Relaxation of the area of the vortex map: a non-parametric Plateau problem for a catenoid containing a segment

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Abstract

Motivated by the study of the non-parametric area $\mathcal{A}$ of the graph of the vortex map u (a two-codimensional singular surface in $\mathbb{R}^4$) over the disc $\Omega \subset \mathbb{R}^2$ of radius l, we perform a careful analysis of the singular part of the relaxation of $\mathcal{A}$ computed at u. The precise description is given in terms of a area-minimizing surface in a vertical copy of $\mathbb{R}^3 \subset \mathbb{R}^4$, which is a sort of "catenoid" containing a segment corresponding to a radius of Ω. The problem involves an area-minimization with a free boundary part; several boundary regularity properties of the minimizer are inspected.

Key words: Relaxation, non-parametric minimal surfaces, Plateau problem, area functional in codimension two.

AMS (MOS) subject classification: 49Q15, 49Q20, 49J45, 58E12.

1 Introduction

Let $l > 0$, and $\Omega = \mathrm{B}_l(0) = \{x = (x_1, x_2) \in \mathbb{R}^2 : 0 \leq |x| < l\} \subset \mathbb{R}^2$. Consider the vortex map

$$u(x) := \frac{x}{|x|}, \qquad x \in \Omega \setminus \{0\} \tag{1.1}$$

The recent results of [2, 3] provide a formula, obtained via a relaxation procedure, for the area of the graph of u, a singular two-dimensional surface in $\mathbb{R}^4$. More specifically, consider the classical expression

$$\mathcal{A}(v, \Omega) = \int_\Omega \sqrt{1 + |\nabla v(x_1, x_2)|^2 + |Jv(x_1, x_2)|^2}\; dx_1 dx_2 \qquad \forall v \in C^1(\Omega; \mathbb{R}^2),$$

where ∇v is the gradient of v, a 2×2 matrix, $|\nabla v|^2$ is the sum of the squares of all elements of ∇v, and Jv is the Jacobian determinant of v, *i.e.*, the determinant of ∇v. Namely, $\mathcal{A}(v, \Omega)$ is the area of the graph of the smooth map v. Now, denote by

$$\overline{\mathcal{A}}(v, \Omega) := \inf \left\{ \liminf_{k \to +\infty} \mathcal{A}(v_k, \Omega) \right\} \qquad \forall v \in L^1(\Omega, \mathbb{R}^2), \tag{1.2}$$

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the sequential relaxation of $\mathcal{A}(\cdot,\Omega)$ in the L^1-convergence. The infimum in (1.2) is computed over all sequences of maps $v_k \in C^1(\Omega,\mathbb{R}^2)$ approaching v in $L^1(\Omega,\mathbb{R}^2)$. Despite $\mathcal{A}(\cdot,\Omega) = \overline{\mathcal{A}}(\cdot,\Omega)$ on $C^1(\Omega,\mathbb{R}^2)$, the computation of $\overline{\mathcal{A}}(v,\Omega)$ for $v \notin C^1(\Omega,\mathbb{R}^2)$ appears to be at the moment out of reach, due essentially to highly nonlocal phenomena created by the mutual interaction of the singularities of the map v, and also by the interaction of such singularities with the boundary of Ω [?, 18, 19]. However, in the case the map v equals the vortex map u, the following is proven in [2, 3]:

$$\overline{\mathcal{A}}(u,\mathrm{B}_l(0)) = \int_{\mathrm{B}_l(0)} \sqrt{1+|\nabla u|^2}dx + 2 \inf_{(h,\psi)\in X_l} \mathcal{F}_l(h,\psi), \tag{1.3}$$

showing the presence of the interesting singular term

$$2 \inf_{(h,\psi)\in X_l} \mathcal{F}_l(h,\psi), \tag{1.4}$$

expressed as an infimum of an appropriate functional $\mathcal{F}_l(h,\psi)$, that we are going to describe, and that will be the argument of the present paper. If we set $R_l = (0,l)\times(-1,1)$, and

$$\begin{aligned} X_l := \Big\{ &(h,\psi) : h \in L^\infty([0,l];[-1,1]), \psi \in \mathrm{BV}(R_l;[0,1]), \\ &\psi = 0 \text{ in } SG_h := \{(w_1,w_2)\in R_l : w_2 < h(w_1)\}\Big\}, \end{aligned} \tag{1.5}$$

the functional $\mathcal{F}_l : X_l \to [0,+\infty)$ in (1.3) is defined as

$$\mathcal{F}_l(h,\psi) := \overline{\mathcal{A}}(\psi,R_l) - \mathcal{H}^2(R_l \setminus SG_h) + \int_{\partial_D R_l} |\psi-\varphi|\, d\mathcal{H}^1 + \int_{(0,l)\times\{1\}} |\psi|\, d\mathcal{H}^1. \tag{1.6}$$

Since the function ψ takes scalar values, the expression $\overline{\mathcal{A}}(\psi,B)$ of the area of its graph in any Borel set $B \subseteq R_l$ (i.e., the L^1-relaxation of the localized area $\mathcal{A}(\cdot,B)$ defined on $C^1(B)$ or on $W^{1,1}(B)$) has the well-known expression

$$\overline{\mathcal{A}}(\psi,B) = \int_B \sqrt{1+|\nabla^a\psi|^2}\, dx + |D^s\psi|(B), \tag{1.7}$$

with $\nabla^a\psi$ the absolutely continuous and $D^s\psi$ the singular part of the measure $D\psi$ (the distributional gradient of ψ). Also, the subgraph of ψ, and the trace of ψ on the boundary of R_l, are well-defined by classical results on BV-functions, and so (1.6) is well-defined. Here the Dirichlet part $\partial_D R_l$ of ∂R_l is given by two of the four sides, $\partial_D R_l = (\{0\}\times[-1,1]) \cup ((0,l)\times\{-1\})$, and the Dirichlet boundary condition $\varphi : \partial_D R_l \to [0,1]$, dictated by the geometry of the vortex map, is given by $\varphi(t,s) := \sqrt{1-s^2}$ if $(t,s)\in\{0\}\times[-1,1]$ and $\varphi(t,s) := 0$ if $(t,s)\in(0,l)\times\{-1\}$; see Fig. 4. The multiplicative factor 2 in front of the infimum in (1.4) is due to the fact that we find convenient to describe the singular term, as we shall see, using a non-parametric Plateau problem, while the original relaxation for the vortex map takes into account also the area contribution of the reflected surface over the horizontal plane.

The aim of this paper is a careful analysis of the functional $\mathcal{F}_l$ and of its domain X_l, and the precise computation of (1.4) and of its minimizers; our results shed light on the geometric meaning of the two-codimensional question posed by the computation of $\overline{\mathcal{A}}(u,\Omega)$, and on its nonlocality with respect to Ω. As we shall see, a non-parametric Plateau problem with partial free boundary pops-up; its solution turns out to be an area-minimizing surface of disc-type, having trace half of a "catenoid" constrained to contain a segment (a radius of Ω). The intuitive reason for this is the following: if we pick in the source domain Ω a generic circle surrounding the origin (the singular point of u), and we look at the corresponding values over it of a minimizing sequence (v_k)

of *smooth* maps approximating u in $L^1(\Omega, \mathbb{R}^2)$, we can reasonably expect that these values almost fill a copy of $\mathbb{S}^1$ in the target plane. The question then becomes: how can v_k approximate u in such a way that the area (in $\mathbb{R}^4$) of their graphs is as small as possible? The answer given here is that the best way is to construct, in a "vertical" copy of $\mathbb{R}^3$ obtained by dropping from $\mathbb{R}^4$ an appropriate coordinate, half of a "catenoid" (together with its mirror reflection with respect to the horizonal plane), hinged exactly on a radius of Ω; morally, this produce a sort of phantom optimal "catenoidal" tube joining the singularity with the boundary of Ω. As we shall see, a technical simplification in the formulation of the problem will consist in doubling the rectangle R_l, in order to make the boundary of Ω disappear in some sense, creating instead another point singularity at distance $2l$ from the origin (a sort of image charge), where assigning a boundary condition similar to the one given above the origin.

To introduce the correct setting we need to fix some notation. Denote $R_{2l} := (0, 2l) \times (-1, 1)$ the doubled rectangle, and let $\partial_D R_{2l} := (\{0, 2l\} \times [-1, 1]) \cup ((0, 2l) \times \{-1\})$, now consisting of three sides of ∂R_{2l}. We also introduce the map $\varphi : \mathbb{R}^2 \to [0, 1]$ as

$$\varphi(w_1, w_2) := \begin{cases} \sqrt{1 - w_2^2} & \text{if } |w_2| \leq 1, \\ 0 & \text{otherwise,} \end{cases} \tag{1.8}$$

which we will employ as Dirichlet boundary datum. Let

$$S_{2l} := \{\sigma : (0, 1) \to R_{2l} \text{ Lipschitz and injective}, \sigma(0^+) = (0, 1),\ \sigma(1^-) = (2l, 1)\},$$
$$\mathcal{X}_{D,\varphi} := \{\psi \in W^{1,1}(R_{2l}) : \psi = \varphi \text{ on } \partial_D R_{2l}\},$$

where we have noted $\sigma(x_0^\pm) := \lim_{x \to x_0^\pm} \sigma(x)$. For any $\sigma \in S_{2l}$ we denote by A_σ the open planar region enclosed between $\sigma((0, 1))$ and $\partial_D R_{2l}$. Let us first consider the following minimum problem:

$$\inf\{\mathcal{A}(\psi, A_\sigma) : (\sigma, \psi) \in S_{2l} \times \mathcal{X}_{D,\varphi},\ \psi = 0 \text{ on } \sigma((0, 1))\}. \tag{1.9}$$

Since this problem in general has not a minimizer, we need a relaxed formulation. However we prefer not to directly relax problem (1.9), but instead we reduce to a cartesian setting, where the free-boundary curve σ becomes the graph of a function h defined on $(0, 2l)$ and A_σ its subgraph SG_h: namely, we introduce

$$\widetilde{\mathcal{H}}_{2l} := \{h : [0, 2l] \to [-1, 1] \text{ continuous},\ h(0) = h(2l) = 1\},$$

and for any $h \in \widetilde{\mathcal{H}}_{2l}$ set $G_h := \{(t, s) \in R_{2l} : s = h(t)\}$ and $SG_h := \{(t, s) \in R_{2l} : s < h(t)\}$ (see Fig. 4). We then consider the minimum problem

$$\inf\{\mathcal{A}(\psi, SG_h) : (h, \psi) \in \widetilde{\mathcal{H}}_{2l} \times \mathcal{X}_{D,\varphi},\ \psi = 0 \text{ on } G_h\}. \tag{1.10}$$

Focusing on this, we will show that it is sufficient to restric attention to convex functions h, and thus we introduce a sort of relaxed functional, namely

$$\mathcal{F}_{2l}(h, \psi) := \overline{\mathcal{A}}(\psi; R_{2l}) - \mathcal{H}^2(R_{2l} \setminus SG_h) + \int_{\partial_D R_{2l}} |\psi - \varphi| d\mathcal{H}^1 + \int_{\partial R_{2l} \setminus \partial_D R_{2l}} |\psi|\, d\mathcal{H}^1, \tag{1.11}$$

defined on

$$X_{2l}^{\text{conv}} := \{(h, \psi) : h \in \mathcal{H}_{2l}, \psi \in BV(R_{2l}, [0, 1]), \psi = 0 \text{ on } R_l \setminus SG_h\}, \tag{1.12}$$

where

$$\mathcal{H}_{2l} = \big\{h : [0, 2l] \to [-1, 1],\ h \text{ convex},\ h(w_1) = h(2l - w_1)\ \forall w_1 \in [0, 2l]\big\}. \tag{1.13}$$

Notice that, with respect to $\widetilde{\mathcal{H}}_{2l} \times \mathcal{X}_{D,\varphi}$, the class $X_{2l}^{\rm conv}$ is obtained specializing the choice of h but generalizing the choice of ψ.

Then we will prove (Theorem 1.1) that

$$\inf_{(h,\psi)\in X_{2l}^{\rm conv}} \mathcal{F}_{2l}(h,\psi) = \inf\{\mathcal{A}(\psi, SG_h) : (h,\psi) \in \widetilde{\mathcal{H}}_{2l} \times \mathcal{X}_{D,\varphi},\ \psi = 0 \text{ on } G_h\}, \tag{1.14}$$

and that, for l large enough, the infimum on the right-hand side is not attained in $\widetilde{\mathcal{H}}_{2l} \times \mathcal{X}_{D,\varphi}$ and equals π. Instead a minimizer $(\overline{h}, \overline{\psi}) \in \widetilde{\mathcal{H}}_{2l} \times \mathcal{X}_{D,\varphi}$ exists for l small, and $\overline{\psi}$ is real analytic in the interior of $SG_{\overline{h}}$; furthermore, we show that $\overline{h}$ is smooth and convex, and $\overline{\psi}$ has vanishing trace on $G_{\overline{h}}$.

Thus, problem (1.10) and the minimization of the functional in (1.11) are related through (1.14); now, let us see how they are related with (1.9). An equivalent formulation of (1.14) is the following: for $\sigma \in S_{2l}$, take the closed curve $\Gamma \subset \mathbb{R}^3$ defined by glueing the trace of σ with the graph of φ over $\partial_D R_{2l}$. We can then consider an area-minimizing disc Σ^+ spanning Γ, solution of the classical parametric Plateau problem. If $\sigma([0,1])$ is the graph of a function $h \in \widetilde{\mathcal{H}}_{2l}$, then

$$\inf\{\mathcal{A}(\psi, SG_h) : (h,\psi) \in \widetilde{\mathcal{H}}_{2l} \times \mathcal{X}_{D,\varphi},\ \psi = 0 \text{ on } G_h\} = \inf \mathcal{H}^2(\Sigma^+), \tag{1.15}$$

where the infimum on the right-hand side is computed over the set of all such curves σ (see Corollary 4.14). In turn, we will prove that

$$\inf \mathcal{H}^2(\Sigma^+) = \inf\{\mathcal{A}(\psi, A_\sigma) : (\sigma,\psi) \in S_{2l} \times \mathcal{X}_{D,\varphi},\ \psi = 0 \text{ on } \sigma((0,1))\}, \tag{1.16}$$

so that the original problem (1.9) is equivalent to (1.10).

Indeed, for l sufficiently small, say $l \in (0, l_0)$, the infimum on the left-hand side of (1.16) is attained by a disc-type surface Σ^+, and $\sigma([0,1])$ coincides with the graph of a smooth convex function $h \in \widetilde{\mathcal{H}}_{2l}$. Also, the surface Σ_+ is cartesian, i.e. is the graph of a function $\psi \in \mathcal{X}_{D,\varphi}$. On the contrary, for $l \geq l_0$, Σ^+ is degenerate, in the sense that if $\sigma_n((0,1)) \subset R_{2l}$ is the free-boundary of Σ_n^+, where (Σ_n^+) is a minimizing sequence of discs for the Plateau problem, then $\sigma_n((0,1))$ converges to $\partial_D R_{2l}$ and Σ_n^+ converges to two distinct half-circles of radius 1, whose total area is π.

We do not know the explicit value of the threshold l_0. However, it is clear that $l_0 > \frac{1}{2}$ (see the discussion at the end of Section 2.2 and Remark 2.1). Furthermore, if we double the surface Σ^+ by considering its symmetric with respect to the plane containing R_{2l}, and then taking the union Σ of these two area-minimizing surfaces, it turns out that Σ solves a non-standard Plateau problem, spanning a nonsimple curve which shows self-intersections (this is the union of Γ with its symmetric with respect to R_{2l}, the obtained curve is the union of two circles connected by a segment, see Section 2.2 and Fig. 1). Again, the obtained area-minimizing surface is a sort of catenoid forced to contain a segment (see Fig. 2, left) for l small, and two distinct discs spanning the two circles for l large (Fig. 2, right).

A related analysis of a general geometric setting for constrained non-parametric Plateau problems can be found in [4]. Notice however that in [4] it is assumed positiveness of the boundary datum, while in our case it is crucial that $\varphi = 0$ on $\{-1\} \times (0, 2l)$; it must be observed that the vanishing of φ a some points is source of a number of difficulties. Furthermore we need here to relate the Plateau problem to the functional $\mathcal{F}_l$ in (1.6), analysis which is missing in [4]. Indeed, in the special setting of the present paper, some more precise description of solutions is necessary. Specifically (see Corollary 4.12) every solution will be continuous and null on its free boundary curve.

Before stating our main results, it is worth to point out that the solution surface that we obtain is related to the problem of relaxation of area functional in codimension > 1. Indeed, the restriction of Σ to the set $\overline{B}_1(0) \times [0, l]$ is a suitable projection in $\mathbb{R}^3$ of the vertical part of an optimal cartesian current with underlying map the vortex map (see the introduction of [3]).

In Proposition 3.4 below, we first prove that the infimum of $\mathcal{F}_l$ on the class X_l can be equivalently computed on pairs $(h,\psi) \in X_l^{\rm conv}$, where the function h is assumed to be convex, nonincreasing and continuous on $[0,l]$. Then, using a symmetry argument, it is easily seen that

$$2 \inf_{(h,\psi)\in X_l^{\rm conv}} \mathcal{F}_l(h,\psi) = \inf_{(h,\psi)\in X_{2l}^{\rm conv}} \mathcal{F}_{2l}(h,\psi).$$

The main results of this paper are contained in the following two theorems.

