GEOMETRIC FEATURES OF NONLOCAL MINIMAL SURFACES

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These notes were born from gathering ideas for a seminar on nonlocal minimal surfaces that I held at Stanford University during the Winter and Spring Quarters of 2024. They intentionally focus on comparing fractional objects with their local counterparts (perimeter, surfaces, mean curvature). For this reason, I think they will be helpful, particularly for readers already familiar with the theory of classical minimal surfaces.

Further material on this topic that I have found useful preparing this notes can be found in [CRS10, BV16, Gar19, Sav22, CS07, CDSV23, CFS23, CCS20].

Contents

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1. The fractional Laplacian and the $H^s(\mathbb{R}^n)$ spaces.

For $s \in (0,1)$ define

$$H^s(\mathbb{R}^n) := \left\{ u \in L^2(\mathbb{R}^n) \, : \, [u]^2_{H^s(\mathbb{R}^n)} := \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} \, dxdy < +\infty \right\}.$$

This is a Hilbert space with its natural norm

$$\|u\|^2_{H^s(\mathbb{R}^n)} = \|u\|^2_{L^2(\mathbb{R}^2)} + [u]^2_{H^s(\mathbb{R}^n)}.$$

One can also give a completely analogous definition on every open $\Omega \subset \mathbb{R}^n$, which is denoted by $H^s(\Omega)$ and we will define in (3). These are called *fractional Sobolev spaces*, and they appear naturally in many places in mathematics. For example

- They are the real interpolation spaces $H^s(\Omega) = (L^2(\Omega), H^1(\Omega))_{s,2}$ in the sense of harmonic analysis.
- They are the trace spaces of classical Sobolev spaces. In particular, $H^{1/2}(\partial\Omega)$ is the trace space of $H^1(\Omega)$, and in general $H^{s-1/2}(\partial\Omega)$ is the trace space of $H^s(\Omega)$ for $s \in (0,1)$.
- They are the Hilbert spaces associated with the fractional Laplacian

$$(-\Delta)^s u(x) = \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}}\, dy, \tag{1}$$

which is a fractional power of the Laplacian $-\Delta$ in a strong and functional-analytic sense.

By the symmetry in x, y one can rewrite

$$[u]^2_{H^s(\mathbb{R}^n)} = \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dxdy = 2\int_{\mathbb{R}^n} u(-\Delta)^s u\, dx\,.$$

For $\Omega \subset \mathbb{R}^n$, we call *localized energy* the following quantity

$$\mathcal{E}^{\mathrm{Sob}}_{s,\Omega}(u) := \frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n\setminus\Omega^c\times\Omega^c} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dxdy\,. \tag{2}$$

Even though we call this the *localized energy in* Ω, this requires u to be defined in the whole $\mathbb{R}^n$ to be computed. Thus, for example, in order to minimize this energy, we need to prescribe u in Ω^c and not only on $\partial\Omega$.

Lemma 1.1. *Let $\Omega \subset \mathbb{R}^n$ open and fix $u_\circ \in H^s(\mathbb{R}^n)$. Let*

$$H^s_g(\Omega) := \left\{u \in H^s(\Omega) \,:\, u = g \text{ in } \Omega^c\right\}.$$

If u is a smooth critical point of $u \mapsto \mathcal{E}^{\mathrm{Sob}}_{s,\Omega}$ on the space $H^s_g(\Omega)$, then $(-\Delta)^s u = 0$ in Ω. That is, $(-\Delta)^s u = 0$ (in Ω) is the Euler-Lagrange equation of $\mathcal{E}^{\mathrm{Sob}}_{s,\Omega}$.

Proof. For every $\varphi \in C^\infty_c(\Omega)$ we have

$$\begin{aligned} 0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{E}^{\mathrm{Sob}}_{s,\Omega}(u+\varepsilon\varphi) &= \iint_{\mathbb{R}^n\times\mathbb{R}^n\setminus\Omega^c\times\Omega^c} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\, dxdy \\ &= \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\, dxdy \\ &= 2\int_{\mathbb{R}^n} \varphi(-\Delta)^s u\, dx = 2\int_{\Omega} \varphi(-\Delta)^s u\, dx\,. \end{aligned}$$

Since this holds for all $\varphi \in C^\infty_c(\Omega)$, this clearly implies that $(-\Delta)^s u$ in Ω. □

Remark 1.2. *If we had defined the localized energy by the (arguably more natural) formula*

$$\frac{1}{2}\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dxdy,$$

the Euler-Lagrange equation would turn out to being

$$\int_\Omega \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy = 0,$$

for every $x \in \Omega$. But the left-hand side is not the fractional Laplacian in the standard sense. For example, with Dirichlet boundary conditions, this operator does not have the same eigenfunctions of the Dirichlet Laplacian, but (1) does. Indeed, (2) is the correct localization of the fractional Laplacian, which is a nonlocal operator.

Exercise 1.3. *Prove that*

$$[u]^2_{H^s(\mathbb{R}^n)} = C_{n,s}\int_{\mathbb{R}^n} |\xi|^{2s}|\widehat{u}(\xi)|^2\,d\xi.$$

In particular, the exercise above implies that

$$H^s(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) \,:\, \int_{\mathbb{R}^n} (1+|\xi|^{2s})|\widehat{u}|^2\,d\xi < +\infty \right\}.$$

We will use this characterization a few times. One important feature of the $H^s(\Omega)$ spaces (this will imply the existence of sets minimizing the s-perimeter and more) is the fractional version of the Rellich–Kondrachov theorem.

For $\Omega \subset \mathbb{R}^n$ set

$$H^s(\Omega) = \left\{ u \in L^2(\Omega) \,:\, \mathcal{E}^{\mathrm{Sob}}_{s,\Omega}(u) < +\infty \right\}, \tag{3}$$

endowed with the norm $\|u\|^2_{H^s(\Omega)} = \|u\|_{L^2(\Omega)} + \mathcal{E}^{\mathrm{Sob}}_{s,\Omega}(u)$.

Theorem 1.4. *Let $s \in (0,1)$. Then, $H^s(B_1) \hookrightarrow L^2(B_1)$ is compact.*

Proof. Let $E : H^s(B_1) \to H^s(\mathbb{R}^n)$ be an extension operator with $Ev = v$ in B and

$$[Ev]^2_{H^s(\mathbb{R}^n)} \le C\|v\|^2_{H^s(B_1)}, \quad \|Ev\|^2_{L^2(\mathbb{R}^n)} \le C\|v\|^2_{L^2(B_1)}, \quad \mathrm{supp}(Ev) \subset B_2.$$

Let $\{u_k\}_k$ be a sequence in $H^s(B_1)$ with $\|u_k\|_{H^s(B_1)} \le C$. Up to subsequences that we do not relabel, there exists $u \in L^2(B_1)$ such that $u_k \rightharpoonup u$ in $L^2(B_1)$ and $Eu_k \rightharpoonup Eu$ in $L^2(B_2)$. We show that the convergence of this subsequence holds strongly in $L^2(B_1)$.

Then, for $R > 0$ we have

$$\begin{aligned}
\|u_k - u_\ell\|^2_{L^2(B_1)} &\le C\|Eu_k - Eu_\ell\|^2_{L^2(\mathbb{R}^n)} \\
&= \int_{\mathbb{R}^n} |\widehat{Eu_k}(\xi) - \widehat{Eu_\ell}(\xi)|^2\,d\xi \\
&= \int_{B_R} |\widehat{Eu_k}(\xi) - \widehat{Eu_\ell}(\xi)|^2\,d\xi + \int_{B_R^c} |\widehat{Eu_k}(\xi) - \widehat{Eu_\ell}(\xi)|^2\,d\xi\,.
\end{aligned}$$

For the second term

$$\begin{aligned}
\int_{B_R^c} |\widehat{Eu_k}(\xi) - \widehat{Eu_\ell}(\xi)|^2\,d\xi &\le C\int_{B_R^c} |\widehat{Eu_k}(\xi)|^2\,d\xi \\
&\le \frac{C}{R^{2s}}\int_{\mathbb{R}^n} |\widehat{Eu_k}(\xi)|^2|\xi|^{2s}\,d\xi \\
&= \frac{C}{R^{2s}}[Eu_k]^2_{H^s(\mathbb{R}^n)} \le \frac{C}{R^{2s}}\|u_k\|^2_{H^s(B)} \le \frac{C}{R^{2s}}.
\end{aligned}$$

On the other hand, for every $\xi \in B_R$ by the weak convergence $Eu_k \rightharpoonup Eu$ in $L^2(B_2)$ we have

$$\widehat{Eu_k}(\xi) = \int_{B_2} e^{-ix\cdot\xi} Eu_k \, dx \to \int_{B_2} e^{-ix\cdot\xi} Eu \, dx \, ,$$

as $k \to \infty$, since $e^{-ix\cdot\xi} \in L^2(B_2)$. Moreover

$$|\widehat{Eu_k}(\xi)| \leq \int_{B_2} |u_k| \, dx \leq \|u_k\|_{L^2(B)} |B_2|^{1/2} \leq C \in L^2(B_R).$$

Hence, by dominated convergence for the first integral, we have

$$\int_{B_R} |\widehat{Eu_k}(\xi) - \widehat{Eu_\ell}(\xi)|^2 \, d\xi \to 0,$$

as $k, \ell \to \infty$. Then

$$\limsup_{k \geq \ell \to \infty} \|u_k - u_\ell\|^2_{L^2(B_1)} \leq \frac{C}{R^{2s}}.$$

Since this holds for all $R > 0$, this implies that $u_k \to u$ strongly in $L^2(B_1)$, and we are done. □

Moreover, the same proof gives the following stronger result.

Theorem 1.5. *Let $s, \sigma \in (0,1)$ with $s < \sigma$. Then $H^\sigma(B_1) \hookrightarrow H^s(B_1)$ is compact.*

Fractional (sub)harmonic functions satisfy a maximum principle similar to the one for classical harmonic functions.

Theorem 1.6. *If $(-\Delta)^s u \geq 0$ in Ω and $u \geq 0$ in Ω^c, then $u \geq 0$ in Ω.*

Proof. The proof is almost trivial. Suppose that $x_\circ \in \overline{\Omega}$ realizing $\min_{\overline{\Omega}} u$ satisfies $u(x_\circ) < 0$. Then $x_\circ$ is a global minimum of u (since $u \geq 0$ outside Ω) and hence $u(x_\circ) \leq u(z)$ for every $z \in \mathbb{R}^n$. Then

$$0 \leq (-\Delta)^s u(x_\circ) = P.V. \int_{\mathbb{R}^n} \frac{u(x_\circ) - u(y)}{|x-y|^{n+2s}} \, dy \leq 0 \, ,$$

and hence $u \equiv u(x_\circ) < 0$, which contradicts that u is nonnegative outside Ω. □

Even though a global maximum principle holds, due to the nonlocality of the operator, any form of local maximum principle does not hold.

Lemma 1.7 (Theorem 2.3.1 in [BV16])**.** *There exists a nonconstant, bounded function $u : \mathbb{R}^n \to \mathbb{R}$ with $(-\Delta)^s u = 0$ in B_1 and $u \geq 0$ in B_1 such that $\inf_{B_1} u = 0$.*

1.1. Riesz potentials.

There is another point of view to look at the fractional Sobolev spaces $H^s(\mathbb{R}^n)$, which is valuable for gathering some intuition, that is the one of Riesz' potentials. We refer to [Gar19] and the references therein for a detailed discussion of Riesz's potential.

Definition 1.8. *(Riesz' potential) Let $0 < \alpha < n$ and $u \in C_c^\infty(\mathbb{R}^n)$. Then we call the operator*

$$I_\alpha u(x) := C_{n,\alpha} \int_{\mathbb{R}^n} \frac{u(y)}{|x-y|^{n-\alpha}} \, dy = u * \Phi_\alpha(x)$$

the Riesz potential of order α, where

$$\Phi_\alpha(x) := \frac{C_{n,\alpha}}{|x|^{n-\alpha}} \, .$$

Here $C_{n,\alpha} > 0$ is a positive constant depending only on n and α, whose value is unimportant to us. The only important property of $C_{n,\alpha}$ is that it is chosen so that (4) holds.

The only property we will care about of the Riesz potentials is that $I_\alpha u$ inverts the fractional $\alpha/2$-Laplacian, as stated in the next proposition.

Proposition 1.9 ([Gar19])**.** *Let $u \in C_c^\infty(\mathbb{R}^n)$. Then for every $0 < \alpha < n$ there holds*

$$I_\alpha((-\Delta)^{\alpha/2}u) = (-\Delta)^{\alpha/2}(I_\alpha u) = u\,. \tag{4}$$

Equivalently, Φ_α is the fundamental solution of the fractional $\alpha/2$-Laplacian.

Remark 1.10. *Note that the constants are set so that, at least formally, for $\alpha = 2$, I_2 inverts the standard Laplacian, and Φ_2 is the Green's function of $\mathbb{R}^n$.*

Now, we are in the position to rewrite the H^s-seminorm in a useful form.

Lemma 1.11. *For $s \in (0,1)$ and $u \in H^s(\mathbb{R}^n)$ there holds*

$$(1-s)[u]^2_{H^s(\mathbb{R}^n)} = (1-s)\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dxdy = c_s\int_{\mathbb{R}^n} |\nabla I_{1-s}u|^2 dx\,,$$

where ∇ on the right-hand side denotes the classical weak gradient, and $\lim_{s\to 1^-} c_s = \pi/2$.

For a proof of a more general fact, we refer to [Š20]. Nevertheless, we first give formal proof and then an actual proof using the Fourier transform.

Formal proof. For $u \in C_c^\infty(\mathbb{R}^n)$ one can easily see that $I_{1-s}u \in H^1_0(\mathbb{R}^n)$, thus

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla I_{1-s}u|^2 dx &= \big\langle \nabla I_{1-s}u, \nabla I_{1-s}u\big\rangle_{L^2(\mathbb{R}^n)} = \big\langle I_{1-s}u, -\Delta(I_{1-s}u)\big\rangle_{L^2(\mathbb{R}^n)}\\ &= c_s(1-s)\big\langle I_{1-s}u, (-\Delta)^{1-\frac{1-s}{2}}u\big\rangle_{L^2(\mathbb{R}^n)}\\ &= c_s(1-s)\big\langle (-\Delta)^{\frac{1-s}{2}}(I_{1-s}u), (-\Delta)^{1-(1-s)}u\big\rangle_{L^2(\mathbb{R}^n)}\\ &= c_s(1-s)\big\langle u, (-\Delta)^s u\big\rangle_{L^2(\mathbb{R}^n)}\\ &= (1-s)[u]^2_{H^s(\mathbb{R}^n)},\end{aligned}$$

where c_s is a constant of s that remains bounded as $s \to 1^-$. □

Proof. We have

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla I_{1-s}u|^2 dx &= \int_{\mathbb{R}^n} |\xi|^2 |\widehat{I_{1-s}u}|^2\, d\xi\\ &= \int_{\mathbb{R}^n} |\xi|^2 |\widehat{u * \Phi_{1-s}}|^2\, d\xi\\ &= \int_{\mathbb{R}^n} |\xi|^2 |\widehat{u}|^2 |\widehat{\Phi_{1-s}}|^2\, d\xi\,.\end{aligned}$$

But we can compute

$$\widehat{\Phi_{1-s}}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\Phi_{1-s}(x)\,dx = c_{n,s}\int_{\mathbb{R}^n} \frac{e^{-ix\cdot\xi}}{|x|^{n-1+s}}\,dx = \frac{c_{n,s}}{|\xi|^{1-s}}\,,$$

where the last equality can be easily seen by scaling and rotational invariance.

