z). \tag{5.40}$$

This implies

$$\begin{aligned}
&\int_X c(x_1, F_2(x_1), \dots, F_n(x_1))\,\mathrm{d}\mu_1 \\
&= \int_X \sum_{i=1}^n \frac{1}{2}\mathsf{d}^2(T_i(z), z)\,\mathrm{d}\bar{\mu}(z) \\
&\ge \int_X \sum_{j\neq i} \frac{1}{2}\mathsf{d}^2(T_j(z), z)\,\mathrm{d}\bar{\mu}(z) + \int_{T_1^{-1}(A)} \frac{1}{2}\mathsf{d}^2(H(z), z)\,\mathrm{d}\bar{\mu}(z) + \int_{X\setminus T_1^{-1}(A)} \frac{1}{2}\mathsf{d}^2(H(z), z)\,\mathrm{d}\bar{\mu}(z) \\
&> \int_X \sum_{j\neq i} \frac{1}{2}\mathsf{d}^2(T_j(z), z)\,\mathrm{d}\bar{\mu}(z) + \int_X \frac{1}{2}\mathsf{d}^2(T_i(z), z)\,\mathrm{d}\bar{\mu}(z) \\
&= \inf_{\pi\in\Pi} \int_{X^n} c(x_1, \dots, x_n)\,\mathrm{d}\pi,
\end{aligned} \tag{5.41}$$

which contradicts to the optimality of $(F_2, \dots, F_n)$, we complete the proof. □

Next, we will prove our main theorem concerning the unique resolvability of the multi-marginal optimal transport problem of Monge type, for absolute continuous marginals, in full generality.

Theorem 5.15 (Existence and uniqueness of multi-marginal optimal transport map)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space. Then the multi-marginal optimal transport problem of Monge type, associated with the cost function $c(x_1, \dots, x_n)$, has a unique solution, if one of the following conditions holds:*

- *(i) $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, N)$ space, and $\mu_1 \ll \mathfrak{m}$.*
- *(ii) $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, \infty)$ space, and $\mu_i \ll \mathfrak{m}, i = 1, \dots, n$.*

Proof. We claim that in order to obtain those consequences, it is sufficiently to prove that any multi-marginal optimal transport plan is actually induced by a multi-marginal optimal transport map. Indeed, if this is not true, then there exists two distinct multi-marginal optimal transport plans, denoted by π_1 and π_2, all induced by a map. Now, by linearity of the Kantorovich functional, the mean $\frac{1}{2}(\pi_1 + \pi_2)$ is also optimal. However, it is not induced by a map, a contradiction.

Now assume by contradiction that there exists a multi-marginal optimal plan, denoted by π, is not induced by a multi-marginal optimal map. This means, there is a set $E \subset X^n$ with positive π-measure, such that any subset $E' \subset E$ with positive measure is not included in the graph of a map from X to X^{n-1}.

For $\mathrm{RCD}(K, N)$ case, since $\mu_1 = \rho_1\mathfrak{m} \ll \mathfrak{m}$, the union $\bigcup_{C>0}\{x_1 \in X : C^{-1} \le \rho_1(x_1) \le C\}$ has full μ_1-measure. Therefore, there exists some $C > 0$, such that the set $\tilde{E} := \{(x_1, \ldots, x_n) \in E : C^{-1} \le \rho_1(x_1) \le C\}$ has positive measure. Consider the plan $\tilde{\pi} := \frac{1}{\pi(\tilde{E})}\pi\llcorner_{\tilde{E}}$. By optimality of π and the linearity of the Kantorovich problem, we can see that $\tilde{\pi}$ is also a multi-marginal optimal transport plan. By Proposition 5.14 (i), we know that $\tilde{\pi}$ is concentrated on the graph of a X^{n-1}-valued map, which contradicts to the choice of E.

For condition $\mathrm{RCD}(K, \infty)$ case, by the same contradiction argument, we can reduce the problem to the multi-marginal optimal transport problem for marginal measures with finite entropy, then by Proposition 5.14 (ii) we get the contradiction. □

With the help of Theorem 5.15, we can also improve Proposition 5.13 by removing the finite entropy condition from the assumption.

Theorem 5.16. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space. Assume $\mu_1, \ldots, \mu_n \in \mathcal{P}_2(X, \mathsf{d})$, then there exists a unique Wasserstein barycenter $\bar{\mu}$ and it is absolutely continuous with respect to $\mathfrak{m}$ if one of the following conditions holds:*

(i) $(X, \mathsf{d}, \mathfrak{m})$ *is an* $\mathrm{RCD}(K, N)$ *space, and* $\mu_1 \ll \mathfrak{m}$.

(ii) $(X, \mathsf{d}, \mathfrak{m})$ *is an* $\mathrm{RCD}(K, \infty)$ *space, and* $\mu_i \ll \mathfrak{m}, i = 1, \ldots, n$.

Proof. Consider the multi-marginal optimal transport problem associated with the given measures $(\mu_1, \ldots, \mu_n)$. We give the existence of Wasserstein barycenter first. For $\mathrm{RCD}(K, N)$ spaces, by Proposition 4.1, we can see the existence of Wasserstein barycenter. For $\mathrm{RCD}(K, \infty)$ spaces, by Theorem 5.15 (ii), there exists a unique multi-marginal optimal transport map starting from μ_1, denoted by $(T_2, \ldots, T_n)$. Assume the Wasserstein barycenter of $(\mu_1, \ldots, \mu_n)$ is non-existent, since $\mu_1 = \rho_1 \mathfrak{m} \ll \mathfrak{m}$, the union $\bigcup_{C>0}\{x_1 \in X : C^{-1} \le \rho_1(x_1) \le C\}$ has full μ_1-measure. Therefore, there exists some $C > 0$, such that the Wasserstein barycenter of $\tilde{\mu}_1 = \frac{1}{\mu_1(E)}\mu_1\llcorner_E, \tilde{\mu}_i = T_{i\sharp}\tilde{\mu}_1, i = 2, \ldots, n$, is non-existent, where $E := \{x_1 \in X : C^{-1} \le \rho_1(x_1) \le C\}$. Note that $\mu_1, \ldots, \mu_n$ have the equal status, by the same idea from Theorem 5.15, we can reduce the problem to $\mu_1, \ldots, \mu_n$ with finite entropy, and the Wasserstein barycenter is still non-existent. However, by Theorem 5.7, we will obtain the existence of Wasserstein barycenter, a contradiction. Now, denote by $\bar{\mu}$ the Wasserstein barycenter. It is sufficient to prove $\bar{\mu} \ll \mathfrak{m}$, then by strict convexity of the Wasserstein distance with respect to the linear interpolation (cf. Proposition 4.4 and Theorem 5.8), we can prove the uniqueness of the Wasserstein barycenter.