Theorem 1.1. *There exists a solution of*

$$\min\left\{\mathcal{F}_{2l}(h,\psi) : (h,\psi) \in X_{2l}^{\rm conv}\right\}. \tag{1.17}$$

Moreover, any minimizing pair (h,ψ) *in* (1.17) *satisfies the following properties:*

(1) ψ *is symmetric with respect to* $\{w_1 = l\} \cap R_{2l}$*;*

(2) If h *is not identically* -1*, then*

(2i) $h \in C^0([0,2l])$ *and is analytic in* $(0,2l)$*,* $h(0) = 1 = h(2l)$*, and* $h > -1$ *in* $(0,2l)$*;*

(2ii) ψ *is analytic and strictly positive in* SG_h*;*

(2iii) ψ *is continuous up to the boundary of* SG_h*, and attains the boundary condition, i.e., for* $(w_1,w_2) \in \partial SG_h$*,*

$$\psi(w_1,w_2) = \begin{cases} 0 & \text{if} \quad w_2 = -1 \\ 0 & \text{if} \quad w_2 = h(w_1), \\ \sqrt{1-{w_2}^2} & \text{if} \quad w_1 = 0 \text{ or } w_1 = 2l, \end{cases} \tag{1.18}$$

hence

$$\mathcal{F}_{2l}(h,\psi) = \mathcal{A}(\psi, SG_h); \tag{1.19}$$

(2iv) $\psi < \varphi$ in R_{2l}.

Theorem 1.2. *Problem* (1.3) *has a solution, and*

$$\begin{aligned} 2 \inf_{(h,\psi)\in X_l} \mathcal{F}_l(h,\psi) &= \inf\{\mathcal{A}(\psi, A_\sigma) : (\sigma,\psi) \in S_{2l} \times \mathcal{X}_{D,\varphi},\ \psi = 0 \text{ on } \sigma((0,1))\} \\ &= \inf\{\mathcal{A}(\psi, SG_h) : (h,\psi) \in \widetilde{\mathcal{H}}_{2l} \times \mathcal{X}_{D,\varphi},\ \psi = 0 \text{ on } G_h\} \\ &= \inf_{(h,\psi)\in X_{2l}^{\rm conv}} \mathcal{F}_{2l}(h,\psi). \end{aligned} \tag{1.20}$$

Furthermore, there is a constant $l_0 > 0$ *such that the following holds:*

(i) for $l \in (0,l_0)$ *there is a minimizer* $(h^\star,\psi^\star) \in X_{2l}^{\rm conv}$ *of* $\mathcal{F}_{2l}$ *satisfying conditions (1) and (2i)-(2iv) of Theorem 1.1, which is also a minimizer of* (1.10)*,* $(h^\star \llcorner [0,l], \psi^\star \llcorner R_l)$ *is a minimizer of* $\mathcal{F}_l$*, and the pair* (σ^*,ψ^*)*, with* $\sigma^\star(t) := (2lt, h^\star(t))$*, is a solution of* (1.9)*;*

(ii) For $l \geq l_0$ *a common solution is* $h^\star \equiv -1$ *and* $\psi \equiv 0$*.*

The organization of the paper is as follows. Section 2 contains some notation used throughout the paper. Section 3 contains some preliminary results on the functional $\mathcal{F}_l$ and its doubled, on their domains and on relaxation. Section 4 contains the proofs of Theorems 1.1 and 1.2; the most difficult part is contained in the regularity Theorem 4.11.

2 Notation and preliminaries

Let $O \subset \mathbb{R}^n$ be an open set, and let $v \in L^1(O)$. We denote by $R(v)$ the set of regular points of v, *i.e.*, the set consisting of all $x \in O$ which are Lebesgue points for v and $v(x)$ coincides with the Lebesgue value of v at x. We denote by $\overline{v}$ a good representative of $v \in L^1(O)$, i.e. a function such that $\overline{v}(x)$ coincides with the Lebesgue value of v at x, for all $x \in R(v)$. If in addition v is a function of bounded variation, we denote by R_v the set of regular points $x \in R(v)$ such that v is approximately differentiable at x. Notice that if $v \in BV(O)$ the set $R(v) \setminus R_v$ is $\mathcal{L}^n$-negligible.

For a function $v \in L^1(O) \setminus BV(O)$ we also set

$$G'_v := \{(x, \overline{v}(x)) \in R(v) \times \mathbb{R}\}, \qquad SG'_v := \{(x, y) \in R(v) \times \mathbb{R} : y < \overline{v}(x)\}. \tag{2.1}$$

If in addition $v \in BV(O)$, the previous sets are classically defined as above with $R(v)$ replaced by R_v, namely

$$G_v := \{(x, v(x)) \in R_v \times \mathbb{R}\}, \qquad SG_v := \{(x, y) \in R_v \times \mathbb{R} : y < v(x)\}.$$

Given a 2-dimensional rectifiable set $S \subset U \subset \mathbb{R}^3$, U open, and a tangent unit simple 2-vector τ to it, we denote by $[\![S]\!]$ the current given by integration over S, namely

$$[\![S]\!](O) = \int_S <\tau(x), \omega(x)> \, d\mathcal{H}^2(x),$$

ω a smooth 2-form with compact support in U, see [14], [10], [9]. We often will identify SG'_v with the integral 3-current $[\![SG'_v]\!] \in \mathcal{D}_3(O \times \mathbb{R})$. If v is a function of bounded variation, $O \setminus R_v$ has zero Lebesgue measure, so that the current $[\![SG_v]\!]$ coincides with the standard integration over the subgraph of v. It is well-known that the perimeter of SG_v in $O \times \mathbb{R}$ coincides with $\overline{\mathcal{A}}(v, O)$.

The support of the boundary of $[\![SG_v]\!]$ includes the graph G_v, but in general consists also of additional parts, called vertical. We denote by

$$\mathcal{G}_v := \partial [\![SG_v]\!] \llcorner (O \times \mathbb{R}),$$

the generalized graph of v, which is a 2-integral current supported on $\partial^* SG_v$, the reduced boundary of SG_v in $O \times \mathbb{R}$.

Let $\widehat{O} \subset \mathbb{R}^2$ be a bounded open set such that $O \subseteq \widehat{O}$, and suppose that $L := \widehat{O} \cap \partial O$ is a rectifiable curve. Given $\psi \in BV(O)$ and a $W^{1,1}$ function $\varphi : \widehat{O} \to \mathbb{R}$, we can consider

$$\overline{\psi} := \begin{cases} f & \text{on } O, \\ \varphi & \text{on } \widehat{O} \setminus O. \end{cases}$$

Then (see [11], [1])

$$\overline{\mathcal{A}}(\overline{\psi}, \widehat{O}) = \overline{\mathcal{A}}(\psi, O) + \int_L |\psi - \varphi| d\mathcal{H}^1 + \overline{\mathcal{A}}(\varphi, \widehat{O} \setminus \overline{O}).$$

2.1 Plateau problem in parametric form

We report here some results about the classical solution to the disc-type Plateau problem. Let D denote the open disc of radius 1 centered at the origin of $\mathbb{R}^2$. If $\Gamma \subset \mathbb{R}^3$ is a closed rectifiable Jordan curve, the Plateau problem consists into minimize the functional

$$\mathcal{P}_\Gamma(X) := \int_{\mathrm{D}} |\partial_{x_1} X \wedge \partial_{x_2} X| dx_1 dx_2, \tag{2.2}$$

on the class of all functions $X \in C^0(\overline{B}_1;\mathbb{R}^3) \cap H^1(\mathrm{D};\mathbb{R}^3)$ with $X \llcorner \partial\mathrm{D}$ being a weakly monotonic parametrization of the curve Γ. The functional (2.2) measures the area (with multiplicity) of the surface $X(\mathrm{D})$.

A solution X_Γ to the Plateau problem exists and satisfies the properties: it is harmonic (hence analytic)

$$\Delta X_\Gamma = 0 \qquad \text{in } D,$$

it is a conformal parametrization

$$|\partial_{x_1} X_\Gamma|^2 = |\partial_{x_2} X_\Gamma|^2, \qquad \partial_{x_1} X_\Gamma \cdot \partial_{x_2} X_\Gamma = 0 \qquad \text{in D},$$

and $X_\Gamma \llcorner \partial\mathrm{D}$ is a strictly monotonic parametrization of Γ. We will say that the surface $X_\Gamma(\mathrm{D})$ has the topology of the disc.

Thanks to the properties above it is always possible, with the aid of a conformal change of variables, to parametrize $X(\mathrm{D})$ over any simply connected bounded domain. In other words, if U is any such domain, and if $\Phi : U \to \mathrm{D}$ is any conformal homeomorphism, then $X \circ \Phi$ is a solution to the Plateau problem on U.

2.2 A Plateau problem for a self-intersecting boundary space curve

The classical disc-type Plateau problem is solved for boundary value a simple Jordan space curve, in particular Γ does not have self-intersections. Here we will treat a specific Plateau problem where the curve Γ has non-trivial intersections, and it overlaps itself on a segment which is parametrized two times with opposite directions.

Specifically, we consider the cylinder $(0, 2l) \times \mathrm{D}$ and two circles $\mathcal{C}_1, \mathcal{C}_2$ which are the boundaries of its two circular bases, namely $\mathcal{C}_1 := \{0\} \times \partial\mathrm{D}$ and $\mathcal{C}_2 := \{2l\} \times \partial\mathrm{D}$. Then we take the segment $(0, 2l) \times \{1\} \times \{0\}$. If γ_0 is a monotonic parametrization of this segment, starting from $(0, 1, 0)$ up to $(2l, 1, 0)$, γ_1 is a monotonic parametrization of $\mathcal{C}_1$ starting from the point $(0, 1, 0)$ and ending at the same point, and γ_2 a parametrization of $\mathcal{C}_2$ with initial and final point $(2l, 1, 0)$ with the same orientation of $\mathcal{C}_1$, then we consider the parametrization

$$\gamma := \gamma_1 \star \gamma_0 \star (-\gamma_2) \star (-\gamma_0), \tag{2.3}$$

(read from left to right) which is a closed curve in $\mathbb{R}^3$ which travels two times across the segment $(0, 2l) \times \{1\} \times \{0\}$ with opposite directions (the orientation of this curve is depicted in Fig. 1). We want to solve the Plateau problem with Γ to be the image of γ.

The existence of solutions to the Plateau problem spanning self-intersecting boundaries has been addressed in [12], whose results have been recently improved in [7]. Without entering deeply into the details, it is known that, depending on the geometry of γ (in this case, depending on the distance between the two circles $\mathcal{C}_1$ and $\mathcal{C}_2$) two kind of solutions are expected:

(a) The solution consists of two discs filling $\mathcal{C}_1$ and $\mathcal{C}_2$, see Fig. 2, right. In this case, a parametrization of it $X : \overline{\mathrm{D}} \to \mathbb{R}^3$ can be chosen so that, if L_1 and L_2 are two parallel chords in D dividing D in three sectors, then X restricted to the sector enclosed between L_1 and L_2 parametrizes the segment γ_0 (and then its resulting area is null), $X(L_1) = P_1$ and $X(L_2) = P_2$ are the two endpoints of γ_0, and X restricted to the sectors between L_i, $i = 1, 2$, and $\partial\mathrm{D}$ parametrizes the disc filling $\mathcal{C}_i$, $i = 1, 2$. Moreover the map X can be still taken Sobolev regular (see [7] for details).

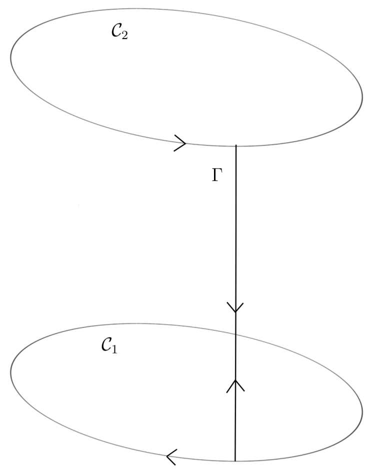

Figure 1: The self-overlapping curve Γ with its orientation.

(b) There is a classical solution, *i.e.*, there is a harmonic and conformal map $X : \mathrm{D} \to \mathbb{R}^3$, continuous up to the boundary of D, such that $X \llcorner \partial \mathrm{D}$ is a weakly monotonic parametrization of Γ. In this case the resulting minimal surface is a sort of catenoid attached to the segment $(0, 2l) \times \{(1, 0)\}$ (see Fig. 2 left).

Remark 2.1. We expect that there is a threshold l_0 such that if $l < l_0$ an area-minimizing disc with boundary γ is of the form (b), and for values $l > l_0$ the two discs have minimal area. We do not find explicitly l_0 but it is easy to see that if $l \le \frac{1}{2}$ an area-minimizing disc with boundary γ has always less area than the solution with two discs. Indeed, the area of the two discs is 2π, whereas we can always compare the area of the surface Σ as in (b) with the area of the lateral surface of the cylinder $(0, 2l) \times \mathrm{D}$, that is $4l\pi$. Hence $\mathcal{H}^2(\Sigma) < 4l\pi \le 2\pi$ for $l \le \frac{1}{2}$.

3 Preliminary results on the functional $\mathcal{F}_l$ and its doubled

For all $\varrho > 0$ we denote $R_\varrho := (0, \varrho) \times (-1, 1)$. For any $h \in L^\infty([0, \varrho]; [-1, 1])$, we denote by G'_h and SG'_h the sets in (2.1). We recall that φ has been defined in (1.8).

Definition 3.1 (The functional $\mathcal{F}_l$). *Let $l > 0$ be fixed. Given $h \in L^\infty([0, l]; [-1, 1])$ and $\psi \in \mathrm{BV}(R_l; [0, 1])$ we define*

$$\mathcal{F}_l(h, \psi) := \overline{\mathcal{A}}(\psi, R_l) - \mathcal{H}^2(R_l \setminus SG'_h) + \int_{\partial_D R_l} |\psi - \varphi| \, d\mathcal{H}^1 + \int_{(0,l)\times\{1\}} |\psi| \, d\mathcal{H}^1. \tag{3.1}$$

We notice that if h is also a function of bounded variation, then we can also write

$$\mathcal{F}_l(h, \psi) := \overline{\mathcal{A}}(\psi, R_l) - \mathcal{H}^2(R_l \setminus SG_h) + \int_{\partial_D R_l} |\psi - \varphi| \, d\mathcal{H}^1 + \int_{(0,l)\times\{1\}} |\psi| \, d\mathcal{H}^1. \tag{3.2}$$

We further remember from (1.5) the definition of X_l:

$$X_l := \{(h, \psi) : h \in L^\infty([0, l]; [-1, 1]), \psi \in \mathrm{BV}(R_l; [0, 1]), \psi = 0 \text{ in } R_l \setminus SG'_h\}. \tag{3.3}$$

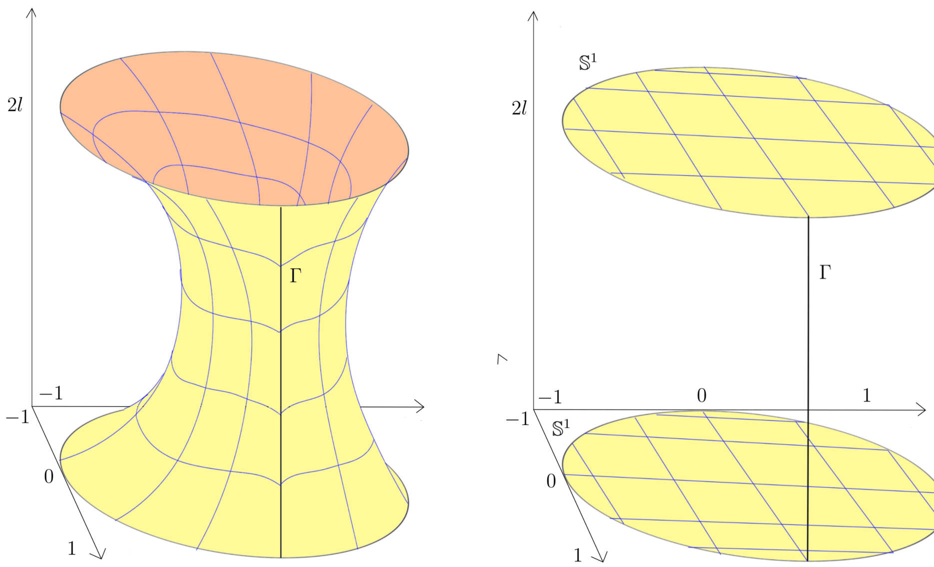

Figure 2: on the left the shape of a possible solution to the Plateau problem with boundary Γ. On the right another solution to the Plateau problem with boundary Γ. See Section 2.2

In this section our concern is the analysis of the minimum problem

$$\inf_{(h,\psi)\in X_l} \mathcal{F}_l(h,\psi). \tag{3.4}$$

Notice that in minimizing $\mathcal{F}_l$ we have a free boundary condition on the edge $\{l\}\times[-1,1]$.

Remark 3.2. Let $(h,\psi)\in X_l$. If $t_0\in(0,l)$ is a regular point for h (i.e. $t_0\in R(h)$) and if $\overline{h}(t_0)<1$, then the trace of ψ over the segment $\{w_1=t_0,\ \overline{h}(t_0)\le w_2\le 1\}$ vanishes. Indeed for any $\eta>0$ we can find $\delta_\eta>0$ such that

$$\frac{1}{2\delta}\int_{t_0-\delta}^{t_0+\delta}|h(w_1)-\overline{h}(t_0)|\,dw_1<\eta \qquad \forall\delta\in(0,\delta_\eta). \tag{3.5}$$

Let now $s_0\in(-1,1)$ be such that $\overline{h}(t_0)<s_0\le 1$ (*i.e.*, $(t_0,s_0)\in\{w_1=t_0,\ w_2>\overline{h}(t_0)\}$), and set $2\Delta:=s_0-\overline{h}(t_0)$. By Chebyschev inequality and (3.5) it follows that

$$\mathcal{H}^1(B_\Delta)\le\frac{2\delta\eta}{\Delta} \qquad \text{where} \quad B_\Delta:=\{w_1\in(t_0-\delta,t_0+\delta):|h(w_1)-\overline{h}(t_0)|>\Delta\}. \tag{3.6}$$

Then, for any $\xi\in(0,\Delta)$ we infer[1]

$$\begin{aligned}\frac{1}{2\delta}\int_{t_0-\delta}^{t_0+\delta}\int_{s_0-\xi}^{s_0+\xi}\psi(w_1,w_2)\,dw_2dw_1 &\le \frac{1}{2\delta}\int_{t_0-\delta}^{t_0+\delta}\int_{s_0-\xi}^{s_0+\xi}\chi_{\{\psi>0\}}(w_1,w_2)\,dw_2dw_1\\ \le\frac{1}{2\delta}\int_{t_0-\delta}^{t_0+\delta}\int_{s_0-\xi}^{s_0+\xi}\chi_{SG_h}(w_1,w_2)\,dw_2dw_1 &\le\frac{\xi}{\delta}\int_{t_0-\delta}^{t_0+\delta}\chi_{B_\Delta}(w_1)dw_1\le\frac{2\xi\eta}{\Delta},\end{aligned} \tag{3.7}$$

where the penultimate inequality follows from the inclusions

$$\begin{aligned}SG_h'\cap\big([t_0-\delta,t_0+\delta]\times[s_0-\xi,s_0+\xi]\big)&\subseteq SG_h'\cap\big([t_0-\delta,t_0+\delta]\times[s_0-\Delta,s_0+\Delta]\big)\\ &\subseteq B_\Delta\times[s_0-\Delta,s_0+\Delta],\end{aligned}$$

[1]In the first inequality we have used that $0\le\psi\le 1$; in the second inequality that SG_h' is the subgraph of h in $(0,l)\times(-1,1)$; in the third inequality we have used that $s_0-h(t_0)=2\Delta$ and that $\xi<\Delta$.

and the last inequality follows from (3.6). Now (3.7) entails the claim by the arbitrariness of $\eta > 0$ and since $\psi \geq 0$.

