The L^1-relaxed area of the graph of the vortex map: optimal upper bound

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Abstract

We compute an upper bound for the value of the L^1-relaxed area of the graph of the vortex map $u : \mathrm{B}_l(0) \subset \mathbb{R}^2 \to \mathbb{R}^2$, $u(x) := x/|x|$, $x \neq 0$, for all values of $l > 0$. Together with a previously proven lower bound, this upper bound turns out to be optimal. Interestingly, for the radius l in a certain range, in particular l not too large, a Plateau-type problem, having as solution a sort of catenoid constrained to contain a segment, has to be solved.

Key words: Area functional, minimal surfaces, Plateau problem, relaxation, Cartesian currents.

AMS (MOS) subject classification: 49Q15, 49Q20, 49J45, 58E12.

1 Introduction

Determining the domain and the expression of the relaxed area functional of graphs of nonsmooth maps in codimension greater than 1 is a challenging problem whose solution is far from being reached. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $v : \Omega \to \mathbb{R}^N$ be a map of class C^1; the graph area of v over Ω is given by

$$\mathcal{A}(v, \Omega) = \int_\Omega |\mathcal{M}(\nabla v)| \, dx, \tag{1.1}$$

where $\mathcal{M}(\nabla v)$ is the vector whose entries are the determinants of the minors of the gradient ∇v of v of all orders[1] k, $0 \leq k \leq \min\{n, N\}$. In order to extend this functional out of $C^1(\Omega, \mathbb{R}^N)$, one is led to define, for any $v \in L^1(\Omega, \mathbb{R}^N)$,

$$\overline{\mathcal{A}}(v, \Omega) := \inf \Big\{ \liminf_{k \to +\infty} \mathcal{A}(v_k, \Omega) \Big\}, \tag{1.2}$$

which is called the *(sequential) relaxed area functional.* The infimum appearing in 1.2 is computed among all possible sequences of maps $v_k \in C^1(\Omega, \mathbb{R}^N)$ tending to v in $L^1(\Omega, \mathbb{R}^N)$. The results of Acerbi and Dal Maso [1] show that $\overline{\mathcal{A}}(\cdot, \Omega)$ extends $\mathcal{A}(\cdot, \Omega)$ and is L^1-lower semicontinuous.

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[1] By convention, the determinant of order 0 is 1.

This procedure of relaxation, besides extending the notion of graph's area to non-smooth maps, is needed also because $\mathcal{A}(\cdot,\Omega)$ is not L^1-lower semicontinuous[2], in contrast with similar polyconvex functionals that enjoy a growth condition of the form $F(u) \geq C|\mathcal{M}(\nabla u)|^p$ for some $C > 0$, and suitable $p > 1$ (see, *e.g.*, [21, 28, 37]).

When $N = 1$ it is possible to characterize the domain of $\overline{\mathcal{A}}(\cdot,\Omega)$ and its expression [22]: $\overline{\mathcal{A}}(v,\Omega)$ is finite if and only if $v \in BV(\Omega)$, in which case

$$\overline{\mathcal{A}}(v,\Omega) = \int_\Omega \sqrt{1+|\nabla v|^2}dx + |D^s v|(\Omega), \tag{1.3}$$

∇v and $D^s v$ representing the absolutely continuous and singular parts of the distributional gradient Dv of v. Formula (1.3) gives a classical example of non-parametric variational integral. This turns out to be a measure when considered as a function of Ω (and the map u being fixed [31]), and has several applications, as for instance in capillarity problems [27] and in the analysis of the Cartesian Plateau problem [30]. The higher codimension case, namely $N > 1$, is much more involved and, once again, has as main motivation the study of the Cartesian Plateau problem in higher codimension; from a theoretical point of view, it is of indipendent interest in Calculus of Variations questions involving nonconvex integrands with nonstandard growth (see, e.g., [3, 21, 29]).

In this paper we restrict our attention to the first non-standard case, namely $n = N = 2$. For a map $v \in C^1(\Omega,\mathbb{R}^2)$ and $\Omega \subset\subset \mathbb{R}^2$, the quantity $\mathcal{A}(v,\Omega)$ coincides with the area of the graph $G_v := \{(x,y) \in \Omega\times\mathbb{R}^2 : y = v(x)\}$ of v seen as a Cartesian surface of codimension 2 in $\Omega\times\mathbb{R}^2$, and is given by

$$\mathcal{A}(v,\Omega) = \int_\Omega \sqrt{1+|\nabla v(x_1,x_2)|^2+|Jv(x_1,x_2)|^2}\; dx_1dx_2.$$

Here ∇v is the gradient of v, a 2×2 matrix, $|\nabla v|^2$ is the sum of the squares of all elements of ∇v, and Jv is the Jacobian determinant of v, *i.e.*, the determinant of ∇v. It is worth to point out once more a couple of relevant difficulties arising when the codimension is greater than 1: the functional $\mathcal{A}(\cdot,\Omega)$ is no longer convex, but just polyconvex; in addition it has a sort of unilateral linear growth, in the sense that it is bounded below, but not necessarily above, by the total variation of v. A characterization of the domain of $\overline{\mathcal{A}}(\cdot,\Omega)$ and of its expression is, at the moment, not available. Specifically, it is only known that the domain of $\overline{\mathcal{A}}(\cdot,\Omega)$ is a proper subset of $BV(\Omega,\mathbb{R}^2)$, and that integral representation formulas such as (1.3) (on the domain of $\overline{\mathcal{A}}(\cdot,\Omega)$) are not possible. This is due to the additional difficulty that in general, for a fixed map v, the set function $A \subseteq \Omega \mapsto \overline{\mathcal{A}}(v,A)$ may be not subadditive and,in such a case, it cannot be a measure (as opposite to what happens in codimension 1 for a large class of non-parametric variational integrals [31]). This interesting phenomenon was conjectured by De Giorgi [23] for the triple junction map $u_T : \Omega = \mathrm{B}_l(0) \to \mathbb{R}^2$, and proved in [1], where the authors exhibited three subsets $\Omega_1, \Omega_2, \Omega_3$ of the open disk $\mathrm{B}_l(0)$ of radius l centered at 0, such that

$$\Omega_1 \subset \Omega_2 \cup \Omega_3 \qquad \text{and} \qquad \overline{\mathcal{A}}(u_T,\Omega_1) > \overline{\mathcal{A}}(u_T,\Omega_2) + \overline{\mathcal{A}}(u_T,\Omega_3). \tag{1.4}$$

The triple junction map $u_T \in BV(\Omega,\mathbb{R}^2)$ takes only three values $\alpha,\beta,\gamma \in \mathbb{R}^2$, the vertices of an equilateral triangle, in three circular 120^o-degree sectors of Ω meeting at 0. The same authors show that the non-locality property (1.4) holds also for the Sobolev map $u(x) = \frac{x}{|x|}$, called here the vortex map, where Ω is the open ball $B_l(0)$ of radius l centered at the origin, the singular point,

[2]When $n = N = 2$, there are sequences $(v_k) \subset W^{1,p}(\Omega,\mathbb{R}^2)$, with $p \in [1,2)$, weakly converging in $W^{1,p}(\Omega,\mathbb{R}^2)$ to a smooth map v for which $\mathcal{A}(v,\Omega) > \limsup_{k\to+\infty}\mathcal{A}(v_k,\Omega)$, where $\mathcal{A}(v_k,\Omega)$ is defined in the same form as for C^1-maps in (1.1), with the determinant of ∇v_k intended in the almost everywhere pointwise sense; see [4, Counterexample 7.4] and [1]. This counterexample must be slightly modified, considering $u_k(x) = kx + \lambda(x/\|x\|_\infty - x)$ for $x \in [-1/k, 1/k]$, with $\lambda > 0$ satisfying $(1+\lambda^2)/2 > \sqrt{1+\lambda^2}$, in order to get the strict inequality above.

and $n = N \geq 3$. For these two maps u_T and u much effort has been done to understand the exact value of the area functional; the corresponding geometric problem stands in finding the optimal way, in terms of area, to “fill the holes” of the graph of u_T and u (two non-smooth 2-dimensional sets of codimension two) with limits of sequences of smooth two-dimensional graphs. In [1] it is proved that both u_T and u have finite relaxed area, but only lower and upper bounds were available for u_T, whereas the sharp estimate for u is provided only for l large enough. For the triple junction map u_T an improvement is obtained in [11], where it is exhibited a sequence (u_k) of Lipschitz maps $u_k : \mathrm{B}_l(0) \to \mathbb{R}^2$ converging to u in $L^1(\Omega, \mathbb{R}^2)$, such that

$$\lim_{k\to+\infty} \mathcal{A}(u_k, \mathrm{B}_l(0)) = |\mathcal{G}_{u_T}| + 3m_l,$$

where $|\mathcal{G}_{u_T}|$ is the area of the graph of u_T out of the jump set, and m_l is the area of an area-minimizing surface, solution of a Plateau-type problem in $\mathbb{R}^3$. Roughly speaking, three entangled area-minimizing surfaces with area m_l (each sitting in a copy of $\mathbb{R}^3 \subset \mathbb{R}^4$, the three $\mathbb{R}^3$'s being mutually nonparallel) are needed in $\mathrm{B}_l(0) \times \mathbb{R}^2$ to “fill the holes” left by the graph $\mathcal{G}_{u_T}$ of u_T, which is not boundaryless (*i.e.*, the boundary as a current is nonzero). The optimality of (u_k) was also conjectured in [11], and proven subsequently in [40], where a crucial tool is a symmetrization technique for boudaryless integral currents.

In the present paper we instead focus on the *vortex map* u in $n = 2$ dimensions, and provide the optimal upper bound for $\overline{\mathcal{A}}(u, \mathrm{B}_l(0))$, for all $l > 0$. The vortex map, that is

$$u(x) := \frac{x}{|x|}, \qquad x \in \Omega \setminus \{0\},\ \Omega = \mathrm{B}_l(0) \subset \mathbb{R}^2, \tag{1.5}$$

belongs to $W^{1,p}(\Omega, \mathbb{R}^2)$ for all $p \in [1, 2)$, but not to $W^{1,2}(\Omega, \mathbb{R}^2)$), and its image is the one-dimensional unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$, so that $Ju(x) = \det(\nabla u(x)) = 0$ for all $x \in \Omega \setminus \{0\}$. In [1, Lemma 5.2], the authors show[3] that, for l large enough,

$$\overline{\mathcal{A}}(u, \mathrm{B}_l(0)) = |\mathcal{G}_u| + \pi = \int_{\mathrm{B}_l(0)} \sqrt{1 + |\nabla u|^2}dx + \pi. \tag{1.6}$$

With the aid of an example, they also show that $\overline{\mathcal{A}}(u, \mathrm{B}_l(0))$ must be strictly smaller than the right-hand side of (1.6), since there is a sequence of C^1-maps approximating u and having, asymptotically, a lower value of $\mathcal{A}(\cdot, \Omega)$. We anticipate here that, when l is small, the above mentioned sequence is not optimal, and the construction of a recovery sequence for $\overline{\mathcal{A}}(u, \mathrm{B}_l)$ is much more involved and requires to solve a sort of Plateau-type problem in $\mathbb{R}^3$ with singular boundary, with a part of multiplicity 2. This has been studied in [9], where with a reflection argument with respect to a plane, it can be seen as a non-parametric Plateau-type problem with a partial free boundary; in the special setting of [9] it is possible to show that, excluding a singular configuration (corresponding to l large), the solution is non-parametric and attains the zero boundary condition on the free part (we refer to [10] for a more general setting where similar results are obtained).

To state our main result we need to fix some notation. For $l > 0$ we denote $R_{2l} := (0, 2l) \times (-1, 1)$ and let $\partial_D R_{2l} := (\{0, 2l\} \times [-1, 1]) \cup ((0, 2l) \times \{-1\})$ be what we call the Dirichlet boundary of R_{2l}. Define $\varphi : \partial_D R_{2l} \to [0, 1]$ as $\varphi(t, s) := \sqrt{1 - s^2}$ if $(t, s) \in \{0, 2l\} \times [-1, 1]$ and $\varphi(t, s) := 0$ if $(t, s) \in (0, 2l) \times \{-1\}$. Let

$$\widetilde{\mathcal{H}}_{2l} := \{h : [0, 2l] \to [-1, 1],\ h \text{ continuous},\ h(0) = h(2l) = 1\},$$
$$\mathcal{X}_{D,\varphi} := \{\psi \in W^{1,1}(R_{2l}) : \psi = \varphi \text{ on } \partial_D R_{2l}\},$$

and for any $h \in \widetilde{\mathcal{H}}_{2l}$ set $G_h := \{(t, s) \in R_{2l} : s = h(t)\}$ and $SG_h := \{(t, s) \in R_{2l} : s \leq h(t)\}$. The main result of the present paper (see Theorem 3.4) reads as follows:

[3] In [1] the proof of (1.6) is given also for $N = n \geq 2$, where now π in (1.6) is replaced by ω_n.

Theorem 1.1. *Let $N = n = 2$, $l > 0$ and $u : \mathrm{B}_l(0) \to \mathbb{R}^2$ be the vortex map defined in* (1.5)*. Then*

$$\overline{\mathcal{A}}(u, \mathrm{B}_l(0)) \le \int_{\mathrm{B}_l(0)} \sqrt{1 + |\nabla u|^2} dx + \inf\{\mathcal{A}(\psi, SG_h) : (h, \psi) \in \widetilde{\mathcal{H}}_{2l} \times \mathcal{X}_{D,\varphi},\ \psi = 0 \text{ on } G_h\}. \quad (1.7)$$

We emphasize that, for l large, the infimum on the right-hand side is π. Further, thanks to the opposite inequality proved in [8], equality holds in Theorem 1.1, for each value of $l > 0$.