For $\mathrm{RCD}(K, N)$ spaces, by Theorem 5.15 (i), there exists a unique multi-marginal optimal transport map, denoted by $(F_2, \ldots, F_n)$. Since $\mu_1 = \rho_1 \mathfrak{m} \ll \mathfrak{m}$, there exists bounded partitions $\mathrm{supp}(\mu_k) = \bigcup_{j \in \mathbb{N}} E^k_j$, $k = 1, 2 \ldots, n$, such that

- $E^k_j, k = 1, \ldots, n$ are bounded;
- $E^1_i \cap E^1_j = \emptyset$, $\forall i \ne j \in \mathbb{N}$;
- $\mu_1(E^1_j) > 0$ and $\|\rho_1\|_{L^\infty(E^1_j, \mathfrak{m})} < \infty$, $\forall j \in \mathbb{N}$;
- for every $j \in \mathbb{N}$, there exists $j_k \in \mathbb{N}$, such that $F_k(E^1_j) \subseteq E^k_{j_k}$.

Let $F \subseteq X$ be such that $\mathfrak{m}(F) = 0$. Assume by contradiction that $\bar{\mu}(F) > 0$. Denoted by S the unique optimal transport map from μ_1 to $\bar{\mu}$. Let $E = S^{-1}(F)$, by measure preserving property of S, $\mu_1(E) > 0$. Since

$$\mu_1(E) = \mu_1(E \cap \bigcup_{j \in \mathbb{N}} E_j^1) = \sum_{j \in \mathbb{N}} \mu_1(E \cap E_j^1) > 0, \tag{5.42}$$

then there exists j_0, such that $\mu_1(E \cap E_{j_0}^1) > 0$.

We construct $\nu_1 = \frac{1}{\mu_1(E \cap E_{j_0}^1)} \mu_1 \llcorner_{E \cap E_{j_0}^1}$, $\nu_k = F_{k\sharp}\nu_1$, $k = 2, \ldots, n$. By construction, $\nu_1, \ldots, \nu_n \in \mathcal{P}(X, \mathsf{d})$, ν_k is concentrated on some $F \cap E_{j_k}^k$, $k = 2, \ldots, n$ and $\nu_1 \ll \mathfrak{m}$ with $\mathrm{Ent}_{\mathfrak{m}}(\nu_1) < \infty$. Consider the Wasserstein barycenter of $\nu_1, \ldots, \nu_n$, there exists a unique Wasserstein barycenter $\bar{\nu} \in \mathcal{P}(X, \mathsf{d})$ of $\nu_1, \ldots, \nu_n$. In particular, $\bar{\nu}(F) = 1$. However, since $\nu_1 \ll \mathfrak{m}$ with $\mathrm{Ent}(\nu_1) < \infty$, by Lemma 5.13 (i), $\bar{\nu} \ll \mathfrak{m}$, which implies $\bar{\nu}(F) = 0$, which is the contradiction. Thus $\bar{\mu}(F) = 0$, and by arbitrariness of F, we have $\mu \ll \mathfrak{m}$.

For $\mathrm{RCD}(K, \infty)$ spaces, the argument is a bit different from the $\mathrm{RCD}(K, N)$ case, we will illustrate it briefly. Note that in this case, $\mu_1, \ldots, \mu_n$ have the equal status. By Theorem 5.15 (ii), for any $i = 1, \ldots, n$, there exists a unique multi-marginal optimal transport map $\mathbf{T}_i$ which maps μ_i to $(\mu_1, \ldots, \mu_n)$. Denote by T_i the i-th component of the map $\mathbf{T}_{i-1}$, which maps μ_{i-1} to μ_i, denotes its inverse map by T_i^{-1}, $i = 2, \ldots, n$.

Denote $\mu_i = \rho_i \mathfrak{m} \ll \mathfrak{m}, i = 1, \ldots, n$. Let $F \subseteq X$ be such that $\mathfrak{m}(F) = 0$, and $\bar{\mu}(F) > 0$. Consider the optimal transport associated with $(\mu_1, \bar{\mu})$. There exists an optimal transport plan π, such that F is transported to $E_1 \subseteq X$ by π and $\mu_1(E_1) > 0$. Similarly, there still exists $E_{j_1}^1 \subset E_1$ with $\|\rho_1\|_{L^\infty(E_{j_1}^1, \mathfrak{m})} < \infty$, and $\mu_1(E_{j_1}^1) > 0$. Denote by $E_2 = T_2(E_{j_1}^1)$. Then $\mu_2(E_2) > 0$, and similarly, there exists $E_{j_2}^2 \subset E_2$ such that $\mu_2(E_{j_2}^2) > 0$ and $\|\rho_2\|_{L^\infty(E_{j_2}^2, \mathfrak{m})} < \infty$. Continuing this construction, for $i = 3, \ldots, n$, $E_i := T_i(E_{j_{i-1}}^{i-1})$, we have $\mu_i(E_i) > 0$, and there exists $E_{j_i}^i \subset E_j$ such that $\mu_i(E_i \cap E_{j_i}^i) > 0$ and $\|\rho_i\|_{L^\infty(E_{j_i}^i, \mathfrak{m})} < \infty$.