Now, we refine the choice of the class of pairs (h, ψ) appearing in the infimum in (3.4).

Definition 3.3 (The classes $\mathcal{H}_l$ and $X_l^{\rm conv}$). *We set*

$$\begin{aligned}\mathcal{H}_l &:= \{h \in L^\infty([0,l];[-1,1]) : \ h \text{ convex and nonincreasing in } [0,l], \ h(0) = 1\},\\ X_l^{\rm conv} &:= \{(h,\psi) \in X_l : \ h \in \mathcal{H}_l\}.\end{aligned}$$

Proposition 3.4 (Convexifying h). *We have*

$$\inf_{(h,\psi)\in X_l} \mathcal{F}_l(h,\psi) = \inf_{(h,\psi)\in X_l^{\rm conv}} \mathcal{F}_l(h,\psi). \tag{3.8}$$

Proof. It is enough to show the inequality "$\geq$". By extending ψ outside the open rectangle R_l as $\psi := 0$ in $((0,l)\times\mathbb{R}) \setminus R_l$, we see that

$$\mathcal{F}_l(h,\psi) = \overline{\mathcal{A}}\left(\psi, \overline{R}_l \setminus \left(\{w_1 = 0\} \cup \{w_1 = l\}\right)\right) - \mathcal{H}^2(R_l \setminus SG'_h) + \int_{\{0\}\times[-1,1]} |\psi^- - \varphi| \, d\mathcal{H}^1, \tag{3.9}$$

where (see (1.7))

$$\overline{\mathcal{A}}\left(\zeta, \overline{R}_l \setminus \left(\{w_1 = 0\} \cup \{w_1 = l\}\right)\right) = \overline{\mathcal{A}}(\zeta, R_l) + \int_{(0,l)\times\{1,-1\}} |\zeta^-| \, d\mathcal{H}^1,$$

ζ^- being the trace of $\zeta \in BV(R_l)$ on $(0,l)\times\{1,-1\}$.

The thesis of the proposition will follow from the next three observations:

(1) If $h \in \mathcal{H}_l$ is such that $\overline{h}(t_0) = -1$ for some regular point $t_0 \in (0,l)$, then the subgraph SG_h of h splits in two mutually disjoint components: $(SG'_h)^- = SG'_h \cap \{w_1 < t_0\}$ and $(SG'_h)^+ = SG'_h \cap \{w_1 > t_0\}$. Let $\psi \in BV(R_l, [0,1])$ be such that

$$\psi = 0 \qquad \text{a.e. in } \ R_l \setminus SG'_h.$$

The trace of ψ over the segment $\{w_1 = t_0, \ \overline{h}(t_0) \leq w_2 \leq 1\}$ is 0, as a consequence of Remark 3.2. Then the function $\psi^\star : R_l \to [0,1]$ defined as

$$\psi^\star(w_1,w_2) := \begin{cases} \psi(w_1,w_2) & \text{if } w_1 < t_0,\\ 0 & \text{otherwise,}\end{cases}$$

still satisfies $(h,\psi^\star) \in X_l$, and

$$\mathcal{F}_l(h,\psi^\star) \leq \mathcal{F}_l(h,\psi).$$

Being $\psi^\star$ identically zero in $\{w_1 > t_0\}$, in particular in $SG'_h \cap \{w_1 > t_0\}$, we can introduce

$$h^\star(w_1) := \begin{cases} h(w_1) & \text{if } w_1 < t_0,\\ -1 & \text{otherwise,}\end{cases}$$

so that $(h^\star, \psi^\star) \in X_l$ and we easily see that $\mathcal{F}_l(h^\star,\psi^\star) \leq \mathcal{F}_l(h,\psi^\star)$; hence

$$\mathcal{F}_l(h^\star,\psi^\star) \leq \mathcal{F}_l(h,\psi).$$

(2) More generally, let $(h,\psi) \in X_l$ and let $t_0 \in (0,l)$ be any regular point of h; we can also suppose that $\overline{h}(t_0) < 1$. Consider

$$h^\star(w_1) := \begin{cases} h(w_1) & \text{if } w_1 < t_0, \\ h(w_1) \wedge \overline{h}(t_0) & \text{otherwise,} \end{cases} \tag{3.10}$$

$$\psi^\star(w_1,w_2) := \begin{cases} \psi(w_1,w_2) & \text{if } w_1 < t_0, \\ \psi(w_1,w_2) & \text{if } w_1 \geq t_0,\ w_2 \leq \overline{h}(t_0), \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $\mathcal{F}_l(h^\star,\psi^\star) \leq \mathcal{F}_l(h,\psi)$. Define

$$U := \{(w_1,w_2) \in (0,l) \times (-1,1) : w_1 > t_0,\ \overline{h}(t_0) < w_2 < h(w_1)\},$$

that is the set where we have replaced ψ by 0. To prove the claim, using (3.9) and the equalities

$$\int_{\{0\}\times[-1,1]} |\psi^- - \varphi|\, d\mathcal{H}^1 = \int_{\{0\}\times[-1,1]} |\psi^{\star -} - \varphi|\, d\mathcal{H}^1,$$

$$\mathcal{H}^2(R_l \setminus SG'_{h^\star}) = \mathcal{H}^2(U \cup (R_l \setminus SG'_h)) = \mathcal{H}^2(U) + \mathcal{H}^2(R_l \setminus SG'_h),$$

we have to show that

$$\overline{\mathcal{A}}\left(\psi^\star, \overline{R}_l \setminus \left(\{w_1=0\} \cup \{w_1=l\}\right)\right) \leq \overline{\mathcal{A}}\left(\psi, \overline{R}_l \setminus \left(\{w_1=0\} \cup \{w_1=l\}\right)\right) + \mathcal{H}^2(U). \tag{3.11}$$

Assume that U is non-empty and that $\mathcal{H}^2(U) > 0$. It is convenient to introduce

$$V := \{(w_1,w_2) \in R_l : t_0 < w_1 < l,\ h(w_1) \vee \overline{h}(t_0) \leq w_2 < 1\},$$

so that $U \cup V = \{(w_1,w_2) : w_1 > t_0,\ \overline{h}(t_0) < w_2 < 1\}$ is an open rectangle. Since we have modified ψ only in U, inequality (3.11) is equivalent to

$$\begin{aligned} &\overline{\mathcal{A}}(\psi^\star, U \cup V) + \int_{(t_0,l)\times\{h(t_0)\}} |\psi^{\star +} - \psi^{\star -}| d\mathcal{H}^1 + \int_{(t_0,l)\times\{1\}} |\psi^{\star -}| d\mathcal{H}^1 \\ \leq &\overline{\mathcal{A}}(\psi, U \cup V) + \int_{(t_0,l)\times\{h(t_0)\}} |\psi^+ - \psi^-| d\mathcal{H}^1 + \int_{(t_0,l)\times\{1\}} |\psi^-| d\mathcal{H}^1 + \mathcal{H}^2(U), \end{aligned} \tag{3.12}$$

with $\psi^\pm$ (resp. $\psi^{\star\pm}$) the external and internal traces of ψ (resp. $\psi^\star$) on $\partial(U \cup V)$; here we have used from Remark 3.2 that the trace of ψ on $\{t_0\} \times (h(t_0),1)$ is zero (hence $\int_{\{t_0\}\times(\overline{h}(t_0),1)} |\psi^+ - \psi^-| d\mathcal{H}^1 = \int_{\{t_0\}\times(\overline{h}(t_0),1)} |\psi^{\star +} - \psi^{\star -}| d\mathcal{H}^1 = 0$) and that the external traces ψ^+, $\psi^{\star +}$ on $(t_0,l) \times \{1\}$ vanish as well. Hence, exploiting that $\psi^\star = 0$ on $U \cup V$, so that $\overline{\mathcal{A}}(\psi^\star, U \cup V) = \mathcal{H}^2(U) + \mathcal{H}^2(V)$, and that $\psi^\star = \psi$ on $R_l \setminus (U \cup V)$, inequality (3.12) is equivalent to

$$\begin{aligned} &\mathcal{H}^2(V) + \int_{(t_0,l)\times\{\overline{h}(t_0)\}} |\psi^+| d\mathcal{H}^1 \\ \leq &\overline{\mathcal{A}}(\psi, U \cup V) + \int_{(t_0,l)\times\{\overline{h}(t_0)\}} |\psi^+ - \psi^-| d\mathcal{H}^1 + \int_{(t_0,l)\times\{1\}} |\psi^-| d\mathcal{H}^1. \end{aligned} \tag{3.13}$$

We split

$$(t_0,l) = H_1 \cup H_2 \cup H_3,$$

with $H_1 := \{w_1 \in (t_0, l) : h(w_1) = 1\}$, $H_2 := \{w_1 \in (t_0, l) : \overline{h}(t_0) \le h(w_1) < 1\}$, and $H_3 := \{w_1 \in (t_0, l) : h(w_1) < \overline{h}(t_0)\}$. Since $\overline{\mathcal{A}}(\psi; U \cup V) = \mathcal{H}^2(\mathcal{G}_\psi \cap ((U \cup V) \times \mathbb{R}))$, by slicing and looking at $\mathcal{G}_\psi$ as an integral current [14], [10], [9], we have[2]

$$\begin{aligned}
\overline{\mathcal{A}}(\psi, U \cup V) &\ge \int_{(t_0,l)} \mathcal{H}^1\Big((\mathcal{G}_\psi)_t \cap ((t_0,l) \times (\overline{h}(t_0),1) \times \mathbb{R})\Big)\, dt \\
&\ge \int_{(t_0,l)} \int_{(\overline{h}(t_0),1)} |D_{w_2}\psi(t,s)| \ \, dt + \mathcal{H}^2(V) \\
&= \int_{H_1 \cup H_2} \int_{(\overline{h}(t_0),1)} |D_{w_2}\psi(t,s)| \ \, dt + \mathcal{H}^2(V) \\
&\ge \int_{H_2} |\psi^-(t, \overline{h}(t_0))|\, dt + \int_{H_1} |\psi^-(t, \overline{h}(t_0)) - \psi^-(t,1)|\, dt + \mathcal{H}^2(V) \\
&\ge \int_{H_1 \cup H_2} |\psi^-(t, \overline{h}(t_0))|\, dt - \int_{H_1} |\psi^-(t,1)|\, dt + \mathcal{H}^2(V) \\
&= \int_{(t_0,l)} |\psi^-(t, \overline{h}(t_0))|\, dt - \int_{H_1} |\psi^-(t,1)|\, dt + \mathcal{H}^2(V) \\
&= \int_{(t_0,l)} |\psi^-(t, \overline{h}(t_0))|\, dt - \int_{(t_0,l)} |\psi^-(t,1)|\, dt + \mathcal{H}^2(V),
\end{aligned}$$

where $(\mathcal{G}_\psi)_t$ is the slice of $\mathcal{G}_\psi$ on the plane $\{w_1 = t\}$, that is the generalized graph of the function $\psi \llcorner \{w_2 = t\}$. From the above expression, the triangular inequality implies (3.13).

(3) Let $(h, \psi) \in X_l$. Let $t_1, t_2 \in (\varepsilon, l)$ be regular points for h with $t_1 < t_2$, and let $r_{12}(t) := \overline{h}(t_1) + \frac{\overline{h}(t_2) - \overline{h}(t_1)}{t_2 - t_1}(t - t_1)$. We consider the following modifications of h and ψ:

$$h^{\#}(w_1) := \begin{cases} h(w_1) & \text{if } 0 < w_1 < t_1 \text{ or } l > w_1 > t_2, \\ h(w_1) \wedge r_{12}(w_1) & \text{otherwise,} \end{cases}$$

and

$$\psi^{\#}(w_1, w_2) := \begin{cases} \psi(w_1, w_2) & \text{if } 0 < w_1 < t_1 \text{ or } l > w_1 > t_2, \\ \psi(w_1, w_2) & \text{if } w_1 \in [t_1, t_2] \text{ and } w_2 \le r_{12}(w_1), \\ 0 & \text{otherwise.} \end{cases}$$

In other words we set ψ equal to 0 above the segment L_{12} connecting $(t_1, \overline{h}(t_1))$ to $(t_2, \overline{h}(t_2))$. Also in this case we have

$$\mathcal{F}_l(h^{\#}, \psi^{\#}) \le \mathcal{F}_l(h, \psi). \tag{3.14}$$

Indeed, if $\overline{h}(t_1) = \overline{h}(t_2)$ the proof is identical to the case (2). Otherwise, it can be obtained by slicing as well, parametrizing L_{12} by an arc length parameter, then slicing the region $\{(w_1, w_2) : w_1 \in (t_1, t_2),\ w_2 \in (\ell_{12}(w_1), 1)\}$[3] by lines perpendicular to L_{12}, and exploiting the fact that ψ equals zero on the segments $\{t_i\} \times (h(t_i), 1)$.

Let $(h, \psi) \in X_l$ be given; from (3) we can always replace h by its convex envelope and modifying accordingly ψ, we get two functions $h^{\#}$ and $\psi^{\#}$ such that (3.14) holds. Moreover, by (2), if $t_0 \in (0, l)$ is a regular point for $h^{\#}$, we can always replace $h^{\#}$ by $h^\star$ in (3.10), so that $h^\star$ turns out to be nonincreasing. The assertion of the proposition follows. □

[2]Here we use that $D_{w_2}\psi = 0$ in V.

[3]ℓ_{12} represents the affine function whose graph is L_{12}.

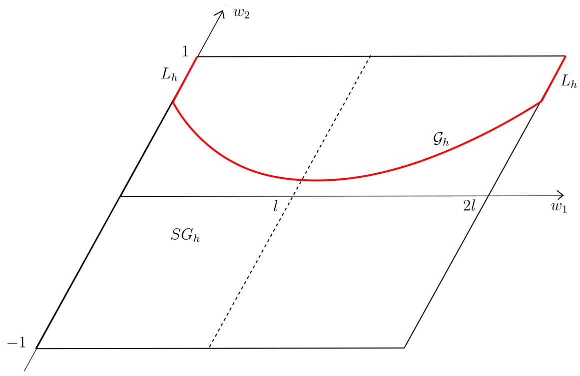

Figure 3: the graph of a convex symmetric function $h \in \mathcal{H}_{2l}$; L_h, defined in (3.16), consists of the two vertical segments over the boundary of $(0, 2l)$, from $h(0) = h(2l)$ to 1.

Let us rewrite the functional $\mathcal{F}_l$ in a convenient way. Let $(h, \psi) \in X_l^{\rm conv}$, and let $G_h = \{(w_1, h(w_1)) : w_1 \in (0, l)\} \subset \overline{R}_l$ be the graph of h. We have, using (3.2),

$$\mathcal{F}_l(h, \psi) = \overline{\mathcal{A}}(\psi, SG_h) + \int_{G_h \setminus \{h=-1\}} |\psi| \, d\mathcal{H}^1 + \int_{\partial_D R_l} |\psi - \varphi| \, d\mathcal{H}^1, \tag{3.15}$$

where, in the integral over G_h, we consider the trace of $\psi \llcorner SG_h$ on G_h.

3.1 Doubling

Now we analyse the minimum problem on the right-hand side of (3.8). To this aim, as explained in the introduction, it is convenient to write the analogue of $\mathcal{F}_l$ in a doubled rectangle, see (1.11).

Remember that R_{2l} denotes the open doubled rectangle, $R_{2l} := (0, 2l) \times (-1, 1)$; we define its Dirichlet boundary [4] $\partial_D R_{2l} \subset \partial R_{2l}$ as

$$\partial_D R_{2l} := \big(\{0, 2l\} \times [-1, 1]\big) \cup \big((0, 2l) \times \{-1\}\big),$$

so that $\partial R_{2l} \setminus \partial_D R_{2l} = (0, 2l) \times \{1\}$.

We recall that $\mathcal{H}_{2l}$ has been defined in (1.13), and that for each $h \in \mathcal{H}_{2l}$,

$$G_h := \{(w_1, h(w_1)) : w_1 \in (0, 2l)\}, \qquad SG_h := \{(w_1, w_2) \in R_{2l} : w_2 < h(w_1)\},$$

where $SG_h := \emptyset$ in the case $h \equiv -1$. Notice that for $h \in \mathcal{H}_{2l}$, SG_h is an open set. We set

$$L_h := \Big(\{0\} \times (h(0), 1)\Big) \cup \Big(\{2l\} \times (h(2l), 1)\Big), \tag{3.16}$$

which is either empty, or the union of two equal intervals, see Fig. 3.

Clearly, the restriction of φ, defined in (1.8), on $\partial_D R_{2l}$ reads as:

$$\varphi(w_1, w_2) := \begin{cases} \sqrt{1 - w_2{}^2} & \text{if } (w_1, w_2) \in \{0, 2l\} \times [-1, 1], \\ 0 & \text{if } (w_1, w_2) \in (0, 2l) \times \{-1\}. \end{cases} \tag{3.17}$$

The graph of φ on $\{0, 2l\} \times [-1, 1]$ consists of two half-circles of radius 1 centered at $(0, 0)$ and $(2l, 0)$ respectively, see Fig. 4.

