The L^1-relaxed area of the graph of the vortex map: optimal lower bound

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Abstract

We prove a lower bound for the value of the L^1-relaxed area of the graph of the map $u : \mathrm{B}_l(0) \setminus \{0\} \subset \mathbb{R}^2 \to \mathbb{R}^2$, $u(x) := x/|x|$, $x \neq 0$, for all values of the radius $l > 0$. In the computation of the singular part of the relaxed area, for l in a certain range, in particular l not too large, a nonparametric Plateau-type problem with partial free boundary, has to be solved. Our lower bound turns out to be optimal, in view of an upper bound proven in a companion paper.

Key words: Relaxation, Cartesian currents, area functional, minimal surfaces, Plateau problem.

AMS (MOS) subject classification: 49Q15, 49Q20, 49J45, 58E12.

1 Introduction

Given a bounded open set $\Omega \subset \mathbb{R}^n$ and a map $v : \Omega \to \mathbb{R}^N$ of class C^1, the area of the graph of v over Ω is given by the classical formula

$$\mathcal{A}(v, \Omega) = \int_\Omega |\mathcal{M}(\nabla v)| \, dx, \tag{1.1}$$

where $\mathcal{M}(\nabla v)$ is the vector whose entries are the determinants of the minors of the gradient ∇v of v of all orders k, $0 \leq k \leq \min\{n, N\}$ (conventionally, the determinant of order 0 is 1). Classical methods of relaxation suggest to consider the functional defined, for any $v \in L^1(\Omega, \mathbb{R}^N)$, as

$$\overline{\mathcal{A}}(v, \Omega) := \inf \Big\{ \liminf_{k \to +\infty} \mathcal{A}(v_k, \Omega) \Big\}, \tag{1.2}$$

and called *(sequential) relaxed area functional.* The infimum is computed over all sequences of maps $v_k \in C^1(\Omega, \mathbb{R}^N)$ approaching v in $L^1(\Omega, \mathbb{R}^N)$. Following [1], it follows that $\overline{\mathcal{A}}(\cdot, \Omega)$ extends $\mathcal{A}(\cdot, \Omega)$ and is L^1-lower semicontinuous. When the codimension $N = 1$, it is well-known both the domain of $\overline{\mathcal{A}}(\cdot, \Omega)$ and its expression [22]: $\overline{\mathcal{A}}(v, \Omega)$ is finite if and only if $v \in BV(\Omega)$, in which case

$$\overline{\mathcal{A}}(v, \Omega) = \int_\Omega \sqrt{1 + |\nabla v|^2} dx + |D^s v|(\Omega), \tag{1.3}$$

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∇v and $D^s v$ representing the absolutely continuous and singular parts of the distributional gradient Dv of v. Formula (1.3) is a basic example of non-parametric variational integral that is a measure when considered as a function of Ω [30], and is crucial, among others, in the study of capillarity problems [26], and in the analysis of the Cartesian Plateau problem [29]. The case $N > 1$ (referred here to as the case of codimension greater than 1) is much more involved, and only a few results are available about the behaviour of $\overline{\mathcal{A}}$. Again, one of its main motivations is the study of the Cartesian Plateau problem in higher codimension; in addition, from the point of view of Calculus of Variations, it is of interest in those vector minimum problems involving nonconvex integrands with nonstandard growth [3], [21], [28].

In this paper we restrict attention to the case $n = 2 = N$, and compute a lower bound for the relaxed area of the graph of the vortex map $u : B_l(0) \subset \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$u(x) := \frac{x}{|x|}, \qquad x \in \mathrm{B}_l(0) \setminus \{0\}. \tag{1.4}$$

which turns out to be *optimal*. Our sharp estimate, together with the upper bound provided in [8,9], gives the explicit value of $\overline{\mathcal{A}}(u, \mathrm{B}_l(0))$. Before stating the main result, observe that u belongs to $W^{1,p}(\Omega, \mathbb{R}^2)$ for all $p \in [1, 2)$, and that the image of u is the one-dimensional unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$, so that $Ju(x) = \det(\nabla u(x)) = 0$ for all $x \in \Omega \setminus \{0\}$. On the other hand, the distributional Jacobian of u is nonzero, and precisely equals the measure $\pi\delta_0$. In [1, Lemma 5.2], the authors show[1] that, for l large enough,

$$\overline{\mathcal{A}}(u, \mathrm{B}_l(0)) = |[\![G_u]\!]| + \pi = \int_{\mathrm{B}_l(0)} \sqrt{1 + |\nabla u|^2}\, dx + \pi. \tag{1.5}$$

Here $[\![G_u]\!]$ represents the current given by integration on the graph of u (see Section 2.2). With the aid of an example, they also show that $\overline{\mathcal{A}}(u, \mathrm{B}_l(0))$ must be strictly smaller than the right-hand side of (1.5), since there is a sequence of C^1-maps approximating u and having, asymptotically, a lower value of $\mathcal{A}(\cdot, \Omega)$. We anticipate here that, when l is small, the above mentioned sequence is not optimal, and the construction of a recovery sequence for $\overline{\mathcal{A}}(u, \mathrm{B}_l)$ is much more involved and requires to solve a sort of Plateau-type problem in $\mathbb{R}^3$ with singular boundary (i.e., with partial overlapping). Equivalently, with a reflection argument with respect to a plane, it can be seen as a non-parametric Plateau-type problem with a partial free boundary; in [9] valid for any $l > 0$, we have analysed this problem, and in particular we show that, excluding a singular configuration, there is a non-parametric solution attaining a zero boundary condition on the free part.

We emphasize that, for l small, the fact that the above mentioned sequence is not optimal is strongly related with the choice of the L^1-convergence in the definition (1.2) of $\overline{\mathcal{A}}$. Even if this is the most natural notion of convergence for the approximating maps v_k of u, one can also opts to choose stronger topologies. Some results are known when one chooses, instead of the convergence in L^1, the strict convergence in $BV(\Omega; \mathbb{R}^2)$ (see [5,6,16,17,37]). With this convergence, it has been shown in [5] that the relaxed area of the vortex map u always equals the right-hand side of (1.5).

In order to construct a recovery sequence for $\mathcal{A}(u, \Omega)$, the necessity of solving a 1-codimensional Plateau problem with partial free boundary in nonparametric form, is not a surprise. A similar construction is done in [11], to show an upper bound for the relaxed graph area of the triple junction map u_T; in [39] it is shown that this sequence is optimal. Together with u_T, the relaxation of the area of the vortex map are the only non-trivial examples in which it is possible to compute explicitly $\overline{\mathcal{A}}$. In other cases, it is only possible to show some specific upper and lower bounds, see for instance [14, 40]. To state our main result we need to fix some notation. For $l > 0$ we denote $R_{2l} := (0, 2l) \times (-1, 1)$ and let $\partial_D R_{2l} := (\{0, 2l\} \times [-1, 1]) \cup ((0, 2l) \times \{-1\})$ be what we call the

[1] In [1] the proof of (1.5) is given also for $N = n \geq 2$, where now π in (1.5) is replaced by ω_n.

Dirichlet boundary of R_{2l}. Define $\varphi : \partial_D R_{2l} \to [0,1]$ as $\varphi(t,s) := \sqrt{1-s^2}$ if $(t,s) \in \{0,2l\} \times [-1,1]$ and $\varphi(t,s) := 0$ if $(t,s) \in (0,2l) \times \{-1\}$. We introduce the functional $\mathcal{F}_l$ in the following way: Given $h \in L^\infty([0,l],[-1,1])$ and $\psi \in \mathrm{BV}(R_l;[0,1])$ we define

$$\mathcal{F}_l(h,\psi) := \overline{\mathcal{A}}(\psi, R_l) - \mathcal{H}^2(R_l \setminus SG_h) + \int_{\partial_D R_l} |\psi - \varphi| \, d\mathcal{H}^1 + \int_{(0,l)\times\{1\}} |\psi| \, d\mathcal{H}^1, \tag{1.6}$$

where we have noted, for any h, its subgraph $SG_h := \{(t,s) \in R_{2l} : s \leq h(t)\}$.

We further define

$$X_l := \{(h,\psi) : h \in L^\infty([0,l],[-1,1]), \psi \in \mathrm{BV}(R_l,[0,1]), \psi = 0 \text{ in } R_l \setminus SG_h\}. \tag{1.7}$$

The main result of the present paper is the following:

Theorem 1.1 (**Lower bound for the area of the vortex map**). *Let $l > 0$ and $u : \mathrm{B}_l(0) \setminus \{0\} \to \mathbb{R}^2$ be the vortex map defined in* (1.4). *Then*

$$\overline{\mathcal{A}}(u, \mathrm{B}_l(0)) \geq \int_{\mathrm{B}_l(0)} \sqrt{1 + |\nabla u|^2} dx + 2 \inf_{(h,\psi)\in X_l} \mathcal{F}_l(h,\psi). \tag{1.8}$$

In [8] it is proved that this estimate is sharp, and as a consequence equality holds in (1.8); it is also proved that for l large enough the infimum is not attained in $\widetilde{\mathcal{H}}_{2l} \times \mathcal{X}_{D,\varphi}$, and it equals π. Instead a minimizer exists for l small, hence ψ is real analytic in the interior of SG_h; furthermore, we show that h is smooth and convex, and ψ has vanishing trace on the graph of h (see [9, Theorems 1.1, 1.2]).

The proof of Theorem 1.1 is extremely involved: we assume (u_k) to be a recovery sequence converging to u, so that $\mathcal{A}(u_k, \mathrm{B}_l(0)) \to \overline{\mathcal{A}}(u, \mathrm{B}_l(0))$, and we analyse the behaviour of the graphs G_{u_k} over two distinct subsets of $\mathrm{B}_l(0)$, respectively one on which u_k converges uniformly to u, and one where concentration phenomena are allowed (let us call this the "bad set", denoted D_k in the sequel). In the former, studied in Section 4, we see that, up to small errors, the contribution of the areas of G_{u_k} gives the first term on the right-hand side of (1.8). In the set D_k, the graphs G_{u_k} might behave very badly. In order to detect their behaviour we introduce suitable projections in $\mathbb{R}^3$ (the maps Ψ_k in Definition 5.1 and the maps π_{λ_k} in Definition 5.3) and use them to reduce the currents carried by the graphs G_{u_k} to integral 2-currents supported in the cylinder $[0,l] \times \overline{\mathrm{B}}_1(0) \subset \mathbb{R}^3$. To this aim, it is necessary to use a cylindrical Steiner-type symmetrization technique for these integral currents, described in Section 3. Afterwards, an additional partition of the domain is needed, and we focus on what happens far from the origin and in a neighbourhood $\mathrm{B}_\varepsilon(0)$ of it. The first analysis is carried on in Sections 5, 6, and 7. The analysis around 0 is instead done in Section 8. Roughly speaking, we construct a cylindrically symmetric integral 2-current in $[0,l] \times \overline{\mathrm{B}}_1(0)$ whose area, up to small errors, is equal to the area of G_{u_k} over D_k. In order to relate the area of this current with the second term on the right-hand side of (1.8), we have to artificially add some rectifiable sets to this current (see Section 9), in such a way to force the new integral current to be a candidate for the minimum of $\mathcal{F}_l$. Some additional rearrangements are needed here, and are described in Section 10. The passage to the limit as $k \to +\infty$, and the conclusion of the proof, are then performed in Section 11, where we also show that all the errors in the estimates of the previous sections are negligible.

2 Preliminaries

2.1 Notation and conventions

For a map $v \in C^1(\Omega, \mathbb{R}^2)$ and $\Omega \subset\subset \mathbb{R}^2$, $\mathcal{A}(v,\Omega)$ coincides with the area of the graph $G_v := \{(x,y) \in \Omega \times \mathbb{R}^2 : y = v(x)\}$ of v seen as a Cartesian surface of codimension 2 in $\Omega \times \mathbb{R}^2$, and is given by

$$\mathcal{A}(v,\Omega)=\int_\Omega \sqrt{1+|\nabla v(x_1,x_2)|^2+|Jv(x_1,x_2)|^2}\;dx_1dx_2.$$

Here ∇v is the gradient of v, a 2×2 matrix, $|\nabla v|^2$ is the sum of the squares of all elements of ∇v, and Jv is the Jacobian determinant of v, *i.e.*, the determinant of ∇v. The relaxed area functional (with respect to the L^1-convergence) is denoted by $\overline{\mathcal{A}}(v,\Omega)$ and is defined in (1.2). We first remark that the infimum in (1.2) can be considered as taken over the class of sequences $v_k\in \mathrm{Lip}(\Omega;\mathbb{R}^2)$. This does not change the value of $\overline{\mathcal{A}}(\cdot,\Omega)$, as observed in [11].

Recall that in formula (1.1) the symbol $\mathcal{M}(\nabla v)$ denotes the vector whose entries are all determinants of the minors of ∇v. Precisely, let α and β be subsets of $\{1,2\}$, let $\bar\alpha$ denote the complementary set of α, namely $\bar\alpha=\{1,2\}\setminus\alpha$, let $|\cdot|$ denote the cardinality, and let $A\in\mathbb{R}^{2\times 2}$ be a matrix. Then, if $|\alpha|+|\beta|=2$, we denote by

$$M^{\beta}_{\bar\alpha}(A) \tag{2.1}$$

the determinant of the submatrix of A whose lines are those with index in β, and columns with index in $\bar\alpha$. By convention $M^{\emptyset}_{\emptyset}(A)=1$ and moreover

$$M^i_j=a_{ij},\qquad i,j\in\{1,2\},\qquad\qquad M^{\{1,2\}}_{\{1,2\}}(A)=\det A,$$

and the vector $\mathcal{M}(A)$ will take the form

$$\mathcal{M}(A)=(M^{\beta}_{\bar\alpha})(A))=(1,a_{11},a_{12},a_{21},a_{22},\det A),$$

where α and β run over all the subsets of $\{1,2\}$ with the constraint $|\alpha|+|\beta|=2$. We will identify α and β as multi-indeces in $\{1,2\}$.

2.1.1 Area in cylindrical coordinates

Polar coordinates in $\mathbb{R}^2_{\text{source}}$ are denoted by (r,α). Polar coordinates in the target space $\mathbb{R}^2_{\text{target}}$ are denoted by (ρ,θ).

Assume that $B=\{(r,\alpha)\in\mathbb{R}^2 : r\in(r_0,r_1),\ \alpha\in(\alpha_0,\alpha_1)\}$; then the area of the graph of $v=(v_1,v_2)$ in polar coordinates in B is given by

$$\mathcal{A}(v,B)=\int_{r_0}^{r_1}\int_{\alpha_0}^{\alpha_1}|\mathcal{M}(\nabla v)|(r,\alpha)\;rdrd\alpha.$$

For $i\in\{1,2\}$, we have

$$\partial_{x_1}v_i=\cos\alpha\partial_r v_i-\frac{1}{r}\sin\alpha\partial_\alpha v_i,\qquad \partial_{x_2}v_i=\sin\alpha\partial_r v_i+\frac{1}{r}\cos\alpha\partial_\alpha v_i.$$

Hence

$$\begin{aligned}|\nabla v_i|^2&=(\partial_r v_i)^2+\frac{1}{r^2}(\partial_\alpha v_i)^2,\qquad i\in\{1,2\},\\ \partial_{x_1}v_1\partial_{x_2}v_2-\partial_{x_2}v_1\partial_{x_1}v_2&=\frac{1}{r}\Big(\partial_r v_1\partial_\alpha v_2-\partial_\alpha v_1\partial_r v_2\Big).\end{aligned} \tag{2.2}$$

Thus the area of the graph of v on B is given by

$$\begin{aligned}&\mathcal{A}(v,B)\\ =&\int_{r_0}^{r_1}\int_{\alpha_0}^{\alpha_1}\sqrt{1+(\partial_r v_1)^2+(\partial_r v_2)^2+\frac{1}{r^2}\left\{(\partial_\alpha v_1)^2+(\partial_\alpha v_2)^2+\Big(\partial_r v_1\partial_\alpha v_2-\partial_\alpha v_1\partial_r v_2\Big)^2\right\}}\;rdrd\alpha.\end{aligned} \tag{2.3}$$

We denote by $\mathrm{B}_r = \mathrm{B}_r(0) \subset \mathbb{R}^2 = \mathbb{R}^2_{\text{source}}$ the open disc centered at 0 with radius $r > 0$ in the source space. Our reference domain is $\Omega = \mathrm{B}_l \subset \mathbb{R}^2_{\text{source}} = \mathbb{R}^2_{(x_1,x_2)}$ where $l > 0$ is fixed once for all. The symbol u will be used to note the vortex map in (1.4), which we assume to be defined on $\mathrm{B}_l \setminus \{0\}$.

For any $\varrho > 0$, it is convenient to introduce the (portion of) cylinder $C_l(\varrho)$, as

$$C_l(\varrho) := (-1, l) \times B_\varrho = \{(t, \rho, \theta) \in (-1, l) \times \mathbb{R}^+ \times (-\pi, \pi] : \rho < \varrho\} \subset \mathbb{R}^3 = \mathbb{R}_t \times \mathbb{R}^2_{\text{target}}, \tag{2.4}$$

where (t, ρ, θ) are cylindrical coordinates in $\mathbb{R}^3$, with the cylinder axis the t-axis. For $\varrho = 1$ we simply write

$$C_l(1) = C_l. \tag{2.5}$$

For a fixed parameter $\varepsilon \in (0, l)$, we also set

$$C_l^\varepsilon(\varrho) := (\varepsilon, l) \times B_\varrho = \{(t, \rho, \theta) \in (0, l) \times \mathbb{R}^+ \times (-\pi, \pi] : \varepsilon < \rho < \varrho\} \subset \mathbb{R}_t \times \mathbb{R}^2_{\text{target}}. \tag{2.6}$$

Also in this case we use the notation

$$C_l^\varepsilon(1) = C_l^\varepsilon. \tag{2.7}$$

The closure of $C_l(\rho)$ (resp. $C_l^\varepsilon(\rho)$) is denoted by $\overline{C}_l(\rho)$ (resp. $\overline{C}_l^\varepsilon(\rho)$), and the lateral boundary of $C_l(\rho)$ (resp. $C_l^\varepsilon(\rho)$) is denoted by $\partial_{\text{lat}} C_l(\rho)$ (resp. $\partial_{\text{lat}} C_l^\varepsilon(\rho)$).

Remark 2.1. We will often deal with integral currents supported in

$$[0, l] \times \overline{B}_1 \subset \overline{C}_l.$$

The choice of $C_l = (-1, l) \times B_1$ covering also certain negative values of the first coordinate t is useful to control and detect the behaviour of these currents on the plane $\{t = 0\}$.

2.1.2 Area formula

Let $f : U \subset \mathbb{R}^k \to \mathbb{R}^n$ be Lipschitz continuous, with $k \leq n$. The area of the image $f(U)$ of U by f is given by

$$\int_U Jf(x)dx,$$

with the Jacobian matrix of f given by

$$Jf = \sqrt{\det\left((\nabla f)^T \nabla f\right)} = \sqrt{\sum (\det A)^2} \qquad \text{a.e. in } U,$$

where, for almost every $x \in U$, the sum is made on all submatrices $A(x)$ of $\nabla f(x)$ of dimension $k \times k$.

2.2 Currents

For the reader convenience we recall some basic notion on currents. We refer to [25], [33] and [28] for an exhaustive discussion.

Given an open set $U \subset \mathbb{R}^n$ we denote by $\mathcal{D}^k(U)$ the space of smooth k-forms compactly supported in U and by $\mathcal{D}_k(U)$ the space of k-dimensional currents, for $0 \leq k \leq n$. If $T \in \mathcal{D}_k(\mathbb{R}^n)$, the symbol $|T|$ denotes the mass of the current T, and if $U \subset \mathbb{R}^n$ is an open set, the symbol $|T|_U$ will denote the mass of T in U, namely

$$|T|_U := \sup T(\omega),$$

the supremum being over all $\omega \in \mathcal{D}^k(U)$ with $\|\omega\| \leq 1$.

For $k \geq 1$ it is defined the boundary $\partial T \in \mathcal{D}_{k-1}(U)$ of a current $T \in \mathcal{D}_k(U)$ by the formula

$$\partial T(\omega) := T(d\omega) \text{ for all } \omega \in \mathcal{D}^{k-1}(U),$$

where $d\omega$ is the external differential of ω. For $T \in \mathcal{D}_0(U)$ one sets $\partial T := 0$.

If $F : U \to V$ a Lipschitz map between open sets, and $T \in \mathcal{D}_k(U)$, we denote by $F_\sharp T \in \mathcal{D}_k(V)$ the push-forward of T by F (see [33, Section 7.4.2]).

Given a k-dimensional rectifiable set $S \subset U$ and a tangent unit simple k-vector τ to it, we denote by $[\![S]\!]$ the current given by integration over S, namely

$$[\![S]\!](\omega) = \int_S <\tau(x), \omega(x)> \, d\mathcal{H}^k(x), \qquad \omega \in \mathcal{D}^k(U).$$

We will often omit specifying which is the vector τ if it is clear from the context. We will often deal with the case $k = 2$, and $U \subset \mathbb{R}^3$ where there are only two possible orientations. Moreover in the case $k = 3$ and $U \subset \mathbb{R}^3$ the current $[\![S]\!]$ reduces to the integration over the 3-dimensional set $S \subset \mathbb{R}^3$, and $\tau = e_1 \wedge e_2 \wedge e_3$.

We call $T \in \mathcal{D}_k(U)$ an integral current if it is rectifiable with integer multiplicity and if both $|T|_U$ and $|\partial T|_U$ are finite. The Federer-Fleming theorem for integral currents then states that a sequence of integral currents $T_i \in \mathcal{D}_k(U)$ with $\sup_i(T_i| + |\partial T_i|) < +\infty$ admits a subsequence converging weakly in the sense of currents to an integral current T.

A finite perimeter set is a subset $E \subset \mathbb{R}^n$ such that the current $[\![E]\!] \in \mathcal{D}_n(U)$ is integral. The symbol $\partial^* E$ denotes the reduced boundary of E. E is unique up to negligible sets, so that we always choose a representative of E for which the closure of the reduced boundary equals the topological boundary [34].

An integral current $T \in \mathcal{D}_k(U)$ is called indecomposable if there is no integral current $R \in \mathcal{D}_k(U)$ such that $R \neq 0 \neq T - R$ with

$$|T|_U + |\partial T|_U = |R|_U + |\partial R|_U + |T - R|_U + |\partial(T - R)|_U.$$

We will often use the following decomposition theorem for integer multiplicity currents: For every integral current $T \in \mathcal{D}_k(U)$ there is a sequence of indecomposable integral currents $T_i \in \mathcal{D}_k(U)$ with $T = \sum_i T_i$ and $|T| + |\partial T| = \sum_i |T_i| + \sum_i |\partial T_i|$ (see [25, Section 4.2.25]). In the case that $T \in \mathcal{D}_n(U)$, $U \subseteq \mathbb{R}^n$, the previous decomposition theorem can be stated as follows: There is a sequence of finite perimeter sets with $\{E_i\}_{i \in \mathbb{Z}}$ such that $T = \sum_{i \geq 0} [\![E_i \cap U]\!] - \sum_{i<0} [\![E_i \cap U]\!]$ with $\sum_i |E_i \cap U| + \sum_i \mathcal{H}^{n-1}(U \cap \partial^* E_i) = |T| + |\partial T|$ (see [33, Theorem 7.5.5] and its proof). Moreover, the decomposition theorem applied to E_i allows us to assume that the sequence $([\![E_i]\!])$ consists of indecomposable currents. In the case of 1-dimensional currents, it is possible also to characterize indecomposable currents, namely $T \in \mathcal{D}_1(\mathbb{R}^n)$ is indecomposable if $T = \gamma_\sharp [\![0, |T|]\!]$ with $\gamma : [0, |T|] \to \mathbb{R}^n$ a 1-Lipschitz simple curve, *i.e.*, injective on $[0, |T|)$. If moreover $\partial T = 0$ then $\gamma(0) = \gamma(|T|)$.

We will exploit the property that any boudaryless current $T \in \mathcal{D}_{n-1}(\mathbb{R}^n)$ is the boundary of a sum of currents given by integration over locally finite perimeter sets E_i, *i.e.*, $T = \sum_i \partial [\![E_i]\!]$. This is a consequence of the cone construction, and for integral currents can be obtained also from the isoperimetric inequality (see [33, Formula (7.26)] and [33, Theorem 7.9.1]).

We need also the concept of slice of an integral current with respect to a Lipschitz function f (see [33, Section 7.6]). Since we only employ it for slices with respect to parallel planes, the function f will be $f(x) = x_h$ where x_h is the coordinate in $\mathbb{R}^n$ whose axis is orthogonal to the considered planes. We denote by $T_t \in \mathcal{D}_{k-1}(\mathbb{R}^n)$ the slices of $T \in \mathcal{D}_k(\mathbb{R}^n)$ on the plane $\{x_h = t\}$, which will be supported on this plane. We will also use that, if T is boundaryless, then

$$\partial(T \llcorner \{x_h < t\}) = T_t \qquad \text{for a.e. } t \in \mathbb{R}.$$

For a map $v \in C^1(\Omega; \mathbb{R}^2)$ the symbol $G_v := \{(x, v(x)) \in \Omega \times \mathbb{R}^2\}$ represents the graph of v, which is a 2-dimensional oriented submanifold which is naturally identified with an integral boundaryless current given by integration over it, denoted by $[\![G_u]\!]$. Its mass, which coincides with the $\mathcal{H}^2$-measure of G_u, is given by

$$|[\![G_v]\!]| = \mathcal{A}(v, \Omega) = \int_\Omega |\mathcal{M}(\nabla v)| \, dx.$$

If $v \in BV(\Omega; \mathbb{R}^2)$ we denote by $R_v \subseteq \Omega$ the set of regular points of v, *i.e.*, the set consisting of points x which are Lebesgue points for v, $v(x)$ coincides with the Lebesgue value of v at x, and v is approximately differentiable at x. We also set

$$G_v^R := \{(x, v(x)) \in R_v \times \mathbb{R}^2\}.$$

The set G_v^R is $\mathcal{H}^2$-rectifiable and is identified with an integral current given by integration over it, denoted by $[\![G_v^R]\!]$. Also,

$$|[\![G_v^R]\!]| = \mathcal{A}(v, \Omega) = \int_\Omega |\mathcal{M}(\nabla v)| \, dx,$$

where now ∇v is the approximate gradient of v. In this case, in general, $[\![G_v]\!]$ is not a boundaryless current.

2.3 Generalized graphs in codimension 1

Let $v \in L^1(\Omega)$. Also in this case we denote by $R_v \subseteq \Omega$ the set of regular points of v, as above. We introduce

$$G_v^R := \{(x, v(x)) \in R_v \times \mathbb{R}\},$$
$$SG_v^R := \{(x, y) \in R_v \times \mathbb{R} : y < v(x)\}.$$

We often will identify SG_v^R with the integral 3-current $[\![SG_v]\!] \in \mathcal{D}_3(\Omega \times \mathbb{R})$. If v is a function of bounded variation, $\Omega \setminus R_v$ has zero Lebesgue measure, so that the current $[\![SG_v]\!]$ coincides with the integration over the subgraph

$$SG_v := \{(x, y) \in \Omega \times \mathbb{R} : y < v(x)\}.$$

For this reason we often identify $SG_v = SG_v^R$. It is well-known that the perimeter of SG_v in $\Omega \times \mathbb{R}$ coincides with $\overline{\mathcal{A}}(v, \Omega)$.

The support of the boundary of $[\![SG_v]\!]$ includes the graph G_v^R, but in general consists also of additional parts, called vertical. We denote by

$$\mathcal{G}_v := \partial [\![SG_v]\!] \llcorner (\Omega \times \mathbb{R}),$$

the generalized graph of u, which is a 2-integral current supported on $\partial^* SG_v$, the reduced boundary of SG_v in $\Omega \times \mathbb{R}$.

Let $\widehat{\Omega} \subset \mathbb{R}^2$ be a bounded open set such that $\Omega \subseteq \widehat{\Omega}$, and suppose that $L := \widehat{\Omega} \cap \partial\Omega$ is a rectifiable curve. Given $\psi \in BV(\Omega)$ and a $W^{1,1}$ function $\varphi : \widehat{\Omega} \to \mathbb{R}$, we can consider

$$\overline{\psi} := \begin{cases} f & \text{on } \Omega, \\ \varphi & \text{on } \widehat{\Omega} \setminus \Omega. \end{cases}$$

Then (see [29], [2])

$$\overline{\mathcal{A}}(\overline{\psi}, \widehat{\Omega}) = \overline{\mathcal{A}}(\psi, \Omega) + \int_L |\psi - \varphi| d\mathcal{H}^1 + \overline{\mathcal{A}}(\varphi, \widehat{\Omega} \setminus \overline{\Omega}).$$

2.4 Polar graphs and subgraphs in a cylinder

Consider the (portion of) cylinder $C_l = (-1, l) \times B_1$ defined in (2.5), endowed with cylindrical coordinates $(t, \rho, \theta) \in (-1, l) \times [0, 1) \times (-\pi, \pi]$. Take the rectangle $H = \{(t, \rho, \theta) \in C_l : \theta = 0\}$, which is endowed with Cartesian coordinates $(t, \rho) \in (-1, l) \times (0, 1)$. If $\Theta : H \to [0, \pi]$ is a function defined on H, we can associate to it the map $\mathrm{id} \bowtie \Theta : H \to C_l$ defined as

$$(t, \rho) \to (t, \rho, \Theta(t, \rho)), \qquad (t, \rho) \in H.$$

The polar graph of Θ is defined as

$$G_\Theta^{\mathrm{pol}} := \{(t, \rho, \Theta(t, \rho)) : t \in (-1, l),\ \rho \in (0, 1)\} = \mathrm{id} \bowtie \Theta(H),$$

where again we have used cylindrical coordinates.

We define a sort of polar subgraph of Θ as

$$SG_\Theta^{\mathrm{pol}} := \{(t, \rho, \theta) : t \in (-1, l),\ \rho \in [0, 1),\ \theta \in (-\eta, \Theta(t, \rho))\}.$$

Here $\eta > 0$ is a small number introduced for convenience, and it will suffice to take $\eta < 1$. If the set SG_Θ^{pol} has finite perimeter, its reduced boundary in $\{-\eta < \theta < \pi + \eta\} \cap C_l$ coincides with the generalized polar graph $\mathcal{G}_\Theta$ of Θ,

$$\mathcal{G}_\Theta = (\partial^* SG_\Theta^{\mathrm{pol}}) \cap (\{-\eta < \theta < \pi + \eta\} \cap C_l). \tag{2.8}$$

This set includes, up to $\mathcal{H}^2$-negligible sets, the polar graph G_Θ^{pol}. When SG_Θ^{pol} has finite perimeter, the current $[\![SG_\Theta^{\mathrm{pol}}]\!] \in \mathcal{D}_3(C_l)$ is integral and its boundary in $\{-\eta < \theta < \pi + \eta\} \cap C_l$ is the integration over the generalized polar graph of Θ, *i.e.*,

$$\partial [\![SG_\Theta^{\mathrm{pol}}]\!] \llcorner (\{-\eta < \theta < \pi + \eta\} \cap C_l) = [\![\mathcal{G}_\Theta]\!],$$

where $\mathcal{G}_\Theta$ is naturally oriented by the outer normal to $\partial^* SG_\Theta^{\mathrm{pol}}$.

Notice also that since $\Theta \in [0, \pi]$ the current $[\![\mathcal{G}_\Theta]\!]$ carried by the generalized polar graph $\mathcal{G}_\Theta$ is supported in $\{0 \le \theta \le \pi\} \cap C_l$.

3 Cylindrical Steiner symmetrization

In this section we introduce the cylindrical Steiner symmetrization of a finite perimeter[2] set $U \subseteq C_l = (-1, l) \times B_1(0)$. This rearrangement is obtained slice by slice by spherical (two dimensional) symmetrization, a technique introduced first by Pòlya. We refer to [15] and references therein for an exhaustive description of the subject. Here we collect the main properties used in the sequel of the paper. Furthermore we will introduce a generalization of this symmetrization in order to apply it to integral 2-currents.

Let us recall that C_l is endowed with cylindrical coordinates $(t, \rho, \theta) \in (-1, l) \times [0, 1) \times (-\pi, \pi]$. If x_1, x_2, x_3 are cartesian coordinates, we have $x_1 = t$, $x_2 = \rho \cos \theta$, $x_3 = \rho \sin \theta$. Sometimes it will be convenient to extend 2π-periodically the values of θ on the whole of $\mathbb{R}$.

[2]Recall that we choose a representative of U such that the closure of its reduced boundary $\partial^* U$ equals the topological boundary.

For every $t \in (-1, l)$ let $U_t := U \cap (\{t\} \times \mathbb{R}^2)$ the slice of U on the plane with first coordinate t, and for every $\rho \in (0,1)$ let $U_t(\rho) := U_t \cap (\{t\} \times \{\rho\} \times (-\pi, \pi])$ be the slice of U_t with the circle of radius ρ.

Definition 3.1 (**Cylindrical Symmetrization of solid sets in C_l**). *Let $U \subseteq C_l$ be a finite perimeter set. For every $t \in (-1, l)$ and $\rho \in (0,1)$ we let*

$$\Theta(t, \rho) = \Theta_U(t, \rho) := \frac{1}{\rho} \mathcal{H}^1(U_t(\rho)), \tag{3.1}$$

and we define the cylindrically symmetrized set $\mathbb{S}(U) \subseteq C_l$ as

$$\mathbb{S}(U) := \big\{(t, \rho, \theta) : t \in (-1, l),\ \rho \in (0,1),\ \theta \in \big(-\Theta(t,\rho)/2, \Theta(t,\rho)/2\big)\big\}. \tag{3.2}$$

Notice that $\Theta_U = \Theta_{\mathbb{S}(U)}$. The set $\mathbb{S}(U)$ enjoys the following properties:

(1) $\mathcal{H}^2(\mathbb{S}(U)_t) = \mathcal{H}^2(U_t)$ and $\mathcal{H}^1(\partial^*(\mathbb{S}(U)_t)) \le \mathcal{H}^1(\partial^*(U_t))$ for every $t \in (-1, l)$;

(2) $|\mathbb{S}(U)| = |U|$ and $\mathcal{H}^2(\partial^* \mathbb{S}(U)) \le \mathcal{H}^2(\partial^* U)$.

A proof of these properties is contained in [15, Theorem 1.4]. In particular, since U has finite perimeter, so is $\mathbb{S}(U)$ and its perimeter cannot increase after symmetrization. We will need to apply symmetrization to integral 3-currents in C_l. That is, (possibly infinite) sums of finite perimeter sets with integer coefficients. For this reason we introduce the following generalization of cylindrical symmetrization.

Let $\mathcal{E} \in \mathcal{D}_3(C_l)$ be an integral 3-current. By Federer decomposition theorem [25, Section 4.2.25, p. 420] (see also [25, Section 4.5.9] and [33, Theorem 7.5.5]) it follows that there is a sequence $(E_i)_{i \in \mathbb{N}}$ of finite perimeter sets such that

$$\mathcal{E} = \sum_i (-1)^{\sigma_i} [\![E_i]\!], \tag{3.3}$$

for suitable $\sigma_i \in \{0, 1\}$. We can also assume the decomposition is done in undecomposable components, so that

$$|\mathcal{E}| = \sum_i |E_i| \qquad \text{and} \qquad |\partial \mathcal{E}| = \sum_i \mathcal{H}^2(\partial^* E_i). \tag{3.4}$$

According to Definition 3.1 , we can symmetrize each set E_i into $\mathbb{S}(E_i)$.

Definition 3.2 (**Cylindrical symmetrization of an integer 3-current**). *Let $E := supp(\mathcal{E})$ denote the support of the current $\mathcal{E} \in \mathcal{D}_3(C_l)$. We let*

$$\mathbb{S}(E) := \bigcup_i \mathbb{S}(E_i),$$

which will be referred to as the symmetrized support of $\mathcal{E}$. The symmetrized current $\mathbb{S}(\mathcal{E}) \in \mathcal{D}_3(C_l)$ is defined as

$$\mathbb{S}(\mathcal{E}) := [\![\mathbb{S}(E)]\!]. \tag{3.5}$$

Notice that the multiplicity of $[\![\mathbb{S}(E)]\!]$ is one, hence $[\![\mathbb{S}(E)]\!]$ is the integration over a finite perimeter set.

3.1 Cylindrical symmetrization of a two-current. Slicings

Let us focus on a slice $\mathcal{E}_t$ of the current $\mathcal{E}$ with respect to a plane $\{t\} \times \mathbb{R}^2_{\rm target}$. Suppose for the moment that $\mathcal{E}$ is the integration over a finite perimeter set (that we identify with E) in C_l; $\mathcal{E}_t$ is the integration over the slice E_t of E, and suppose that the boundary of E_t is the trace σ of a rectifiable Jordan curve. Applying Definition 3.2 to the set E we see that E_t is transformed into the symmetrized set $\mathbb{S}(E_t)$ whose boundary is again[3] the trace σ_s of a Jordan curve. By the properties of the symmetrization we infer $\mathcal{H}^1(\sigma) \geq \mathcal{H}^1(\sigma_s)$.

However, if the boundary of E_t is the trace σ of a nonsimple curve, then the procedure is more involved. More generally, from Definition 3.2, we see that for a.e. $t \in (-1, l)$ the slice $\mathcal{E}_t$ of $\mathcal{E}$ is an integral 2-current, and it can be represented by integration over finite perimeter sets $(E_i)_t$ (with suitable signs) which are exactly the slices of the sets E_i in (3.3). Moreover for a.e. $t \in (-1, l)$ the boundary of $\mathcal{E}_t$ is a 1-dimensional integral current with finite mass, and it coincides with the integration (with suitable signs) over the boundaries of $(E_i)_t$, namely

$$\partial \mathcal{E}_t = \sum_i (-1)^{\sigma_i} \partial [\![(E_i)_t]\!].$$

Let us call this boundary σ (which, with a little abuse of notation, we identify with an integral 1-current, an at most countable sum of simple curves), and set $\sigma_s := \partial [\![\mathbb{S}(E)_t]\!]$. By Definition 3.2 it then follows that $\mathbb{S}(\mathcal{E})_t = [\![\mathbb{S}(E)_t]\!]$. Now, by the properties of the symmetrization, we see that $\mathcal{H}^1(\mathrm{supp}(\sigma)) \geq \mathcal{H}^1(\mathrm{\ spt\ }(\sigma_s))$. Also in this case it turns out that σ_s is the integration over countable many simple curves (with suitable orientation).