Thus

$$\int_{\mathbb{R}^n} |\nabla I_{1-s}u|^2 = c_{n,s}\int_{\mathbb{R}^n} |\xi|^{2s}|\mathcal{F}u|^2\, d\xi = c_{n,s}[u]^2_{H^s(\mathbb{R}^n)}\,.$$

□

Let us take a moment to comment on this formula. Assume for simplicity that that $\mathrm{supp}(u) \subset B_R$, then for every $x \in B_{2R}$ one can rewrite (up to a constant that tends to 1 as $s \to 1$)

$$I_{1-s}u(x) = \frac{1-s}{|\partial B_1|} \int_{\mathbb{R}^n} \frac{u(y-x)}{|y|^{n-1+s}}\, dy = \frac{1-s}{|\partial B_1|} \int_{B_{3R}} \frac{u(y)}{|y|^{n-1+s}}\, dy = u * \mu_s\,,$$

where

$$\mu_s(x) = \frac{1-s}{|\partial B_1|} \frac{\chi_{B_{3R}}}{|x|^{n-1+s}}$$

is a sequence of kernels such that, as $s \to 1^-$

$$\mu_s(\mathbb{R}^n) = (1-s) \int_0^{3R} \rho^{-s}\, d\rho = (3R)^{1-s} \to 1\,,$$

and for every $\delta > 0$

$$\mu_s(\mathbb{R}^n \setminus B_\delta) = (1-s) \int_\delta^{3R} \rho^{-s}\, d\rho = (3R)^{1-s} - \delta^{1-s} \to 0\,.$$

Remark 1.12. *Be careful of the following: since we are convolving against a kernel that is not compactly supported in $\mathbb{R}^n$, it is not true that $I_{1-s}u$ has compact support even if u does. This is one point where one can see the non-locality of the seminorms, as the convolution instantly sees u everywhere from every point in $\mathbb{R}^n$. Nevertheless, even if the kernel has tails in the whole $\mathbb{R}^n$, since*

$$\frac{1-s}{|\partial B_1|} \int_{B_{(1-s)^\theta}} \frac{1}{|x|^{n-(1-s)}} = (1-s)^{\theta(1-s)} \to 1\,, \quad \textit{as} \;\; s \to 1^-\,,$$

the "effective support" of the kernel we are convolving with is any power of $(1-s)$. This means that, as $s \to 1^-$, all the mass of the kernel is concentrating near zero at velocity polynomial in $(1-s)$, and all the effects that happen at a scale smaller than this will be neglected.

Hence (say, in B_{2R}) $I_{1-s}u$ is a convolution of u against an approximation of the identity, and thus $I_{1-s}u \to u$ suitably. Now, looking again at (neglecting constants)

$$[u]^2_{H^s(\mathbb{R}^n)} = \|\nabla(I_{1-s}u)\|^2_{L^2(\mathbb{R}^n)}\,.$$

This formula allows us to write the H^s-seminorm of a fixed function as an H^1-seminorm of a (nonlocal, that is, against a kernel without compact support) approximation of u. From this formula, you can probably gather intuition from what you already know about standard Sobolev spaces.

2. The fractional perimeter.

We start with a motivating example to see why the $H^s(\mathbb{R}^n)$ spaces naturally induce a notion of perimeter. Let $E \subset \mathbb{R}^n$ be smooth and bounded and $\{\rho_\varepsilon\}_{\varepsilon>0}$ fe a family of standard mollifiers. For every $p \in [1, \infty)$ It is not hard to show that

$$\varepsilon^{p-1}[\chi_E * \rho_\varepsilon]^p_{W^{1,p}} = \varepsilon^{p-1} \int_{\mathbb{R}^n} |\nabla(\chi_E * \rho_\varepsilon)|^p dx \to \mathcal{H}^{n-1}(\partial E)\,,$$

as $\varepsilon \to 0$. Indeed, at least formally, as $\varepsilon \to 0$ all the mass concentrates around an ε-neighborhood Σ_ε of ∂E and $|\nabla(\chi_E * \rho_\varepsilon)| \sim 1/\varepsilon$ there. Moreover $|\Sigma_\varepsilon| \sim \varepsilon \mathcal{H}^{n-1}(\partial E)$, hence

$$\varepsilon^{p-1} \int_{\mathbb{R}^n} |\nabla(\chi_E * \rho_\varepsilon)|^p dx \sim \varepsilon^{p-1} |\Sigma_\varepsilon| \varepsilon^{-p} \sim \mathcal{H}^{n-1}(\partial E)\,. \tag{5}$$

Surprisingly, the same works with the fractional Sobolev seminorms of $H^s(\mathbb{R}^n)$, but the situation is very different for $s \in (0, 1/2)$ or $s \in (1/2, 1)$.

Case $s \in (1/2, 1)$**.** Since $H^s(\mathbb{R}^n)$ is associated with an operator (the fractional Laplacian) of order $2s$, for the same reason in (5) the power of ε to expect in front of the seminorm is ε^{2s-1}. Indeed, it turns out that for $s \in (1/2, 1)$ there holds (up to dimensional constants)

$$\varepsilon^{2s-1}[\chi_E * \rho_\varepsilon]^2_{H^s} \to \mathcal{H}^{n-1}(\partial E) .$$

Hence, in this range, this recipe converges to the *classical perimeter* of E again.

Case $s \in (0, 1/2)$**.** In this case, the situation is very different. First, one can see that $\chi_E \in H^s(\mathbb{R}^n)$ and we no longer need the mollifier! We will prove in Lemma 2.3 that characteristic functions of regular sets are in $H^s(\mathbb{R}^n)$ for $s \in (0, 1/2)$. Thus, for $s \in (0, 1)$ one defines the s-perimeter $\mathrm{Per}_s(E)$ just as

$$\mathrm{Per}_s(E) := [\chi_E]^2_{H^{s/2}(\mathbb{R}^n)} ,$$

without any regularization of the characteristic function.

Definition 2.1. *For $s \in (0, 1)$, the fractional perimeter (or s-perimeter) of a measurable set $E \subset \mathbb{R}^n$ is defined as*

$$\mathrm{Per}_s(E) = \frac{1}{2}[\chi_E]^2_{H^{s/2}(\mathbb{R}^n)} = \iint_{E \times E^c} \frac{1}{|x-y|^{n+s}} \, dxdy .$$

This object is *not* the classical perimeter and, indeed, is structurally different. For example, it scales as

$$\mathrm{Per}_s(\lambda E) = \lambda^{n-s}\mathrm{Per}_s(E) \qquad \forall \lambda > 0 ,$$

instead of scaling as λ^{n-1} as the classical perimeter does.

One can even define a localized version of the nonlocal perimeter in a bounded open set $\Omega \subset \mathbb{R}^n$, in the same spirit of the localized fractional Sobolev spaces $H^s(\Omega)$. This will be useful because, for example, we would like to say that a hyperplane in $\mathbb{R}^n$ is an s-minimal surface even though a half-space has infinite s-perimeter for the definition above.

Definition 2.2. *For $s \in (0, 1)$, the fractional perimeter (or s-perimeter) of a measurable set $E \subset \mathbb{R}^n$ in a bounded, open set Ω is defined as*

$$\mathrm{Per}_s(E, \Omega) := \mathcal{E}^{\mathrm{Sob}}_{s/2,\Omega}(\chi_E) = \frac{1}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n \setminus \Omega^c \times \Omega^c} \frac{|\chi_E(x) - \chi_E(y)|^2}{|x-y|^{n+s}} \, dxdy .$$

Letting

$$\mathcal{J}_s(A, B) = \iint_{A \times B} \frac{1}{|x-y|^{n+s}} \, dxdy$$

be the interaction between the sets A and B, one can write

$$\begin{aligned} &\mathrm{Per}_s(E) \\ &= J_s(E \cap \Omega, E^c \cap \Omega) + J_s(E \cap \Omega, E^c \cap \Omega^c) + J_s(E \cap \Omega^c, E^c \cap \Omega) + J_s(E \cap \Omega^c, E^c \cap \Omega^c) . \end{aligned} \tag{6}$$

Among the four terms on the right-hand side, the first three terms consider interactions in which at least one contribution comes from Ω, but the last term is different since it only considers contributions coming from outside Ω. It would, therefore, be natural to define the localized version of the fractional perimeter as the sum of the first three terms.

Indeed, this is what we have done in Definition 2.2 since, with our definition

$$\begin{aligned}\mathrm{Per}_s(E,\Omega) &= \frac{1}{2}\mathcal{E}^{\mathrm{Sob}}_{s/2,\Omega}(\chi_E)\\ &= J_s(E\cap\Omega, E^c\cap\Omega) + J_s(E\cap\Omega, E^c\cap\Omega^c) + J_s(E\cap\Omega^c, E^c\cap\Omega)\end{aligned}$$

is exactly the sum of the first three terms in (6).

It is important to notice that if $E\subset\Omega$ or $E^c\subset\Omega$ then $\mathrm{Per}_s(E,\Omega)=\mathrm{Per}_s(E)$.

Let us start by proving that smooth sets have finite s-perimeter for every $s\in(0,1)$.

Lemma 2.3. *Let $E\subset\mathbb{R}^n$ be bounded and smooth and $s\in(0,1)$. Then*

(i) $\chi_E\in H^{s/2}(\mathbb{R}^n)$. That is, E has finite s-perimeter.

(ii) There holds

$$\limsup_{s\to1^-}(1-s)\mathrm{Per}_s(E)\le\omega_{n-1}\mathrm{Per}(E)\,,$$

for some dimensional $C>0$.

Proof. We have

$$\begin{aligned}\frac{1}{2}[\chi_E]^2_{H^{s/2}(\mathbb{R}^n)} = \iint_{E\times E^c}\frac{1}{|x-y|^{n+s}}dxdy &= \int_E dx\int_{E^c}\frac{1}{|x-y|^{n+s}}dy\\ &\le \int_E dx\int_{B_R(x)^c}\frac{1}{|x-y|^{n+s}}dy\,,\end{aligned}$$

where $R:=\mathrm{dist}(x,\partial E)$ and this is well defined since ∂E is a compact set. But

$$\int_{B_R(x)^c}\frac{1}{|x-y|^{n+s}}dy=\omega_{n-1}\int_R^\infty\frac{1}{\rho^{n+s}}\rho^{n-1}\,d\rho=\omega_{n-1}\int_R^\infty\frac{d\rho}{\rho^{1+s}}=\frac{\omega_{n-1}R^{-s}}{s}\,.$$

For $t\in(0,\mathrm{diam}(E))$ set $E_t:=\{x\in E\,:\,\mathrm{dist}(x,\partial E)<t\}$, this is an open set with Lipschitz boundary. Fix $\delta>0$, then there is t_* (depending on E and δ) such that $|\partial E_t|\le(1+\delta)|\partial E|$ for $t\in(0,t_*)$. Then, by the coarea formula

$$\begin{aligned}[\chi_E]^2_{H^{s/2}(\mathbb{R}^n)} &\le \frac{\omega_{n-1}}{s}\int_E\frac{1}{\mathrm{dist}(x,\partial E)^s}dx\\ &= \frac{\omega_{n-1}}{s}\int_{E_{t_*}}\frac{1}{\mathrm{dist}(x,\partial E)^s}dx+\frac{\omega_{n-1}}{s}\int_{E\setminus E_{t_*}}\frac{1}{\mathrm{dist}(x,\partial E)^s}dx\\ &\le \frac{\omega_{n-1}}{s}\int_0^{t_*}t^{-s}|\partial E_t|\,dt+\frac{\omega_{n-1}}{s}t_*^{-s}|E|\\ &\le \frac{\omega_{n-1}}{s(1-s)}(1+\delta)|\partial E|t_*^{1-s}+\frac{\omega_{n-1}}{s}t_*^{-s}|E|<+\infty.\end{aligned}$$

This proves (i). Moreover, multiplying by $(1-s)$ and taking $s\to1^-$ gives

$$\limsup_{s\nearrow1}(1-s)\mathrm{Per}_s(E)\le(1+\delta)\omega_{n-1}\mathrm{Per}(E)\,,$$

and letting $\delta\to0^+$ gives (ii). □

With more effort, one can strengthen part (ii) into the convergence of the s-perimeter to the classical one as $s\to1^-$. This result is originally due to Dávila in [D02].

Theorem 2.4. *Let $E \subset \mathbb{R}^n$ be a set of finite perimeter and Ω be a bounded open set with regular boundary. Then, there exists a dimensional constant $c_n > 0$ such that*
$$\lim_{s\to 1^-} (1-s)\mathrm{Per}_s(E,\Omega) = c_n \mathrm{Per}(E,\Omega)\,.$$

Sketch of the proof in [D0́2]. The proof is elementary but in full details not "easy". To simplify the argument, let us assume $\Omega \equiv \mathbb{R}^n$ and E is bounded, say $E \subset B_R$. Actually, we will show in Step 1 in the proof given after this sketch that this is not restrictive and actually implies the general result. Then

$$\begin{aligned}
(1-s)\mathrm{Per}_s(E) &= \frac{1-s}{2}\iint \frac{|\chi_E(x)-\chi_E(y)|^2}{|x-y|^{n+s}}\,dxdy\\
&= \frac{1-s}{4}\iint \frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^{n+s}}\,dxdy\\
&= \frac{\omega_{n-1}}{4}\iint \frac{|\chi_E(x)-\chi_E(y)|}{|x-y|}\cdot\frac{1-s}{\omega_{n-1}|x-y|^{n+s-1}}\,dxdy\\
&= \frac{\omega_{n-1}}{4}\int\left(\int_{B_{10R}} \frac{|\chi_E(y+z)-\chi_E(y)|}{|z|}\cdot\frac{1-s}{\omega_{n-1}|z|^{n+s-1}}\,dz\right)dy\,.
\end{aligned}$$

Note that the sequence of measures
$$\mu_s := \frac{(1-s)\chi_{B_{10R}}}{\omega_{n-1}|z|^{n+s-1}}\,dz$$
satisfies, as $s \to 1^-$, that
$$\mu_s(\mathbb{R}^n) = (1-s)\int_0^{10R} \rho^{-s}\,d\rho = (10R)^{1-s} \to 1, \tag{7}$$
and for every $\delta > 0$
$$\mu_s(\mathbb{R}^n\setminus B_\delta) = (1-s)\int_\delta^{10R} \rho^{-s}\,d\rho = (10R)^{1-s} - \delta^{1-s} \to 0. \tag{8}$$
Then, the result follows from proving an even more general fact on BV functions: for every $u \in \mathrm{BV}(\mathbb{R}^n)$ and $\{\mu_s\}_{s>0}$ sequence of radial kernels satisfying (7) and (8) there holds
$$\left(\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|}{|x-y|}\mu_s(x-y)\,dx\right)dy \rightharpoonup c_n|Du|\,,$$
weakly in the sense of Radon measures. □

To carry out the proof in full generality, we will need the following approximation result. See Proposition 15 in [ADPM11] for a proof.

Lemma 2.5. *Let $E \subset \mathbb{R}^n$ be a set with finite perimeter in Ω, and let $s \in (0,1)$. Then, for every $\varepsilon > 0$ there exists a polyhedral set $\Pi \subset \mathbb{R}^n$ (whose choice is independent of s) such that*
$$|(E\triangle\Pi)\cap\Omega| < \varepsilon, \quad |\mathrm{Per}(E,\Omega) - \mathrm{Per}(\Pi,\Omega)| < \varepsilon, \quad \dim_{\mathcal{H}}(\partial\Pi\cap\partial\Omega) = n-2\,, \tag{9}$$
and
$$(1-s)|\mathrm{Per}_s(E,\Omega) - \mathrm{Per}_s(\Pi,\Omega)| < \varepsilon\,. \tag{10}$$

Proof. All the estimates in (9) are Proposition 15 in [ADPM11]; then (10) follows by interpolation using (9). □

Proof of Theorem 2.4. We divide the proof into four steps.

Step 1. We can assume $\Omega \equiv \mathbb{R}^n$ and E bounded.

Assuming we have proved the result for $\Omega \equiv \mathbb{R}^n$ and E bounded, we show how it implies the real one. Fix $\varepsilon > 0$, and let Π be the polyhedral approximation of E given by Lemma 2.5.