Finally, we construct $\nu_n = \mu_n(E_{j_n}^n)^{-1} \mu_n \llcorner_{E_{j_n}^n}$, $\nu_k = (T_{k+1}^{-1})_\sharp \nu_{k+1}, k = n-1, \ldots, 1$. By construction, $\nu_1, \ldots, \nu_n \in \mathcal{P}(X, \mathsf{d})$, ν_i is concentrated on $E_i \cap E_{j_i}^i$ and $\nu_i \ll \mathfrak{m}$, $\mathrm{Ent}_{\mathfrak{m}}(\nu_i) < \infty$ for any $i = 1, \ldots, n$. Consider the Wasserstein barycenter problem associate with $\nu_1, \ldots, \nu_n$. By Lemma 5.13(ii), there exists a unique $\bar{\nu} \in \mathcal{P}(X)$ and $\bar{\nu}$ is the Wasserstein barycenter of $\nu_1, \ldots, \nu_n$ with $\bar{\nu} \ll \mathfrak{m}$. This implies $\bar{\nu}(F) = 0$. However, by construction and the uniqueness of $\bar{\nu}$, we known $\bar{\nu}$ is concentrated on F, i.e. $\bar{\nu}(F) = 1$ which is a contradiction. Thus we have $\bar{\mu}(F) = 0$, by arbitrariness of F, we have $\bar{\mu} \ll \mathfrak{m}$. □

6 BCD condition

6.1 Definition and examples

There is no doubt that curvature is one of the most important concepts in geometry. In particular, in the study of general metric measure spaces, it has been a long history to give a synthetic notion of upper and lower curvature bounds. Metric spaces with sectional curvature lower bound, referring to Alexandrov spaces, has been widely studied and greatly achieved from the last century. In recent years, Sturm and Lott–Villani introduce a synthetic notion of lower Ricci curvature bounds, called Lott–Sturm–Villani curvature–dimension condition today, in the general framework of metric measure spaces. This Lott–Sturm–Villani curvature-dimension condition is defined in terms of the displacement convexity, of certain functionals on the Wasserstein space. In his celebrated paper [McC97], McCann introduced the notion of displacement convexity, on the Wasserstein space over $\mathbb{R}^n$. Later, Cordero-Erausquin, McCann and Schmuckenschläger [CEMS01] extended this result to Riemannian manifolds with Ricci curvature lower bound. Conversely, von Renesse and Sturm [vRS05] proved that the displacement convexity actually implies lower Ricci curvature bounds. Thus, in the setting of Riemannian manifolds, lower Ricci curvature bounds can be identified with the displacement convexity of certain functionals. This can be seen as one of the starting points of Lott–Sturm–Villani theory on non-smooth metric measure spaces.

In this section, based on the barycenter convexity of certain functionals on the Wasserstein space, which was proved in the Section 5, we introduce a new curvature-dimension condition, called *barycenter curvature-dimension condition*.

Definition 6.1 ($\mathrm{BCD}(K,\infty)$ condition)**.** Let $K \in \mathbb{R}$. We say that an **extended metric measure space** $(X, \mathsf{d}, \mathfrak{m})$ verifies $\mathrm{BCD}(K,\infty)$ condition, if for any probability measure $\Omega \in \mathcal{P}_2(\mathcal{P}(X), W_2)$, concentrated on finitely many measures, there exists a barycenter $\bar{\mu}$ of Ω such that the following Jensen's inequality holds:

$$\mathrm{Ent}_{\mathfrak{m}}(\bar{\mu}) \leq \int_{\mathcal{P}(X)} \mathrm{Ent}_{\mathfrak{m}}(\mu)\,\mathrm{d}\Omega(\mu) - \frac{K}{2}\mathrm{Var}(\Omega). \tag{6.1}$$

Remark 6.2. If we take $\Omega = (1-t)\delta_{\mu_0} + t\delta_{\mu_1}$ and (X, d) is a length space, (6.1) implies the Lott–Sturm–Villani curvature-dimension condition.

In general, this condition can be weaker than Lott–Sturm–Villani's condition, even if (X, d) is a metric space and $\Omega = (1-t)\delta_{\mu_0} + t\delta_{\mu_1}$.

Definition 6.3 ($\mathrm{BCD}(K,N)$ condition)**.** Let $K \in \mathbb{R}, N > 0$. We say that a **metric measure space** $(X, \mathsf{d}, \mathfrak{m})$ verifies $\mathrm{BCD}(K,N)$ condition, if for any probability measure $\Omega \in \mathcal{P}_2(\mathcal{P}(X,\mathsf{d}), W_2)$, concentrated on finitely many measures, there exists a barycenter $\bar{\mu}$ of Ω such that the following Jensen-type inequality holds:

$$\int \frac{W_2(\bar{\mu},\mu)}{s_{K/N}(W_2(\bar{\mu},\mu))} U_N(\mu)\,\mathrm{d}\Omega(\mu) \leq U_N(\bar{\mu}) \int \frac{W_2(\bar{\mu},\mu)}{t_{K/N}(W_2(\bar{\mu},\mu))}\,\mathrm{d}\Omega(\mu), \tag{6.2}$$

where $U_N(\mu) = e^{-\frac{\mathrm{Ent}_{\mathfrak{m}}(\mu)}{N}}$.

6.2 Stability under measured Gromov–Hausdorff convergence

Similar to [LV09, Theorem 4.15], we can prove that the BCD condition is stable under the measured Gromov–Hausdorff convergence. For simplicity, in this paper we focus on compact $\mathrm{BCD}(K,\infty)$ metric measure spaces, general BCD spaces will be studied in near future.

We will adopt the following definition of measured Gromov–Hausdorff convergence of compact metric measure spaces (see [BBI01], [GMS15] and [Vil09] for equivalent ways to define this convergence). Note that in order to study the convergence of possibly non-compact (extended) metric measure spaces in the future, one needs to consider pointed (extended) metric measure spaces, see [GMS15] and Gromov's book [Gro99] for more discussion about other topologies on the class of metric-measure spaces.