We have described so far how the boundary of $\mathcal{E}$ is trasformed slice by slice. In general if $\mathcal{E}$ is a 3-integral current in C_l, then the current $\mathcal{S} := \partial \mathcal{E}$ has the property that

$$|\mathcal{S}| \geq \mathcal{H}^2(\partial^* \mathbb{S}(E)).$$

There is also a viceversa. Precisely assume that $\mathcal{S}$ is any boundaryless integral 2-current in C_l. Then there is an integral 3-current $\mathcal{E}$ whose boundary is $\mathcal{S}$. So that we can define the symmetrization of $\mathcal{S}$ by symmetrizing $\mathcal{E}$.

Definition 3.3 (**Cylindrical symmetrization of the boundary of a three-current**). *The symmetrization of $\mathcal{S} = \partial \mathcal{E}$ is defined as*

$$\mathbb{S}(\mathcal{S}) := \partial \mathbb{S}(\mathcal{E}).$$

The next lemma will be useful in Section 8.

Lemma 3.4. *Let $\mathcal{S}$ be a boundaryless integral 2-current in C_l. Let $t \in (-1, l)$ be such that $\mathcal{S} \llcorner (\{t\} \times \mathbb{R}^2) = 0$. Then*

$$\mathbb{S}(\mathcal{S}) \llcorner (\{t\} \times \mathbb{R}^2) = 0. \tag{3.6}$$

Proof. We know that $\mathcal{S} = \partial \mathcal{E}$. By the properties of the cylindrical symmetrization (see item (2) above) for each set E_i we have

$$\mathcal{H}^2\Big((\{t\} \times \mathbb{R}^2) \cap \partial^* E_i \Big) \geq \mathcal{H}^2\Big((\{t\} \times \mathbb{R}^2) \cap \partial^* \mathbb{S}(E_i) \Big).$$

From our assumption it follows[4] that for all i we have $\mathcal{H}^2((\{t\} \times \mathbb{R}^2) \cap \partial^* E_i) = 0$, and thus

$$\mathcal{H}^2((\{t\} \times \mathbb{R}^2) \cap \partial^* \mathbb{S}(E_i)) = 0, \qquad i \in \mathbb{N}.$$

[3] $\mathbb{S}(E_t)$ is simply connected. Indeed the support of $\rho \mapsto \Theta(t, \rho)$ is a connected subset of $(0, 1)$.

[4] This follows since the decomposition is done in undecomposable components: if there is some boundary of some E_i then it cannot cancel with some other boundary (oppositely oriented) of some E_j.

To conclude the proof we have to show that

$$\mathcal{H}^2\Big((\{t\}\times\mathbb{R}^2)\cap\partial^*\mathbb{S}(E)\Big)=\mathcal{H}^2\Big((\{t\}\times\mathbb{R}^2)\cap\partial^*(\cup_i\mathbb{S}(E_i))\Big)=0. \tag{3.7}$$

The conclusion easily follows if the family $\{E_i\}$ is finite, since in this case $\partial(\cup_i\mathbb{S}(E_i))\subseteq\cup_i\partial\mathbb{S}(E_i)$. If this family is not finite we argue as follows: fix $\varepsilon>0$ and $N_\varepsilon\in\mathbb{N}$ so that (see (3.4))

$$\sum_{i=N_\varepsilon+1}^{+\infty}\mathcal{H}^2(\partial^*E_i)\le\varepsilon. \tag{3.8}$$

We have

$$\mathbb{S}(E)=\cup_i\mathbb{S}(E_i)=\big(\cup_{i=1}^{N_\varepsilon}\mathbb{S}(E_i)\big)\cup\big(\cup_{i=N_\varepsilon+1}^{+\infty}\mathbb{S}(E_i)\big)=:A\cup B,$$

thus

$$(\{t\}\times\mathbb{R}^2)\cap\partial\mathbb{S}(E)\subseteq\Big((\{t\}\times\mathbb{R}^2)\cap\partial^*A\Big)\cup\Big((\{t\}\times\mathbb{R}^2)\cap\partial^*B\Big).$$

By the previous observations $\mathcal{H}^2((\{t\}\times\mathbb{R}^2)\cap\partial^*A)=0$; we will prove that

$$\mathcal{H}^2((\{t\}\times\mathbb{R}^2)\cap\partial^*B)=\mathcal{H}^2\Big((\{t\}\times\mathbb{R}^2)\cap\partial^*(\cup_{i=N_\varepsilon+1}^{+\infty}\mathbb{S}(E_i))\Big)\le\varepsilon,$$

so that (3.7) follows by arbitrariness of $\varepsilon>0$. To do so, it suffices to write

$$\mathcal{H}^2\Big((\{t\}\times\mathbb{R}^2)\cap\partial^*(\cup_{i=N_\varepsilon+1}^{+\infty}\mathbb{S}(E_i))\Big)\le\mathcal{H}^2\big(\partial^*(\cup_{i=N_\varepsilon+1}^{+\infty}\mathbb{S}(E_i))\big)\le\sum_{i=N_\varepsilon+1}^{+\infty}\mathcal{H}^2(\partial^*\mathbb{S}(E_i))\le\varepsilon.$$

The last inequality follows from (3.8) and from the fact that symmetrization does not increase the perimeter. As for the second inequality, it follows from the lower semicontinuity of the perimeter. Indeed, setting $F_k:=\cup_{i=N_\varepsilon+1}^{k}\mathbb{S}(E_i)$ for $k\ge N_\varepsilon+1$, we see that $F_k\to F_\infty:=\cup_{i=N_\varepsilon+1}^{\infty}\mathbb{S}(E_i)$ in $L^1(C_l)$, and since F_k has finite perimeter we infer

$$\mathcal{H}^2(\partial^*F_\infty)\le\liminf_{k\to+\infty}\mathcal{H}^2(\partial^*F_k)\le\liminf_{k\to+\infty}\sum_{i=N_\varepsilon+1}^{k}\mathcal{H}^2(\partial^*\mathbb{S}(E_i)).$$

□

As before, we can look at what happens to the current $\mathcal{S}$ slice by slice. If $\partial\mathcal{E}=\mathcal{S}$, then $\mathcal{S}_t=-\partial(\mathcal{E}_t)$ for a.e. $t\in(-1,l)$. Assume that $\mathcal{E}$ decomposes as in (3.3), then

$$\mathcal{E}_t=\sum_i(-1)^{\sigma_i}[\![(E_i)_t]\!]\qquad\text{for a.e. } t\in(-1,l). \tag{3.9}$$

Now the sets $(E_i)_t$ are symmetrized as before, and their union, denoted $\mathbb{S}(E_t)$ (so that $\mathbb{S}(\mathcal{E})_t=[\![\mathbb{S}(E_t)]\!]$) satisfies

$$\partial[\![\mathbb{S}(E_t)]\!]=-\mathbb{S}(\mathcal{S})_t$$

and

$$|\mathcal{S}_t|\ge\mathcal{H}^1(\partial^*\mathbb{S}(E)_t).$$

Let us go back to (3.9). In general

$$|\mathcal{E}_t|\le\sum_i\mathcal{H}^2((E_i)_t); \tag{3.10}$$

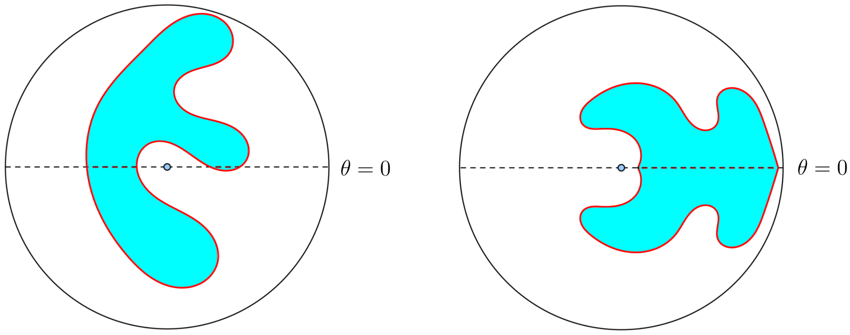

Figure 1: The symmetrization of a subset of B_1 bounded by a Jordan curve, with the respect to the radius $\{\theta = 0\}$; see formula (3.1).

however, since the decomposition is made of undecomposable components, (3.4) holds and hence

$$|\mathcal{E}_t| = \sum_i \mathcal{H}^2((E_i)_t) \qquad \text{for a.e. } t \in (-1, l). \tag{3.11}$$

This can be seen integrating in t formula (3.10), so that if strict inequality holds for a positive measured set of $t \in (-1, l)$ we would get the strict inequality in the first equation of (3.4), which is a contradiction.

Moreover, by construction, $\mathcal{H}^2((E_i)_t) = \mathcal{H}^2(\mathbb{S}(E_i)_t)$ for all i, and since $\mathbb{S}(E)_t = \cup_i \mathbb{S}(E_i)_t$ it also follows

$$|\mathcal{E}_t| = \sum_i \mathcal{H}^2((E_i)_t) = \sum_i \mathcal{H}^2(\mathbb{S}(E_i)_t) \geq \mathcal{H}^2(\mathbb{S}(E)_t).$$

Now we fix t such that (3.11) holds and set $F_i := (E_i)_t$, $\mathcal{F} := \mathcal{E}_t$, $F := \text{spt}\,(\mathcal{F})$, $\mathbb{S}(F) = \mathbb{S}(E)_t$. The set $F_i \in B_1$ can be sliced with respect to the radial coordinate $\rho \in (0, 1)$, so that exploiting that

$$(\mathcal{E}_t)_\rho = \sum_i (-1)^{\sigma_i} [\![((E_i)_t)_\rho]\!]$$

holds for a.e. ρ, we can repeat the same argument as before to obtain

$$|\mathcal{F}_\rho| = \sum_i \mathcal{H}^1((F_i)_\rho) \qquad \text{for a.e. } \rho \in (0, 1).$$

Again we have $\sum_i \mathcal{H}^1((F_i)_\rho) \geq \mathcal{H}^1(\mathbb{S}(F)_\rho)$. Recalling that $\mathbb{S}(F)_\rho = \mathbb{S}(E)_t \cap \partial B_\rho$, we conclude that, for a.e. $t \in (-1, l)$ and for a.e. $\rho \in (0, 1)$ the slice $(\mathcal{E}_t)_\rho$ satisfies

$$|(\mathcal{E}_t)_\rho| \geq \mathcal{H}^1(\mathbb{S}(E)_t \cap \partial B_\rho) = \rho\Theta(t, \rho), \tag{3.12}$$

where we have defined $\Theta(t, \rho) := \rho^{-1}\mathcal{H}^1(\mathbb{S}(E)_t \cap \partial B_\rho)$ the measure in radiants of the arc $\mathbb{S}(E)_t \cap \partial B_\rho$.

Remark 3.5. In the sequel we are going to apply the cylindrical symmetrization to a current supported in the portion of the cylinder $(0, l) \times B_1 \subset C_l$. The fact that we symmetrize in $C_l = (-1, l) \times B_1$ is useful to avoid possible creation of boundary on the disc $\{0\} \times B_1$.

4 Lower bound: first reductions on a recovery sequence

Let $u(x) = x/|x|$, $x \neq 0$, be the vortex map and $\Omega = \mathrm{B}_l$. In the aim of proving (1.8), consider a recovery $(u_k) \subset C^1(\Omega, \mathbb{R}^2)$ for the area of the graph of u, *i.e.*, $u_k \to u$ in $L^1(\Omega, \mathbb{R}^2)$ and

$$\liminf_{k\to+\infty} \mathcal{A}(u_k, \Omega) = \overline{\mathcal{A}}(u, \Omega);$$

with no loss of generality we can suppose that $u_k \to u$ almost everywhere in Ω and

$$\liminf_{k\to+\infty} \mathcal{A}(u_k, \Omega) = \lim_{k\to+\infty} \mathcal{A}(u_k, \Omega) < +\infty. \tag{4.1}$$

If

$$\Pi : \mathbb{R}^2_{\mathrm{target}} \to \overline{B}_1 \subset \mathbb{R}^2_{\mathrm{target}}, \qquad \Pi(x) := \begin{cases} \frac{x}{|x|} & \text{if } |x| > 1 \\ x & \text{if } |x| \leq 1, \end{cases} \tag{4.2}$$

is the projection map onto $\overline{B}_1$, then

$$\mathcal{A}(v, \Omega) \geq \mathcal{A}(\Pi \circ v, \Omega) \qquad \forall v \in C^1(\Omega, \mathbb{R}^2).$$

Notice that in general $\Pi \circ v \notin C^1(\Omega, \mathbb{R}^2)$; however $\Pi \circ v$ is of class C^1 on the set $\{x \in \Omega : |v(x)| < 1\}$ and Lipschitz continuous in Ω. Therefore, possibly replacing u_k by $\Pi \circ u_k$, we may assume that

$$u_k \text{ takes values in } \overline{B}_1 \text{ for all } k \in \mathbb{N}. \tag{4.3}$$

We start by dividing the source disc Ω in several suitable subsets. First we observe that from (4.1) there exists a constant $C > 0$ such that

$$C \geq \int_\Omega |\nabla u_k| \, dx = \int_0^l \int_{\partial \mathrm{B}_r} |\nabla u_k(r, \alpha)| \, d\mathcal{H}^1(\alpha) dr \qquad \forall k \in \mathbb{N}. \tag{4.4}$$

By Fatou's lemma, we then infer

$$\int_0^l L(r) \, dr \leq C,$$

where

$$L(r) := \liminf_{k\to+\infty} \int_{\partial \mathrm{B}_r} |\nabla u_k(r, \alpha)| \, d\mathcal{H}^1(\alpha) \qquad \text{for a.e. } r \in (0, l).$$

In particular, $L(r)$ is finite for almost every $r \in (0, l)$. Since $u_k \to u$ almost everywhere in Ω, we have that for almost every $r \in (0, l)$

$$u_k(r, \alpha) \to u(r, \alpha) \qquad \text{for } \mathcal{H}^1 - \text{a.e. } \alpha \in \partial \mathrm{B}_r.$$

Thus we can choose $\varepsilon \in (0, 1)$ arbitrarily small such that the two following properties are satisfied:

$$L(\varepsilon) \leq C_\varepsilon \quad \text{for a constant } C_\varepsilon > 0 \text{ depending on } \varepsilon; \tag{4.5}$$

$$\lim_{k\to+\infty} u_k(\varepsilon, \alpha) = u(\varepsilon, \alpha) \quad \text{for } \mathcal{H}^1 - \text{a.e. } \alpha \in \partial \mathrm{B}_\varepsilon. \tag{4.6}$$

4.1 The functions d_k, the subdomains A_n and D_k^δ, and selection of (λ_k)

By Egorov lemma, there exists a sequence (A_n) of measurable subsets of Ω such that, for any $n \in \mathbb{N}$, $A_{n+1} \subseteq A_n$,

$$|A_n| < \frac{1}{n}, \tag{4.7}$$

and

$$u_k \to u \ \text{ in } L^\infty(\Omega \setminus A_n, \mathbb{R}^2) \text{ as } k \to +\infty. \tag{4.8}$$

Definition 4.1 (**The function d_k and the set D_k^δ**). *We indicate by $d_k : \Omega \setminus \{0\} \to [0, 2]$ the function*

$$d_k := \big|u_k - u\big|, \tag{4.9}$$

and for any $\delta > 0$ we introduce

$$D_k^\delta := \{x \in \Omega \setminus \{0\} : d_k(x) > \delta\} =: \{d_k > \delta\}. \tag{4.10}$$

Notice that

$$\partial D_k^\delta \subseteq \{x \in \Omega \setminus \{0\} : d_k(x) = \delta\} =: \{d_k = \delta\}. \tag{4.11}$$

For $\varepsilon \in (0, 1)$ satisfying (4.5) and (4.6), we have $d_k \in \mathrm{Lip}(\Omega \setminus \overline{\mathrm{B}}_\varepsilon; \mathbb{R}^2) \cap W^{1,1}(\Omega; \mathbb{R}^2)$. Given $n \in \mathbb{N}$, from (4.8) it follows that for any $\delta > 0$ there exists $k_{\delta,n} \in \mathbb{N}$ such that $d_k < \frac{\delta}{2}$ in $\Omega \setminus A_n$ for any $k \geq k_{\delta,n}$, and thus

$$\Omega \setminus A_n \subseteq \Big\{d_k < \frac{\delta}{2}\Big\} \subseteq \Omega \setminus D_k^\delta \qquad \forall k > k_{\delta,n}.$$

Passing to the complement, from (4.11) and the inclusion $\{d_k = \delta\} \subseteq \{d_k \geq \delta/2\}$, we get

$$D_k^\delta \subseteq A_n \qquad \text{and} \qquad \partial D_k^\delta \subseteq A_n \qquad \forall k > k_{\delta,n}. \tag{4.12}$$

Lemma 4.2 (**Choice of λ_k and estimates on $D_k^{\lambda_k}$**). *Let $\varepsilon \in (0, 1)$ satisfy (4.5) and (4.6). Let $n > 0$ and $A_n \subset \Omega$ be a measurable set satisfying (4.7) and (4.8). Then there are a (not relabelled) subsequence of (u_k) and a decreasing infinitesimal sequence (λ_k) of positive numbers, both depending on n and ε, such that the following properties hold:*

(i) for all $k \in \mathbb{N}$ we have $\lambda_k \neq 1 - |u_k(0)|$ and the boundary of the set $D_k^{\lambda_k} = \{d_k > \lambda_k\}$ consists of an at most countable number of continuous curves which are either closed or with endpoints on $\partial\Omega$, and whose total length is finite;

(ii) $D_k^{\lambda_k} \cup \partial D_k^{\lambda_k} \subseteq A_n$ for all $k \in \mathbb{N}$;

(iii) $\displaystyle\lim_{k\to+\infty} \int_{\partial D_k^{\lambda_k}} d_k \, d\mathcal{H}^1 = 0 = \lim_{k\to+\infty} \Big(\lambda_k \mathcal{H}^1(\partial D_k^{\lambda_k})\Big)$;

(iv) $\partial D_k^{\lambda_k} \cap \partial \mathrm{B}_\varepsilon$ consists of a finite set of points. Hence[5], also the relative boundary of $D_k^{\lambda_k} \cap \partial\mathrm{B}_\varepsilon$ in $\partial\mathrm{B}_\varepsilon$ consists of a finite set $\{x_i\}_{i\in I_k}$ of points which are the endpoints of the corresponding finite number of arcs forming $D_k^{\lambda_k} \cap \partial\mathrm{B}_\varepsilon$, and

$$\lim_{k\to+\infty} \sum_{i \in I_k} d_k(x_i) = 0; \tag{4.13}$$

[5] The relative boundary of $D_k^{\lambda_k} \cap \partial\mathrm{B}_\varepsilon$ is contained in $\partial D_k^{\lambda_k} \cap \partial\mathrm{B}_\varepsilon$.

(v) $\mathcal{H}^1(D_k^{\lambda_k} \cap \partial\mathrm{B}_\varepsilon) \leq \frac{1}{n}$ *for all* $k \in \mathbb{N}$.

Proof. Let

$$I := (0,2) \setminus \bigcup_{k\in\mathbb{N}} \{1 - |u_k(0)|\},$$

which is of full measure in $(0,2)$.

We have, for an absolute positive constant α, recalling the definition of d_k in (4.9),

$$\int_\Omega |\nabla u_k - \nabla u|\ dx \geq \alpha \int_\Omega |\nabla d_k|\ dx = \alpha \int_0^2 \mathcal{H}^1(\{d_k = \lambda\})\ d\lambda, \tag{4.14}$$

where the last equality follows from the coarea formula, recalling also that u_k takes values in $\overline{B}_1$. The left-hand side is uniformly bounded with respect to k, thanks to (4.4) and the fact that $\nabla u \in L^1(\Omega, \mathbb{R}^2)$. Thus, denoting

$$\varphi_k(\cdot) := \mathcal{H}^1(\{d_k = \cdot\}), \qquad \varphi := \liminf_{k\to+\infty} \varphi_k, \tag{4.15}$$

we get, from Fatou's lemma,

$$\int_0^2 \varphi(\lambda)\ d\lambda = \int_I \varphi(\lambda)\ d\lambda \leq C_1, \tag{4.16}$$

for some constant $C_1 > 0$.

Let us now focus attention on the set $\partial\mathrm{B}_\varepsilon$. We apply the tangential coarea formula to $\partial\mathrm{B}_\varepsilon$ (see for instance [34, Theorems 11.4, 18.8]) so that, if ∂_{tg} stands for the tangential derivative along $\partial\mathrm{B}_\varepsilon$, we have

$$\int_{\partial\mathrm{B}_\varepsilon} \left|\partial_{\mathrm{tg}} d_k\right|\ d\mathcal{H}^1 = \int_0^2 \mathcal{H}^0(\{d_k = \lambda\} \cap \partial\mathrm{B}_\varepsilon)\ d\lambda.$$

Arguing in a similar manner as before, denoting

$$\psi_k(\cdot) := \mathcal{H}^0(\{d_k = \cdot\} \cap \partial\mathrm{B}_\varepsilon), \qquad \psi := \liminf_{k\to+\infty} \psi_k, \tag{4.17}$$

it follows that, exploiting condition (4.5), there exists a constant $C'_\varepsilon > 0$ such that

$$\int_I \psi(\lambda)\ d\lambda \leq C'_\varepsilon. \tag{4.18}$$

We now claim that

$$\exists (\lambda_m) \subset I : \lim_{m\to+\infty} \lambda_m = 0, \quad \lim_{m\to+\infty} (\varphi(\lambda_m)\lambda_m) = 0 = \lim_{m\to+\infty} (\psi(\lambda_m)\lambda_m). \tag{4.19}$$

Recalling that I is of full measure, assume (4.19) is false, so that either there are $c_0 > 0$ and $\delta_0 > 0$ such that

$$\varphi(\lambda) > \frac{c_0}{\lambda} \qquad \forall \lambda \in (0, \delta_0) \cap I, \tag{4.20}$$

or there are $c'_0 > 0$ and $\delta'_0 > 0$ such that

$$\psi(\lambda) > \frac{c'_0}{\lambda} \qquad \forall \lambda \in (0, \delta'_0) \cap I. \tag{4.21}$$

Suppose for instance we are in case (4.20): since I has full measure, this contradicts (4.16); the same argument applied to (4.21) leads to contradict (4.18). Hence claim (4.19) is proven, and therefore, upon passing to a (not relabelled) subsequence we might assume that (λ_m) is decreasing, and

$$\varphi(\lambda_m)\lambda_m < \frac{1}{m}, \qquad \psi(\lambda_m)\lambda_m < \frac{1}{m} \qquad\qquad \forall m \in \mathbb{N}.$$

Thus, recalling (4.15) and (4.17), for any $m \in \mathbb{N}$ there are infinitely many $l \in \mathbb{N}$ such that

$$\varphi_l(\lambda_m)\lambda_m < \frac{2}{m}, \qquad \psi_l(\lambda_m)\lambda_m < \frac{2}{m}. \tag{4.22}$$

Moreover, for any $n \in \mathbb{N}$ and $m \in \mathbb{N}$ there exists $k(n, \lambda_m) \in \mathbb{N}$ such that

$$D_h^{\lambda_m} \cup \partial D_h^{\lambda_m} \subseteq A_n \qquad \text{and} \qquad \mathcal{H}^1(D_h^{\lambda_m} \cap \partial \mathrm{B}_\varepsilon) \leq \frac{1}{n} \qquad \forall h \geq k(n, \lambda_m), \tag{4.23}$$

where the inclusion follows from (4.12) and the inequality being a consequence of (4.6). For any $m \in \mathbb{N}$ we can choose $h_m \in \mathbb{N}$ (depending also on n) such that $h_m < h_{m+1}$, $h_m \geq k(n, \lambda_m)$, and (4.22) is verified for $l = h_m$. Therefore

$$\lim_{m \to +\infty} (\varphi_{h_m}(\lambda_m)\lambda_m) = 0, \tag{4.24}$$

$$D_{h_m}^{\lambda_m} \cup \partial D_{h_m}^{\lambda_m} \subseteq A_n \quad \text{for all } n, m \in \mathbb{N}, \tag{4.25}$$

$$\lim_{m \to +\infty} (\psi_{h_m}(\lambda_m)\lambda_m) = 0, \tag{4.26}$$

$$\mathcal{H}^1(D_{h_m}^{\lambda_m} \cap \partial \mathrm{B}_\varepsilon) \leq \frac{1}{n} \quad \text{for all } n, m \in \mathbb{N}. \tag{4.27}$$

Notice also that from (4.22) we have $\psi_{h_m}(\lambda_m) < +\infty$, so that $\{d_{h_m} = \lambda_m\} \cap \partial \mathrm{B}_\varepsilon$ is a finite set $\{\widetilde{x}_i\}$ of points. The relative boundary $\partial(D_{h_m}^{\lambda_m} \cap \partial \mathrm{B}_\varepsilon)$ of $D_{h_m}^{\lambda_m} \cap \partial \mathrm{B}_\varepsilon$ in $\partial \mathrm{B}_\varepsilon$ must belong to $\partial D_{h_m}^{\lambda_m} \cap \partial \mathrm{B}_\varepsilon \subseteq \{d_{h_m} = \lambda_m\} \cap \partial \mathrm{B}_\varepsilon = \{\widetilde{x}_i\}$. Hence, let $\{x_i\} \subseteq \{\widetilde{x}_i\}$ be the set of boundary points of $D_{h_m}^{\lambda_m} \cap \partial \mathrm{B}_\varepsilon$ in $\partial \mathrm{B}_\varepsilon$.

Since $D_{h_m}^{\lambda_m} \cap \partial \mathrm{B}_\varepsilon$ is open in $\partial \mathrm{B}_\varepsilon$, we have that (whenever it is nonempty) it consists either of the union of arcs with endpoints $\{x_i\}$ or is the whole of $\partial \mathrm{B}_\varepsilon$, and statements (ii) and (v) follow. Notice also that

$$\sum_{x \in \partial(D_{h_m}^{\lambda_m} \cap \partial \mathrm{B}_\varepsilon)} d_{h_m}(x) = \lambda_m \mathcal{H}^0(\partial(D_{h_m}^{\lambda_m} \cap \partial \mathrm{B}_\varepsilon)) \leq \lambda_m \psi_{h_m}(\lambda_m),$$

and (4.13) follows from (4.26).

To prove (iii) we see that, by definition of φ_k in (4.15) and recalling (4.24), we obtain

$$\lim_{m \to +\infty} \int_{\{d_{h_m} = \lambda_m\}} d_{h_m} \, d\mathcal{H}^1 = \lim_{m \to +\infty} \left(\lambda_m \mathcal{H}^1(\{d_{h_m} = \lambda_m\})\right) = \lim_{m \to +\infty} (\lambda_m \varphi_{h_m}(\lambda_m)) = 0.$$

A similar argument holds for ψ_k using (4.26), and also (v) follows.

It remains to prove (i). The first assertion follows since $\lambda_m \in I$ from (4.19). As for the second assertion, we see that $D_{h_m}^{\lambda_m}$ is a subset of $\Omega \setminus \{0\}$ whose perimeter is finite: indeed, by definition the reduced boundary of $D_{h_m}^{\lambda_m}$ is a subset of $\{d_{h_m} = \lambda_m\}$, which has finite $\mathcal{H}^1$ measure by (4.22). Thus $\partial D_{h_m}^{\lambda_m}$ is a closed 1-integral current in $\Omega \setminus \{0\}$ and by the decomposition theorem for 1-dimensional currents it is the sum of integration on simple curves [25, pag. 420, 421], either closed or with endopoints on the boundary of $\Omega \setminus \{0\}$, *i.e.*, $\{0\} \cup \partial\Omega$. The finiteness of the total length of these curves follows, since $D_{h_m}^{\lambda_m}$ is a set of finite perimeter. This concludes the proof of (i), and of the lemma. $\square$

Corollary 4.3. *Let ε, n and (λ_k) be as in Lemma 4.2. Then*

$$\lim_{k\to+\infty}\big(\lambda_k\mathcal{H}^1(\{d_k=\lambda_k\})\big)=0,\qquad \lim_{k\to+\infty}\big(\lambda_k\mathcal{H}^0\big(\{d_k=\lambda_k\}\cap\partial\mathrm{B}_\varepsilon(0)\big)\big)=0.$$

Proof. It follows from the proof of Lemma 4.2. □

Once for all we fix the sequence (λ_k) as in Lemma 4.2 and, in order to shorten the notation, we give the following:

Definition 4.4 (Definite choice of D_k). *We set*

$$D_k := D_k^{\lambda_k}. \tag{4.28}$$

Let us recall that

$$\partial D_k \subseteq \{d_k=\lambda_k\}. \tag{4.29}$$

Also, observe that, upon extracting a further (not relabelled) subsequence, we might assume that the characteristic functions χ_{D_k} converge weakly* in $L^\infty(\Omega)$ to some $\zeta_n\in L^\infty(\Omega;[0,1])$ (the sequence (D_k) depends on n, and so ζ_n depends on n). Since the limit holds also weakly in $L^1(\Omega)$ we see that

$$\|\zeta_n\|_{L^1(\Omega)}\le\liminf_{k\to+\infty}\|\chi_{D_k}\|_{L^1(\Omega)}\le\frac{1}{n}. \tag{4.30}$$

Recalling the definition of $M^\beta_{\bar\alpha}(A)$ in (2.1), we prove the following statement.

Lemma 4.5 (The currents T_k and the limit current $\mathcal{T}_n$). *Let $n\in\mathbb{N}$ be fixed and let $A_n\subset\Omega$ satisfy* (4.7) *and* (4.8)*. For any $k\in\mathbb{N}$ define the current $T_k\in\mathcal{D}_2(\Omega\times\mathbb{R}^2)$ as*

$$T_k(\omega):=\begin{cases}\displaystyle\int_{\Omega\setminus D_k}\varphi(x,u_k(x))M^\beta_{\bar\alpha}(\nabla u_k(x))\,dx & \text{if } |\beta|\le 1,\\ 0 & \text{if } |\beta|=2,\end{cases}$$

where $\omega\in\mathcal{D}^2(\Omega\times\mathbb{R}^2)$ is a 2-form that writes as

$$\omega(x,y)=\varphi(x,y)dx^\alpha\wedge dy^\beta,\quad \varphi\in C^\infty_c(\Omega\times\mathbb{R}^2),\quad |\alpha|+|\beta|=2. \tag{4.31}$$

Then

$$\lim_{k\to+\infty}T_k=\mathcal{T}_n\in\mathcal{D}_2(\Omega\times\mathbb{R}^2)\qquad\text{weakly in the sense of currents,}$$

where

$$\mathcal{T}_n(\omega):=\int_\Omega\varphi(x,u(x))M^\beta_{\bar\alpha}(\nabla u(x))(1-\zeta_n(x))\,dx\qquad\forall\omega\text{ as in (4.31).}$$

Proof. Since the Jacobian of u vanishes almost everywhere it follows that $\mathcal{T}_n(\varphi dy^1\wedge dy^2)=0$ for all φ as in (4.31). Then for 2-forms $\omega=\varphi dy^1\wedge dy^2$ the convergence $T_k(\omega)\to\mathcal{T}_n(\omega)$ is achieved. We are then left to prove that for all 2-forms ω with $\omega(x,y)=\varphi(x,y)dx^\alpha\wedge dy^\beta$, $\varphi\in C^\infty_c(\Omega\times\mathbb{R}^2)$, $|\alpha|+|\beta|=2$, and $|\beta|\le 1$, it holds

$$\lim_{k\to+\infty}\int_{\Omega\setminus D_k}\varphi(x,u_k(x))M^\beta_{\bar\alpha}(\nabla u_k(x))\,dx=\int_\Omega\varphi(x,u(x))M^\beta_{\bar\alpha}(\nabla u)(1-\zeta_n(x))\,dx. \tag{4.32}$$

To simplify the argument we treat separately the cases $\omega = \varphi(x,y)dx^1 \wedge dx^2$ and $\omega = \varphi(x,y)dx^i \wedge dy^j$ for some $i,j \in \{1,2\}$. In the former case we simply have

$$\int_{\Omega\setminus D_k} \varphi(x,u_k(x))\ dx = \int_{\Omega} \varphi(x,u_k(x))\chi_{\Omega\setminus D_k}(x)\ dx.$$

Then, using that $u_k \to u$ uniformly in $\Omega \setminus D_k$ (see (4.8), Lemma 4.2(ii) and (4.28)) and $\chi_{\Omega\setminus D_k} \to \chi_\Omega - \zeta_n$ weakly* in $L^\infty(\Omega)$, it follows

$$\lim_{k\to+\infty} \int_{\Omega\setminus D_k} \varphi(x,u_k(x))\ dx = \int_{\Omega} \varphi(x,u(x))(1-\zeta_n(x))\ dx = \mathcal{T}_n(\omega).$$

Assume now $\omega = \varphi(x,y)dx^i \wedge dy^j$, $i,j \in \{1,2\}$, $i \neq j$. In this case (4.32) reads as

$$\lim_{k\to+\infty} \Big(\int_{\Omega\setminus D_k} \varphi(x,u_k(x))D_{\bar{i}}[(u_k(x))_j]\ dx - \int_{\Omega} \varphi(x,u(x))D_{\bar{i}}u_j(x)(1-\zeta_n(x))\ dx \Big) = 0,$$

with $\bar{i} = \{1,2\} \setminus \{i\}$. Since $\chi_{D_k} \to \zeta_n$ weakly* in $L^\infty(\Omega)$, this is equivalent to proving

$$\lim_{k\to+\infty} \Big(\int_{\Omega\setminus D_k} \varphi(x,u_k(x))D_{\bar{i}}[(u_k(x))_j]\ dx - \int_{\Omega\setminus D_k} \varphi(x,u(x))D_{\bar{i}}u_j(x)\ dx \Big) = 0.$$

The quantity between parentheses on the left-hand side can be written as

$$\int_{\Omega\setminus D_k} \Big(\varphi(x,u_k(x)) - \varphi(x,u(x))\Big) D_{\bar{i}}[(u_k(x))_j]\ dx + \int_{\Omega\setminus D_k} \varphi(x,u(x)) \Big(D_{\bar{i}}[(u_k(x))_j] - D_{\bar{i}}u_j(x) \Big)\ dx,$$

and we see that the first integral tends to zero as $k \to +\infty$, since $u_k \to u$ uniformly in $\Omega \setminus D_k$, φ is Lipschitz continuous, and the $L^1(\Omega)$-norm of $D_{\bar{i}}[(u_k)_j]$ is uniformly bounded with respect to k. The second integral can be instead integrated by parts[6], obtaining

$$\begin{aligned}
&\int_{\Omega\setminus D_k} \varphi(x,u(x))(D_{\bar{i}}[(u_k(x))_j] - D_{\bar{i}}u_j(x))\ dx \\
&= \int_{\partial D_k} \varphi(x,u(x))((u_k(x))_j - u_j(x))\nu_{\bar{i}}(x)\ d\mathcal{H}^1(x) - \int_{\Omega\setminus D_k} D_{\bar{i}}(\varphi(x,u(x)))((u_k(x))_j - u_j(x))\ dx \\
&=: \mathrm{I}_k + \mathrm{II}_k.
\end{aligned}$$

Thanks to the fact that φ is bounded and that $|(u_k)_j(x) - u_j(x)| \leq d_k(x) = \lambda_k$ on ∂D_k, we conclude by Corollary 4.3 that $\lim_{k\to+\infty} \mathrm{I}_k = 0$. Moreover

$$\begin{aligned}
\mathrm{II}_k = &- \int_{\Omega\setminus D_k} \partial_{x_{\bar{i}}}\varphi(x,u(x))((u_k(x))_j - u_j(x))dx \\
&- \sum_{l=1}^{2} \int_{\Omega\setminus D_k} \partial_{y_l}\varphi(x,u(x))D_{\bar{i}}u_l(x)((u_k)_j(x) - u_j(x))dx =: \mathrm{II}_{k,1} + \mathrm{II}_{k,2}.
\end{aligned}$$

Then $\lim_{k\to+\infty} \mathrm{II}_{k,1} = \lim_{k\to+\infty} \mathrm{II}_{k,2} = 0$, since the partial derivatives of φ are bounded, $D_{\bar{i}}u \in L^1(\Omega \setminus D_k, \mathbb{R}^2)$, $|(u_k)_j - u_j| \leq d_k \leq \lambda_k$ on $\Omega \setminus D_k$, and $\lim_{k\to+\infty} \lambda_k = 0$. □

[6]From Lemma 4.2(i), D_k has rectifiable boundary; moreover, $\varphi(\cdot, u_k(\cdot))$ is Lipschitz. We can then apply a version of the Gauss-Green theorem, see for instance [34, pag. 124, exercise 12.12].