[4]It is worth noticing once more that $\partial_D R_{2l}$ consists of three edges of ∂R_{2l}, while $\partial_D R_l$ (see (??)) consists of two edges of ∂R_l.

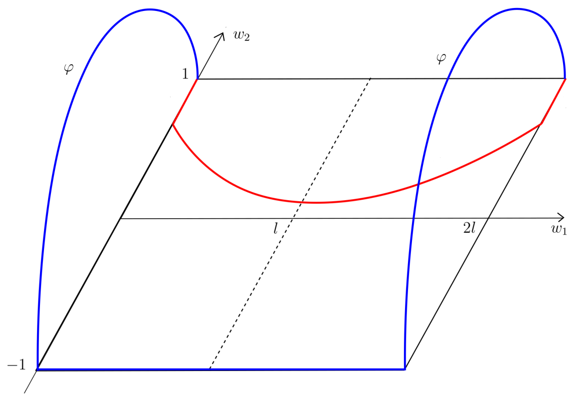

Figure 4: the graph of the boundary condition function φ in (3.17) on the Dirichlet boundary of R_{2l}. We also draw the graph of a function $h \in \mathcal{H}_{2l}$, and the two segments L_h.

We further recall that $X_{2l}^{\rm conv}$ has been defined in (1.12) and that, and for any $(h, \psi) \in X_{2l}^{\rm conv}$, $\mathcal{F}_{2l}(h, \psi)$ has been defined in (1.11).

Remark 3.5. (i) The only case in which the last addendum on the right-hand side of (1.11) may be positive is when h is identically 1 on $\partial R_{2l} \setminus \partial_D R_{2l}$;

(ii) We claim that

$$\mathcal{F}_{2l}(h, \psi) = \overline{\mathcal{A}}(\psi, SG_h) + \int_{\partial_D SG_h} |\psi - \varphi| \, d\mathcal{H}^1 + \int_{G_h \setminus \{w_2=-1\}} |\psi^-| \, d\mathcal{H}^1 + \int_{L_h} \varphi \, d\mathcal{H}^1, \quad (3.18)$$

where

$$\partial_D SG_h := (\partial_D R_{2l}) \cap \partial SG_h, \quad (3.19)$$

and ψ^- denotes the trace of ψ from the side of SG_h.

To show (3.18), we start to observe that, using that $\psi = 0$ on $R_{2l} \setminus SG_h$, it follows $\int_{\partial R_{2l} \setminus \partial_D R_{2l}} |\psi| \, d\mathcal{H}^1 = \int_{G_h \cap \{w_2=1\}} |\psi| \, d\mathcal{H}^1$. This last term is nonzero only if $h \equiv 1$, in which case L_h is empty, and the equivalence between (1.11) and (3.18) easily follows. If instead h is not identically 1, then, using again that $\psi = 0$ on $R_{2l} \setminus SG_h$, we see that the last term on the right-hand side of (1.11) is null, and from (1.7),

$$\overline{\mathcal{A}}(\psi, SG_h) = \overline{\mathcal{A}}(\psi, R_{2l}) - \mathcal{H}^2(R_{2l} \setminus SG_h) - \int_{G_h \cap R_{2l}} |\psi^-| d\mathcal{H}^1; \quad (3.20)$$

hence, inserting (3.20) into (1.11), we obtain, splitting $\partial_D R_{2l} = (\partial_D SG_h) \cup L_h \cup (G_h \cap \{w_2 =$

$-1\}$), and using that $\varphi = 0$ on $(0, 2l) \times \{-1\}$,

$$
\begin{aligned}
\mathcal{F}_{2l}(h,\psi) =& \overline{\mathcal{A}}(\psi, R_{2l}) - \mathcal{H}^2(R_{2l} \setminus SG_h) + \int_{\partial_D R_{2l}} |\psi - \varphi| d\mathcal{H}^1 \\
=& \overline{\mathcal{A}}(\psi, SG_h) + \int_{\partial_D R_{2l}} |\psi - \varphi| d\mathcal{H}^1 + \int_{G_h \cap R_{2l}} |\psi^-| d\mathcal{H}^1 \\
=& \overline{\mathcal{A}}(\psi, SG_h) + \int_{\partial_D SG_h} |\psi - \varphi| d\mathcal{H}^1 + \int_{L_h} |\varphi| d\mathcal{H}^1 + \int_{G_h \cap \{w_2=-1\}} |\varphi| d\mathcal{H}^1 \\
&+ \int_{G_h \cap R_{2l}} |\psi^-| d\mathcal{H}^1 \\
=& \overline{\mathcal{A}}(\psi, SG_h) + \int_{\partial_D SG_h} |\psi - \varphi| d\mathcal{H}^1 + \int_{G_h \setminus \{w_2=-1\}} |\psi^-| d\mathcal{H}^1 + \int_{L_h} \varphi d\mathcal{H}^1.
\end{aligned}
$$

(iii) We have

$$
\begin{aligned}
&\inf_{(h,\psi) \in X_{2l}^{\text{conv}}} \mathcal{F}_{2l}(h,\psi) \\
=& \inf \Big\{ \overline{\mathcal{A}}(\psi, SG_h) + \int_{\partial_D SG_h} |\psi - \varphi| \; d\mathcal{H}^1 + \int_{G_h \setminus \{w_2=-1\}} |\psi^-| \; d\mathcal{H}^1 + \int_{L_h} \varphi \; d\mathcal{H}^1 \\
&: h \in \mathcal{H}_{2l} \setminus \{h \equiv -1\}, \; \psi \in \mathrm{BV}(SG_h, [0,1]) \Big\}.
\end{aligned} \tag{3.21}
$$

(iv) If $h > -1$ everywhere, then SG_h is connected, $\partial_D SG_h = \partial_D R_{2l} \setminus L_h$, and the sum of the first three terms on the right-hand side of (3.18) gives the area of the graph of ψ on $\overline{SG_h}$, with the boundary condition φ set to be 0 on G_h.

(v) Our aim is to have a surface in $\overline{R_{2l}} \times \mathbb{R} \subset \mathbb{R}^3 = \mathbb{R}^2_{(w_1,w_2)} \times \mathbb{R}$ of graph type, whose boundary consists of the union of the graph of φ and the graph of a convex function $h \in \mathcal{H}_{2l}$. The last three terms in (3.18) are an area penalization to force the solution to attain these boundary conditions by filling, with vertical walls, the gap between the boundary of any competitor surface (the generalized graph of ψ) and the required boundary conditions. In particular the presence of the last term of (3.18) is explained as follows: when $h(0) < 1$, *i.e.*, $L_h \neq \emptyset$, the graph of any $\psi \in \mathrm{BV}(SG_h, [0,1])$ does not reach the graph of $\varphi|_{L_h}$ (simply because $L_h \cap \overline{SG_h} = \emptyset$). To overcome this, the graph of ψ is glued to the wall consisting of the subgraph of $\varphi|_{L_h}$ (inside $\overline{R}_{2l}$).

(vi) Take $h_n := -1 + \frac{1}{n}$, and $\psi_n := c > 0$ on SG_{h_n}, then $\lim_{n\to+\infty} \overline{\mathcal{A}}(\psi_n, SG_{h_n}) = 0$, $\lim_{n\to+\infty} \int_{\partial_D SG_{h_n}} |\psi_n - \varphi| \; d\mathcal{H}^1 = 2cl$, and $\lim_{n\to+\infty} \int_{G_{h_n} \setminus \{h_n=-1\}} |\psi| \; d\mathcal{H}^1 = 2cl$, $\lim_{n\to+\infty} \int_{L_{h_n}} \varphi \; d\mathcal{H}^1 = \pi$, hence

$$
\mathcal{F}_{2l}(-1, 0) = \pi < \lim_{n\to+\infty} \mathcal{F}_{2l}(h_n, \psi_n) = 4cl + \pi,
$$

that is the functional $\mathcal{F}_{2l}$ in some sense forces a minimizing sequence to attain the boundary conditions as much as possible.

By symmetry, we easily infer

$$
2 \inf_{(h,\psi) \in X_l^{\text{conv}}} \mathcal{F}_l(h,\psi) = \inf_{(h,\psi) \in X_{2l}^{\text{conv}}} \mathcal{F}_{2l}(h,\psi), \tag{3.22}
$$

therefore, by (3.8),

$$\inf_{(h,\psi)\in X_l} \mathcal{F}_l(h,\psi) = \frac{1}{2} \inf_{(h,\psi)\in X_{2l}^{\mathrm{conv}}} \mathcal{F}_{2l}(h,\psi). \tag{3.23}$$

Next we analyse the latter minimization problem.

Remark 3.6 (**Two explicit estimates from above**). Let $h \equiv 1$ and $\psi(w_1, w_2) := \sqrt{1 - w_2{}^2} = \varphi(w_1, w_2)$, for any $(w_1, w_2) \in R_{2l}$. Then (h, ψ) is one of the competitors in (3.22) and therefore

$$\inf_{(h,\psi)\in X_{2l}^{\mathrm{conv}}} \mathcal{F}_{2l}(h,\psi) \le \mathcal{F}_{2l}(1,\psi) = 2\pi l \qquad \forall l > 0,$$

which is the lateral area of the cylinder $(0, 2l) \times \mathrm{D}$. Also, $\mathcal{F}_{2l}$ is well-defined for $h \equiv -1$, in which case $SG_h = \emptyset$, $\psi \equiv 0$ in R_{2l}, and therefore

$$\mathcal{F}_{2l}(-1, 0) = \int_{\{0,2l\}\times(-1,1)} \varphi \, d\mathcal{H}^1 = \pi, \tag{3.24}$$

which is the area of the two half-discs joined by the segment $(0, 2l) \times \{-1\}$, see Fig. 4. In particular

$$\inf_{(h,\psi)\in X_{2l}^{\mathrm{conv}}} \mathcal{F}(h,\psi) \le \pi \qquad \forall l > 0. \tag{3.25}$$

In the next section we shall prove the existence and regularity of minimizers for the minimum problem on the right-hand side of (3.23).

4 Proof of Theorems 1.1 and 1.2

This section is devoted to the proof of our main results, Theorems 1.1 and 1.2, and it is splitted into various subsections for clarity of the presentation.

4.1 Existence of a minimizer of $\mathcal{F}_{2l}$ in X_{2l}^{conv}

We start by analysing the features of the space $\mathcal{H}_{2l}$ defined in (1.12). Clearly the graph of $h \in \mathcal{H}_{2l}$ is symmetric with respect to $\{w_1 = l\}$; also, the convexity of h implies $h \in \mathrm{Lip}_{\mathrm{loc}}((0, 2l))$, and h has a continuous extension on $[0, 2l]$.

Definition 4.1 (**Convergence in X_{2l}^{conv}**). *We say that a sequence $((h_n, \psi_n)) \subset X_{2l}^{\mathrm{conv}}$ converges to $(h, \psi) \in X_{2l}^{\mathrm{conv}}$, if*

- *(h_n) converges to h uniformly on compact subsets of $(0, 2l)$;*
- *(ψ_n) converges to ψ in $L^1(R_{2l})$.*

Lemma 4.2 (**Compactness of $\mathcal{H}_{2l}$**). *Every sequence $(h_k) \subset \mathcal{H}_{2l}$ has a subsequence converging uniformly on compact subsets of $(0, 2l)$ to some element of $\mathcal{H}_{2l}$.*

Proof. See for instance [13, Sec. 1.1]. □

Lemma 4.3 (**Closedness of X_{2l}^{conv}**). *Let $((h_n, \psi_n)) \subset X_{2l}^{\mathrm{conv}}$ be a sequence such that (h_n) converges to $h \in \mathcal{H}_{2l}$ uniformly on compact subsets of $(0, 2l)$, and (ψ_n) converges to $\psi \in \mathrm{BV}(R_{2l})$ in $L^1(R_{2l})$. Then $(h, \psi) \in X_{2l}^{\mathrm{conv}}$.*

Proof. Possibly passing to a (not relabelled) subsequence, we can assume that (ψ_n) converges to ψ pointwise in $A \subseteq R_{2l}$, with $\mathcal{H}^2(R_{2l} \setminus A) = 0$, and $\psi_n = 0$ in $A \cap (R_{2l} \setminus SG_{h_n})$ for all $n \in \mathbb{N}$. We only have to show that $\psi = 0$ in $A \cap (R_{2l} \setminus SG_h)$. We may assume that A does not intersect the graph of h. If $(w_1, w_2) \in A \cap (R_{2l} \setminus SG_h)$, then $w_2 > h(w_1)$. From the local uniform convergence of (h_n) to h in $(0, 2l)$ it follows that $w_2 > h_n(w_1)$ for n large enough, *i.e.,* $(w_1, w_2) \in A \cap (R_{2l} \setminus SG_{h_n})$, and the assertion follows. □

Lemma 4.4 (Lower semicontinuity of $\mathcal{F}_{2l}$). *Let $((h_n, \psi_n)) \subset X_{2l}^{\mathrm{conv}}$ be a sequence converging to $(h, \psi) \in X_{2l}^{\mathrm{conv}}$ in the sense of Definition 4.1. Then*

$$\mathcal{F}_{2l}(h, \psi) \leq \liminf_{n \to +\infty} \mathcal{F}_{2l}(h_n, \psi_n). \tag{4.1}$$

Proof. It is standard[5] to show that the functional

$$\psi \in BV(R_{2l}, [0,1]) \to \overline{\mathcal{A}}(\psi, R_{2l}) + \int_{\partial_D R_{2l}} |\psi - \varphi| \, d\mathcal{H}^1 + \int_{\partial R_{2l} \setminus \partial_D R_{2l}} |\psi| \, d\mathcal{H}^1 \tag{4.2}$$

is $L^1(R_{2l})$-lower semicontinuous. Since (h_n) converges to h pointwise in $(0, 2l)$, we also have $\lim_{n \to +\infty} \mathcal{H}^2(R_{2l} \setminus SG_{h_n}) = \mathcal{H}^2(R_{2l} \setminus SG_h)$. The assertion follows. □

The existence statement of Theorem 1.1 is given by the following

Proposition 4.5 (Existence of a minimizer of (1.17)). *The minimum problem (1.17) has a solution.*

Proof. The pair (h_0, ψ_0) given by

$$h_0(w_1) := -1, \quad \psi_0(w_1, w_2) := 0, \qquad (w_1, w_2) \in R_{2l},$$

is a competitor in (1.17). Hence, for a minimizing sequence $((h_n, \psi_n)) \subset X_{2l}^{\mathrm{conv}}$, recalling (3.24) we have

$$\lim_{n \to +\infty} \mathcal{F}_{2l}(h_n, \psi_n) = \inf \left\{ \mathcal{F}_{2l}(h, \psi) : (h, \psi) \in X_{2l}^{\mathrm{conv}} \right\} \leq \pi. \tag{4.3}$$

Thus $\sup_{n \in \mathbb{N}} |D\psi_n|(R_{2l}) < +\infty$, and there exists $\psi \in BV(R_{2l}, [0,1])$ such that, up to a (not relabelled) subsequence, (ψ_n) converges to ψ in $L^1(\Omega)$.

Using Lemmas 4.2 and 4.3, we may assume that (h_n) converges locally uniformly to some $h \in \mathcal{H}_{2l}$, and $\psi = 0$ in $R_{2l} \setminus SG_h$. The assertion then follows from Lemma 4.4. □

Now, we turn to the regularity and qualitative properties of minimizers.

[5] Indeed, let $\tilde{\varphi} : \partial R_{2l} \to [0,1]$ be defined as $\tilde{\varphi} := \varphi$ on $\partial_D R_{2l}$, and $\tilde{\varphi} := 0$ on $\partial R_{2l} \setminus \partial_D R_{2l}$. Let $B \subset \mathbb{R}^2$ be an open disc containing $\overline{R_{2l}}$. We extend $\tilde{\varphi}$ to a $W^{1,1}$ function in $B \setminus \overline{R_{2l}}$, [11, Thm. 2.16], and we still denote by $\tilde{\varphi}$ such an extension. For every $\psi \in BV(R_{2l})$, define $\widehat{\psi} := \psi$ in R_{2l} and $\widehat{\psi} := \tilde{\varphi}$ in $B \setminus R_{2l}$. We have

$$\overline{\mathcal{A}}(\psi, R_{2l}) + \int_{\partial_D R_{2l}} |\psi - \varphi| d\mathcal{H}^1 + \int_{\partial R_{2l} \setminus \partial_D R_{2l}} |\psi| d\mathcal{H}^1 = \overline{\mathcal{A}}(\psi, R_{2l}) + \int_{\partial R_{2l}} |\psi - \tilde{\varphi}| d\mathcal{H}^1 = \overline{\mathcal{A}}(\widehat{\psi}, B) - \overline{\mathcal{A}}(\tilde{\varphi}, B \setminus \overline{R_{2l}}),$$

where the last equality follows from [11, (2.15)]. Thus the lower semicontinuity of the functional in (4.2) follows from the $L^1(B)$-lower semicontinuity of the area functional.

4.2 Regularity of minimizers of $\mathcal{F}_{2l}$ in $X_{2l}^{\rm conv}$

The next proposition shows (2ii) in Theorem 1.1.