Write

$$\mathrm{Per}_s(\Pi,\Omega) = \frac{1}{2}\Big(\mathrm{Per}_s(\Pi\cap\Omega)+\mathrm{Per}_s(\Pi^c\cap\Omega)-\mathrm{Per}_s(\Omega)\Big)+\mathcal{J}_s(\Pi\cap\Omega,\Pi^c\cap\Omega^c)+\mathcal{J}_s(\Pi\cap\Omega^c,\Pi^c\cap\Omega)\,.$$

Since all the sets $\Pi\cap\Omega, \Pi^c\cap\Omega$ and Ω are bounded, we have

$$\begin{aligned}\lim_{s\to 1^-} \frac{1-s}{2}&\Big(\mathrm{Per}_s(\Pi\cap\Omega)+\mathrm{Per}_s(\Pi^c\cap\Omega)-\mathrm{Per}_s(\Omega)\Big)\\ &=\frac{1}{2}\Big(\mathrm{Per}(\Pi\cap\Omega)+\mathrm{Per}(\Pi^c\cap\Omega)-\mathrm{Per}(\Omega)\Big)=c_n\mathrm{Per}(\Pi,\Omega),\end{aligned}$$

where we have used the transversality condition of the boundaries to infer $\mathcal{H}^{n-1}(\partial\Pi,\partial\Omega)=0$.

Moreover, again, by the transversality, one can check that

$$\lim_{s\to 1^-}(1-s)\mathcal{J}_s(\Pi\cap\Omega,\Pi^c\cap\Omega^c)=\lim_{s\to 1^-}(1-s)\mathcal{J}_s(\Pi\cap\Omega^c,\Pi^c\cap\Omega)=0.$$

Indeed, in both integrals, the singularity of the kernel happens only on points in $\partial\Pi\times\partial\Omega$, which is $(n-2)$-dimensional. Then, an argument with the coarea formula similar to the proof of Lemma 2.3 shows that both limits tend to zero.

Then, we have proved that for Π there holds

$$\lim_{s\to 1^-}(1-s)\mathrm{Per}_s(\Pi,\Omega)=c_n\mathrm{Per}(\Pi,\Omega)\,.$$

But since

$$(1-s)|\mathrm{Per}_s(E,\Omega)-\mathrm{Per}_s(\Pi,\Omega)|<\varepsilon\,.$$

holds for every $s\in(0,1)$, we get by the triangle inequality

$$\limsup_{s\to 1^-}\big|(1-s)\mathrm{Per}_s(E,\Omega)-c_n\mathrm{Per}(E,\Omega)\big|\le\varepsilon+c_n\varepsilon\,.$$

As this holds for every $\varepsilon>0$, this proved the result.

Step 2. Proof for $n=1$.

We have to show: for $F\in\mathrm{BV}(\mathbb{R})$ (i.e. a finite union of intervals) there holds

$$\lim_{s\to 1^-}(1-s)\mathrm{Per}_s(F)=\#\{\text{points in }\partial F\}.$$

Indeed, for just one interval $I=[a,b]\subset\mathbb{R}$ a direct computation shows

$$(1-s)\int_I\int_{I^c}\frac{1}{|x-y|^{1+s}}\,dxdy=\frac{2(b-a)^{1-s}}{s}\to 2,$$

as $s \to 1^-$. Moreover, if $F \in \mathrm{BV}(\mathbb{R})$ then F is a finite union of disjoint intervals $F = \bigcup_{k=1}^N I_k$. Then

$$\begin{aligned}\mathrm{Per}_s(F) = \mathcal{J}_s(F, F^c) &= J_s\left(\bigcup_{k=1}^N I_k, \left(\bigcup_{k=1}^N I_k\right)^c\right)\\ &= \sum_{k=1}^N \mathcal{J}_s(I_k, I_k^c \setminus \bigcup_{k\neq\ell} I_\ell) = \sum_{k=1}^N J_s(I_k, I_k^c) - 2\sum_{k\neq\ell} J_s(I_k, I_\ell)\\ &= \sum_{k=1}^N \mathrm{Per}_s(I_k) - 2\sum_{k\neq\ell} J_s(I_k, I_\ell)\,.\end{aligned}$$

Multiplying by $(1-s)$ and sending $s \to 1^-$, all the cross interactions with $k \neq \ell$ tend to zero since the intervals are at a positive distance, and we get

$$\lim_{s\to 1^-}(1-s)\mathrm{Per}_s(F) = 2N,$$

which is the number of points in ∂F as desired.

Step 3. Slice the fractional perimeter along lines.

Since $\chi_E \in \mathrm{BV}(\mathbb{R}^n)$ then $\chi_E \in H^{s/2}(\mathbb{R}^n)$, hence

$$\mathrm{Per}_s(E) = C_{n,s}\int_{\mathcal{G}_1^n} dL \int_{L^\perp} \mathrm{Per}_s^{L+h}(E)\, d\mathcal{H}^{n-1}(h), \tag{11}$$

where $\mathcal{G}_1^n$ is the linear Grassmanian of lines, $C_{n,s} > 0$ is a constant that stays bounded as $s \to 1^-$ and $\mathrm{Per}_s^{L+h}(E)$ is the 1-dimensional s-perimeter of E restricted to $L+h$. This follows from a more general formula allowing to slice fractional seminorms with k-dimensional subspaces, see for example, [CFP24, Theorem 2.12].

In the specific case of lines ($k = 1$), it can also be proved using just polar coordinates that

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} f(x,y)\,dxdy = C_n \int_{\mathcal{G}_1^n} dL \int_{L^\perp} dh \iint_{(L+h)\times(L+h)} f(x,y)|x-y|^{n-1}\, d\mathcal{H}^1(x)d\mathcal{H}^1(y)$$

for every $f \geq 0$. When $f(x,y) = \chi_E(x)\chi_{E^c}(y)$, this formula also gives (11).

Step 4. Reconstruct the perimeter of E with a Crofton formula.

For example (Theorem 3.2.26 in [Fed87]): if M is an $(n-1)$-dimensional rectifiable set, then

$$\mathcal{H}^{n-1}(M) = \beta_n \int_{\mathcal{G}_1^n} dL \int_{L^\perp} \#\{M \cap (L+y)\}\, d\mathcal{H}^{n-1}(y)\,.$$

Combining these three steps, we can conclude the proof. By Step 1, we can assume that E is bounded and $\Omega \equiv \mathbb{R}^n$. Since E is bounded, by translating and scaling, we can, with no loss of generality, assume that $E \subset B_1$.

Let dL be the standard measure on the linear Grassmanian $\mathcal{G}_1^n$. Since E has finite s-perimeter, it follows from (11) that almost every restriction $E|_{(L+h)}$, with respect to the tensor product measure $dP := dL \otimes \mathcal{H}^{n-1}(L^\perp)$, has finite (1-dimensional) s-perimeter in $L+h$.

Hence, for almost every $L+h$, the set $E|_{(L+h)}$ is a finite union of intervals. For every such $L+h$ in this set of full measure, since E is bounded, by the interpolation inequality between $W^{s,1}(\mathbb{R})$ and $\mathrm{BV}(\mathbb{R})$ (or, essentially by the proof of Lemma 2.3 for $n = 1$) we have, for some constant $C > 0$ independent of s that

$$(1-s)\mathrm{Per}_s^{L+h}(E|_{(L+h)}) \leq C\,\mathrm{Per}(E|_{(L+h)}, L+h) = C\#\{\partial E \cap (L+h)\} \in L^1(dP)\,,$$

since by the definition of dP and Crofton's formula

$$\int_{\mathcal{G}_1^n} dL \int_{L^\perp} \#\{\partial E \cap (L+y)\}\, d\mathcal{H}^{n-1}(y) = C_n \mathrm{Per}(E) < +\infty\,.$$

Moreover, for every $L+h$ in this set of full measure, by Step 1

$$\lim_{s\to 1^-} (1-s)\mathrm{Per}_s^{L+h}(E|_{(L+h)}) = \#\{\partial E \cap (L+h)\}\,.$$

Thus, by dominated convergence

$$\begin{aligned}\lim_{s\to 1^-}(1-s)\mathrm{Per}_s(E) &= \lim_{s\to 1^-} C_{n,s}\int_{\mathcal{G}_1^n} dL \int_{L^\perp} (1-s)\mathrm{Per}_s^{L+h}(E)\, d\mathcal{H}^{n-1}(h)\\ &= C_{n,1}\int_{\mathcal{G}_1^n} dL \int_{L^\perp} \#\{\partial E\cap P\}\, d\mathcal{H}^{n-1}(h)\\ &= c_n \mathrm{Per}(E)\,,\end{aligned}$$

as desired. □

Remark 2.6. *It was proved by Ambrosio, De Philippis, and Martinazzi in [ADPM11] that the s-perimeter even Γ-converges to the classical perimeter as $s\to 1^-$, that is*

$$\Gamma - \lim_{s\to 1^-}(1-s)\mathrm{Per}_s(E,\Omega) = c_n\mathrm{Per}(E,\Omega)\,.$$

2.1. **Perimeter minimizers.** This subsection is entirely devoted to the properties of perimeter minimizers, that is, sets that locally minimize the fractional perimeter.

It will be useful in a few places that the fractional perimeter satisfies an isoperimetric inequality analogous to the classical one. To prove it, it is convenient to name the local part of the fractional perimeter. Let

$$\mathrm{Per}_s|_\Omega(E) := \iint_{\Omega\times\Omega} \frac{|\chi_E(x)-\chi_E(y)|^2}{|x-y|^{n+s}}\, dxdy\,.$$

We stress that $\mathrm{Per}_s|_\Omega(E) \neq \mathrm{Per}_s(E,\Omega)$, and actually $\mathrm{Per}_s|_\Omega(E) \le \mathrm{Per}_s(E,\Omega)$, with equality iff $E\cap\Omega=\emptyset$ or $E^c\cap\Omega=\emptyset$.

Theorem 2.7 (Fractional isoperimetric inequality)**.** *There is a constant $C=C(n,s)>0$ such that, for every E set of finite s-perimeter there holds*

$$\min\{|E|,|E^c|\}^{\frac{n-s}{n}} \le C\,\mathrm{Per}_s(E)\,.$$

Proof. The proof adapts to the fractional case of the classical isoperimetric inequality. See, for example, [AFP00, Theorem 3.46].

By the fractional Poincaré inequality (which, for example, can be proved by compactness as the classical Poincaré inequality using Theorem 1.4), for every $Q_R = Q_R(x)$ hypercube of side R we have that

$$\int_{Q_R} |u-u_{Q_R}|^2\, dx \le CR^s \iint_{Q_R\times Q_R} \frac{|u(x)-u(y)|^2}{|x-y|^{n+s}}\, dxdy\,,$$

where $u_{Q_R} = \frac{1}{|Q_R|}\int_{Q_R} u\, dx$.

Plugging $u=\chi_E$ gives

$$\int_{Q_R}\left|\chi_E - \frac{|E\cap Q_R|}{|Q_R|}\right|^2 dx \le CR^s\mathrm{Per}_s|_{Q_R}(E)\,.$$

Denoting $m_x(R) := \frac{|E\cap Q_R(x)|}{|Q_R|}$, the left hand side equals to

$$\begin{aligned}|E\cap Q_R(x)|(1-m_x(R))^2 &+ (|Q_R| - |E\cap Q_R(x)|)m_x(R)^2 \\ &= |Q_R|\Big(m_x(R)(1-m_x(R)^2) + (1-m_x(R))m_x(R)^2\Big) \\ &= |Q_R| m_x(R)(1-m_x(R)),\end{aligned}$$

thus

$$m_x(R)(1-m_x(R)) \leq \frac{C_\circ}{R^{n-s}} \mathrm{Per}_s|_{Q_R}(E),$$

for some $C_\circ > 0$ dimensional constant.

Now choose $R_* > 0$, depending on E and $C_\circ$, such that

$$\frac{C_\circ}{R_*^{n-s}} \mathrm{Per}_s(E) = \frac{1}{100}\,. \tag{12}$$

With this choice, $m_x^* := m_x(R_*)$ satisfies $m_x^*(1-m_x^*) \in [0, 1/10) \cup (9/10, 1]$. Moreover, since the map $x \mapsto m_x^*(R_*)$ is continuous, either $m_x^* \in [0, 1/10)$ for all $x \in \mathbb{R}^n$ or $m_x^* \in (9/10, 1]$ for all $x \in \mathbb{R}^n$. Up to changing E with E^c, we can assume $m_x^* \in [0, 1/10)$. Covering almost all of $\mathbb{R}^n$ with disjoint open cubes $\{Q_{R_*}(x_i)\}_i$ of side R_*, we get

$$\begin{aligned}|E| &= \sum_i |E \cap Q_{R_*}(x_i)| = R_*^n \sum_i m_{x_i}(R_*) \\ &\leq C R_*^s \sum_i \mathrm{Per}_s|_{Q_{R_*}(x_i)}(E) \leq C R_*^s \mathrm{Per}_s(E) = C \mathrm{Per}_s(E)^{\frac{n}{n-s}}\,,\end{aligned}$$

where in the last equality, we have used the definition of R_* from (12). □

Definition 2.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, we say that $E \subset \mathbb{R}^n$ is a minimizer of the s-perimeter (or an s-minimizer) in Ω if*

$$\mathrm{Per}_s(E, \Omega) \leq \mathrm{Per}_s(F, \Omega)\,,$$

for every measurable F such that $E = F$ outside Ω.

Now, we list here some classical properties of fractional perimeter minimizers for the reader to compare with the classical ones. We omit the proof of most; we only include few proofs to serve as a comparison with the proofs in the classical case. Most of the proofs can be found in [Sav22], [CF17], and [CRS10].

- (Lower semicontinuity) Let $E_k \to E$ in $L^1_{\mathrm{loc}}(B_1)$, then
$$\mathrm{Per}_s(E, B_1) \leq \liminf_{k\to\infty} \mathrm{Per}_s(E_k, B_1)\,.$$
- (Compactness, see also Theorem 1.4) If $\mathrm{Per}_s(E_k, B_1) \leq C$ uniformly in k, then there is a subsequence of the E_k that converges in $L^1(B_1)$.
- (Existence) Let $E_\circ \subset \mathbb{R}^n$ be a set with locally finite s-perimeter in B_1. Then, there exists a set E minimizing the s-perimeter $\mathrm{Per}_s(\cdot, B_1)$ among all sets E with $E \setminus B_1 = E_\circ \setminus B_1$.
- (Density estimate) Let $E \subset B_1$ be a minimizer of the s-perimeter in B_1 and[1] $0 \in \partial E$. Then there is $c = c(n, s) > 0$ such that
$$|E \cap B_R(0)| \geq c|B_R| \;\text{ and }\; |B_R(0) \setminus E| \geq c|B_R|\,.$$

[1] Here we understand ∂E as the complement of points of density 1 and 0.