For l small, the fact that the sequence leading to the value in (1.6) is not optimal is strongly related with the choice of the L^1-convergence in the definition (1.2) of $\overline{\mathcal{A}}(\cdot, \Omega)$. Even if this seems the most natural notion of convergence for the approximating maps v_k of u, one can also opts to choose stronger topologies. Some results are known when one chooses, instead of the L^1-convergence, the strict convergence in $BV(\Omega; \mathbb{R}^2)$ (see [5, 6, 16, 17, 38]). With this convergence, it has been shown in [5] that the relaxed area of the vortex map u always equals the right-hand side of (1.6).

In order to give an idea of how the value π in (1.6) pops up (and then how it appears in (1.7) for l large), it is convenient to introduce the tool of Cartesian currents. One can regard the graphs $G_v = \{(x, y) \in \Omega \times \mathbb{R}^2 : y = v(x)\}$ of a C^1 map $v : \Omega \to \mathbb{R}^2$ as an integer multiplicity 2-current in $\Omega \times \mathbb{R}^2$. It is seen that a sequence (G_{u_k}) with u_k approaching u and with $\sup_k \mathcal{A}(u_k, \Omega) < +\infty$, converges[4], up to subsequences, to a Cartesian current T which splits as $T = \mathcal{G}_u + S$, with S a vertical integral current such that $\partial S = -\partial \mathcal{G}_u$. A direct computation shows that

$$\partial \mathcal{G}_u = -\delta_0 \times \partial [\![B_1]\!]$$

(see [29, Section 3.2.2]), so that the problem of determining the value of $\overline{\mathcal{A}}(u, \Omega)$ is somehow related to the computation of the mass of a mass-minimizing vertical current $S_{\min} \in \mathcal{D}_2(\Omega \times \mathbb{R}^2)$ satisfying

$$\partial S_{\min} = \delta_0 \times \partial [\![B_1]\!] \qquad \text{in } \mathcal{D}_1(\Omega \times \mathbb{R}^2). \quad (1.8)$$

In some cases, and in particular for l large, these two problems are related, and it turns out that $S_{\min} = \delta_0 \times [\![B_1]\!]$, whose mass is π. However $S_{\min} \neq \delta_0 \times [\![B_1]\!]$ for l small. Moreover, the two problems of determining $S_{\min}$ and the value of the relaxed area functional are, unfortunately, not related in general. This is mainly due to the following two obstructions:

- we have to guarantee that the current $\mathcal{G}_u + S_{\min}$ is obtained as a limit of smooth graphs, that is not easy to establish, since not all Cartesian currents can be obtained as such limits (see [29, Section 4.2.2]);
- even if $\mathcal{G}_u + S_{\min}$ is limit of graphs $\mathcal{G}_{u_k}$ of smooth maps u_k, nothing ensures that $\mathcal{A}(u_k, \Omega) \to \overline{\mathcal{A}}(u, \Omega)$, due to possible cancellations of the currents $\mathcal{G}_{u_k}$ that, in the limit, might overlap with opposite orientation.

Actually, in many cases, as in the one considered in this paper, for an optimal sequence (u_k) realizing the value of $\overline{\mathcal{A}}(u, \Omega)$, it holds

$$\mathcal{G}_{u_k} \rightharpoonup \mathcal{G}_u + S_{\mathrm{opt}} \neq \mathcal{G}_u + S_{\min}, \quad (1.9)$$

and the limit vertical part S_{opt} satisfies $|S_{\mathrm{opt}}| > |S_{\min}|$. For instance, if l is small, it is possible to construct a sequence $(\widehat{u}_k)$ approaching u which is not a recovery sequence for $\overline{\mathcal{A}}(u, \Omega)$, but whose limit vertical part $S_{\min}$ has mass strictly smaller than the mass of S_{opt} (see Section 4.2). In this case, a suitable projection of $S_{\min}$ in $\mathbb{R}^3$ is half of a classical area-minimizing catenoid between two unit circles at distance $2l$ from each other.

[4] This is a consequence of Federer-Fleming closure theorem.

An additional source of difficulties in the computation of $\overline{\mathcal{A}}(u,\Omega)$ is due to an example [40] valid for the triple junction map u_T, and showing that the equality

$$\overline{\mathcal{A}}(u_T,\Omega) = |\mathcal{G}_{u_T}| + 3m_l \tag{1.10}$$

holds only under some additional requirements; for instance if the triple junction point is exactly located at the origin 0 and the domain is a disc $\Omega = \mathrm{B}_l(0)$ around it. In particular, for different domains, (1.10) is no longer valid, and S_{opt} is a vertical current whose support projection on Ω is a set connecting the triple point with $\partial\Omega$, and which does not coincide with (neither is a subset of) the jump set of u_T (see [40, Example in Section 6] and also [7] for other non-symmetric settings).

A similar behaviour of the vertical part S_{opt} holds for u: when l is small, the projection of S_{opt} on $\mathrm{B}_l(0)$ concentrates over a radius connecting 0 to $\partial\mathrm{B}_l(0)$. However, if the domain Ω loses its symmetry, almost nothing is known about S_{opt}.

This kind of phenomena have been observed also in other cases, as in [12, 13] where BV-maps $u : \Omega \to \mathbb{R}^2$ with a prescribed discontinuity on a curve (jump set) are considered. The creation of such "phantom bridges" between the singularities of the map u and the boundary of the domain is very specific of the choice of the L^1-convergence in the computation of $\overline{\mathcal{A}}(\cdot,\Omega)$. As already said, other choices are possible, giving rise to different relaxed functionals[5] [12, 13].

The nonlocality and the uncontrollability of S_{opt} are more and more evident if we try to generalize (1.10) dropping the assumption that the range of u_T consists of the vertices of an equilateral triangle. If we assume that u_T takes values in $\{\alpha,\beta,\gamma\}$, three generic (not aligned) points in $\mathbb{R}^2$ then, also if the domain of u_T is symmetric, there is no sharp computation of $\overline{\mathcal{A}}(u_T,\Omega)$. In this case, the analysis is related to an entangled Plateau problem, where three area-minimizing discs have as partial free boundary three curves connecting the couples of points in $\{\alpha,\beta,\gamma\}$, respectively, and where these three curves are forced to overlap. Some partial results had been obtained in [7], where the authors find an upper bound for $\overline{\mathcal{A}}(u_T,\Omega)$. However the question of finding the value of $\overline{\mathcal{A}}(\cdot,\Omega)$ for this piecewise constant maps seems to be difficult. In the case that u is piecewise costant and takes three values vertices of an equilateral triangle, as for the triple junction map but in general domains, some upper bounds have been provided [41]. The singular contribution of the area is related with the flat norm of the distributional Jacobian of such maps [24]. Similarly, when $u \in W^{1,1}(\Omega;\mathbb{S}^1)$ is circle-valued, it is possible to show that the singular contribution of the area is bounded above by (a suitable multiple) of the flat norm of $\det(\nabla u)$ (see [14]).

Let us go back to the minimum problem

$$\inf\{\mathcal{A}(\psi, SG_h) : (h,\psi) \in \widetilde{\mathcal{H}}_{2l} \times \mathcal{X}_{D,\varphi},\ \psi = 0 \text{ on } G_h\}. \tag{1.11}$$

Following [9], this problem has many formulations and it is proved that the infimum in (1.11) coincides with

$$\min\left\{\mathcal{F}_{2l}(h,\psi) : (h,\psi) \in X_{2l}^{\mathrm{conv}}\right\}. \tag{1.12}$$

Here we refer to Section 3 for the notation and definition of $\mathcal{F}_{2l}$. Also, a solution to this minimum problem has been proven to exist and satisfies suitable regularity property if $l \le l_0$, for some threshold $l_0 > 0$ (see Theorem 3.2). If instead $l > l_0$, the unique solution to (1.12) is given by the two constants maps $h \equiv 1$ and $\psi \equiv 0$, corresponding to the case where $\mathcal{F}_{2l}$ measures the area of two half-discs of radius 1, namely providing the value π appearing in (1.6).

[5] Relaxing $\mathcal{A}(\cdot,\Omega)$ in stronger topologies τ is possible (see, e.g., [5, 38]); however, this would make more difficult to prove, eventually, τ-coercivity of $\overline{\mathcal{A}}(\cdot,\Omega)$. In addition, it could destroy the interesting nonlocal phenomena related to the appearence of certain nonstandard Plateau problems, which are the focus of this paper.

We do not know the explicit value of the threshold l_0. However, it is clear that $l_0 > \frac{1}{2}$ (see [9]). Furthermore, let the surface Σ^+ be the graph of a regular solution ψ when $l < l_0$. Doubling the surface Σ^+ by considering its symmetric with respect to the plane containing R_{2l}, and then taking the union Σ of these two area-minimizing surfaces, it turns out that Σ solves a non-standard Plateau problem, spanning a nonsimple curve which shows self-intersections (this is the union of Γ with its symmetric with respect to R_{2l}, the obtained curve is the union of two circles connected by a segment [9]). Again, the obtained area-minimizing surface is a sort of catenoid forced to contain a segment for l small, and two distinct discs spanning the two circles for l large. The restriction of Σ to the set $\overline{B}_1 \times [0, l]$ is a suitable projection in $\mathbb{R}^3$ of the aforementioned vertical current $S_{\rm opt}$.

In order to prove our main result, the analysis consists in a careful definition of a recovery sequence (u_k) converging to the vortex map, and thus such that $\mathcal{A}(u_k, \mathrm{B}_l(0))$ approaches the value on the right-hand side of (1.7) as $k \to +\infty$. To explicitely construct u_k, we need first to relate the minimum problem stated in (1.12) with the non-parametric Plateau-type problem in (1.11); this is obtained in [9], where we exploit the convexity of the domain together with some well-known regularity results for the solution of the Plateau problem in this setting. This analysis leads us to Theorem 3.2, which characterizes the solution of (1.12), and which is based on a regularity result for the minimizing pair $(h^\star, \psi^\star) \in \widetilde{\mathcal{H}}_{2l} \times \mathcal{X}_{D,\varphi}$. Finally, thanks to the regularity results that we have obtained (especially, boundary regularity), in Section 5 we define explicitely the maps u_k, making a crucial use of rescaled versions of the area-minimizing surface Σ in a vertical copy of $\mathbb{R}^3$ inside $\mathbb{R}^4$, and prove the upper bound in Theorem 3.4.

The paper is organized as follows: in Sections 2 and 3 we introduce some notation and the setting of the problem. In Section 4 we provide some examples of potential recovery sequences, one of which is optimal in the case l large. Finally, in Section 5 we construct a recovery sequence in the more involved case $l \le l_0$.

2 Preliminaries

The symbol $\mathcal{A}(v, \Omega)$ denotes the classical area of the graph of a smooth map $v : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$, given by the right hand side of (1.1). We will deal with the case $n = 2$ and mostly with the cases $(n, N) = (2, 1)$ and $(n, N) = (2, 2)$. The L^1-relaxed area functional is denoted by $\overline{\mathcal{A}}(v, \Omega)$ and is defined in (1.2).

We first remark that the infimum in (1.2) can be equivalently considered as taken over the class of sequences $(v_k) \subset \mathrm{Lip}(\Omega; \mathbb{R}^2)$. This does not change the value of $\overline{\mathcal{A}}(\cdot, \Omega)$, as observed in [11].

Recall that in formula (1.1) the symbol $\mathcal{M}(\nabla v)$ denotes the vector whose entries are all determinants of the minors of ∇v. Precisely, let α and β be subsets of $\{1, 2\}$, let $\bar{\alpha}$ denote the complementary set of α, namely $\bar{\alpha} = \{1, 2\} \setminus \alpha$, let $|\cdot|$ denote the cardinality, and let $A \in \mathbb{R}^{2\times 2}$ be a matrix. Then, if $|\alpha| + |\beta| = 2$, we denote by $M^\beta_{\bar{\alpha}}(A)$ the determinant of the submatrix of A whose lines are those with index in β, and columns with index in $\bar{\alpha}$. By convention $M^\emptyset_\emptyset(A) = 1$ and moreover

$$M^i_j = a_{ij}, \qquad i, j \in \{1, 2\}, \qquad\qquad M^{\{1,2\}}_{\{1,2\}}(A) = \det A,$$

and the vector $\mathcal{M}(A)$ takes the form

$$\mathcal{M}(A) = (M^\beta_{\bar{\alpha}})(A) = (1, a_{11}, a_{12}, a_{21}, a_{22}, \det A),$$

where α and β run over all the subsets of $\{1, 2\}$ with the constraint $|\alpha| + |\beta| = 2$. We identify α and β as multi-indices in $\{1, 2\}$.

2.0.1 Area in cylindrical coordinates

Polar coordinates in the source space $\mathbb{R}^2_{\text{source}}$ are denoted by (r, α). Polar coordinates in the target space $\mathbb{R}^2_{\text{target}}$ are denoted by (ρ, θ).