Proposition 4.6 (Analyticity and positivity of a minimizer). *Suppose that (h, ψ) is a minimizer of (1.17), and that h is not identically -1. Then ψ is analytic in SG_h, and*

$$\mathrm{div}\Big(\frac{\nabla \psi}{\sqrt{1+|\nabla\psi|^2}}\Big) = 0 \qquad \text{in } SG_h. \tag{4.4}$$

Moreover

$$\psi > 0 \quad \text{in } SG_h. \tag{4.5}$$

Proof. Since by assumption h is not identically -1, we have that SG_h is nonempty. Moreover minimality ensures

$$\int_{SG_h} \sqrt{1+|D\psi|^2} \leq \int_{SG_h} \sqrt{1+|D\psi_1|^2}$$

for any $\psi_1 \in BV(SG_h)$ with $\mathrm{spt}(\psi - \psi_1) \subset\subset SG_h$. Thus, by [11, Thm 14.13], ψ is locally Lipschitz, and hence analytic, in SG_h, and (4.4) follows. Now, let $z \in SG_h$ and take an open disc $B_\eta(z) \subset\subset SG_h$. Since $\psi \geq 0$ on $\partial B_\eta(z)$ we find, by the strong maximum principle [11, Thm. C.4], that either ψ is identically zero in $B_\eta(z)$, or $\psi > 0$ in $B_\eta(z)$. Hence from the analyticity of ψ and the arbitrariness of z, we have that either ψ is identically zero in SG_h or $\psi > 0$ in SG_h. Now $\mathcal{F}_{2l}(h, 0) = |SG_h| + \pi > \mathcal{F}_{2l}(-1, 0) = \pi$, see (3.24). Thus $(h, 0)$ is not a minimizer, and the positivity of ψ in SG_h is achieved.

□

Now, we show (1) of Theorem 1.1; moreover, we prove in particular that the (symmetric) convex function h cannot touch and detouch the value -1.

Lemma 4.7. *Suppose that (h, ψ) is a minimizer of (1.17) such that:*

(i) h is not identically -1;

(ii) ψ is symmetric with respect to $\{w_1 = l\} \cap R_{2l}$.

Then

$$h(w_1) > -1 \qquad \forall w_1 \in [0, 2l].$$

Proof. Since $h \in \mathcal{H}_{2l}$, it is symmetric with respect to $\{w_1 = l\} \cap R_{2l}$; hence, by assumption (ii), we may restrict our argument to $[0, l]$. Assume by contradiction that there exists $\overline{w}_1 \in (0, l]$ such that $h(\overline{w}_1) = -1$. Recall that h is convex, nonincreasing in $[0, l]$ and continuous at l. Let

$$w_1^0 := \min\{w_1 \in (0, l] : h(w_1) = -1\}.$$

By assumption (i) we have $w_1^0 > 0$ and, by convexity, h is strictly decreasing in $(0, w_1^0)$. We have, using (3.18) and (3.19),

$$\begin{aligned}
\frac{1}{2}\mathcal{F}_{2l}(h,\psi) &= \frac{1}{2}\left[\overline{\mathcal{A}}(\psi, SG_h) + \int_{\partial_D SG_h} |\psi - \varphi| \, d\mathcal{H}^1 + \int_{G_h \setminus \{w_2 = -1\}} |\psi^-| \, d\mathcal{H}^1 + \int_{L_h} \varphi \, d\mathcal{H}^1\right] \\
&= \frac{1}{2}\overline{\mathcal{A}}(\psi, SG_h) + \int_{(-1,h(0))} |\psi(0, w_2) - \varphi(0, w_2)| \, dw_2 + \int_{(0,w_1^0)} |\psi(w_1, -1) - \varphi(w_1, -1)| \, dw_1 \\
&\quad + \int_{G_{h \llcorner (0, w_1^0)}} |\psi^-| d\mathcal{H}^1 + \int_{(h(0),1)} \varphi(0, w_2) dw_2.
\end{aligned} \tag{4.6}$$

Recalling Proposition 4.6, we have

$$\frac{1}{2}\mathcal{F}_{2l}(h,\psi) = \int_0^{w_1^0}\int_{-1}^{h(w_1)} \sqrt{1+|\nabla\psi|^2}\; dw_2 dw_1.$$

Now, we argue by slicing the rectangle $R_l = (0,l)\times(-1,1)$ with lines $\{w_1 = \tau\}, \tau \in (0,l)$. Recalling the expression of SG_h (which is non empty by assumption (i)), and neglecting the third addendum in (4.6),

$$\begin{aligned}
\frac{1}{2}\mathcal{F}_{2l}(h,\psi) &= \int_0^{w_1^0}\int_{-1}^{h(w_1)} \sqrt{1+|\nabla\psi|^2}\; dw_2 dw_1 \\
&\quad + \int_{(-1,h(0))} |\psi(0,w_2)-\varphi(0,w_2)| dw_2 + \int_{(0,w_1^0)} |\psi(w_1,-1)-\varphi(w_1,-1)|\; dw_1 \\
&\quad + \int_{G_{h\llcorner(0,w_1^0)}} |\psi^-| d\mathcal{H}^1 + \int_{(h(0),1)} \varphi(0,w_2) dw_2 \\
&\geq \int_0^{w_1^0}\int_{-1}^{h(w_1)} \sqrt{1+|\nabla\psi|^2} dw_2 dw_1 + \int_{(-1,h(0))} |\psi(0,w_2)-\varphi(0,w_2)| dw_2 \qquad (4.7)\\
&\quad + \int_{G_{h\llcorner(0,w_1^0)}} |\psi^-| d\mathcal{H}^1 + \int_{(h(0),1)} \varphi(0,w_2) dw_2 \\
&> \int_0^{w_1^0}\int_{-1}^{h(w_1)} |\nabla_{w_1}\psi(w_1,w_2)| dw_2 dw_1 + \int_{(-1,h(0))} |\psi(0,w_2)-\varphi(0,w_2)| dw_2 \\
&\quad + \int_{G_{h\llcorner(0,w_1^0)}} |\psi^-| d\mathcal{H}^1 + \int_{(h(0),1)} \varphi(0,w_2) dw_2,
\end{aligned}$$

where ∇_{w_1} stands for the partial derivative with respect to w_1.

Now, let

$$h^{-1} : [-1,h(0)] \to [0,w_1^0]$$

be the inverse of $h\llcorner[0,w_1^0]$. Neglecting $\sqrt{1+(\frac{d}{dw_2}h^{-1})^2}$ in the third addendum on the right-hand side of (4.7), using also that $\varphi \geq 0$ and $\psi \geq 0$ (Proposition 4.6), we deduce

$$\begin{aligned}
\frac{1}{2}\mathcal{F}_{2l}(h,\psi) &> \int_{-1}^{h(0)}\int_0^{h^{-1}(w_2)} |\nabla_{w_1}\psi(w_1,w_2)|\; dw_1 dw_2 + \int_{(-1,h(0))} |\psi(0,w_2)-\varphi(0,w_2)| dw_2 \\
&\quad + \int_{(-1,h(0))} \psi^-(h^{-1}(w_2),w_2) dw_2 + \int_{(h(0),1)} \varphi(0,w_2) dw_2 \\
&\geq \int_{-1}^{h(0)} \Big| \int_0^{h^{-1}(w_2)} \nabla_{w_1}\psi(w_1,w_2) dw_1 \Big| dw_2 - \int_{(-1,h(0))} \psi(0,w_2) dw_2 \\
&\quad + \int_{(-1,h(0))} \psi^-(h^{-1}(w_2),w_2) dw_2 + \int_{(-1,1)} \varphi(0,w_2) dw_2 \\
&\geq \int_{(-1,h(0))} |\psi(h^{-1}(w_2),w_2)-\psi(0,w_2)| dw_2 - \int_{(-1,h(0))} \psi(0,w_2) dw_2 \\
&\quad + \int_{(-1,h(0))} \psi^-(h^{-1}(w_2),w_2) dw_2 + \int_{(-1,1)} \varphi(0,w_2) dw_2 \\
&\geq \int_{(-1,1)} \varphi(0,w_2) dw_2 = \frac{1}{2}\mathcal{F}_{2l}(-1,0).
\end{aligned}$$

Hence the value of $\mathcal{F}_{2l}$ on the pair $(h \equiv -1, \psi \equiv 0)$ is smaller than $\mathcal{F}_{2l}(h, \psi)$, thus contradicting the minimality of (h, ψ). ☐

The next lemma concludes, in particular, the proof of the first statement in Theorem 1.1.

Lemma 4.8 (**Symmetry of minimizers**). *Every minimizer (h, ψ) of (1.17) is such that ψ is symmetric with respect to $\{w_1 = l\} \cap R_{2l}$.*

Proof. Let $I \subset (0, 2l)$ be an open interval; consistently with (1.11), and since ψ is continuous in SG_h, we set

$$\mathcal{F}_{2l}(h, \psi; I) := \overline{\mathcal{A}}(\psi, I \times (-1,1)) - \mathcal{H}^2\Big(I \times (-1,1) \setminus SG_h\Big) + \int_{(\partial_D R_{2l}) \cap (\overline{I} \times [-1,1))} |\psi - \varphi| \, d\mathcal{H}^1 + \int_{(\partial R_{2l} \setminus \partial_D R_{2l}) \cap (\overline{I} \times (-1,1])} |\psi| \, d\mathcal{H}^1.$$

Recall that $h \in \mathcal{H}_{2l}$, hence its graph is symmetric with respect to $\{w_1 = l\} \cap R_{2l}$. Define $\tilde{\psi} := \psi$ on $(0, l) \times (-1, 1)$ and $\tilde{\psi}(w_1, w_2) := \psi(2l - w_1, w_2)$ for $(w_1, w_2) \in (l, 2l) \times (-1, 1)$, in particular the graph of $\tilde{\psi}$ is symmetric with respect to $\{w_1 = l\} \cap R_{2l}$. Since $\mathcal{F}_{2l}(h, \psi; (0, l)) = \mathcal{F}_{2l}(h, \psi; (l, 2l))$, it follows $\mathcal{F}_{2l}(h, \tilde{\psi}) = \mathcal{F}_{2l}(h, \psi)$ for, if $\mathcal{F}_{2l}(h, \psi; (0, l)) < \mathcal{F}_{2l}(h, \psi; (l, 2l))$, then $\mathcal{F}_{2l}(h, \tilde{\psi}) < \mathcal{F}_{2l}(h, \psi)$ which contradicts the minimality of (h, ψ).

Now, if $h \equiv -1$ then the minimizer $\psi = 0$ is symmetric. On the other hand, if h is not identically -1, due to Lemma 4.7, we have that $h(l) > -1$ so that SG_h is open and connected, and thus the two analytic functions ψ and $\tilde{\psi}$, coinciding on $SG_h \cap R_l$, must coincide. Hence, $\psi = \tilde{\psi}$ and ψ is symmetric. ☐

Now, we prove items (2i) and (2iii) of Theorem 1.1: the proof will be a consequence of the next lemma and Theorem 4.11. Recall the definition of $\partial_D SG_h$ in (3.19).

Lemma 4.9. *Let (h, ψ) be a minimizer of (1.17) with h not identically -1. Then ψ attains the boundary condition on $\partial_D SG_h$.*

Proof. The result follows from [11, Theorem 15.9], since $\partial_D SG_h$ is union of three segments. ☐

Remark 4.10. In the hypotheses of Lemma 4.7, if $h \equiv 1$ then the graph of h is a segment and, as in Lemma 4.9, $\psi = 0$ on G_h.

The conclusion of the proof of Theorem 1.1 (2iii) is given by the following delicate result.

Theorem 4.11 (**Boundary regularity**). *Assume there is a minimizer $(h, \psi) \in X_{2l}^{\rm conv}$ of (1.17) with h not identically -1. Then there exists another minimizer $(\widetilde{h}, \widetilde{\psi}) \in X_{2l}^{\rm conv}$ of (1.17) having the following properties:*

(i) $\widetilde{h}(0) = 1 = \widetilde{h}(1)$,

(ii) $\widetilde{\psi}$ *is continuous up to the boundary of* $SG_{\widetilde{h}}$,

(iii) $\widetilde{\psi} = 0$ on $G_{\widetilde{h}}$.

Proof. By Remark 4.10, we can assume that h is not identically 1 and, by Lemma 4.7, also that $h(w_1) \geq h(l) > -1$ for any $w_1 \in [0, 2l]$. We start to fix a number $\bar{s} \in (-1, h(l))$ and to set

$$K := (0, 2l) \times (\bar{s}, 1) \subset R_{2l}.$$

The usefulness of $\bar{s}$ stands on the fact that, by (4.5) and Lemma 4.9, the graph of the restriction of ψ over $\partial K \setminus \{w_2 = 1\}$ is *strictly positive* (in particular, excluding the two points $(0,1)$ and $(2l,1)$). We shall see at the end of the arguments, that the proof will be independent of the choice of $\bar{s}$.

Let us extend ψ in $\mathbb{R}^2 \setminus R_{2l}$ as follows: we define $\widehat{\psi} : \mathbb{R}^2 \to [0,1]$, $\widehat{\psi} := \psi$ in R_{2l}, and

$$\widehat{\psi}(w_1, w_2) := \begin{cases} \varphi(w_1, w_2) & \text{if } w_1 < 0 \text{ or } w_1 > 2l, \text{ and } |w_2| \leq 1, \\ 0 & \text{if } |w_2| > 1, \end{cases} \tag{4.8}$$

(see (1.8)). In this way $\widehat{\psi}$ is continuous in $\mathbb{R}^2 \setminus \overline{R}_{2l}$.

Now, we divide the proof into eight steps. In step 1 we start by regularizing $\widehat{\psi}$ in order that the regularized functions have smooth graphs over the sets K_n defined in (4.9), and so these graphs are of disc-type. We expect the graph of $\widehat{\psi}$ over the sets K_n, considering also a possible vertical part over the graph of h, to be a surface of disc-type; however, we miss the proof of this fact, mainly due to possible irregularity of the trace of $\widehat{\psi}$ over G_h. The information on the topological type of these graphs will be crucial in our proof.

Also, an appropriate approximation of h will be needed; this latter approximation depends on the approximation of $\widehat{\psi}$. Next (step 2), we will compare these graphs with the solution of a suitable disc-type Plateau problem.

Step 1, part 1: Approximation of $\widehat{\psi}$.

Let $n > 0$ be a natural number (that will be sent to $+\infty$ later) such that $\bar{s} + \frac{1}{n} < h(l)$, and consider the enlarged rectangle

$$K_n := \left(-\frac{1}{n}, 2l + \frac{1}{n}\right) \times \left(\bar{s}, 1 + \frac{1}{n}\right), \tag{4.9}$$

see Fig. 5. Note that

$$\widehat{\psi} \text{ is continuous on } \partial K_n. \tag{4.10}$$

Given $n \in \mathbb{N}$, we claim that we can build a sequence $(\psi_k^n)_{k \in \mathbb{N}}$ depending on n, which satisfies the following properties:

$$\begin{aligned} &\psi_k^n \in C^\infty(K_n, [0,1]) \cap C(\overline{K}_n, [0,1]) \qquad \forall k \in \mathbb{N},\ k > 0, \\ &\psi_k^n = \widehat{\psi} \text{ on } \partial K_n \qquad \forall k \in \mathbb{N},\ k > 0, \\ &\psi_k^n \rightharpoonup \widehat{\psi} \text{ weakly}^\star \text{ in } BV(K_n) \text{ as } k \to +\infty, \\ &\int_{K_n} |\nabla \psi_k^n| \, dw \to |D\widehat{\psi}|(K_n) \text{ as } k \to +\infty. \end{aligned} \tag{4.11}$$

In order to obtain (4.11) we use standard arguments (details can be found in [1, Thm. 3.9] or [10, Thm. 1, Section 4.1.1]). To the aim of our discussion, we just recall that we proceed by constructing an increasing sequence $(U_{i,n})_{i \geq 1}$ of open subsets of K_n, $U_{i,n} \subset\subset U_{i+1,n} \subset\subset K_n$, $\cup_i U_{i,n} = K_n$ (for $i \geq 1$ we take $U_{i,n} := \{x \in \mathbb{R}^2 : \text{dist}(x, \mathbb{R}^2 \setminus K_n) > \frac{1}{i+n}\}$ for definitiveness) and with the aid of a partition of unity $(\eta_{i,n})$ associated to $V_{1,n} := U_{2,n}$, $V_{i,n} := U_{i+1,n} \setminus \overline{U}_{i-1,n}$ for $i \geq 2$, we mollify $\widehat{\psi}$ accordingly in $V_{i,n}$. For our purpose we choose[6] $\eta_{i,n}$ in such a way that

$$\text{supp } (\eta_{i,n}) = \overline{V}_{i,n}. \tag{4.12}$$

Since ψ_k^n is obtained by mollification we have $\psi_k^n \in C^\infty(K_n)$ and moreover $\psi_k^n \in C(\overline{K}_n)$ because it attains the continuous boundary datum $\widehat{\psi}$ on ∂K_n. Here we use the same standard mollifier

[6]We need the full set $\overline{V}_{i,n}$ as support in order that the argument to detect the behaviour of h_n (defined in (4.22)) in $[-\frac{1}{n}, 0]$ applies.