Remark 4.6. The mass of the current T_k is given by

$$|T_k| = \int_{\Omega\setminus D_k} \sqrt{1+|\nabla u_k|^2}dx. \tag{4.33}$$

To see (4.33) we choose a 2-form $\omega \in \mathcal{D}^2(\Omega \times \mathbb{R}^2)$ as

$$\omega := \sum_{|\alpha|+|\beta|=2} \varphi_{\bar{\alpha}\beta} dx^\alpha \wedge dy^\beta, \qquad \|\omega\| \le 1,$$

set[7] $\widehat{\omega}(x,y) =: (\varphi_{\bar{\alpha}\beta}(x,y)) \in \mathbb{R}^6$, and

$$\widetilde{\mathcal{M}}(\nabla u_k(x)) := (1, D_1[(u_k(x))_1], D_2[(u_k(x))_1], D_1[(u_k(x))_2], D_2[(u_k(x))_2], 0) \in \mathbb{R}^6 = \mathbb{R}\times\mathbb{R}^4\times\mathbb{R},$$

so that

$$\begin{aligned} T_k(\omega) &= \int_{\Omega\setminus D_k} \langle \widehat{\omega}(x, u_k(x)), \widetilde{\mathcal{M}}(\nabla u_k(x))\rangle dx \\ &\le \|\widehat{\omega}\| \int_{\Omega\setminus D_k} |\widetilde{\mathcal{M}}(\nabla u_k(x))| dx \le \int_{\Omega\setminus D_k} \sqrt{1+|\nabla u_k|^2}dx. \end{aligned} \tag{4.34}$$

To prove the converse inequality, choosing $\widehat{\omega}(x,y) = \frac{\widetilde{\mathcal{M}}(\nabla u_k(x))}{|\widetilde{\mathcal{M}}(\nabla u_k(x))|}$ would give the equality in (4.34). However, $\frac{\widetilde{\mathcal{M}}(\nabla u_k)}{|\widetilde{\mathcal{M}}(\nabla u_k)|}$ is not necessarily of class C_c^∞, so we need to use the density of $C_c^\infty(\Omega\times\mathbb{R}^2)$ in $L^1(\Omega\times\mathbb{R}^2)$ (here we use that $\widetilde{\mathcal{M}}(\nabla u_k) \in L^\infty(\Omega, \mathbb{R}^6)$ since u_k is Lipschitz continuous).

With a similar argument, setting

$$\widetilde{\mathcal{M}}(\nabla u(x)) := (1-\zeta_n(x))\mathcal{M}(\nabla u(x)) \in \mathbb{R}^6, \quad x \in \Omega\setminus \overline{\mathrm{B}}_\varepsilon$$

we can show that the total mass of $\mathcal{T}_n$ in $(\Omega\setminus\overline{\mathrm{B}}_\varepsilon)\times\mathbb{R}^2$ is given by

$$|\mathcal{T}_n|_{(\Omega\setminus\overline{\mathrm{B}}_\varepsilon)\times\mathbb{R}^2} = \int_{\Omega\setminus\overline{\mathrm{B}}_\varepsilon} |\mathcal{M}(\nabla u)||1-\zeta_n|\ dx. \tag{4.35}$$

4.2 Estimate of the mass of $[\![G_{u_k}]\!]$ over $\Omega\setminus D_k$

We denote by $\Phi_k = \Phi_{u_k} = \mathrm{Id} \bowtie u_k : \Omega \to \Omega\times\mathbb{R}^2$ the map

$$\Phi_k(x) = (x, u_k(x)), \tag{4.36}$$

in such a way that $\Phi_k(\Omega) = G_{u_k}$, with $G_{u_k} = \{(x,y)\in\Omega\times\mathbb{R}^2 : y = u_k(x)\}$ the graph of u_k.

We denote as usual by

$$[\![G_{u_k}]\!] \in \mathcal{D}_2(\Omega\times\mathbb{R}^2) \tag{4.37}$$

the integral current supported by the graph of u_k.

We now want to estimate the area of the graph of u_k over the set $(\Omega\setminus\overline{\mathrm{B}}_\varepsilon)\setminus D_k$.

Proposition 4.7. *Let $\varepsilon\in(0,l)$ satisfy* (4.5) *and* (4.6)*, $n\in\mathbb{N}$, (λ_k) be as in Lemma 4.2, and let D_k be as in* (4.28)*. Then*

$$\liminf_{k\to+\infty} \int_{\Omega\setminus D_k} |\mathcal{M}(\nabla u_k)|\ dx \ge \int_{\Omega\setminus\overline{\mathrm{B}}_\varepsilon} |\mathcal{M}(\nabla u)|\ dx - \frac{1}{n} - \frac{2}{\varepsilon n}. \tag{4.38}$$

[7]Here α and β run over all the multi-indeces in $\{1,2\}$ with the constraint $|\alpha|+|\beta| = 2$.

Proof. Set $\Omega_\varepsilon := \Omega \setminus \overline{\mathrm{B}}_\varepsilon$. Since, by definition, T_k vanishes on smooth 2-forms supported in $(D_k \cap \Omega_\varepsilon) \times \mathbb{R}^2$, we employ (4.33) to obtain

$$\begin{aligned} \liminf_{k\to+\infty} \int_{\Omega\setminus D_k} |\mathcal{M}(\nabla u_k)|\ dx \geq & \liminf_{k\to+\infty} \int_{\Omega\setminus D_k} \sqrt{1+|\nabla u_k|^2}\ dx \geq \liminf_{k\to+\infty} |T_k|_{(\Omega_\varepsilon\setminus D_k)\times\mathbb{R}^2} \\ = \liminf_{k\to+\infty} |T_k|_{\Omega_\varepsilon\times\mathbb{R}^2} \geq |\mathcal{T}_n|_{\Omega_\varepsilon\times\mathbb{R}^2}, & \end{aligned} \tag{4.39}$$

where we use that (T_k) weakly converges to $\mathcal{T}_n$ (Lemma 4.5), and the weak lower semicontinuity of the mass. In turn, from (4.35) and (4.30),

$$\begin{aligned} |\mathcal{T}_n|_{\Omega_\varepsilon\times\mathbb{R}^2} = \int_{\Omega_\varepsilon} |\mathcal{M}(\nabla u)||1-\zeta_n|\ dx &\geq \int_{\Omega_\varepsilon} |\mathcal{M}(\nabla u)|\ dx - \int_{\Omega_\varepsilon} |\mathcal{M}(\nabla u)||\zeta_n|\ dx \\ &\geq \int_{\Omega_\varepsilon} |\mathcal{M}(\nabla u)|\ dx - \|\mathcal{M}(\nabla u)\|_{L^\infty(\Omega_\varepsilon)}\|\zeta_n\|_{L^1(\Omega_\varepsilon)} \\ &\geq \int_{\Omega_\varepsilon} |\mathcal{M}(\nabla u)|\ dx - \frac{1}{n}\|\mathcal{M}(\nabla u)\|_{L^\infty(\Omega_\varepsilon)}. \end{aligned} \tag{4.40}$$

Next, using $\sqrt{1+z^2} \leq 1+|z|$ and[8] $|\nabla u(x)| \leq \frac{2}{|x|}$ which, on Ω_ε, is bounded by $2/\varepsilon$, we also get

$$\|\mathcal{M}(\nabla u)\|_{L^\infty(\Omega_\varepsilon)} = \|\sqrt{1+|\nabla u|^2}\|_{L^\infty(\Omega_\varepsilon)} \leq 1 + \frac{2}{\varepsilon}.$$

We deduce

$$|\mathcal{T}_n|_{\Omega_\varepsilon\times\mathbb{R}^2} \geq \int_{\Omega_\varepsilon} |\mathcal{M}(\nabla u)|\ dx - \frac{1}{n} - \frac{2}{\varepsilon n}.$$

From (4.40) and (4.39) inequality (4.38) follows. □

5 The maps Ψ_k, π_{λ_k}, and the currents $\mathfrak{D}_k$, $\widehat{\mathfrak{D}}_k$, $\mathcal{E}_k$

Recalling that D_k is defined in (4.28) and (4.10), in Section 4.2 we have estimated the area of the graph of u_k over $\Omega \setminus D_k$. The next step, which is considerably more difficult, is to estimate the area over D_k, and this will be splitted in several parts (Sections 6-9). After introducing some preliminaries in Section 5.1, the first step is to reduce the graph of u_k (a surface of codimension 2 in $\mathbb{R}^4$) to a suitable rectifiable set ($\Psi_k(D_k)$ and its projections) of codimension 1 sitting in $\overline{C_l} \subset \mathbb{R}^3$, with C_l defined in (2.5). In this section we introduce all various objects needed to prove the lower bound.

Definition 5.1 (The map Ψ_k). *For all $k \in \mathbb{N}$, we define the map $\Psi_k = \Psi_{u_k} : \Omega \to \mathbb{R}^3 = \mathbb{R}_{|x|} \times \mathbb{R}^2_{\text{target}}$ as*

$$\Psi_k(x) := (|x|, u_k(x)) \qquad \forall x \in \Omega. \tag{5.1}$$

Notice that Ψ_k takes values in $\overline{C_l}$, and is Lipschitz continuous. Moreover $\Psi_k = R \circ \Phi_k$, where $\Phi_k = \mathrm{Id} \bowtie u_k : \Omega \to \mathbb{R}^4$ is defined in (4.36), and $R : \mathbb{R}^4 \ni (x,y) \mapsto (|x|, y) \in \mathbb{R}^3$. By the area formula and since $\mathrm{Lip}(R) = 1$ we have

$$|[\![G_{u_k}]\!]|_{B\times\mathbb{R}^2} = \int_B |\mathcal{M}(\nabla u_k)|\ dx = \int_B (\nabla\Phi_k^T\nabla\Phi_k)^{\frac{1}{2}}\ dx \geq \int_B (\nabla\Psi_k^T\nabla\Psi_k)^{\frac{1}{2}}\ dx,$$

for any Borel set $B \subseteq \Omega$.

[8] $D_i u_j(x) = \frac{\delta_{ij}}{|x|} - \frac{x_i x_j}{|x|^3}$, hence $\sum_{ij}(D_i u_j(x))^2 = \frac{2}{|x|^2} + 2\frac{x_1^2 x_2^2}{|x|^6} \leq \frac{4}{|x|^2}$.

5.1 The sets $\Psi_k(D_k)$ and the currents $(\Psi_k)_\sharp[\![D_k]\!]$

We start noticing that

$$\Psi_k(\Omega \setminus D_k) \subset \overline{C}_l \setminus C_l(1-\lambda_k), \qquad k \in \mathbb{N}, \tag{5.2}$$

where we recall that $C_l(1-\lambda_k)$ is defined in (2.4). Indeed, since $\Omega \setminus D_k \subseteq \{d_k \leq \lambda_k\}$ for any $k \in \mathbb{N}$ we have

$$\lambda_k \geq |u_k(x) - \frac{x}{|x|}| \geq \mathrm{dist}(u_k(x), \mathbb{S}^1) = 1 - |u_k(x)|, \qquad x \in \Omega \setminus D_k, \tag{5.3}$$

so that $|u_k(x)| \geq 1 - \lambda_k$ (and $|u_k(x)| \leq 1$ by (4.3)). In particular

$$\Psi_k(\partial D_k) \subset \overline{C}_l \setminus C_l(1-\lambda_k), \qquad k \in \mathbb{N}. \tag{5.4}$$

As a consequence, since the map Ψ_k is Lipschitz continuous, we have:

Corollary 5.2. *For all $k \in \mathbb{N}$ the integral 2-current $(\Psi_k)_\sharp[\![D_k]\!]$ is boundaryless in $C_l(1-\lambda_k)$.*

Observe that the set $\Psi_k(D_k)$ is rectifiable and contains[9] the support of $(\Psi_k)_\sharp[\![D_k]\!]$; also $\Psi_k(D_k)$ is contained in $[0,l) \times \overline{B}_1$. Specifically, the fact that C_l has axial coordinate in $(-1,l)$ and not in $(0,l)$ will be convenient in order to control the behaviour of $(\Psi_k)_\sharp[\![D_k]\!]$ on $\{0\} \times \mathbb{R}^2$.

Definition 5.3 (The projection π_{λ_k}). *We let*

$$\pi_{\lambda_k} = \pi_{\lambda_k} : \mathbb{R}^3 = \mathbb{R}_{|x|} \times \mathbb{R}^2_{\mathrm{target}} \to \overline{C}_l(1-\lambda_k) \tag{5.5}$$

be the orthogonal projection onto the compact convex set $\overline{C}_l(1-\lambda_k)$.

In Section 5.2 we project $\Psi_k(D_k)$ on $\overline{C}_l(1-\lambda_k)$ in order to get a rectifiable set (and its associated current) whose area, counted with multiplicity, is less than or equal to the area of the original set; the area of the projected set, in turn, gives a lower bound for the mass of $[\![G_{u_k}]\!]$ over D_k (see formulas (5.7) and (5.11)). Then, as a second step, we symmetrize $\pi_{\lambda_k} \circ \Psi_k(D_k)$ using the cylindrical rearrangement introduced in Section 3 to get a still smaller (in area) object. The estimate of the area of the symmetrized object is divided in two parts: the first one (Section 7) deals with $\pi_{\lambda_k} \circ \Psi_k(D_k \cap (\Omega \setminus \mathrm{B}_\varepsilon))$ whose symmetrized set can be seen as the generalized graph of a suitable polar function. In Section 8 we deal with the second part, where we estimate the area of the symmetrization obtained from $\pi_{\lambda_k} \circ \Psi_k(D_k \cap \mathrm{B}_\varepsilon)$. In Sections 9 and 10, we collect our estimates and we utilize the symmetrized object as a competitor for a suitable non-parametric Plateau problem. To do this we need to glue to the obtained rectifiable set some artificial surfaces, whose areas are controlled and are infinitesimal in the limit as $k \to +\infty$. This limit is taken only at the end of Section 10, allowing us to analyse a non-parametric Plateau problem whose boundary condition does not depend on k, so that also its solution does not depend on k. The area of such a solution will be the lower bound for the area of the rectifiable set $\pi_{\lambda_k} \circ \Psi_k(D_k \cap (\Omega \setminus \overline{\mathrm{B}}_\varepsilon)) \cup \pi_{\lambda_k} \circ \Psi_k(D_k \cap \mathrm{B}_\varepsilon)$, and then finally for the area of the graph of u_k on D_k.

5.2 Construction of the current $\widehat{\mathfrak{D}}_k$ via the currents $\mathcal{D}_k$ and $\mathcal{W}_k$

We are interested in the part of the set $\Psi_k(D_k)$ included in $\overline{C}_l(1-\lambda_k)$; we need an explicit description of the boundary of $\Psi_k(D_k)$, and to this aim we compose Ψ_k with the projection π_{λ_k} in (5.5).

Definition 5.4 (Projection of $\Psi_k(D_k)$: the current $\mathfrak{D}_k$). *We define the current $\mathfrak{D}_k \in \mathcal{D}_2(C_l)$ as*

$$\mathfrak{D}_k := (\pi_{\lambda_k} \circ \Psi_k)_\sharp[\![D_k]\!]. \tag{5.6}$$

[9] $\Psi_k(D_k)$ could properly contain the support of $(\Psi_k)_\sharp[\![D_k]\!]$, due to possible cancellations.

Remark 5.5. In general $\Psi_k(D_k) \subseteq \overline{C_l}(1-\lambda_k) \cup (\overline{C_l} \setminus C_l(1-\lambda_k))$, while spt $(\mathfrak{D}_k) \subseteq \overline{C_l}(1-\lambda_k)$.

Since $\mathrm{Lip}(\pi_{\lambda_k}) = 1$, the map π_{λ_k} does not increase the area, and therefore

$$|[\![G_{u_k}]\!]|_{D_k\times\mathbb{R}^2} \geq \int_{D_k} |J(\Psi_k)|\ dx \geq \int_{D_k} |J(\pi_{\lambda_k}\circ\Psi_k)|\ dx, \tag{5.7}$$

$$|[\![G_{u_k}]\!]|_{(D_k\cap(\Omega\setminus\overline{\mathrm{B}}_\varepsilon))\times\mathbb{R}^2} \geq \int_{D_k\cap(\Omega\setminus\overline{\mathrm{B}}_\varepsilon)} |J(\Psi_k)|\ dx \geq \int_{D_k\cap(\Omega\setminus\overline{\mathrm{B}}_\varepsilon)} |J(\pi_{\lambda_k}\circ\Psi_k)|\ dx. \tag{5.8}$$

The same holds for the mass of the current $\mathfrak{D}_k$, *i.e.*,

$$|(\Psi_k)_\sharp[\![D_k]\!]| \geq |\mathfrak{D}_k|$$

and, recalling also the definition of C_l^ε in (2.7),

$$|[\![G_{u_k}]\!]|_{(D_k\cap(\Omega\setminus\overline{\mathrm{B}}_\varepsilon))\times\mathbb{R}^2} \geq |(\Psi_k)_\sharp[\![D_k]\!]|_{\overline{C}_l^\varepsilon} \geq |\mathfrak{D}_k|_{\overline{C}_l^\varepsilon}. \tag{5.9}$$

Remark 5.6. The area, counted with multiplicity, of the 2-rectifiable set $\pi_{\lambda_k}\circ\Psi_k(D_k)$ is greater than or equal to the mass of the current $\mathfrak{D}_k$, more specifically

$$\int_{D_k} |J(\pi_{\lambda_k}\circ\Psi_k)|\ dx \geq |\mathfrak{D}_k|_{\overline{C}_l(1-\lambda_k)} \qquad \text{and} \qquad \int_{D_k\cap(\Omega\setminus\overline{\mathrm{B}}_\varepsilon)} |J(\pi_{\lambda_k}\circ\Psi_k)|\ dx \geq |\mathfrak{D}_k|_{\overline{C}_l^\varepsilon(1-\lambda_k)}. \tag{5.10}$$

This is due to the fact that $\pi_{\lambda_k}\circ\Psi_k(D_k)$ might overlap with opposite orientations so that the multiplicity of $\mathfrak{D}_k$ vanishes, and the overlappings do not contribute to its mass. In particular, spt $(\mathfrak{D}_k) \subseteq \pi_{\lambda_k}\circ\Psi_k(D_k)$.

From (5.7) and (5.10) it follows

$$|[\![G_{u_k}]\!]|_{D_k\times\mathbb{R}^2} \geq |\mathfrak{D}_k|_{\overline{C}_l(1-\lambda_k)}, \qquad |[\![G_{u_k}]\!]|_{(D_k\cap(\Omega\setminus\overline{\mathrm{B}}_\varepsilon))\times\mathbb{R}^2} \geq |\mathfrak{D}_k|_{\overline{C}_l^\varepsilon(1-\lambda_k)}. \tag{5.11}$$

We now analyse the boundary of $\mathfrak{D}_k$. Up to small modifications, we will prove that $\mathfrak{D}_k$ is boundaryless in $C_l(1-\lambda_k')$ (see (5.20) and (5.23), where λ_k' are suitable small numbers in $(0, \lambda_k)$ chosen below in Definition 5.12) and so $\mathfrak{D}_k$ can be symmetrized according to Definition 3.3. Before proceeding to the symmetrization we need some preliminaries. We build suitable currents $\mathcal{W}_k$, with their support sets denoted by W_k (see (5.17) and (5.16)), with $\partial\mathcal{W}_k$ coinciding with $\partial\mathfrak{D}_k$ (see (5.21), (5.22), and (5.23)).

Remark 5.7. By (5.4), $\pi_{\lambda_k}\circ\Psi_k(\partial D_k)$ is contained in $\partial_{\mathrm{lat}}C_l(1-\lambda_k)$. By Lemma 4.2(i), $\pi_{\lambda_k}\circ\Psi_k(\partial D_k)$ is the union of the image of at most countably many curves, and this union, counted with multiplicities, has finite $\mathcal{H}^1$ measure: specifically, if we define

$$M(\pi_{\lambda_k}\circ\Psi_k(\partial D_k)) := \int_{\partial D_k} \left|\partial_{\mathrm{tg}}\ (\pi_{\lambda_k}\circ\Psi_k)\right|\ d\mathcal{H}^1,$$

where ∂_{tg} stands for the tangential derivative along ∂D_k, then $M(\pi_{\lambda_k}\circ\Psi_k(\partial D_k)) < +\infty$ since $\mathcal{H}^1(\partial D_k) < +\infty$ (still by Lemma 4.2(i)) and u_k is Lipschitz continuous.

Moreover

$$\partial\mathfrak{D}_k = (\pi_{\lambda_k}\circ\Psi_k)_\sharp\partial[\![D_k]\!] \in \mathcal{D}_1(C_l) \qquad \text{in } C_l. \tag{5.12}$$

It is convenient to introduce a suitable map τ parametrizing the region $\overline{C}_l\setminus C_l(1-\lambda_k)$ in between the two concentric cylinders; this map can then be pulled back by $\pi_{\lambda_k}\circ\Psi_k$, but only in $\Omega\setminus D_k$, to get the map $\widetilde{\tau}$.

Definition 5.8 (The maps τ, $\widetilde{\tau}$). *We set*

$$\begin{aligned}&\tau = \tau_{\lambda_k} : [1-\lambda_k, 1] \times \partial C_l(1-\lambda_k) \to \overline{C}_l \setminus C_l(1-\lambda_k) \subset \mathbb{R}^3,\\ &\tau(\rho, t, y) := \Big(t, \frac{y}{|y|}\rho\Big), \qquad \rho \in [1-\lambda_k, 1], \quad (t,y) \in \partial C_l(1-\lambda_k) = [-1, l] \times \partial B_{1-\lambda_k}.\end{aligned} \tag{5.13}$$

By (5.2) *it follows* $\pi_{\lambda_k} \circ \Psi_k(\Omega \setminus D_k) \subset \partial C_l(1-\lambda_k)$*, hence we can also set*

$$\widetilde{\tau}(\rho, x) = \widetilde{\tau}_{u_k,\lambda_k}(\rho, x) := \tau(\rho, \pi_{\lambda_k} \circ \Psi_k(x)), \qquad \rho \in [1-\lambda_k, 1],\ x \in \Omega \setminus D_k. \tag{5.14}$$

Notice that $\tau(\rho, \cdot, \cdot)$ takes values in $\partial C_l(\rho)$ for any $\rho \in [1-\lambda_k, 1]$, that $\tau(\cdot, t, y)$ moves along the normal to the lateral boundary of $\partial C_l(1-\lambda_k)$ at the point (t, y), and $\tau(1-\lambda_k, \cdot, \cdot)$ is the identity. We also observe that, due to the fact that $\pi_{\lambda_k} \circ \Psi_k$ takes values in $[0, l) \times \overline{B}_1$, the same holds for $\widetilde{\tau}$.

Remark 5.9. If $\lambda_k > 0$ is small enough (which is true for k large enough), the Jacobian of τ is close to 1 so that the $\mathcal{H}^1$-measure, counted with multiplicities, of the set $\tau(\rho, \pi_{\lambda_k} \circ \Psi_k(\partial D_k))$ is, for fixed ρ, bounded by two times the $\mathcal{H}^1$-measure of $\pi_{\lambda_k} \circ \Psi_k(\partial D_k)$, still counted with multiplicities. More precisely,

$$2\int_{\partial D_k} \Big|\partial_{\rm tg}\ (\pi_{\lambda_k} \circ \Psi_k)\Big|\ d\mathcal{H}^1 \geq \int_{\partial D_k} \Big|\partial_{\rm tg}\ \tau(\rho, \pi_{\lambda_k} \circ \Psi_k)\Big|\ d\mathcal{H}^1, \qquad \rho \in [1-\lambda_k, 1], \tag{5.15}$$

$$2\int_{(\Omega \setminus \overline{\mathrm{B}}_\varepsilon) \cap \partial D_k} \Big|\partial_{\rm tg}\ (\pi_{\lambda_k} \circ \Psi_k)\Big|\ \ d\mathcal{H}^1 \geq \int_{(\Omega \setminus \overline{\mathrm{B}}_\varepsilon) \cap \partial D_k} \Big|\partial_{\rm tg}\ \tau(\rho, \pi_{\lambda_k} \circ \Psi_k)\Big|\ d\mathcal{H}^1,$$

for all $\rho \in [1-\lambda_k, 1]$ and $k \in \mathbb{N}$ large enough, where we recall that, from Lemma 4.2(i), ∂D_k is rectifiable.

Now we take a sequence[10] of numbers $\lambda'_k \in (0, \lambda_k)$, which will be fixed in the sequel (see Definition 5.13).

Definition 5.10 (The set W_k and the current $\mathcal{W}_k$). *We define the* 2*-rectifiable set*[11]

$$W_k := \tau\big([1-\lambda_k, 1-\lambda'_k] \times \pi_{\lambda_k} \circ \Psi_k(\partial D_k)\big) = \widetilde{\tau}\big([1-\lambda_k, 1-\lambda'_k] \times \partial D_k\big), \tag{5.16}$$

and the 2*-current*

$$\mathcal{W}_k := \widetilde{\tau}_\sharp [\![[1-\lambda_k, 1-\lambda'_k] \times \partial D_k]\!] \in \mathcal{D}_2(C_l). \tag{5.17}$$

Clearly spt $(\mathcal{W}_k) \subseteq W_k$; Again, although $\mathcal{W}_k$ is defined as a current in C_l, it is supported in $[0, l] \times B_1$.

Remark 5.11 (Use of $[\![\cdot]\!]$ for not top-dimensional currents). ∂D_k is endowed with a natural orientation, inherited from the fact that it is the boundary of the set D_k; consistently, we sometimes use the identification $[\![\partial D_k]\!] = \partial [\![D_k]\!]$. With a little abuse of notation we have noted the current integration over $[1-\lambda_k, 1-\lambda'_k] \times \partial D_k$, meaning that ∂D_k is endowed with this natural orientation. Finally, recalling that $\tilde{\tau}(\rho, \cdot)$ takes values in $\partial C_l(\rho)$, we can do the following identification:

$$\widetilde{\tau}_\sharp [\![[1-\lambda_k, 1-\lambda'_k] \times \partial D_k]\!] = \widetilde{\tau}_\sharp \partial [\![[1-\lambda_k, 1-\lambda'_k] \times D_k]\!] \llcorner \Big(C_l(1-\lambda'_k) \setminus \overline{C}_l(1-\lambda_k)\Big).$$

[10]The sequence (λ'_k) depends on ε and n.

[11]The set W_k consists of "vertical" walls, normal to $\partial C_l(1-\lambda_k)$, build on $\pi_{\lambda_k} \circ \Psi_k(\partial D_k)$, with height $\lambda'_k - \lambda_k$: see Fig. 9.

We denote

$$M(W_k) := \int_{[1-\lambda_k, 1-\lambda'_k]\times\partial D_k} |J(\widetilde{\tau}(\rho, x))| \, d\rho \, d\mathcal{H}^1(x) \tag{5.18}$$

the area of W_k counted with multiplicities. By the area formula and using (5.15) we infer

$$|\mathcal{W}_k| \le M(W_k) \le 2(\lambda_k - \lambda'_k) \int_{\partial D_k} \left|\partial_{\rm tg}\ (\pi_{\lambda_k} \circ \Psi_k)\right| \, d\mathcal{H}^1 = 2(\lambda_k - \lambda'_k) M(\pi_{\lambda_k} \circ \Psi_k(\partial D_k)). \tag{5.19}$$

Then we are led to the following

Definition 5.12 (**The sequence** (λ'_k)). *We select* $\lambda'_k \in (0, \lambda_k)$ *so that*

$$2(\lambda_k - \lambda'_k) M(\pi_{\lambda_k} \circ \Psi_k(\partial D_k)) \le \frac{1}{n} \qquad \forall k \in \mathbb{N}. \tag{5.20}$$

Finally we observe that

$$\partial\mathcal{W}_k = \tau(1 - \lambda'_k, \cdot, \cdot)_\sharp \Big((\pi_{\lambda_k} \circ \Psi_k)_\sharp \partial[\![D_k]\!]\Big) - (\pi_{\lambda_k} \circ \Psi_k)_\sharp \partial[\![D_k]\!]. \tag{5.21}$$

Definition 5.13 (**The current** $\widehat{\mathfrak{D}}_k$). *We define*

$$\widehat{\mathfrak{D}}_k := \mathfrak{D}_k + \mathcal{W}_k \in \mathcal{D}_2(C_l). \tag{5.22}$$

The next result will be useful to select a primitive of $\widehat{\mathfrak{D}}_k$.

Corollary 5.14. *The current* $\widehat{\mathfrak{D}}_k$ *is supported in* $[0, l] \times \overline{B}_{1-\lambda'_k}$ *and*

$$\widehat{\mathfrak{D}}_k \quad \text{is boundaryless in the open cylinder } C_l(1 - \lambda'_k). \tag{5.23}$$

In particular $\partial\widehat{\mathfrak{D}}_k = 0$ *in* $\mathcal{D}_1((-\infty, l) \times B_{1-\lambda'_k})$.

Proof. The statement follows by construction, and noticing that, since $\tau(1-\lambda'_k, \cdot, \cdot)_\sharp \Big((\pi_{\lambda_k} \circ \Psi_k)_\sharp \partial[\![D_k]\!]\Big)$ has support in $\partial_{\rm lat} C_l(1 - \lambda'_k)$, one can use (5.12) to deduce (5.23). □

5.3 The 3-current $\mathcal{E}_k$ and the symmetrization of $\widehat{\mathfrak{D}}_k$

Since we want to symmetrize $\widehat{\mathfrak{D}}_k$ according to Definition 3.3, we need to identify a unique primitive 3-current $\mathcal{E}_k$ such that $\partial\mathcal{E}_k = \widehat{\mathfrak{D}}_k$.

The restriction of the map $\pi_{\lambda_k} \circ \Psi_k$ to $\Omega \setminus D_k$ takes $\Omega \setminus D_k$ into $\partial C_l(1 - \lambda_k)$ (see (5.2)), and can also be written as

$$\pi_{\lambda_k} \circ \Psi_k(x) = \Big(|x|, \frac{u_k(x)}{|u_k(x)|}(1 - \lambda_k)\Big), \qquad x \in \Omega \setminus D_k. \tag{5.24}$$

The current $(\pi_{\lambda_k} \circ \Psi_k)_\sharp [\![\Omega \setminus D_k]\!]$ has boundary

$$\partial(\pi_{\lambda_k} \circ \Psi_k)_\sharp [\![\Omega \setminus D_k]\!] = -(\pi_{\lambda_k} \circ \Psi_k)_\sharp \partial[\![D_k]\!]. \tag{5.25}$$

Definition 5.15 (**The currents** $\mathcal{Y}_k$ **and** $\mathcal{X}_k$). *Recalling the definition of* τ *(see* (5.14), (5.13)*) we set*

$$\mathcal{Y}_k := \widetilde{\tau}_\sharp [\![[1 - \lambda_k, 1 - \lambda'_k] \times (\Omega \setminus D_k)]\!] \in \mathcal{D}_3(C_l), \tag{5.26}$$

$$\mathcal{X}_k := [\![C_l(1 - \lambda'_k) \setminus \overline{C_l}(1 - \lambda_k)]\!] - \mathcal{Y}_k \in \mathcal{D}_3(C_l). \tag{5.27}$$

Notice that $\mathcal{X}_k$ cannot be directly defined as a push-forward via the map $\widetilde{\tau}$, for part of $\Psi_k(D_k)$ could be contained in $C_l(1-\lambda_k)$, and for this reason we are led to define it as a difference.

The current $\mathcal{Y}_k$ could have multiplicity different from 0 and 1, and in particular could not be the integration over a finite perimeter set. This depends on the fact that the map Ψ_k could generate overlappings and self-intersections of the set $\pi_{\lambda_k} \circ \Psi_k(\Omega \setminus D_k)$. If the multiplicity of $\mathcal{Y}_k$ is only 1 or 0 then the same holds for $\mathcal{X}_k$. Also, $\mathcal{Y}_k$ might be null, and in this case $\mathcal{X}_k$ coincides with the integration over the region $C_l(1-\lambda'_k) \setminus \overline{C_l}(1-\lambda_k)$. A finer description of these two currents will be necessary later, and this will be done by a slicing argument in Lemma 6.4 below.

Recalling (5.17),

$$\partial \mathcal{Y}_k = -\mathcal{W}_k = -\partial \mathcal{X}_k \qquad \text{in } \ C_l(1-\lambda'_k) \setminus \overline{C_l}(1-\lambda_k),$$

as it can be seen by considering the push-forward by τ of (5.25). We proceed to the symmetrization in $C_l(1-\lambda'_k)$ of the current $\widehat{\mathfrak{D}}_k$ in (5.22). By (5.23) it follows the existence of an integer multiplicity 3-current $\mathcal{E}_k \in \mathcal{D}_3(C_l(1-\lambda'_k))$ such that

$$\partial \mathcal{E}_k = \widehat{\mathfrak{D}}_k \qquad \text{in } \ C_l(1-\lambda'_k). \tag{5.28}$$

The current $\mathcal{E}_k$ is unique up to a constant, that we might assume to be integer, since $\mathcal{E}_k$ has integer multiplicity. Hence we choose such a constant[12] so that

$$\mathcal{E}_k \llcorner \Big(C_l(1-\lambda'_k) \setminus \overline{C_l}(1-\lambda_k) \Big) = \mathcal{X}_k. \tag{5.29}$$

Let E_k denote the support of $\mathcal{E}_k$; by decomposition,

$$\mathcal{E}_k = \sum_i (-1)^{\sigma_i} [\![E_{k,i}]\!] \qquad \text{in } \ C_l(1-\lambda'_k), \tag{5.30}$$

with $E_{k,i} \subset C_l(1-\lambda'_k)$ finite perimeter sets, the decomposition done with undecomposable components, see (3.3), (3.4). We denote

$$\mathbb{S}(E_k) := \cup_i \mathbb{S}(E_{k,i})$$

the union of the cylindrical symmetrizations of the sets $E_{k,i}$, see (3.2). Recalling (5.28), Definition 3.3 and (3.5), the symmetrization of the current $\widehat{\mathfrak{D}}_k$ is

$$\mathbb{S}(\widehat{\mathfrak{D}}_k) = \partial \mathbb{S}(\mathcal{E}_k) = \partial [\![\mathbb{S}(E_k)]\!] \llcorner C_l(1-\lambda'_k). \tag{5.31}$$

Formula (5.31) contains the needed information about the symmetrization of $\Psi_k(D_k)$, since by construction $\mathfrak{D}_k = \widehat{\mathfrak{D}}_k \llcorner \overline{C_l}(1-\lambda_k)$ (recall (5.6)).

We have

$$\widehat{\mathfrak{D}}_k = \sum_i (-1)^{\sigma_i} [\![\partial^* E_{k,i}]\!] \qquad \text{in } C_l(1-\lambda'_k), \tag{5.32}$$

and since the decomposition in (5.30) is done by undecomposable components, by (3.4) it follows, in $C_l(1-\lambda'_k)$,

$$|\widehat{\mathfrak{D}}_k| = \sum_i \mathcal{H}^2(\partial^* E_{k,i}) \qquad \text{and} \qquad \partial^* E_{k,i} \subseteq \text{spt } (\widehat{\mathfrak{D}}_k).$$

[12]The fact that this choice is possible is a consequence of the constancy theorem (see for instance [33, Proposition 7.3.1]). Indeed, let $\widehat{\mathcal{E}}_k$ have the same boundary (*i.e.*, $\mathcal{W}_k$) of $\mathcal{X}_k$ in $C_l(1-\lambda'_k) \setminus \overline{C_l}(1-\lambda_k)$. Thus $\widehat{\mathcal{E}}_k - \mathcal{X}_k$ is boundaryless, and must be an integer multiple of the integration over $C_l(1-\lambda'_k) \setminus \overline{C_l}(1-\lambda_k)$, *i.e.*, $\widehat{\mathcal{E}}_k - \mathcal{X}_k = h[\![C_l(1-\lambda'_k) \setminus \overline{C_l}(1-\lambda_k)]\!]$. We then set $\mathcal{E}_k := \widehat{\mathcal{E}}_k - h[\![C_l(1-\lambda'_k)]\!]$ so that $\mathcal{E}_k = \mathcal{X}_k$ in $C_l(1-\lambda'_k) \setminus \overline{C_l}(1-\lambda_k)$.

Remark 5.16 (Nonuniqueness of the decomposition). Once the decomposition (5.30) is fixed, the symmetrization is uniquely determined. However, the decomposition might not be unique, and the resulting symmetrized current in general depends on the choice of the decomposition. This will not be an issue, since our procedure leads to a minimization problem independent of this step.

Since

$$\mathcal{H}^2(\partial^*\mathbb{S}(E_{k,i})) \le \mathcal{H}^2(\partial^* E_{k,i}) \qquad \text{for all } i \in \mathbb{N},$$

and $\mathbb{S}(E_k) = \cup_i \mathbb{S}(E_{k,i})$, we also have

$$\mathcal{H}^2(\partial^*\mathbb{S}(E_k)) \le \sum_i \mathcal{H}^2(\partial^* E_{k,i}) = |\widehat{\mathfrak{D}}_k|. \tag{5.33}$$

The same inequalities hold if we restrict the mass to the set C_l^ε, namely

$$|\mathbb{S}(\widehat{\mathfrak{D}}_k)|_{C_l^\varepsilon} \le |\widehat{\mathfrak{D}}_k|_{C_l^\varepsilon}. \tag{5.34}$$

Now we want to understand whether $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ has some boundary on $\{0\}\times\mathbb{R}^2$. We have already observed (Corollary 5.14) that $\widehat{\mathfrak{D}}_k$ has no boundary in $C_l(1-\lambda_k')$. The same holds for the symmetrized current:

Corollary 5.17 (Closedness of $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ in $(-\infty, l) \times B_{1-\lambda_k'}$). *The current $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ is supported in $[0, l] \times \overline{B}_{1-\lambda_k'}$ and $\partial\mathbb{S}(\widehat{\mathfrak{D}}_k) = 0$ in $\mathcal{D}_1((-\infty, l) \times B_{1-\lambda_k'})$.*

Proof. By definition, $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ is the boundary of the current carried by the integration over the finite perimeter set $\mathbb{S}(E_k)$ in $C_l(1-\lambda_k')$. Hence $\partial\mathbb{S}(\widehat{\mathfrak{D}}_k) = 0$ in $\mathcal{D}_1(C_l(1-\lambda_k'))$. The conclusion then follows from the fact that $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ is supported in $[0, l) \times B_{1-\lambda_k'} \subset C_l(1-\lambda_k')$. □

6 Towards an estimate of $|\mathbb{S}(\widehat{\mathfrak{D}}_k)|$: two useful lemmas

Now that the symmetrization $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ of the current $\widehat{\mathfrak{D}}_k$ in $C_l(1-\lambda_k')$ is obtained (see (5.31)), we need to estimate its mass. This will be done separately in $\overline{C}_l^\varepsilon(1-\lambda_k) = [\varepsilon, l] \times \overline{B}_{1-\lambda_k}$ and in $\overline{C}_\varepsilon(1-\lambda_k) = [-1, \varepsilon] \times \overline{B}_{1-\lambda_k}$. In formula (7.4) of Section 7 we express the restriction of $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ to C_l^ε as generalized graph of suitable functions $\vartheta_{k,\varepsilon}$ and $-\vartheta_{k,\varepsilon}$ and estimate the area of these graphs (see Proposition 7.9, below). In addition, we need a fine description of the trace of the symmetrized set boundary $\partial\mathbb{S}(E_k)$ on the lateral part of $\partial C_l(1-\lambda_k')$; this will be done in Section 8.