Assume that $B = \{(r, \alpha) \in \mathbb{R}^2 : r \in (r_0, r_1),\ \alpha \in (\alpha_0, \alpha_1)\}$; then the area of the graph of the smooth map $v = (v_1, v_2)$ in polar coordinates over B is given by

$$\mathcal{A}(v, B) = \int_{r_0}^{r_1} \int_{\alpha_0}^{\alpha_1} |\mathcal{M}(\nabla v)|(r, \alpha)\ r dr d\alpha.$$

Recall that, for $i \in \{1, 2\}$, we have

$$\partial_{x_1} v_i = \cos\alpha \partial_r v_i - \frac{1}{r} \sin\alpha \partial_\alpha v_i, \qquad \partial_{x_2} v_i = \sin\alpha \partial_r v_i + \frac{1}{r} \cos\alpha \partial_\alpha v_i.$$

Hence

$$\begin{aligned} |\nabla v_i|^2 &= (\partial_r v_i)^2 + \frac{1}{r^2}(\partial_\alpha v_i)^2, \qquad i \in \{1, 2\}, \\ \partial_{x_1} v_1 \partial_{x_2} v_2 - \partial_{x_2} v_1 \partial_{x_1} v_2 &= \frac{1}{r}\Big(\partial_r v_1 \partial_\alpha v_2 - \partial_\alpha v_1 \partial_r v_2\Big). \end{aligned} \tag{2.1}$$

Thus the area of the graph of v over B is given by

$$\begin{aligned} &\mathcal{A}(v, B) \\ = &\int_{r_0}^{r_1} \int_{\alpha_0}^{\alpha_1} \sqrt{1 + (\partial_r v_1)^2 + (\partial_r v_2)^2 + \frac{1}{r^2}\left\{(\partial_\alpha v_1)^2 + (\partial_\alpha v_2)^2 + \Big(\partial_r v_1 \partial_\alpha v_2 - \partial_\alpha v_1 \partial_r v_2\Big)^2\right\}}\ r dr d\alpha. \end{aligned} \tag{2.2}$$

We denote by $\mathrm{B}_r = \mathrm{B}_r(0) \subset \mathbb{R}^2 = \mathbb{R}^2_{\text{source}}$ the open disc centered at 0 with radius $r > 0$ in the source space. Our reference domain is $\Omega = \mathrm{B}_l \subset \mathbb{R}^2_{\text{source}} = \mathbb{R}^2_{(x_1, x_2)}$ where $l > 0$ is fixed once for all.

2.1 Graphs in codimension 1

Let Ω be an open bounded set and let $v \in L^1(\Omega)$. If $v \in C^1(\Omega)$ the classical area of its graph is given by

$$\mathcal{A}(v, \Omega) := \int_\Omega \sqrt{1 + |\nabla v|^2} dx.$$

This notion is extended to every function $v \in L^1(\Omega)$ by relaxation as in (1.2), and $\overline{\mathcal{A}}(v, \Omega)$ coincides with (1.3). For all $v \in L^1(\Omega)$ we denote by $R_v \subseteq \Omega$ the set of regular points of v, *i.e.*, the set consisting of points x which are Lebesgue points for v, $v(x)$ coincides with the Lebesgue value of v at x, and v is approximately differentiable at x. We also set

$$\begin{aligned} G_v^R &:= \{(x, v(x)) \in R_v \times \mathbb{R}\}, \\ SG_v^R &:= \{(x, y) \in R_v \times \mathbb{R} : y < v(x)\}. \end{aligned}$$

We often will identify SG_v^R with the integral 3-current $[\![SG_v]\!] \in \mathcal{D}_3(\Omega \times \mathbb{R})$. If v is a function of bounded variation, $\Omega \setminus R_v$ has zero Lebesgue measure, so that the current $[\![SG_v]\!]$ coincides with the integration over the subgraph

$$SG_v := \{(x, y) \in \Omega \times \mathbb{R} : y < v(x)\}.$$

For this reason we often identify $SG_v = SG_v^R$. It is well-known that the perimeter of SG_v in $\Omega \times \mathbb{R}$ coincides with $\overline{\mathcal{A}}(v, \Omega)$.

The support of the boundary of $[\![SG_v]\!]$ includes the graph G_v^R, but in general consists also of additional parts, called vertical. We denote by

$$\mathcal{G}_v := \partial[\![SG_v]\!] \llcorner (\Omega \times \mathbb{R}),$$

the generalized graph of v, which is a 2-integral current supported on $\partial^* SG_v$, the reduced boundary of SG_v in $\Omega \times \mathbb{R}$.

Let $\widehat{\Omega} \subset \mathbb{R}^2$ be a bounded open set such that $\Omega \subseteq \widehat{\Omega}$, and suppose that $L := \widehat{\Omega} \cap \partial\Omega$ is a rectifiable curve. Given $\psi \in BV(\Omega)$ and a $W^{1,1}$ function $\varphi : \widehat{\Omega} \to \mathbb{R}$, we can consider

$$\overline{\psi} := \begin{cases} f & \text{on } \Omega, \\ \varphi & \text{on } \widehat{\Omega} \setminus \Omega. \end{cases}$$

Then (see [30], [2])

$$\overline{\mathcal{A}}(\overline{\psi}, \widehat{\Omega}) = \overline{\mathcal{A}}(\psi, \Omega) + \int_L |\psi - \varphi| d\mathcal{H}^1 + \overline{\mathcal{A}}(\varphi, \widehat{\Omega} \setminus \overline{\Omega}).$$

3 Setting of the problem

Let us focus on the minimum problem on the right hand side of (1.7), i.e.,

$$\inf\{\mathcal{A}(\psi, SG_h) : (h, \psi) \in \widetilde{\mathcal{H}}_{2l} \times \mathcal{X}_{D,\varphi},\ \psi = 0 \text{ on } G_h\}. \tag{3.1}$$

Definition 3.1 (The functional $\mathcal{F}_{2l}$). *We define*

$$X_{2l}^{\mathrm{conv}} := \{(h, \psi) : h \in \mathcal{H}_{2l}, \psi \in BV(R_{2l}, [0,1]), \psi = 0 \text{ on } R_{2l} \setminus SG_h\}, \tag{3.2}$$

$$\mathcal{H}_{2l} = \big\{h : [0, 2l] \to [-1, 1],\ h \text{ convex},\ h(w_1) = h(2l - w_1)\ \forall w_1 \in [0, 2l]\big\}. \tag{3.3}$$

and for any $(h, \psi) \in X_{2l}^{\mathrm{conv}}$,

$$\mathcal{F}_{2l}(h, \psi) := \mathcal{A}(\psi; R_{2l}) - \mathcal{H}^2(R_{2l} \setminus SG_h) + \int_{\partial_D R_{2l}} |\psi - \varphi| d\mathcal{H}^1 + \int_{\partial R_{2l} \setminus \partial_D R_{2l}} |\psi|\ d\mathcal{H}^1. \tag{3.4}$$

In [9, Theorem 1.2] it is shown that

$$\inf\{\mathcal{A}(\psi, SG_h) : (h, \psi) \in \widetilde{\mathcal{H}}_{2l} \times \mathcal{X}_{D,\varphi},\ \psi = 0 \text{ on } G_h\} = \inf\big\{\mathcal{F}_{2l}(h, \psi) : (h, \psi) \in X_{2l}^{\mathrm{conv}}\big\}, \tag{3.5}$$

and for this reason it is necessary to investigate existence and regularity of minimizers of $\mathcal{F}_{2l}$. To this aim it is first convenient to extend φ in the doubled rectangle $\overline{R}_{2l}$ by defining the extension $\widehat{\varphi}$ as:

$$\widehat{\varphi}(w_1, w_2) = \widehat{\varphi}(0, w_2) := \sqrt{1 - w_2^2} \qquad \forall (w_1, w_2) \in \overline{R}_{2l}. \tag{3.6}$$

From [9, Theorem 1.1] the following result follows:

Theorem 3.2 (Minimizing pairs). *There exists $(h^\star, \psi^\star) \in X_{2l}^{\mathrm{conv}}$ such that*

$$\mathcal{F}_{2l}(h^\star, \psi^\star) = \min\big\{\mathcal{F}_{2l}(h, \psi) : (h, \psi) \in X_{2l}^{\mathrm{conv}}\big\}, \tag{3.7}$$

and $\psi^\star$ is symmetric with respect to $\{w_1 = l\} \cap R_{2l}$. Moreover there exists a threshold $l_0 > 0$ such that, for $l > l_0$ the above minimizer is $(h^\star, \psi^\star) = (1, 0)$, two constant functions, whereas for $0 < l \le l_0$ the above minimizer satisfies the following features: $h^\star$ is not identically -1 and *(i)* $h^\star(0) = 1 = h^\star(2l)$*, and* $h^\star > -1$ *in* $(0, 2l)$*;*

(ii) $\psi^\star$ *is locally Lipschitz, analytic, and strictly positive in* $SG_{h^\star}$*;*

(iii) $\psi^\star$ *is continuous up to the boundary of* $SG_{h^\star}$*, and attains the boundary conditions, i.e., for* $(w_1, w_2) \in \partial SG_{h^*}$*,*

$$\psi^\star(w_1, w_2) = \begin{cases} 0 & \text{if} \quad w_2 = -1 \text{ or } w_2 = h^\star(w_1), \\ \sqrt{1 - w_2{}^2} & \text{if} \quad w_1 = 0 \text{ or } w_1 = 2l, \end{cases} \tag{3.8}$$

hence

$$\mathcal{F}_{2l}(h^\star, \psi^\star) = \mathcal{A}(\psi^\star, SG_{h^\star}); \tag{3.9}$$

(iv) we have

$$\psi^\star < \widehat{\varphi} \text{ in } R_{2l}. \tag{3.10}$$

A minimizer $(h^\star, \psi^\star)$ of (3.7) is needed for constructing a recovery sequence $(u_k) \subset \mathrm{Lip}(\Omega, \mathbb{R}^2)$, see formulas (5.21) and (5.23): we know that $\psi^\star$ is locally Lipschitz, but not Lipschitz, in R_{2l}, therefore we need first a regularization procedure. This is made in Lemma 3.3 below, that will be used in the proof of step 2 of Theorem 3.4.

Let $(h^\star, \psi^\star)$ be a minimizer provided by Theorem 3.2, and assume that $h^\star$ is not identically -1 (namely, we are in the case $l \leq l_0$). We fix an integer $m > 0$ and, recalling the definition of $\widehat{\varphi}$ in (3.6), define

$$\varphi_m := \left(\widehat{\varphi} - \frac{2}{m}\right) \vee 0 \qquad \text{in } \overline{R}_{2l}. \tag{3.11}$$

We observe that φ_m is Lipschitz continuous in $\overline{R}_{2l}$. We then set

$$\psi^\star_m := \left(\left(\psi^\star - \frac{1}{m}\right) \vee 0\right)\right) \wedge \varphi_m \qquad \text{in } R_{2l}. \tag{3.12}$$

Since $\psi^\star$ is locally Lipschitz in R_{2l}, an easy check shows that $\psi^\star_m$ is Lipschitz continuous in R_{2l} for any m (with an unbounded Lipschitz constant as $m \to +\infty$). This follows from the fact that $\psi^\star$ is continuous up to the boundary of R_{2l} (see Theorem 3.2 (iii)) and hence $\psi^\star_m$ coincides with either 0 or φ_m in a neighborhood of $(\partial_D R_{2l}) \cup G_{h^\star}$ in R_{2l}. Furthermore, still $\psi^\star_m = 0$ on the upper graph $\overline{R_{2l}} \setminus SG_{h^\star} = \{(w_1, w_2) \in \overline{R_{2l}} : w_2 \geq h^\star(w_1)\}$ of $h^\star$.

Lemma 3.3 (Properties of $\psi^\star_m$). *Let* $(h^\star, \psi^\star)$ *be a minimizer of* $\mathcal{F}_{2l}$ *as in Theorem 3.2 and assume* $h^\star$ *is not identically* -1*. For all* $m > 0$ *let* $\psi^\star_m$ *be defined as in* (3.12)*. Then:*

(i) $\psi^\star_m$ *is Lipschitz continuous in* $\overline{SG_{h^\star}}$*,* $\psi^\star_m = 0$ *on* $([0, 2l] \times \{-1\}) \cup (\overline{R_{2l}} \setminus SG_{h^\star})$*, and* $\psi^\star_m(0, \cdot) = \varphi_m(0, \cdot)$*, so that* $|\partial_{w_2} \psi^\star_m(0, \cdot)| \leq |\partial_{w_2} \varphi(0, \cdot)| = |\partial_{w_2} \psi^\star(0, \cdot)|$ *a.e. in* $[-1, 1]$*;*

(ii) $(\psi^\star_m)$ *converges to* $\psi^\star$ *uniformly on* $\{0, 2l\} \times [-1, 1]$ *as* $m \to +\infty$*;*

(iii) we have

$$\lim_{m \to +\infty} \mathcal{A}(\psi^\star_m, SG_{h^\star}) = \mathcal{A}(\psi^\star, SG_{h^\star}). \tag{3.13}$$

As a consequence $\mathcal{F}_{2l}(h^\star, \psi^\star_m) \to \mathcal{F}_{2l}(h^\star, \psi^\star)$ *as* $m \to +\infty$*.*

Proof. (i) and (ii) are direct consequences of the definitions. To show (iii) we start to observe that $\psi^\star_m \to \psi^\star$ pointwise in R_{2l}: indeed, this follows from the definitions of $\varphi^\star_m$ and $\psi^\star_m$ up to noticing that $\varphi_m \to \widehat{\varphi}$ pointwise in R_{2l} as $m \to +\infty$, and $\psi^\star \le \widehat{\varphi}$ on R_{2l}. From Theorem 3.2 (iv) it follows that, at any point $(w_1, w_2) \in R_{2l}$, for m large enough $\varphi_m(w_1, w_2) > \psi^\star(w_1, w_2)$ (since $\widehat{\varphi}(w_1, w_2) > \psi^\star(w_1, w_2)$), so that $\psi^\star_m(w_1, w_2) = \psi^\star(w_1, w_2) - \frac{1}{m}$. As a consequence the set $A_m := \{0 < \psi^\star - \frac{1}{m} < \varphi_m\}$ satisfies

$$\lim_{m\to+\infty} \mathcal{H}^2(SG_{h^\star} \setminus A_m) = 0,$$

and on A_m it holds $\psi^\star_m = \psi^\star - \frac{1}{m}$ and $\nabla\psi^\star_m = \nabla\psi^\star$. Moreover, on $SG_{h^\star} \setminus A_m$, either $\psi^\star_m = 0$ (and hence $\nabla\psi^\star_m = 0$) or $\psi^\star_m = \varphi_m$ (and hence $\nabla\psi^\star_m = \nabla\varphi_m$). Therefore

$$\int_{SG_{h^\star}\setminus A_m} \sqrt{1 + |\nabla\psi^\star_m|^2}\, dx \le \int_{SG_{h^\star}\setminus A_m} \sqrt{1 + |\nabla\varphi_m|^2}\, dx$$

and

$$\lim_{m\to+\infty} \int_{SG_{h^\star}\setminus A_m} \sqrt{1 + |\nabla\psi^\star_m|^2}\, dx \le \lim_{m\to+\infty} \int_{SG_{h^\star}\setminus A_m} \sqrt{1 + |\nabla\varphi_m|^2}\, dx = 0,$$

because $|\nabla\varphi_m|$ are uniformly bounded in $L^1(R_{2l})$. Also

$$\mathcal{A}(\psi^\star_m, SG_{h^\star}) = \int_{A_m} \sqrt{1 + |\nabla\psi^\star|^2}\, dx + \int_{SG_{h^\star}\setminus A_m} \sqrt{1 + |\nabla\psi^\star_m|^2}\, dx,$$

and (3.13) follows. □

The main result of this paper reads as follows.