Proof. The proof is taken from [Sav22]. By scaling invariance, we prove the statement just for $R = 1$. Moreover, since being an s-minimal surface is invariant under complementation, we just show the first inequality. Let

$$V(r) := |E \cap B_r| \ \text{ and } \ A(r) := \mathcal{H}^{n-1}(E \cap \partial B_r) = V'(r)\,.$$

By the fractional isoperimetric inequality of Theorem 2.7

$$|E \cap B_r|^{\frac{n-s}{n}} = V(r)^{\frac{n-s}{n}} \le C\mathrm{Per}_s(E \cap B_r)\,.$$

Testing the minimality of E with the competitor $E \setminus B_r$ gives

$$\mathcal{J}(E \cap B_r, E^c) \le \mathcal{J}(E \cap B_r, E \setminus B_r)\,,$$

hence

$$\begin{aligned}\mathrm{Per}_s(E \cap B_r) = \mathcal{J}(E \cap B_r, (E \cap B_r)^c) &= \mathcal{J}(E \cap B_r, E^c) + \mathcal{J}(E \cap B_r, E \setminus B_r)\\ &\le 2\mathcal{J}(E \cap B_r, E \setminus B_r)\\ &\le 2\mathcal{J}(E \cap B_r, B_r^c)\,.\end{aligned}$$

Now for $x \in E \cap B_r$ note that $B_{r-|x|}(x) \subset B_r$, thus we have

$$\begin{aligned}V(r)^{\frac{n-s}{n}} \le C\mathcal{J}(E \cap B_r, B_r^c) &= \int_{E\cap B_r}\int_{B_r^c} \frac{1}{|x-y|^{n+s}}dydx\\ &\le C\int_{E\cap B_r}\int_{B_{r-|x|}(x)^c} \frac{1}{|x-y|^{n+s}}dxdy\\ &= C\int_{E\cap B_r}\int_{r-|x|}^{\infty} \frac{1}{\rho^{n+s}}\rho^{n-1}dxdy = C\int_{E\cap B_r} \frac{1}{(r-|x|)^s}dy\,,\end{aligned}$$

and by polar coordinates again

$$V(r)^{\frac{n-s}{n}} \le C\int_0^r \frac{A(\rho)}{(r-\rho)^s}d\rho\,. \tag{13}$$

(Here see Remark 2.9). Integrating on $[0, R]$ gives

$$\int_0^R V(r)^{\frac{n-s}{n}}dr \le C\int_0^R A(r)(R-r)^{1-s}dr \le CR^{1-s}V(R)\,. \tag{14}$$

Now set $R_k := \frac{1}{2} + 2^{-k}$ and apply (14) to $R = R_k$ to get

$$\int_0^{R_k} V(r)^{\frac{n-s}{n}}dr \le CV(R_k)\,,$$

but since $V(R) \ge V(R_{k+1})$ on $[R_{k+1}, R_k]$

$$2^{-(k+1)}V(R_{k+1})^{\frac{n-s}{n}} = (R_k - R_{k+1})V(R_{k+1})^{\frac{n-s}{n}} \le \int_{R_{k+1}}^{R_k} V(r)^{\frac{n-s}{n}}dr \le CV(R_k)\,,$$

or

$$V(R_{k+1}) \le C(2^kV(R_k))^{1+\frac{1}{n-s}} = C2^{\theta k}V(R_k)^{\theta}\,,$$

for some $\theta > 1$ depending on n and s. Iterating this inequality, one can get

$$V(R_k) \le C(C_\circ V(R_0))^{\theta^k} = C(C_\circ V(1))^{\theta^k}\,,$$

for some $C_\circ = C_\circ(n, s)$. This implies that if $V(1) = |E \cap B_1| \le 1/(2C_\circ)$ then $V(R_k) \to 0$ as $k \to \infty$, and since $R_\infty = 1/2$ we would get

$$V(1/2) = |E \cap B_{1/2}| = 0\,,$$

and this implies $0 \notin \partial E$. This concludes the proof with $c = 1/(2C_\circ)$.

□

Remark 2.9. *Note that* (13) *is the nonlocal version of the differential inequality*

$$V(r)^{\frac{n-1}{n}} \le C_n A(r)\,,$$

which is used in the proof of the density estimate for classical minimal sets and is obtained by testing minimality against $\partial B_R(0) \cap E$*.*

- (Compactness of minimizers, [CRS10]) Let E_k be a sequence of s-perimeter minimizers in B_1, and assume that

$$E_k \to E \ \text{ in } L^1_{\mathrm{loc}}(\mathbb{R}^n)\,.$$

Then E in s-minimizer in B_1.

Proof. Fix F any competitor for E, that is a set $F \subset \mathbb{R}^n$ such that $F \setminus B_1 = E \setminus B_1$. Define

$$F_k := \begin{cases} F & \text{in } B_1, \\ E_k & \text{in } B_1^c. \end{cases}$$

This is a competitor for E_k, and by the minimality of E_k we know $\mathrm{Per}_s(E_k, B_1) \le \mathrm{Per}_s(F_k, B_1)$. We claim that

$$\mathrm{Per}_s(F_k, B_1) \to \mathrm{Per}_s(F, B_1)\,. \tag{15}$$

Indeed, using the definition of F_k and the fact that $F = E$ outside B_1 we get

$$\begin{aligned} \Big|\mathrm{Per}_s(F_k, B_1) - \mathrm{Per}_s(F, B_1)\Big| &\le \mathcal{J}_s((E_k \triangle E) \cap B_1^c, B_1) \\ &= \iint_{B_1 \times B_1^c} \frac{\chi_{E_k \triangle E}(y)}{|x-y|^{n+s}}\, dxdy\,. \end{aligned}$$

Since $E_k \to E$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, up to subsequences, we have that $\chi_{E_k \triangle E} \to 0$ almost everywhere. Moreover, the convergence is dominated since

$$\frac{\chi_{E_k \triangle E}(y)}{|x-y|^{n+s}} \le \frac{1}{|x-y|^{n+s}} \in L^1(B_1 \times B_1^c)\,.$$

Thus, by dominated convergence, we get (15). Then, by the lower semicontinuity of the s-perimeter (along the chosen subsequence)

$$\mathrm{Per}_s(E, B_1) \le \liminf_{k\to\infty} \mathrm{Per}_s(E_k, B_1) \le \liminf_{k\to\infty} \mathrm{Per}_s(F_k, B_1) = \mathrm{Per}_s(F, B_1)\,.$$

That is, E is an s-minimizer in B_1. □

- (Maximum principle, [DSV23]) Let $E, F \subset \mathbb{R}^n$ be s-minimal sets in B_1 with $E \subseteq F$. If there is x such that $x \in \partial E \cap \partial F$, then $E \equiv F$.

Remark 2.10. *This maximum principle is the nonlocal version of the classical result by Leon Simon for classical perimeter minimizers. If both E and F were known to be smooth near the touching point x, then the proof of this would be elementary using the first variation formula. Nevertheless, the result in [DSV23] is nontrivial since it allows the surfaces to touch at an irregular boundary point.*

Moreover, there is also a complete analog of the monotonicity formula and tangent cones to minimal sets. All this was developed in [CRS10]. We refer to Section 2.5 for a precise statement of the monotonicity formula and definition of the Caffarelli-Silvestre extension.

- (Monotonicity formula) Let U_E be the Caffarelli-Silvestre extension of $\chi_E - \chi_{E^c}$ as in Theorem 2.22. If E is a local minimizer of the s-perimeter and is smooth, then for every $x \in \partial E$ the function
$$\Phi_E(r) := \frac{1}{r^{n-s}} \int_{B_r^+(x,0)} z^{1-s} |\nabla U_E|^2 \, dx dz \quad \text{is nondecreasing in } r \,,$$
and is constant if and only if E is a cone.
- (Tangent cones, see Corollary 3.3 in [Sav22]) If E minimizes the s-perimeter in B_1 and $x \in \partial E \cap B_1$, then a subsequence of the blow-ups $\frac{E-x}{r}$ as $r \to 0^+$ converges in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ to a minimal cone.
- (Regularity of flat minimizers) Let E be an s-perimeter minimizer and $x \in \partial E$ be such that the blow-up cone at x is a half-space. Then E is smooth in a neighborhood of x.

  Proof. As the blow-up cone is a half-space, the fact that then E is $C^{1,\alpha}$ graph (in a neighborhood of x) for some α is proved in [CRS10]. Then, a combination of the two works [FV17] (E Lipschitz $\implies$ E is $C^{1,\alpha}$ for every $\alpha < s$) and [BFV14] (E is $C^{1,\alpha}$ for some $\alpha > s/2 \implies E$ is C^∞) shows that it is actually smooth. □
- (Classification of minimal cones) There is $\delta = \delta(n) > 0$ such that, for $s \in (1-\delta, 1)$ and $2 \le n \le 7$ the only s-minimizing cones are the half-spaces. Moreover, this is sharp as the Simons cone is minimizing the s-perimeter in dimension $n = 8$.

  Proof. For $n = 2$ and every $s \in (0,1)$ this was shown by Savin and Valdinoci in [SV13a]. Then, it was proved for $2 \le n \le 7$ and s close to 1 by Caffarelli and Valdinoci in [CV13]. □
- (Full regularity of minimizers) Let $n \ge 2$ and $E \subset \mathbb{R}^n$ be an s-minimal surface. There is $\delta = \delta(n) > 0$ such that for $s \in (1-\delta, 1)$, then:
  (1) if $n \le 7$ then ∂E is smooth.
  (2) if $n \ge 8$ then $\Sigma := \partial E$ is smooth outside a closed set $\mathrm{sing}(\Sigma) \subset \partial E$ with $\mathcal{H}^\alpha(\mathrm{sing}(\Sigma)) = 0$ for every $\alpha > n-8$.

2.1.1. *Minimal cones vs stable cones.* As pointed out above, for s close to 1 and $2 \le n \le 7$, the classification of s-minimizing cones in $\mathbb{R}^n$ is well understood. Nevertheless, the classification of stable s-minimal cones in $\mathbb{R}^n$ in this dimension range is (expected, but) a much more complex issue. Indeed, the classification of stable cones in low dimensions turns out to be more challenging for $s \in (0,1)$ than in the classical case $s = 1$.

At present, the fact that every stable s-minimal cone in $\mathbb{R}^n$ is flat is known for $n = 2$ and every $s \in (0,1)$ by a result of Savin and Valdinoci in [SV13b], for $n = 3$ and s close to 1 by Cabré, Cinti and Serra in [CCS20], and for $n = 4$ and s close to 1 by a recent preprint [CDSV23] by Chan, Dipierro, Serra and Valdinoci. Moreover, these three results use very different techniques for the proof.

Evidence that classifying stable s-minimal cones is a very hard problem can be taken from the following result. It is proved by Dávila, del Pino, and Wei in [DdPW18] that there exist nonflat stable, s-minimal Lawson cones in $\mathbb{R}^7$ for small s. This suggests that unlike the classical theory (or the case s close to 1), the flatness of s-minimizing cones may not hold in dimension 7 for s small. Hence, all the proofs of the classification of stable cones must be sharp to take into account that s small and s close to 1 could (and, in many cases, will) have different behavior regarding this classification. For this reason, the proofs presented above of the classification of stable s-minimal cones require technical and sharp estimates with respect to the dependence on s.

2.2. **First variation and nonlocal mean curvature.** Now we are in the position to define the nonlocal mean curvature.

Definition 2.11 (Nonlocal mean curvature)**.** *Let $x \in \partial E$, we call*

$$H_{s,E}(x) := P.V. \int_{\mathbb{R}^n} \frac{\chi_{E^c}(y) - \chi_E(y)}{|x-y|^{n+s}}\, dy$$

the s-mean curvature of E at x.

We will sometimes denote $H_{s,E}$ just by H_s when the dependence on the set E is clear from the context.

Note that, since the kernel $|x-y|^{-(n+s)}$ is singular near the diagonal, the principal value is needed even for smooth sets, since for $x \in \partial E$

$$\int_E \frac{1}{|x-y|^{n+s}}\, dy = \int_{E^c} \frac{1}{|x-y|^{n+s}}\, dy = +\infty\,.$$

2.2.1. *Nonlocal mean curvature and excess.* Let $E \subset \mathbb{R}^n$ and denote by $\Sigma := \partial E$ its boundary. This subsection describes how the nonlocal mean curvature $H_s(x)$ at a boundary point $x \in \Sigma$ relates to the classical mean curvature $H_\Sigma(x)$.

Definition 2.12 (Excess)**.** *For a boundary point $x \in \Sigma = \partial E$ we call*

$$\mathrm{ex}_E(x,r) := \frac{1}{|\partial B_1| r^{n-1}} \int_{\partial B_r(x)} (\chi_{E^c} - \chi_E)\, d\sigma$$

the excess of E at x.

One can easily see that $|\mathrm{ex}(x,r)| \le 1$ and actually it expands as

$$\mathrm{ex}_E(x,r) = H_\Sigma(x) r + o(r^2)\,, \tag{16}$$

where H_Σ is (classical) the mean curvature of Σ with respect ot the inner normal. In particular, this implies that

$$H_\Sigma(x) = \lim_{r\to 0^+} \frac{\mathrm{ex}_E(x,r)}{r}\,.$$

Exercise 2.13. *Prove the expansion formula (16) for the excess arguing as follows. Since this is a local statement, up to translation and rotation, assume that in a neighborhood of x, ∂E is a graph of a function $u : \mathbb{R}^{n-1} \to \mathbb{R}$ with $u(0) = 0$ and $\nabla u(0) = 0$. Then, write the desired integral in terms of u, use a Taylor expansion in r and that $H_{\mathrm{graph}(u)}(0) = \Delta u(0)$ to conclude.*

By the very definition of $H_{s,E}$, one can observe that

$$\begin{aligned} H_{s,E}(x) &= P.V. \int_{\mathbb{R}^n} \frac{\chi_{E^c}(y) - \chi_E(y)}{|x-y|^{n+s}}\, dy \\ &= \int_0^\infty \int_{\partial B_r(x)} \frac{\chi_{E^c} - \chi_E}{r^{n+s}}\, d\sigma dr = |\partial B_1| \int_0^\infty \frac{\mathrm{ex}_E(x,r)}{r^{1+s}} dr\,, \end{aligned}$$

provided that the limit in the principal value exists.

So, the nonlocal mean curvature is just a weighted average of the excess with arbitrarily big radii. As the following proposition shows, this formula makes it easy to conclude that the nonlocal mean curvature converges (at least pointwise) to the classical mean curvature.

Proposition 2.14. *There is a dimensional constant $c_n > 0$ such that, for every $x \in \partial E$*

$$\lim_{s \nearrow 1}(1-s)H_{s,E}(x) = c_n H_{\partial E}(x).$$

Proof. We have (we neglect dimensional constants in front of the integrals)

$$\begin{aligned}(1-s)H_{s,E}(x) &= \int_0^\infty \frac{\mathrm{ex}_E(x,r)}{r}\frac{(1-s)}{r^s}\,dr\\ &= \int_0^1 \frac{\mathrm{ex}_E(x,r)}{r}\frac{(1-s)}{r^s}\,dr + (1-s)\int_1^\infty \frac{\mathrm{ex}_E(x,r)}{r^{1+s}}\,dr.\end{aligned}$$

The second integral tends to zero as $s \to 1^-$ since, just by using $|\mathrm{ex}(x,r)| \le 1$ we get

$$\left|(1-s)\int_1^\infty \frac{\mathrm{ex}_E(x,r)}{r^{1+s}}dr\right| \le (1-s)\int_1^\infty r^{-1-s}dr = 1-s \to 0.$$

For the first integral note that $f_s(r) := (1-s)r^{-s}$ is a sequence of functions on $(0,1]$ with mass equals 1 for every s, and that converges to zero everywhere on $(0,1]$. Then, by a standard argument of weak compactness of sequence of measures with bounded mass, we have $f_s \rightharpoonup \delta_{\{0\}}$ as $s \to 1^-$ in duality against $C^0([0,1])$, thus

$$\lim_{s\to 1^-}(1-s)H_{s,E}(x) = \lim_{s\to 1^-}\int_0^1 \frac{\mathrm{ex}_E(x,r)}{r} f_s(r)\,dr = \lim_{r \to 0^+}\frac{\mathrm{ex}_E(x,r)}{r} = H_{\partial E}(x)\,,$$

where in the last line, we have used formula (16) for the expansion of the excess. □

Actually, more than Proposition 2.14 is true. Indeed, if ∂E is the graph of some $C^{2,\alpha}$ function u in B_1 in some direction, then the convergence in Proposition 2.14 holds uniformly in x as

$$\sup_{B_{1/2}}\left|c_n H_{\partial E}(x) - (1-s)H_{s,E}(x)\right| \le C(1-s)[u]_{C^{2,\alpha}(B_1)}\,,$$

for some $C = C(\alpha, n) > 0$. See [CV13, Lemma 9] for a proof of this fact.

2.2.2. *First variation formula.* Recall the definition of the s-perimeter

$$\begin{aligned}\mathrm{Per}_s(E) = [\chi_E]^2_{H^{s/2}} &= \iint_{E\times E^c}\frac{1}{|x-y|^{n+s}}\,dxdy\\ &:= \iint_{E\times E^c}\mathcal{K}_s(x,y)\,dxdy\,,\end{aligned}$$

where we have let $\mathcal{K}_s(x,y) := \frac{1}{|x-y|^{n+s}}$.