We start by collecting in Lemma 6.3 and Lemma 6.4 two important preliminary estimates; we need to introduce the functions $|u_k|^-$, $|u_k|^+$.

For any $r \in (\varepsilon, l)$ we consider the closed curve[13] $\alpha \in (0, 2\pi] \mapsto \Psi_k(r, \alpha) \in \{r\} \times \overline{B}_1$; the image of $\Psi_k(r, \cdot)$ is the slice of $\Psi_k(\Omega \setminus \overline{\mathrm{B}}_\varepsilon)$ with the plane $\{t = r\}$.

Definition 6.1 (The functions $|u_k|^\pm$). *For all $r \in (\varepsilon, l)$ we define*

$$|u_k|^-(r) := \min_{\alpha\in(0,2\pi]} |u_k(r, \alpha)|, \qquad |u_k|^+(r) := \max_{\alpha\in(0,2\pi]} |u_k(r, \alpha)|, \tag{6.1}$$

Thus the map $\Psi_k(r, \cdot)$ defined in (5.1) takes values in

$$\{r\} \times (\overline{B}_{|u_k|^+(r)} \setminus B_{|u_k|^-(r)}).$$

[13]We use here polar coordinates (r, α).

Let us remark that $|u_k|^-(r)$ might be equal to 0, that $|u_k|^+(r) \leq 1$, and that it might happen that $|u_k|^+(r) = |u_k|^-(r)$, see Fig. 2. Moreover, from (5.4),

$$|u_k| \geq 1 - \lambda_k \qquad \text{in } \Omega \setminus D_k, \tag{6.2}$$

so that

$$|u_k|^+(r) \geq 1 - \lambda_k \qquad \text{if } r \text{ is such that } (\Omega \setminus D_k) \cap \partial \mathrm{B}_r \neq \emptyset,$$

whereas it might happen that

$$|u_k|^+(r) < 1 - \lambda_k \qquad \text{if } r \text{ is such that } (\Omega \setminus D_k) \cap \partial \mathrm{B}_r = \emptyset. \tag{6.3}$$

In such a case, since $D_k \subseteq A_n$ (Lemma 4.2 (ii)), this can happen only if $\partial \mathrm{B}_r \subseteq A_n$.

Definition 6.2 **(The set** $Q_{k,\varepsilon}$**).** *We define*

$$Q_{k,\varepsilon} := \{r \in (\varepsilon, l) : |u_k|^+(r) < 1 - \lambda_k\}. \tag{6.4}$$

Then

$$Q_{k,\varepsilon} \subseteq \{r \in (\varepsilon, l) : \partial \mathrm{B}_r \subseteq A_n\}. \tag{6.5}$$

The next lemma, that will be used in Section 9, shows that the measure of $Q_{k,\varepsilon}$ is small (see Fig. 2).

Lemma 6.3 **(Estimate of** $Q_{k,\varepsilon}$**).** *We have*

$$\mathcal{H}^1(Q_{k,\varepsilon}) < \frac{1}{2\pi\varepsilon n}.$$

Proof. If $t \in Q_{k,\varepsilon}$ then $\partial \mathrm{B}_t \subseteq A_n$. Then

$$\mathcal{H}^1(Q_{k,\varepsilon}) = \int_{Q_{k,\varepsilon}} 1 dt \leq \frac{1}{2\pi\varepsilon} \int_{Q_{k,\varepsilon}} 2\pi t \, dt = \frac{1}{2\pi\varepsilon} \int_{Q_{k,\varepsilon}} \mathcal{H}^1(\partial \mathrm{B}_t) dt \leq \frac{1}{2\pi\varepsilon} |A_n|,$$

where the last inequality is a consequence of the coarea formula and (6.5). The thesis then follows recalling that $|A_n| < \frac{1}{n}$, see (4.7). □

By slicing and from (5.29), (5.30), we have for almost every $t \in (0, l)$ and almost every $\rho \in (1 - \lambda_k, 1 - \lambda'_k)$,

$$(\mathcal{X}_k)_{t,\rho} = \sum_i (-1)^{\sigma_i} [\![E_{k,i} \cap (\{t\} \times \partial B_\rho)]\!],$$

and

$$\mathcal{H}^1(\mathbb{S}(E_k) \cap (\{t\} \times \partial B_\rho)) \leq \sum_i \mathcal{H}^1(E_{k,i} \cap (\{t\} \times \partial B_\rho)) = |(\mathcal{X}_k)_{t,\rho}|, \tag{6.6}$$

since the decomposition is done in undecomposable components (see (3.12)).

Recalling the definition of Θ in (3.1) we have, for fixed $t \in (0, l)$ and for any $\rho \in (0, 1 - \lambda'_k]$,

$$\Theta_k(t, \rho) := \Theta_{\mathbb{S}(E_k)}(t, \rho) = \frac{1}{\rho} \mathcal{H}^1(\mathbb{S}(E_k) \cap (\{t\} \times \partial B_\rho)) \tag{6.7}$$

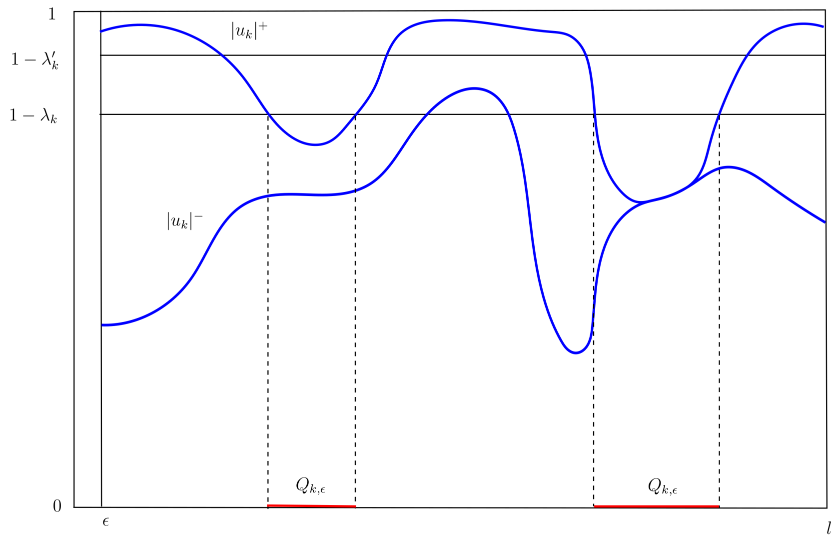

Figure 2: The graphs of the functions $|u_k|^+$ and $|u_k|^-$ defined in (6.1), and the set $Q_{k,\varepsilon}$ in (6.4).

denotes the measure (in radiants) of the slice $\mathbb{S}(E_k) \cap (\{t\} \times \partial B_\rho)$. By construction,

$$\Theta_k(t,\rho) = \Theta_k(t,\varrho) \qquad \text{for any } \rho, \varrho \in (1-\lambda_k, 1-\lambda_k'),$$

since the slices of $\mathcal{X}_k$, and hence of the sets $E_{k,i}$, are radially symmetric[14] in $C_l(1-\lambda_k') \setminus \overline{C_l}(1-\lambda_k)$.

We now look for an estimate of $\Theta_k(t,\rho)$, for $t \in (\varepsilon, l)$ and $\rho \in (1-\lambda_k, 1-\lambda_k')$: the next lemma will be used in Section 9.

Lemma 6.4 (**L^1-estimate of the angular slices**). *We have*

$$\int_\varepsilon^l \Theta_k(t,\rho)\, dt \le \frac{1}{\varepsilon n} + o_k(1) \qquad \forall \rho \in (1-\lambda_k, 1-\lambda_k'), \tag{6.8}$$

where $o_k(1)$ is a nonnegative function, depending on ε and n, and infinitesimal as $k \to +\infty$.

Proof. It is convenient to set

$$H_{k,t} := D_k \cap \partial \mathrm{B}_t, \qquad H_{k,t}^c := (\Omega \setminus D_k) \cap \partial \mathrm{B}_t \qquad \forall t \in (\varepsilon, l). \tag{6.9}$$

Observe that the relative boundary of $H_{k,t}$, *i.e.*, the boundary of $H_{k,t}$ when considered as a subset of $\partial \mathrm{B}_t$, is contained in $\partial D_k \cap \partial \mathrm{B}_t$.

We fix $t \in (\varepsilon, l)$ such that the relative boundary of $H_{k,t}$ is a finite set of points (this happens for $\mathcal{H}^1$-a.e. t, since $\mathcal{H}^1(\partial D_k) < +\infty$ from Lemma 4.2(i)) and fix any $\rho \in (1-\lambda_k, 1-\lambda_k')$. By inequality (6.6) we have

$$\Theta_k(t,\rho) \le \frac{1}{\rho} |(\mathcal{X}_k)_{t,\rho}|, \tag{6.10}$$

so it is sufficient to estimate the mass of a (slice of a) slice of the 3-current $\mathcal{X}_k$ defined in (5.27). We recall that by (5.27) we have[15]

$$(\mathcal{X}_k)_{t,\rho} = [\![\{t\} \times \partial B_\rho]\!] - (\mathcal{Y}_k)_{t,\rho}, \tag{6.11}$$

[14] Each radial section is (suitably rescaled) the same since, by definition, function τ in (5.13) is radial.
[15] The orientation of ∂B_ρ is taken counterclockwise.

where $[\![\{t\}\times\partial B_\rho]\!]$ has a natural orientation[16] inherited by the fact that it is the boundary of $[\![\{t\}\times B_\rho]\!]$ in $\{t\}\times\mathbb{R}^2$, which in turn is a slice of $[\![C_l(\rho)]\!]$. By (5.26)

$$(\mathcal{Y}_k)_{t,\rho} = \widetilde{\tau}_\sharp[\![\{\rho\}\times((\Omega\setminus D_k)\cap\partial \mathrm{B}_t)]\!] = \tau(\rho,\pi_{\lambda_k}\circ\Psi_k(\cdot))_\sharp[\![H^c_{k,t}]\!], \tag{6.12}$$

see (5.13), (5.14), (5.24), and Remark 5.11 for the orientation of $[\![\{\rho\}\times((\Omega\setminus D_k)\cap\partial\mathrm{B}_t)]\!]$. As for $[\![H^c_{k,t}]\!]$ we endow the set $H^c_{k,t}\subset\partial\mathrm{B}_t$ with the orientation inherited by $\partial\mathrm{B}_t$, *i.e.*, by a counterclockwise tangent unit vector. Now, since the restriction of $\tau(\rho,\pi_{\lambda_k}\circ\Psi_k(\cdot))$ to $\partial\mathrm{B}_t$ takes values in $\{t\}\times\partial B_\rho$, the current $(\mathcal{Y}_k)_{t,\rho}$ is the integration over arcs[17] in $\{t\}\times\partial B_\rho$. To identify these arcs we distinguish the following three cases (A), (B), (C):

(A) $H^c_{k,t}=\emptyset$. From (6.12) it follows $(\mathcal{Y}_k)_{t,\rho}=0$ and $(\mathcal{X}_k)_{t,\rho}=[\![\{t\}\times\partial B_\rho]\!]$ from (6.11). Thus

$$\Theta_k(t,\rho)=2\pi\le 2\pi\frac{t}{\varepsilon}=\frac{1}{\varepsilon}\mathcal{H}^1(H_{k,t}). \tag{6.13}$$

(B) $H^c_{k,t}=\partial\mathrm{B}_t\subset\Omega\setminus D_k$, hence $(\mathcal{Y}_k)_{t,\rho}=\tau(\rho,\pi_{\lambda_k}\circ\Psi_k(\cdot))_\sharp[\![\partial\mathrm{B}_t]\!]$ from (6.12). Then

$$(\mathcal{Y}_k)_{t,\rho}=[\![\{t\}\times\partial B_\rho]\!]. \tag{6.14}$$

Indeed, fix three points $x_1,x_2,x_3\in\partial\mathrm{B}_t$ in counterclockwise order such that $|\frac{x_i}{|x_i|}-\frac{x_j}{|x_j|}|>4\lambda_k$ for $i\neq j$. Since $d_k(x)=|\frac{x}{|x|}-u_k(x)|<\lambda_k$ for $x\in\Omega\setminus D_k$, $x\neq 0$, the points $z_i:=\pi_{\lambda_k}\circ\Psi_k(x_i)$ are still in counterclockwise order in $\{t\}\times\partial B_{1-\lambda_k}$ (the image of the arc $\overline{x_1x_2}$ covers the arc $\overline{z_1z_2}$ that does not contain z_3). Therefore $(\pi_{\lambda_k}\circ\Psi_k(\cdot))_\sharp[\![\overline{x_ix_{i+1}}]\!]=[\![\overline{z_iz_{i+1}}]\!]$ for $i=1,2,3$[18] (with the convention $x_4=x_1$, $z_4=z_1$), and hence

$$(\pi_{\lambda_k}\circ\Psi_k(\cdot))_\sharp[\![\partial\mathrm{B}_t]\!]=\sum_{i=1}^3(\pi_{\lambda_k}\circ\Psi_k(\cdot))_\sharp[\![\overline{x_ix_{i+1}}]\!]=\sum_{i=1}^3[\![\overline{z_iz_{i+1}}]\!]=[\![\{t\}\times\partial B_{1-\lambda_k}]\!].$$

Taking the push-forward by τ we get (6.14). From this and (6.11) we deduce $(\mathcal{X}_k)_{t,\rho}=0$, and $\Theta_k(t,\rho)=0$.

Before passing to case (C), we anticipate an observation which will be useful to deal with it. Let $\overline{x_1x_2}\subset(\Omega\setminus D_k)\cap\partial\mathrm{B}_t$ be an arc oriented counterclockwise. We want to identify the current $(\pi_{\lambda_k}\circ\Psi_k)_\sharp[\![\overline{x_1x_2}]\!]$; to do that we consider three different cases for $\overline{x_1x_2}$. Case 1: $|\frac{x_1}{|x_1|}-\frac{x_2}{|x_2|}|>2\lambda_k$. Hence $z_1:=\pi_{\lambda_k}\circ\Psi_k(x_1)$ and $z_2:=\pi_{\lambda_k}\circ\Psi_k(x_2)$ must have the same order on $\partial B_{1-\lambda_k}$ of x_1 and x_2, moreover $(\pi_{\lambda_k}\circ\Psi_k)_\sharp[\![\overline{x_1x_2}]\!]=[\![\overline{z_1z_2}]\!]$, where $\overline{z_1z_2}$ is the arc connecting z_1,z_2, starting from z_1 and oriented counterclockwise. Case 2: $|\frac{x_1}{|x_1|}-\frac{x_2}{|x_2|}|\le 2\lambda_k$ (that implies $|z_1-z_2|\le 4\lambda_k$, and z_1,z_2 could have reversed order of x_1 and x_2). Let z_1,z_2 have the same order of x_1,x_2, then $(\pi_{\lambda_k}\circ\Psi_k)_\sharp[\![\overline{x_1x_2}]\!]=[\![\overline{z_1z_2}]\!]$ where $\overline{z_1z_2}$ is the arc connecting z_1,z_2, starting from z_1 and oriented counterclockwise. Now let z_1,z_2 have the reversed order of x_1,x_2. If $\overline{x_1x_2}$ is the short path arc connecting x_1,x_2, then $(\pi_{\lambda_k}\circ\Psi_k)_\sharp[\![\overline{x_1x_2}]\!]=[\![\overline{z_1z_2}]\!]$, where $\overline{z_1z_2}$ is the (short path) arc connecting z_1, z_2, starting from z_1, and oriented clockwise. If $\overline{x_1x_2}$ is instead the long path arc joining x_1,x_2, then $(\pi_{\lambda_k}\circ\Psi_k)_\sharp[\![\overline{x_1x_2}]\!]=[\![\{t\}\times\partial B_{1-\lambda_k}]\!]+[\![\overline{z_1z_2}]\!]$, where $[\![\{t\}\times\partial B_{1-\lambda_k}]\!]$ is oriented counterclockwise, and $\overline{z_1z_2}$ is the (short path) arc starting from z_1 and oriented counterclockwise. Notice also that in case 2 we always have $\mathcal{H}^1(\overline{z_1z_2})<8\lambda_k$.

Now, we analyse the third case.

[16] The orientation of the 3-current $[\![C_l(\rho)]\!]$ induces an orientation of its slice $[\![\{t\}\times B_\rho]\!]$. This orientation induces an orientation of $[\![\{t\}\times\partial B_\rho]\!]$, which coincides with the orientation of $[\![\{t\}\times\mathbb{R}^2]\!]_\rho$ induced by the slicing by ρ.

[17] Such arcs could overlap, since in general the multiplicity of $\mathcal{Y}_k$ might be different from 1.

[18] The boundary of $(\pi_{\lambda_k}\circ\Psi_k(\cdot))_\sharp[\![\overline{x_ix_{i+1}}]\!]$ is $\delta_{z_{i+1}}-\delta_{z_i}$, hence $(\pi_{\lambda_k}\circ\Psi_k(\cdot))_\sharp[\![\overline{x_ix_{i+1}}]\!]$ is an arc connecting z_i and z_{i+1}. Since this arc cannot contain the third point, it must be $[\![\overline{z_iz_{i+1}}]\!]$, counterclockwise oriented.

(C) $H^c_{k,t}$ is union of finitely many arcs. Let us denote by $\left\{\overline{x^i_1 x^i_2}\right\}_i$ these distinct arcs with endpoints[19] $x^i_j = (t, \alpha^i_j) \in \partial \mathrm{B}_t$, with the index $i \in \{1, \dots, h = h_{k,t}\}$ varying in a finite set, so that

$$[\![H^c_{k,t}]\!] = \sum_{i=1}^{h} [\![\overline{x^i_1 x^i_2}]\!] \qquad \text{and} \qquad [\![H_{k,t}]\!] = \sum_{i=1}^{h} [\![\overline{x^i_2 x^{i+1}_1}]\!],$$

where again the orientation of $\overline{x^i_1 x^i_2}$ is the one inherited by the counterclockwise orientation of $\partial \mathrm{B}_t$ and, by convention, $h+1 = 1$. Being $H^c_{k,t}$ relatively closed set in $\partial \mathrm{B}_t$, it might happen that $x^i_1 = x^i_2$ for some i. Notice that x^i_j belongs to the relative boundary of $H_{k,t}$ which, in turn, is a subset of $\partial D_k \cap \partial \mathrm{B}_t$.

We denote

$$z^i_j := \pi_{\lambda_k} \circ \Psi_k(x^i_j) \in \{t\} \times \partial B_{1-\lambda_k}.$$

After applying $\pi_{\lambda_k} \circ \Psi_k(\cdot)$, the points x^i_j might also reverse their order, *i.e.*, the orientation of the arc $\pi_{\lambda_k} \circ \Psi_k \left(\overline{x^i_1 x^i_2}\right)$ could be the opposite of the orientation of $\overline{x^i_1 x^i_2}$.

In order to describe the current $(\mathcal{X}_k)_{t,\rho}$ we need first to extend $\tau(\rho, \pi_{\lambda_k} \circ \Psi_k(\cdot))$ to $H_{k,t}$: note carefully that $\pi_{\lambda_k} \circ \Psi_k(\cdot)$ is well-defined in $H^c_{k,t}$, but not necessarily in $H_{k,t}$, since $\pi_{\lambda_k} \circ \Psi_k(H_{k,t}) \cap C_l(1-\lambda_k)$ may not be empty, and in such a case it is not in the domain of $\tau(\rho, \cdot)$. The extension we get (see (6.16)) will allow to write a specific double slice of $\mathcal{X}_k$ as push-forward, see (6.25). We stress that this extension is done for a fixed slice $\{t\} \times \mathbb{R}^2$ and in general it cannot be done globally[20] for all $t \in (\varepsilon, l)$.

For t fixed such that case (C) holds, we extend the function $\pi_{\lambda_k} \circ \Psi_k(\cdot)$ to $H_{k,t}$ as follows. Let $\overline{x^i_2 x^{i+1}_1}$ be an arc of $H_{k,t}$; we want to map this arc on an arc in $\{t\} \times \partial B_{1-\lambda_k}$ joining the two image points z^i_2, z^{i+1}_1, with the orientation from z^i_2 to z^{i+1}_1. However there are infinitely many[21] choices of an arc connecting z^i_2 to z^{i+1}_1. To specify which arc we choose we distinguish two possibilities: $|z^i_2 - z^{i+1}_1| \le 2\lambda_k$, and $|z^i_2 - z^{i+1}_1| > 2\lambda_k$. Notice that $|z^i_2 - z^{i+1}_1| \le 2\lambda_k$ is the only case in which the points x^i_2 and x^{i+1}_1 could have images z^i_2 and z^{i+1}_1 with a reversed order on $\{t\} \times \partial B_{1-\lambda_k}$. Indeed, since $x^i_2, x^{i+1}_1 \in \partial \mathrm{B}_t \cap \partial D_k$, we have $d_k(x^i_j) = |\frac{x^i_j}{|x^i_j|} - u_k(x^i_j)| = \lambda_k$. In particular, if the distance between z^i_2 and z^{i+1}_1 is larger than $2\lambda_k$, it means that the distance between $u_k(x^i_2)$ and $u_k(x^{i+1}_1)$ were larger than $2\lambda_k$ (π_{λ_k} does not increase the distance), so that z^i_2 and z^{i+1}_1 must have the same order of $\frac{x^i_2}{|x^i_2|}$ and $\frac{x^{i+1}_1}{|x^{i+1}_1|}$ on ∂B_1, which is the same order of x^i_2 and x^{i+1}_1 on $\partial \mathrm{B}_t$.

We are now in a position to specify the arc: when $|z^i_2 - z^{i+1}_1| > 2\lambda_k$ we define $\overline{z^i_2 z^{i+1}_1}$ to be the counterclockwise oriented arc[22] from z^i_2 to z^{i+1}_1. When $|z^i_2 - z^{i+1}_1| \le 2\lambda_k$ we argue as follows: Let β_i be the angular amplitude of the arc $\overline{x^i_2 x^{i+1}_1}$. We define $\overline{z^i_2 z^{i+1}_1}$ as the unique oriented arc from z^i_2 to z^{i+1}_1 satisfying the following property: If $\widehat{\beta}_i$ is its oriented angular amplitude (positive if counterclockwise oriented, negative otherwise), then

$$|\widehat{\beta}_i - \beta_i| \le 2\widehat{\lambda}_k, \tag{6.15}$$

where $\widehat{\lambda}_k$ is the angular amplitude of a chord on $\partial B_{1-\lambda_k}$ of length λ_k (see Fig. 3). It is easy to check that there is a unique arc $\overline{z^i_2 z^{i+1}_1}$ satisfying this property. Moreover the same property holds

[19] In polar coordinates.

[20] We do not need a global extension since we aim to obtain an estimate which holds for a fixed t.

[21] We can for instance join z^i_2 to z^{i+1}_1 travelling along an oriented arc connecting them, and then travelling along the whole circle an arbitrary number of times (thus considering a self-overlapping arc).

[22] Likewise the orientation from x^i_2 to x^{i+1}_1.

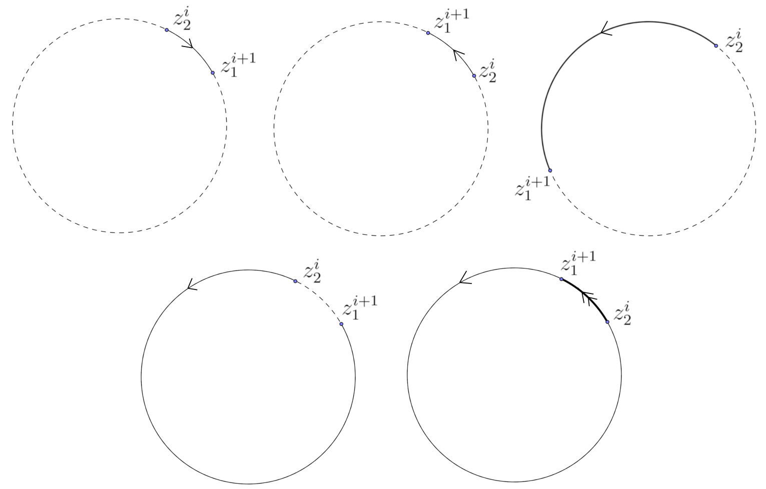

Figure 3: The choice of the arc between z_2^i and z_1^{i+1}. The correct arc is the one in bold on the dashed circle $\{t\} \times \partial B_{1-\lambda_k}$. On top left the case $|z_2^i - z_1^{i+1}| \le 2\lambda_k$ and the arc is clockwise oriented; on top center again case $|z_2^i - z_1^{i+1}| \le 2\lambda_k$ and the arc counterclockwise oriented; on top right the case $|z_2^i - z_1^{i+1}| > 2\lambda_k$; on bottom left again the case $|z_2^i - z_1^{i+1}| \le 2\lambda_k$ when the oriented arc $\overline{z_2^i z_1^{i+1}}$ is the long one. Finally on bottom right it is depicted again the case $|z_2^i - z_1^{i+1}| \le 2\lambda_k$ but the counterclockwise arc between z_2^i and z_1^{i+1} has reversed order with respect to α_2^i and α_1^{i+i}, so that β_i is the long arc; in this case the correct arc such that $|\widehat{\beta}_i - \beta_i| \le 2\widehat{\lambda}_k$ is the short one connecting z_2^i and z_1^{i+1} (double bold) together with a complete turn around the circle.

for β_i and $\widehat{\beta}_i$ in the case that $|z_2^i - z_1^{i+1}| > 2\lambda_k$, since $\pi_{\lambda_k} \circ \Psi_k(\cdot)$ does not change the angular coordinate of a point x_j^i of a quantity larger than $\widehat{\lambda}_k$.

Once we have specified the image arc, we can define $\widehat{P}_{k,i} : \overline{x_2^i x_1^{i+1}} \to \overline{z_2^i z_1^{i+1}}$ to be the affine (with respect to the angular coordinate) function mapping x_2^i to z_2^i and x_1^{i+1} to z_1^{i+1}. We then introduce $P_k = P_{k,t} : \partial \mathrm{B}_t \to \{t\} \times \partial B_{1-\lambda_k}$ as follows:

$$P_k(x) := \begin{cases} \pi_{\lambda_k} \circ \Psi_k(x) & \text{if } x \in H_{k,t}^c, \\ \\ \widehat{P}_{k,i}(x) & \text{if } x \in \overline{x_2^i x_1^{i+1}} \text{ for some } i. \end{cases} \tag{6.16}$$

We claim that

$$\tau(\rho, P_k(\cdot))_\sharp [\![\partial \mathrm{B}_t]\!] = [\![\{t\} \times \partial B_\rho]\!]. \tag{6.17}$$

Since the map $\tau(\rho, \cdot)$ is an orientation preserving homeomorphism between $\partial B_{1-\lambda_k}$ and ∂B_ρ, it is sufficient to show that

$$P_k(\cdot)_\sharp [\![\partial \mathrm{B}_t]\!] = [\![\{t\} \times \partial B_{1-\lambda_k}]\!]. \tag{6.18}$$

Equivalently, we will prove that

$$\sum_{i=1}^h ([\![\overline{z_1^i z_2^i}]\!] + [\![\overline{z_2^i z_1^{i+1}}]\!]) = [\![\{t\} \times \partial B_{1-\lambda_k}]\!]. \tag{6.19}$$

Let ω_i (resp. β_i) be the angular amplitude, in counterclockwise order, of the arc $\overline{x_1^i x_2^i}$ (resp. $\overline{x_2^i x_1^{i+1}}$). Trivially we have $\sum_{i=1}^h(\omega_i+\beta_i)=2\pi$. If $\widehat{\omega}_i$ (resp. $\widehat{\beta}_i$) is the angular amplitude of $\overline{z_1^i z_2^i}$ (resp. $\overline{z_2^i z_1^{i+1}}$), taken with sign ± 1 according to their orientation, we see that to prove (6.19) it suffices to show

$$\sum_{i=1}^h (\widehat{\omega}_i+\widehat{\beta}_i)=2\pi. \tag{6.20}$$

To do this we use (6.15); notice first that the counterpart of (6.15) holds for the arc between x_1^i and x_2^i: Namely the map $\pi_{\lambda_k}\circ\Psi_k$ transforms the arc $\overline{x_1^i x_2^i}$ of angular amplitude ω_i, in the arc $\overline{z_1^i z_2^i}$ of amplitude $\widehat{\omega}_i$ in such a way that

$$|\widehat{\omega}_i-\omega_i|\le 2\widehat{\lambda}_k. \tag{6.21}$$

Now, if θ_j^i is the angular coordinate of z_j^i, and α_j^i is the angular coordinate of x_j^i, we know that

$$\theta_j^i=\alpha_j^i+r_j^i, \qquad \text{with } |r_j^i|\le\widehat{\lambda}_k. \tag{6.22}$$

Here again $\widehat{\lambda}_k$ is the angle of a chord of length λ_k on $\partial B_{1-\lambda_k}$. To prove (6.20) we reduce ourselves to show that

$$\widehat{\omega}_i=\omega_i+r_2^i-r_1^i, \tag{6.23}$$
$$\widehat{\beta}_i=\beta_i+r_1^{i+1}-r_2^i, \tag{6.24}$$

for all i. Fix i; we can assume $\alpha_2^i=\alpha_1^i+\omega_i$, and by (6.22) we get

$$\widehat{\omega}_i=\omega_i+r_2^i-r_1^i+2k_i\pi,$$

with $k_i\in\mathbb{Z}$ accordingly to the number of oriented complete turns around the circle $\partial B_{1-\lambda_k}$. From (6.21) we have $k_i=0$ for all i, and (6.23) follows. A similar argument, using (6.15), leads to (6.24), hence (6.20) is proved, and (6.17) follows at once. Define

$$y_j^i:=\tau(\rho,z_j^i)\in\{t\}\times\partial B_\rho.$$

From (6.17), (6.11), and (6.12) it follows that

$$(\mathcal{X}_k)_{t,\rho}=\tau(\rho,P_k(\cdot))_\sharp[\![\partial \mathrm{B}_t]\!]-\tau(\rho,\pi_{\lambda_k}\circ\Psi_k(\cdot))_\sharp[\![H_{k,t}^c]\!]=\tau(\rho,P_k(\cdot))_\sharp[\![H_{k,t}]\!], \tag{6.25}$$

so that, since the maps $\tau(\rho,\cdot)$ send the arcs $\overline{z_2^i z_1^{i+1}}$ onto $\overline{y_2^i y_1^{i+1}}$, we have

$$(\mathcal{X}_k)_{t,\rho}=\sum_{i=1}^h[\![\overline{y_2^i y_1^{i+1}}]\!], \tag{6.26}$$

hence

$$|(\mathcal{X}_k)_{t,\rho}|\le\sum_{i=1}^h\mathcal{H}^1(\overline{y_2^i y_1^{i+1}}). \tag{6.27}$$

We now estimate the length of the arcs $\overline{y_2^i y_1^{i+1}}$. For simplicity we fix i and set $Y_1:=y_2^i$, $Y_2:=y_1^{i+1}$, $X_1:=x_2^i$ and $X_2:=x_1^{i+1}$. Let $\mathrm{d}(\cdot,\cdot)$ denote the distance between points of $\{t\}\times\partial B_\rho$ (*i.e.*, the length of the minimal arc connecting the two points), let π_ρ be the orthogonal projection of $\mathbb{R}^2_{\mathrm{target}}$ onto the convex set $\overline{B}_\rho$, and write $Y_i = (t, \widetilde{Y}_i)$ with $\widetilde{Y}_i \in \overline{B}_\rho$, for $i = 1, 2$. Then, setting $\widehat{X}_i := \frac{X_i}{|X_i|}$ and denoting $\overline{\widehat{X}_1\widehat{X}_2}$ the arc between $\widehat{X}_1$ and $\widehat{X}_2$ on $\{t\} \times \partial B_1$, we have

$$
\begin{aligned}
\mathcal{H}^1(\overline{Y_1Y_2}) &\leq \mathcal{H}^1\Big(\overline{\pi_\rho(X_1)\pi_\rho(X_2)}\Big) + \mathrm{d}(\pi_\rho(X_1), \widetilde{Y}_1) + \mathrm{d}(\pi_\rho(X_2), \widetilde{Y}_2) \\
&= \rho\mathcal{H}^1\Big(\overline{\widehat{X}_1\widehat{X}_2}\Big) + \mathrm{d}(\pi_\rho(X_1), \widetilde{Y}_1) + \mathrm{d}(\pi_\rho(X_2), \widetilde{Y}_2) \\
&\leq \rho\mathcal{H}^1\Big(\overline{\widehat{X}_1\widehat{X}_2}\Big) + \frac{\pi}{2}|\pi_\rho(X_1) - \widetilde{Y}_1| + \frac{\pi}{2}|\pi_\rho(X_2) - \widetilde{Y}_2| \\
&\leq \rho\mathcal{H}^1\Big(\overline{\widehat{X}_1\widehat{X}_2}\Big) + \frac{\pi}{2}|\pi_\rho(X_1) - \pi_\rho \circ u_k(X_1)| + \frac{\pi}{2}|\pi_\rho \circ u_k(X_1) - \widetilde{Y}_1| \\
&\quad + \frac{\pi}{2}|\pi_\rho(X_2) - \pi_\rho \circ u_k(X_2)| + \frac{\pi}{2}|\pi_\rho \circ u_k(X_2) - \widetilde{Y}_2| \\
&\leq \frac{\rho}{\varepsilon}\mathcal{H}^1\Big(\overline{X_1X_2}\Big) + \frac{\pi}{2}\big(d_k(X_1) + d_k(X_2)\big) + \pi(\lambda_k - \lambda'_k),
\end{aligned}
\tag{6.28}
$$

where we use that, for $x \neq 0$,

$$
d_k(x) = |\frac{x}{|x|} - u_k(x)| = |u(x) - u_k(x)| \geq |\pi_\rho \circ u(x) - \pi_\rho \circ u_k(x)|,
$$

because $\mathrm{Lip}(\pi_\rho) = 1$, $|\pi_\rho \circ u_k(X_i) - \widetilde{Y}_i| \leq \lambda_k - \lambda'_k$ for $i = 1, 2$, and $X_i \in \partial \mathrm{B}_t$, $t > \varepsilon$. By (6.10) (6.27) and (6.28), we infer

$$
\Theta_k(t, \rho) \leq \frac{1}{\varepsilon}\mathcal{H}^1(H_{k,t}) + \frac{\pi}{2\rho}\sum_{x \in \partial H_{k,t}}(d_k(x) + \lambda_k).
\tag{6.29}
$$

Estimate (6.29) holds for $\mathcal{H}^1$-almost every $t \in (\varepsilon, l)$ such that neither case (A) nor (B) happens. Moreover, by (6.13) it holds also in case (A). Case (B) does not contribute to the L^1 norm of $\Theta_k(\cdot, \rho)$, and therefore (6.29) holds for $\mathcal{H}^1$-almost every $t \in (\varepsilon, l)$.

Denoting by $m(x) = |x|$, so that $|\nabla m| = 1$ out of the origin, the coarea formula allows us to write

$$
\int_{\partial D_k} d_k(\sigma) d\mathcal{H}^1(\sigma) \geq \int_{\partial D_k} |\frac{\partial m}{\partial \sigma}| d_k(\sigma) d\mathcal{H}^1(\sigma) = \int_\varepsilon^l \sum_{x \in m^{-1}(t) \cap \partial D_k} d_k(x) dt = \int_\varepsilon^l \sum_{x \in \partial H_{k,t}} d_k(x) dt.
$$

Similarly

$$
\int_{\partial D_k} \lambda_k \, d\mathcal{H}^1(\sigma) \geq \int_\varepsilon^l \lambda_k \mathcal{H}^0(\{x \in \partial H_{k,t}\}) \, dt.
$$

Recalling (4.7), from (6.29) we finally get

$$
\int_\varepsilon^l \Theta_k(t, \rho) dt \leq \frac{1}{\varepsilon}|D_k| + \frac{\pi}{2(1 - \lambda_k)} \int_{\partial D_k} (d_k(\sigma) + 2\lambda_k) \, d\mathcal{H}^1(\sigma) \leq \frac{1}{\varepsilon n} + o_k(1),
$$

where $o_k(1)$ depends on ε and n (since λ_k does) and vanishes as $k \to +\infty$, thanks to Lemma 4.2 (iii). □

7 Estimate from below of the mass of $[\![G_{u_k}]\!]$ over $D_k \cap (\Omega \setminus \overline{\mathrm{B}}_\varepsilon)$

Now we want to identify the current $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ in (5.31) as sum of polar graphs (Section 2.4), and to do this we need some preliminaries.