Theorem 3.4 (Upper bound for the area of the vortex map). *The relaxed area of the graph of the vortex map u satisfies*

$$\overline{\mathcal{A}}(u, \Omega) \le \int_\Omega |\mathcal{M}(\nabla u)|\, dx + \inf\left\{\mathcal{F}_{2l}(h, \psi) : (h, \psi) \in X^{\mathrm{conv}}_{2l}\right\}. \tag{3.14}$$

Notice that by (3.5) this result is equivalent to Theorem 1.1.

4 Some examples

Before going into the details of Theorem 3.4, it is worth making some nontrivial examples, which are also useful for the understanding of the proof of the theorem.

4.1 An approximating sequence of maps with degree zero: cylinder

In [1] the authors describe a sequence (u_k) of Lipschitz maps converging to u and taking values in $\mathbb{S}^1$; in our context, u_k is defined in polar coordinates as follows:

$$u_k(r,\alpha) := \begin{cases} u(r,\alpha) = (\cos\alpha, \sin\alpha) & \text{in } \Omega_1 := \Omega \setminus (\mathrm{B}_{r_k} \cup \{\alpha \in (-\alpha_k, \alpha_k)\}), \\ (\cos(\frac{r}{r_k}(\alpha - \pi) + \pi), \sin(\frac{r}{r_k}(\alpha - \pi) + \pi)) & \text{in } \mathrm{B}_{r_k} \setminus \{\alpha \in (-\alpha_k, \alpha_k)\}, \\ (\cos(\frac{\alpha_k - \pi}{\alpha_k}\alpha + \pi), \sin(\frac{\alpha_k - \pi}{\alpha_k}\alpha + \pi)) & \text{in } \{\alpha \in [0, \alpha_k)\} \setminus \mathrm{B}_{r_k}, \\ (\cos(\frac{-\alpha_k + \pi}{-\alpha_k}\alpha + \pi), \sin(\frac{-\alpha_k + \pi}{-\alpha_k}\alpha + \pi)) & \text{in } \{\alpha \in (-\alpha_k, 0)\} \setminus \mathrm{B}_{r_k}, \\ (\cos(\frac{r}{r_k}(\frac{\alpha_k - \pi}{\alpha_k}\alpha) + \pi), \sin(\frac{r}{r_k}(\frac{\alpha_k - \pi}{\alpha_k}\alpha) + \pi)) & \text{in } \mathrm{B}_{r_k} \cap \{\alpha \in [0, \alpha_k)\}, \\ (\cos(\frac{r}{r_k}(\frac{-\alpha_k + \pi}{-\alpha_k}\alpha) + \pi), \sin(\frac{r}{r_k}(\frac{-\alpha_k + \pi}{-\alpha_k}\alpha) + \pi)) & \text{in } \mathrm{B}_{r_k} \cap \{\alpha \in (-\alpha_k, \alpha_k)\}, \end{cases} \tag{4.1}$$

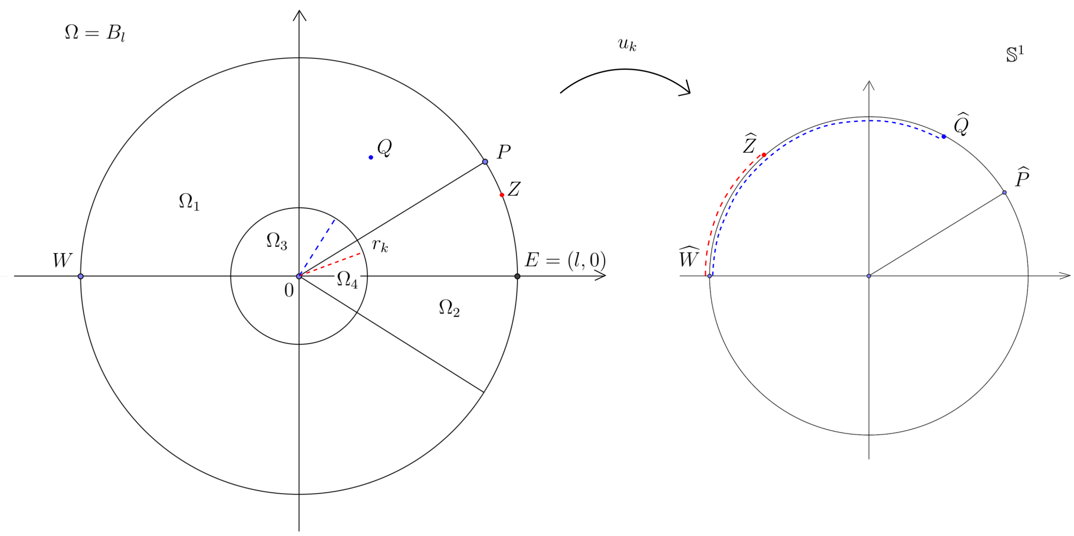

Figure 1: The map u_k in (4.1). We set $\hat{P} := P/|P| = \alpha_k$, $\hat{Q} := Q/|Q|$, $\hat{Z} := Z/|Z|$, $\hat{W} := W/|W|$. All points in $\Omega_1 \cup \Omega_2$ are retracted on $\mathbb{S}^1$ and suitably interpolated. The image of Ω_3 through u_k is as follows: u_k sends the generic dotted segment onto the (long) dotted arc on $\mathbb{S}^1$. Finally, the image of Ω_4 through u_k is as follows: u_k sends the generic dotted segment onto the (short) dotted arc on $\mathbb{S}^1$: Thus a short arc centered at $E/|E|$ remains uncovered.

where (r_k) and (α_k) are two infinitesimal sequences of positive numbers; see Fig. 1. Notice that $u_k(0,0) = (-1,0) = u_k(r,0)$ for $r \in (0,l)$. Moreover for $t \in (0,l)$ we have $u_k(\partial \mathrm{B}_t) = \partial B_1 \setminus \{\alpha \in (-\alpha_k, \alpha_k)\}$, and the degree of u_k is zero.

Remark 4.1. (u_k) is not a recovery sequence, due to Theorem 3.4. It is proven in [1] that

$$\lim_{k \to +\infty} \mathcal{A}(u_k, \Omega) = \int_\Omega |\mathcal{M}(\nabla u)| \, dx + 2\pi l,$$

and $2\pi l$ has the meaning of the lateral area of the cylinder of height l and basis the unit disc. This surface is not a minimizer of the problem on the right-hand side of (3.5) (where it corresponds to $h \equiv 1$).

4.2 A non-optimal approximating sequence of maps: catenoid union a flap

In this section we discuss another example of a sequence (u_k) converging to u. We replace the cylinder lateral surface[6] $[0,l] \times \{1\} \times (-\pi, \pi]$, which contains the image of $(r_k, l) \times (-\alpha_k, \alpha_k)$ through the map $\Psi_k(x) = (|x|, u_k(x))$ in the example of Section 4.1, with half[7] of a catenoid union a flap (see Fig. 3): calling this union $CF \llcorner (0,l) \times \mathbb{R}^2$, we have

$$CF =: \{(t, \overline{\rho}(t), \theta) : t \in [0, 2l],\ \theta \in (-\pi, \pi]\} \cup \{(t, r, 0) : t \in (0, 2l),\ r \in [\overline{\rho}(t), 1]\},$$

where $\overline{\rho}(t) := a \cosh(\frac{t-l}{a})$, and $a > 0$ is such that $\overline{\rho}(0) = 1$ (and $\overline{\rho}(2l) = 1$).

[6] In polar coordinates.

[7] For convenience, we consider the doubled segment $[0, 2l]$, in order to define the catenoid; then we restrict the construction to $(0, l)$.

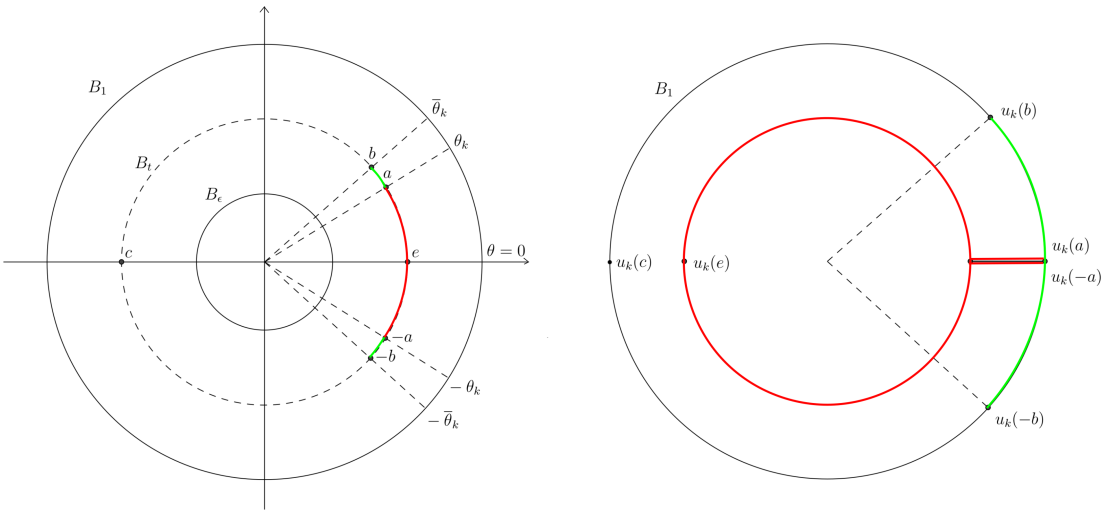

Figure 2: Source and target of the map u_k in the example of Section 4.2. The small interior circle in the right figure is a t-slice of a catenoid, whereas the horizontal segment is the t-section of the flap. The radius of the small circle is $\overline{\rho}(l)$.

Notice that CF "spans" $\Big(\{0,2l\} \times \{1\} \times (-\pi,\pi]\Big) \cup \Big([0,2l] \times \{1\} \times \{0\}\Big)$, which is the union of two unit circles joined by a segment.

Let $r_k > 0, \theta_k > 0$, $\overline{\theta}_k > \theta_k$ be such that $r_k, \theta_k, (\overline{\theta}_k - \theta_k) \to 0^+$ as $k \to +\infty$. Set

$$\rho(t) := \overline{\rho}\left(\frac{t-r_k}{l-r_k} l\right), \qquad t \in (r_k, l).$$

We define $u_k := u$ in $\Omega \setminus \Big(\mathrm{B}_{r_k} \cup \{\alpha \in (-\overline{\theta}_k, \overline{\theta}_k)\}\Big)$, in particular

$$u_k(\partial \mathrm{B}_t \setminus \{\alpha \in (-\overline{\theta}_k, \overline{\theta}_k)\}) = \partial B_1 \setminus \{\theta \in (-\overline{\theta}_k, \overline{\theta}_k)\}, \qquad t \in (r_k, l).$$

On $\{\alpha \in (-\overline{\theta}_k, \overline{\theta}_k)\} \setminus \mathrm{B}_{r_k}$ we define u_k in such a way that, for each $t \in (r_k, l)$, one has

$$u_k\left(\partial \mathrm{B}_t \cap \{\pm\alpha \in (\theta_k, \overline{\theta_k})\}\right) = \partial B_1 \cap \{\pm\theta \in (0, \overline{\theta}_k)\},$$
$$u_k\left(\partial \mathrm{B}_t \cap \{\pm\alpha \in (0, \theta_k)\}\right) = \{(r,0) \in \overline{\mathrm{B}}_1 : r \in [\rho(t), 1]\} \cup \Big(\partial B_{\rho(t)} \cap \{\theta \neq 0\}\Big).$$

See Fig. 2 for a representation of the map u_k. The several parts of the image are run so that the winding number around the origin is always null. To define u_k on B_{r_k} we adopt a construction similar to the one in (4.1). First of all, $u_k(0,0) := (-1,0)$. Then, in $\mathrm{B}_{r_k} \cap \{\alpha \in (-\pi,\pi) \setminus (-\overline{\theta}_k, \overline{\theta}_k)\}$ we impose u_k as in (4.1) with $\overline{\theta}_k$ replacing α_k. In $\mathrm{B}_{r_k} \cap \{\alpha \in (-\overline{\theta}_k, \overline{\theta}_k)\}$ we require

$$u_k([0,r_k], \alpha) := \partial B_1 \cap \{\pm\theta \in ((u_k)_2(r_k,\alpha), \pi)\}, \qquad \pm\alpha \in (0,\pi],$$

where $(u_k)_2$ is the second (angular) coordinate of u_k. Hence

$$u_k(\partial \mathrm{B}_t) \begin{cases} \subsetneq \partial B_1 & \text{if } t \in (0, r_k], \\ = (\partial B_1) \cup \{(r,0) \in \overline{B}_1 : r \in [\rho(t), 1]\} \cup (\partial B_{\rho(t)}) & \text{if } t \in (r_k, l). \end{cases}$$

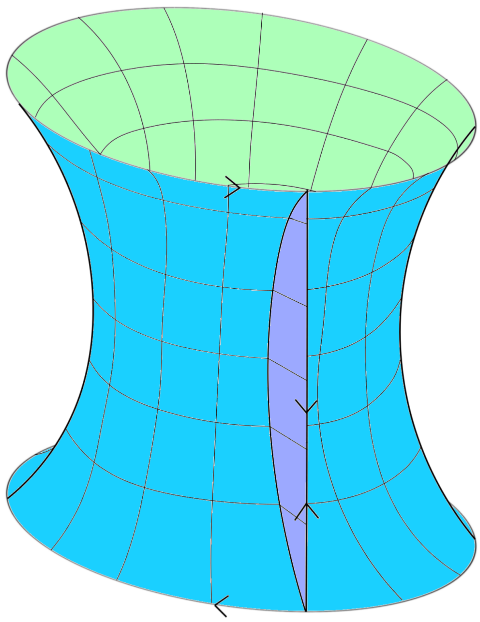

Figure 3: Catenoid union a flap, namely the set CF (Section 4.2).