$\rho \in C_c^\infty(\mathrm{D})$ in each $V_{i,n}$, choosing, for $w = (w_1, w_2)$, $\rho_{i,n,k}(w) := \rho(w/r_{i,n,k})$ with $r_{i,n,k} := r_{i,n}/k > 0$, $r_{i,n}$ decreasing with respect to $i \geq 1$, with $r_{i,n} \to 0^+$ as $i \to +\infty$; we take

$$r_{i,n} = \frac{1}{i+2+n} \tag{4.13}$$

for definiteness. Finally, $[0, 2l] \times [\bar{s} + \frac{1}{n}, 1] \subset U_{1,n} \subset V_{1,n}$, and $V_{i,n} \cap \big([0, 2l] \times [\bar{s} + \frac{1}{n}, 1]\big) = \emptyset$ for $i \geq 2$. It follows

$$\psi_k^n = \widehat{\psi} \star \rho_{1,n,k} \qquad \text{in } [0, 2l] \times \left[\bar{s} + \frac{1}{n}, 1\right] \qquad \forall n \in \mathbb{N}. \tag{4.14}$$

Using [10, Prop. 3 Sec. 4.2.4 pag. 408, and Th. 1 Sec. 4.1.5 pag. 331] we infer

$$\mathcal{A}(\psi_k^n, K_n) \to \overline{\mathcal{A}}(\widehat{\psi}, K_n) \qquad \text{as } k \to +\infty. \tag{4.15}$$

Now that properties (4.11) are achieved, by a diagonal argument we select functions

$$\psi_n := \psi_{k_n}^n \in (\psi_k^n) \qquad \forall n \in \mathbb{N}, \tag{4.16}$$

such that

$$\begin{aligned} &\psi_n = \widehat{\psi} \text{ on } \partial K_n \qquad \forall n \in \mathbb{N}, \\ &\psi_n \rightharpoonup \widehat{\psi} \text{ weakly}^* \text{ in } BV(K) \text{ as } n \to +\infty, \\ &\int_{K_n} |\nabla \psi_n| \, dw \to |D\widehat{\psi}|(\overline{K}) \text{ as } n \to +\infty, \end{aligned} \tag{4.17}$$

where $\overline{K}$ is the closed rectangle $\overline{K} := \cap_n K_n$. On the basis of (4.15) and (4.17), we can also ensure[7] that

$$\mathcal{A}(\psi_n, K_n) \to \overline{\mathcal{A}}(\widehat{\psi}, \overline{K}) \qquad \text{as } n \to +\infty. \tag{4.18}$$

Here, by $\overline{\mathcal{A}}(\widehat{\psi}, \overline{K})$ we mean the area of the graph of $\widehat{\psi}$ relative to $\overline{K}$ which, recalling also Proposition 4.6, reads as

$$\overline{\mathcal{A}}(\widehat{\psi}, \overline{K}) = \overline{\mathcal{A}}(\widehat{\psi}, K) + \int_{\{0\} \times (\bar{s}, 1)} |\widehat{\psi}^- - \varphi| \, d\mathcal{H}^1 + \int_{\{2l\} \times (\bar{s}, 1)} |\widehat{\psi}^- - \varphi| \, d\mathcal{H}^1, \tag{4.19}$$

where $\widehat{\psi}^-$ denotes the trace of $\widehat{\psi}$ on ∂K. This concludes the proof of the first part of step 1.

Before passing to the second part, for any $n \in \mathbb{N}$ we define

$$\widehat{h}(w_1) := \sup\left\{ w_2 \in \left(\bar{s}, 1 + \frac{1}{n}\right) : \widehat{\psi}(w_1, w_2) > 0 \right\} \qquad \forall w_1 \in \left(-\frac{1}{n}, 2l + \frac{1}{n}\right).$$

Notice that

$$\begin{aligned} &\widehat{h} = h \quad \text{in } [0, 2l], \\ &\widehat{h} = 1 \quad \text{in } (-1/n, 0) \cup (2l, 2l + 1/n). \end{aligned}$$

[7] To prove claim (4.18), fix $m \in \mathbb{N}$, and set $\tilde{\psi}_n := \widehat{\psi}$ outside K_n and $\tilde{\psi}_n = \psi_n$ in K_n, so that

$$\begin{aligned} &\tilde{\psi}_n \rightharpoonup \widehat{\psi} \text{ weakly}^* \text{ in } BV(K_m) \text{ as } n \to +\infty, \\ &|\nabla \tilde{\psi}_n|(K_m) \to |D\widehat{\psi}|(K_m) = |D\widehat{\psi}|(\overline{K}) + |D\widehat{\psi}|(K_m \setminus \overline{K}) \text{ as } n \to +\infty. \end{aligned}$$

Then $\limsup_{n \to +\infty} \overline{\mathcal{A}}(\psi_n, K_n) \leq \limsup_{n \to +\infty} \overline{\mathcal{A}}(\tilde{\psi}_n, K_m) = \overline{\mathcal{A}}(\widehat{\psi}, K_m) = \overline{\mathcal{A}}(\widehat{\psi}, \overline{K}) + \overline{\mathcal{A}}(\widehat{\psi}, K_m \setminus \overline{K})$, the first equality following from the strict convergence of $\tilde{\psi}_n$ to $\widehat{\psi}$ [10, Prop. 3 Sec. 4.2.4 pag. 408 and Thm. 1 Sec. 4.1.5 pag. 371]. Taking the limit as $m \to +\infty$, since $\widehat{\psi} \in W^{1,1}(K_m \setminus \overline{K})$ we conclude $\limsup_{n \to +\infty} \overline{\mathcal{A}}(\psi_n, K_n) \leq \overline{\mathcal{A}}(\psi, \overline{K})$. Then (4.18) follows by lower semicontinuity.

Step 1, part 2: Approximation of h. We construct functions $h_n : (-\frac{1}{n}, 2l + \frac{1}{n}) \to (\bar{s}, 1 + \frac{1}{n})$ such that

$$\begin{aligned} &h_n(\cdot) = h_n(2l - \cdot),\\ &\psi_n = 0 \text{ in } K_n \setminus SG_{h_n},\\ &h_n \in BV\big(\big(-\frac{1}{n}, l\big)\big), \end{aligned} \tag{4.20}$$

and, setting

$$SG_{h_n,\bar{s}} := \left\{(w_1, w_2) : w_1 \in \left(-\frac{1}{n}, 2l + \frac{1}{n}\right),\ w_2 \in (\bar{s}, h_n(w_1))\right\}, \tag{4.21}$$

also such that

$$\begin{aligned} &\lim_{n\to+\infty} \mathcal{H}^2(SG_{h_n,\bar{s}}) = \mathcal{H}^2(K \cap SG_h),\\ &\lim_{n\to+\infty} \overline{\mathcal{A}}(\psi_n, SG_{h_n,\bar{s}}) = \mathcal{F}_{2l}(h, \psi) - \overline{\mathcal{A}}(\psi, R_{2l} \setminus K). \end{aligned}$$

To this aim, for any $n \in \mathbb{N}$ we define

$$h_n(w_1) := \sup\left\{w_2 \in \left(\bar{s}, 1 + \frac{1}{n}\right) : \psi_n(w_1, w_2) > 0\right\} \qquad \forall w_1 \in \Big(-\frac{1}{n}, 2l + \frac{1}{n}\Big), \tag{4.22}$$

Since (see (4.5) of Proposition 4.6 and (4.8)) $\widehat{\psi}$ is positive in $SG_h \cup ((-\frac{1}{n}, 0) \times (\bar{s}, 1)) \cup ((2l, 2l + \frac{1}{n}) \times (\bar{s}, 1))$ it turns out, recalling also that the function ψ_n in (4.16) is obtained by mollification, that

$$\begin{aligned} -1 < h(w_1) < h_n(w_1) < 1 + \frac{1}{n} \qquad &\forall w_1 \in (0, 2l),\\ 1 < h_n(w_1) < 1 + \frac{1}{n} \qquad &\forall w_1 \in \Big(-\frac{1}{n}, 0\Big] \cup \Big[2l, 2l + \frac{1}{n}\Big). \end{aligned} \tag{4.23}$$

The validity of (4.23) is due to the fact that ψ is positive in the subgraph of $\widehat{h}$ and vanishes on the epigraph of $\widehat{h}$. Therefore, when mollifying ψ, the positivity set must increase (and the mollified function must vanish at points at distance from the subgraph of $\widehat{h}$ of the order of the mollification radius. Thus $h_n > h$; also $h_n < h + \frac{1}{n}$ due to our choice of $r_{i,n}$ in (4.13), since the mollification radius is smaller than $\frac{1}{n}$.

Moreover, again the positivity of $\widehat{\psi}$ implies that

$$\psi_n > 0 \quad \text{in} \quad SG_{h_n,\bar{s}} \subset K_n, \tag{4.24}$$

whereas

$$\psi_n(w_1, w_2) = 0 \qquad \text{if } w_1 \in \Big(-\frac{1}{n}, 2l + \frac{1}{n}\Big),\ w_2 \in \Big[h_n(w_1), 1 + \frac{1}{n}\Big), \tag{4.25}$$

because $\widehat{\psi}(w_1, w_2) = 0$ if $w_1 \in [0, 2l]$, $w_2 > h(w_1)$ and if $w_2 > 1$. Exploiting (4.14), and the fact that h is nonincreasing (resp. nondecreasing) in $[0, l]$ (resp. in $[l, 2l]$), one checks[8] that also h_n is nonincreasing in $[0, l]$ (resp. nondecreasing in $[l, 2l]$). Concerning the behaviour of h_n in $(-\frac{1}{n}, 0]$

[8]Let us show for instance that h_n is decreasing in $[0, l]$. Recall that the function $\widehat{\psi}$ vanishes above the graph of h, which is decreasing in $[0, l]$. Now, take a point $(w_1, w_2) \in K_n$, $w_1 \in [0, l)$, $w_2 > h(w_1)$; suppose first that $w_1 \geq r_{1,n}$. If $\text{dist}((w_1, w_2), \text{graph}(h)) > r_{1,n}$, then $\psi_n(w_1, w_2) = \widehat{\psi} \star \rho_{1,n}(w_1, w_2) = 0$, and if $\text{dist}((w_1, w_2), \text{graph}(h)) < r_{1,n}$, then $\psi_n(w_1, w_2) = \widehat{\psi} \star \rho_{1,n}(w_1, w_2) > 0$. Hence, if $\widehat{\psi} \star \rho_{1,n}(w_1, w_2) = 0$ then also $\widehat{\psi} \star \rho_{1,n}(w_1 + \varepsilon, w_2) = 0$ for $\varepsilon > 0$ small enough, because $\text{dist}((w_1 + \varepsilon, w_2), \text{graph}(h)) > \text{dist}((w_1, w_2), \text{graph}(h))$, being h decreasing in $[0, l]$. This argument applies also when $w_1 \in [0, r_{1,n})$ by (4.14), since $\overline{h}$ is nonincreasing also in $(-1/n, l)$.

(and similarly in $[2l, 2l+\frac{1}{n})$), we see that in $V_{i,n}$ ($i > 1$), we are mollifying with ρ_{i,n,k_n} whose radius of mollification is $r_{i,n}/k_n$, so that $\widehat{\psi}\star\rho_{i,n,k_n}$ equals 0 on the line $\{w_2 = 1+\frac{r_{i,n}}{k_n}\}$, and nonzero below inside K_n: this follows from the fact that $\widehat{\psi}$ is 0 on the line $\{w_2 = 1\}\cap K$ and nonzero below. We have defined the radii $r_{i,n}$ to be decreasing with respect to i, so that, ψ_n being the sum of $\widehat{\psi}\star\rho_{i,n,k_n}$ (whose support is $\overline{V}_{i,n}$ by (4.12)), it turns out that ψ_n is 0 on $\{w_2 = 1+\frac{r_{i,n}}{k_n}\}$ and nonzero below[9] in $V_{i,n}\setminus V_{i-1,n}$. As a consequence, h_n is nondecreasing[10] in $(-\frac{1}{n}, 0)$. In particular

$$h_n \in BV\Big((-\frac{1}{n}, 2l+\frac{1}{n})\Big). \tag{4.26}$$

Finally, it is not difficult to see that the functions h_n converge to h in $L^1((0, 2l))$ as $n\to\infty$, and

$$\lim_{n\to+\infty}\mathcal{H}^2(SG_{h_n,\bar{s}}) = H^2(K\cap SG_h). \tag{4.27}$$

From this, (4.18), Lemma 4.9, (4.19) and (1.11) we deduce

$$\overline{\mathcal{A}}(\psi_n, SG_{h_n,\bar{s}}) = \overline{\mathcal{A}}(\psi_n, K_n) - \mathcal{H}^2(K_n\setminus SG_{h_n,\bar{s}}) \to \mathcal{F}_{2l}(h,\psi) - \overline{\mathcal{A}}(\psi, R_{2l}\setminus K). \tag{4.28}$$

Step 2: The curves Γ_n, and the surfaces Σ_n and $\mathcal{G}_{\psi_n}$. Comparison with a Plateau problem.

In this step we compare the graph of ψ_n over K_n with the solution of a disc-type Plateau problem. In particular we will obtain a disc-type surface Σ_n^+ whose area is smaller than or equal to the area of the graph of ψ_n, see (4.29). In step 3 (see (4.32)) we will compare this surface with the graph of ψ on K.

We recall that ψ_n is continuous in $\overline{K}_n$, it is positive on the bottom edge $[-\frac{1}{n}, 2l+\frac{1}{n}]\times\{\bar{s}\}$ of K_n (see (4.17)), it is zero on the top edge $[-\frac{1}{n}, 2l+\frac{1}{n}]\times\{1+\frac{1}{n}\}$ by (4.23), and on the lateral edges of K_n it coincides with $\widehat{\psi}$; more specifically

$$\psi_n\Big(-\frac{1}{n}, w_2\Big) = \psi_n\Big(2l+\frac{1}{n}, w_2\Big) = \varphi(0, w_2) > 0 \qquad \text{for } w_2\in[\bar{s}, 1),$$
$$\psi_n\Big(-\frac{1}{n}, w_2\Big) = \psi_n\Big(2l+\frac{1}{n}, w_2\Big) = 0 \qquad \text{for } w_2\in\Big[1, 1+\frac{1}{n}\Big).$$

Define

$$\partial_D K_n := \Big(\Big[-\frac{1}{n}, 2l+\frac{1}{n}\Big]\times\{\bar{s}\}\Big)\cup\Big(\Big\{-\frac{1}{n}, 2l+\frac{1}{n}\Big\}\times[\bar{s}, 1]\Big).$$

From (4.17), we see that ψ_n coincides with $\widehat{\psi}$ over $\partial_D K_n$, and its graph over this set is a curve, that we denote by Γ_n^+. This curve, excluding its endpoints $P_n = (-\frac{1}{n}, 1, 0)$ and $Q_n = (2l+\frac{1}{n}, 1, 0)$, is contained in the half-space $\{w_3 > 0\}$, while $P_n, Q_n\in\{w_3 = 0\}$. We further denote by Γ_n^- the symmetric of Γ_n^+ with respect to the plane $\{w_3 = 0\}$, so that

$$\Gamma_n := \Gamma_n^+\cup\Gamma_n^-$$

is a Jordan curve in $\mathbb{R}^3$, see Fig. 5. Thus we can solve the disc-type Plateau problem with boundary Γ_n [8] and call $\Sigma_n\subset\mathbb{R}^3$ one of its solutions[11]. In addition, we may assume that Σ_n is symmetric with respect to the plane $\{w_3 = 0\}$ and that

$$\mathcal{H}^2(\Sigma_n^+) = \mathcal{H}^2(\Sigma_n^-),$$

with $\Sigma_n^\pm := \Sigma_n\cap\{w_3 \gtrless 0\}$, respectively (see Fig. 5).

[9] Notice that in $V_{i,n}\setminus V_{i-1,n}$ only $\widehat{\psi}\star\rho_{i,n,k_n}$ and $\widehat{\psi}\star\rho_{i+1,n,k_n}$, are nonzero (from this it follows that $h_n = 1+\frac{r_{i,n}}{k_n}$ in $(-\frac{1}{n}+\frac{1}{i+n+1}, -\frac{1}{n}+\frac{1}{i+n}]$).

[10] Precisely, h_n is piecewise constant and nondecreasing in $(-\frac{1}{n}, 0]$, but these two properties are not needed in the proof.

[11] Σ_n is the image of an area-minimizing map from the unit disc into $\mathbb{R}^3$.

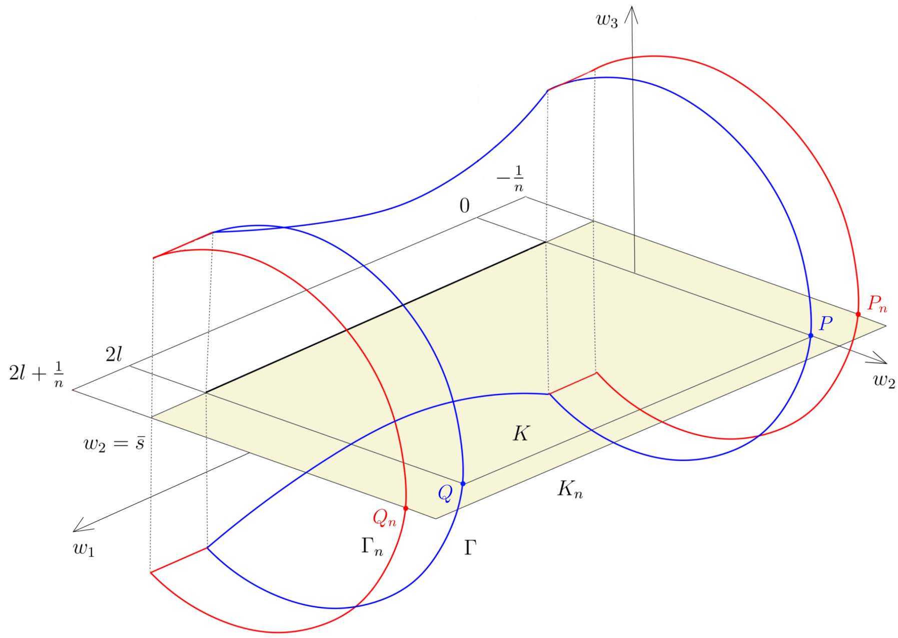

Figure 5: The rectangle K_n in (4.9) in dark, and the rectangle K inside. Γ is the curve passing through Q and P, the curves Γ_n (which pass through Q_n and P_n) approach Γ (Γ and Γ_n coincide and overlap on the graph of ψ over the bold segment $\{w_2 = \bar{s}\} \cap K$).

Now, we want to compare the area of the graph of ψ_n in $SG_{h_n,\bar{s}}$ with $\mathcal{H}^2(\Sigma_n^+)$. To this aim we start by observing that ψ_n, being smooth in K_n and continuous in $\overline{K}_n$ (see (4.11)), is such that its graph over $SG_{h_n,\bar{s}}$ has the topology of $SG_{h_n,\bar{s}}$, that is the topology of the disc. Indeed, $SG_{h_n,\bar{s}}$ is bounded by construction, and it is open from (4.24), (4.25). In addition, it is connected and simply connected. Indeed, take any continuous curve $\gamma : S^1 \to SG_{h_n,\bar{s}}$. Using (4.23), let $\widehat{s} \in (\bar{s}, 1)$ be such that $\{w_2 = \widehat{s}\} \cap K_n \subset SG_{h_n,\bar{s}}$; hence we can (vertically) contract γ continuously to its projection on the line $\{w_2 = \widehat{s}\}$, and then contract it continuously to the middle point of $\{w_2 = \widehat{s}\} \cap K_n$, showing that γ is homotopic to the constant curve. Hence, by the Riemann mapping theorem, $SG_{h_n,\bar{s}}$ is biholomorphic to the open unit disc, and $\overline{S}G_{h_n,\bar{s}}$ is homeomorphic to the closure of the disc, thanks to the fact that $\partial SG_{h_n,\bar{s}}$ is a Jordan curve, due to the BV-regularity of h_n (see (4.26)).