Definition 7.1 (The function $\vartheta_{k,\varepsilon}$). *Recalling the definition of* $\Theta_k = \Theta_{\mathbb{S}(E_k)}$ *in* (6.7), *we set*

$$\vartheta_{k,\varepsilon} : (\varepsilon, l) \times (0, 1-\lambda'_k] \times \{0\} \to [0, \pi], \qquad \vartheta_{k,\varepsilon}(t,\rho,0) := \frac{\Theta_k(t,\rho)}{2}. \tag{7.1}$$

Note that $\mathrm{dom}(\vartheta_{k,\varepsilon}) \subsetneq \mathrm{dom}(\Theta_k)$. The polar graph of $\vartheta_{k,\varepsilon}$ is the set $G^{\mathrm{pol}}_{\vartheta_{k,\varepsilon}} = \{(t,\rho,\vartheta_{k,\varepsilon}(t,\rho,0)) : (t,\rho,0) \in (\varepsilon,l) \times (0,1-\lambda'_k] \times \{0\}\}$. By construction $\mathbb{S}(E_k)$ is the polar subgraph of $\vartheta_{k,\varepsilon}$ restricted to the half-cylinder $\{(t,\rho,\theta) : t \in (\varepsilon,l), \theta \in (0,\pi)\}$. More precisely, let η be any number[23] with $0 < \eta < \frac{\pi}{4}$; then the polar subgraph

$$SG^{\mathrm{pol}}_{\vartheta_{k,\varepsilon}} := \{(t,\rho,\theta) \in (\varepsilon,l) \times (0,1-\lambda'_k] \times (-\pi/4,\pi) : \theta \in (-\eta, \vartheta_{k,\varepsilon}(t,\rho,0))\}$$

satisfies

$$SG^{\mathrm{pol}}_{\vartheta_{k,\varepsilon}} \cap \{\theta \in (0,\pi)\} = \mathbb{S}(E_k) \cap \{\theta \in (0,\pi)\}, \tag{7.2}$$

and similarly (for the polar epigraph), setting

$$UG^{\mathrm{pol}}_{-\vartheta_{k,\varepsilon}} := \{(t,\rho,\theta) \in (\varepsilon,l) \times (0,1-\lambda'_k] \times (-\pi,\pi/4) : \theta \in (-\vartheta_{k,\varepsilon}(t,\rho,0), \eta)\},$$

we have

$$UG^{\mathrm{pol}}_{-\vartheta_{k,\varepsilon}} \cap \{\theta \in (-\pi,0)\} = \mathbb{S}(E_k) \cap \{\theta \in (-\pi,0)\}. \tag{7.3}$$

Remark 7.2 (The sets $\vartheta_{k,\varepsilon} = 0$, $\vartheta_{k,\varepsilon} = \pi$). Careful attention must be paid to the sets $\{\vartheta_{k,\varepsilon} = 0\}$ and $\{\vartheta_{k,\varepsilon} = \pi\}$. Indeed on such sets the two graphs of $\vartheta_{k,\varepsilon}$ and $-\vartheta_{k,\varepsilon}$ overlap and then, when considered as integral currents, they cancel each other. Moreover the set $\partial^*\mathbb{S}(E_k)$ includes the two graphs of $\vartheta_{k,\varepsilon}$ and $-\vartheta_{k,\varepsilon}$ with the exception of these two sets. In other words, from (7.2) and (7.3) we have

$$\mathbb{S}(E_k) \cap C_l^\varepsilon = \big(SG^{\mathrm{pol}}_{\vartheta_{k,\varepsilon}} \cap \{\theta \in (0,\pi)\}\big) \cup \big(UG^{\mathrm{pol}}_{-\vartheta_{k,\varepsilon}} \cap \{\theta \in (-\pi,0)\}\big) \tag{7.4}$$

up to $\mathcal{H}^3$-negligible sets. From this formula it is evident that the graphs of $\vartheta_{k,\varepsilon}$ and $-\vartheta_{k,\varepsilon}$ over $\{\Theta_k = 0\} \cup \{\Theta_k = 2\pi\}$ cancel each other, and thus they do not belong to the reduced boundary of $\mathbb{S}(E_k)$. Moreover, the polar subgraph and the polar epigraph are sets of finite perimeter, as is their union in (7.4).

Definition 7.3 (The polar projection map π_0^{pol}). *We let* $\pi_0^{\mathrm{pol}} = \pi^{\mathrm{pol}}_{0,\lambda'_k,\varepsilon} : \overline{C}_l^\varepsilon(1-\lambda'_k) \to \overline{C}_l^\varepsilon(1-\lambda'_k) \cap \{\theta = 0\}$ *be the polar projection defined by*

$$\pi_0^{\mathrm{pol}}(t,\rho,\theta) := (t,\rho,0). \tag{7.5}$$

[23] $\eta = 0$ is not allowed, since in this case the boundary of the subgraph (as a current) does not include the set where $\theta = 0$.

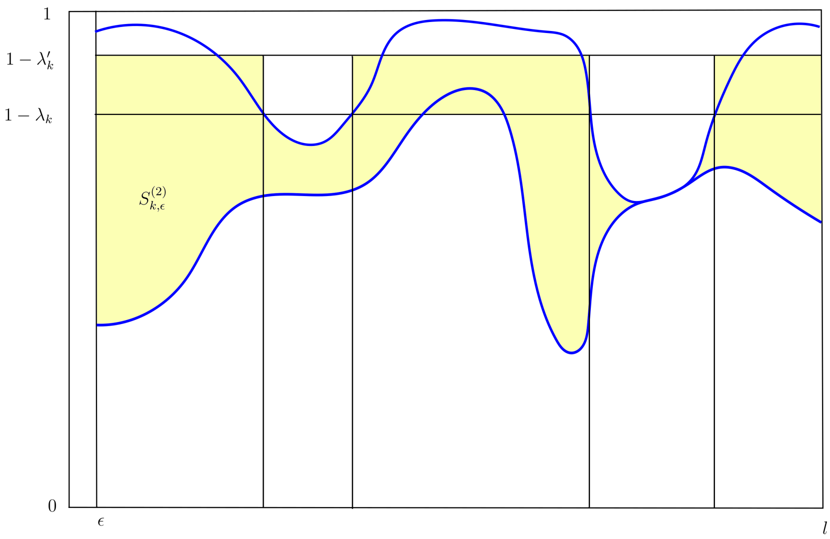

Figure 4: The graphs of the functions $|u_k|^+$ and $|u_k|^-$ and the set $S^{(2)}_{k,\varepsilon}$ in Definition 7.4. See also Fig. 2.

We now introduce various subsets of $(0,l) \times (0,1) \times \{0\}$ in cylindrical coordinates, namely $\pi_0^{\rm pol}(\pi_{\lambda_k} \circ \Psi_k(\Omega \setminus \overline{\rm B}_\varepsilon)) \subseteq S^{(2)}_{k,\varepsilon} \subseteq S^{(2)}_{k,\varepsilon} \cup J_{Q_{k,\varepsilon}} \subseteq S^{(4)}_{k,\varepsilon}$. We start with $S^{(2)}_{k,\varepsilon}$ (see also formulas (7.16) and (9.4) below), and note preliminarly that

$$\pi_0^{\rm pol}(\pi_{\lambda_k} \circ \Psi_k(\Omega \setminus \overline{\rm B}_\varepsilon)) \tag{7.6}$$

coincides with

$$\left\{(t,\rho,0) \in C_l : t \in (\varepsilon,l),\ \rho \in [|u_k|^-(t) \wedge (1-\lambda_k), |u_k|^+(t) \wedge (1-\lambda_k)]\right\},$$

$|u_k|^-, |u_k|^+$ being the functions introduced in (6.1).

Definition 7.4 (The set $S^{(2)}_{k,\varepsilon}$). *Recalling the expression of $Q_{k,\varepsilon}$ in (6.4), we define*

$$S^{(2)}_{k,\varepsilon} := \pi_0^{\rm pol}(\pi_{\lambda_k} \circ \Psi_k(\Omega \setminus \overline{\rm B}_\varepsilon)) \cup \Big(((\varepsilon,l) \setminus Q_{k,\varepsilon}) \times [1-\lambda_k, 1-\lambda_k'] \times \{0\}\Big), \tag{7.7}$$

see Fig. 4.

We have $S^{(2)}_{k,\varepsilon} = \pi_0^{\rm pol}\Big(\pi_{\lambda_k} \circ \Psi_k(\Omega \setminus \overline{\rm B}_\varepsilon) \cup \tau\big([1-\lambda_k, 1-\lambda_k'] \times A\big)\Big)$, where τ is defined in (5.13), and $A := \{(t,y) \in \pi_{\lambda_k} \circ \Psi_k(\Omega \setminus \overline{\rm B}_\varepsilon) : t \in (\varepsilon,l), y \in \partial B_{1-\lambda_k}\}$, since $(t,y) \in \pi_{\lambda_k} \circ \Psi_k(\Omega \setminus \overline{\rm B}_\varepsilon)$ and $y \in \partial B_{1-\lambda_k}$ implies that $t \in (\varepsilon,l) \setminus Q_{k,\varepsilon}$, see Fig. 2 and (5.5).

Remark 7.5. (i) It might happen that $\pi_{\lambda_k} \circ \Psi_k(D_k \setminus \overline{\rm B}_\varepsilon) = \emptyset$. By construction we have

$$\pi_0^{\rm pol}(\pi_{\lambda_k} \circ \Psi_k(\Omega \setminus \overline{\rm B}_\varepsilon)) \cap \Big(((\varepsilon,l) \setminus Q_{k,\varepsilon}) \times [1-\lambda_k, 1-\lambda_k'] \times \{0\}\Big) = ((\varepsilon,l) \setminus Q_{k,\varepsilon}) \times \{1-\lambda_k\} \times \{0\}.$$

The two functions $|u_k|^-$ and $|u_k|^+$ could coincide in large portions of (ε,l) (and even everywhere), so that $\pi_0^{\rm pol}(\pi_{\lambda_k} \circ \Psi_k(\Omega \setminus \overline{\rm B}_\varepsilon))$ could collapse to a curve (for instance if $\Psi_k(\Omega) \subset \partial C_l$). On the other hand, $\mathcal{H}^2(S^{(2)}_{k,\varepsilon}) > 0$ (see Lemma 6.3).

(ii) Notice that $A \supseteq \pi_{\lambda_k} \circ \Psi_k(\partial D_k \setminus \overline{\mathrm{B}}_\varepsilon) \cup \pi_{\lambda_k} \circ \Psi_k((\Omega \setminus \overline{\mathrm{B}}_\varepsilon) \setminus D_k)$. Moreover $A \cap \pi_{\lambda_k} \circ \Psi_k(D_k)$ may not be empty.

(iii) Inside the cylinder $C_l^\varepsilon(1-\lambda_k)$, $S^{(2)}_{k,\varepsilon}$ is exactly the π_0^{pol}-projection of $\pi_{\lambda_k} \circ \Psi_k(D_k \setminus \overline{\mathrm{B}}_\varepsilon)$; remember also that $(\pi_{\lambda_k} \circ \Psi_k)_\sharp(\llbracket D_k \setminus \overline{\mathrm{B}}_\varepsilon \rrbracket) = (\partial \mathcal{E}_k) \llcorner C_l^\varepsilon(1-\lambda_k)$, by (5.6), (5.22) and (5.28).

(iv) Recalling the definition of W_k in (5.16),

$$\pi_0^{\mathrm{pol}}\Big(\pi_{\lambda_k} \circ \Psi_k(D_k \setminus \overline{\mathrm{B}}_\varepsilon) \cup W_k\Big) \subseteq S^{(2)}_{k,\varepsilon}, \tag{7.8}$$

and the above inclusion might be strict.

(v) If $\vartheta_{k,\varepsilon}(t,\rho,0) \in (0,\pi)$ then $(t,\rho,0) \in S^{(2)}_{k,\varepsilon}$, by (7.8). Indeed in this case the circle $(\pi_0^{\mathrm{pol}})^{-1}(t,\rho,0)$ intersects both some sets in $\{E_{k,i}\}$ (see (5.30)) and their complement, so in particular $(\pi_0^{\mathrm{pol}})^{-1}(t,\rho,0)$ must intersect the reduced boundary of some of the sets in $\{E_{k,i}\}$, namely $\pi_{\lambda_k} \circ \Psi_k(D_k \setminus \overline{\mathrm{B}}_\varepsilon) \cup W_k$, for $\mathcal{H}^2$-a.e. $(t,\rho,0) \in S^{(2)}_{k,\varepsilon}$. Furthermore

$$\{(t,\rho,0) : \vartheta_{k,\varepsilon}(t,\rho,0) \in (0,\pi)\} = \pi_0^{\mathrm{pol}}\big(\mathrm{spt}\ (\mathbb{S}(\widehat{\mathfrak{D}}_k))\big),$$

up to $\mathcal{H}^2-$ negligible sets[24]

Remark 7.6. (i) $\Theta_k = 2\pi$ on $\{(t,\rho,0) : t \in Q_{k,\varepsilon}, \rho \in (|u_k|^+(t), 1-\lambda'_k)\}$. Notice that the part of the cylinder $\{(t,\rho,\theta) : t \in Q_{k,\varepsilon}, \rho \in (|u_k|^+(t), 1-\lambda'_k), \theta \in (-\pi,\pi]\}$ does not intersect $\pi_{\lambda_k} \circ \Psi_k(D_k)$, and neither W_k, by construction. As a consequence it does not intersect $\mathrm{spt}\ (\mathbb{S}(\widehat{\mathfrak{D}}_k))$.

(ii) We write $\{(t,\rho,0) \in S^{(2)}_{k,\varepsilon} :$ either $\Theta_k(t,\rho) = 0$ or $\Theta_k(t,\rho) = 2\pi\} = S^{(2)}_{k,\varepsilon} \cap \{\Theta_k \in \{0,2\pi\}\}$. Then $S^{(2)}_{k,\varepsilon} \cap \{\Theta_k \in \{0,2\pi\}\}$ corresponds to the values of t and ρ for which $(t,\rho,0) \in S^{(2)}_{k,\varepsilon}$ and the slice $\mathbb{S}(E_k)_{(t,\rho)} = \mathbb{S}(E_k) \cap (\{t\} \times \partial B_\rho)$ is either empty or the whole circle $\{t\} \times \partial B_\rho$ (up to $\mathcal{H}^1$-negligible sets). Notice also that the intersection $\pi_0^{\mathrm{pol}}\Big(\pi_{\lambda_k} \circ \Psi_k(D_k \setminus \overline{\mathrm{B}}_\varepsilon) \cup W_k\Big) \cap S^{(2)}_{k,\varepsilon} \cap \{\Theta_k \in \{0,2\pi\}\}$ may not be empty on a set of positive $\mathcal{H}^2-$measure. Indeed in the proof of Proposition 7.9, we show that the π_0^{pol}-projection of[25] $\Big(\pi_{\lambda_k} \circ \Psi_k(D_k)\Big) \setminus \mathrm{spt}\ (\widehat{\mathfrak{D}}_k)$ is contained in $S^{(2)}_{k,\varepsilon} \cap \{\Theta_k \in \{0,2\pi\}\}$.

7.1 The current $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ as sum of a polar subgraph and a polar epigraph

Let $G^{\mathrm{pol}}_{\pm\vartheta_{k,\varepsilon} \llcorner \left(S^{(2)}_{k,\varepsilon} \cap \{\Theta_k \in \{0,2\pi\}\}\right)}$ be the polar graph of $\pm\vartheta_{k,\varepsilon} \llcorner \Big(S^{(2)}_{k,\varepsilon} \cap \{\Theta_k \in \{0,2\pi\}\}\Big)$; these two sets, by symmetry, overlap, and

$$\llbracket G^{\mathrm{pol}}_{-\vartheta_{k,\varepsilon} \llcorner \left(S^{(2)}_{k,\varepsilon} \cap \{\Theta_k \in \{0,2\pi\}\}\right)} \rrbracket + \llbracket G^{\mathrm{pol}}_{\vartheta_{k,\varepsilon} \llcorner \left(S^{(2)}_{k,\varepsilon} \cap \{\Theta_k \in \{0,2\pi\}\}\right)} \rrbracket = 0,$$

due to the fact that $\llbracket G^{\mathrm{pol}}_{\vartheta_{k,\varepsilon} \llcorner \left(S^{(2)}_{k,\varepsilon} \cap \{\Theta_k \in \{0,2\pi\}\}\right)} \rrbracket$ and $\llbracket G^{\mathrm{pol}}_{-\vartheta_{k,\varepsilon} \llcorner \left(S^{(2)}_{k,\varepsilon} \cap \{\Theta_k \in \{0,2\pi\}\}\right)} \rrbracket$ are oriented in opposite way. Indeed we endow $\llbracket G^{\mathrm{pol}}_{\vartheta_{k,\varepsilon} \llcorner \left(S^{(2)}_{k,\varepsilon} \cap \{\Theta_k \in \{0,2\pi\}\}\right)} \rrbracket$ with the orientation inherited by looking

[24]There could be $(t,\rho,0) \in \pi_0^{\mathrm{pol}}\big(\mathrm{spt}\ (\mathbb{S}(\widehat{\mathfrak{D}}_k))\big)$ such that $\vartheta_{k,\varepsilon}(t,\rho,0) \notin (0,\pi)$. Indeed take $t \in (\varepsilon, l)$ such that $\partial \mathrm{B}_t \subset D_k$ and assume that $\Psi_k(\partial \mathrm{B}_t) = \{t\} \times \partial B_\rho$, $\rho < 1-\lambda_k$. Then $\vartheta_{k,\varepsilon}(t,\rho) = \pi$ and $\{t\} \times \partial B_\rho \subset \partial^* \mathbb{S}(E_k)$; however this can only happen for (t,ρ) in a negligible $\mathcal{H}^2$-set.

[25]This is the set where $\pi_{\lambda_k} \circ \Psi_k(D_k)$ overlaps itself with opposite orientation; this set might have positive area, see Fig. 8.

at it as the boundary of the polar subgraph of $\vartheta_{k,\varepsilon}$, and we endow $[\![G^{\rm pol}_{-\vartheta_{k,\varepsilon}\llcorner\left(S^{(2)}_{k,\varepsilon}\cap\{\Theta_k\in\{0,2\pi\}\}\right)}]\!]$ with the opposite orientation, since we look at it as boundary of an epigraph.

Definition 7.7 (**The currents $\mathcal{G}^{\pm}_{k,\varepsilon}$**). *We set*

$$\begin{aligned}\mathcal{G}^{+}_{k,\varepsilon} &:= (\partial[\![SG^{\rm pol}_{\vartheta_{k,\varepsilon}}]\!])\llcorner\big(\{\theta\in(0,\pi)\}\cap C^{\varepsilon}_{l}(1-\lambda'_k)\big) + [\![G^{\rm pol}_{\vartheta_{k,\varepsilon}\llcorner\left(S^{(2)}_{k,\varepsilon}\cap\{\Theta_k\in\{0,2\pi\}\}\right)}]\!],\\ \mathcal{G}^{-}_{k,\varepsilon} &:= (\partial[\![UG^{\rm pol}_{-\vartheta_{k,\varepsilon}}]\!])\llcorner\big(\{\theta\in(-\pi,0)\}\cap C^{\varepsilon}_{l}(1-\lambda'_k)\big) + [\![G^{\rm pol}_{-\vartheta_{k,\varepsilon}\llcorner\left(S^{(2)}_{k,\varepsilon}\cap\{\Theta_k\in\{0,2\pi\}\}\right)}]\!].\end{aligned} \tag{7.9}$$

The non standard orientation of $\mathcal{G}^{-}_{k,\varepsilon}$ is chosen in such a way that condition (7.10) in Proposition 7.9 below takes place. In this proposition we will also see that, being $\mathbb{S}(E_k)$ a finite perimeter set in C^{ε}_{l}, its reduced boundary, seen as a current, has finite mass. In turn, the integration on its boundary is exactly $\mathcal{G}^{+}_{k,\varepsilon}+\mathcal{G}^{-}_{k,\varepsilon}$ (see also (7.4)).

Remark 7.8. The generalized polar graph of $\vartheta_{k,\varepsilon}$ might have large parts on which $\vartheta_{k,\varepsilon}\in\{0,\pi\}$; for this reason we neglected this part in the currents introduced in (7.9) by restricting the boundary of the subgraph in $\{\theta\in(0,\pi)\}\cap C^{\varepsilon}_{l}(1-\lambda'_k)$ (and similarly for the epigraph). However we want to consider the graph above the set $\vartheta_{k,\varepsilon}\in\{0,\pi\}$ on the strip $S^{(2)}_{k,\varepsilon}$, in particular the projection of the set where $\pi_{\lambda_k}\circ\Psi_k(D_k)$ overlaps itself (which may have positive area), for this reason we have to add the term $[\![G^{\rm pol}_{\vartheta_{k,\varepsilon}\llcorner\left(S^{(2)}_{k,\varepsilon}\cap\{\Theta_k\in\{0,2\pi\}\}\right)}]\!]$ in formulas (7.9). The reason why we have to get rid of the graph of $\vartheta_{k,\varepsilon}$ on $\{\vartheta_{k,\varepsilon}\in\{0,\pi\}\}$ outside $S^{(2)}_{k,\varepsilon}$ is that this term is not controlled by the area of $(\pi_{\lambda_k}\circ\Psi_k(\Omega\setminus\overline{\rm B}_\varepsilon))$ (see also Remark 7.6 (iv)).

Proposition 7.9 (**Estimate of the mass of $\mathcal{G}^{\pm}_{k,\varepsilon}$**). *Let ε be fixed as in* (4.5) *and* (4.6)*, and recall the definition* (5.31) *of $\mathbb{S}(\widehat{\mathfrak{D}}_k)$. Then the following properties hold:*

$$\mathcal{G}^{+}_{k,\varepsilon}+\mathcal{G}^{-}_{k,\varepsilon}=\mathbb{S}(\widehat{\mathfrak{D}}_k)\llcorner C^{\varepsilon}_{l}(1-\lambda'_k), \tag{7.10}$$

$$|\mathcal{G}^{+}_{k,\varepsilon}|+|\mathcal{G}^{-}_{k,\varepsilon}|=|\mathbb{S}(\widehat{\mathfrak{D}}_k)|_{C^{\varepsilon}_{l}(1-\lambda'_k)}+2\mathcal{H}^2\left(S^{(2)}_{k,\varepsilon}\cap\{\Theta_k\in\{0,2\pi\}\}\right), \tag{7.11}$$

$$|\mathcal{G}^{+}_{k,\varepsilon}|+|\mathcal{G}^{-}_{k,\varepsilon}|\le\int_{D_k\cap(\Omega\setminus\overline{\rm B}_\varepsilon)}|J(\pi_{\lambda_k}\circ\Psi_k)|\,dx+\frac{1}{n}+o_k(1), \tag{7.12}$$

where $o_k(1)$ is a nonnegative infinitesimal sequence as $k\to+\infty$, depending on n and ε.

Proof. Identity (7.10) follows by definition and from (7.4). Concerning (7.11), setting for simplicity

$$J^{0,2\pi}_{k,\varepsilon}:=S^{(2)}_{k,\varepsilon}\cap\{\Theta_k\in\{0,2\pi\}\}, \tag{7.13}$$

it is sufficient to observe that ${\rm spt}\,([\![\mathcal{G}^{+}_{k,\varepsilon}]\!])$ and ${\rm spt}\,([\![\mathcal{G}^{-}_{k,\varepsilon}]\!])$ coincide on the set $\vartheta_{k,\varepsilon}(J^{0,2\pi}_{k,\varepsilon})$ (whose measure is equal[26] to the measure of $J^{0,2\pi}_{k,\varepsilon}$). Thus, the currents $\mathcal{G}^{+}_{k,\varepsilon}$ and $\mathcal{G}^{-}_{k,\varepsilon}$ cancel each other on this set, since they are endowed with opposite orientation. Hence

$$|\mathcal{G}^{+}_{k,\varepsilon}|+|\mathcal{G}^{-}_{k,\varepsilon}|=|\mathcal{G}^{+}_{k,\varepsilon}+\mathcal{G}^{-}_{k,\varepsilon}|+2\mathcal{H}^2(J^{0,2\pi}_{k,\varepsilon}), \tag{7.14}$$

and (7.11) follows from (7.10).

[26]Indeed $\vartheta_{k,\varepsilon}$ restricted to $J^{0,2\pi}_{k,\varepsilon}\cap\{\Theta_k=0\}$ is the identity map and $\vartheta_{k,\varepsilon}$ restricted to $J^{0,2\pi}_{k,\varepsilon}\cap\{\Theta_k=2\pi\}$ is a π-rotation.

Let us prove (7.12). We recall that the rectifiable set $\pi_{\lambda_k} \circ \Psi_k(D_k) \cup W_k$ includes the support of the current $\widehat{\mathfrak{D}}_k$. There might be parts of $\pi_{\lambda_k} \circ \Psi_k(D_k) \cup W_k$ where the multiplicity of $\widehat{\mathfrak{D}}_k$ is zero, and this happens for instance where two pieces of $\pi_{\lambda_k} \circ \Psi_k(D_k) \cup W_k$ overlap with opposite orientations. We decompose $\pi_{\lambda_k} \circ \Psi_k(D_k) \cup W_k$ as follows:

$$\pi_{\lambda_k} \circ \Psi_k(D_k) \cup W_k = Z_k^0 \cup \text{spt } (\widehat{\mathfrak{D}}_k) = Z_k^0 \cup \text{spt } (\mathfrak{D}_k) \cup \text{spt } (\mathcal{W}_k), \tag{7.15}$$

where

$$Z_k^0 := \Big(\pi_{\lambda_k} \circ \Psi_k(D_k) \setminus \text{spt } (\mathfrak{D}_k)\Big) \cup \Big(W_k \setminus \text{spt } (\mathcal{W}_k)\Big)$$

is the set where $\widehat{\mathfrak{D}}_k$ has vanishing multiplicity. It is convenient to introduce the following notation for the set in (7.6):

$$S_{k,\varepsilon}^{(1)} := \pi_0^{\text{pol}}(\pi_{\lambda_k} \circ \Psi_k(\Omega \setminus \overline{\text{B}}_\varepsilon)). \tag{7.16}$$

We claim that

$$S_{k,\varepsilon}^{(1)} \cap J_{k,\varepsilon}^{0,2\pi} \subseteq \pi_0^{\text{pol}}\big(Z_k^0\big), \tag{7.17}$$

where π_0^{pol} is the projection introduced in (7.5) (again, here the inclusion is intended up to $\mathcal{H}^2$-negligible sets). To prove this we argue by slicing: for $t \in (\varepsilon, l)$ set

$$(S_{k,\varepsilon}^{(1)} \cap J_{k,\varepsilon}^{0,2\pi})_t := (S_{k,\varepsilon}^{(1)} \cap J_{k,\varepsilon}^{0,2\pi}) \cap (\{t\} \times \mathbb{R}^2).$$

It is sufficient to show that

$$(S_{k,\varepsilon}^{(1)} \cap J_{k,\varepsilon}^{0,2\pi})_t \subseteq \pi_0^{\text{pol}}\big(Z_k^0\big) \qquad \text{for } \mathcal{H}^1 - \text{a.e. } t \in (\varepsilon, l). \tag{7.18}$$

In turn, denoting $(Z_k^0)_t := Z_k^0 \cap (\{t\} \times \mathbb{R}^2)$ we will prove[27]

$$(S_{k,\varepsilon}^{(1)} \cap J_{k,\varepsilon}^{0,2\pi})_t \subseteq \pi_0^{\text{pol}}\big((Z_k^0)_t\big) \qquad \text{for } \mathcal{H}^1 - \text{a.e. } t \in (\varepsilon, l). \tag{7.19}$$

Now, $(Z_k^0)_t$ is, for $\mathcal{H}^1$-a.e. $t \in (\varepsilon, l)$, the set where the coefficient of the integral current $(\widehat{\mathfrak{D}}_k)_t$ is zero. Recalling (7.13), we have that[28] $\Theta_k(t, \rho) \in \{0, 2\pi\}$ for $\rho \in [|u_k|^-(t) \wedge (1 - \lambda_k), |u_k|^+(t) \wedge (1 - \lambda_k)]$ such that $(t, \rho, 0) \in J_{k,\varepsilon}^{0,2\pi}$. This means that either

- for all i the intersection between $E_{k,i}$ (see (5.30)) and $\{t\} \times \partial B_\rho$ is empty (up to $\mathcal{H}^1$-negligible sets), or
- for at least one i, it happens $E_{k,i} \cap (\{t\} \times \partial B_\rho) = \{t\} \times \partial B_\rho$ (up to $\mathcal{H}^1$-negligible sets).

In both cases, for $\mathcal{H}^1$-a.e. $\rho \in (S_{k,\varepsilon}^{(1)} \cap J_{k,\varepsilon}^{0,2\pi})_t$, the current $(\mathbb{S}(\widehat{\mathfrak{D}}_k))_t$ is null on the set

$$\{(t, \rho, \theta) : \theta \in (-\pi, \pi), \rho \in (S_{k,\varepsilon}^{(1)} \cap J_{k,\varepsilon}^{0,2\pi})_t\}.$$

Indeed, recalling that $\mathbb{S}(\widehat{\mathfrak{D}}_k) = \partial[\![\mathbb{S}(E_k)]\!]$, in the first case this is obvious, in the second one it is sufficient to remember that $E_k = \cup_i E_{k,i}$. In other words, the set $(\pi_{\lambda_k} \circ \Psi_k(D_k))_t$ must overlap itself with opposite directions in this set, because the multiplicity of $(\widehat{\mathfrak{D}}_k)_t$ is null there. Hence we have proved (7.19), and claim (7.17) follows.

[27] The only fact we will use is that the π_0^{pol}-projection of the set $\pi_{\lambda_k} \circ \Psi_k(D_k)$ is surjective on $S_{k,\varepsilon}^{(2)}$ (essentially by definition) and then the inverse image of a point where $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ is null is covered at least two times.

[28] See Fig. 8, the two bold segments: on one $\Theta_k = 0$ and on the other one $\Theta_k = 2\pi$

As a consequence of (7.17) and of its proof, we have

$$
\begin{aligned}
2\mathcal{H}^2(J^{0,2\pi}_{k,\varepsilon}) &\leq 2\mathcal{H}^2\big(\pi_0^{\rm pol}(Z_k^0\cap C_l^\varepsilon(1-\lambda_k))\big)+2(\lambda_k-\lambda_k')l\\
&\leq \int_{D_k\cap(\Omega\setminus\overline{\rm B}_\varepsilon)}|J(\pi_{\lambda_k}\circ\Psi_k)|dx+M(W_k\cap C_l^\varepsilon(1-\lambda_k'))-|\widehat{\mathfrak{D}}_k|_{C_l^\varepsilon(1-\lambda_k')}+o_k(1), \qquad (7.20)
\end{aligned}
$$

see (5.18). Indeed, the first inequality is easy to see, recalling that $J^{0,2\pi}_{k,\varepsilon}$ is the union of $S^{(1)}_{k,\varepsilon}\cap J^{0,2\pi}_{k,\varepsilon}$ and $J^{0,2\pi}_{k,\varepsilon}\setminus S^{(1)}_{k,\varepsilon}$, and the latter has measure less than $(\lambda_k-\lambda_k')l$ that is infinitesimal as $k\to+\infty$ (we denote it by $o_k(1)$). To see the second inequality we use decomposition (7.15). Since Z_k^0 is covered at least two times (with opposite directions), the area $M((\pi_{\lambda_k}\circ\Psi_k(D_k)\cup W_k)\cap C_l^\varepsilon(1-\lambda_k'))$ of $\pi_{\lambda_k}\circ\Psi_k(D_k)\cup W_k$ in $C_l^\varepsilon(1-\lambda_k')$ counted with multiplicity, *i.e.*,

$$
M((\pi_{\lambda_k}\circ\Psi_k(D_k)\cup W_k)\cap C_l^\varepsilon(1-\lambda_k')):=\int_{D_k\cap(\Omega\setminus\overline{\rm B}_\varepsilon)}|J(\pi_{\lambda_k}\circ\Psi_k)|dx+M(W_k\cap C_l^\varepsilon(1-\lambda_k')),
$$

satisfies

$$
\begin{aligned}
M((\pi_{\lambda_k}\circ\Psi_k(D_k)\cup W_k)\cap C_l^\varepsilon(1-\lambda_k')) \geq& 2\mathcal{H}^2(Z_k^0\cap C_l^\varepsilon(1-\lambda_k'))+|\widehat{\mathfrak{D}}_k|_{C_l^\varepsilon(1-\lambda_k')}\\
\geq& 2\mathcal{H}^2\big(\pi_0^{\rm pol}(Z_k^0\cap C_l^\varepsilon(1-\lambda_k'))\big)+|\widehat{\mathfrak{D}}_k|_{C_l^\varepsilon(1-\lambda_k')},\\
\geq& 2\mathcal{H}^2\big(\pi_0^{\rm pol}(Z_k^0\cap C_l^\varepsilon(1-\lambda_k))\big)+|\widehat{\mathfrak{D}}_k|_{C_l^\varepsilon(1-\lambda_k')},
\end{aligned}
$$

and (7.20) follows.

In order to prove (7.12) it is now sufficient to observe that

$$
\begin{aligned}
&|\mathcal{G}^+_{k,\varepsilon}|+|\mathcal{G}^-_{k,\varepsilon}|=|\mathbb{S}(\widehat{\mathfrak{D}}_k)|_{C_l^\varepsilon(1-\lambda_k')}+2\mathcal{H}^2(J^{0,2\pi}_{k,\varepsilon})\\
\leq&|\mathbb{S}(\widehat{\mathfrak{D}}_k)|_{C_l^\varepsilon(1-\lambda_k')}+\int_{D_k\cap(\Omega\setminus\overline{\rm B}_\varepsilon)}|J(\pi_{\lambda_k}\circ\Psi_k)|dx+M(W_k\cap C_l^\varepsilon(1-\lambda_k'))-|\widehat{\mathfrak{D}}_k|_{C_l^\varepsilon(1-\lambda_k')}+o_k(1)\\
\leq&\int_{D_k\cap(\Omega\setminus\overline{\rm B}_\varepsilon)}|J(\pi_{\lambda_k}\circ\Psi_k)|dx+\frac{1}{n}+o_k(1),
\end{aligned}
$$

where we have used (5.20) and (5.34) localized in the cylinder $C_l^\varepsilon(1-\lambda_k')$.

□

Corollary 7.10. *We have*

$$
|[\![G_{u_k}]\!]|_{D_k\cap(\Omega\setminus\overline{\rm B}_\varepsilon))\times\mathbb{R}^2}\geq|\mathcal{G}^+_{k,\varepsilon}|+|\mathcal{G}^-_{k,\varepsilon}|-\frac{1}{n}-o_k(1).
$$

Proof. It follows from (7.12) and (5.8). □

Now we restrict our attention to the rectifiable sets spt $(\mathcal{G}^\pm_{k,\varepsilon})$, the supports of the currents in (7.9). We recall that the function $\vartheta_{k,\varepsilon}$ might take values in $(0,\pi)$ only in the "strip" $S^{(2)}_{k,\varepsilon}$, see Remark 7.5 (v), and

$$
S^{(2)}_{k,\varepsilon}\subset(\varepsilon,l)\times[0,1]\times\{0\}\subset C_l.
$$

Now we add to $\mathcal{G}^+_{k,\varepsilon}$ a graph on some additional set outside $S^{(2)}_{k,\varepsilon}$, see Fig.6.

Definition 7.11. *We let*

$$
J_{Q_{k,\varepsilon}}:=\{(t,\rho,0)\in C_l: t\in Q_{k,\varepsilon},\ \rho\in[|u_k|^+(t),1-\lambda_k']\}. \qquad (7.21)
$$

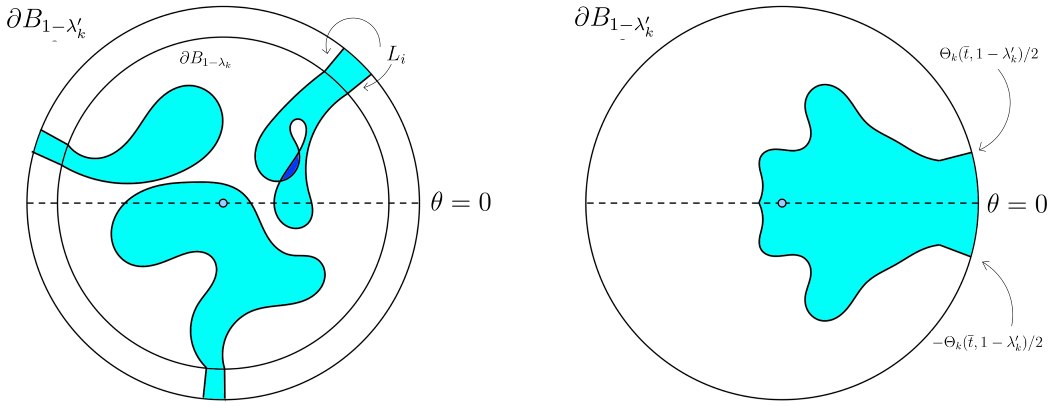

Figure 5: Intersection of the cylinder $C_l(1-\lambda_k)$ with $\{t = \bar{t}\} \times \mathbb{R}^2$. The symmetrization of a closed current in $B_{1-\lambda'_k}$, which on the left is emphasized in grey, and with dark grey the region in which the multiplicity of the current is 2. The set is bounded by a generic curve with endpoints on $\partial B_{1-\lambda_k}$, in turn these endpoints have been joined with $\partial B_{1-\lambda'_k}$ by radial segments L_i. The area emphasized has been symmetrized with the respect to the radius $\{\theta = 0\}$ in the right picture, where we have indicated the angles $\pm\Theta_k(\bar{t}, 1-\lambda'_k)/2$.

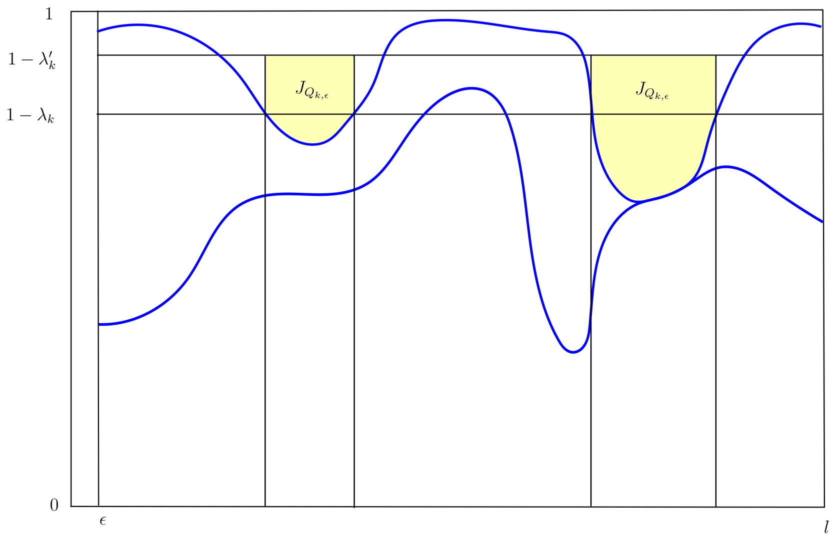

Figure 6: The graphs of the functions $|u_k|^+$ and $|u_k|^-$ and the set $J_{Q_{k,\varepsilon}}$ in (7.21). See also Fig. 2.