To prove that $H_{s,E}$ is actually the first variation of the s-perimeter, we will need to approximate the kernel $\mathcal{K}_s$ with a sequence $\mathcal{K}_s^\varepsilon$ of non-singular ones and then send $\varepsilon \to 0^+$ just at the end. Let, for example

$$\mathcal{K}_s^\varepsilon(x,y) = \frac{\chi_{\{|x-y|\ge\varepsilon\}}}{|x-y|^{n+s}}.$$

Proposition 2.15 (First variation)**.** *Let $E \subset \mathbb{R}^n$ be a set with finite s-perimeter and $\phi_t^X : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth one-parameter family of diffeomorphisms such that $\phi_0^X(x) = x$. Let $X := \frac{\partial}{\partial t}\big|_{t=0}\phi_t^X$ be its velocity field, and assume X has compact support. Then*

$$\frac{d}{dt}\bigg|_{t=0} \mathrm{Per}_s(\phi_t^X(E)) = \int_{\partial E} H_{s,E}(N \cdot X)\, d\sigma,$$

where N is the outer unit normal from E on ∂E.

Proof. Let $\mathcal{K}_s^\varepsilon$ be a non-singular approximation of $\mathcal{K}_s$ as above, and let

$$\mathrm{Per}_s^\varepsilon(E) := \iint_{E\times E^c} \mathcal{K}_s^\varepsilon(x,y)\, dxdy\,.$$

Write $E_t := \phi_t^X(E)$. Then, by Lemma 4.1 we get

$$\begin{aligned}
\frac{d}{dt}\bigg|_{t=0} \mathrm{Per}_s^\varepsilon(E_t) &= \frac{d}{dt}\bigg|_{t=0} \int_{E_t}\left(\int_{E_t^c} \mathcal{K}_s^\varepsilon(x,y)\, dx\right) dy \\
&= \int_E \frac{d}{dt}\bigg|_{t=0}\left(\int_{E_t^c} \mathcal{K}_s^\varepsilon(x,y)\, dx\right) dy + \int_{\partial E}\left(\int_{E^c} \mathcal{K}_s^\varepsilon(x,y)\, dx\right) N\cdot X\, d\sigma_y \\
&= \int_E \left(\int_{\partial E} \mathcal{K}_s^\varepsilon(x,y)\, d\sigma_x\right)(-N)\cdot X\, dy + \int_{\partial E}\left(\int_{E^c} \mathcal{K}_s^\varepsilon(x,y)\, dx\right) N\cdot X\, d\sigma_y \\
&= \int_{\partial E}\left(\int_{\mathbb{R}^n} \mathcal{K}_s^\varepsilon(x,y)\big(\chi_{E^c}(y) - \chi_E(y)\big) dy\right)(N\cdot X)\, d\sigma_x,
\end{aligned}$$

where in the last line, we have used in an essential way that the kernel $\mathcal{K}_s^\varepsilon$ is symmetric.

Now, we would like to take $\varepsilon \to 0^+$ inside the derivative to get

$$\begin{aligned}
\frac{d}{dt}\bigg|_{t=0} \mathrm{Per}_s(E_t) &= \int_{\partial E}\left(P.V. \int_{\mathbb{R}^n} \mathcal{K}_s(x,y)\big(\chi_{E^c}(y) - \chi_E(y)\big) dy\right)(N\cdot X)\, d\sigma_x \\
&= \int_{\partial E} H_{s,E}(x)(N\cdot X)\, d\sigma_x.
\end{aligned}$$

but this requires a bit of care since we are exchanging the limit in ε and the derivative in t.

Set

$$\varphi_\varepsilon(t) := \mathrm{Per}_s^\varepsilon(E_t)\,, \quad \text{and} \quad \varphi(t) := \mathrm{Per}_s(E_t),$$

so that $\varphi_\varepsilon \uparrow \varphi$ monotonically, since the approximated kernels $\mathcal{K}_s^\varepsilon$ monotonically converge to $\mathcal{K}_s$ from below. Moreover, for a smooth set F and $x \in \partial F$ let also

$$H_{s,F}^\varepsilon(x) := \int_{\mathbb{R}^n} \mathcal{K}_s^\varepsilon(x,y)\big(\chi_{F^c}(y) - \chi_F(y)\big) dy.$$

Here, there is no need for the principal value since $\mathcal{K}_s^\varepsilon$ is nonsingular.

We know by the first part of this proof, applied at time t instead of only $t = 0$, that

$$\varphi_\varepsilon'(t) = \int_{\partial E_t} H_{s,E_t}^\varepsilon(N_t\cdot X)\, d\sigma, \tag{17}$$

where N_t is the outer unit normal to E_t. If we can show that the derivatives φ_ε' converge uniformly in a neighborhood of zero, that is

$$\lim_{\varepsilon\to 0^+} \sup_{|t|\le\delta} \left|\varphi_\varepsilon'(t) - \int_{\partial E_t} H_{s,E_t}(N_t\cdot X)\, d\sigma\right| = 0. \tag{18}$$

Then, we can conclude by using standard arguments from real analysis.

By (17) we have

$$\varphi'_\varepsilon(t) - \int_{\partial E_t} H_{s,E_t}(N_t\cdot X)\,d\sigma = \int_{\partial E_t} \big(H^\varepsilon_{s,E_t} - H_{s,E_t}\big)(N_t\cdot X)\,d\sigma\,.$$

Moreover, analogously to the proof of Proposition 2.14

$$H^\varepsilon_{s,E_t}(x) - H_{s,E_t}(x) = P.V.\int_{B_\varepsilon(x)} \frac{\chi_{E_t^c}(y)-\chi_{E_t}(y)}{|x-y|^{n+s}}\,dy = |\partial B_1|\int_0^\varepsilon \frac{\mathrm{ex}_{E_t}(x,r)}{r^{1+s}}\,dr\,.$$

Note that, by the proof of (16) (that is, following Exercise 2.13 and using Taylor with the exact remainder at second order), if ∂F is graphical around $z\in\partial F$ in $B_R(z)$, then for $r\le R/2$

$$\sup_{x\in B_{R/2}(z)\cap\partial F} \frac{\mathrm{ex}_F(x,r)}{r} \le C(\partial F,R,z)\,,$$

where $C(\partial F,R,z)$ depends only on the C^2 norm of ∂F in $B_R(z)$.

Let K be a compact set such that $\mathrm{supp}(X)\subset\subset K$. Then, taking $\delta=\delta(X)>0$ sufficiently small so that all the C^2 norms of ∂E_t are bounded by (a constant times) the C^2 norm of E for $|t|\le\delta$, we get

$$\sup_{|t|\le\delta}\,\sup_{x\in\partial E_t\cap K}\big|H^\varepsilon_{s,E_t}(x)-H_{s,E_t}(x)\big| \le C\int_0^\varepsilon r^{-s}\,dr = C\varepsilon^{1-s}\,, \tag{19}$$

where C depends on s and the C^2 norm of ∂E in K.

Hence, letting $\varepsilon\to0^+$, we get the desired uniform convergence (18). Thus, we can exchange limit and derivative to get

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\mathrm{Per}_s(E_t) = \frac{d}{dt}\Big|_{t=0}\lim_{\varepsilon\to0^+}\varphi_\varepsilon(t) &= \lim_{\varepsilon\to0^+}\varphi'_\varepsilon(0)\\ &= \lim_{\varepsilon\to0^+}\int_{\partial E} H^\varepsilon_{s,E}(N\cdot X)\,d\sigma\\ &= \int_{\partial E} H_{s,E}(N\cdot X)\,d\sigma\,,\end{aligned}$$

where in the last line, we have used again (19) for the uniform convergence of the approximate nonlocal mean curvature to the nonlocal mean curvature of E. This concludes the proof. □

Definition 2.16 (s-minimal surface)**.** *Let Ω be a bounded open set and $E\subset\mathbb{R}^n$ be a set with locally finite s-perimeter in Ω. Then, E is said to be an s-minimal surface in Ω if, for every vector field X with compact support in Ω there holds*

$$\frac{d}{dt}\Big|_{t=0}\mathrm{Per}_s(\phi^X_t(E),\Omega) = 0\,.$$

By the first variation formula of Proposition 2.15 we see that, if E is a smooth s-minimal surface in Ω then

$$H_{s,E}(x) = P.V.\int_{\mathbb{R}^n} \frac{\chi_{E^c}(y)-\chi_E(y)}{|x-y|^{n+s}}\,dy = 0\,,$$

for every $x\in\partial E\cap\Omega$. Heuristically, this is saying that

$$\int_E \frac{dy}{|x-y|^{n+s}} = \int_{E^c}\frac{dy}{|x-y|^{n+s}}\,,$$

that is E and E^c have equal average interaction with $x \in \partial E$. However, this formula cannot be interpreted since, even for smooth sets, these two integrals always do not converge.

Exercise 2.17. *For $n = 1$ and $\lambda > 0$, check that the nonlocal mean curvature of a segment $[0, \lambda]$ at 0 equals $\frac{2}{s\lambda^s}$.*

Exercise 2.18. *Using the first variation formula, prove the strict maximum principle for smooth s-minimal surfaces, that is, the following.*

Let $R > 0$ and $E, F \subset \mathbb{R}^n$ be smooth s-minimal surfaces in $B_R(x)$ where $x \in \partial E \cap \partial F$. Then, if $E \subseteq F$ we have $E \equiv F$.

2.3. **Second variation and stability.** Next, we state the second variation formula for the nonlocal perimeter without proof (see Theorem 6.1 in [FFM+15] for detailed proof).

Proposition 2.19. *Let E be a smooth s-minimal surface and $\phi_t^X : \mathbb{R}^n \to \mathbb{R}^n$ be a family of diffeomorphisms as in Proposition 2.15. Then*

$$\begin{aligned}
&\frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{Per}_s(\phi_t^X(E)) \\
&= \iint_{\partial E \times \partial E} \frac{|(X \cdot N)(x) - (X \cdot N)(y)|^2}{|x-y|^{n+s}}\, d\sigma_x d\sigma_y - \iint_{\partial E \times \partial E} \frac{|N(x) - N(y)|^2}{|x-y|^{n+s}} (X \cdot N)^2(x)\, d\sigma_x d\sigma_y \\
&= \iint_{\partial E \times \partial E} \frac{|f(x) - f(y)|^2}{|x-y|^{n+s}}\, d\sigma_x d\sigma_y - \iint_{\partial E \times \partial E} \frac{|N(x) - N(y)|^2}{|x-y|^{n+s}} f^2(x)\, d\sigma_x d\sigma_y\,,
\end{aligned}$$

where N is the outer unit normal from E on ∂E, and $f := N \cdot X$ is the tangent part of X to N.

Observe that, by symmetry

$$\iint_{\partial E \times \partial E} \frac{|f(x) - f(y)|^2}{|x-y|^{n+s}}\, d\sigma_x d\sigma_y = 2 \int_{\partial E} f(x) \left(\int_{\partial E} \frac{f(x) - f(y)}{|x-y|^{n+s}}\, d\sigma_y \right) d\sigma_x$$

and

$$\iint_{\partial E \times \partial E} \frac{|N(x) - N(y)|^2}{|x-y|^{n+s}} f^2(x)\, d\sigma_x d\sigma_y = 2 \int_{\partial E} f^2(x) \left(\int_{\partial E} \frac{(N(x) - N(y)) \cdot N(x)}{|x-y|^{n+s}}\, d\sigma_y \right) d\sigma_x.$$

Hence, the second variation can be rewritten as

$$\frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{Per}_s(\phi_t^X(E)) = 2 \int_{\partial E} J_{s,\partial E}[f] f\, d\sigma,$$

where

$$J_{s,\partial E}[f](x) := P.V. \int_{\partial E} \frac{f(x) - f(y)}{|x-y|^{n+s}}\, d\sigma_y - f(x) \int_{\partial E} \frac{(N(x) - N(y)) \cdot N(x)}{|x-y|^{n+s}}\, d\sigma_y$$

is called the *fractional Jacobi operator*.

Note that this is very reminiscent of the second variation formula for classical minimal surfaces, which, with our notations, would be

$$\frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{Per}(\phi_t^X(E)) = \int_{\partial E} |\nabla f|^2 - |\mathcal{A}_\Sigma|^2 f^2\, d\sigma,$$

where $\mathcal{A}_\Sigma = \nabla N$ is the second fundamental form of $\Sigma = \partial E$.

Nevertheless, the principal term

$$P.V. \int_\Sigma \frac{f(x) - f(y)}{|x-y|^{n+s}}\, d\sigma_y$$

in the fractional Jacobi operator is *not* the fractional s-Laplacian of Σ, but since Σ is $(n-1)$-dimensional it has the same singularity type of the $\frac{1+s}{2}$-Laplacian $(-\Delta_\Sigma)^{\frac{1+s}{2}}$ of Σ.

Indeed, it can be shown that

$$(1-s)\left[P.V.\int_\Sigma \frac{f(x)-f(y)}{|x-y|^{n+s}}\,d\sigma_y\right] \to -\Delta_\Sigma f(x),$$

as $s\to 1^-$. See [DdPW18] for a proof of this fact.

2.4. **An instructive example on nonlocal stability.** We now discuss a simple but instructive example to understand what nonlocal stability and instability mean. Consider the set $E\subset\mathbb{R}^n$ defined by

$$E := \bigcup_{k\in\mathbb{Z}}\left\{2kC_\circ\sqrt{1-s}\le x_n\le (2k+1)C_\circ\sqrt{1-s}\right\}, \tag{20}$$

whose boundary consists of parallel hyperplanes at distance $d:=C_\circ\sqrt{1-s}$.

Clearly, by symmetry

$$H_{s,E}(x)=0$$

for every $x\in\partial E$, and hence E is an s-minimal surface. The crucial property regarding this set is that, depending on the value of $C_\circ$, E can be stable or unstable.

Proposition 2.20. *Let E be as in (20) and $Q=[-1,1]^n$. Then, for every $s\in(9/10,1)$*

(i) E is stable in Q provided $C_\circ\gg 1$ is large (depending only on n).

(ii) E is unstable in Q if $C_\circ\ll 1$ is small (depending only on n).

Proof. In what follows, $C,c>0$ denote dimensional constants where, in general, C is big and c is small.

Set $\Sigma:=\partial E=\cup_{i\in\mathbb{Z}}\Sigma_i$, where each Σ_i has the induced orientation from E, and we denote by N_i the outer unit normal to Σ_i from E. Moreover, we call $\Sigma_1,\dots,\Sigma_m$ the hyperplanes that intersect Q, that is

$$\bigcup_{i=1}^m \Sigma_i\cap Q=\Sigma\cap Q.$$

We want to show that E is stable provided we take $C_\circ$ large, that is (recall Proposition 2.19) for every $f\in C_c^\infty(\Sigma\cap Q)$

$$\iint_{\Sigma\times\Sigma}\frac{|N(x)-N(y)|^2}{|x-y|^{n+s}}f^2(x)\,d\sigma_x d\sigma_y \le \iint_{\Sigma\times\Sigma}\frac{|f(x)-f(y)|^2}{|x-y|^{n+s}}\,d\sigma_x d\sigma_y. \tag{Stab}$$

Hence, we estimate the left-hand side from above to show (i).

For $i=1,2,\dots m$ let $J(i)=\{\,j\,:\,j\ge 1\,,\,j-i \text{ odd}\}$. We have

$$\begin{aligned}\iint_{\Sigma\times\Sigma}\frac{|N(x)-N(y)|^2}{|x-y|^{n+s}}f^2(x)\,d\sigma_x d\sigma_y &= \sum_{i,j}\iint_{\Sigma_i\times\Sigma_j}\frac{|N_i(x)-N_j(y)|^2}{|x-y|^{n+s}}f^2(x)\,d\sigma_x d\sigma_y\\ &= \sum_{i=1}^m\sum_{j\in J(i)}\iint_{\Sigma_i\times\Sigma_j}\frac{4}{|x-y|^{n+s}}f(x)^2\,d\sigma_x d\sigma_y\\ &= 4\sum_{i=1}^m\sum_{j\in J(i)}\int_{\Sigma_i\cap Q}f(x)^2\left(\int_{\Sigma_j}\frac{d\sigma_y}{|x-y|^{n+s}}\right)d\sigma_x,\end{aligned}$$

where we have used that $\mathrm{supp}(f) \subset\subset Q$.