Definition 6.4. Given $\varepsilon \in (0,1)$. A map $\varphi : (X_1, \mathsf{d}_1) \to (X_2, \mathsf{d}_2)$ between two metric spaces is called an ε-approximation if

(i) $\Big|\mathsf{d}_2\big(\varphi(x), \varphi(y)\big) - \mathsf{d}_1(x,y)\Big| \le \epsilon \qquad \forall x, y \in X_1$

(ii) For all $x_2 \in X_2$, there is an $x_1 \in X_1$ so that $\mathsf{d}_2\big(\varphi(x_1), x_2\big) \le \epsilon$.

where $B_r(x)$ denotes the open geodesic ball with radius r and center x, and $N_\epsilon(A)$ denotes the open ϵ-neighbourhood of a subset A.

Now we can introduce a notion of convergence for metric measure spaces.

Definition 6.5 (Measured Gromov–Hausdorff convergence)**.** A sequence of metric spaces (X_n, d_n), $n \in \mathbb{N}$ converges in Gromov–Hausdorff sense to (X, d) if there exists a sequence of ϵ_n-approximations $\varphi_n : X_n \to X$ with $\varepsilon_n \downarrow 0$.

Moreover, we say that a sequence of metric measure spaces $(X_n, \mathsf{d}_n, \mathfrak{m}_n)$ converges in measured Gromov–Hausdorff sense (mGH for short) to a metric measure space $(X, \mathsf{d}, \mathfrak{m})$, if additionally $(\varphi_n)_\sharp \mathfrak{m}_n \to \mathfrak{m}$ weakly as measures, i.e.

$$\lim_{n\to\infty} \int_X g \, \mathrm{d}\big((\varphi_n)_\sharp \mathfrak{m}_n\big) = \int_X g \, \mathrm{d}\mathfrak{m} \qquad \forall g \in C_b(X),$$

where $C_b(X)$ denotes the set of real valued bounded continuous functions with bounded support in X and $(\varphi_n)_\sharp \mathfrak{m}_n(A) = \mathfrak{m}_n(\varphi_n^{-1}(A))$ for any $A \subset X$ Borel.

It is known that this measured Gromov–Hausdorff convergence comes from a metrizable topology on the space of all compact metric spaces modulo isometries (i.e. a metric on the isomorphism classes of metric measure spaces).

Moreover, if X_n is a sequence of barycenter spaces that converges to X in the Gromov–Hausdorff topology, it is not hard to prove that X is also a barycenter space (cf. [BBI01, Theorem 7.5.1]).

Theorem 6.6. *Let $\{(X_i, \mathsf{d}_i, \nu_i)\}_{i=1}^\infty$ be a sequence of compact* $\mathrm{BCD}(K, \infty)$ *spaces and $K \in \mathbb{R}$. If $\{(X_i, \mathsf{d}_i, \nu_i)\}$ converges to (X, d, ν) in the measured Gromov–Hausdorff sense as $n \to \infty$, then (X, d, ν) is also a* $\mathrm{BCD}(K, \infty)$ *space.*

Proof. It is well known that compactness and length property is stable under the Gromov–Hausdorff limit. Thus, (X, d) is a geodesic and barycenter space. Given $\Omega \in \mathcal{P}_0(\mathcal{P}_2(X, \mathsf{d}))$ with finite support and let $\bar\mu$ be a barycenter of Ω. We write

$\Omega = \sum_{k=1}^m \lambda_k \delta_{\mu_k}$. By [LV09, Lemma 3.24], we may assume that the support of Ω is in

$$\Big\{\gamma \in \mathcal{P}_2(X, \mathsf{d}) : \mathrm{Ent}_\nu(\gamma) < \infty, \gamma \text{ has continuous density with respect to } \nu\Big\}.$$

Write $\mu_k = \rho_k \nu_\infty$. Let $f_i : X_i \to X$ be an ϵ_i-approximation and $f_i' : X \to X_i$ be an inverse ϵ_i-approximation, with $\lim_{i\to\infty} \epsilon_i = 0$ and $\lim_{i\to\infty} (f_i)_\sharp \nu_i = \nu_\infty$. For i sufficiently large, we know $\int_X \rho_k \, \mathrm{d}(f_i)_\# \nu_i > 0$. For such i, put $\mu_{i,k} = \frac{(f_i^\sharp \rho_k)\nu_i}{\int_X \rho_k \, \mathrm{d}(f_i)_\sharp \nu_i}$ and $\Omega_i = \sum_{k=1}^m \lambda_k \delta_{\mu_{i,k}}$, where $f_i^\sharp \rho_k := \rho_k \circ f_i$ denotes the pull-back of the function ρ_k. Let $\bar{\omega}_i$ is a barycenter of Ω_i. Here, we list some basic properties, whose proof can be found in [LV09, Corollary 4.3 and Theorem 4.15]:

1. for any $\gamma \in \mathcal{P}_2(X, \mathsf{d})$, $\lim_{i\to\infty} W_2(\mu_{i,k}, (f_i')_\sharp \gamma) = W_2(\mu_k, \gamma)$;
2. $\lim_{i\to\infty} \mathrm{Ent}_{\nu_i}(\mu_{i,k}) = \mathrm{Ent}_\nu(\mu_k)$;
3. $\mathrm{Ent}_{(f_i)_\sharp \nu_i}((f_i)_\sharp \bar{\omega}_i) \le \mathrm{Ent}_{\nu_i}(\bar{\omega}_i)$;
4. the functional $\mathrm{Ent}_\nu(\mu)$ is lower semi-continuous with respect to (ν, μ).