By definition of $Q_{k,\varepsilon}$ in (6.4), we have that for $\mathcal{H}^2$-a.e. $(t,\rho,0) \in J_{Q_{k,\varepsilon}}$ it holds $(\pi_0^{\rm pol})^{-1}((t,\rho,0)) \cap$ spt $(\mathbb{S}(\widehat{\mathfrak{D}}_k)) = \emptyset$, so that $\vartheta_{k,\varepsilon} \in \{0,\pi\}$ on $J_{Q_{k,\varepsilon}}$. Recalling (5.29) and (5.27), it is not difficult to see that $(\pi_0^{\rm pol})^{-1}(J_{Q_{k,\varepsilon}}) \subseteq \mathbb{S}(E_k)$. Hence

$$\vartheta_{k,\varepsilon} = \pi \qquad \text{in } J_{Q_{k,\varepsilon}}$$

(see also Remark 7.6).

Now, we want to add to the currents $\mathcal{G}^{\pm}_{k,\varepsilon}$ in (7.7) a new part above a region that becomes, in Section 10, the subgraph of the function $h_{k,\varepsilon}$.

Definition 7.12 (The currents $\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}}$ and $\mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}}$). *We define*

$$\begin{aligned}\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} &:= \mathcal{G}^{+}_{k,\varepsilon} + [\![G^{\rm pol}_{\vartheta_{k,\varepsilon} \llcorner J_{Q_{k,\varepsilon}}}]\!] \in \mathcal{D}_2(C_l^{\varepsilon}(1-\lambda_k')),\\ \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} &:= \mathcal{G}^{-}_{k,\varepsilon} + [\![G^{\rm pol}_{-\vartheta_{k,\varepsilon} \llcorner J_{Q_{k,\varepsilon}}}]\!] \in \mathcal{D}_2(C_l^{\varepsilon}(1-\lambda_k')).\end{aligned}$$

Lemma 7.13. *The following holds:*

$$|\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}}| = |\mathcal{G}^{+}_{k,\varepsilon}| + \mathcal{H}^2(J_{Q_{k,\varepsilon}}); \tag{7.22}$$

$$\mathcal{H}^2(J_{Q_{k,\varepsilon}}) \leq |Q_{k,\varepsilon}| \leq \frac{1}{2\pi\varepsilon n}; \tag{7.23}$$

$$\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} = \partial [\![\mathbb{S}(E_k)]\!] \llcorner C_l^{\varepsilon}(1-\lambda_k'). \tag{7.24}$$

Proof. Formula (7.22) follows from the fact that $\mathcal{G}^{+}_{k,\varepsilon}$ and $[\![G_{\vartheta_{k,\varepsilon} \llcorner J_{Q_{k,\varepsilon}}}]\!]$ have disjoint supports, and $|[\![G_{\vartheta_{k,\varepsilon} \llcorner J_{Q_{k,\varepsilon}}}]\!]| = \mathcal{H}^2(J_{Q_{k,\varepsilon}})$. Inequality (7.23) follows from

$$\mathcal{H}^2(J_{Q_{k,\varepsilon}}) = \int_{Q_{k,\varepsilon}} (1-\lambda_k' - |u_k|^+(t))\ dt \leq |Q_{k,\varepsilon}| \leq \frac{1}{2\pi\varepsilon n},$$

where the last inequality is a consequence of Lemma 6.3. Formula (7.24) follows as in Proposition 7.9, using the fact that $[\![G_{\vartheta_{k,\varepsilon} \llcorner J_{Q_{k,\varepsilon}}}]\!]$ and $[\![G_{-\vartheta_{k,\varepsilon} \llcorner J_{Q_{k,\varepsilon}}}]\!]$ have opposite orientation. □

Current $\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}}$ is closed in $C_l^{\varepsilon}(1-\lambda_k')$. We can look at its boundary as a current in $\mathcal{D}_2((\varepsilon,l)\times\mathbb{R}^2)$, which has support on the lateral boundary of the cylinder $C_l^{\varepsilon}(1-\lambda_k')$. To this aim we study the trace of $\vartheta_{k,\varepsilon}$ (that is $\Theta_k(t,\rho)/2$) on the segment

$$(\varepsilon,l)\times\{1-\lambda_k'\}\times\{0\}. \tag{7.25}$$

Observe that by definition

$$\vartheta_{k,\varepsilon} = \pi \quad \text{on } Q_{k,\varepsilon}\times\{1-\lambda_k'\}\times\{0\} \subseteq (\varepsilon,l)\times\{1-\lambda_k'\}\times\{0\},$$

whereas on $((\varepsilon,l)\setminus Q_{k,\varepsilon})\times\{1-\lambda_k'\}\times\{0\}$ we have

$$\vartheta_{k,\varepsilon}(t,1-\lambda_k',0) = \Theta_k(t,1-\lambda_k')/2 = \Theta_k(t,\rho)/2, \qquad t\in(\varepsilon,l)\setminus Q_{k,\varepsilon},$$

for all $\rho \in (1-\lambda_k, 1-\lambda_k')$.

Definition 7.14 (The 2-rectifiable set $\Sigma_{k,\varepsilon}$). *We let*

$$\Sigma_{k,\varepsilon} := \Big\{(t,\rho,\theta) : t\in(\varepsilon,l),\ \rho = 1-\lambda_k',\ \theta\in(-\Theta_k(t,1-\lambda_k')/2, \Theta_k(t,1-\lambda_k')/2)\Big\}. \tag{7.26}$$

Referring to the right picture in Figure 5, the section of $\Sigma_{k,\varepsilon}$ is the short arc connecting the points $(\bar{t}, 1-\lambda'_k, -\Theta_k(\bar{t}, 1-\lambda'_k)/2)$ and $(\bar{t}, 1-\lambda'_k, \Theta_k(\bar{t}, 1-\lambda'_k)/2)$; see also Fig. 9.

If we denote by $[\![\Sigma_{k,\varepsilon}]\!]$ the current given by integration over $\Sigma_{k,\varepsilon}$ (suitably oriented), its boundary coincides with the boundary of $\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}}$ on $\partial_{\rm lat} C^{\varepsilon}_l(1-\lambda'_k)$.

Lemma 7.15 (Properties of $\Sigma_{k,\varepsilon}$). *$\Sigma_{k,\varepsilon}$, oriented by the outward unit normal to the lateral boundary of $C_l(1-\lambda'_k)$, is such that*

$$\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} + [\![\Sigma_{k,\varepsilon}]\!] \in \mathcal{D}_2((\varepsilon, l)\times\mathbb{R}^2) \qquad \text{is boundaryless.}$$

Moreover

$$\mathcal{H}^2(\Sigma_{k,\varepsilon}) \le \frac{1}{\varepsilon n} + o_k(1), \tag{7.27}$$

where the sequence $o_k(1)\ge 0$ depends on n and ε, and is infinitesimal as $k\to+\infty$. Finally

$$(\partial[\![\Sigma_{k,\varepsilon}]\!])\llcorner(\{\varepsilon\}\times\mathbb{R}^2) = [\![\{\varepsilon\}\times\{1-\lambda'_k\}\times\left[\frac{-\Theta_k(\varepsilon,1-\lambda'_k)}{2}, \frac{\Theta_k(\varepsilon,1-\lambda'_k)}{2}\right]]\!], \tag{7.28}$$

oriented counterclockwise[29].

Proof. The fact that the current $\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} + [\![\Sigma_{k,\varepsilon}]\!]$ is boundaryless in $\mathcal{D}_2((\varepsilon,l)\times\mathbb{R}^2)$ is a consequence of the fact that $\Sigma_{k,\varepsilon}$ is a subset of the polar subgraph of the trace of $\vartheta_{k,\varepsilon}$ on $(\varepsilon,l)\times\{1-\lambda'_k\}\times\{0\}$. Concerning (7.27) we have

$$\mathcal{H}^2(\Sigma_{k,\varepsilon}) = \int_{\varepsilon}^{l}\int_{-\Theta_k(t,1-\lambda'_k)/2}^{\Theta_k(t,1-\lambda'_k)/2} (1-\lambda'_k)\,d\theta dt \le \frac{1}{\varepsilon n} + o_k(1), \tag{7.29}$$

where the last inequality follows from Lemma 6.4. As for the last assertion, we have to understand which is the orientation of $[\![\Sigma_{k,\varepsilon}]\!]$, which has been chosen in such a way that $\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} + [\![\Sigma_{k,\varepsilon}]\!] = (\partial[\![\mathbb{S}(E_k)]\!])\llcorner((\varepsilon,l)\times[0,1-\lambda'_k]\times\{\theta\in(-\pi,\pi]\})$. Hence, since $\mathbb{S}(E_k)$ is contained in $C_l(1-\lambda'_k)$, the orientation of $[\![\Sigma_{k,\varepsilon}]\!]$ is the one inherited by the external normal to $\partial[\![\mathbb{S}(E_k)]\!]$, namely the outward unit normal to the lateral boundary of $C_l(1-\lambda'_k)$. $\square$

8 Estimate from below of the mass of $[\![G_{u_k}]\!]$ over $D_k\cap \mathrm{B}_\varepsilon$

We now analyse the image of $D_k\cap\mathrm{B}_\varepsilon$ through Ψ_k. We want to reduce this set to a current $\mathcal{V}_k\in\mathcal{D}_2(\{\varepsilon\}\times\mathbb{R}^2)$ (defined in (8.12)), in order that it contains the necessary information on the area of $\Psi_k(D_k\cap\mathrm{B}_\varepsilon)$. To this aim we need first to describe the boundary of $\mathcal{V}_k$ and then show that its mass gives a lower bound for the area of the graph of u_k (see formula (8.13)).

Borrowing the notation from the proof of Lemma 6.4, the set $\partial\mathrm{B}_\varepsilon$ is splitted as:

$$\partial\mathrm{B}_\varepsilon = (D_k\cap\partial\mathrm{B}_\varepsilon)\cup((\Omega\setminus D_k)\cap\partial\mathrm{B}_\varepsilon) =: H_{k,\varepsilon}\cup H^c_{k,\varepsilon}. \tag{8.1}$$

We denote by

$$\{x_i\}_{i=1}^{I_k}\subseteq\{\widehat{x}_i\}_{i=1}^{J_k} := \partial\mathrm{B}_\varepsilon\cap\partial D_k, \tag{8.2}$$

the finite family of points (see Lemma 4.2 (v)) which represents the relative boundary of $H_{k,\varepsilon}$ in $\partial\mathrm{B}_\varepsilon$. Recall that $\{\widehat{x}_i\}_{i=1}^{J_k}$ is finite as well by Lemma 4.2 (iv). For notational simplicity, we skip the dependence on ϵ.

Recalling the definition of W_k in (5.16), the following crucial lemma states that $(\pi_{\lambda_k}\circ\Psi_k(D_k))\cup W_k$ does not intersect the plane $\{\varepsilon\}\times\mathbb{R}^2$ in a set of positive $\mathcal{H}^2$-measure.

[29] Looking at the plane $\{\varepsilon\}\times\mathbb{R}^2$ from the side $t>\varepsilon$.

Lemma 8.1. *The rectifiable set* $(\pi_{\lambda_k} \circ \Psi_k(D_k)) \cup W_k$ *satisfies*

$$\mathcal{H}^2\Big(\big(\pi_{\lambda_k} \circ \Psi_k(D_k) \cup W_k\big) \cap \{t = \varepsilon\}\Big) = 0.$$

Proof. It is sufficient to show that $\mathcal{H}^2((\pi_{\lambda_k} \circ \Psi_k(D_k)) \cap \{t = \varepsilon\}) = 0$ and $\mathcal{H}^2(W_k \cap \{t = \varepsilon\}) = 0$. To show the first equality, suppose $\mathcal{H}^2(\pi_{\lambda_k} \circ \Psi_k(D_k) \cap \{t = \varepsilon\}) > 0$. Since $\mathrm{Lip}(\pi_{\lambda_k}) = 1$ and π_{λ_k} takes the plane $\{t = \varepsilon\}$ into itself, we have $\mathcal{H}^2(\Psi_k(D_k) \cap \{t = \varepsilon\}) > 0$. Again, being Ψ_k Lipschitz continuous, we deduce that $\Psi_k^{-1}(\Psi_k(D_k) \cap \{t = \varepsilon\})$ has positive measure. But $\Psi_k^{-1}(\Psi_k(D_k) \cap \{t = \varepsilon\}) \subset \Psi_k^{-1}(\{t = \varepsilon\}) = \partial \mathrm{B}_\varepsilon$ which has obviously $\mathcal{H}^2$ null measure.

Let us prove that $\mathcal{H}^2(W_k \cap \{t = \varepsilon\}) = 0$. Recalling (see (5.16)) that $W_k = \tau([1 - \lambda_k, 1 - \lambda'_k] \times \gamma_k)$ with $\gamma_k := \pi_{\lambda_k} \circ \Psi_k(\partial D_k)$, and since $\tau(\cdot, z)$ in (5.13) does not change the axial coordinate of z, we see[30] that $\tau([1 - \lambda_k, 1 - \lambda'_k] \times \gamma_k) \cap \{t = \varepsilon\}$ has positive $\mathcal{H}^2$ measure only if $\gamma_k \cap \{t = \varepsilon\}$ has positive $\mathcal{H}^1$ measure. Again, since also π_{λ_k} does not change the axial coordinate, as before this happens only if $\Psi_k^{-1}(\widehat{\gamma}_k \cap \{t = \varepsilon\})$ has positive $\mathcal{H}^1$-measure, where $\widehat{\gamma}_k := \Psi_k(\partial D_k)$; by Lemma 4.2, this is not possible, since we know that $\widehat{\gamma}_k \cap \{t = \varepsilon\} = \{\Psi_k(\widehat{x}_i)\}$ (see (8.2)), and then $\Psi_k^{-1}(\widehat{\gamma}_k \cap \{t = \varepsilon\}) = \{\widehat{x}_i\}$ which is a finite set. □

We recall from (5.6) and (5.22) that

$$\widehat{\mathfrak{D}}_k = (\pi_{\lambda_k} \circ \Psi_k)_\sharp [\![D_k]\!] + \mathcal{W}_k. \tag{8.3}$$

An immediate consequence of Lemma 8.1, formula (8.3), and the fact that $\widehat{\mathfrak{D}}_k$ is boundaryless in $C_l(1 - \lambda'_k)$, is the following:

Corollary 8.2. *We have* $\widehat{\mathfrak{D}}_k \llcorner \{t = \varepsilon\} = 0$. *In particular*

$$\partial(\widehat{\mathfrak{D}}_k \llcorner \{\varepsilon < t < l\}) \llcorner (\{t = \varepsilon\}) = -\partial(\widehat{\mathfrak{D}}_k \llcorner \{-1 < t < \varepsilon\}) \llcorner (\{t = \varepsilon\}) \qquad \textit{in } C_l(1 - \lambda'_k).$$

If $\{E_{k,i}\}_{i \in \mathbb{N}}$, $E_{k,i} \in C_l$ are the sets which we have symmetrized (see (5.30)), $\mathbb{S}(E_k)$ is the symmetrized set, and $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ is the symmetrized current, we have to understand the behaviour of $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ on $\{\varepsilon\} \times \mathbb{R}^2$. We have observed that $\widehat{\mathfrak{D}}_k \llcorner (\{\varepsilon\} \times \mathbb{R}^2) = 0$ because $\mathcal{H}^2\left((\pi_{\lambda_k} \circ \Psi_k(D_k) \cup W_k) \cap \{\varepsilon\} \times \mathbb{R}^2\right) = 0$. The same holds for the symmetrized current, as a particular consequence of Lemma 3.4:

$$\mathbb{S}(\widehat{\mathfrak{D}}_k) \llcorner (\{\varepsilon\} \times \mathbb{R}^2) = 0.$$

8.1 Description of the boundary of the current $\mathbb{S}(\widehat{\mathfrak{D}}_k) \llcorner ((-1, \varepsilon) \times B_{1-\lambda'_k})$

Our first aim is to describe the boundary of $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ on $\{\varepsilon\} \times \mathbb{R}^2$ (Corollary 8.5). To do so, let us recall that π_{λ_k} is given in Definition 5.3 and that the points x_i are defined in (8.2).

Definition 8.3 (**The current** $\mathcal{H}_{k,\varepsilon}$)**.** *Recalling* (8.1), *we set*

$$\mathcal{H}_{k,\varepsilon} := (\pi_{\lambda_k} \circ \Psi_k)_\sharp [\![H_{k,\varepsilon}]\!] \in \mathcal{D}_1(\{\varepsilon\} \times B_1), \tag{8.4}$$

where $H_{k,\varepsilon}$ *is oriented counterclockwise.*

Let us denote by $\{\widetilde{x}_i\} \subseteq \{x_i\}$ the points which represent the support of the current $\partial [\![H_{k,\varepsilon}]\!]$. We can consider the orthogonal projection[31] onto the lateral boundary of $C_l(1 - \lambda'_k)$, and we denote

[30] For instance, using the coarea formula.

[31] Defined at least in the region $C_l(1 - \lambda'_k) \setminus C_l(1 - \lambda_k)$.

by $L_{k,i}$ the segment connecting $\pi_{\lambda_k}(\Psi_k(\widetilde{x}_i))$ (which belongs to the lateral boundary of $C_l(1-\lambda_k)$) to the image point of $\Psi_k(\widetilde{x}_i)$ through this projection.

We consider the 1-integral current in $\{\varepsilon\} \times B_{1-\lambda'_k}$ given by

$$\mathcal{H}_{k,\varepsilon} + \sum_i [\![L_{k,i}]\!] \in \mathcal{D}_1(\{\varepsilon\} \times B_{1-\lambda'_k}), \tag{8.5}$$

where $[\![L_{k,i}]\!]$ are the integrations over the segments $L_{k,i}$ taken with suitable orientation in order that

$$\partial\Big(\mathcal{H}_{k,\varepsilon} + \sum_i [\![L_{k,i}]\!]\Big) = 0 \qquad \text{in } \{\varepsilon\} \times B_{1-\lambda'_k}. \tag{8.6}$$

Before stating the following crucial lemma, we recall that the current $\widehat{\mathfrak{D}}_k$ is defined in C_l but is supported in $[0,l] \times \overline{B}_{1-\lambda'_k}$.

Lemma 8.4 (Boundary of $\widehat{\mathfrak{D}}_k \llcorner ((-1,\varepsilon) \times \mathbb{R}^2)$ in $\{\varepsilon\} \times B_{1-\lambda'_k}$). *We have*

$$\partial\big(\widehat{\mathfrak{D}}_k \llcorner ((-1,\varepsilon) \times \mathbb{R}^2)\big) = \mathcal{H}_{k,\varepsilon} + \sum_i [\![L_{k,i}]\!] \qquad \text{in } \mathcal{D}_1(\{\varepsilon\} \times B_{1-\lambda'_k}). \tag{8.7}$$

Proof. We recall that

$$\widehat{\mathfrak{D}}_k = \mathfrak{D}_k + \mathcal{W}_k,$$

where $\mathfrak{D}_k$ is defined in (5.6) and, by (5.17), $\mathcal{W}_k = \widetilde{\tau}_\sharp[\![[1-\lambda_k, 1-\lambda'_k] \times \partial D_k]\!]$. Observe that

$$\partial\left(\mathcal{W}_k \llcorner ((-1,\varepsilon) \times \mathbb{R}^2)\right) = \sum_i [\![L_{k,i}]\!] \qquad \text{in the annulus } \{\varepsilon\} \times (B_{1-\lambda'_k} \setminus \overline{B}_{1-\lambda_k}).$$

Indeed, this follows from the definition of $L_{k,i}$, the equality[32]

$$\begin{aligned}\mathcal{W}_k \llcorner ((-1,\varepsilon) \times \mathbb{R}^2) =& \widetilde{\tau}_\sharp[\![[1-\lambda_k, 1-\lambda'_k] \times (\mathrm{B}_\varepsilon \cap \partial D_k)]\!] \\ =& \widetilde{\tau}_\sharp[\![[1-\lambda_k, 1-\lambda'_k] \times \partial(D_k \cap \mathrm{B}_\varepsilon)]\!] - \widetilde{\tau}_\sharp[\![[1-\lambda_k, 1-\lambda'_k] \times H_{k,\varepsilon}]\!],\end{aligned}$$

and (8.6). Moreover, from (5.6),

$$\begin{aligned}\partial\Big(\mathfrak{D}_k \llcorner ((-1,\varepsilon) \times \mathbb{R}^2)\Big) \llcorner (\{\varepsilon\} \times B_1) &= \partial\Big(((\pi_{\lambda_k} \circ \Psi_k)_\sharp [\![D_k]\!]) \llcorner ((-1,\varepsilon) \times \mathbb{R}^2)\Big) \llcorner (\{\varepsilon\} \times B_1) \\ &= \partial\Big((\pi_{\lambda_k} \circ \Psi_k)_\sharp [\![D_k \cap \mathrm{B}_\varepsilon]\!]\Big) \llcorner (\{\varepsilon\} \times B_1) \\ &= \Big((\pi_{\lambda_k} \circ \Psi_k)_\sharp \partial [\![D_k \cap \mathrm{B}_\varepsilon]\!]\Big) \llcorner (\{\varepsilon\} \times B_1) \\ &= \mathcal{H}_{k,\varepsilon} \qquad \text{on } \{\varepsilon\} \times B_1,\end{aligned}$$

where in the last equality we use[33] $[\![\partial(D_k \cap \mathrm{B}_\varepsilon)]\!] = [\![D_k \cap \partial\mathrm{B}_\varepsilon]\!] = [\![H_{k,\varepsilon}]\!]$ on $\partial\mathrm{B}_\varepsilon$. □

Thanks to Corollary 5.17, both $\widehat{\mathfrak{D}}_k$ and $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ have no boundary in $(-\infty, l) \times B_{1-\lambda'_k}$. Now, we need to describe the boundary of the symmetrized current $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ restricted to $(-1,\varepsilon) \times B_{1-\lambda'_k}$, see (8.11). We recall the definitions of $\mathcal{X}_k$ and $\mathcal{Y}_k$ in (5.27) and (5.26), and for $t \in (-1,\varepsilon]$ and $\rho \in (1-\lambda_k, 1-\lambda'_k)$ the function $\Theta_k(t,\rho)$ defined in (6.10). Also in this case

$$\Theta_k(t,\rho) = \Theta_k(t,\varrho) \qquad \text{for all } \rho, \varrho \in (1-\lambda_k, 1-\lambda'_k).$$

[32]Notice that $\partial(D_k \cap \mathrm{B}_\varepsilon) = (\partial D_k \cap \mathrm{B}_\varepsilon) \cup (\partial D_k \cap \partial\mathrm{B}_\varepsilon) \cup (D_k \cap \partial\mathrm{B}_\varepsilon)$; recall also that, by Lemma 4.2 (iv), $\partial D_k \cap \partial\mathrm{B}_\varepsilon$ consists of a finite set of points.

[33]Here we take the boundary of D_k in $\partial\mathrm{B}_\varepsilon$ in the sense of currents, so that isolated points are neglected.

In cylindrical coordinates, if

$$X_1 := (\varepsilon, 1-\lambda_k, \Theta_k(\varepsilon, 1-\lambda_k)/2), \qquad X_2 := (\varepsilon, 1-\lambda_k, -\Theta_k(\varepsilon, 1-\lambda_k)/2),$$

we denote the two 1-currents

$$\mathbb{S}(L)_1 := \tau(\cdot, X_1)_\sharp[\![(1-\lambda_k, 1-\lambda_k')]\!] \qquad \text{and} \qquad \mathbb{S}(L)_2 := \tau(\cdot, X_2)_\sharp[\![(1-\lambda_k, 1-\lambda_k')]\!], \tag{8.8}$$

see Fig. 7. Set

$$Y_1 = \tau(1-\lambda_k', X_1), \qquad Y_2 = \tau(1-\lambda_k', X_2). \tag{8.9}$$

We know, by construction and definition of Θ_k, that

$$\partial\big(\mathbb{S}(\widehat{\mathfrak{D}}_k)\llcorner((-1,\varepsilon)\times(B_{1-\lambda_k'}\setminus \overline{B}_{1-\lambda_k}))\big)\llcorner(\{\varepsilon\}\times\mathbb{R}^2) = \mathbb{S}(L)_1 - \mathbb{S}(L)_2$$

in $\mathcal{D}_1(\{\varepsilon\}\times(B_{1-\lambda_k'}\setminus\overline{B}_{1-\lambda_k}))$. We define

$$\mathbb{S}(\mathcal{H}_{k,\varepsilon}) := \partial\big(\mathbb{S}(\widehat{\mathfrak{D}}_k)\llcorner((-1,\varepsilon)\times B_{1-\lambda_k'})\big)\llcorner(\{\varepsilon\}\times\mathbb{R}^2) - \mathbb{S}(L)_1 + \mathbb{S}(L)_2 \tag{8.10}$$

in $\mathcal{D}_1(\{\varepsilon\}\times B_{1-\lambda_k})$, see again Fig. 7. With these definitions at our disposal we can now write

$$\begin{aligned}
&\partial\Big(\mathbb{S}(\widehat{\mathfrak{D}}_k)\llcorner((-1,\varepsilon)\times B_{1-\lambda_k'})\Big)\\
=&\mathbb{S}(\mathcal{H}_{k,\varepsilon}) + \mathbb{S}(L)_1 - \mathbb{S}(L)_2 + \partial\big(\mathbb{S}(\widehat{\mathfrak{D}}_k)\llcorner\{t\in(-1,\varepsilon)\}\big)\llcorner(\{-1\}\times B_{1-\lambda_k'})\\
=&\mathbb{S}(\mathcal{H}_{k,\varepsilon}) + \mathbb{S}(L)_1 - \mathbb{S}(L)_2.
\end{aligned} \tag{8.11}$$

Here we have used once again that $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ is supported in $[0,l]\times B_1$, and then its boundary on $\{t=-1\}$ is always null.

We can clarify the meaning of the last term in formula (8.11).

Corollary 8.5. *We have*

$$\mathbb{S}(\mathcal{H}_{k,\varepsilon}) + \mathbb{S}(L)_1 - \mathbb{S}(L)_2 = -\partial\big(\mathbb{S}(\widehat{\mathfrak{D}}_k)\llcorner((\varepsilon,l)\times B_{1-\lambda_k'})\big)\llcorner(\{\varepsilon\}\times\mathbb{R}^2).$$

Proof. It follows from (8.11), Corollary 8.2, and Lemma 3.4. □

8.2 Construction of the current $\mathcal{V}_{k,\varepsilon}$

Let $\Pi_\varepsilon : \mathbb{R}^3 \to \{\varepsilon\}\times\mathbb{R}^2$ be the orthogonal projection on $\{\varepsilon\}\times\mathbb{R}^2$.

Definition 8.6. *We set*

$$\mathcal{V}_{k,\varepsilon} := (\Pi_\varepsilon)_\sharp\Big(\mathbb{S}(\widehat{\mathfrak{D}}_k)\llcorner((-1,\varepsilon)\times B_{1-\lambda_k'})\Big)\in\mathcal{D}_2(C_l). \tag{8.12}$$

Lemma 8.7. *We have*

$$|[\![G_{u_k}]\!]|_{(D_k\cap \mathrm{B}_\varepsilon)\times\mathbb{R}^2} \geq |\mathcal{V}_{k,\varepsilon}| - 2\pi(\lambda_k - \lambda_k').$$

Proof. By (8.12), since $\mathrm{Lip}(\Pi_\varepsilon)=1$, we have, using (8.3), (5.17),

$$\begin{aligned}
|\mathcal{V}_{k,\varepsilon}| =&|\mathcal{V}_{k,\varepsilon}|_{(-1,\varepsilon)\times(B_{1-\lambda_k'}\setminus B_{1-\lambda_k})} + |\mathcal{V}_{k,\varepsilon}|_{(-1,\varepsilon)\times B_{1-\lambda_k}}\\
\leq&|\mathcal{V}_{k,\varepsilon}|_{(-1,\varepsilon)\times(B_{1-\lambda_k'}\setminus B_{1-\lambda_k})} + |\widehat{\mathfrak{D}}_k|_{(-1,\varepsilon)\times B_{1-\lambda_k}}\\
=&|\mathcal{V}_{k,\varepsilon}|_{(-1,\varepsilon)\times(B_{1-\lambda_k'}\setminus B_{1-\lambda_k})} + |\mathfrak{D}_k|_{(-1,\varepsilon)\times B_{1-\lambda_k}}\\
\leq&2\pi(\lambda_k-\lambda_k') + |[\![G_{u_k}]\!]|_{(D_k\cap\mathrm{B}_\varepsilon)\times\mathbb{R}^2},
\end{aligned} \tag{8.13}$$

where we have also used a localized version of (5.34) in $(-1,\varepsilon)\times B_{1-\lambda_k}$. □

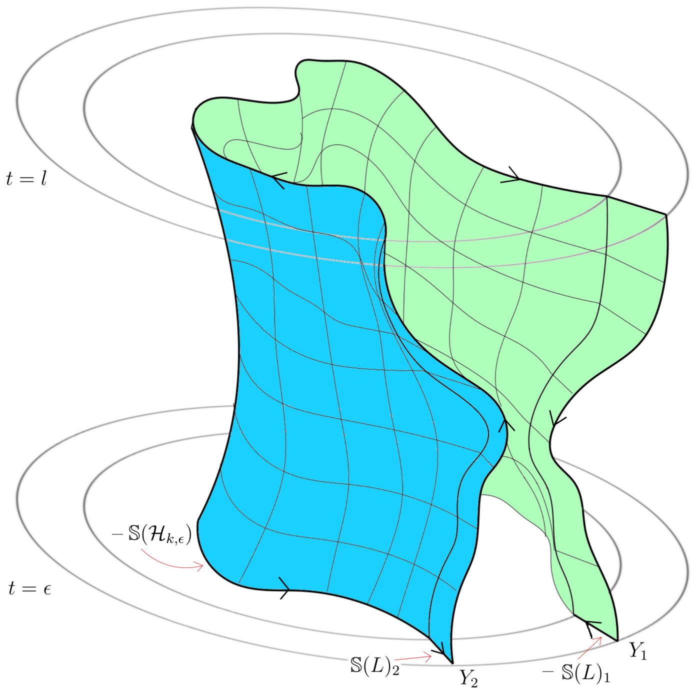

Figure 7: The current $\mathbb{S}(\widehat{\mathfrak{D}}_k)\llcorner((\varepsilon, l) \times B_{1-\lambda'_k})$ is depicted. At $t = \varepsilon$ we emphasized the various objects composing its boundary, taken with their orientation.

By Corollary 8.5 it holds[34]

$$\partial \mathcal{V}_{k,\varepsilon} = \mathbb{S}(\mathcal{H}_{k,\varepsilon}) + \mathbb{S}(L)_1 - \mathbb{S}(L)_2 \qquad \text{in } \{\varepsilon\} \times B_{1-\lambda'_k}. \tag{8.14}$$

Clearly the above current is boundaryless in $\{\varepsilon\} \times B_{1-\lambda'_k}$; more precisely it is an oriented curve connecting Y_2 to Y_1 (defined in (8.9)) as soon as $Y_2 \neq Y_1$, with $\mathbb{S}(\mathcal{H}_{k,\varepsilon})$ clockwise oriented[35]. If we extend $\mathcal{V}_{k,\varepsilon}$ to 0 on the whole plane $\{\varepsilon\} \times \mathbb{R}^2$ (keeping the same notation) we have

$$\partial \mathcal{V}_{k,\varepsilon} = \mathcal{L}_k + \mathbb{S}(\mathcal{H}_{k,\varepsilon}) + \mathbb{S}(L)_1 - \mathbb{S}(L)_2 \qquad \text{on } \{\varepsilon\} \times \mathbb{R}^2, \tag{8.15}$$

for some current $\mathcal{L}_k$ supported on $\{\varepsilon\} \times \partial B_{1-\lambda'_k}$ and whose boundary is two deltas, with suitable signs, on Y_1 and Y_2. In particular $\mathcal{L}_k$ is the integration between Y_1 to Y_2 on the circle $\{\varepsilon\} \times \partial B_{1-\lambda'_k}$.

However there are two arcs which connect these two points, namely (in cylindrical coordinates)

$$\{\varepsilon\} \times \{1 - \lambda'_k\} \times \left[\frac{-\Theta_k(\varepsilon, 1 - \lambda'_k)}{2}, \frac{\Theta_k(\varepsilon, 1 - \lambda'_k)}{2} \right] \tag{8.16}$$

oriented clockwise and

$$\Big(\{\varepsilon\} \times \partial B_{1-\lambda'_k} \Big) \setminus \left\{ \{\varepsilon\} \times \{1 - \lambda'_k\} \times \left[\frac{-\Theta_k(\varepsilon, 1 - \lambda'_k)}{2}, \frac{\Theta_k(\varepsilon, 1 - \lambda'_k)}{2} \right] \right\} \tag{8.17}$$

oriented counterclockwise. We have to identify $\mathcal{L}_k$ with the integration over one of these two arcs.

[34] Recall that $\partial(\Pi_\varepsilon)_\sharp\Big(\mathbb{S}(\widehat{\mathfrak{D}}_k)\llcorner\{t < \varepsilon\}\Big) = (\Pi_\varepsilon)_\sharp\partial\Big(\mathbb{S}(\widehat{\mathfrak{D}}_k)\llcorner\{t < \varepsilon\}\Big)$ and that the map Π_ε does not move the plane where $\partial(\mathbb{S}(\widehat{\mathfrak{D}}_k)\llcorner\{t < \varepsilon\})$ is supported.

[35] When looking at the plane $\{\varepsilon\} \times \mathbb{R}^2$ from $t > \varepsilon$.

Proposition 8.8. *$\mathcal{L}_k$ is the counterclockwise integration over the arc connecting Y_1 and Y_2 given by* (8.17)*.*

Before proving this proposition we anticipate a useful observation.

Remark 8.9. We set

$$\mathbb{S}(E_k)_\varepsilon := \mathbb{S}(E_k) \cap \{t = \varepsilon\}.$$

Since $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ is the boundary of the integration over $\mathbb{S}(E_k)$, the current $\mathbb{S}(\widehat{\mathfrak{D}}_k) \llcorner ((-1, \varepsilon) \times B_{1-\lambda'_k}) + [\![\mathbb{S}(E_k)_\varepsilon]\!]$ is boundaryless in $\mathcal{D}_2(C_l(1 - \lambda'_k))$ (with $\mathbb{S}(E_k)_\varepsilon$ suitably oriented). It follows, invoking Corollary 8.5, that

$$\partial [\![\mathbb{S}(E_k)_\varepsilon]\!] = -\mathbb{S}(\mathcal{H}_{k,\varepsilon}) - \mathbb{S}(L)_1 + \mathbb{S}(L)_2 \qquad \text{in } \{\varepsilon\} \times B_{1-\lambda'_k}.$$

The fact that

$$\partial \mathcal{V}_{k,\varepsilon} = \mathcal{L}_k + \mathbb{S}(\mathcal{H}_{k,\varepsilon}) + \mathbb{S}(L)_1 - \mathbb{S}(L)_2 \qquad \text{in } \{\varepsilon\} \times \mathbb{R}^2, \tag{8.18}$$

(where $\mathcal{L}_k$ is as in Proposition 8.8) means that $\mathcal{V}_{k,\varepsilon}$ is the integration over the set

$$B_{1-\lambda'_k} \setminus \mathbb{S}(E_k)_\varepsilon.$$

In particular $\mathcal{V}_{k,\varepsilon}$ has coefficient 1 in $B_{1-\lambda'_k} \setminus \mathbb{S}(E_k)_\varepsilon$ and zero in $\mathbb{S}(E_k)_\varepsilon$. On the other hand, if $\mathcal{L}_k$ were the integration over (8.16) oriented clockwise, then we would have that $\mathcal{V}_{k,\varepsilon}$ had coefficient -1 in $\mathbb{S}(E_k)_\varepsilon$ and 0 in $B_{1-\lambda'_k} \setminus \mathbb{S}(E_k)_\varepsilon$.

We can now prove Proposition 8.8.

Proof. Appealing to Remark 8.9, it is sufficient to show that the coefficient of $\mathcal{V}_{k,\varepsilon}$ is 1 in $B_{1-\lambda'_k} \setminus \mathbb{S}(E_k)_\varepsilon$. Equivalently we can show that this coefficient is zero in $B_{1-\lambda'_k} \cap \mathbb{S}(E_k)_\varepsilon$.

Let us recall, by definitions (5.26) and (5.27),

$$(\mathcal{Y}_k)_t = \widetilde{\tau}_\sharp [\![[1 - \lambda_k, 1 - \lambda'_k] \times ((\Omega \setminus D_k) \cap \partial \mathrm{B}_t)]\!] \qquad \text{for a.e. } t \in (0, \varepsilon], \tag{8.19}$$

$$(\mathcal{X}_k)_t = [\![\{t\} \times (B_{1-\lambda'_k} \setminus \overline{B}_{1-\lambda_k})]\!] - (\mathcal{Y}_k)_t \qquad \text{for a.e. } t \in (0, \varepsilon]. \tag{8.20}$$

Recalling Lemma 4.2(i), we now divide our analysis in two cases:

(1) $|u_k(0)| < 1 - \lambda_k$.

(2) $|u_k(0)| > 1 - \lambda_k$.