Write the coordinates in $\mathbb{R}^n$ as $x = (\widetilde{x}, x_n)$, and for $x \in \Sigma_i$ set $d_{ij} := \mathrm{dist}(\Sigma_i, \Sigma_j) = d|i-j|$. We get, for every $j \in J(i)$, that

$$\int_{\Sigma_j} \frac{d\sigma_y}{|x-y|^{n+s}} = \int_{\mathbb{R}^{n-1}} \frac{d\widetilde{y}}{(|\widetilde{x}-\widetilde{y}|^2 + |x_n - y_n|^2)^{\frac{n+s}{2}}} = \int_{\mathbb{R}^{n-1}} \frac{d\widetilde{y}}{(|\widetilde{x}-\widetilde{y}|^2 + d_{ij}^2)^{\frac{n+s}{2}}}$$

$$= \frac{1}{d_{ij}^{1+s}} \int_{\mathbb{R}^{n-1}} \frac{d\widetilde{z}}{(|\widetilde{z}|^2+1)^{\frac{n+s}{2}}} = \frac{C}{d_{ij}^{1+s}}, \qquad (21)$$

where we have made the substitution $\widetilde{x} - \widetilde{y} = d_{ij}\widetilde{z}$. Thus

$$\iint_{\Sigma\times\Sigma} \frac{|N(x)-N(y)|^2}{|x-y|^{n+s}} f^2(x)\, d\sigma_x d\sigma_y = 4\sum_{i=1}^m \sum_{j\in J(i)} \int_{\Sigma_i} f(x)^2 \left(\int_{\Sigma_j} \frac{d\sigma_y}{|x-y|^{n+s}} \right) d\sigma_x$$

$$= C \sum_{i=1}^m \sum_{j\in J(i)} \frac{1}{d_{ij}^{1+s}} \int_{\Sigma_i} f(x)^2\, d\sigma_x$$

$$= \frac{C}{d^{1+s}} \sum_{i=1}^m \sum_{j\in J(i)} \frac{1}{|i-j|^{1+s}} \int_{\Sigma_i} f^2\, d\sigma.$$

Clearly, for every $i = 1, 2, \ldots, m$ and $s \in (9/10, 1)$ there holds

$$c \le \sum_{j\in J(i)} \frac{1}{|i-j|^{1+s}} \le C, \qquad (22)$$

for $C, c > 0$ absolute constants. This gives, for $s \in (9/10, 1)$ and $C_\circ \ge 1$ that

$$\iint_{\Sigma\times\Sigma} \frac{|N(x)-N(y)|^2}{|x-y|^{n+s}} f^2(x)\, d\sigma_x d\sigma_y \le \frac{C}{d^{1+s}} \sum_{i=1}^m \int_{\Sigma_i} f^2\, d\sigma$$

$$= \frac{C}{C_\circ^{1+s}(1-s)^{\frac{1+s}{2}}} \sum_{i=1}^m \int_{\Sigma_i} f^2\, d\sigma\,.$$

By the fractional Poincaré inequality for the $H^{\frac{1+s}{2}}([-1,1]^{n-1})$-seminorm, applied to each restriction $f|_{\Sigma_i} \in C_c^\infty(\Sigma_i \cap Q)$, we have

$$\int_{\Sigma_i} f^2\, d\sigma \le C(1-s) \iint_{\Sigma_i\times\Sigma_i} \frac{|f(x)-f(y)|^2}{|x-y|^{n+s}} d\sigma_x d\sigma_y. \qquad (23)$$

Then

$$\iint_{\Sigma\times\Sigma} \frac{|N(x)-N(y)|^2}{|x-y|^{n+s}} f^2(x)\, d\sigma_x d\sigma_y \le \frac{C(1-s)^{\frac{1-s}{2}}}{C_\circ} \sum_{i=1}^m \iint_{\Sigma_i\times\Sigma_i} \frac{|f(x)-f(y)|^2}{|x-y|^{n+s}} d\sigma_x d\sigma_y$$

$$\le \frac{C(1-s)^{\frac{1-s}{2}}}{C_\circ} \iint_{\Sigma\times\Sigma} \frac{|f(x)-f(y)|^2}{|x-y|^{n+s}} d\sigma_x d\sigma_y\,,$$

which, since $(1-s)^{\frac{1-s}{2}} \to 0$ as $s \to 1^-$, implies stability (Stab) if $C_\circ \ge C_n$ for some dimensional $C_n > 0$. This concludes the proof of (i).

Now, we show (ii). We want to choose $C_\circ$ small to make (Stab) fail. With no loss of generality, assume $C_\circ \le 1/100$.

On the one hand, following the same lines above using the lower bound in (22) and choosing each $f|_{\Sigma_i} = \varphi \in C_c^\infty(\Sigma_i \cap Q)$ equal to the minimizer of the fractional Poincaré inequality

(23) (this is the first eigenfunction of the fractional $\frac{1+s}{2}$-Laplacian in $\mathbb{R}^{n-1}$, with zero Dirichlet boundary condition outside $[-1,1]^{n-1}$) one gets for the left-hand side of the stability inequality (Stab) that

$$\begin{aligned}\iint_{\Sigma\times\Sigma}&\frac{|N(x)-N(y)|^2}{|x-y|^{n+s}}\varphi^2(x)\,d\sigma_x d\sigma_y\\&\geq \frac{c(1-s)^{\frac{1-s}{2}}}{C_\circ^{1+s}}\sum_{i=1}^m\iint_{\Sigma_i\times\Sigma_i}\frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+s}}d\sigma_x d\sigma_y\geq\frac{c\,m}{C_\circ^{1+s}}\mathcal{E}(\varphi),\end{aligned}\tag{24}$$

where we have set $\mathcal{E}(\varphi):=[\varphi]^2_{H^{\frac{1+s}{2}}(\mathbb{R}^{n-1})}$.

On the other hand, for the right-hand side of the stability inequality

$$\begin{aligned}&\iint_{\Sigma\times\Sigma}\frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+s}}\,d\sigma_x d\sigma_y\\&=\sum_{i=1}^m\iint_{\Sigma_i\times\Sigma_i}\frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+s}}\,d\sigma_x d\sigma_y+\sum_{i\neq j}\iint_{\Sigma_i\times\Sigma_j}\frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+s}}\,d\sigma_x d\sigma_y\\&=m\mathcal{E}(\varphi)+\sum_{i\neq j}\iint_{\Sigma_i\times\Sigma_j}\frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+s}}\,d\sigma_x d\sigma_y.\end{aligned}\tag{25}$$

Moreover

$$\iint_{\Sigma_i\times\Sigma_j}\frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+s}}\,d\sigma_x d\sigma_y\leq 2\int_{\Sigma_i\cap Q}\left(\int_{\Sigma_j}\frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+s}}\,d\sigma_y\right)d\sigma_x$$

and, similarly to (21), for fixed i and $j\neq i$ we have

$$\begin{aligned}\int_{\Sigma_j}\frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+s}}\,d\sigma_y&=\int_{\mathbb{R}^{n-1}}\frac{|\varphi(\widetilde{x},x_n)-\varphi(\widetilde{y},y_n)|^2}{(|\widetilde{x}-\widetilde{y}|^2+d_{ij}^2)^{\frac{n+s}{2}}}\,d\widetilde{y}\\&\leq\int_{\mathbb{R}^{n-1}}\frac{\min\{C,|\widetilde{x}-\widetilde{y}|^2\}}{(|\widetilde{x}-\widetilde{y}|^2+d^2|i-j|^2)^{\frac{n+s}{2}}}\,d\widetilde{y}\\&=\frac{C}{d^{1+s}}\int_{\mathbb{R}^{n-1}}\frac{\min\{1,d^2|\widetilde{z}|^2\}}{(|\widetilde{z}|^2+|i-j|^2)^{\frac{n+s}{2}}}\,d\widetilde{z}.\end{aligned}$$

Claim 1. There holds

$$\int_{\mathbb{R}^{n-1}}\frac{\min\{1,d^2|\widetilde{z}|^2\}}{(|\widetilde{z}|^2+|i-j|^2)^{\frac{n+s}{2}}}\,d\widetilde{z}\leq C\min\left\{\frac{d^{1+s}}{1-s},\frac{1}{|i-j|^{1+s}}\right\}.\tag{26}$$

Proof of Claim 1. We have

$$\begin{aligned}\int_{\mathbb{R}^{n-1}}\frac{\min\{1,d^2|\widetilde{z}|^2\}}{(|\widetilde{z}|^2+|i-j|^2)^{\frac{n+s}{2}}}\,d\widetilde{z}&\leq d^2\int_{B_{1/d}}\frac{|\widetilde{z}|^2}{(|\widetilde{z}|^2+1)^{\frac{n+s}{2}}}\,d\widetilde{z}+\int_{B_{1/d}^c}\frac{1}{(|\widetilde{z}|^2+1)^{\frac{n+s}{2}}}\,d\widetilde{z}\\&=Cd^2\int_1^{1/d}\frac{1}{\rho^{n+s-2}}\rho^{n-2}\,d\rho+C\int_{1/d}^\infty\frac{1}{\rho^{n+s}}\rho^{n-2}\,d\rho\\&\leq d^2\frac{C}{d^{1-s}(1-s)}+Cd^{1+s}\\&\leq\frac{Cd^{1+s}}{1-s}.\end{aligned}$$

On the other hand, bounding trivially $\min\{1, d^2|\widetilde{z}|^2\} \le 1$ we get arguing exactly as in (21)

$$\int_{\mathbb{R}^{n-1}} \frac{\min\{1, d^2|\widetilde{z}|^2\}}{(|\widetilde{z}|^2 + |i-j|^2)^{\frac{n+s}{2}}}\, d\widetilde{z} \le \int_{\mathbb{R}^{n-1}} \frac{1}{(|\widetilde{z}|^2 + |i-j|^2)^{\frac{n+s}{2}}}\, d\widetilde{z} = \frac{C}{|i-j|^{1+s}}.$$

These two last inequalities prove (26).

□

Lastly, by the very definition of d and since $s \in (9/10, 1)$ note that

$$\frac{d^{1+s}}{1-s} = \frac{C_\circ^{1+s}}{(1-s)^{\frac{1-s}{2}}} \le 2C_\circ\,.$$

Hence, putting together the estimates above

$$\begin{aligned}\sum_{i=1}^{m}\sum_{j\neq i}\iint_{\Sigma_i\times\Sigma_j} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+s}}\, d\sigma_x d\sigma_y &\le 2\sum_{i=1}^{m}\sum_{j\neq i}\int_{\Sigma_i\cap Q}\left(\int_{\Sigma_j}\frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+s}}\, d\sigma_y\right) d\sigma_x \\ &\le \frac{C}{d^{1+s}}\sum_{i=1}^{m}\sum_{j\neq i}\min\left\{C_\circ, \frac{1}{|i-j|^{1+s}}\right\} \\ &\le \frac{Cm}{d^{1+s}}\sum_{j=1}^{\infty}\min\left\{C_\circ, \frac{1}{j^{1+3/2}}\right\}.\end{aligned}$$

Claim 2. As $C_\circ \to 0^+$ we have

$$\sum_{j=1}^{\infty}\min\left\{C_\circ, \frac{1}{j^{1+3/2}}\right\} \to 0\,.$$

Proof of Claim 2. We have

$$\begin{aligned}\sum_{j=1}^{\infty}\min\left\{C_\circ, \frac{1}{j^{1+3/2}}\right\} &= \sum_{1\le j\le C_\circ^{-2/3}}\min\left\{C_\circ, \frac{1}{j^{3/2}}\right\} + \sum_{j\ge C_\circ^{-2/3}}\min\left\{C_\circ, \frac{1}{j^{3/2}}\right\} \\ &= C_\circ^{1/3} + \sum_{j\ge C_\circ^{-2/3}}\frac{1}{j^{3/2}} \to 0\,.\end{aligned}$$

□

Let us denote by $o(1)$ anything that tends to zero as $C_\circ \to 0^+$ uniformly in $s \in (9/10, 1)$. With this notation, we have shown

$$\sum_{i=1}^{m}\sum_{j\neq i}\iint_{\Sigma_i\times\Sigma_j} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+s}}\, d\sigma_x d\sigma_y \le \frac{Cm}{d^{1+s}}o(1)\,. \tag{27}$$

Now, suppose by contradiction that E was stable for $C_\circ$ arbitrarily small. Then stability (Stab) would hold for $f = \varphi$. Putting together the estimates (24), (25) and (27) it would imply

$$\frac{c\,m}{C_\circ^{1+s}}\mathcal{E}(\varphi) \le m\mathcal{E}(\varphi) + \frac{Cm}{d^{1+s}}o(1)\,,$$

or, multiplying by $C_\circ^{1+s}(1-s)$ and recalling $d = C_\circ\sqrt{1-s}$, equivalently

$$c(1-s)\mathcal{E}(\varphi) \le C_\circ^{1+s}(1-s)\mathcal{E}(\varphi) + C(1-s)^{\frac{1-s}{2}}o(1)\,.$$

But since $c \le (1-s)\mathcal{E}(\varphi) \le C$ for every $s \in (9/10, 1)$ (as φ is the minimizer in the fractional Poincaré inequality (23)), sending $C_\circ \to 0$ gives $0 < c \le 0$, contradiction.

Hence, φ is an unstable direction for E for $C_\circ$ small depending only on n, which concludes the proof of (ii). □

2.5. **The Caffarelli-Silvestre extension and the monotonicity formula.**

Definition 2.21. *We define the weighted Sobolev space*

$$\widetilde{H}^1(\mathbb{R}^n \times (0,\infty)) = H^1(\mathbb{R}^n \times (0,\infty), z^{1-s} dxdz)$$

as the completion of $C_c^\infty(\mathbb{R}^n \times [0,\infty))$ with the norm

$$\|U\|^2_{\widetilde{H}^1} := \|U\|^2_{L^2(\mathbb{R}^n\times(0,\infty), z^{1-s}dxdz)} + \|\nabla U\|^2_{L^2(\mathbb{R}^n\times(0,\infty), z^{1-s}dxdz)},$$

where $\nabla U = \left(\frac{\partial U}{\partial x^1}, \dots, \frac{\partial U}{\partial x^n}, \frac{\partial U}{\partial z}\right)$ denotes the Euclidean gradient in $\mathbb{R}^{n+1}$. This is a Hilbert space with the natural inner product that induces the norm above. It is a known fact that any $U \in \widetilde{H}^1(\mathbb{R}^n \times (0,\infty))$ has a well defined trace in $L^2(\mathbb{R}^n)$ that we denote by $U(x,\cdot)$.

The following essential result by Caffarelli and Silvestre shows that fractional powers of the Laplacian on $\mathbb{R}^n$ can be realized as a Dirichlet-to-Neumann map via an extension problem. See [CFS23] for a proof of this fact that also holds on Riemannian manifolds.

Theorem 2.22 ([CS07])**.** *Let $s \in (0,2)$ and $u \in H^{s/2}(\mathbb{R}^n)$. Then, there is a unique solution $U = U(x,z) : \mathbb{R}^n \times [0,+\infty) \to \mathbb{R}$ among functions in $\widetilde{H}^1(\mathbb{R}^n \times (0,\infty))$ to the problem*

$$\begin{cases} \operatorname{div}(z^{1-s}\nabla U) = 0\,, & \text{on } \mathbb{R}^n \times (0,\infty) \\ U(x,0) = u(x) & \text{for } x \in \mathbb{R}^n\,, \end{cases}$$

and it satisfies

$$\lim_{z\to 0^+} z^{1-s}\frac{\partial U}{\partial z}(x,z) = -c_s^{-1}(-\Delta)^{s/2}u(x)\,, \tag{28}$$

where c_s is a positive constant that depends only on s.

Remark 2.23. *There is a quite easy heuristic on why this produces a fractional power of the Laplacian for $s = 1/2$. In this case, the extension is just the harmonic extension $\Delta U = 0$ in $\mathbb{R}^n \times (0,\infty)$ and the left-hand side of (28) is just the normal derivative. Call $T := \lim_{z\to 0^+} U_z$ the operator that takes the extension and then the normal derivative. Since the extension of $U_z(\cdot,0)$ is U_z itself there holds*

$$T^2 = T \circ T = T(U_z(\cdot,0)) = \lim_{z\to 0^+} U_{zz}\,,$$

but since U is harmonic

$$\lim_{z\to 0^+} U_{zz} = \lim_{z\to 0^+}(-\Delta_x U) = -\Delta u\,.$$

Thus, T is a first-order operator whose square is the Laplacian.