Now, we claim that $(f_i)_\sharp \bar{\omega}_i$ converges to a barycenter $\bar{\omega}$ of Ω after passing to a subsequence. In fact, by the properties above, we know that for any $\gamma \in \mathcal{P}_2(X, \mathsf{d})$,

$$\sum_{k=1}^m \lambda_k W_2^2(\mu_k, \gamma) = \lim_{i\to\infty} \sum_{k=1}^m \lambda_k W_2^2(\mu_{k,i}, (f_i')_\sharp \gamma) \ge \limsup_{i\to\infty} \mathrm{Var}(\Omega_i).$$

This implies that $\mathrm{Var}(\Omega) \ge \limsup_{i\to\infty} \mathrm{Var}(\Omega_i)$. Note that

$$\lim_{i\to\infty} \left| \sum_{k=1}^m \lambda_k W_2^2(\mu_k, (f_i)_\sharp \bar{\omega}_i) - \sum_{k=1}^m \lambda_k W_2^2(\mu_{k,i}, \bar{\omega}_i) \right| = 0.$$

Thus, after passing to a subsequence, $(f_i)_\sharp(\bar{\omega}_i)$ converge to a barycenter of Ω, denoted by $\bar{\omega}$, and

$$\begin{aligned} \mathrm{Var}(\Omega) \ge \limsup_{i\to\infty} \mathrm{Var}(\Omega_i) &= \lim_{i\to\infty} \sum_{k=1}^m \lambda_k W_2^2(\mu_k, (f_i)_\sharp \bar{\omega}_i) \\ &= \sum_{k=1}^m \lambda_k W_2^2(\mu_k, \bar{\omega}) \\ &\ge \mathrm{Var}(\Omega). \end{aligned}$$

In conclusion, we get

$$\begin{aligned} \mathrm{Ent}_{\nu_\infty}(\bar{\omega}) &\overset{4.}{\le} \liminf_{i\to\infty} \mathrm{Ent}_{(f_i)_\sharp \nu_i}(f_i)_\sharp(\bar{\omega}_i) \overset{3.}{\le} \liminf_{i\to\infty} \mathrm{Ent}_{\nu_i}(\bar{\omega}_i) \\ &\overset{(6.1)}{\le} \liminf_{i\to\infty} \int_{\mathcal{P}_2(X_i, \mathsf{d}_i)} \mathrm{Ent}_{\nu_i}(\mu_i) \, \mathrm{d}\Omega_i(\mu_i) - \frac{K}{2} \mathrm{Var}(\Omega_i) \\ &\overset{2.}{=} \int_{\mathcal{P}_2(X)} \mathrm{Ent}_\nu(\mu) \, \mathrm{d}\Omega(\mu) - \frac{K}{2} \mathrm{Var}(\Omega). \end{aligned} \tag{6.3}$$

This concludes the proof. □

6.3 Wasserstein barycenter in $\mathrm{BCD}(K,\infty)$ spaces

In this section, we continue studying the existence of Wasserstein barycenter in $\mathrm{BCD}(K,\infty)$ spaces under mild assumptions.

Theorem 6.7. *Let $(X, \mathrm{d}, \mathfrak{m})$ be an extended metric measure space satisfying $\mathrm{BCD}(K,\infty)$ curvature-dimension condition, and the exponential growth condition*

$$\int_X e^{-c\mathsf{d}(x_0,x)^2}\,\mathrm{d}\mathfrak{m}(x) < \infty, \qquad \text{for all } x_0 \in X \text{ and } c > 0,$$

then any probability measure $\Omega \in \mathcal{P}_2(\mathcal{P}_2(X,\mathsf{d}), W_2)$ satisfying

$$\int_{\mathcal{P}_2(X,\mathsf{d})} \mathrm{Ent}_{\mathfrak{m}}(\mu)\,\mathrm{d}\Omega(\mu) < \infty$$

has a Wasserstein barycenter.

Proof. By [Vil09, Theorem 6.18], for any $\Omega \in \mathcal{P}_2(\mathcal{P}_2(X,\mathsf{d}), W_2)$, there exists a sequence of finitely supported probability measures Ω_j, such that $\Omega_j \to \Omega$ in the Wasserstein space over $(\mathcal{P}_2(X,\mathsf{d}), W_2)$ as $j \to \infty$. By Definition 6.1, Ω_j admits a Wasserstein barycenter $\mu_j \in \mathcal{P}_2(X,\mathsf{d})$. By stability of the Wasserstein barycenters (cf. [LGL17, Theorem 3]), it is sufficient to prove that $(\mu_j)_{j\in\mathbb{N}}$ has a narrow convergent subsequence. We divide the proof into three steps.

Step 1: We claim there exists a sequence of finitely supported probabilities Ω_j, such that $\Omega_j \to \Omega$ and $\overline{\lim}_{j\to\infty} \int_{\mathcal{P}_2(X,\mathsf{d})} \mathrm{Ent}_{\mathfrak{m}}(\mu)\,\mathrm{d}\Omega_j(\mu) \le \int_{\mathcal{P}_2(X,\mathsf{d})} \mathrm{Ent}_{\mathfrak{m}}(\mu)\,\mathrm{d}\Omega(\mu)$. This was inspired by [Vil09, Theorem6.18].

To prove this, for any $j \ge 1$, set $\epsilon = 1/j$, and let $\mu_0 \in \mathcal{P}_2(X,\mathsf{d})$ be such that $\mathrm{Ent}_{\mathfrak{m}}(\mu_0) \le 1$. Then there exists a compact set $K \subseteq \mathcal{P}_2(X,\mathsf{d})$, such that $\Omega(\mathcal{P}_2(X,\mathsf{d})\backslash K) < \epsilon$ and

$$\int_{\mathcal{P}_2(X,\mathsf{d})\backslash K} W_2^2(\mu_0,\mu)\,\mathrm{d}\Omega(\mu) < \epsilon.$$