We notice that, in both cases, by continuity of u_k, for all $\delta \in (0, 1)$ there is $t_k^\delta > 0$ such that

$$u_k(\mathrm{B}_t) \subset B_\delta(u_k(0)) \qquad \forall t \in (0, t_k^\delta]. \tag{8.21}$$

Case (1): If δ is sufficiently small, we can also assume that

$$B_\delta(u_k(0)) \subset B_{1-\lambda_k}(0), \tag{8.22}$$

and therefore

$$u_k(\mathrm{B}_t) \subset B_{1-\lambda_k}(0) \qquad \forall t \in (0, t_k^\delta]. \tag{8.23}$$

In this case it turns out that if $t \le t_k^\delta$ then the current $(\mathcal{Y}_k)_t$ in (8.19) is null, because $|u_k|^+(t) < 1-\lambda_k$, hence $(\Omega \setminus D_k) \cap \partial \mathrm{B}_t = \emptyset$ by (6.2). In particular, by (8.20),

$$(\mathcal{X}_k)_t = [\![\{t\} \times (B_{1-\lambda'_k} \setminus \overline{B}_{1-\lambda_k})]\!] \qquad \text{for a.e. } t \le t_k^\delta.$$

Eventually, since $\overline{\mathrm{B}}_t \subset D_k$ for any $t \in [0, t_k^\delta]$, from (8.21), we also deduce $u_k(\overline{\mathrm{B}}_{t_k^\delta} \cap D_k) = u_k(\overline{\mathrm{B}}_{t_k^\delta}) \subset B_\delta(u_k(0))$, so that

$$\Psi_k(D_k) \cap ([0, t_k^\delta] \times B_1) = \pi_{\lambda_k} \circ \Psi_k(D_k) \cap ([0, t_k^\delta] \times B_1) \subset [0, t_k^\delta] \times B_\delta(u_k(0)). \tag{8.24}$$

Now, consider the decomposition (5.30) of $\mathcal{E}_k$. By the crucial identification (5.29) and (8.22) we infer that there must be a set $E_{k,h} \in \{E_{k,i}\}_{i\in\mathbb{N}}$ with[36]

$$\begin{aligned}\mathcal{X}_k \llcorner ((-1, t_k^\delta) \times B_{1-\lambda'_k}) &= [\![(-1, t_k^\delta) \times (B_{1-\lambda'_k} \setminus \overline{B}_{1-\lambda_k})]\!] \\ &= [\![E_{k,h} \cap \big((-1, t_k^\delta) \times (B_{1-\lambda'_k} \setminus \overline{B}_{1-\lambda_k})\big)]\!].\end{aligned}$$

Therefore

$$E_{k,h} \cap \big((-1, t_k^\delta) \times (B_{1-\lambda'_k} \setminus \overline{B}_{1-\lambda_k})\big) = (-1, t_k^\delta) \times (B_{1-\lambda'_k} \setminus \overline{B}_{1-\lambda_k}).$$

This has the following consequence: denoting as usual $\mathbb{S}(E_{k,h})$ the cylindrical symmetrization of $E_{k,h}$ we infer

$$\mathbb{S}(E_{k,h}) \cap \big((-1, t_k^\delta) \times (B_{1-\lambda'_k} \setminus \overline{B}_{1-\lambda_k})\big) = (-1, t_k^\delta) \times (B_{1-\lambda'_k} \setminus \overline{B}_{1-\lambda_k}),$$

and since $\mathbb{S}(E_{k,h}) \subset \mathbb{S}(E_k)$ we also have

$$\mathbb{S}(E_k) \cap \big((-1, t_k^\delta) \times (B_{1-\lambda'_k} \setminus \overline{B}_{1-\lambda_k})\big) = (-1, t_k^\delta) \times (B_{1-\lambda'_k} \setminus \overline{B}_{1-\lambda_k}). \tag{8.25}$$

We now consider two subcases.

(1A) $\mathcal{H}^2\Big(\big(\{\varepsilon\} \times (B_{1-\lambda'_k} \setminus B_{1-\lambda_k})\big) \setminus \mathbb{S}(E_k)_\varepsilon\Big) > 0$. To conclude the proof it is sufficient to show that

$$\text{the multiplicity of } \mathcal{V}_{k,\varepsilon} \text{ on } \big(\{\varepsilon\} \times (B_{1-\lambda'_k} \setminus B_{1-\lambda_k})\big) \setminus \mathbb{S}(E_k)_\varepsilon \text{ is } 1, \tag{8.26}$$

because $(\{\varepsilon\} \times B_{1-\lambda'_k}) \setminus \mathbb{S}(E_k)_\varepsilon$ is, by definition, outside the finite perimeter set $\mathbb{S}(E_k)$.

We argue by slicing, and consider the lines $l_{\rho,\theta}$ in $\mathbb{R}^3$ given by $l_{\rho,\theta} = \mathbb{R} \times \{\rho\} \times \{\theta\}$, with ρ and θ fixed. Consider any point p_0 of coordinates $\rho \in (1-\lambda_k, 1-\lambda'_k)$ and $\theta \in (-\pi, \pi]$ such that

$$p_0 \in (\{\varepsilon\} \times B_{1-\lambda'_k}) \setminus \mathbb{S}(E_k)_\varepsilon. \tag{8.27}$$

For a.e. such $\rho \in (1-\lambda_k, 1-\lambda'_k)$ and $\theta \in (-\pi, \pi]$ the slice of $\widehat{\mathfrak{D}}_k \llcorner ((-1, \varepsilon) \times B_{1-\lambda'_k})$ with respect to this line is the sum of some Dirac deltas with suitable signs, according to the orientation of $\widehat{\mathfrak{D}}_k$. Indeed $\widehat{\mathfrak{D}}_k$ is the integration over the boundary of the finite perimeter set $\mathbb{S}(E_k)$, so it turns out that, for a.e. $\rho \in (1-\lambda_k, 1-\lambda'_k)$ and $\theta \in (-\pi, \pi]$ the slice of $[\![\mathbb{S}(E_k)]\!]$ with respect to the line $l_{\rho,\theta}$ is exactly

$$[\![\mathbb{S}(E_k)_{\rho,\theta}]\!] = [\![\mathbb{S}(E_k) \cap l_{\rho,\theta}]\!], \tag{8.28}$$

[36]Since the decomposition in (5.30) is done in undecomposable components, such a set is unique.

that is the integration over some disjoint intervals. If $p_1, p_2, \dots p_m$ are the intervals endpoints (written in order[37] on $l_{\rho,\theta}$) and if we assume that the last interval between the points p_1 and $p_0 = (\varepsilon, \rho, \theta)$ is outside $\mathbb{S}(E_k)$, then it results

$$\partial[\![\mathbb{S}(E_k)_{\rho,\theta}]\!] = -\sum_{\substack{i>0 \\ i \text{ even}}} \delta_{p_i} + \sum_{\substack{i>0 \\ i \text{ odd}}} \delta_{p_i}. \tag{8.29}$$

If instead the last interval $[p_1, p_0]$ is inside $\mathbb{S}(E_k)$ we have

$$\partial[\![\mathbb{S}(E_k)_{\rho,\theta}]\!] = \sum_{\substack{i>0 \\ i \text{ even}}} \delta_{p_i} - \sum_{\substack{i>0 \\ i \text{ odd}}} \delta_{p_i}. \tag{8.30}$$

Let us now prove claim (8.26). We have obtained that, for a.e. $\rho \in (1-\lambda_k, 1-\lambda'_k)$ and any $\theta \in (-\pi, \pi]$ such that (8.27) holds, the slice $\partial[\![\mathbb{S}(E_k)_{\rho,\theta}]\!]$ is the sum in (8.29), and thanks to (8.25) we deduce that the total number of points involved in (8.29) must be odd. As a consequence, the push-forward by Π_ε of $\partial[\![\mathbb{S}(E_k)_{\rho,\theta}]\!]$ is a Dirac delta with coefficient -1. Since this holds for a.e. $\rho \in (1-\lambda_k, 1-\lambda'_k)$ and any $\theta \in (-\pi, \pi]$, the conclusion follows.

(1B) Suppose $\mathcal{H}^2\Big(\big(\{\varepsilon\}\times(B_{1-\lambda'_k}\setminus B_{1-\lambda_k})\big)\setminus\mathbb{S}(E_k)_\varepsilon\Big) = 0$. In this case we pass to the complementary set; namely, if $\{\varepsilon\} \times (B_{1-\lambda'_k} \setminus B_{1-\lambda_k}) = \mathbb{S}(E_k)_\varepsilon \cap \big(\{\varepsilon\} \times (B_{1-\lambda'_k} \setminus B_{1-\lambda_k})\big)$, up to $\mathcal{H}^2$-negligible sets, we show that the multiplicity of $\mathcal{V}_{k,\varepsilon}$ on this set is null. To do so it is sufficient to repeat the slicing argument above for a.e. (ρ, θ) such that $p_0 = (\varepsilon, \rho, \theta) \in \{\varepsilon\} \times (B_{1-\lambda'_k} \setminus B_{1-\lambda_k})$. For these points (8.30) takes place, since by (8.25) the number of points involved in the sum is even. The conclusion follows.

Case (2): Choosing $\delta \in (0,1)$ small enough,

$$\Psi_k(\mathrm{B}_{t_k^\delta}) \subset [0, t_k^\delta] \times B_\delta(u_k(0)), \tag{8.31}$$

and, using $|u_k(0)| > 1 - \lambda_k$,

$$\pi_{\lambda_k} \circ \Psi_k(D_k \cap \mathrm{B}_{t_k^\delta}) \subset [0, t_k^\delta] \times (B_{1-\lambda'_k} \setminus B_{1-\lambda_k}).$$

Recalling the definition of $\widehat{\mathfrak{D}}_k$, it is not difficult to see that the current $\widehat{\mathfrak{D}}_k \llcorner ((-1, t_k^\delta) \times B_{1-\lambda'_k})$ is supported in $[0, t_k^\delta] \times (\overline{B}_{1-\lambda'_k} \setminus B_{1-\lambda_k})$. By the properties of cylindrical symmetrization, we have also that $\mathbb{S}(\widehat{\mathfrak{D}}_k) \llcorner ((-1, t_k^\delta) \times B_{1-\lambda'_k})$ is supported in $[0, t_k^\delta] \times (\overline{B}_{1-\lambda'_k} \setminus B_{1-\lambda_k})$.

Obviously, being $\mathcal{Y}_k$ null on $(-1, 0) \times B_{1-\lambda'_k}$, we have

$$\mathcal{X}_k \llcorner \Big((-1,0) \times (B_{1-\lambda'_k} \setminus B_{1-\lambda_k})\Big) = [\![(-1,0) \times (B_{1-\lambda'_k} \setminus B_{1-\lambda_k})]\!],$$

and we find a set $E_{k,h}$, such that

$$\mathcal{X}_k \llcorner \Big((-1,0) \times (B_{1-\lambda'_k} \setminus B_{1-\lambda_k})\Big) = [\![E_{k,h} \cap ((-1,0) \times (B_{1-\lambda'_k} \setminus B_{1-\lambda_k}))]\!].$$

If we pass to the symmetrized set, arguing as in case (1A), we infer

$$\mathbb{S}(E_k) \cap \big((-1,0) \times (B_{1-\lambda'_k} \setminus B_{1-\lambda_k})\big) = (-1,0) \times (B_{1-\lambda'_k} \setminus B_{1-\lambda_k}).$$

[37] p_1 is the point closer to $\{\varepsilon\} \times \mathbb{R}^2$

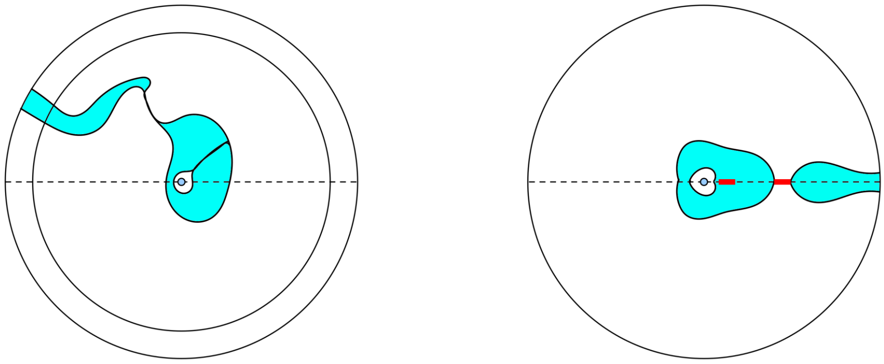

Figure 8: We represent the symmetrization of a general closed current in B_1. On the left it is visible that on two parts the curve overlaps itself in such a way that the multiplicity of the associated current is zero. In the symmetrized set, on the right picture, we have emphasized in bold the corresponding set $J^{0,2\pi}_{k,\varepsilon}$ in (7.13).

In other words, $(-1,0) \times (B_{1-\lambda'_k} \setminus B_{1-\lambda_k})$ is contained in $\mathbb{S}(E_k)$, and since the support of $\partial^*\mathbb{S}(E_k)$ does not intersect the set $(-1,0) \times B_{1-\lambda'_k}$, we infer that also

$$(-1,0) \times B_{1-\lambda'_k} \subset \mathbb{S}(E_k). \tag{8.32}$$

We now decompose $\{\varepsilon\} \times B_{1-\lambda_k}$ as

$$\{\varepsilon\} \times B_{1-\lambda_k} = \big((\{\varepsilon\} \times B_{1-\lambda_k}) \cap \mathbb{S}(E_k)_\varepsilon\big) \cup \big((\{\varepsilon\} \times B_{1-\lambda_k}) \setminus \mathbb{S}(E_k)_\varepsilon\big),$$

and one of these two sets on the right-hand side must have positive $\mathcal{H}^2$-measure. Assume that $\mathcal{H}^2((\{\varepsilon\} \times B_{1-\lambda_k}) \setminus \mathbb{S}(E_k)_\varepsilon) > 0$. Then we will prove that the multiplicity of $\mathcal{V}_{k,\varepsilon}$ on this set is 1 (if instead $(\{\varepsilon\} \times B_{1-\lambda_k}) \setminus \mathbb{S}(E_k)_\varepsilon$ has zero measure then it is sufficient to prove that $\mathcal{V}_{k,\varepsilon}$ has zero multiplicity on $(\{\varepsilon\} \times B_{1-\lambda_k}) \cap \mathbb{S}(E_k)_\varepsilon$; we drop this case being completely similar to the former).

Therefore we now proceed as in case (1), slicing with respect to lines $l_{\rho,\theta}$ with $(\varepsilon, \rho, \theta) \in \{\varepsilon\} \times (\{\varepsilon\} \times B_{1-\lambda_k}) \setminus \mathbb{S}(E_k)_\varepsilon$. Since the last point $p_0 = (\varepsilon, \rho, \theta)$ does not belong to $\mathbb{S}(E_k)_\varepsilon$, we are concerned with the sum in (8.29), and by (8.32) we infer that the number of $\{p_i\}$ involved in the sum is odd. The conclusion follows as in case (1). □

9 Gluing rectifiable sets

In this section we show that, up to adding to $\partial\mathbb{S}(E_k)$ a rectifiable set with small $\mathcal{H}^2$-measure, $\partial\mathbb{S}(E_k)$ can be described as a polar graph of a suitable modification of the function $\vartheta_{k,\varepsilon}$ over a subset[38] of the rectangle[39] $(0,l) \times [0,1] \times \{0\} \subset \mathbb{R}^3$, and with Dirichlet boundary conditions independent of k. In Section 10 we will reduce the estimate of the area of the graph of u_k to an estimate for a non-parametric Plateau problem which in turn will be independent of k.

First we remark that $\mathbb{S}(E_k) \subseteq C_l(1-\lambda'_k)$ and $\mathbb{S}(\widehat{\mathfrak{D}}_k) = \partial\mathbb{S}(\mathcal{E}_k)$ in $C_l(1-\lambda'_k)$, see (5.31). If we look at $\mathbb{S}(E_k)$ as a subset of C_l, we cannot conclude $\partial^*\mathbb{S}(E_k) = \mathbb{S}(\widehat{\mathfrak{D}}_k)$ in C_l, and $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ is not a closed current in C_l. For this reason we have to identify the boundary of $\mathbb{S}(\widehat{\mathfrak{D}}_k)$ in C_l.

[38]called $S^{(4)}_{k,\varepsilon}$, see (9.4).
[39]In cartesian coordinates.

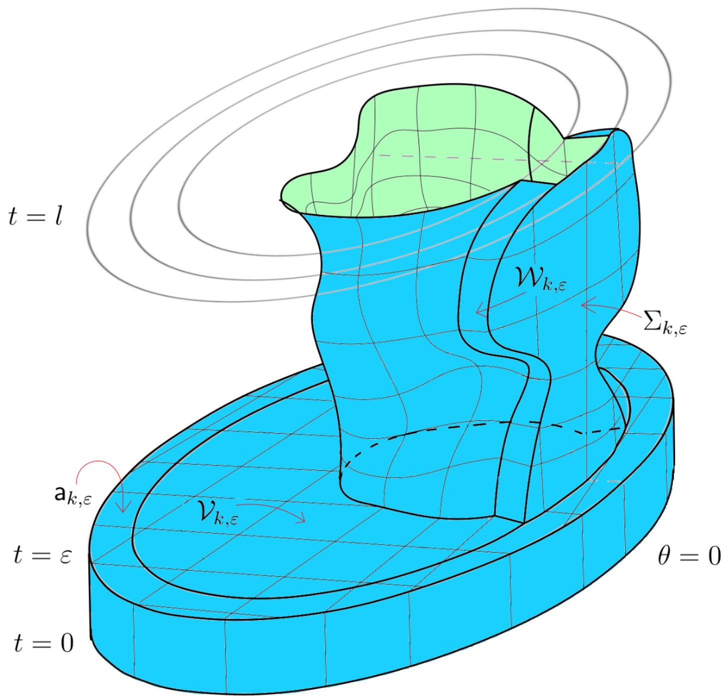

Figure 9: The largest (resp. smaller) basis circle has radius 1 (resp. $1 - \lambda'_k$). The smallest top circle has radius $1 - \lambda_k$. The symbol $\mathcal{W}_{k,\varepsilon}$ denotes the restriction of $\mathcal{W}_k$ to $\overline{C}^l_\varepsilon(1 - \lambda'_k)$, after symmetrization. Note that $\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}}$ does not include $\Sigma_{k,\varepsilon}$ and $\mathcal{V}_{k,\varepsilon}$; see (7.24).

Recalling Corollary 8.5 and Definition 7.14,

$$\begin{aligned}&\partial\Big(\big(\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} + [\![\Sigma_{k,\varepsilon}]\!]\big)\llcorner((\varepsilon, l) \times \mathbb{R}^2)\Big)\llcorner(\{\varepsilon\} \times \mathbb{R}^2)\\ =&\partial\big(\mathbb{S}(\widehat{\mathfrak{D}}_k)\llcorner((\varepsilon, l) \times \mathbb{R}^2)\big)\llcorner(\{\varepsilon\} \times \mathbb{R}^2) = -\mathbb{S}(\mathcal{H}_{k,\varepsilon}) - \mathbb{S}(L)_1 + \mathbb{S}(L)_2 - [\![\overline{Y_1Y_2}]\!],\end{aligned}$$

in $\mathcal{D}_2((-\infty, l) \times \mathbb{R}^2)$, where $[\![\overline{Y_1Y_2}]\!]$ is the integration on $\overline{Y_1Y_2}$ (see (8.16)) oriented from Y_1 to Y_2. As a consequence, from (8.18), we obtain

$$\partial\big(\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} + [\![\Sigma_{k,\varepsilon}]\!] + \mathcal{V}_{k,\varepsilon}\big)\llcorner(\{\varepsilon\} \times \mathbb{R}^2) = [\![\{\varepsilon\} \times \partial B_{1-\lambda'_k}]\!] \qquad \text{in } \mathcal{D}_2((-\infty, l) \times \mathbb{R}^2),$$

where $\partial B_{1-\lambda'_k}$ is counterclockwisely oriented.

9.1 Enforcing boundary conditions at $\{0\} \times \mathbb{R}^2$; a modification $\widehat{\vartheta}_{k,\varepsilon}$ of $\vartheta_{k,\varepsilon}$

Let $\mathsf{a}_{k,\varepsilon}$ denote the integration over the annulus $\{\varepsilon\} \times (B_1 \setminus B_{1-\lambda'_k})$, in such a way that

$$\partial\mathsf{a}_{k,\varepsilon} = [\![\{\varepsilon\} \times \partial B_1]\!] - [\![\{\varepsilon\} \times \partial B_{1-\lambda'_k}]\!],$$

see Fig. 9. Then

$$|\mathsf{a}_{k,\varepsilon}| = \pi(1 - (1 - \lambda'_k)^2) \le 2\pi\lambda'_k, \tag{9.1}$$

and

$$\partial\big(\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} + [\![\Sigma_{k,\varepsilon}]\!] + \mathcal{V}_{k,\varepsilon} + \mathsf{a}_{k,\varepsilon}\big) = [\![\{\varepsilon\} \times \partial B_1]\!] \qquad \text{in } \mathcal{D}_2((-\infty, l) \times \mathbb{R}^2).$$

Finally, we add to the current $\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} + [\![\Sigma_{k,\varepsilon}]\!] + \mathcal{V}_{k,\varepsilon} + \mathsf{a}_{k,\varepsilon}$ the integration over the lateral boundary of the cylinder $(0, \varepsilon) \times B_1$, so that the resulting current

$$\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} + [\![\Sigma_{k,\varepsilon}]\!] + \mathcal{V}_{k,\varepsilon} + \mathsf{a}_{k,\varepsilon} + [\![(0, \varepsilon) \times \partial B_1]\!] \in \mathcal{D}_2((0, l) \times \mathbb{R}^2), \tag{9.2}$$

satisfies

$$\partial\Big(\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} + [\![\Sigma_{k,\varepsilon}]\!] + \mathcal{V}_{k,\varepsilon} + \mathsf{a}_{k,\varepsilon} + [\![(0, \varepsilon) \times \partial B_1]\!]\Big) = [\![\{0\} \times \partial B_1]\!] \text{ in } \mathcal{D}_2((-\infty, l) \times \mathbb{R}^2);$$

in particular it is boundaryless in $\mathcal{D}_2((0, l) \times \mathbb{R}^2)$.

Now, we want to identify the solid region that we may call the “inside” of the current in (9.2).

Definition 9.1 (**The sets $O_{k,\varepsilon}$**). *We let*

$$O_{k,\varepsilon} := \big(\mathbb{S}(E_k) \cap ((\varepsilon, l) \times \mathbb{R}^2)\big) \cup ((0, \varepsilon] \times B_1) \subset [0, l] \times \mathbb{R}^2. \tag{9.3}$$

A direct check shows that the current built in (9.2) is the integration over the boundary of $[\![O_{k,\varepsilon}]\!]$. Indeed, by (7.24) and Definition 7.14 we see that the integration over $\mathbb{S}(E_k) \cap ((\varepsilon, l) \times \mathbb{R}^2)$ has as boundary $\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} + [\![\Sigma_{k,\varepsilon}]\!]$ in $(\varepsilon, l) \times \mathbb{R}^2$, whereas $(0, \varepsilon] \times B_1$ trivially has boundary $(0, \varepsilon) \times \partial B_1$ in $(0, \varepsilon) \times \mathbb{R}^2$. The current $\mathcal{V}_{k,\varepsilon} + \mathsf{a}_{k,\varepsilon}$ represents the boundary of $[\![O_{k,\varepsilon}]\!]$ concentrated on the plane $\{\varepsilon\} \times \mathbb{R}^2$. In turn we will see (formulas (9.6), (9.7)) that $O_{k,\varepsilon}$ is the polar subgraph of a suitable modification of $\vartheta_{k,\varepsilon}$. Thus we are going to introduce the new extra “strip” (recalling the definition of $S^{(2)}_{k,\varepsilon}$ in (7.7) and of $J_{Q_{k,\varepsilon}}$ in (7.21)):

$$\begin{aligned} S^{(4)}_{k,\varepsilon} :&= S^{(2)}_{k,\varepsilon} \cup J_{Q_{k,\varepsilon}} \cup ((\varepsilon, l) \times [1 - \lambda'_k, 1] \times \{0\}) \\ &= \{(t, \rho, \theta) : t \in (\varepsilon, l), \rho \in [|u_k|^-(t) \wedge (1 - \lambda_k), 1], \theta = 0\}, \end{aligned} \tag{9.4}$$

see Fig. 10 (and also Figs. 2, 6).

Definition 9.2 (**The function $\widehat{\vartheta}_{k,\varepsilon}$**). *We define $\widehat{\vartheta}_{k,\varepsilon} : (0, l) \times [0, 1] \times \{0\} \to \mathbb{R}$ as*

$$\widehat{\vartheta}_{k,\varepsilon} := \begin{cases} \vartheta_{k,\varepsilon} & \textit{in } (\varepsilon, l) \times [0, 1 - \lambda'_k] \times \{0\} \\ 0 & \textit{in } (\varepsilon, l) \times [1 - \lambda'_k, 1] \times \{0\} \\ \pi & \textit{in } (0, \varepsilon] \times [0, 1] \times \{0\}. \end{cases} \tag{9.5}$$

Accordingly, we extend the currents $\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}}$ and $\mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}}$ as follows: As in (7.2) we fix $\eta \in (0, \frac{\pi}{4})$, and set

$$\begin{aligned} SG^{\mathrm{pol}}_{\widehat{\vartheta}_{k,\varepsilon}} &:= \{(t, \rho, \theta) \in (0, l) \times [0, 1] \times \{0\} : \theta \in (-\eta, \widehat{\vartheta}_{k,\varepsilon}(t, \rho, 0))\}, \\ UG^{\mathrm{pol}}_{-\widehat{\vartheta}_{k,\varepsilon}} &:= \{(t, \rho, \theta) \in (0, l) \times [0, 1] \times \{0\} : \theta \in (-\widehat{\vartheta}_{k,\varepsilon}(t, \rho, 0), \eta)\}. \end{aligned}$$

Remark 9.3. By construction,

$$SG^{\mathrm{pol}}_{\widehat{\vartheta}_{k,\varepsilon}} \cap \{\theta \in (0, \pi)\} = O_{k,\varepsilon} \cap \{\theta \in (0, \pi)\}, \tag{9.6}$$

$$UG^{\mathrm{pol}}_{-\widehat{\vartheta}_{k,\varepsilon}} \cap \{\theta \in (-\pi, 0)\} = O_{k,\varepsilon} \cap \{\theta \in (-\pi, 0)\}, \tag{9.7}$$

where the set $O_{k,\varepsilon}$ is defined in (9.3).

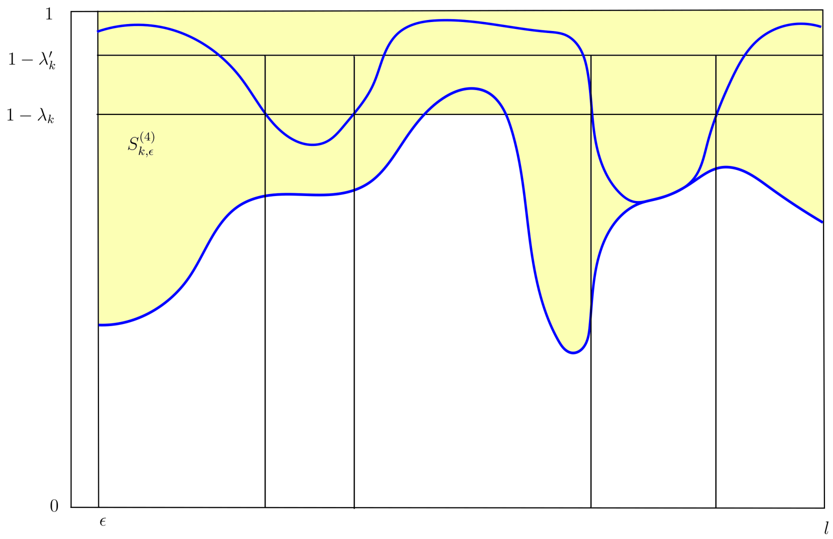

Figure 10: The graphs of the functions $|u_k|^+$ and $|u_k|^-$ and the set $S^{(4)}_{k,\varepsilon}$ in (9.4). See also Fig. 2.

The next currents are constructed to reach the segment $(0,l)\times\{1\}\times\{0\}$.

Definition 9.4 (The currents $\mathcal{G}^{(4)}_{\pm\widehat{\vartheta}_{k,\varepsilon}}$). *We define the currents*

$$\begin{aligned}\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}} &:= (\partial [\![SG^{\rm pol}_{\widehat{\vartheta}_{k,\varepsilon}}]\!]) \llcorner \{\theta\in(0,\pi)\} + [\![G^{\rm pol}_{\widehat{\vartheta}_{k,\varepsilon}\llcorner \left(\left\{\widehat{\vartheta}_{k,\varepsilon}\in\{0,\pi\}\right\}\cap S^{(4)}_{k,\varepsilon}\right)}]\!],\\ \mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}} &:= (\partial [\![UG^{\rm pol}_{-\widehat{\vartheta}_{k,\varepsilon}}]\!]) \llcorner \{\theta\in(-\pi,0)\} + [\![G^{\rm pol}_{-\widehat{\vartheta}_{k,\varepsilon}\llcorner \left(\left\{\widehat{\vartheta}_{k,\varepsilon}\in\{0,\pi\}\right\}\cap S^{(4)}_{k,\varepsilon}\right)}]\!].\end{aligned} \tag{9.8}$$

In other words, the support of $\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}$ coincides with the generalized polar graph of $\widehat{\vartheta}_{k,\varepsilon}$ restricted to $S^{(4)}_{k,\varepsilon}\times[0,\pi]$. Notice that also in this case $[\![G^{\rm pol}_{-\widehat{\vartheta}_{k,\varepsilon}\llcorner \left(\left\{\widehat{\vartheta}_{k,\varepsilon}\in\{0,\pi\}\right\}\cap S^{(4)}_{k,\varepsilon}\right)}]\!] + [\![G^{\rm pol}_{\widehat{\vartheta}_{k,\varepsilon}\llcorner \left(\left\{\widehat{\vartheta}_{k,\varepsilon}\in\{0,\pi\}\right\}\cap S^{(4)}_{k,\varepsilon}\right)}]\!] = 0$, and

$$\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}} + \mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}} = [\![\partial^* O_{k,\varepsilon}]\!] \text{ in } (0,l)\times\mathbb{R}^2. \tag{9.9}$$

Moreover, by (9.8) and (7.9),

$$|\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}| + |\mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}| = |\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}} + \mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}| + 2\mathcal{H}^2\left(\left\{\widehat{\vartheta}_{k,\varepsilon}\in\{0,\pi\}\right\}\cap S^{(4)}_{k,\varepsilon}\right). \tag{9.10}$$

Finally

$$\begin{aligned}\mathcal{G}^{(4)}_{\vartheta_{k,\varepsilon}} =& \mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + [\![(\varepsilon,l)\times[1-\lambda_k',1]\times\{0\}]\!] + [\![\Sigma_{k,\varepsilon}\cap\{0\le\theta\le\pi\}]\!] + \mathcal{V}_{k,\varepsilon}\llcorner\{0\le\theta\le\pi\}\\ &+ \mathsf{a}_{k,\varepsilon}\llcorner\{0\le\theta\le\pi\} + [\![((0,\varepsilon)\times\partial B_1)\cap\{0\le\theta\le\pi\}]\!],\end{aligned} \tag{9.11}$$

and

$$\begin{aligned}\mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}} =& \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} - [\![(\varepsilon,l)\times[1-\lambda_k',1]\times\{0\}]\!] + [\![\Sigma_{k,\varepsilon}\cap\{-\pi\le\theta\le 0\}]\!] + \mathcal{V}_{k,\varepsilon}\llcorner\{-\pi\le\theta\le 0\}\\ &+ \mathsf{a}_{k,\varepsilon}\llcorner\{-\pi\le\theta\le 0\} + [\![((0,\varepsilon)\times\partial B_1)\cap\{-\pi\le\theta\le 0\}]\!],\end{aligned}$$

so that

$$\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}} + \mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}} = \mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}} + \mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}} + [\![\Sigma_{k,\varepsilon}]\!] + \mathcal{V}_{k,\varepsilon} + \mathsf{a}_{k,\varepsilon} + [\![(0,\varepsilon)\times \partial B_1]\!].$$

Remark 9.5. The function $\widehat{\vartheta}_{k,\varepsilon}$ is defined on the whole domain $(0,l)\times[0,1]\times\{0\}$, but it might take values in $(0,\pi)$ only in $S^{(2)}_{k,\varepsilon}$, see Remark 7.5(v). Moreover, referring also to Remark 7.8, we see that the currents $\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}$ and $\mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}$ neglect the generalized polar graph of $\widehat{\vartheta}_{k,\varepsilon}$ (defined in (2.8)) on $((0,l)\times[0,1]\times\{0\})\setminus S^{(4)}_{k,\varepsilon}$, with the only exception of the "vertical" part $[\![(0,\varepsilon)\times\partial B_1]\!]$.

An important step in the proof of Theorem 1.1 is given by the next inequality.

Proposition 9.6 (Area estimate from below in terms of $|\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}|+|\mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}|$). *Let $\varepsilon\in(0,1)$ be fixed as in (4.5), (4.6), and let $n\in\mathbb{N}$. Then*

$$|[\![G_{u_k}]\!]|_{D_k\times\mathbb{R}^2} \geq |\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}| + |\mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}| - \pi\varepsilon - \frac{C}{\varepsilon n} - o_k(1), \tag{9.12}$$

for an absolute constant $C>0$, where the sequence $o_k(1)\geq 0$ depends on ε and n, and is infinitesimal as $k\to+\infty$.

Proof. By (9.11) we get

$$\begin{aligned}|\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}| \leq & |\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}}| + \lambda'_k l + |[\![\Sigma_{k,\varepsilon}\cap\{0<\theta<\pi\}]\!]| + |\mathcal{V}_{k,\varepsilon}\llcorner\{0<\theta<\pi\}| \\ & + |\mathsf{a}_{k,\varepsilon}\llcorner\{0\leq\theta\leq\pi\}| + |[\![((0,\varepsilon)\times\partial B_1)\cap\{0\leq\theta\leq\pi\}]\!]|.\end{aligned}$$

A similar estimate holds for $|\mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}|$, so that

$$|\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}| + |\mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}| \leq |\mathcal{G}^{(3)}_{\vartheta_{k,\varepsilon}}| + |\mathcal{G}^{(3)}_{-\vartheta_{k,\varepsilon}}| + |[\![\Sigma_{k,\varepsilon}]\!]| + |\mathcal{V}_{k,\varepsilon}| + |\mathsf{a}_{k,\varepsilon}| + |[\![(0,\varepsilon)\times\partial B_1]\!]| + 2\lambda'_k l.$$

Coupling the above inequality with (7.22) and (7.23) gives

$$\begin{aligned}&|\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}| + |\mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}| \\ \leq & |\mathcal{G}^{+}_{k,\varepsilon}| + |\mathcal{G}^{-}_{k,\varepsilon}| + |[\![\Sigma_{k,\varepsilon}]\!]| + |\mathcal{V}_{k,\varepsilon}| + |\mathsf{a}_{k,\varepsilon}| + |[\![(0,\varepsilon)\times\partial B_1]\!]| + 2\lambda'_k l + \frac{1}{\pi\varepsilon n} \\ \leq & \int_{D_k\cap(\Omega\setminus\overline{\mathrm{B}}_\varepsilon)} |J(\pi_{\lambda_k}\circ\Psi_k)|dx + |[\![\Sigma_{k,\varepsilon}]\!]| + |\mathcal{V}_{k,\varepsilon}| + |\mathsf{a}_{k,\varepsilon}| + |[\![(0,\varepsilon)\times\partial B_1]\!]| \\ & \quad + 2\lambda'_k l + \frac{1}{\pi\varepsilon n} + \frac{1}{n} \\ \leq & \int_{D_k\cap(\Omega\setminus\overline{\mathrm{B}}_\varepsilon)} |J(\pi_{\lambda_k}\circ\Psi_k)|dx + |[\![G_{u_k}]\!]|_{(D_k\cap\mathrm{B}_\varepsilon)\times\mathbb{R}^2} + \pi\varepsilon + \frac{C}{\varepsilon n} + o_k(1),\end{aligned}$$

where the second inequality follows from (7.12), the last inequality follows from (7.27), (9.1) and (8.13), and $C>0$ is an absolute constant. Here $o_k(1)$ is a nonnegative quantity, infinitesimal as $k\to+\infty$, and depending on ε and n. In conclusion

$$\begin{aligned}&|\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}| + |\mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}| - \pi\varepsilon - \frac{C}{\varepsilon n} - o_k(1) \\ \leq & |[\![G_{u_k}]\!]|_{(D_k\cap(\Omega\setminus\mathrm{B}_\varepsilon))\times\mathbb{R}^2} + |[\![G_{u_k}]\!]|_{(D_k\cap\mathrm{B}_\varepsilon)\times\mathbb{R}^2} = |[\![G_{u_k}]\!]|_{D_k\times\mathbb{R}^2}.\end{aligned}$$

□

10 Estimate of $|\mathcal{G}^{(4)}_{\pm\widehat{\vartheta}_{k,\varepsilon}}|$ and a minimum problem with partial free boundary

In this section we reduce the analysis of $\mathcal{G}^{(4)}_{\pm\widehat{\vartheta}_{k,\varepsilon}}$ in Definition 9.4 to a non-parametric Plateau-type problem with a sort of free boundary. Precisely, after suitable projections, we will arrive to a Plateau-type problem on the closed rectangle $\overline{R}_l$, where

$$R_l := (0,l) \times (-1,1) \times \{0\}$$

in Cartesian coordinates, equivalently $R_l = \{t \in (0,l),\ \rho \in [0,1),\ \theta = 0\} \cup \{t \in (0,l),\ \rho \in [0,1),\ \theta = \pi\}$ in cylindrical coordinates. The rectangle R_l will be often identified with $(0,l) \times (-1,1)$, thus neglecting the third coordinate. We will impose a Dirichlet boundary condition φ on a part

$$\partial_D R_l := (\{0\} \times [-1,1]) \cup ([0,l] \times \{-1\}) \tag{10.1}$$

of ∂R_l, while no conditions will be imposed on $\{l\} \times (-1,1)$; more involved conditions will be assigned on $(0,l) \times \{1\}$, see the mutual relations between ψ and h in (10.23) (see also the problem on the right-hand side of (10.25)).