Denoting by $\mathcal{G}^+_{\psi_n}$ the graph of ψ_n over $SG_{h_n,\bar{s}}$, we consider the graph $\mathcal{G}^-_{\psi_n}$ of $-\psi_n$ over $SG_{h_n,\bar{s}}$, and observe that the closure of $\mathcal{G}^+_{\psi_n} \cup \mathcal{G}^-_{\psi_n}$ is a disc-type surface with boundary Γ_n. Therefore, by minimality,

$$\overline{\mathcal{A}}(\psi_n, SG_{h_n,\bar{s}}) = \mathcal{H}^2(\mathcal{G}^+_{\psi_n}) \geq \mathcal{H}^2(\Sigma_n^+). \tag{4.29}$$

Step 3: Passing to the limit as $n \to +\infty$: the curve Γ and the surface Σ.

The graph of ψ over the segment $[0, 2l] \times \{\bar{s}\}$ and the graph of φ over the two segments $\{0, 2l\} \times [\bar{s}, 1]$ form a simple continuous curve Γ^+ which, excluding the two endpoints

$$P = (0, 1, 0), \qquad Q = (2l, 1, 0), \tag{4.30}$$

is contained in the half-space $\{w_3 > 0\}$, while $P, Q \in \{w_3 = 0\}$ (see Fig. 5). If we consider

$$\Gamma := \Gamma^+ \cup \Gamma^-,$$

with Γ^- the symmetric of Γ^+ with respect to the plane $\{w_3 = 0\}$, a direct check shows that the curves Γ_n converge to the curve Γ in the sense of Frechet [17], as $n \to +\infty$. As a consequence, the area-minimizing disc-type surfaces Σ_n defined in step 2 satisfy $\mathcal{H}^2(\Sigma_n) \to \mathcal{H}^2(\Sigma)$ (see [17, Paragraphs 301, 305]), with Σ a disc-type area-minimizing surface spanned by Γ. It follows

$$\mathcal{H}^2(\Sigma_n^+) \to \mathcal{H}^2(\Sigma^+) \qquad \text{as } n \to +\infty, \tag{4.31}$$

where $\Sigma^+ := \Sigma \cap \{w_3 > 0\}$. From (4.31), (4.29), (4.28) we deduce

$$\begin{aligned}\mathcal{H}^2(\Sigma^+) &= \lim_{n\to+\infty} \mathcal{H}^2(\Sigma_n^+) \le \lim_{n\to+\infty} \overline{\mathcal{A}}(\psi_n, SG_{h_n,\bar{s}}) \\ &= \lim_{n\to+\infty} \big(\overline{\mathcal{A}}(\psi_n, K_n) - \mathcal{H}^2(K_n \setminus SG_{h_n,\bar{s}})\big) = \mathcal{F}_{2l}(h,\psi) - \overline{\mathcal{A}}(\psi, R_{2l} \setminus K).\end{aligned}$$

Since $\psi_n = \psi$ on $R_{2l} \setminus K$, we get

$$\lim_{n\to+\infty} \big(\overline{\mathcal{A}}(\psi_n, SG_{h_n,\bar{s}}) + \overline{\mathcal{A}}(\psi, R_{2l} \setminus K)\big) = \mathcal{F}_{2l}(h,\psi) \ge \mathcal{H}^2(\Sigma^+) + \overline{\mathcal{A}}(\psi, R_{2l} \setminus K). \tag{4.32}$$

Let $\Phi = (\Phi_1, \Phi_2, \Phi_3) : \overline{\mathrm{D}} \subset \mathbb{R}^2 \to \Sigma \subset \mathbb{R}^3$ be an analytic and conformal parametrization of Σ in the open unit disc D, continuous up to ∂D, with $\Phi(\partial \mathrm{D}) = \Gamma$. Exploiting the results in [16] (see also [8, pag. 343]) we know that

$$\Phi \text{ is an embedding,} \tag{4.33}$$

since Γ is a simple curve on the boundary of the convex set $K \times \mathbb{R}$.

Now, we need to prove several qualitative properties of Σ: this will be achieved in steps 4,5 and 6.

Step 4: $\Sigma \cap \{w_3 = 0\}$ is a simple curve Γ_0 connecting the two points P and Q in (4.30).

This can be seen as follows: Assume $\Phi(p_0) = P$ and $\Phi(q_0) = Q$ for two distinct points $p_0, q_0 \in \partial$D. By standard arguments[12], the open unit disc D is splitted into two connected components $\{x \in \mathrm{D} : \Phi_3(x) \ge 0\}$ and $\{x \in \mathrm{D} : \Phi_3(x) < 0\}$ and the set $\{\Phi_3 = 0\}$ must be a simple curve in D connecting p_0 and q_0 (here we use that the points p_0 and q_0 are, by the definition of Γ and the properties of Φ, the unique points on ∂D where $\Phi_3 = 0$ and that the two relatively open arcs on ∂D with extreme points p_0 and q_0 are mapped in $\{w_3 > 0\}$ and $\{w_3 < 0\}$ respectively). By the injectivity of Φ (property (4.33)) we conclude that

$$\Gamma_0 := \Phi(\{\Phi_3 = 0\}) \tag{4.34}$$

is a simple curve connecting P and Q on the plane $\{w_3 = 0\}$, and more specifically $\Gamma_0 \subset K$.

In the next two steps 5 and 6 we define the functions $\widetilde{h}$ and $\widetilde{\psi}$ which appear in the statement of the theorem. In step 5 we show that, due to the particular shape of Γ, the surface Σ admits a semicartesian parametrization [6], namely that if we slice Σ with a plane orthogonal to the first coordinate $w_1 \in (0, 2l)$ then the intersection is a curve connecting the two corresponding points on Γ; in addition, in this present case, this curve turns out to be simple. We will also show that the free part Γ_0 of Σ leaves a trace on R_{2l} which is the graph of a convex function $\widetilde{h}$ (of one variable).

Step 5: The projection $\pi_3(\Sigma)$ of Σ on the plane $\{w_3 = 0\}$ is the subgraph of a function $\widetilde{h} \in \mathcal{H}_{2l}$, where we recall that $\mathcal{H}_{2l}$ is defined in (1.13). *In particular, $\widetilde{h}$ is convex.*

We first show that $\pi_3(\Sigma)$ is the subgraph of a function $\widetilde{h}$, and then we prove that $\widetilde{h} \in \mathcal{H}_{2l}$. Take a point $W = (W_1, W_2, W_3) \in \Sigma \setminus \Gamma$, $W_1 \in (0, 2l)$; by the strong maximum principle, $\pi_3(W) \notin \partial K$:

[12] See also step 5 where a similar statement is proved.

this follows since points in $\Sigma \setminus \Gamma$ are in the interior of the convex envelope of Γ, see [8]. Consider the unique point $x \in \mathrm{D}$ such that $\Phi(x) = W$. Due to the particular structure of Γ, one checks that $\partial\mathrm{D} = \Phi^{-1}(\Gamma)$ splits into two connected components, $\Phi_1^{-1}((W_1, 2l]) \cap \partial\mathrm{D}$ and $\Phi_1^{-1}([0, W_1]) \cap \partial\mathrm{D}$, since $\Phi_1^{-1}(\{W_1\}) \cap \partial\mathrm{D}$ consists of two distinct points q_1, q_2 in $\partial\mathrm{D}$. In particular, the continuous function $\Phi_1(\cdot) - W_1$ changes sign only twice on $\partial\mathrm{D}$, namely in correspondence of q_1 and q_2. From Rado's lemma [8, Lemma 2, pag. 295] it follows that there are no points on $\Sigma \cap \{w_1 = W_1\}$ where the two area-minimizing surfaces Σ and the plane $\{w_1 = W_1\}$ are tangent to each other[13]. It follows that, if $\mathcal{P} \in (\Sigma \setminus \Gamma) \cap \{w_1 = W_1\}$, then the set $(\Sigma \setminus \Gamma) \cap \{w_1 = W_1\}$ is, in a neighbourhood of $\mathcal{P}$, an analytic curve, see again [8, Lemma 2, pag. 295]. Hence, $\{\Phi_1 = W_1\} \cap \mathrm{D}$ is, in a neighbourhood of $\Phi^{-1}(\mathcal{P})$, an analytic curve. If $\gamma_I : I \to \mathrm{D}$ is a parametrization of this curve, $I = (a, b)$ a bounded open interval, we see that the limits as $t \to a^+$ and $t \to b^-$ of $\gamma_I(t)$ exist[14] and belong to $\overline{\mathrm{D}}$. If $\lim_{t \to a^+} \gamma_I(t)$ belongs to $\partial\mathrm{D}$, it must be either q_1 or q_2; if instead it is in D, then we can always extend γ_I in a neighbourhood of a and find a larger interval $J \supset I$ on which γ_I can be extended. A similar argument applies for $\lim_{t \to b^-} \gamma_I(t)$. Let now $I_m = (a_m, b_m)$ be a maximal interval on which γ_I is defined, so that, by maximality, the limits as $t \to a_m^+$ and $t \to b_m^-$ are q_1 and q_2, respectively. We can then consider the closure $\overline{I}_m$ of I_m and we have that $\gamma_{\overline{I}_m}(\overline{I}_m)$ is a curve in $\overline{\mathrm{D}}$ joining q_1 and q_2. Thus we have proved that $\sigma_W := \Sigma \cap \{w_1 = W_1\}$ equals $\Phi(\gamma_{\overline{I}_m}(\overline{I}_m))$. In particular σ_W is a curve in $\mathbb{R}^3$ contained in the plane $\{w_1 = W_1\}$ and connecting the points $\Phi(q_1) \in \Gamma$ and $\Phi(q_2) \in \Gamma$. But we know that $\pi_3(\Phi(q_1)) = \pi_3(\Phi(q_2)) = (W_1, \bar{s}, 0)$, so $\pi_3(\sigma_W)$ is a segment in R_{2l} with endpoints $(W_1, \bar{s}, 0)$ and $(W_1, s^+, 0)$ for some $s^+ > \bar{s}$, and $s^+ \geq W_2$. In particular the whole segment "below" $\pi_3(W)$, namely the one with endpoints $(W_1, \bar{s}, 0)$ and $(W_1, W_2, 0)$, belongs to $\pi_3(\Sigma)$, and $\pi_3(\Sigma)$ is then the subgraph of some function $\widetilde{h}$. As a remark, due to the symmetry of the curve Γ, we can assume $\widetilde{h}$ is symmetric with respect to $\{w_1 = l\}$, namely $\widetilde{h}(\cdot) = \widetilde{h}(2l - \cdot)$.

Now we show that $\widetilde{h}$ is convex. Assume it is not, and take two points $(t_1, \widetilde{h}(t_1), 0), (t_2, \widetilde{h}(t_2), 0) \in R_{2l}$, $t_1 < t_2$, and a third point $(t_*, \widetilde{h}(t_*), 0)$, with $t_1 < t_* < t_2$, which is strictly above the segment l_{12} in R_{2l} joining $(t_1, \widetilde{h}(t_1), 0)$ and $(t_2, \widetilde{h}(t_2), 0)$. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a nonzero affine function[15] vanishing on the plane passing through l_{12} and orthogonal to $\{w_3 = 0\}$, and assume that f is positive at $(t_*, \widetilde{h}(t_*), 0)$. Let $\mathcal{Q} \in \Sigma$ be such that $\pi_3(\mathcal{Q}) = (t_*, \widetilde{h}(t_*), 0)$. Then $f \circ \Phi : \mathrm{D} \to \mathbb{R}$ is harmonic, and by the maximum principle there is a continuous curve[16] $\gamma_{\mathcal{Q}}$ in D joining $\Phi^{-1}(\mathcal{Q})$ to $\partial\mathrm{D}$ such that $f \circ \Phi$ is always positive on $\gamma_{\mathcal{Q}}$. But now, the continuous curve $\pi_3 \circ \Phi(\gamma_{\mathcal{Q}})$ joins $(t_*, \widetilde{h}(t_*), 0)$ to $\pi_3(\Gamma)$ and remains, in R_{2l}, strictly above the segment l_{12}. This is a contradiction, because $\pi_3 \circ \Phi(\gamma_{\mathcal{Q}})$ must be in the interior of the subgraph of $\widetilde{h}$.

Before passing to step 6, recall the definition of Γ_0 in (4.34), and observe that the Jordan curve $\Gamma^+ \cup \Gamma_0$ is the boundary of the disc-type surface Σ^+.

Let us denote by $U \subset K$ the connected component of $K \setminus \Gamma_0$ with boundary $\Gamma_0 \cup (\{0\} \times [\bar{s}, 1]) \cup ([0, 2l] \times \{\bar{s}\}) \cup (\{2l\} \times [\bar{s}, 1])$.

[13] If $\mathcal{P}$ is a tangency point, then the differential of Φ_1 must vanish at $\Phi^{-1}(\mathcal{P}) \in \mathrm{D}$.

[14] $\overline{\mathrm{D}}$ is compact, hence $\gamma_I(t)$ has some accumulation point as $t \to a^+$. Notice that I and $\gamma_I(I)$ are homeomorphic by contruction; in turn $\gamma_I(I)$ is homeomorphic to the analytic curve $\Phi \circ \gamma_I(I)$. Assume x is an accumulation point for $\gamma_I(t)$ as $t \to a^+$. If $x \in \mathrm{D}$, there is a neighborhood U of x such that $\sigma := \Phi(U) \cap \{w_1 = W_1\}$ is an analytic curve. Then γ_I, in a right neighbourhood J of a, is homeomorphic to the analytic curve $\Phi \circ \gamma_I(J) \in \mathbb{R}^3$ emanating from $\Phi(x)$, which in turn is the restriction of σ. In particular $\gamma_I(I)$ is a curve emanating from x and the limit as $t \to a^+$ of $\gamma_I(t)$ is x. If instead $x \in \partial\mathrm{D}$ then x must be the unique accumulation point. Indeed, $\lim_{t \to a^+} \Phi_1 \circ \gamma_I(t) = W_1$, and then $x = q_1$ or $x = q_2$, say $x = q_1$. Assume there is another accumulation point y as $t \to a^+$; then $y \notin \mathrm{D}$, otherwise we fall in the previous case, and therefore necessarily $y = q_2$. But in this case, we see that there must be another accumulation point $z \in \mathrm{D}$ (as $t \to a^+$, we move between a neighbourhood U of x and a neighbourhood V of y frequently, so that there should be some other accumulation point in $\overline{\mathrm{D}} \setminus (U \cup V)$) leading us to the previous case again.

[15] Take the signed distance from the plane.

[16] The set $(f \circ \Phi)^{-1}((0, \infty))$ is open, and cannot have connected components not intersecting $\partial\mathrm{D}$, by harmonicity.

We are now in a position to show that Σ^+ admits a non-parametric description over the plane $\{w_3 = 0\}$.

Step 6: The disc-type surface Σ^+ can be written as a graph over the plane $\{w_3 = 0\}$ of a $W^{1,1}$ function $\widetilde{\psi} : U \to [0, +\infty)$.

At first we observe that if Σ^+ is not Cartesian with respect to $\{w_3 = 0\}$, then there is some point $\mathcal{P} \in \Sigma^+ \setminus \partial\Sigma^+$ where the tangent plane to Σ^+ is vertical, that is, it contains the line $\{\mathcal{P} + (0, 0, w_3) : \ w_3 \in \mathbb{R}\}$. This can be seen as follows: as shown in step 5, the intersection between Σ^+ and any plane $\{w_1 = \text{cost}\}$, cost $\in (0, 2l)$, is a simple curve with endpoints in $\partial\Sigma^+$. If Σ^+ is not Cartesian, one of these curves γ is not Cartesian, and then there is a point where the tangent vector to γ is vertical. At such a point the tangent plane to Σ^+ is vertical.

Claim: If Π is a vertical plane tangent to Σ, then there is at most one point where Π and Σ are tangent.

We use an argument similar to the one needed to prove Rado's Lemma [8, Lemma 2, pag. 295]. Assume Π intersects the relative interior of Σ. It is easy to see that the intersection between Π and the Jordan curve Γ consists at most of four points[17] p_i, $i = 1, 2, 3, 4$. Let f be a linear function on $\mathbb{R}^3$ vanishing on Π. Then $f \circ \Phi$ is harmonic in D and continuous in $\overline{\mathrm{D}}$; in addition, it vanishes at $\{p_i, \ i = 1, 2, 3, 4\}$, and alternates its sign on the relatively open four arcs $\overline{p_i p_{i+1}}$ on $\partial\mathrm{D}$ with endpoints p_i. With no loss of generality, we may assume $f \circ \Phi > 0$ on $\overline{p_1 p_2}$ and $\overline{p_3 p_4}$. By harmonicity of $f \circ \Phi$, any connected component of the region $\{x \in \overline{\mathrm{D}} : f \circ \Phi(x) > 0\}$ must contain part of $\overline{p_1 p_2}$ or $\overline{p_3 p_4}$, so that we deduce that these connected components are at most two.