Cover K by finite balls $B(\mu_k, \frac{\epsilon}{2}), 1 \le k \le N$, with centers $\mu_k \in \mathcal{P}_2(X,\mathsf{d})$ (this means, $\mu \in B(\mu_k, \frac{\epsilon}{2})$if and only if $W_2(\mu_k,\mu) \le \frac{\epsilon}{2}$). Then define

$$B'_k = B(\mu_k,\epsilon)\backslash \bigcup_{j<k} B(\mu_j,\epsilon).$$

By construction, all B'_k are disjoint and still cover K. Without loss of generality, we assume that $\Omega(B'_k \cap K) > 0$ for any k. Note that for any $B'_k \cap K$, there exists $\mu'_k \in B'_k \cap K$, such that

$$\mathrm{Ent}_{\mathfrak{m}}(\mu'_k) \le \frac{1}{\Omega(B'_k\cap K)} \int_{B'_k\cap K} \mathrm{Ent}_{\mathfrak{m}}(\mu)\,\mathrm{d}\Omega(\mu). \tag{6.4}$$

Define a map $f : \mathcal{P}_2(X,\mathsf{d}) \to \mathcal{P}_2(X,\mathsf{d})$ so that

$$f(B'_k \cap K) = \{\mu'_k\}, \quad f(\mathcal{P}_2(X,\mathsf{d})\backslash K) = \{\mu_0\}. \tag{6.5}$$

Then, for any $\mu \in K$, we have $W_2(\mu, f(\mu)) \le W_2(\mu, \mu_k) + W_2(\mu_k, f(\mu)) \le 2\epsilon$. So

$$\begin{aligned}\int_{\mathcal{P}_2(X)} W_2^2(\mu, f(\mu))\,\mathrm{d}\Omega(\mu) &= \int_K W_2^2(\mu, f(\mu))\,\mathrm{d}\Omega(\mu) + \int_{\mathcal{P}_2(X,\mathsf{d})\setminus K} W_2^2(\mu, f(\mu))\,\mathrm{d}\Omega(\mu)\\ &\le (2\epsilon)^2 \int_K \mathrm{d}\Omega(\mu) + \int_{\mathcal{P}_2(X,\mathsf{d})\setminus K} W_2^2(\mu, \mu_0)\,\mathrm{d}\Omega(\mu)\\ &\le 4\epsilon^2 + \epsilon.\end{aligned} \tag{6.6}$$

This implies $W_2(\Omega, f_\sharp\Omega) \le 4\epsilon^2 + \epsilon$, where $f_\sharp\Omega \in \mathcal{P}_0(\mathcal{P}_2(X, \mathsf{d}))$. Moreover, note that

$$\begin{aligned}\int_{\mathcal{P}_2(X,\mathsf{d})} \mathrm{Ent}_{\mathfrak{m}}(\mu)\,\mathrm{d}f_\sharp\Omega(\mu) &= \int_K \mathrm{Ent}_{\mathfrak{m}}(f(\mu))\,\mathrm{d}\Omega(\mu) + \int_{\mathcal{P}_2(X,\mathsf{d})\setminus K} \mathrm{Ent}_{\mathfrak{m}}(f(\mu))\,\mathrm{d}\Omega(\mu)\\ &= \sum_{k=1}^N \int_{B'_k\cap K} \mathrm{Ent}_{\mathfrak{m}}(f(\mu))\,\mathrm{d}\Omega(\mu) + \int_{\mathcal{P}_2(X,\mathsf{d})\setminus K} \mathrm{Ent}_{\mathfrak{m}}(f(\mu))\,\mathrm{d}\Omega(\mu)\\ &= \sum_{k=1}^N \int_{B'_k\cap K} \mathrm{Ent}_{\mathfrak{m}}(\mu'_k)\,\mathrm{d}\Omega(\mu) + \int_{\mathcal{P}_2(X,\mathsf{d})\setminus K} \mathrm{Ent}_{\mathfrak{m}}(\mu_0)\,\mathrm{d}\Omega(\mu)\\ &\le \sum_{k=1}^N \int_{B'_k\cap K} \mathrm{Ent}_{\mathfrak{m}}(\mu)\,\mathrm{d}\Omega(\mu) + \int_{\mathcal{P}_2(X,\mathsf{d})\setminus K} \mathrm{Ent}_{\mathfrak{m}}(\mu_0)\,\mathrm{d}\Omega(\mu)\\ &\le \int_{\mathcal{P}_2(X,\mathsf{d})} \mathrm{Ent}_{\mathfrak{m}}(\mu)\,\mathrm{d}\Omega(\mu) + \epsilon.\end{aligned} \tag{6.7}$$

Therefore, we can construct $\Omega_j = f_\sharp\Omega$. As $j \to \infty$, we have $W_2(\Omega_j, \Omega) \to 0$ and $\overline{\lim}_{j\to\infty} \int_{\mathcal{P}_2(X,\mathsf{d})} \mathrm{Ent}_{\mathfrak{m}}(\mu)\,\mathrm{d}\Omega_j(\mu) \le \int_{\mathcal{P}_2(X,\mathsf{d})} \mathrm{Ent}_{\mathfrak{m}}(\mu)\,\mathrm{d}\Omega(\mu)$.

Step 2: Since Ω_j is concentrated on finite number of probability measures, by Definition 6.1, any Ω_j admits a Wasserstein barycenter μ_j, such that

$$\int_{\mathcal{P}_2(X,\mathsf{d})} W_2^2(\mu, \mu_j)\,\mathrm{d}\Omega_j(\mu) = \min_{\nu\in\mathcal{P}_2(X,\mathsf{d})} \int_{\mathcal{P}_2(X,\mathsf{d})} W_2^2(\mu, \nu)\,\mathrm{d}\Omega_j(\mu).$$

Let $\mu_0 \in \mathcal{P}(X, \mathsf{d})$, we claim that

$$\overline{\lim_{j\to\infty}} \int_{\mathcal{P}_2(X,\mathsf{d})} W_2^2(\mu, \mu_j)\,\mathrm{d}\Omega_j(\mu) \le 2\int_{\mathcal{P}_2(X,\mathsf{d})} W_2^2(\mu_0, \mu)\,\mathrm{d}\Omega(\mu) < +\infty. \tag{6.8}$$

Note that by triangle inequality, for any $\mu \in \mathcal{P}_2(X, \mathsf{d})$,

$$W_2^2(\mu_0, f(\mu)) \le 2(W_2^2(\mu_0, \mu) + W_2^2(\mu, f(\mu))). \tag{6.9}$$