Then the strategy to estimate from below the relaxed area of the graph of the vortex map u will be the following (see Section 11): We split

$$\mathcal{A}(u_k,\Omega) = \int_{\Omega\setminus D_k} |\mathcal{M}(\nabla u_k)|\ dx + \int_{D_k} |\mathcal{M}(\nabla u_k)|\ dx.$$

In order to estimate the $\liminf\limits_{k\to+\infty} \displaystyle\int_{\Omega\setminus D_k} |\mathcal{M}(\nabla u_k)|\ dx$ we will employ (4.38), whereas, to evaluate $\liminf\limits_{k\to+\infty} \displaystyle\int_{D_k} |\mathcal{M}(\nabla u_k)|\ dx$ we will use (9.12), so that we first want to render the right-hand side of this latter inequality independent of k. This will be done with the aid of the non-parametric Plateau-type problem (see also [9]).

Definition 10.1 (**The projection** p). *We let $p : C_l \cap \{t \geq 0\} \to \overline{R}_l$ be the othogonal projection.*

Recall that $\mathcal{G}^{(4)}_{\pm\widehat{\vartheta}_{k,\varepsilon}}$ are defined in (9.8), that $\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}$ is the generalized polar graph of $\widehat{\vartheta}_{k,\varepsilon}$ on its domain of definition (see (9.5)), and that $\widehat{\vartheta}_{k,\varepsilon}$ takes values in $[0,\pi]$. We first prove the following preliminary result:

Lemma 10.2. *Let $\varepsilon \in (0,1)$ be as in (4.5), (4.6), and $k \in \mathbb{N}$. Then there is a negligible set $C_{k,\varepsilon} \subset (0,l)$ such that, for all $t \in (0,l) \setminus C_{k,\varepsilon}$,*

$$p\Big(\operatorname{spt}\,(\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})\Big) \cap (\{t\} \times \mathbb{R}^2) \tag{10.2}$$

is a subinterval of the segment $\overline{R}_l \cap (\{t\} \times \mathbb{R}^2) = \{t\} \times [-1,1] \times \{0\}$ with one endpoint $(t,1,0)$. Moreover $p(\operatorname{spt}\,(\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})) = p(\operatorname{spt}\,(\mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}))$.

Proof. The latter assertion follows by symmetry. To prove the former, we argue by slicing. For a.e. $t \in (0,l)$ the set $\operatorname{spt}\,(\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}) \cap (\{t\} \times \mathbb{R}^2)$ coincides with the support of the current $(\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})_t$, see [33, Def. 7.6.2]. First notice that for all $t \in (0,\varepsilon)$ the conclusion follows by construction[40].

[40] In this set we have $\vartheta_{k,\varepsilon} = \pi$ and the current $(\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})_t$ is the integration over the half-circle $\{t\} \times ((\partial B_1) \cap \{\theta \in (0,\pi)\})$, whose projection through p is the whole interval with endpoints $(t,1,0)$ and $(t,1,0)$ (in cylindrical coordinates).

It remains to consider the case $t \in (\varepsilon, l)$. Recall that the set $S^{(4)}_{k,\varepsilon}$ in (9.4) has the form

$$\{t \in (\varepsilon, l),\ \rho \in [|u_k|^-(t) \wedge (1-\lambda_k), 1],\ \theta = 0\}.$$

Therefore, for a.e. $t \in (\varepsilon, l)$ the slice $(\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})_t$ is the integration over spt $(\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})$ restricted to the plane $\{t\} \times \mathbb{R}^2$, which in turn is the integration over the generalized polar graph (see (2.8)) of $\widehat{\vartheta}_{k,\varepsilon}$ restricted to the closed set

$$\{t\} \times [|u_k|^-(t) \wedge (1-\lambda_k), 1] \times [0, \pi].$$

Namely

$$(\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})_t = [\![\mathrm{spt}\ (\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}) \cap \big(\{t\} \times [|u_k|^-(t) \wedge (1-\lambda_k), 1] \times [0,\pi]\big)]\!],$$

so that the support σ_t of $(\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})_t$ can also be obtained as

$$\sigma_t = \bigcap_{h=1}^{+\infty} \sigma_t^h, \tag{10.3}$$

where

$$\sigma_t^h := \mathrm{spt}\ (\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}) \cap \Big(\{t\} \times [\big(|u_k|^-(t) \wedge (1-\lambda_k)\big) - \frac{1}{h}, 1] \times [0,\pi]\Big).$$

For $h \in \mathbb{N}$ large enough, let

$$U_h := \{t\} \times \big((|u_k|^-(t) \wedge (1-\lambda_k)) - \frac{1}{h}, 1\big) \times (-\frac{1}{h}, \pi + \frac{1}{h}),$$

which is a relatively open set in $\{t\} \times B_1$, and let $(\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})_t$ be the slice of $\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}$ on $\{t\} \times B_1$. We have

$$\mathrm{spt}\ ((\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})_t \llcorner U_h) \subset \sigma_t^h \subset \{t\} \times [\big(|u_k|^-(t) \wedge (1-\lambda_k)\big) - \frac{1}{h}, 1] \times [0,\pi]. \tag{10.4}$$

On the other hand since $\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}} \llcorner (\{t\} \times B_1)$ is the boundary of the subgraph of $\widehat{\vartheta}_{k,\varepsilon}$ in $\{t\} \times B_1$, it is a closed 1-integral current in U_h and in $(\{t\} \times (B_1 \setminus \overline{B}_{(|u_k|^-(t) \wedge (1-\lambda_k)) - \frac{1}{h}})$, so that the boundary $\partial((\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})_t \llcorner U_h)$ in $\mathcal{D}_1(\{t\} \times B_1)$ is supported on $(\partial U_h) \cap ((\partial B_1) \cup \partial B_{(|u_k|^-(t) \wedge (1-\lambda_k)) - \frac{1}{h}})$. From (10.4), the fact that $\widehat{\vartheta}_{k,\varepsilon} = 0$ at $(t, 1)$ and that $\widehat{\vartheta}_{k,\varepsilon}$ is constant on the segment $\big((|u_k|^-(t) \wedge (1-\lambda_k)) - \frac{1}{h}, |u_k|^-(t) \wedge (1-\lambda_k)\big)$ with value either 0 or π, we deduce that

$$\begin{aligned} &\mathrm{spt}\ (\partial((\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})_t \llcorner U_h)) \\ &\subset \big(\{t\} \times \{(|u_k|^-(t) \wedge (1-\lambda_k)) - \frac{1}{h}\} \times \{0, \pi\}\big) \bigcup \big(\{t\} \times \{1\} \times \{0\}\big). \end{aligned} \tag{10.5}$$

Moreover, if we set

$$P_1 := (t, 1, 0) \text{ and } P_2^h := \big(t, (|u_k|^-(t) \wedge (1-\lambda_k)) - \frac{1}{h}, \widehat{\vartheta}_{k,\varepsilon}((|u_k|^-(t) \wedge (1-\lambda_k)) - \frac{1}{h})\big),$$

from (10.5) it follows that

$$\partial((\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})_t \llcorner U_h) = \delta_{P_2^h} - \delta_{P_1}.$$

By decomposition of the integral 1-current $(\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})_t \llcorner U_h$ (see [25, Section 4.2.25]), there are at most countable Lipschitz curves $\{\alpha_i^h\}$ such that α_0^h connects P_2^h to P_1, and α_i^h is closed for $i > 0$. We claim that there cannot be closed curves α_i^h, namely $\{\alpha_i^h\}_{i\in\mathbb{N}} = \{\alpha_0^h\}$. Indeed, since α_0^h connects P_1 and P_2^h, we see that $(\{t\} \times \partial B_\rho) \cap \alpha_0^h$ consists of at least one point for $\mathcal{H}^1$-a.e. $\rho \in \big((|u_k|^-(t) \wedge (1-\lambda_k)) - \frac{1}{h}, 1\big)$. On the other hand, $(\{t\} \times \partial B_\rho) \cap \sigma_t^h$ consists of only one point[41] for $\mathcal{H}^1$-a.e. $\rho \in \big((|u_k|^-(t) \wedge (1-\lambda_k)) - \frac{1}{h}, 1\big)$. So there cannot be other curves α_i^h otherwise the last condition will be violated.

From the claim we deduce that the current $(\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})_t \llcorner U_h$ is the integration over a simple curve α_0^h connecting P_2^h and P_1, and its support coincides with σ_t^h. Now, from (10.3) and the fact that σ_t is a segment on $\{t\} \times \big((|u_k|^-(t) \wedge (1-\lambda_k)) - \frac{1}{h}, |u_k|^-(t) \wedge (1-\lambda_k)\big)$, we conclude that also σ_t must be a unique curve, say α_0, connecting P_1 to $P_2 := \lim_{h\to\infty} P_2^h$. By continuity of the projection by p, α_0 is an interval with one endpoint in $p(P_1) = (t, 1, 0)$, for a.e. $t \in (\varepsilon, l)$. □

The new coordinates (w_1, w_2, w_3). In what follows, it is convenient to revert the rectangle with respect to its second coordinate: if $(t, \rho, \theta) \in [0, l] \times [0, 1] \times (-\pi, \pi]$ are the cylindrical coordinates in the cylinder C_l exploited so far, we introduce Cartesian coordinates $(w_1, w_2, w_3) \in [0, l] \times [-1, 1] \times [-1, 1]$ defined as

$$w_1 := t, \quad w_2 := -\rho \cos\theta, \quad w_3 := -\rho \sin\theta, \tag{10.6}$$

in such a way that the segment $\{0 \le t \le l, \rho = 1, \theta = 0\}$ coincides with the bottom edge $[0, l] \times \{-1\} \times \{0\}$ of the rectangle $\overline{R}_l$.

10.1 The two functions $h_{k,\varepsilon}$ and $\psi_{k,\varepsilon}$

Thanks to Lemma 10.2 we are allowed to give the following

Definition 10.3 (The function $h_{k,\varepsilon}$). *Let $\varepsilon \in (0, 1)$ be as in (4.5), (4.6), and $k \in \mathbb{N}$. We define $h_{k,\varepsilon} : [0, l] \to [-1, 1]$ as*

$$h_{k,\varepsilon}(w_1) := \mathcal{H}^1\Big(p\big(\operatorname{spt}\, (\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})\big) \cap (\{w_1\} \times \mathbb{R}^2)\Big) - 1.$$

For all $w_1 \in (0, l)$ for which Lemma 10.2 is valid, we have that $1 + h_{k,\varepsilon}(w_1)$ equals the length of the interval in (10.2). Now the content of Lemma 10.2 is that the p-projection of $\operatorname{spt}\, (\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})$ on $\overline{R}_l$ is of the form

$$p\Big(\operatorname{spt}\, (\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})\Big) = SG_{h_{k,\varepsilon}} := \{(w_1, w_2) \in R_l : w_1 \in (0, l), w_2 \in (-1, h_{k,\varepsilon}(w_1))\}, \tag{10.7}$$

up to a set of zero $\mathcal{H}^2$-measure. The function $h_{k,\varepsilon}$ is built in such a way that $(w_1, -1)$ and $(w_1, h_{k,\varepsilon}(w_1))$ are the endopoints of the interval $p(\operatorname{spt}\, (\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}})) \cap (\{w_1\} \times \mathbb{R}^2)$ for almost every $w_1 \in (0, l)$. Observe that

$$h_{k,\varepsilon} \ge -1 + \lambda'_k \qquad \text{in } (\varepsilon, l),$$

and

$$h_{k,\varepsilon} = 1 \qquad \text{in } (0, \varepsilon).$$

[41] Because σ_t^h is the support of a polar graph; the points where this intersection is not a singleton coincide with the values of ρ where $\widehat{\vartheta}_{k,\varepsilon}$ has a jump.

Indeed, from Definition 9.2, equation (9.4) and Definition 9.4, we see that the set $\Big((0,l) \times [1-\lambda_k', 1] \times \{0\}\Big) \cup \Big((0,\varepsilon) \times [-1,1] \times \{0\}\Big)$ is contained in $p(\,\mathrm{spt}\,(\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}))$.

We have built $O_{k,\varepsilon}$ in (9.3) as the set enclosed between $\mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}$ and $\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}$, see formula (9.9). We now perform a (classical) Steiner symmetrization[42] of the set $O_{k,\varepsilon}$ with respect to the plane $\{w_3 = 0\}$. We denote by $\mathbb{S}_{\rm cl}(O_{k,\varepsilon})$ the symmetrized set.

Remark 10.4. We emphasize that the set $O_{k,\varepsilon}$ in $(0,\varepsilon) \times \mathbb{R}^2$ is exactly $(0,\varepsilon) \times B_1$, and is already symmetric with respect to the plane containing R_l. For this reason $O_{k,\varepsilon}$ does not change (in that region) after Steiner symmetrization,

$$O_{k,\varepsilon} \cap \{w_1 \in (0,\varepsilon)\} = \mathbb{S}_{\rm cl}(O_{k,\varepsilon}) \cap \{w_1 \in (0,\varepsilon)\}. \tag{10.8}$$

Since the perimeter does not increase when symmetrizing, from (9.9) we conclude

$$|\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}} + \mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}| \geq \mathcal{H}^2(\partial^* \mathbb{S}_{\rm cl}(O_{k,\varepsilon}) \cap ((0,l) \times \mathbb{R}^2)). \tag{10.9}$$

Now, using the sets $O_{k,\varepsilon}$, we can define the functions $\psi_{k,\varepsilon}$ which, together with the functions $h_{k,\varepsilon}$ given in Definition 10.3, will allow to express the singular part of the relaxed area using a Cartesian Plateau-type problem with partial free boundary (see Remark 10.10).

Definition 10.5 (The function $\psi_{k,\varepsilon}$). *We introduce the function $\psi_{k,\varepsilon} : R_l \to [0,+\infty)$ as*

$$\psi_{k,\varepsilon}(w_1,w_2) := \frac{1}{2}\mathcal{H}^1(\{w_3 : (w_1,w_2,w_3) \in O_{k,\varepsilon}\}), \qquad (w_1,w_2) \in R_l. \tag{10.10}$$

We stress that the set where $\psi_{k,\varepsilon} > 0$ is contained, up to $\mathcal{H}^2$-negligible sets, in the region $SG_{h_{k,\varepsilon}}$ defined in (10.7). Notice also that $\psi_{k,\varepsilon}$ may take the value 0 in $SG_{h_{k,\varepsilon}}$ on a set of positive $\mathcal{H}^2$-measure.

Remark 10.6. (i) By definition of classical Steiner symmetrization,

$$\begin{aligned}\mathbb{S}_{\rm cl}(O_{k,\varepsilon}) &= \{w = (w_1,w_2,w_3) \in R_l \times \mathbb{R} : w_3 \in (-\psi_{k,\varepsilon}(w_1,w_2), \psi_{k,\varepsilon}(w_1,w_2))\} \\ &= \{w = (w_1,w_2,w_3) \in SG_{h_{k,\varepsilon}} \times \mathbb{R} : w_3 \in (-\psi_{k,\varepsilon}(w_1,w_2), \psi_{k,\varepsilon}(w_1,w_2))\},\end{aligned}$$

up to Lebesgue-negligible sets, the second equality following from the fact that $\psi_{k,\varepsilon} = 0$ almost everywhere in $R_l \setminus SG_{h_{k,\varepsilon}}$;

(ii) since $O_{k,\varepsilon}$ has finite perimeter, it follows that $\psi_{k,\varepsilon} \in BV(R_l)$;

(iii) since $O_{k,\varepsilon} \llcorner ([0,\varepsilon) \times \mathbb{R}^2) = C_l \llcorner ([0,\varepsilon) \times \mathbb{R}^2)$ and $O_{k,\varepsilon} \llcorner ([\varepsilon, l) \times \mathbb{R}^2)$ is contained in $C_l(1-\lambda_k') \llcorner ([\varepsilon,l) \times \mathbb{R}^2)$ (as a consequence of (9.5)), it follows that $\psi_{k,\varepsilon}$ has null trace on the segments $(0,l) \times \{-1\}$ and $(0,l) \times \{1\}$.

We can split $\partial^* \mathbb{S}_{\rm cl}(O_{k,\varepsilon})$ as

$$\partial^* \mathbb{S}_{\rm cl}(O_{k,\varepsilon}) = ((\partial^* \mathbb{S}_{\rm cl}(O_{k,\varepsilon})) \cap \{w_3 > 0\}) \bigcup ((\partial^* \mathbb{S}_{\rm cl}(O_{k,\varepsilon})) \cap \{w_3 < 0\}) =: (\partial^* \mathbb{S}_{\rm cl}(O_{k,\varepsilon}))^+ \cup (\partial^* \mathbb{S}_{\rm cl}(O_{k,\varepsilon}))^- \tag{10.11}$$

[42]Despite $O_{k,\varepsilon}$ is obtained by cylindrical symmetrization, it still can have "holes" (see Fig. 8 for a slice), that disappear when further performing the Steiner symmetrization.

up to a set of $\mathcal{H}^2$-measure zero, in such a way that

$$(\partial^*\mathbb{S}_{\rm cl}(O_{k,\varepsilon}))^+ = (\partial^* SG_{\psi_{k,\varepsilon}})\cap\big(R_l\times(0,+\infty)\big), \qquad (\partial^*\mathbb{S}_{\rm cl}(O_{k,\varepsilon}))^- = (\partial^* UG_{-\psi_{k,\varepsilon}})\cap\big(R_l\times(-\infty,0)\big), \tag{10.12}$$

where $SG_{\psi_{k,\varepsilon}}$ and $UG_{-\psi_{k,\varepsilon}}$ are, respectively, the (standard) generalized subgraph and epigraph of $\pm\psi_{k,\varepsilon}$ in $R_l\times\mathbb{R}$. Notice that, since $\psi_{k,\varepsilon}\geq 0$,

$$\begin{aligned}
&(\partial^* SG_{\psi_{k,\varepsilon}})\cap(R_l\times[0,+\infty))\\
&\qquad = (\partial^*\mathbb{S}_{\rm cl}(O_{k,\varepsilon}))^+\cup\{(w_1,w_2,0)\in SG_{h_{k,\varepsilon}}:\psi_{k,\varepsilon}=0\}\cup(R_l\setminus SG_{h_{k,\varepsilon}}),\\
&(\partial^* UG_{-\psi_{k,\varepsilon}})\cap(R_l\times(-\infty,0])\\
&\qquad = (\partial^*\mathbb{S}_{\rm cl}(O_{k,\varepsilon}))^-\cup\{(w_1,w_2,0)\in SG_{h_{k,\varepsilon}}:\psi_{k,\varepsilon}=0\}\cup(R_l\setminus SG_{h_{k,\varepsilon}}),
\end{aligned} \tag{10.13}$$

up to $\mathcal{H}^2$-negligible sets.

We are ready to prove the following:

Lemma 10.7 (Lower bound for $|\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}|+|\mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}|$). *We have*

$$\begin{aligned}
|\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}|+|\mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}| \geq & \mathcal{H}^2\left((\partial^* SG_{\psi_{k,\varepsilon}})\cap(R_l\times[0,+\infty))\right)+\mathcal{H}^2\left((\partial^* UG_{-\psi_{k,\varepsilon}})\cap(R_l\times(-\infty,0])\right)\\
&-2\mathcal{H}^2(R_l\setminus SG_{h_{k,\varepsilon}}).
\end{aligned} \tag{10.14}$$

Moreover, $\mathcal{H}^2\left((\partial^* SG_{\psi_{k,\varepsilon}})\cap(R_l\times[0,+\infty))\right)=\mathcal{H}^2\left((\partial^* UG_{-\psi_{k,\varepsilon}})\cap(R_l\times(-\infty,0])\right)$.

Proof. The last assertion follows by symmetry. Let us prove the former: By (10.13) we have

$$\begin{aligned}
\mathcal{H}^2(\partial^* SG_{\psi_{k,\varepsilon}}\cap(R_l\times[0,+\infty))) =&\mathcal{H}^2\big((\partial^*\mathbb{S}_{\rm cl}(O_{k,\varepsilon}))^+\big)+\mathcal{H}^2(\{(w_1,w_2)\in SG_{h_{k,\varepsilon}}:\psi_{k,\varepsilon}=0\})\\
&+\mathcal{H}^2(R_l\setminus SG_{h_{k,\varepsilon}}),\\
\mathcal{H}^2(\partial^* UG_{-\psi_{k,\varepsilon}}\cap(R_l\times(-\infty,0])) =&\mathcal{H}^2\big((\partial^*\mathbb{S}_{\rm cl}(O_{k,\varepsilon}))^-\big)+\mathcal{H}^2(\{(w_1,w_2)\in SG_{h_{k,\varepsilon}}:\psi_{k,\varepsilon}=0\})\\
&+\mathcal{H}^2(R_l\setminus SG_{h_{k,\varepsilon}}).
\end{aligned}$$

Taking the sum of these two expressions and using (10.9), (10.11), we obtain

$$\begin{aligned}
&\mathcal{H}^2(\partial^* SG_{\psi_{k,\varepsilon}}\cap(R_l\times[0,+\infty)))+\mathcal{H}^2(\partial^* UG_{-\psi_{k,\varepsilon}}\cap(R_l\times(-\infty,0]))\\
&\leq|\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}+\mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}|+2\mathcal{H}^2(\{(w_1,w_2)\in SG_{h_{k,\varepsilon}}:\psi_{k,\varepsilon}=0\})+2\mathcal{H}^2(R_l\setminus SG_{h_{k,\varepsilon}}).
\end{aligned}$$

Recalling (9.4), we now claim that, up to $\mathcal{H}^2$-negligible sets,

$$\{(w_1,w_2)\in SG_{h_{k,\varepsilon}}:\psi_{k,\varepsilon}(w_1,w_2)=0\}\subset\big\{\widehat{\vartheta}_{k,\varepsilon}=0\big\}\cap S^{(4)}_{k,\varepsilon}, \tag{10.15}$$

see Fig. 8. From the claim it follows that

$$\mathcal{H}^2(\{(w_1,w_2)\in SG_{h_{k,\varepsilon}}:\psi_{k,\varepsilon}=0\})\leq\mathcal{H}^2\big(\{\widehat{\vartheta}_{k,\varepsilon}\in\{0,\pi\}\}\cap S^{(4)}_{k,\varepsilon}\big),$$

and hence by (9.10) we conclude

$$\begin{aligned}
&\mathcal{H}^2(\partial^* SG_{\psi_{k,\varepsilon}}\cap(R_l\times[0,+\infty)))+\mathcal{H}^2(\partial^* UG_{-\psi_{k,\varepsilon}}\cap(R_l\times(-\infty,0]))\\
&\leq|\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}|+|\mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}|+2\mathcal{H}^2(R_l\setminus SG_{h_{k,\varepsilon}}),
\end{aligned}$$

that is (10.14). It remains to show (10.15). As usual, we argue by slicing; hence for almost all $w_1 \in (0, l)$ we will show that (10.15) holds (up to $\mathcal{H}^1$-negligible sets). Notice that both the left and right-hand sides of (10.15) are empty for $w_1 < \varepsilon$, so we assume $w_1 > \varepsilon$. Therefore, fix $(\widetilde{w}_1, \widetilde{w}_2) \in SG_{h_{k,\varepsilon}}$ (with $\widetilde{w}_1 > \varepsilon$) such that $\psi_{k,\varepsilon}(\widetilde{w}_1, \widetilde{w}_2) = 0$ and assume, by contradiction, that $\widehat{\vartheta}_{k,\varepsilon}(\widetilde{w}_1, \widetilde{w}_2) > 0$. In a first step we will suppose $\widetilde{w}_2 < 0$. We might further assume that $\widetilde{w}_2$ is a Lebesgue point for the function $\widehat{\vartheta}_{k,\varepsilon}(\widetilde{w}_1, \cdot)$. Hence in any left-neighbourhood of this point $\widehat{\vartheta}_{k,\varepsilon}$ is strictly positive on a set of positive meausure, *i.e.*, we can find positive numbers δ_1, δ_2 such that for all $\delta \in (0, \delta_1)$, there exists a set $B \subset (\widetilde{w}_2 - \delta, \widetilde{w}_2)$, of positive measure such that

$$\widehat{\vartheta}_{k,\varepsilon}(\widetilde{w}_1, w) > \delta_2 > 0 \qquad \forall w \in B. \tag{10.16}$$

If $\pi_0^{\rm pol}$ is the projection in Definition (7.5), since $\widetilde{w}_2 < 0$ for $\delta_3 > 0$ small enough the segment $I := \{(\widetilde{w}_1, \widetilde{w}_2, w_3) : w_3 \in (0, \delta_3)\}$ satisfies

$$I_0 := \pi_0^{\rm pol}(I) \subset \{(\widetilde{w}_1, w_2, 0) : w_2 \in (\widetilde{w}_2 - \delta_2, \widetilde{w}_2)\}.$$

We have that $\pi_0^{\rm pol} : I \to I_0$ is a homeomorphism. Now, if $\psi_{k,\varepsilon}(\widetilde{w}_1, \widetilde{w}_2) = 0$ the segment I cannot intersect the subgraph of $\widehat{\vartheta}_{k,\varepsilon}$ (on a set of positive $\mathcal{H}^1$-measure), and thus

$$\widehat{\vartheta}_{k,\varepsilon}(\widetilde{w}_1, w_2) \le \theta\big((\pi_0^{\rm pol}{}_{|I_0})^{-1}(\widetilde{w}_1, w_2, 0)\big) \qquad \text{for } \mathcal{H}^1\text{-a.e. } (\widetilde{w}_1, w_2, 0) \in I_0, \tag{10.17}$$

where θ represents the usual angular coordinate. Since $\theta\big((\pi_0^{\rm pol}{}_{|I_0})^{-1}(\widetilde{w}_1, w_2, 0)\big)$ is infinitesimal as $w_2 \to \widetilde{w}_2^-$, condition (10.17) contradicts (10.16).

Let us now treat the case $\widetilde{w}_2 > 0$. This is much simpler to deal with, up to noticing that $\widehat{\vartheta}_{k,\varepsilon}$ is defined on $S^{(4)}_{k,\varepsilon} \subset \{(w_1, w_2, w_3) : w_2 \in [-1, 0]\}$. The fact that $\psi_{k,\varepsilon}(\widetilde{w}_1, \widetilde{w}_2) = 0$ means that the line $(\widetilde{w}_1, \widetilde{w}_2) \times \mathbb{R}$ does not intersect $O_{k,\varepsilon}$ on a set of positive $\mathcal{H}^1$-measure but this contradicts the fact that $(\widetilde{w}_1, \widetilde{w}_2) \in SG_{h_{k,\varepsilon}}$. Indeed since $(\widetilde{w}_1, \widetilde{w}_2) \in SG_{h_{k,\varepsilon}}$ hence there exists $w_2 > \widetilde{w}_2$ such that $\psi_{k,\varepsilon}(\widetilde{w}_1, w_2) > 0$. Let $A := O_{k,\varepsilon} \cap (\widetilde{w}_1, w_2) \times \mathbb{R})$ then a suitable rotation of A around the axis of the cylinder shall meet $(\widetilde{w}_1, \widetilde{w}_2) \times \mathbb{R}$ on a set $\widetilde{A}$ of positive $\mathcal{H}^1$-measure (note that $\widetilde{A} \subset O_{k,\varepsilon}$), which contradicts $\psi_{k,\varepsilon}(\widetilde{w}_1, \widetilde{w}_2) = 0$. □

Remark 10.8. By (10.8), (10.10) and (9.3), we deduce

$$\text{the trace of } \psi_{k,\varepsilon} \text{ on } \overline{R}_l \cap \{w_1 = 0\} \text{ is } \sqrt{1 - w_2^2}, \quad \text{for } w_2 \in [-1, 1]. \tag{10.18}$$

Moreover, by construction and by Remark 10.6 (iii),

$$\psi_{k,\varepsilon}(w_1, -1) = 0 \quad \text{and} \quad \psi_{k,\varepsilon}(w_1, 1) = 0, \qquad w_1 \in (0, l). \tag{10.19}$$

Remark 10.9. We can write (see [29])

$$\mathcal{H}^2\Big((\partial^* SG_{\psi_{k,\varepsilon}}) \cap (R_l \times [0, \infty))\Big) = \overline{\mathcal{A}}(\psi_{k,\varepsilon}, R_l), \tag{10.20}$$

where

$$\overline{\mathcal{A}}(\psi_{k,\varepsilon}, R_l) = \int_{R_l} \sqrt{1 + |\nabla \psi_{k,\varepsilon}|^2}\, dx + |D^s \psi_{k,\varepsilon}|(R_l)$$

is the area of the graph of the scalar BV-function $\psi_{k,\varepsilon}$ in R_l. Moreover, by (10.19), it follows $|D^s \psi_{k,\varepsilon}|(R_l) = |D^s \psi_{k,\varepsilon}|\big(\overline{R}_l \setminus (\{w_1 = 0\} \cup \{w_1 = l\})\big)$ and hence

$$\overline{\mathcal{A}}(\psi_{k,\varepsilon}, R_l) = \overline{\mathcal{A}}\big(\psi_{k,\varepsilon}, \overline{R}_l \setminus (\{w_1 = 0\} \cup \{w_1 = l\})\big).$$

Recalling the expression (10.1) of $\partial_D R_l$, define $\varphi : \partial_D R_l \to [0, 1]$ as

$$\varphi(w_1, w_2) := \begin{cases} \sqrt{1 - w_2^2} & \text{if } (w_1, w_2) \in \{0\} \times [-1, 1], \\ 0 & \text{if } (w_1, w_2) \in (0, l) \times \{-1\}. \end{cases} \tag{10.21}$$

10.2 A minimum problem with partial free boundary

For convenience, we recall the following definitions, introduced in (10.22), (10.23): for $h \in L^\infty([0,l],[-1,1])$ and $\psi \in \mathrm{BV}(R_l;[0,1])$ we define

$$\mathcal{F}_l(h,\psi) := \overline{\mathcal{A}}(\psi,R_l) - \mathcal{H}^2(R_l \setminus SG_h) + \int_{\partial_D R_l} |\psi-\varphi|\, d\mathcal{H}^1 + \int_{(0,l)\times\{1\}} |\psi|\, d\mathcal{H}^1. \tag{10.22}$$

and

$$X_l := \{(h,\psi) : h \in L^\infty([0,l],[-1,1]), \psi \in \mathrm{BV}(R_l,[0,1]), \psi = 0 \text{ in } R_l \setminus SG_h\}. \tag{10.23}$$

Remark 10.10. (i) The Borel function $h_{k,\varepsilon} : [0,l] \to [-1,1]$ satisfies $h_{k,\varepsilon} = 1$ in $[0,\varepsilon)$, and $\psi_{k,\varepsilon} \in BV([0,l]\times[-1,1])$ is such that:

- (i1) $\psi_{k,\varepsilon} = 0$ almost everywhere in $R_l \setminus SG_{h_{k,\varepsilon}}$;
- (i2) $\psi_{k,\varepsilon}(w_1,w_2) = \sqrt{1-w_2^2}$ for $(w_1,w_2) \in (0,\varepsilon)\times[-1,1]$;
- (i3) $\psi_{k,\varepsilon}(\cdot,-1) = 0$ in $[0,l]$.

In particular

$$(h_{k,\varepsilon},\psi_{k,\varepsilon}) \in X_l.$$

(ii) if $(h,\psi) \in X_l$, and if h is smaller than 1 almost everywhere on $(0,l)$ then the last addendum on the right-hand side of (10.22) vanishes.

(iii) Thanks to (10.18) and (10.19), it follows from Remark 10.9 that

$$\begin{aligned}&\mathcal{H}^2\big((\partial^* SG_{\psi_{k,\varepsilon}}) \cap (R_l \times [0,+\infty))\big) - \mathcal{H}^2(R_l \setminus SG_{h_{k,\varepsilon}})\\ =&\overline{\mathcal{A}}(\psi_{k,\varepsilon}, \overline{R}_l \setminus (\{w_1=0\}\cup\{w_1=l\})) - \mathcal{H}^2(R_l \setminus SG_{h_{k,\varepsilon}}) = \mathcal{F}_l(h_{k,\varepsilon},\psi_{k,\varepsilon}).\end{aligned}$$

As a consequence, from Lemma 10.7 we have

$$|\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}| + |\mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}| \ \geq\ 2\mathcal{F}_l(h_{k,\varepsilon},\psi_{k,\varepsilon}). \tag{10.24}$$

Notice that in minimizing $\mathcal{F}_l$ we have a free boundary condition on the edge $\{l\}\times[-1,1]$. By Remark 10.10 (i) and inequality (10.24) we have

$$|\mathcal{G}^{(4)}_{\widehat{\vartheta}_{k,\varepsilon}}| + |\mathcal{G}^{(4)}_{-\widehat{\vartheta}_{k,\varepsilon}}| \geq 2 \inf_{(h,\psi)\in X_l} \mathcal{F}_l(h,\psi), \tag{10.25}$$

which leads to the investigation of the minimum problem on the right-hand side performed in [9].

Let us rewrite the functional $\mathcal{F}_l$ in a convenient way. Let $(h,\psi) \in X_l$, and let $G_h = \{(w_1,h(w_1)) : w_1 \in (0,l)\} \subset \overline{R}_l$ be the graph of h. We have, using (10.22),

$$\mathcal{F}_l(h,\psi) = \overline{\mathcal{A}}(\psi,SG_h) + \int_{G_h\setminus\{h=-1\}} |\psi|\, d\mathcal{H}^1 + \int_{\partial_D R_l} |\psi-\varphi|\, d\mathcal{H}^1, \tag{10.26}$$

where, in the integral over G_h, we consider the trace of $\psi \llcorner SG_h$ on G_h. Combining (10.25) with (9.12), we readily infer:

Corollary 10.11. *Let $\varepsilon \in (0,1)$ and $n \in \mathbb{N}$. Then for any $k \in \mathbb{N}$ we have*

$$|[\![G_{u_k}]\!]|_{D_k\times\mathbb{R}^2} \geq 2 \inf_{(h,\psi)\in X_l} \mathcal{F}_l(h,\psi) - \pi\varepsilon - \frac{C}{\varepsilon n} - o_k(1), \tag{10.27}$$

for an absolute constant $C > 0$, and where the sequence $o_k(1)$ depends on ε and n and is infinitesimal as $k \to +\infty$.

11 Conclusion of the proof of Theorem 1.1

We are finally in the position to conclude the proof of Theorem 1.1. We write

$$\mathcal{A}(u_k, \Omega) = \mathcal{A}(u_k, \Omega \setminus D_k) + \mathcal{A}(u_k, D_k) = \int_{\Omega \setminus D_k} |\mathcal{M}(\nabla u_k)| \, dx + \int_{D_k} |\mathcal{M}(\nabla u_k)| \, dx.$$

Therefore

$$\overline{\mathcal{A}}(u, \Omega) \geq \liminf_{k \to +\infty} \int_{\Omega \setminus D_k} |\mathcal{M}(\nabla u_k)| \, dx + \liminf_{k \to +\infty} \int_{D_k} |\mathcal{M}(\nabla u_k)| \, dx. \tag{11.1}$$

Given $\varepsilon \in (0, l)$ satisfying (4.5) and (4.6), and given $n \in \mathbb{N}$, from (4.38) it follows

$$\liminf_{k \to +\infty} \int_{\Omega \setminus D_k} |\mathcal{M}(\nabla u_k)| \, dx \geq \int_{\Omega \setminus \overline{\mathrm{B}}_\varepsilon} |\mathcal{M}(\nabla u)| \, dx - \frac{1}{n} - \frac{2}{\varepsilon n}. \tag{11.2}$$

Furthermore, from (10.27) we have

$$\int_{D_k} |\mathcal{M}(\nabla u_k)| \, dx = |[\![G_{u_k}]\!]|_{D_k \times \mathbb{R}^2} \geq 2 \inf_{(h,\psi) \in X_l} \mathcal{F}_l(h, \psi) - \pi\varepsilon - \frac{C}{\varepsilon n} - o_k(1). \tag{11.3}$$

We can pass to the liminf as $k \to +\infty$ in the above expression, to obtain

$$\liminf_{k \to +\infty} \int_{D_k} |\mathcal{M}(\nabla u_k)| \, dx \geq 2 \inf_{(h,\psi) \in X_l} \mathcal{F}_l(h, \psi) - \pi\varepsilon - \frac{C}{\varepsilon n}. \tag{11.4}$$

From (11.1), (11.2) and (11.4) we obtain

$$\overline{\mathcal{A}}(u, \Omega) \geq \int_{\Omega \setminus \overline{\mathrm{B}}_\varepsilon} |\mathcal{M}(\nabla u)| \, dx + 2 \inf_{(\psi,h) \in X_l} \mathcal{F}_l(\psi, h) - \pi\varepsilon - \frac{C+2}{\varepsilon n} - \frac{1}{n}, \tag{11.5}$$

for all $n \in \mathbb{N}$ and $\varepsilon \in (0, l)$. Letting $n \to +\infty$ and then $\varepsilon \to 0^+$ (keeping the validity of (4.5) and (4.6)), by the dominated convergence theorem (since $\Omega \setminus \overline{\mathrm{B}}_\varepsilon \to \Omega$ as $\varepsilon \to 0^+$) we get

$$\begin{aligned} \overline{\mathcal{A}}(u, \Omega) &\geq \liminf_{\varepsilon \to 0^+} \Big(\int_{\Omega \setminus \overline{\mathrm{B}}_\varepsilon} |\mathcal{M}(\nabla u)| \, dx + 2 \inf_{(h,\psi) \in X_l} \mathcal{F}_l(h, \psi) - \pi\varepsilon \Big) \\ &= \int_\Omega |\mathcal{M}(\nabla u)| \, dx + 2 \inf_{(h,\psi) \in X_l} \mathcal{F}_l(h, \psi). \end{aligned}$$

This concludes the proof. □

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