Remark 4.2. Also in this case (u_k) is not a recovery sequence, due to Theorem 3.4, and the results in [8, 9]. For this particular sequence we have

$$\lim_{k\to+\infty} \mathcal{A}(u_k,\Omega) = \int_\Omega |\mathcal{M}(\nabla u)|\; dx + \mathcal{H}^2(\text{catenoid}) + 2\,\mathcal{H}^2(\text{flap}).$$

This surface is not a minimizer of problem on the right-hand side of (3.5). However it is worth noticing that, by minimality property of the catenoid, it can be proved that the set CF, treated as an integral current, is $S_{\min}$, the minimal vertical current closing the graph $\mathcal{G}_u$ of the vortex map in Ω (see the discussion in the Introduction).

4.3 The case of two discs

In [1], the authors describe a sequence (u_k) of maps converging to the vortex map u, simply defined as follows:

$$u_k(r,\alpha) := \phi_k(r)u(r,\alpha), \tag{4.2}$$

where $\phi_k : [0,l] \to [0,1]$ is a smooth function such that $\phi_k = 0$ in $[0,\frac{1}{k^2}]$, $\phi_k = 1$ in $[\frac{1}{k},l]$, and $0 \le \phi_k' \le 2k$. In this case (u_k) is a recovery sequence for l sufficiently large, due to [1, Lemma 4.2]. We have

$$\lim_{k\to+\infty} \mathcal{A}(u_k,\Omega) = \int_\Omega |\mathcal{M}(\nabla u)|\; dx + \pi,$$

and π has the meaning of the area of the unit disc. This surface, for l sufficiently large, is a minimizer of problem on the right-hand side of (3.5) (where it corresponds to $h \equiv -1$).

5 Proof of Theorem 3.4

In this section we prove Theorem 3.4. To this aim, we need to construct a sequence $(u_k) \subset \mathrm{Lip}(\Omega, \mathbb{R}^2)$ converging to u in $L^1(\Omega, \mathbb{R}^2)$ such that

$$\lim_{k \to +\infty} \mathcal{A}(u_k, \Omega) \leq \int_\Omega |\mathcal{M}(\nabla u)| dx + \mathcal{F}_{2l}(h^\star, \psi^\star),$$

where $(h^\star, \psi^\star)$ is a pair minimizing $\mathcal{F}_{2l}$ as in Theorem 3.2. We may assume that $h^\star$ is not identically -1, otherwise the result follows from [1] (and a recovery sequence is provided as in (4.2)).

We will specify various subsets of Ω and define the sequence (u_k) on each of these sets (see Fig. 4). More precisely, we will define u_k as a map taking values in $\mathbb{S}^1$ in the largest sector (step 1). This construction is similar to the one in [1] (see also Remark 5.1 below). The contribution of the area in this sector will equal, as $k \to +\infty$, the first term in (3.14). The second term will be instead provided by the contribution of u_k in region $C_k \setminus B_{r_k}$ (step 2), where we will need the aid of the functions $(h^\star, \psi^\star)$ (suitably regularized, in order to render u_k Lipschitz continuous). The other regions surrounding $C_k \setminus B_{r_k}$ are needed to glue u_k between the aforementioned regions. This is done in steps 3, 4 and 5, where it is also proven that the corresponding area contribution is negligible. Finally, in steps 6 and 7 we show the crucial estimates to prove (3.14). In Fig. 4 this subdivion of the domain Ω is drawn.

Remark 5.1. Our construction differs from the one in [1], even when in place of $(h^\star, \psi^\star)$ we use $(1, \sqrt{1-s^2})$ (*i.e.*, the one in Section 4.1) in the following sense. We use the full graph of $\pm\psi^\star$ to construct u_k and therefore, in the case when $(h^\star, \psi^\star)$ is replaced by $(1, \sqrt{1-s^2})$, the image of u_k covers the whole cylinder and not only a part of it. Since $h^\star$ *may be not identically* 1 (and actually is not explicit in general), the presence of a new set T_k is now needed, as an intermediate region to glue the trace of u_k along the two segments $\{\alpha = \pm\overline{\theta}_k\}$. The image set $u_k(T_k)$ covers a small part of the unit circle. See Fig. 4, where T_k is represented as the union of the two thin sectors in Ω. To glue all the piecese in order that u_k is Lipschitz, it will be useful to have two transition regions, one in a ball $B_{r_k/2}$ and one in the annulus $B_{r_k} \setminus B_{r_k/2}$. It is worth noticing that the curve $u_k \llcorner \partial \mathrm{B}_t$ has null winding number around the origin, for all $t \in (0, l)$.

Let $k \in \mathbb{N}$ and let $(r_k), (\theta_k), (\overline{\theta}_k)$ be infinitesimal sequences of positive numbers such that $\overline{\theta}_k - \theta_k =: \delta_k > 0$. We suppose[8]

$$\lim_{k \to +\infty} (\theta_k k) = 0. \tag{5.1}$$

Let B_{r_k} be the open disc centered at the origin with radius r_k, and

$$C_k := \{(r, \alpha) \in [0, l) \times [0, 2\pi) : \alpha \in [0, \theta_k] \cup [2\pi - \theta_k, 2\pi)\}, \tag{5.2}$$

be the half-cone in Ω, with vertex at the origin and aperture equal to $2\theta_k$, see Fig. 4. We set

$$C_k^+ := C_k \cap \{\alpha \in [0, \theta_k]\}, \quad C_k^- := C_k \cap \{\alpha \in [2\pi - \theta_k, 2\pi]\},$$

and we divide $C_k \cap (\Omega \setminus \mathrm{B}_{r_k})$ into two sets

$$C_k \setminus \mathrm{B}_{r_k} := \big(C_k^+ \setminus \mathrm{B}_{r_k}\big) \cup \big(C_k^- \setminus \mathrm{B}_{r_k}\big). \tag{5.3}$$

Finally, let

$$T_k := \{(r, \alpha) \in [0, l) \times [0, 2\pi) : \alpha \in [\theta_k, \overline{\theta}_k] \cup [2\pi - \overline{\theta}_k, 2\pi - \theta_k]\}. \tag{5.4}$$

[8] This assumption is used only in step 7.

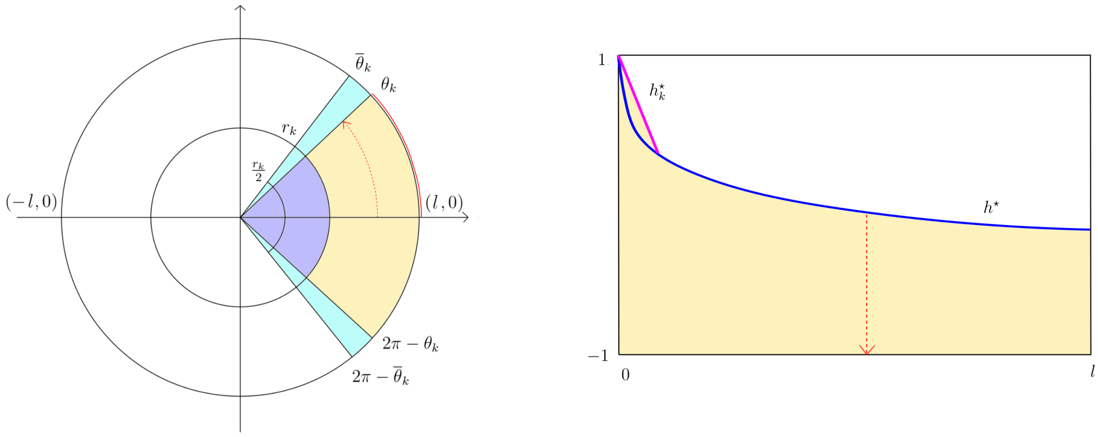

Figure 4: On the left the subdivision of B_l in sectors. Specifically, the sectors $C_k^+ \setminus \mathrm{B}_{r_k}$ and $C_k^- \setminus \mathrm{B}_{r_k}$ are emphasized in light grey. The map $\mathcal{T}_k$ defined in (5.18) sends $C_k^+ \setminus \mathrm{B}_{r_k}$ in the (reflected) subgraph of $h_k^\star$ in R_l, depicted on the right; it maps the segment joining $(r_k, 0)$ to $(1, 0)$ onto the graph of $h_k^\star$, and the radius corresponding to $\alpha = \theta_k$ onto the basis of R_l, following the orientation emphasized by the dashed arrow. The graph of $h_k^\star$ starts linearly from the point $(0, 1)$ in the interval $[0, 1/k)$ with negative derivative, then joins (and next coincides) with the graph of $h^\star$. The definition of u_k in $C_k^+ \setminus \mathrm{B}_{r_k}$ makes use of this parametrization of $SG_{h_k^\star} \cap \overline{R}_l$ (see (5.21)). This parametrization needs a reflection, in order to glue u_k on the horizontal segment $\{\alpha = 0\}$ with the definition of u_k in $C_k^- \setminus \mathrm{B}_{r_k}$.

Step 1. Definition of u_k on $\overline{\Omega \setminus (C_k \cup T_k)}$.

In this step our construction is similar to the one in [1, Lem. 5.3], see also (4.1); in order to define u_k, in the source we use polar coordinates (r, α) and Cartesian coordinates in the target. Define

$$u_k(r,\alpha) := \begin{cases} u(r,\alpha) = (\cos\alpha, \sin\alpha), & r \in (r_k/2, l), \alpha \in [\overline{\theta}_k, 2\pi - \overline{\theta}_k], \\ \Big(\cos(\frac{2r}{r_k}(\alpha - \pi) + \pi), \sin(\frac{2r}{r_k}(\alpha - \pi) + \pi)\Big), & r \in [0, r_k/2], \alpha \in [\overline{\theta}_k, 2\pi - \overline{\theta}_k]. \end{cases} \tag{5.5}$$

Obviously

$$\begin{aligned} &u_k(0,0) = (-1,0) = u_k(r,\pi), \qquad r \in [0,l), \\ &u_k(r,\overline{\theta}_k) = (\cos\overline{\theta}_k, \sin\overline{\theta}_k), \qquad u_k(r, 2\pi - \overline{\theta}_k) = (\cos\overline{\theta}_k, \sin(-\overline{\theta}_k)), \qquad r \in (r_k/2, l), \\ &u_k(r,\overline{\theta}_k) = \Big(\cos(\frac{2r}{r_k}(\overline{\theta}_k - \pi) + \pi)\ ,\ \sin(\frac{2r}{r_k}(\overline{\theta}_k - \pi) + \pi)\Big), \qquad r \in [0, r_k/2], \\ &u_k(r, 2\pi - \overline{\theta}_k) = \Big(\cos(\frac{2r}{r_k}(\pi - \overline{\theta}_k) + \pi)\ ,\ \sin(\frac{2r}{r_k}(\pi - \overline{\theta}_k) + \pi)\Big), \qquad r \in [0, r_k/2]. \end{aligned} \tag{5.6}$$

The relevant contribution to the area of the graph of u_k is the one in region C_k, and more specifically in $C_k \setminus B_{r_k}$; it is in this region that we need to use a minimizing pair of $\mathcal{F}_{2l}$.

Step 2. Definition of u_k on $C_k \setminus \mathrm{B}_{r_k}$.

We first need a regularization of $h^\star$: assuming without loss of generality $1/k < l$, we define

$$h_k^\star(w_1) := \begin{cases} h^\star(w_1) & \text{for } w_1 \in [\frac{1}{k}, l], \\ k\left(h^\star(\frac{1}{k}) - h^\star(0)\right) w_1 + h^\star(0) & \text{for } w_1 \in [0, \frac{1}{k}), \end{cases} \tag{5.7}$$

where we recall that $h^\star(0) = 1$ (see Theorem 3.2), and we set $h^\star_k(w_1) := h^\star_k(2l - w_1)$ for $w_1 \in [l, 2l]$ (see Fig. 4, right). Notice that $h^\star_k(0) = 1$, $h^\star_k \in \mathrm{Lip}([0, 2l])$ and the convexity of $h^\star$ implies that also $h^\star_k$ is convex, $h^\star_k \geq h^\star$, and therefore by Lemma 3.3 (i) we see that $(h^\star_k, \psi^\star_k) \in X^{\mathrm{conv}}_{2l}$, where $\psi^\star_k$ is the approximation of $\psi^\star$ considered in Lemma 3.3 (with $k = m$), see formula (3.12). Again by Lemma 3.3, $\mathcal{F}_{2l}(h^\star_k, \psi^\star_k) = \mathcal{F}_{2l}(h^\star, \psi^\star_k) + \int_0^{2l} (h^\star_k(w_1) - h^\star(w_1))\, dw_1 \to \mathcal{F}_{2l}(h^\star, \psi^\star)$ as $k \to +\infty$.