Integrating (6.9) with respect to Ω, we obtain

$$\begin{aligned}
&\int_{\mathcal{P}_2(X,\mathsf{d})} W_2^2(\mu_j,\mu)\,\mathrm{d}\Omega_j(\mu) \\
&\le \int_{\mathcal{P}_2(X,\mathsf{d})} W_2^2(\mu_0,\mu)\,\mathrm{d}\Omega_j(\mu) = \int_{\mathcal{P}_2(X,\mathsf{d})} W_2^2(\mu_0,f(\mu))\,\mathrm{d}\Omega(\mu) \\
&\le 2\left(\int_{\mathcal{P}_2(X,\mathsf{d})} W_2^2(\mu_0,\mu)\,\mathrm{d}\Omega(\mu) + \int_{\mathcal{P}_2(X,\mathsf{d})} W_2^2(\mu,f(\mu))\,\mathrm{d}\Omega(\mu)\right) \\
&\overset{(6.6)}{\le} 2\left(\int_{\mathcal{P}_2(X,\mathsf{d})} W_2^2(\mu_0,\mu)\,\mathrm{d}\Omega(\mu) + 4\epsilon^2+\epsilon\right).
\end{aligned} \tag{6.10}$$

Notice that $\Omega \in \mathcal{P}_2(\mathcal{P}_2(X,\mathsf{d}),W_2)$, so

$$\int_{\mathcal{P}_2(X,\mathsf{d})} W_2^2(\mu_0,\mu)\,\mathrm{d}\Omega(\mu) < +\infty.$$

Letting $j\to\infty$ and $\epsilon\to 0^+$ in (6.10), we prove this claim.

Step 3: By the exponential growth condition

$$\int_X e^{-c\mathsf{d}(x_0,x)^2}\,\mathrm{d}\mathfrak{m}(x) < \infty, \qquad \text{for all } x_0\in X \text{ and } c>0,$$

define $z = \int_X e^{-c\mathsf{d}^2(x_0,x)}\,\mathrm{d}\mathfrak{m}$ and $\tilde{\mathfrak{m}} = \frac{1}{z}e^{-c\mathsf{d}^2(x_0,x)}\mathfrak{m} \in \mathcal{P}(X,\mathsf{d})$. By using the well-known formula for the change of the reference measure,

$$\mathrm{Ent}_{\mathfrak{m}}(\mu) = \mathrm{Ent}_{\tilde{\mathfrak{m}}}(\mu) - c\int_X \mathsf{d}^2(x_0,x)\,\mathrm{d}\mu - \ln z. \tag{6.11}$$

Note that by Jensen's inequality (6.1), for every $j\in\mathbb{N}$,

$$\mathrm{Ent}_{\mathfrak{m}}(\mu_j) \le \int_{\mathcal{P}_2(X,\mathsf{d})} \mathrm{Ent}_{\mathfrak{m}}(\mu)\,\mathrm{d}\Omega_j(\mu) - \frac{K}{2}\int_{\mathcal{P}_2(X,\mathsf{d})} W_2^2(\mu_j,\mu)\,\mathrm{d}\Omega_j(\mu),$$

and by Step 2,

$$\overline{\lim_{j\to\infty}} \int_{\mathcal{P}_2(X,\mathsf{d})} W_2^2(\mu_j,\mu)\,\mathrm{d}\Omega_j(\mu) \le 2\int_{\mathcal{P}_2(X,\mathsf{d})} W_2^2(\mu_0,\mu)\,\mathrm{d}\Omega(\mu) < +\infty.$$

Therefore, for $K\ge 0$ we have

$$\mathrm{Ent}_{\mathfrak{m}}(\mu_j) \le \int_{\mathcal{P}_2(X,\mathsf{d})} \mathrm{Ent}_{\mathfrak{m}}(\mu)\,\mathrm{d}\Omega_j(\mu),$$

and for $K<0$ we have

$$\mathrm{Ent}_{\mathfrak{m}}(\mu_j) \le \int_{\mathcal{P}_2(X,\mathsf{d})} \mathrm{Ent}_{\mathfrak{m}}(\mu)\,\mathrm{d}\Omega_j(\mu) - K\int_{\mathcal{P}_2(X,\mathsf{d})} W_2^2(\mu_0,\mu)\,\mathrm{d}\Omega(\mu).$$

Combining with Step 1 and Step 2, we then obtain, there is $C = C(c,K,\mu_0) > 0$, so that

$$\mathrm{Ent}_{\tilde{\mathfrak{m}}}(\mu_j) \le C < \infty, \tag{6.12}$$

for every $j\in\mathbb{N}$. This surely implies (μ_j) is tight. By Prokhorov's theorem, there exists a narrow convergent subsequence of $(\mu_j)_{j\ge1}$. Then by [LGL17, Theorem 3], the limit of the narrow convergent subsequence is a Wasserstein barycenter of Ω. We complete the proof. □

6.4 Applications

In the last part of this paper, we provide some new geometric inequalities, as simple but interesting applications of our BCD theory. For simplicity, we will only deal with BCD metric measure spaces. More functional and geometric inequalities on BCD extended metric measure spaces, will be studied in a forthcoming paper.