With this result, it follows immediately that the fractional Sobolev seminorm can be expressed as the infimum of the energy of the extensions with fixed trace. Here and onwards we denote $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0,\infty)$.

Proposition 2.24. *The fractional Sobolev seminorm is equal to*

$$[u]^2_{H^{s/2}(\mathbb{R}^n)} = c_s \inf_{V\in\widetilde{H}^1(\mathbb{R}^{n+1}_+)} \left\{ \int_{\mathbb{R}^{n+1}_+} |\nabla V|^2 z^{1-s}\,dxdz \,:\, V(x,0) = u(x) \right\}.$$

Moreover, this infimum is attained by the unique $U \in \widetilde{H}^1(\mathbb{R}^{n+1}_+)$ *given by Theorem 2.22, and hence also*

$$[u]^2_{H^{s/2}(\mathbb{R}^n)} = c_s \int_{\mathbb{R}^{n+1}_+} |\nabla U|^2 z^{1-s}\, dxdz\,,$$

where c_s *is the constant in* (28).

We are ready to state the monotonicity formula for nonlocal minimal surfaces. This monotonicity formula was proved for minimizers in the original work [CRS10] by Caffarelli, Roquejeoffre, and Savin. Still, the proof only uses infinitesimal perturbations, so it's essentially a proof for stable critical points.

Then, for general critical points, the monotonicity formula was shown in [MSW19] on $\mathbb{R}^n$ and in [CFS23] on general Riemannian manifolds.

Let

$$B^+_r(x,0) := B^{n+1}_r(x,0) \cap \{z \geq 0\}\,.$$

Theorem 2.25 (Monotonicity formula, [CRS10,MSW19,CFS23])**.** *Let* E *be an* s*-minimal surface in* $\mathbb{R}^n$ *(meaning a critical point of the* s*-perimeter under inner variations), and let* $U_E : \mathbb{R}^{n+1}_+ \to (-1,1)$ *be the Caffarelli-Silvestre extension of* $\chi_E - \chi_{E^c}$ *in the sense of Theorem 2.22. Then, for every* $x \in \partial E$ *the function*

$$\Phi_E(r) := \frac{1}{r^{n-s}} \int_{B^+_r(x,0)} z^{1-s} |\nabla U_E|^2\, dxdz \tag{29}$$

in nondecreasing in r*, and is constant if and only if* E *is a cone.*

Note that the quantity in (29) is scale-invariant as

$$\Phi_{\lambda E}(\lambda r) = \Phi_E(r)\,.$$

3. Uniform estimates for the fractional Allen-Cahn equation

3.1. The BV-estimate for stable solutions.

In this subsection, we want to sketch the proof of the BV-estimate for stable solutions of the fractional Allen-Cahn equation

$$(-\Delta)^{s/2} u + W'(u) = 0\,. \tag{AC_s}$$

First, we point out that (AC_s) is the Euler-Lagrange equation of the localized Allen-Cahn functional

$$\mathcal{E}_\Omega(u) := \mathcal{E}^{\mathrm{Sob}}_\Omega(u) + \mathcal{E}^{\mathrm{Pot}}_\Omega(u),$$

where

$$\mathcal{E}^{\mathrm{Sob}}_\Omega(u) = \frac{1}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n \setminus \Omega^c \times \Omega^c} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}}\, dxdy,$$

and

$$\mathcal{E}^{\mathrm{Pot}}_\Omega(u) = \int_\Omega W(u)\, dx.$$

Usually, in the setting of the (classical and) fractional Allen-Cahn equation, one takes W to be the double-well potential $W(u) = (1-u^2)/4$. Nevertheless, this choice is not crucial for the estimates that follow since these hold for general nonnegative potentials.

Definition 3.1 (Stability). *Let $\Omega \subset \mathbb{R}^n$ open and $u : \mathbb{R}^n \to (-1,1)$ be a solution of (AC_s) in Ω. We say that u is stable in Ω if*

$$\mathcal{E}''_\Omega(u)[\eta,\eta] = \frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|\eta(x)-\eta(y)|^2}{|x-y|^{n+s}}\,dxdy + \int_\Omega W''(u)\eta^2\,dx \geq 0\,, \quad \forall\,\eta \in C_c^\infty(\Omega)\,.$$

Theorem 3.2 (Stable BV, [CCS21]). *Let $s \in (0,1)$ and $u : \mathbb{R}^n \to (-1,1)$ be a solution of (AC_s) in B_1 that is stable in B_1. Then, there exists $C = C(n,s) > 0$, such that*

$$\int_{B_{1/4}} |\nabla u|\,dx \leq C.$$

A similar result can be obtained working only in the setting of fractional minimal surfaces, in contrast to solutions to the fractional Allen-Cahn equation. This holds even for a perimeter estimate in the case of finite-index fractional minimal surfaces, as it was shown in [FS24]. Nevertheless, we believe that the setting of smooth functions that are solutions to the fractional Allen-Cahn equation is easier to follow, so we present here the results in this case.

Remark 3.3. *We emphasize that in the* BV*-estimate above (and the same will hold for the* BV*-estimate for finite index solutions), the constant C on the right-hand side is independent of the potential W. This implies that if one replaces W by $\varepsilon^{-s}W$ (that is, one takes a solution to the fractional Allen-Cahn equation with ε and not $\varepsilon = 1$), the same bound holds with a uniform constant as $\varepsilon \to 0^+$.*

Before diving into the proof of the BV-estimate, let us state and prove an abstract (but very useful) result due to Leon Simon that we will need at the end of the proof.

We will apply this to the subadditive function defined on balls $B \mapsto \|\nabla u\|_{L^1(B)}$.

Lemma 3.4. *Let $\beta \in \mathbb{R}$, $M_\circ > 0$ and $\mathcal{S} : \mathfrak{B} \to [0,+\infty)$ be a nonnegative function defined on the family $\mathfrak{B}$ of open balls contained in the Euclidean ball $B_1(0) \subset \mathbb{R}^n$ that is subadditive for finite unions, meaning that whenever $B \subset \bigcup_i B_i$ a finite union then $\mathcal{S}(B) \leq \sum_i \mathcal{S}(B_i)$. Then, there exists a constant $\delta_\circ = \delta_\circ(n,\beta) > 0$ such that, if*

$$r^\beta \mathcal{S}(B_{r/4}(x_0)) \leq \delta_\circ r^\beta \mathcal{S}(B_r(x_0)) + M_\circ \quad \text{whenever} \quad B_r(x_0) \subset B_1(0),$$

then

$$\mathcal{S}(B_{1/4}(0)) \leq CM_\circ\,,$$

for some constant $C = C(n,\beta) > 0$.

Proof. Let

$$Q := \sup_{B_r(z)\subset B_1} \left(\frac{r}{2}\right)^\beta \mathcal{S}(B_{r/2}(z))\,,$$

and let me assume $Q < +\infty$. This will indeed be true in our case with $\mathcal{S}(B) = \|\nabla u\|_{L^1(B)}$ for u is smooth. By assumption for every $B_r(z) \subset B_1$ we have

$$\left(\frac{r}{2}\right)^\beta \mathcal{S}(B_{r/8}(z)) \leq \delta \left(\frac{r}{2}\right)^\beta \mathcal{S}(B_{r/2}(z)) + M_\circ \leq \delta Q + M_\circ\,,$$

and taking the supremum over all $B_r(z) \subset B_1$ gives

$$\widetilde{Q} := \sup_{B_r(z)\subset B_1} \left(\frac{r}{2}\right)^\beta \mathcal{S}(B_{r/8}(z)) \leq \delta Q + M_\circ\,.$$

From here, if we knew that

$$Q \le C\widetilde{Q}\,, \tag{30}$$

we would get

$$Q \le C\widetilde{Q} \le C\delta Q + CM_\circ \quad \Longrightarrow \quad Q \le CM_\circ\,,$$

having taken $\delta = 1/(10C)$, and this would conclude the proof taking $B_{1/2}(0)$ as a competitor in the definition of Q.

Hence, we are left to prove (30). Fix a covering of $B_{1/2} \subset \bigcup B_{1/40}(x_i)$ of $B_{1/2}(0)$ of balls with radius $1/40$, with the centers $x_i \in B_{1/2}(0)$. Let N be the number of balls in this covering (this is a dimensional constant). For every $B_r(z) \subset B_1$, scaling and translating this covering we get

$$B_{r/2}(z) \subset \bigcup_{i\le N} B_{r/40}(z_i)\,, \quad z_i \in B_{r/2}(z) \ \text{ for all } i \le N\,.$$

In particular $B_{r/5}(z_i) \subset B_1$ for all i, thus by the very definition of $\widetilde{Q}$ we get

$$\left(\frac{r}{10}\right)^\beta \mathcal{S}(B_{r/40}(z_i)) \le \widetilde{Q}\,.$$

By the subadditivity of $\mathcal{S}$, we obtain

$$\left(\frac{r}{2}\right)^\beta \mathcal{S}(B_{r/2}(z)) \le \left(\frac{r}{2}\right)^\beta \sum_{i\le N} \mathcal{S}(B_{r/40}(z_i)) \le 5^\beta N\widetilde{Q}\,,$$

and taking the supremum over all $B_r(z) \subset B_1$ we get (30).

□

Remark 3.5. *As we will see shortly, the standard situation where this lemma is of use is when one can obtain, for some $\theta \in (0,1)$ and $C > 0$, an inequality like*

$$\|\nabla u\|_{L^q(B_1)} \le C + C\|\nabla u\|^\theta_{L^q(B_4)}.$$

Indeed, if this holds, then Young's inequality gives, for every $\delta > 0$, that

$$\begin{aligned}\|\nabla u\|_{L^q(B_1)} &\le C + \delta\|\nabla u\|_{L^q(B_4)} + C(\delta,\theta)\\ &= \delta\|\nabla u\|_{L^q(B_4)} + C_1(\delta,\theta)\,.\end{aligned}$$

Then, just by scaling and translating for every $B_r(x) \subset \mathbb{R}^n$ we get

$$r^{q-n}\|\nabla u\|_{L^q(B_r(x))} \le \delta r^{q-n}\|\nabla u\|_{L^q(B_{4r(x)})} + C_1(\delta,\theta)\,.$$

From here, choosing $\delta = \delta_\circ(n, q-n)$ the one of Lemma 3.4 one concludes a uniform bound

$$\|\nabla u\|_{L^q(B_{1/2})} \le CC_1(\delta_\circ,\theta).$$

We can now prove the BV estimate for stable solutions Proposition 3.2. The proof presented here is a slight modification of the one given in [CCS21].

Proof of Theorem 3.2. Let $\nu \in \mathbb{S}^{n-1}$. The idea is essentially to compare the energy of u with one of the translations $u(\cdot + t\nu)$. To do so, let φ be a smooth, radial cut-off function with $\varphi = 1$ in $B_{1/2}$ and with $\varphi = 0$ outside $B := B_1$.

Moreover, we also take φ to be linear as a function of $|x|$ in $B_1 \setminus B_{1/2}$. Let also $\phi_t(x) := x + t\varphi(x)\nu$ and $v_t := u \circ \phi_t^{-1}$. Taking φ linear in $|x|$ for $1/2 < |x| < 1$ is not strictly necessary, but it will make the computations a bit cleaner since it will avoid higher order terms in t in the Jacobians $J\phi_{\pm t}$. Indeed, by the very definition of $\phi_{\pm t}$ we have

$$\nabla\phi_{\pm t} = \mathrm{id} + t\nu \otimes \nabla\varphi.$$

Hence (this is somewhere called the *"matrix determinant lemma"*)

$$J\phi_{\pm t} = \det(\nabla \phi_{\pm t}) = 1 \pm t\frac{\partial \varphi}{\partial \nu}. \tag{31}$$

Claim 1. For *every function* v and $t \le 1/100$ there holds

$$\mathcal{E}_B(v_t) + \mathcal{E}_B(v_{-t}) - 2\mathcal{E}_B(v) \le Ct^2 \mathcal{E}_B^{\rm Sob}(v)\,,$$

for some $C = C(n) > 0$.

Proof of Claim 1. The full proof can be found in [CCS21, Lemma 3.1]. We sketch the main steps of the argument.

The idea is just to change variables inside the integrals in $\mathcal{E}_B(v_{\pm t})$ and estimate the Jacobians. For the Potential part, since $\phi_{\pm t}$ sends B to B, we have

$$\mathcal{E}_B^{\rm Pot}(v_{\pm t}) = \int_B W(v_{\pm t})\,dx = \int_B W(v)\,J\phi_{\pm t}\,dy\,.$$

Hence, using (31) for the Jacobians

$$\mathcal{E}_B^{\rm Pot}(v_t) + \mathcal{E}_B^{\rm Pot}(v_{-t}) = \int_B W(v)\big(J\phi_t + J\phi_{-t}\big)\,dy = 2\mathcal{E}_B^{\rm Pot}(v)\,,$$

or equivalently

$$\mathcal{E}_B^{\rm Pot}(v_t) + \mathcal{E}_B^{\rm Pot}(v_{-t}) - 2\mathcal{E}_B^{\rm Pot}(v) = 0\,. \tag{32}$$

Regarding the Sobolev part of the energy, we have (again since $\phi_{\pm t}$ sends B and B^c to themselves)

$$\begin{aligned}
\mathcal{E}_B^{\rm Sob}(v_{\pm t}) &= \frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n\setminus B^c\times B^c} \frac{|v_{\pm t}(x) - v_{\pm t}(y)|^2}{|x-y|^{n+s}}\,dxdy \\
&= \frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n\setminus B^c\times B^c} \frac{|v(x) - v(y)|^2}{|\phi_{\pm t}(x) - \phi_{\pm t}(y)|^{n+s}}\,J\phi_{\pm t}(x)J\phi_{\pm t}(y)\,dxdy\,.
\end{aligned}$$

Expanding at the second order with Taylor's formula gives

$$g(t) := \frac{1}{|\phi_{\pm t}(x) - \phi_{\pm t}(y)|^{n+s}} = \frac{1}{|x-y|^{n+s}} \pm tF(x,y) + t^2 R(x,y)\,,$$

where $R(x,y) = \frac{1}{2}g''(\xi)$ for some $\xi \in [0,t]$. In particular, R satisfies

$$|R(x,y)| \le C\frac{|x-y|^2}{|\phi_{\pm\xi}(x) - \phi_{\pm\xi}(y)|^{n+s+2}} \le \frac{C}{|x-y|^{n+s}},$$

if we choose $t \le 1/100$. Here we have used that $|\nabla\varphi| \le 2$.

Hence (note that the first order terms cancel)

$$\begin{aligned}
&\frac{J\phi_t(x)J\phi_t(y)}{|\phi_t(x) - \phi_t(y)|^{n+s}} + \frac{J\phi_{-t}(x)J\phi_{-t}(y)}{|\phi_{-t}(x) - \phi_{-t}(y)|^{n+s}} \\
&\qquad = \frac{(1 + t\partial_\nu\varphi(x))(1 + t\partial_\nu\varphi(y))}{|\phi_t(x) - \phi_t(y)|^{n+s}} + \frac{(1 - t\partial_\nu\varphi(x))(1 - t\partial_\nu\varphi(y))}{|\phi_{-t}(x) - \phi_{-t}(y)|^{n+s}} \\
&\qquad = \frac{2}{|x-y|^{n+s}} + Ct^2 R(x,y) \\
&\qquad \le \frac{2}{|x-y|^{n+s}} + Ct^2\frac{1}{|x-y|^{n+s}}\,.
\end{aligned}$$

Thus, regarding the Sobolev part of the energy

$$\mathcal{E}_B^{\rm Sob}(v_t) + \mathcal{E}_B^{\rm Sob}(v_{-t}) - 2\mathcal{E}_B^{\rm Sob}(v) \le Ct^2 \mathcal{E}_B^{\rm Sob}(v)\,. \tag{33}$$

Putting together (32) and (33) finishes the proof of Claim 1. □

Observe that dividing by t^2 in what we have shown in Claim 1 and sending $t \to 0^+$ gives

$$\mathcal{E}_B''(u)[\nabla_X u, \nabla_X u] \le C\mathcal{E}_B^{\rm Sob}(u)\,, \tag{34}$$

where X is the vector field $X := \nu\varphi$.