Assume now by contradiction that there are two distinct points $\mathcal{P}$ and $\mathcal{Q}$ of Σ such that Π is tangent to Σ at $\mathcal{P}$ and $\mathcal{Q}$. Since $f \circ \Phi$ has null differential at $\Phi^{-1}(\mathcal{P})$ and $\Phi^{-1}(\mathcal{Q})$, the set $\{f \circ \Phi = 0\}$, in a neighbourhood of $\Phi^{-1}(\mathcal{P})$, consists of $2m_p$ analytic curves crossing at $\Phi^{-1}(\mathcal{P})$, whereas in a neighbourhood of $\Phi^{-1}(\mathcal{Q})$, it consists of $2m_q$ analytic curves crossing at $\Phi^{-1}(\mathcal{Q})$. Therefore, in a neighbourhood of $\Phi^{-1}(\mathcal{P})$, the set $\{f \circ \Phi > 0\}$ counts at least 2 open regions (and similarly at $\Phi^{-1}(\mathcal{Q})$). Let us call A_1 and A_2 two of these regions around $\Phi^{-1}(\mathcal{P})$, and B_1, B_2 two of these regions around $\Phi^{-1}(\mathcal{Q})$. By harmonicity each A_i and B_i must be connected to one of the arcs $\overline{p_1 p_2}$ or $\overline{p_3 p_4}$. Hence some of these regions must belong to the same connected component of $\{f \circ \Phi > 0\}$. Then we are reduced to two following cases (see Fig. (6)):

(Case A) A_1 and A_2 belong to the same connected component, say the one containing $\overline{p_1 p_2}$. Hence we can construct two disjoint curves in $\{f \circ \Phi > 0\}$, both joining $\Phi^{-1}(\mathcal{P})$ to a point in $\overline{p_1 p_2}$, emanating from $\Phi^{-1}(\mathcal{P})$, one in region A_1 and one in region A_2. This contradicts the maximum principle, because these two curves would enclose a region where $f \circ \Phi$ takes also negative values, whereas its boundary is in $\{f \circ \Phi > 0\}$.

(Case B) A_1 and B_1 are joined to $\overline{p_1 p_2}$ and A_2 and B_2 are joined to $\overline{p_3 p_4}$. In this case we can construct four curves in $\{f \circ \Phi > 0\}$: σ_1 and σ_2 emanating from $\Phi^{-1}(\mathcal{P})$ in regions A_1 and A_2 and reaching $\overline{p_1 p_2}$ and $\overline{p_3 p_4}$, respectively; β_1 and β_2 emanating from $\Phi^{-1}(\mathcal{Q})$ in regions B_1 and B_2 and reaching $\overline{p_1 p_2}$ and $\overline{p_3 p_4}$, respectively. The region enclosed between these 4 curves has boundary contained in $\{f \circ \Phi > 0\}$ and, inside it, necessarily the function $f \circ \Phi$ takes also negative values, again in contrast with the maximum principle.

[17] A vertical plane Π intersects K on a straight segment. In turn, this segment intersects ∂K in two points. If Π intersects Γ in a point (W_1, W_2, W_3), then $(W_1, W_2, 0) \in \partial K$. Moreover, Π intersects Γ also at $(W_1, W_2, -W_3)$. Thus, the points of intersection are at most four. The degenerate cases in which Π contains a full $\mathcal{H}^1$-measured part of Γ are excluded by this analysis, because in these cases Π does not intersect the interior of Σ. Instead, the cases in which the intersection consists of 2 or 3 points are easier to treat, and we detail only the 4-points case (notice that by the geometry of Γ, the case of 3 points occurs when this plane is tangent to Γ at one of the points $(0, 1, 0)$ or $(2l, 1, 0)$).

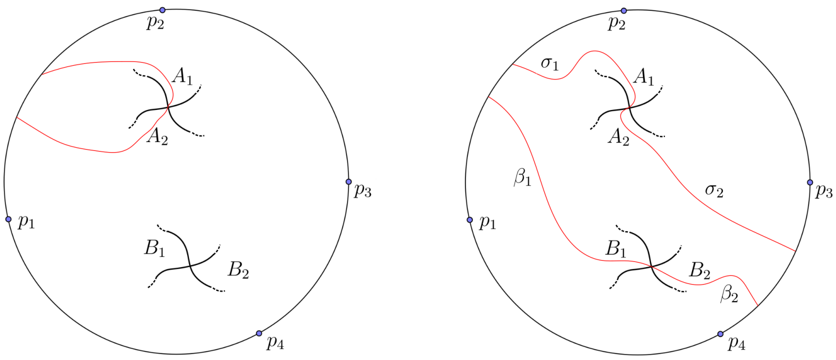

Figure 6: On the left it is represented case A in step 6 of the proof of Theorem 4.11. The point $\Phi^{-1}(\mathcal{P})$ is in the cross where the two emphasized paths start from. These curves stand in the region $\{f \circ \Phi > 0\}$ and join $\Phi^{-1}(\mathcal{P})$ with the arc $\overline{p_1 p_2} \subset \partial \mathrm{D}$. The picture on the right represents instead case B. The two cross points are $\Phi^{-1}(\mathcal{P})$ and $\Phi^{-1}(\mathcal{Q})$ and the paths σ_1, σ_2, β_1, β_2 are depicted.

From the above discussion our claim follows.

We are now ready to conclude the proof of step 6: suppose by contradiction that Σ^+ is not Cartesian with respect to $\{w_3 = 0\}$, and take a point $P^+ \in \Sigma^+ \setminus \Gamma$ where the tangent plane Π to Σ^+ at P^+ is vertical. By symmetry of Σ, the point P^-, defined as the symmetric of P^+ with respect to the rectangle R_{2l}, belongs to Σ^-, and the tangent plane to Σ^- at P^- is the same plane Π. This contradicts the claim. We eventually observe that $\widetilde{\psi}$ is analytic on the subgraph of $\widetilde{h}$, since its graph is Σ^+. We conclude that $\widetilde{\psi}$ belongs to $W^{1,1}(SG_{\widetilde{h}})$, since its total variation is bounded by the area of its graph, which is finite.

Step 7: the pair $(\widetilde{h}, \widetilde{\psi}) \in X_{2l}^{\mathrm{conv}}$.

We recall that in step 5 we proved that $\widetilde{h}$ is convex and $\widetilde{h}(\cdot) = \widetilde{h}(2l - \cdot)$, *i.e.* $\widetilde{h} \in \mathcal{H}_{2l}$. Furthermore Σ^+ is the graph of $\widetilde{\psi}$, and its projection on the plane $\{w_3 = 0\}$ is the subgraph of $\widetilde{h}$. It follows that the area of the graph of $\widetilde{\psi}$ is exactly the area of Σ^+ upon $SG_{\widetilde{h}}$. Let us also recall the $W^{1,1}$ regularity of $\widetilde{\psi}$ proved in step 6. Setting

$$\widetilde{\psi} := \psi \qquad \text{in } R_{2l} \setminus K,$$

we infer $(\widetilde{h}, \widetilde{\psi}) \in X_{2l}^{\mathrm{conv}}$. Since ψ is analytic, this construction turns out to be independent of the choice of $\overline{s}$.

Step 8: Conclusion of the proof.

From (4.32) we deduce

$$\mathcal{F}_{2l}(h, \psi) \geq \mathcal{H}^2(\Sigma^+) + \overline{\mathcal{A}}(\psi, R_{2l} \setminus K) = \mathcal{F}_{2l}(\widetilde{h}, \widetilde{\psi}), \tag{4.35}$$

where the last equality follows from the fact that $\widetilde{\psi}$ is continuous on $\partial_D R_{2l}$. Hence, also $(\widetilde{h}, \widetilde{\psi})$ is a minimizer for $\mathcal{F}_{2l}$. Now, we show (ii) and (iii), namely that $\widetilde{\psi}$ is continuous and equals 0 on $G_{\widetilde{h}}$. Indeed $\Sigma = \Sigma^+ \cup \Sigma^-$ is analytic, hence the graph of $\widetilde{h}$ coincide with the intersection of the analytic surface Σ with the plane $\{w_3 = 0\}$ (which is not tangent to Σ); it follows that $\widetilde{h}$ is continuous. Moreover we know that $\widetilde{\psi}$ is smooth in $SG_{\widetilde{h}}$. If its trace $\widetilde{\psi}^+$ on the boundary of $SG_{\widetilde{h}}$ is strictly positive somewhere, say at $Z \in G_{\widetilde{h}} \times \{0\}$, we infer that the vertical segment defined as

$$\{(Z_1, Z_2, w_3) : |w_3| \in (0, \widetilde{\psi}^+(w_1, w_2))\},$$

is contained in $\Sigma \cap \{w_1 = Z_1\}$, which is an analytic curve. This would imply that $\Sigma \cap \{w_1 = Z_1\}$ is contained in the straight line $(Z_1, Z_2) \times \mathbb{R}$, which is a contradiction, because $(Z_1, -1, 0) \in \Sigma \cap \{w_1 = Z_1\}$, and $Z_2 > -1$. We conclude $\widetilde{\psi}^+ = 0$ on $G_{\widetilde{h}}$.

Finally, let us prove (i), i.e., that $L_{\widetilde{h}} = \emptyset$. For, if not, the vertical part of Σ^+ obtained on $L_{\widetilde{h}}$ is flat and then, by analyticity, also Σ^+ is, a contradiction. This completes the proof. □

A direct consequence of Theorem 4.11 is the following which, coupled with Corollary 4.13 and its subsequent discussion, concludes the proof of Theorem 1.1.

Corollary 4.12. *Let $(h, \psi) \in X_{2l}^{\mathrm{conv}}$ be a solution of* (1.17)*. Then, either (h, ψ) is degenerate, i.e. $h \equiv -1$, or:*

(i) $h(0) = 1 = h(2l)$,

(ii) ψ *is continuous up to the boundary of* SG_h,

(iii) $\psi = 0$ *on* G_h.

Proof. As in the proof of Theorem 4.11, if (h, ψ) is not degenerate, then (h, ψ) is a solution as in Lemma 4.7 and we can choose $\overline{s} \in (-1, h(l))$. In step 7 of that proof we found that $(\widetilde{h}, \widetilde{\psi})$ is another minimizer, with $\widetilde{\psi}$ coinciding with ψ on $R_{2l} \setminus K$. Now, we claim that $h = \widetilde{h}$ and $\psi = \widetilde{\psi}$. By analyticity of both ψ and $\widetilde{\psi}$ in their domain SG_h and $SG_{\widetilde{h}}$ of definition respectively, they must coincide on $SG_h \cap SG_{\widetilde{h}}$. Hence, if $\overline{w}_1 \in (0, 2l)$ is such that $\widetilde{h}(\overline{w}_1) < h(\overline{w}_1)$, we deduce that $\psi(\overline{w}_1, \widetilde{h}(\overline{w}_1)) = \widetilde{\psi}(\overline{w}_1, \widetilde{h}(\overline{w}_1))$ contradicting the maximum principle because $(\overline{w}_1, \widetilde{h}(\overline{w}_1)) \in SG_h$. Therefore, necessarily $h \leq \widetilde{h}$. This implies that the trace of ψ on G_h coincides with the restriction of $\widetilde{\psi}$ on G_h, and thus this trace is continuous. If, by contradiction $h(\overline{w}_1) < \widetilde{h}(\overline{w}_1)$ for some $\overline{w}_1 \in (0, 2l)$, we find a contradiction as follows: Let (a, b) be a maximal interval containing $\overline{w}_1$ where $h < \widetilde{h}$ on it. Suppose for simplicity that $(a, b) = (0, 2l)$ (otherwise a similar argument applies).

We consider the curve Γ^+ obtained by glueing the graph of $\psi = \widetilde{\psi}$ over G_h with the graph of φ on the two segments L_h, and we consider also Γ^-, the symmetric of Γ^+ with respect to the plane $\{w_3 = 0\}$. Since both ψ and φ are strictly positive on these domains, the curve $\Gamma := \Gamma^+ \cup \Gamma^-$ is a Jordan curve. The surface

$$S := \{w \in \mathbb{R}^3 : (w_1, w_2) \in G_h,\ |w_3| \leq \psi(w_1, w_2)\} \cup \{w \in \mathbb{R}^3 : (w_1, w_2) \in L_h,\ |w_3| \leq \varphi(w_1, w_2)\}$$

is a disc-type surface spanning Γ, and thus $\mathcal{H}^2(S) \geq \mathcal{H}^2(\Sigma_\Gamma)$ where Σ_Γ is a disc-type solution of the Plateau problem spanning Γ. By definition of $\widetilde{\psi}$, its graph $\mathcal{G}_{\widetilde{\psi}}$ over $SG_{\widetilde{h}} \setminus SG_h$ enjoys the property that, denoting by $\mathcal{G}^-_{\widetilde{\psi}}$ its symmetric with respect to the plane $\{w_3 = 0\}$, the surface $\widetilde{\Sigma} := \mathcal{G}_{\widetilde{\psi}} \cup \mathcal{G}^-_{\widetilde{\psi}}$ is a solution to the Plateau problem for discs spanning Γ. Hence we deduce that $\mathcal{H}^2(S) \geq \mathcal{H}^2(\widetilde{\Sigma})$. Since however (h, ψ) and $(\widetilde{h}, \widetilde{\psi})$ are both minimizers of $\mathcal{F}_{2l}$, the same argument in Step 8 of Theorem 4.11 implies $\mathcal{H}^2(S) = \mathcal{H}^2(\widetilde{\Sigma})$, and S is a solution to the Plateau problem for discs spanning Γ. However, unless $h \equiv 1$, this contradicts the strong maximum principle, because Γ is a non-planar curve contained in the boundary of the convex set

$$C := \{w \in \mathbb{R}^3 : w_2 > h(w_1)\},$$

and so the interior of S cannot lie on ∂C. This contradiction leads us to our claim, namely $h = \widetilde{h}$, and $\psi = \widetilde{\psi}$. □

Now, we discuss the smoothness of h:

Corollary 4.13. *If a solution* (h,ψ) *of* (1.17) *is not degenerate (i.e. h is not constantly* -1*), then* $h \in C([0,2l])$ *and it is analytic in* $(0,2l)$.

Proof. The continuity of h at 0 and $2l$ is clear since $L_h = \varnothing$. Going back to Step 4 of the proof of Theorem 4.11, we have seen that the graph G_h of h coincides with the curve $\Gamma_0 = \Sigma \cap \{w_3 = 0\}$ (see (4.34)). Let $t_0 \in (0,2l)$, and let $Z = (t_0, h(t_0), 0) \in \Sigma \cap \{w_3 = 0\}$. Let $\gamma_0 \subset \overline{\mathrm{D}}$ be defined as $\gamma_0 := \Phi^{-1}(\Sigma \cap \{w_3 = 0\}) = \{x \in \overline{\mathrm{D}} : \Phi_3(x) = 0\}$, which is a simple curve in D connecting two points on ∂D, and let $z := \Phi^{-1}(Z) \in \gamma_0$. Setting $\partial_i = \frac{\partial}{\partial w_i}$, we have that $\partial_1\Phi(z)$ and $\partial_2\Phi(z)$ are distinct vectors generating the tangent plane to Σ at Z, which is a vertical plane; hence $\partial_1\Phi_3(z)$ and $\partial_2\Phi_3(z)$ cannot be both 0 (say, $\partial_2\Phi_3(z) \neq 0$). Therefore, by the implicit function theorem, in a neighborhood of z, γ_0 can be parametrized by a function $\sigma : (-\delta,\delta) \to \gamma_0$, $\sigma(s) = (s, f(s))$ with $f'(s) = -\frac{\partial_1\Phi_3(s,f(s))}{\partial_2\Phi_3(s,f(s))}$; as Φ is analytic in D, we deduce that f, and therefore σ, are analytic in a neighborhood of 0. Now $s \mapsto \Phi(\sigma(s))$ parametrizes Γ_0 in a neighborhood of Z. As we know that Γ_0 is the graph of h, we see that $\frac{d}{ds}\Phi_1(\sigma(s))$ is non-zero in a neighborhood of 0. Defining the parameter

$$t(s) := t_0 + \int_0^s |\nabla\Phi_1(\sigma(s))\sigma'(s)| ds,$$

if $s(t)$ denotes its inverse, we have

$$\frac{d}{dt}s(t) = \frac{1}{|\nabla\Phi_1(\sigma(s(t)))\sigma'(s(t))|},$$

and $s(\cdot)$ is analytic in a neighborhood of t_0. Then we have $h(t) = \Phi_2(\sigma(s(t)))$, which is the composition of analytic maps, hence analytic in a neighborhood of t_0. As this holds for all $t_0 \in (0,2l)$, the assertion follows. □

To conclude the proof of Theorem 1.1, it remains to show (2iv). The pair $(h \equiv 1, \varphi)$, where the function φ is as in (1.8), is one of the competitors for problem (1.17) (notice that φ attains the boundary condition); in addition, its subgraph is strictly convex (see Fig. 4), hence[18] necessarily $\psi \leq \widehat{\varphi}$ in $\overline{R}_{2l}$, where we have taken $\psi = \widetilde{\psi}$, the solution given by Theorem 4.11.

Eventually, the strict inequality in Theorem 1.1 (2iv) is a consequence of the strong maximum principle: indeed, points in $\Sigma \setminus \partial\Sigma$ are always strictly inside the convex hull of $\partial\Sigma$, with the only exception when $\partial\Sigma$ is planar (see [17, pag 63, section 70]); so that points of $\Sigma^+ \setminus \partial\Sigma$ are strictly inside the graph G_φ of φ (that is half of the lateral boundary of a cylinder). □

Now, we point out another consequence of Theorem 4.11, which gives the proof of Theorem 1.2. Let G_w be the graph in R_{2l} of a function $w \in C([0,2l],(-1,1])$ such that $w(0) = w(2l) = 1$, and consider the curve Γ_w obtained by concatenation of G_w with the graph of φ over $\partial_D R_{2l}$.

Corollary 4.14. *We have*

$$\mathcal{F}_{2l}(h,\psi) = \inf \mathcal{P}_{\Gamma_w}(X_{\min}), \tag{4.36}$$

where $(h,\psi) \in X_{2l}^{\mathrm{conv}}$ *is a minimizer of* $\mathcal{F}_{2l}$*,* $X_{\min}$ *is a parametrization of a disc-type area-mininizing solution of the Plateau problem spanning* Γ_w *(see* (2.2)*), and the infimum is computed over all functions* w *as above.*

The proof of this corollary can be achieved by adapting the proof of Theorem 4.11, which shows that the solution to the Plateau problem in (4.36) is Cartesian and the optimal w is convex.

[18] As already observed, the minimal surface Σ^+ is the graph of $\psi = \widetilde{\psi}$, and it must be contained in the convex envelope of Γ, *i.e.*, inside the subgraph of φ.

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