Proposition 6.8 (Multi-marginal Brunn–Minkowski inequality)**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a* BCD$(0, N)$ *metric measure space,* $E_1, ..., E_n$ *be bounded measurable sets with positive measure and* $\lambda_1, \dots, \lambda_n \in (0, 1)$ *with* $\sum_i \lambda_i = 1$*. Then*

$$\mathfrak{m}(E) \geq \left(\sum_{i=1}^{n} \lambda_i \big(\mathfrak{m}(E_i)\big)^{\frac{1}{N}} \right)^N$$

where

$$E := \left\{ x \text{ is the barycenter of } \sum_{i=1}^{n} \lambda_i \delta_{x_i} : x_i \in E_i, i = 1, \dots, n \right\}.$$

Proof. Set $\mu_i := \frac{1}{\mathfrak{m}(E_i)} \mathfrak{m} \llcorner_{E_i}, i = 1, \dots, n$ and $\Omega := \sum_{i=1}^n \lambda_i \delta_{\mu_i} \in \mathcal{P}_0(\mathcal{P}_2(X, \mathsf{d}))$. By Definition 6.3, (6.2) we get

$$\int U_N(\mu) \, \mathrm{d}\Omega(\mu) \leq U_N(\bar{\mu}), \tag{6.13}$$

where $\bar{\mu}$ is the barycenter of Ω. This implies

$$\sum_{i=1}^{n} \lambda_i (\mathfrak{m}(E_i))^{\frac{1}{N}} \leq U_N(\bar{\mu}).$$

Note that $\bar{\mu}$ is concentrated on E, by Jensen's inequality, we have

$$\mathrm{Ent}_{\mathfrak{m}}(\bar{\mu}) \geq -\ln(\mathfrak{m}(E))$$

and

$$U_N(\bar{\mu}) \leq (\mathfrak{m}(E))^{\frac{1}{N}}.$$

In conclusion, we obtain

$$\sum_{i=1}^{n} \lambda_i (\mathfrak{m}(E_i))^{\frac{1}{N}} \leq (\mathfrak{m}(E))^{\frac{1}{N}}.$$

which is the thesis. □

Proposition 6.9 (Multi-marginal logarithmic Brunn–Minkowski inequality)**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a* BCD$(0, \infty)$ *metric measure space,* $E_1, ..., E_n$ *be bounded measurable sets with positive measure and* $\lambda_1, \dots, \lambda_n \in (0, 1)$ *with* $\sum_i \lambda_i = 1$*. Then*

$$\mathfrak{m}(E) \geq \mathfrak{m}(E_1)^{\lambda_1} ... \mathfrak{m}(E_n)^{\lambda_n}$$

where

$$E := \left\{ x \text{ is the barycenter of } \sum_{i=1}^{n} \lambda_i \delta_{x_i} : x_i \in E_i, i = 1, \dots, n \right\}.$$

Proof. Set $\mu_i := \frac{1}{\mathfrak{m}(E_i)}\mathfrak{m}\llcorner_{E_i}, i = 1, \ldots, n$ and $\Omega := \sum_{i=1}^n \lambda_i \delta_{\mu_i} \in \mathcal{P}_0(\mathcal{P}_2(X, \mathsf{d}))$. By Definition 6.1, (6.1) we get

$$\mathrm{Ent}_{\mathfrak{m}}(\bar{\mu}) \leq -\sum_{i=1}^n \lambda_i \ln(\mathfrak{m}(E_i)).$$

where $\bar{\mu}$ is the barycenter of Ω. Note that $\bar{\mu}$ is concentrated on E, by Jensen's inequality, we have

$$\mathrm{Ent}_{\mathfrak{m}}(\bar{\mu}) \geq -\ln(\mathfrak{m}(E)).$$

In conclusion, we obtain

$$-\ln(\mathfrak{m}(E)) \leq -\sum_{i=1}^n \lambda_i \ln(\mathfrak{m}(E_i))$$

which is the thesis. □

Proposition 6.10 (A functional Blaschke–Santaló type inequality)**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a* $\mathrm{BCD}(1, \infty)$ *metric measure space. Then we have*

$$\prod_{i=1}^k \int_X e^{f_i} \,\mathrm{d}\mathfrak{m} \leq 1$$

for any measurable functions f_i *on* X *such that* $\frac{e^{f_i}}{\int e^{f_i}\,\mathrm{d}\mathfrak{m}} \in \mathcal{P}_2(X, \mathsf{d})$ *and*

$$\sum_{i=1}^k f_i(x_i) \leq \frac{1}{2} \inf_{x \in X} \sum_{i=1}^k \mathrm{d}(x, x_i)^2 \qquad \forall x_i \in X, i = 1, 2, ..., k.$$

Proof. Let $\mu_i := \frac{e^{f_i}}{\int e^{f_i}\,\mathrm{d}\mathfrak{m}}, i = 1, 2, ..., k$ be probability measures on X. Let μ be the Wasserstein barycenter of the probability measure $\sum_{i=1}^n \frac{1}{k}\delta_{\mu_i}$ on $\mathcal{P}(X, \mathsf{d})$. By Definition 6.1, the relative entropy $\mathrm{Ent}_{\mathfrak{m}}$ satisfies the following Wasserstein Jensen's inequality

$$\mathrm{Ent}_{\mathfrak{m}}(\mu) \leq \sum_{i=1}^k \frac{1}{k}\mathrm{Ent}_{\mathfrak{m}}(\mu_i) - \frac{1}{2k}\sum_{i=1}^k W_2^2(\mu, \mu_i).$$

Therefore

$$\begin{aligned} \sum_{i=1}^k \int f_i(x_i)\,\mathrm{d}\mu_i(x_i) &\leq \frac{1}{2} \inf_{\pi \in \Pi(\mu_1, \ldots, \mu_k)} \int \inf_{x \in X} \sum_{i=1}^k \mathrm{d}(x, x_i)^2 \,\mathrm{d}\pi \\ = \quad & \frac{1}{2}\sum_{i=1}^k W_2^2(\mu, \mu_i) \leq \sum_{i=1}^k \mathrm{Ent}_{\mathfrak{m}}(\mu_i) - k\mathrm{Ent}_{\mathfrak{m}}(\mu) \\ \leq \quad & \sum_{i=1}^k \Big(\int \frac{e^{f_i}}{\int e^{f_i}} \ln \frac{e^{f_i}}{\int e^{f_i}} \,\mathrm{d}\mathfrak{m} \Big) \\ = \quad & \sum_{i=1}^k \Big(\int f_i(x_i)\,\mathrm{d}\mu_i(x_i) - \ln \int e^{f_i} \Big). \end{aligned}$$

So $\sum_{i=1}^k \ln \int e^{f_i} \leq 0$ which is the thesis. □

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