We start with the construction of u_k on $C^+_k \setminus \mathrm{B}_{r_k}$. Set

$$\tau_k : [r_k, l] \to [0, l], \qquad \tau_k(r) := \frac{l}{l - r_k}(r - r_k), \tag{5.8}$$

$$s_k : [r_k, l] \times [0, \theta_k] \to [-1, 1], \qquad s_k(r, \alpha) := \frac{1 + h^\star_k(\tau_k(r))}{\theta_k}\alpha - h^\star_k(\tau_k(r)). \tag{5.9}$$

Note that $s_k(r, \cdot) : [0, \theta_k] \to [-h^\star_k(\tau_k(r)), 1]$ is a bijective increasing function, for any $r \in [r_k, l]$, and

$$s_k(r, 0) = -h^\star_k(\tau_k(r)) \quad \text{for any } r \in [r_k, l], \text{ in particular } s_k(r_k, 0) = -1, \tag{5.10}$$

$$s_k(r, \theta_k) = 1, \qquad r \in [r_k, l], \tag{5.11}$$

$$s_k(r_k, \alpha) = \frac{2\alpha}{\theta_k} - 1, \qquad \alpha \in [0, \theta_k]. \tag{5.12}$$

We have, for all $r \in [r_k, l]$ and $\alpha \in [0, \theta_k]$,

$$\tau'_k(r) = \frac{l}{l - r_k}, \tag{5.13}$$

$$\partial_\alpha s_k(r, \alpha) = \frac{1 + h^\star_k(\tau_k(r))}{\theta_k}, \tag{5.14}$$

and, for almost every $r \in [r_k, l]$ and all $\alpha \in [0, \theta_k]$,

$$\partial_r s_k(r, \alpha) = \left(\frac{\alpha}{\theta_k} - 1\right) \tau'_k(r) h^{\star\prime}_k(\tau_k(r)) = \frac{l}{l - r_k}\left(\frac{\alpha}{\theta_k} - 1\right) h^{\star\prime}_k(\tau_k(r)). \tag{5.15}$$

Moreover we define

$$H_k : [0, l] \to [r_k, l], \qquad H_k(w_1) := \frac{l - r_k}{l} w_1 + r_k \tag{5.16}$$

to be the inverse of τ_k and, recalling that $\overline{R}_l = [0, l] \times [-1, 1]$,

$$\Theta_k : SG_{h^\star_k} \cap \overline{R}_l \to [0, \theta_k], \qquad \Theta_k(w_1, w_2) := \frac{\theta_k}{1 + h^\star_k(w_1)}(h^\star_k(w_1) - w_2). \tag{5.17}$$

Notice that $\Theta_k(w_1, \cdot) : [-1, h^\star_k(w_1)] \to [0, \theta_k]$ is a linearly decreasing bijective function[9]

The map

$$\mathcal{T}_k : C^+_k \setminus \mathrm{B}_{r_k} \to SG_{h^\star_k} \cap \overline{R}_l, \qquad \mathcal{T}_k(r, \alpha) := (\tau_k(r), -s_k(r, \alpha)), \tag{5.18}$$

is invertible, and its inverse is the map

$$\mathcal{T}_k^{-1} : SG_{h^\star_k} \cap \overline{R}_l \to C^+_k \setminus \mathrm{B}_{r_k}, \qquad \mathcal{T}_k^{-1}(w_1, w_2) := (H_k(w_1), \Theta_k(w_1, w_2)). \tag{5.19}$$

The modulus of the determinant of the Jacobian of $\mathcal{T}_k^{-1}$ is given by

$$|J_{\mathcal{T}_k^{-1}}| = \left(\frac{l - r_k}{l}\right) \frac{\theta_k}{1 + h^\star_k(w_1)}. \tag{5.20}$$

[9]We recall that in our hypothesis $h^*_k > -1$ by Theorem 3.2 (i).

We set

$$u_k(r,\alpha) := \Big(s_k(r,\alpha), \psi_k^\star\big(\mathcal{T}_k(r,\alpha)\big)\Big) = \Big(u_{k1}(r,\alpha), u_{k2}(r,\alpha)\Big), \qquad r\in[r_k,l], \alpha\in[0,\theta_k]. \tag{5.21}$$

Observe that, using the definition of $\psi_k^\star$,

$$\begin{aligned} &u_k \in \mathrm{Lip}(C_k^+ \setminus \mathrm{B}_{r_k}, \mathbb{R}^2),\\ &u_k(r,\theta_k) = (s_k(r,\theta_k), \psi_k^\star(\mathcal{T}_k(r,\theta_k))) = (1,0),\\ &u_k(r,0) = (-h_k^\star(\tau_k(r)), \psi_k^\star(\tau_k(r), h_k^\star(\tau_k(r)))) = (-h_k^\star(\tau_k(r)),0),\\ &u_k(r_k,\alpha) = (s_k(r_k,\alpha), \psi_k^\star(0,-s_k(r_k,\alpha))) = (s_k(r_k,\alpha), \varphi_k(0,-s_k(r_k,\alpha))), \end{aligned} \tag{5.22}$$

for $r \in [r_k, l]$ and $\alpha \in [0, \theta_k]$, as it follows from (5.8), (5.10), (5.11), and (3.8), where φ_k is defined in (3.11) (with $k = m$).

Eventually we define u_k on $C_k^- \setminus \mathrm{B}_{r_k}$ as

$$u_k(r,\alpha) := (u_{k1}(r, 2\pi-\alpha), -u_{k2}(r, 2\pi-\alpha)), \qquad r\in[r_k,l], \alpha\in[2\pi-\theta_k, 2\pi). \tag{5.23}$$

It turns out

$$\begin{aligned} &u_k \in \mathrm{Lip}(C_k^- \setminus \mathrm{B}_{r_k}, \mathbb{R}^2),\\ &u_k(r, 2\pi-\theta_k) = (1,0),\\ &u_k(r,2\pi) = (-h_k^\star(\tau_k(r)), -\psi_k^\star(\tau_k(r), h_k^\star(\tau_k(r)))) = (-h_k^\star(\tau_k(r)),0),\\ &u_k(r_k,\alpha) = (s_k(r_k, 2\pi-\alpha), -\psi_k^\star(0,-s_k(r_k,2\pi-\alpha))), \end{aligned}$$

for $r \in [r_k, l]$, $\alpha \in [2\pi-\theta_k, 2\pi)$.

The area of the graph of u_k on $C_k \setminus B_{r_k}$ will be computed in step 7.

Step 3. Definition of u_k on $C_k \cap (\overline{B}_{r_k} \setminus \mathrm{B}_{r_k/2})$ and its area contribution.

Let $G_{\psi_k^\star(0,\cdot)} \subset \mathbb{R}^2$ (resp. $G_{\psi^\star(0,\cdot)} \subset \mathbb{R}^2$) denote the graph of $\psi_k^\star(0,\cdot)$ (resp. of $\psi^\star(0,\cdot)$) on $[-1,1]$. We introduce the retraction map $\Upsilon : (\mathbb{R}\times[0,+\infty)) \setminus O \subset \mathbb{R}^2_{\mathrm{target}} \to G_{\psi^\star(0,\cdot)} \subset \mathbb{R}^2_{\mathrm{target}}$, $O = (0,0)$, defined by

$$\Upsilon(p) = q := G_{\psi^\star(0,\cdot)} \cap \ell_{Op} \qquad \forall p \in (\mathbb{R}\times[0,+\infty))\setminus O,$$

where ℓ_{Op} is the line passing through O and p. Then Υ is well-defined and it is Lipschitz continuous in a neighbourhood of $G_{\psi^\star(0,\cdot)}$ in $\mathbb{R}\times[0,+\infty)$. We also define

$$\Upsilon_k : G_{\psi_k^\star(0,\cdot)} \to G_{\psi^\star(0,\cdot)}$$

as the restriction of Υ to $G_{\psi_k^\star(0,\cdot)}$; see Fig. 5. As a consequence, since for $k\in\mathbb{N}$ large enough $G_{\psi_k^\star(0,\cdot)}$ is contained in a neighbourhood of $G_{\psi^\star(0,\cdot)}$, we have that Υ_k is Lipschitz continuous with Lipschitz constant independent of k. Notice also that $\Upsilon_k((-1,0)) = (-1,0)$ and $\Upsilon_k((1,0)) = (1,0)$.

We define u_k on $C_k^+ \cap (\overline{\mathrm{B}}_{r_k} \setminus \mathrm{B}_{r_k/2})$ setting, for $r \in [\frac{r_k}{2}, r_k]$ and $\alpha\in[0,\theta_k]$,

$$u_k(r,\alpha) := \Big(2 - \frac{2r}{r_k}\Big)\Upsilon_k\big(s_k(r_k,\alpha), \psi_k^\star(0,-s_k(r_k,\alpha))\big) + \Big(\frac{2r}{r_k} - 1\Big)\big(s_k(r_k,\alpha), \psi_k^\star(0,-s_k(r_k,\alpha))\big).$$

We have

$$u_k(r_k,\alpha) = (s_k(r_k,\alpha), \psi_k^\star(0,-s_k(r_k,\alpha))),$$

so that u_k glues, on $C_k^+ \cap \partial \mathrm{B}_{r_k}$, with the values obtained in step 2 (last formula in (5.22)), and

$$u_k(r,\theta_k) = (1,0), \qquad u_k(r,0) = (-1,0).$$

This formula shows that u_k also glues, on $C_k^+ \cap \{(r,\alpha) : r \in [r_k/2, r_k], \alpha \in \{0, \theta_k\}\}$, with the values obtained in step 2 (second and third formula in (5.22)). Moreover

$$u_k(r_k/2, \alpha) = \Upsilon_k\big(s_k(r_k,\alpha), \psi_k^\star(0, -s_k(r_k,\alpha))\big), \qquad \alpha \in [0, \theta_k]. \tag{5.24}$$

In addition, using (5.12), the derivatives of u_k satisfy, for $r \in (\frac{r_k}{2}, r_k)$ and $\alpha \in (0, \theta_k)$,

$$\begin{aligned}
\partial_r u_k(r,\alpha) &= -\frac{2}{r_k}\Upsilon_k\big(s_k(r_k,\alpha), \psi_k^\star(0,-s_k(r_k,\alpha))\big) + \frac{2}{r_k}\big(s_k(r_k,\alpha), \psi_k^\star(0,-s_k(r_k,\alpha))\big),\\
\partial_\alpha u_k(r,\alpha) &= \Big(2 - \frac{2r}{r_k}\Big)\nabla\Upsilon_k\big(s_k(r_k,\alpha), \psi_k^\star(0,-s_k(r_k,\alpha))\big)\cdot\Big(\frac{2}{\theta_k}, -\frac{2}{\theta_k}\partial_{w_2}\psi_k^\star(0,-s_k(r_k,\alpha))\Big)\\
&\quad + \Big(\frac{2r}{r_k} - 1\Big)\Big(\frac{2}{\theta_k}, -\frac{2}{\theta_k}\partial_{w_2}\psi_k^\star(0,-s_k(r_k,\alpha))\Big),
\end{aligned}$$

so that

$$\begin{aligned}
|\partial_r u_k(r,\alpha)| &\le \frac{4}{r_k},\\
|\partial_\alpha u_k(r,\alpha)| &\le \frac{2(\widehat{C}+1)}{\theta_k}(|\partial_{w_2}\psi_k^\star(0,-s_k(r_k,\alpha))| + 1),
\end{aligned}$$

where $\widehat{C}$ is a positive constant independent of k, which bounds the gradient of Υ_k. Since $\psi_k^\star$ is Lipschitz, we deduce that u_k is Lipschitz continuous[10] on $C_k^+ \cap (\mathrm{B}_{r_k} \setminus \mathrm{B}_{r_k/2})$.

Furthermore the image of $(\frac{r_k}{2}, r_k) \times (0, \theta_k)$ through the map $(r,\alpha) \mapsto u_k(r,\alpha)$ is the region enclosed by $G_{\psi_k^\star}$ and $G_{\psi^\star}$ (with multiplicity 1). The area of this region is infinitesimal as $k \to +\infty$, so that, by the area formula,

$$\int_{r_k/2}^{r_k}\int_0^{\theta_k} r|Ju_k(r,\alpha)|d\alpha dr = o(1) \qquad \text{as } k \to +\infty.$$

Hence, using the fact that the gradient in polar coordinates is $(\partial_r, \frac{1}{r}\partial_\alpha)$, we eventually estimate

$$\begin{aligned}
\int_{r_k/2}^{r_k}\int_0^{\theta_k} r|\mathcal{M}(\nabla u_k)|d\alpha dr &\le \int_{r_k/2}^{r_k}\int_0^{\theta_k}\Big(r + \frac{4r}{r_k} + \frac{C}{\theta_k}|\partial_{w_2}\psi_k^\star(0, 1 - \frac{2\alpha}{\theta_k})| + \frac{C}{\theta_k}\Big)\, d\alpha dr + o(1),\\
&= o(1) + C\frac{r_k}{2\theta_k}\int_0^{\theta_k}|\partial_{w_2}\psi_k^\star(0, 1 - \frac{2\alpha}{\theta_k})|d\alpha = o(1)
\end{aligned} \tag{5.25}$$

as $k \to +\infty$. In the last equality we use that $|\partial_{w_2}\psi_k^\star(0,\cdot)| \le |\partial_{w_2}\psi^\star(0,\cdot)|$, which is integrable via the change of variables $w_2 = 1 - \frac{2\alpha}{\theta_k}$ (it also makes θ_k disappear at the denominator in front of the integral in (5.25)).

This proves that the contribution of area of the graph of u_k over $C_k^+ \cap (\mathrm{B}_{r_k} \setminus \mathrm{B}_{r_k/2})$ is infinitesimal as $k \to +\infty$.

Eventually, for $r \in [r_k/2, r_k]$, $\alpha \in [2\pi - \theta_k, 2\pi)$, we set

$$u_k(r,\alpha) := (u_{k1}(r, 2\pi - \alpha), -u_{k2}(r, 2\pi - \alpha)). \tag{5.26}$$

Observe that, thanks to (5.23), u_k is continuous on $\partial\mathrm{B}_{r_k}$, and similar estimates as in (5.25) hold on $(\mathrm{B}_{r_k} \setminus \mathrm{B}_{r_k/2}) \cap C_k^-$.

Step 4. Definition of u_k on $C_k \cap \mathrm{B}_{r_k/2}$ and its area contribution.

[10]The Lipschitz constant of u_k on this set turns out to be unbounded with respect to k.

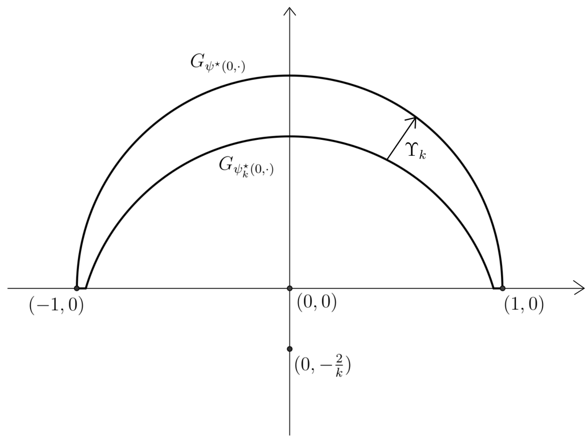

Figure 5: the graphs of the functions $\psi_k^\star(0,\cdot)$ and $\psi^\star(0,\cdot)$; these contain arcs of circle centered at $(0,0)$ and $(0,-\frac{2}{k})$ respectively. The map Υ_k is emphasized, and turns out to be the restriction of $x \mapsto \frac{x}{|x|}$ on $\psi_k^\star(0,\cdot)$.

We start with the construction of u_k on $C_k^+ \cap \mathrm{B}_{r_k/2}$. For $r \in [0, r_k/2]$ and $\alpha \in [0, \theta_k]$ we set

$$u_k(r,\alpha) := \Upsilon_k\Big(\frac{4r\alpha}{r_k\theta_k} - 1, \psi_k^\star\big(0, 1 - \frac{4r\alpha}{r_k\theta_k}\big)\Big). \tag{5.27}$$

First we observe that

$$u_k\Big(\frac{r_k}{2},\alpha\Big) = \Upsilon_k\Big(\frac{2\alpha}{\theta_k} - 1, \psi_k^\star\big(0, 1 - \frac{2\alpha}{\theta_k}\big)\Big), \qquad \alpha \in (0,\theta_k),$$

so that u_k is continuous on $C_k^+ \cap \partial\mathrm{B}_{r_k/2}$ (see (5.24) and (5.12)), and

$$u_k(r,\theta_k) = \Upsilon_k\Big(\frac{4r}{r_k} - 1, \psi_k^\star\big(0, 1 - \frac{4r}{r_k}\big)\Big), \tag{5.28}$$

$$u_k(r,0) = (-1, \psi_k^\star(0,1)) = (-1,0). \tag{5.29}$$

Direct computations lead to the following estimates:

$$|\partial_r u_k(r,\alpha)| \le \widehat{C}\frac{4\alpha}{r_k\theta_k}\Big(1 + |\partial_{w_2}\psi_k^\star(0, 1 - \frac{4\alpha r}{r_k\theta_k})|\Big), \tag{5.30}$$

$$|\partial_\alpha u_k(r,\alpha)| \le \widehat{C}\frac{4r}{r_k\theta_k}\Big(1 + |\partial_{w_2}\psi_k^\star(0, 1 - \frac{4\alpha r}{r_k\theta_k})|\Big), \tag{5.31}$$

where $\widehat{C}$ is the constant bounding the gradient of Υ_k as in step 3. Finally, since by (5.27) u_k takes values in $\mathbb{S}^1 \subset \mathbb{R}^2$, we have $Ju_k(r,\alpha) = 0$ for all $r \in (0, r_k/2)$, $\alpha \in [0,\theta_k]$. Hence, the area of the graph of u_k on $C_k^+ \cap \mathrm{B}_{r_k/2}$ is

$$\int_0^{r_k/2}\int_0^{\theta_k} r|\mathcal{M}(\nabla u_k)(r,\alpha)|\, d\alpha dr \le \int_0^{r_k/2}\int_0^{\theta_k} (r+C) + \frac{C}{\theta_k} + \frac{Cr}{r_k}(1 + \frac{1}{\theta_k})|\partial_{w_2}\psi_k^\star(0, 1 - \frac{4\alpha r}{r_k\theta_k})| d\alpha dr,$$

where C is a positive constant independent of k. Exploiting that $|\partial_{w_2}\psi_k^\star(0,\cdot)| \le |\partial_{w_2}\psi^\star(0,\cdot)|$, we can estimate the right-hand side of the previous formula as follows:

$$
\begin{aligned}
&C\int_0^{r_k/2}\int_0^{\theta_k}\frac{r}{r_k}\Big(1+\frac{1}{\theta_k}\Big)|\partial_{w_2}\psi^\star(0,1-\frac{4\alpha r}{r_k\theta_k})|d\alpha dr+o(1)\\
\le&C\int_0^{r_k/2}\int_{-1}^{1}\theta_k\Big(1+\frac{1}{\theta_k}\Big)|\partial_{w_2}\psi^\star(0,w_2)|dw_2dr+o(1)\\
\le&C\int_0^{r_k/2}(\theta_k+1)dr+o(1)=o(1),
\end{aligned}
\tag{5.32}
$$

where $o(1)\to 0$ as $k\to+\infty$, and C is a positive constant independent of k which might change from line to line.

In $C_k^-\cap \mathrm{B}_{r_k/2}$ we set, for $r\in[0,r_k/2)$, $\alpha\in[2\pi-\theta_k,2\pi)$,

$$
u_k(r,\alpha):=(u_{k1}(r,2\pi-\alpha),-u_{k2}(r,2\pi-\alpha)).
$$

Similar estimates as in (5.32) for the area of the graph of u_k hold on $C_k^-\cap\mathrm{B}_{r_k/2}$.

Step 5. Definition of u_k on T_k and its area contribution.

We first construct u_k on $T_k\cap\{(r,\alpha):r\in[0,r_k/2],\alpha\in[\theta_k,\overline{\theta}_k]\}$. We define $\beta_k:[0,r_k/2]\times[\theta_k,\overline{\theta}_k]\to[0,\pi]$ as

$$
\beta_k(r,\alpha):=\frac{\overline{\theta}_k-\alpha}{\overline{\theta}_k-\theta_k}\alpha_k(r)+(1-\frac{\overline{\theta}_k-\alpha}{\overline{\theta}_k-\theta_k})\big(\frac{2r}{r_k}(\overline{\theta}_k-\pi)+\pi\big),
$$

where

$$
\alpha_k(r):=\arccos\big(\Upsilon_{k1}(\frac{4r}{r_k}-1,\psi_k^\star(0,1-\frac{4r}{r_k}))\big),\qquad r\in[0,r_k/2].
$$

Notice that α_k is decreasing and takes values in $[0,\pi]$. Therefore we set

$$
u_k(r,\alpha):=\big(\cos(\beta_k(r,\alpha)),\sin(\beta_k(r,\alpha))\big),\qquad (r,\alpha)\in[0,r_k/2]\times[\theta_k,\overline{\theta}_k].
\tag{5.33}
$$

One checks that $\beta_k(r,\theta_k)=\alpha_k(r)$, $\beta_k(r,\overline{\theta}_k)=\frac{2r}{r_k}(\overline{\theta}_k-\pi)+\pi$ (see also (5.5)), and

$$
\begin{aligned}
&\alpha_k(r_k/2)=0,\\
&u_k(r_k/2,\alpha)=\big(\cos((1-\frac{\overline{\theta}_k-\alpha}{\overline{\theta}_k-\theta_k})\overline{\theta}_k),\sin((1-\frac{\overline{\theta}_k-\alpha}{\overline{\theta}_k-\theta_k})\overline{\theta}_k)\big),\\
&u_k(r,\theta_k)=\big(\cos(\alpha_k(r)),\sin(\alpha_k(r))\big)=\Upsilon_k(\frac{4r}{r_k}-1,\psi_k^\star(0,1-\frac{4r}{r_k})),\\
&u_k(r,\overline{\theta}_k)=\big(\cos(\frac{2r}{r_k}(\overline{\theta}_k-\pi)+\pi),\sin(\frac{2r}{r_k}(\overline{\theta}_k-\pi)+\pi)\big),
\end{aligned}
$$

so that u_k is continuous on $\{\alpha\in\{\theta_k,\overline{\theta}_k\},\ r\in[0,r_k/2]\}\cap\Omega$, see (5.6) and (5.28).

Notice also that u_k is continuous at $(0,0)\in\mathbb{R}^2$ and $u_k(0,0)=(-1,0)$. Finally, since u_k in (5.33) takes values in $\mathbb{S}^1$, the determinant of its Jacobian vanishes, so that in order to estimate the area contribution of the graph of u_k in $T_k\cap\{(r,\alpha):r\in[0,r_k/2],\ \alpha\in[\theta_k,\overline{\theta}_k]\}$ it is sufficient to estimate the gradient of u_k. We have

$$
\begin{aligned}
&|\partial_r u_k(r,\alpha)|=|\partial_r\beta_k(r,\alpha)|\le|\partial_r\alpha_k(r)|+\frac{2\pi}{r_k},\\
&|\partial_\alpha u_k(r,\alpha)|=|\partial_\alpha\beta_k(r,\alpha)|\le\frac{|\alpha_k(r)|}{\overline{\theta}_k-\theta_k}+\frac{\pi}{\overline{\theta}_k-\theta_k}\le\frac{2\pi}{\overline{\theta}_k-\theta_k}.
\end{aligned}
$$

Therefore

$$\int_0^{r_k/2}\int_{\theta_k}^{\overline{\theta}_k} r|\mathcal{M}(\nabla u_k)(r,\alpha)|d\alpha dr \le \int_0^{r_k/2}\int_{\theta_k}^{\overline{\theta}_k} \Big[\frac{r_k}{2}(1+|\partial_r\beta_k(r,\alpha)|)+|\partial_\alpha\beta_k(r,\alpha)|\Big]\, d\alpha dr$$
$$\le o(1)+\int_0^{r_k/2}\int_{\theta_k}^{\overline{\theta}_k}\Big(\frac{r_k}{2}|\partial_r\alpha_k(r)|+\pi+\frac{2\pi}{\overline{\theta}_k-\theta_k}\Big)\, d\alpha dr = o(1), \tag{5.34}$$

with $o(1)\to 0$ as $k\to+\infty$. Notice that the integral of $|\partial_r\alpha_k(r)|$ with respect to r can be computed via the fundamental integration theorem, since α_k is monotone.

In $T_k\cap\{(r,\alpha): r\in[0,r_k/2], \alpha\in[2\pi-\overline{\theta}_k, 2\pi-\theta_k]\}$ we set

$$u_k(r,\alpha) := (u_{k1}(r,2\pi-\alpha), -u_{k2}(r,2\pi-\alpha)).$$

We now define u_k on $T_k\cap\{(r,\alpha): r\in(r_k/2,l), \alpha\in[\theta_k,\overline{\theta}_k]\}$. We set

$$u_k(r,\alpha) := \big(\cos((1-\frac{\overline{\theta}_k-\alpha}{\overline{\theta}_k-\theta_k})\overline{\theta}_k), \sin((1-\frac{\overline{\theta}_k-\alpha}{\overline{\theta}_k-\theta_k})\overline{\theta}_k)\big).$$

Then $u_k\in \mathrm{Lip}(T_k,\mathbb{S}^1)$, and

$$\begin{aligned}
&u_k(r,\theta_k)=(1,0), \qquad u_k(r,\overline{\theta}_k)=(\cos\overline{\theta}_k,\sin\overline{\theta}_k) \quad \text{for } r\in(r_k/2,l),\\
&\partial_r u_k(r,\alpha)=0,\\
&\partial_\alpha u_k(r,\alpha)=\frac{\overline{\theta}_k}{\overline{\theta}_k-\theta_k}\Big(-\sin((1-\frac{\overline{\theta}_k-\alpha}{\overline{\theta}_k-\theta_k})\overline{\theta}_k), \cos((1-\frac{\overline{\theta}_k-\alpha}{\overline{\theta}_k-\theta_k})\overline{\theta}_k)\Big).
\end{aligned}$$

Hence

$$\int_{r_k/2}^{l}\int_{\theta_k}^{\overline{\theta}_k} r|\mathcal{M}(\nabla u_k)(r,\alpha)|d\alpha dr \le \int_{r_k/2}^{l}\int_{\theta_k}^{\overline{\theta}_k}\Big(r+\frac{\overline{\theta}_k}{\overline{\theta}_k-\theta_k}\Big)\, d\alpha dr = o(1) \tag{5.35}$$

as $k\to+\infty$.

Finally in $T_k\cap\{(r,\alpha): r\in(r_k/2,l), \alpha\in[2\pi-\overline{\theta}_k, 2\pi-\theta_k]\}$ we set

$$u_k(r,\alpha) := (u_{k1}(r,2\pi-\alpha), -u_{k2}(r,2\pi-\alpha)).$$

Similar estimates as in (5.34), (5.35) for the area of the graph of u_k hold on $T_k\cap\{(r,\alpha): r\in(0,r_k/2), \alpha\in[2\pi-\overline{\theta}_k, 2\pi-\theta_k]\}$, $T_k\cap\{(r,\alpha): r\in(r_k/2,l), \alpha\in[2\pi-\overline{\theta}_k,2\pi-\theta_k]\}$, respectively.

Step 6. We claim that

$$\int_{\Omega\setminus(C_k\cup T_k)} |\mathcal{M}(\nabla u_k)|dx \longrightarrow \int_\Omega |\mathcal{M}(\nabla u)|\, dx \qquad \text{as } k\to+\infty, \tag{5.36}$$

where we recall that $C_k\cup T_k=\{(r,\alpha)\in\Omega: r\in[0,l), \alpha\in[0,\overline{\theta}_k]\cup[2\pi-\overline{\theta}_k,2\pi)\}$.

Indeed, on $\Omega\setminus(C_k\cup T_k)$ the maps u_k and u take values in the circle $\mathbb{S}^1$, hence

$$\det(\nabla u_k)=0, \qquad \det(\nabla u)=0, \qquad \text{in } \Omega\setminus(C_k\cup T_k).$$

Thus

$$\int_{\Omega\setminus(C_k\cup T_k)} |\mathcal{M}(\nabla u_k)-\mathcal{M}(\nabla u)|\, dx \le \sum_{i=1,2}\int_{\Omega\setminus(C_k\cup T_k)} |\nabla(u_{ki}-u_i)|\, dx.$$

From (5.5) we have

$$\begin{aligned}
&|\partial_r(u_k - u)| = 0 && \text{in} \quad \Omega \setminus (\mathrm{B}_{r_k} \cup C_k \cup T_k),\\
&|\partial_r(u_k - u)| \le \frac{\pi}{r_k} && \text{in} \quad \mathrm{B}_{r_k} \setminus (C_k \cup T_k),\\
&|\partial_\alpha(u_k - u)| = 0 && \text{in} \quad \Omega \setminus (\mathrm{B}_{r_k} \cup C_k \cup T_k),\\
&|\partial_\alpha(u_k - u)| \le 2 && \text{in} \quad \mathrm{B}_{r_k} \setminus (C_k \cup T_k).
\end{aligned} \tag{5.37}$$

Our previous remarks and the fact that $r_k, \theta_k, (\overline{\theta}_k - \theta_k) \to 0^+$ as $k \to +\infty$, imply (5.36).

Step 7. We know from (5.25), (5.32), (5.34), and (5.35), that the integral of $|\mathcal{M}(\nabla u_k)|$ is infinitesimal as $k \to +\infty$, on the region $(\mathrm{B}_{r_k} \cap C_k) \cup T_k$. Therefore it remains to compute the area of the graphs of u_k in the region $C_k \setminus \mathrm{B}_{r_k}$. We claim that this contribution gives

$$\lim_{k\to+\infty} \int_{C_k \setminus \mathrm{B}_{r_k}} |\mathcal{M}(\nabla u_k)| \, dx \le 2\mathcal{F}_l(h^\star, \psi^\star) = \mathcal{A}(\psi^\star, SG_{h^\star}). \tag{5.38}$$

To prove this, we start to compute the area of the graph of u_k restricted to $C_k^+ \setminus \mathrm{B}_{r_k}$. From (5.21), (5.13), (5.15) and (5.14), we have

$$\begin{aligned}
\partial_r u_{k1} &= \Big(\frac{\alpha}{\theta_k} - 1\Big)\tau_k' h_k^{\star\prime} = \frac{l}{l - r_k}\Big(\frac{\alpha}{\theta_k} - 1\Big)h_k^{\star\prime},\\
\partial_\alpha u_{k1} &= \frac{1 + h_k^\star}{\theta_k},\\
\partial_r u_{k2} &= \tau_k'\Big[\Big(1 - \frac{\alpha}{\theta_k}\Big)h_k^{\star\prime}\partial_{w_2}\psi_k^\star + \partial_{w_1}\psi_k^\star\Big] = \frac{l}{l - r_k}\Big[\Big(1 - \frac{\alpha}{\theta_k}\Big)h_k^{\star\prime}\partial_{w_2}\psi_k^\star + \partial_{w_1}\psi_k^\star\Big],\\
\partial_\alpha u_{k2} &= -\Big[\frac{1 + h_k^\star}{\theta_k}\Big]\partial_{w_2}\psi_k^\star,\\
\partial_r u_{k1}\partial_\alpha u_{k2} - \partial_\alpha u_{k1}\partial_r u_{k2} &= -\left(\frac{1 + h_k^\star}{\theta_k}\right)\frac{l}{l - r_k}\partial_{w_1}\psi_k^\star,
\end{aligned} \tag{5.39}$$

where $h_k^{\star\prime}$ denotes the derivative of $h_k^\star$ with respect to w_1, $h_k^\star, h_k^{\star\prime}$ are evaluated at $\tau_k(r)$, and the two partial derivatives $\partial_{w_2}\psi_k^\star$, $\partial_{w_1}\psi_k^\star$ of $\psi_k^\star$ with respect to w_2, w_1 are evaluated at $(\tau_k(r), -s_k(r,\alpha))$. Note carefully that, in the computation of the Jacobian, the terms containing $\partial_{w_2}\psi_k^\star$ cancel each other.

Since $h_k^\star$ is convex, its derivative is nonincreasing, and therefore $\int_{r_k}^l |h_k^{\star\prime}| \, dr < +\infty$. As a consequence of (5.39), from (2.2), we have

$$\begin{aligned}
&\mathcal{A}(u_k, C_k^+ \setminus \mathrm{B}_{r_k})\\
&= \int_{r_k}^l \int_0^{\theta_k} r\Bigg\{1 + \left(\frac{l}{l - r_k}\right)^2 \left(\frac{\alpha}{\theta_k} - 1\right)^2 (h_k^{\star\prime})^2\\
&\quad + \left(\frac{l}{l - r_k}\right)^2 \Big[\big(\frac{\alpha}{\theta_k} - 1\big)^2 (h_k^{\star\prime})^2 (\partial_{w_2}\psi_k^\star)^2 + 2\big(1 - \frac{\alpha}{\theta_k}\big)h_k^{\star\prime}\partial_{w_2}\psi_k^\star\partial_{w_1}\psi_k^\star + (\partial_{w_1}\psi_k^\star)^2\Big]\\
&\quad + \frac{1}{r^2}\left(\frac{1 + h_k^\star}{\theta_k}\right)^2 \left(1 + (\partial_{w_2}\psi_k^\star)^2 + \left(\frac{l}{l - r_k}\right)^2 (\partial_{w_1}\psi_k^\star)^2\right)\Bigg\}^{\frac{1}{2}} \, dr d\alpha,
\end{aligned}$$

where $\partial_{w_2}\psi_k^\star$, $\partial_{w_1}\psi_k^\star$ are evaluated at $(\tau_k(r), -s_k(r,\alpha))$, and $h_k^\star$, $h_k^{\star\prime}$ are evaluated at $\tau_k(r)$. Now we use the change of variable (5.18): from (5.20), we have

$$
\begin{aligned}
&\mathcal{A}(u_k, C_k^+ \setminus \mathrm{B}_{r_k}) \\
&= \int_0^l \int_{-1}^{h_k^\star(w_1)} \left(\frac{l-r_k}{l}\right)\left(\frac{\theta_k}{1+h_k^\star}\right) H_k(w_1) \Bigg\{ 1 + \big(\frac{l}{l-r_k}\big)^2 \left(\frac{\Theta_k(w_1,w_2)}{\theta_k} - 1\right)^2 (h_k^{\star\prime})^2 \\
&+ \left(\frac{l}{l-r_k}\right)^2 \Big[\big(1 - \frac{\Theta_k(w_1,w_2)}{\theta_k}\big)^2 (h_k^{\star\prime})^2 (\partial_{w_2}\psi_k^\star)^2 + 2\big(1 - \frac{\Theta_k(w_1,w_2)}{\theta_k}\big) h_k^{\star\prime} \partial_{w_2}\psi_k^\star \partial_{w_1}\psi_k^\star + (\partial_{w_1}\psi_k^\star)^2 \Big] \\
&+ \frac{1}{(H_k(w_1))^2} \big(\frac{1+h_k^\star}{\theta_k}\big)^2 \Big(1 + (\partial_{w_2}\psi_k^\star)^2 + \big(\frac{l}{l-r_k}\big)^2 (\partial_{w_1}\psi_k^\star)^2\Big) \Bigg\}^{\frac{1}{2}} \, dw_2 dw_1,
\end{aligned}
$$

where $H_k(w_1)$, $\Theta_k(w_1, w_2)$ are defined in (5.16), (5.17), $h_k^{\star\prime}$ is evaluated at w_1, and $\partial_{w_1}\psi_k^\star$ and $\partial_{w_2}\psi_k^\star$ are evaluated at (w_1, w_2). Therefore

$$
\mathcal{A}(u_k, C_k^+ \setminus \mathrm{B}_{r_k}) = \int_0^l \int_{-1}^{h_k^\star(w_1)} \Big\{ \mathrm{I}_k + \mathrm{II}_k + \mathrm{III}_k + \mathrm{IV}_k + \mathrm{V}_k + \mathrm{VI}_k \Big\}^{\frac{1}{2}} \, dw_2 dw_1, \tag{5.40}
$$

where

$$
\begin{cases}
\mathrm{I}_k = \left(\frac{l-r_k}{l}\right)^2 \left(\frac{\theta_k}{1+h_k^\star}\right)^2 (H_k(w_1))^2, \\
\mathrm{II}_k = \left(\frac{\theta_k}{1+h_k^\star}\right)^2 \left(1 - \frac{\Theta_k(w_1,w_2)}{\theta_k}\right)^2 (H_k(w_1))^2 (h_k^{\star\prime})^2, \\
\mathrm{III}_k = \left(\frac{\theta_k}{1+h_k^\star}\right)^2 (H_k(w_1))^2 \Big[\big(1 - \frac{\Theta_k(w_1,w_2)}{\theta_k}\big)^2 (h_k^{\star\prime})^2 (\partial_{w_2}\psi_k^\star)^2 \\
\qquad\qquad + 2\big(1 - \frac{\Theta_k(w_1,w_2)}{\theta_k}\big) h_k^{\star\prime} \partial_{w_2}\psi_k^\star \partial_{w_1}\psi_k^\star + (\partial_{w_1}\psi_k^\star)^2 \Big], \\
\mathrm{IV}_k = \big(\frac{l-r_k}{l}\big)^2, \\
\mathrm{V}_k = \big(\frac{l-r_k}{l}\big)^2 (\partial_{w_2}\psi_k^\star)^2, \\
\mathrm{VI}_k = (\partial_{w_1}\psi_k^\star)^2.
\end{cases}
$$

Since $\lim_{k\to\infty} \frac{l-r_k}{l} = 1$ and $\lim_{k\to+\infty} \theta_k = 0$, we deduce from (5.16), (5.17),

$$
\lim_{k\to+\infty} H_k(w_1) = w_1, \qquad \lim_{k\to+\infty} \frac{\Theta_k(w_1,w_2)}{\theta_k} = \frac{h^\star(w_1) - w_2}{1 + h^\star(w_1)}.
$$

Therefore we see that

$$
\int_0^l \int_{-1}^{h_k^\star(w_1)} (\mathrm{I}_k)^{\frac{1}{2}} + (\mathrm{II}_k)^{\frac{1}{2}} dw_2 dw_1 = o(1),
$$

as $k \to +\infty$. Moreover

$$
\int_0^l \int_{-1}^{h_k^\star(w_1)} (\mathrm{III}_k)^{\frac{1}{2}} dw_2 dw_1 = o(1) \tag{5.41}
$$

as $k \to +\infty$. Indeed we may estimate

$$\int_0^l \int_{-1}^{h_k^\star(w_1)} (\mathrm{III}_k)^{\frac{1}{2}} dw_2 dw_1 \leq C\theta_k \int_0^l \int_{-1}^{h_k^\star(w_1)} |h_k^{\star\prime}(w_1)||\partial_{w_2}\psi_k^\star(w_1,w_2)| + |\partial_{w_2}\psi_k^\star(w_1,w_2)| dw_2 dw_1,$$

and using that $|h_k^{\star\prime}(w_1)| \leq 2k$ (see (5.7)), if we assume (5.1), *i.e.*, $\theta_k k \to 0$, then (5.41) follows, since the BV-norm of $\psi_k^\star$ is bounded uniformly with respect to k.

Hence, from (5.40),

$$\begin{aligned} \mathcal{A}(u_k, C_k^+ \setminus \mathrm{B}_{r_k}) &\leq \int_0^l \int_{-1}^{h_k^\star(w_1)} \Big\{ \mathrm{IV}_k + \mathrm{V}_k + \mathrm{VI}_k \Big\}^{\frac{1}{2}} dw_2 dw_1 + o(1) \\ &\leq \int_0^l \int_{-1}^{h_k^\star(w_1)} \sqrt{1 + (\partial_{w_1}\psi_k^\star)^2 + (\partial_{w_2}\psi_k^\star)^2}\, dw_2 dw_1 + o(1) \\ &= \mathcal{A}(\psi_k^\star, SG_{h^\star} \cap R_l) + o(1) = \frac{1}{2}\mathcal{A}(\psi_k^\star, SG_{h^\star}) + o(1) \end{aligned} \tag{5.42}$$

as $k \to +\infty$. Then taking the limit as $k \to +\infty$ in (5.42), and using Lemma 3.3 (iii), we get

$$\lim_{k\to+\infty} \mathcal{A}(u_k, C_k^+ \setminus \mathrm{B}_{r_k}) \leq \mathcal{A}(\psi^\star, SG_{h^\star}) = \mathcal{F}_{2l}(h^\star, \psi^\star), \tag{5.43}$$

where the last equality follows from (3.9).

Step 8. Conclusion. Notice that $u_k \in \mathrm{Lip}(\Omega, \mathbb{R}^2)$, and $u_k \to u$ in $L^1(\Omega, \mathbb{R}^2)$. Inequality (3.14) follows from (5.36) (which gives the term $\int_\Omega |\mathcal{M}(\nabla u)| dx$), from (5.38) (which gives the second term in (3.14)), and from estimates (5.25), (5.32), (5.34), and (5.35), showing that all the other contributions are negligible.

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