Moreover, since u is stable in B testing the stability inequality with $|\nabla_X u|$ gives

$$\mathcal{E}_B''(u)[|\nabla_X u|, |\nabla_X u|] \ge 0\,.$$

On the other hand, it follows writing $\nabla_X u = (\nabla_X u)_+ - (\nabla_X u)_-$ that

$$\begin{aligned} 0 &\le \mathcal{E}_B''(u)[|\nabla_X u|, |\nabla_X u|] \\ &= \mathcal{E}_B''(u)[\nabla_X u, \nabla_X u] - 4\iint_{B\times B} \frac{(\nabla_X u)_+(x)(\nabla_X u)_-(y)}{|x-y|^{n+s}}\,dxdy\,. \end{aligned}$$

Together with (34), we have proved

$$\iint_{B\times B} \frac{(\nabla_X u)_+(x)(\nabla_X u)_-(y)}{|x-y|^{n+s}}\,dxdy \le C\mathcal{E}_B^{\rm Sob}(u)\,,$$

and since $\frac{1}{|x-y|^{n+s}} \ge c$ on $B \times B$ we get

$$\|(\nabla_X u)_+\|_{L^1(B)}\|(\nabla_X u)_-\|_{L^1(B)} \le C\mathcal{E}_B^{\rm Sob}(u)\,. \tag{35}$$

Moreover, by the divergence theorem

$$\int_B \nabla_X u = \int_B \nabla u \cdot X = \int_{\partial B} u(X\cdot z)\,d\sigma_z - \int_B u{\rm div}(X)\,dz.$$

Hence, using that $(\nabla_X u)_+ - (\nabla_X u)_- = \nabla_X u$ and that $|u| \le 1$ we can estimate

$$\Big|\|(\nabla_X u)_+\|_{L^1(B)} - \|(\nabla_X u)_-\|_{L^1(B)}\Big| = \left|\int_B \nabla_X u\right| \le C. \tag{36}$$

Thus, putting together (35) and (36)

$$\begin{aligned} \|\nabla_X u\|^2_{L^1(B)} &= \Big(\|(\nabla_X u)_+\|_{L^1(B)} + \|(\nabla_X u)_-\|_{L^1(B)}\Big)^2 \\ &= \Big(\|(\nabla_X u)_+\|_{L^1(B)} - \|(\nabla_X u)_-\|_{L^1(B)}\Big)^2 + 4\|(\nabla_X u)_+\|_{L^1(B)}\|(\nabla_X u)_-\|_{L^1(B)} \\ &\le C\mathcal{E}_B^{\rm Sob}(u) + C. \end{aligned}$$

Moreover, by interpolation (see, for example, [CFS23, Proposition 2.22]) and Young's inequality

$$\mathcal{E}_B^{\rm Sob}(u) \le C\|\nabla u\|^s_{L^1(B)}\|u\|^{1-s}_{L^1(B)} \le C\|\nabla u\|_{L^1(B)} + C\,,$$

hence

$$\|\nabla_X u\|^2_{L^1(B)} \le C + \|\nabla u\|_{L^1(B)}\,.$$

Since $X = \nu$ in $B_{1/2}$, taking $\nu = e_1, e_2, \dots, e_n$ and adding up all these inequalities gives

$$\|\nabla u\|^2_{L^1(B_{1/2})} \le C + \|\nabla u\|_{L^1(B_1)}\,.$$

Note that we are controlling $\|\nabla u\|^2_{L^1(B_{1/2})}$ with $\|\nabla u\|_{L^1(B_{1/2})}$, but in a bigger ball. Nevertheless, one can conclude by Young's inequality and Lemma 3.4 as explained in Remark 3.5 to conclude a uniform bound on $\|\nabla u\|_{L^1(B_{1/2})}$. This concludes the proof.

□

3.2. **The BV-estimate for finite Morse index.** The entire discussion in this section is essentially copy-paste from [CFS23]. The goal is to show how to extend the ideas in the proof of the BV-estimate for stable solutions to finite Morse index, ultimately proving:

Theorem 3.6 (Finite index BV). *Let $s \in (0,1)$ and $u : \mathbb{R}^n \to (-1,1)$ be a solution of (AC_s) with Morse index $\le m$. Then, there exists $C = C(n,m,s) > 0$, such that*

$$\int_{B_{1/4}} |\nabla u|\, dx \le C.$$

3.2.1. *Finite Morse index and almost stability.* For critical points of local functionals, it is well known that having Morse index bounded by m implies stability in one out of every $m+1$ disjoint open sets. In the nonlocal case, this is not the case anymore, but in one of the sets, it is possible to obtain a weaker, quantitative lower bound on the second variation, which we will refer to as *almost stability.*

Definition 3.7 (Almost stability). *Let $\Omega \subset \mathbb{R}^n$ open, $\Lambda \ge 0$, and $u : \mathbb{R}^n \to (-1,1)$ be a solution of (AC_s) in Ω. We say that u is Λ-almost stable in Ω if*

$$\mathcal{E}''_\Omega(u)[\eta,\eta] \ge -\Lambda \|\eta\|^2_{L^1(\Omega)}\,, \quad \forall\, \eta \in C^\infty_c(\Omega)\,.$$

Compare the above definition with the one of stability (i.e. $\Lambda = 0$) of Definition 3.1.

Lemma 3.8 (Finite Morse index ⇒ almost stability). *Let $u : \mathbb{R}^n \to (-1,1)$ be a solution of (AC_s) with Morse index $\le m$. Consider a collection $\mathcal{U}_1, \dots, \mathcal{U}_{n+1}$ of $(n+1)$ disjoint open sets, and set*

$$\Lambda_m := m \cdot \max_{i \ne j} \sup_{\mathcal{U}_i \times \mathcal{U}_j} \frac{1}{|x-y|^{n+s}}\,.$$

Then, there is (at least) one set $\mathcal{U}_k$ among $\mathcal{U}_1, \dots, \mathcal{U}_{n+1}$ such that u is Λ_m-almost stable in $\mathcal{U}_k$, in the sense of Definition 3.7.

Proof. We prove the Lemma just for $m = 1$ for the sake of clarity; the proof goes on exactly the same for general m. Let $\eta_1 \in C^\infty_c(\mathcal{U}_1)$ and $\eta_2 \in C^\infty_c(\mathcal{U}_2)$. Testing the second variation of the Allen-Cahn energy with linear combinations of η_1 and η_2 gives

$$\mathcal{E}''(u)[a\eta_1 + b\eta_2] = a^2\mathcal{E}''(u)[\eta_1,\eta_1] + b^2\mathcal{E}''(u)[\eta_2,\eta_2] - 2ab \iint_{\mathcal{U}_1\times\mathcal{U}_2} \frac{\eta_1(x)\eta_2(y)}{|x-y|^{n+s}}\, dxdy\,.$$

Since $\frac{1}{|x-y|^{n+s}} \le \Lambda$ for all $(x,y) \in \mathcal{U}_1 \times \mathcal{U}_2$ (this holds by the very definition of Λ), the interaction term can be bounded as

$$\begin{aligned} -2ab \iint_{\mathcal{U}_1\times\mathcal{U}_2} \frac{\eta_1(x)\eta_2(y)}{|x-y|^{n+s}}\, dxdy &\le 2\Lambda |ab| \|\eta_1\|_{L^1(\mathcal{U}_1)} \|\eta_2\|_{L^1(\mathcal{U}_2)} \\ &\le a^2\Lambda\|\eta_1\|^2_{L^1(\mathcal{U}_1)} + b^2\Lambda\|\eta_2\|^2_{L^1(\mathcal{U}_2)}. \end{aligned}$$

Hence

$$\mathcal{E}''(u)[a\eta_1 + b\eta_2] \le a^2\Big(\underbrace{\mathcal{E}''(u)[\eta_1,\eta_1] + \Lambda\|\eta_1\|^2_{L^1(\mathcal{U}_1)}}_{=:F_1(\eta_1)}\Big) + b^2\Big(\underbrace{\mathcal{E}''(u)[\eta_2,\eta_2] + \Lambda\|\eta_2\|^2_{L^1(\mathcal{U}_2)}}_{=:F_2(\eta_2)}\Big). \quad (37)$$

We want to show that either $F_1(\eta_1) \ge 0$ for all $\eta_1 \in C^\infty_c(\mathcal{U}_1)$ or $F_2(\eta_2) \ge 0$ for all $\eta_2 \in C^\infty_c(\mathcal{U}_2)$. Suppose neither of these two held, then there would exist $\eta_1 \in C^\infty_c(\mathcal{U}_1)$, $\eta_2 \in C^\infty_c(\mathcal{U}_2)$ such that $F_1(\eta_1) < 0$ and $F_2(\eta_2) < 0$. This would imply, however, that (37) is negative for all $(a,b) \ne (0,0)$, thus contradicting that the Morse index of u is at most one. □

The proof of the BV-estimate for finite index is made of two essential results.

The first one says that Λ-almost stability actually implies the BV-estimate, in a smaller ball, if Λ is small. Since the case $\Lambda = 0$ is the one of stable solutions, this result is a slight improvement of Theorem 3.2.

Proposition 3.9 (Almost stability ⇒ BV, [CFS23])**.** *Let $s \in (0,1)$ and $u : \mathbb{R}^n \to (-1,1)$ be a solution of (AC_s) which is Λ-almost stable in $B_1(x)$, in the sense of Definition 3.7. Then, there exist constants $\Lambda_\circ, C > 0$ (depending only on n and s) such that: if $\Lambda \le \Lambda_\circ$ then*

$$\int_{B_{1/4}(x)} |\nabla u|\, dx \le C.$$

Proof. The proof is extremely similar to the one of the BV estimate for stable solutions Proposition 3.2. The point is just to realize that for Λ small, the bad term (coming from the almost stability instead of stability) absorbs to the left, and one can conclude in the same way. See [CFS23] for all the details. □

The second result is a covering theorem tailored for this kind of situation. Note that, by Lemma 3.8 we know that given a solution u with Morse index at most m, then u is Λ-almost stable in one out of $(m+1)$ sets with Λ decreasing as the distance between the sets gets bigger.

On the other hand, Proposition 3.9 above says that if Λ is small, i.e., if the distance between the sets is big, then almost stability implies the BV-estimate. Hence, combining these two results, we know the BV-estimate in one out of $(m+1)$ sets if they are sufficiently far from each other. It remains to prove that this is sufficient to conclude a uniform estimate in a smaller ball. This is what Lemma 3.10 is all about.

In the following, we denote by $\mathcal{Q}_r(x) \subset \mathbb{R}^n$ the (hyper)cube of center x and side r.

Lemma 3.10 ([CFS23])**.** *Let $n \ge 1$, $m \ge 0$, $\theta \in (0,1)$, $D_0 > 0$ and $\beta > 0$. Let $\mathcal{S} : \mathfrak{B} \to [0,+\infty)$ be a subadditive[2] function defined on the family $\mathfrak{B}$ of the (hyper)cubes contained in $\mathcal{Q}_1(0) \subset \mathbb{R}^n$, such that*

(i) $\displaystyle\sup_{\{x \,|\, \mathcal{Q}_r(x)\in\mathfrak{B}\}} \mathcal{S}(\mathcal{Q}_r(x)) \to 0 \quad \text{as } r \to 0.$

(ii) Whenever $\mathcal{Q}_r(x_0), \mathcal{Q}_r(x_1), \dots, \mathcal{Q}_r(x_m) \subset \mathcal{Q}_1(0)$ are $(m+1)$ disjoint cubes of the same side at pairwise distance at least $D_0 r$, then

$$\exists\, i \in \{0,1,\dots,m\} \ \text{ such that } \ \mathcal{S}(\mathcal{Q}_{\theta r}(x_i)) \le r^\beta M_0\,.$$

Then

$$\mathcal{S}(\mathcal{Q}_{1/2}(0)) \le C m M_0\,,$$

[2] Meaning subadditive for finite unions of (hyper)cubes.

for some $C = C(n,\theta,\beta,D_0) > 0$.

Proof. Let $\rho = 2^{-k}$, for a fixed integer $k > 1$, and consider the regular partition of $\mathcal{Q}_\theta(0)$ into 2^{kn} cubes of sidelength $\theta\varrho$. Let us call $\mathfrak{F}_1 := \{\mathcal{Q}_i^1\}$ the family of cubes in this partition. In this way, clearly $\#\mathfrak{F}_1 \le \rho^{-n}$. Let x_i^1 denote the center of the cube $\mathcal{Q}_i^1$ and, for every $\lambda > 0$ and cube $\mathcal{Q}$ of side r, let $\lambda\mathcal{Q}$ be the cube with the same center and side λr.

Now, we split the family $\mathfrak{F}_1$ as $\mathfrak{F}_1 = \mathfrak{G}_1 \cup \mathfrak{B}_1$ into the families of *good* and *bad* cubes as follows. Start by considering $\mathcal{Q}_1^1$, if there holds

$$\mathcal{S}(\mathcal{Q}_1^1) \le M_0\rho^\beta \tag{38}$$

then it is considered a good cube, we assign it to $\mathfrak{G}_1$, and we remove it from $\mathfrak{F}_1$. On the other hand, if $\mathcal{Q}_1^1$ does not satisfy (38), then we assign it to the bad cubes $\mathfrak{B}_1$ and remove it from $\mathfrak{F}_1$. Moreover, if this happens, also all the cubes $\mathcal{Q} \in \mathfrak{F}_1$ such that the distance of $\frac{1}{\theta}\mathcal{Q}$ from $\frac{1}{\theta}\mathcal{Q}_1^1$ is less than $D_0\rho$ are considered bad as well, so they are assigned to $\mathfrak{B}_1$ and removed from $\mathfrak{F}_1$. By a simple count, there are at most $(2 + 2D_0 + 4\sqrt{n}/\theta)^n$ such cubes. We continue this procedure of splitting $\mathfrak{F}_1$ into good cubes and bad cubes until there are no cubes left.

By property (ii), we may have assigned cubes to the bad set $\mathfrak{B}_1$ at most at m steps. Since at each of these steps we removed at most $(2 + 2D_0 + 4\sqrt{n}/\theta)^n$ cubes, this means that $\#\mathfrak{B}_1 \le m(2 + 2D_0 + 4\sqrt{n}/\theta)^n =: N_0$.

Regarding the good set $\mathfrak{G}_1$, we know it contains at most $\#\mathfrak{F}_1 \le \rho^{-n}$ cubes since this is just the total number of cubes in the cover. Moreover, by construction in every $\mathcal{Q} \in \mathfrak{G}_1$ we have

$$\mathcal{S}(\mathcal{Q}) \le M_0\rho^\beta\,.$$

Hence

$$\mathcal{S}(\mathcal{Q}_\theta(0)) \le \sum_{\mathcal{Q}\in\mathfrak{G}_1} \mathcal{S}(\mathcal{Q}) + \sum_{\mathcal{Q}\in\mathfrak{B}_1} \mathcal{S}(\mathcal{Q}) \le M_0\rho^{\beta-n} + \sum_{\mathcal{Q}\in\mathfrak{B}_1} \mathcal{S}(\mathcal{Q})\,.$$

The argument continues iteratively under the same scheme, on the union of the at most N_0 bad cubes that are in $\mathfrak{B}_1$. Consider the partition $\mathfrak{F}_2 := \{\mathcal{Q}_i^2\}$ of the cubes in $\mathfrak{B}_1$ obtained splitting each cube into 2^{kn} smaller cubes of side $\theta\rho^2$. Notice that $\#\mathfrak{F}_2 \le N_0\rho^{-n}$. Now assign cubes in $\mathfrak{F}_2$ to the good cubes $\mathfrak{G}_2$ or bad cubes $\mathfrak{B}_2$ as before: starting from $\mathcal{Q}_1^2$, assign it to $\mathfrak{G}_2$ if

$$\mathcal{S}(\mathcal{Q}_1^2) \le M_0\rho^{2\beta}\,,$$

and then remove it from $\mathfrak{F}_2$. Else, if this is not the case we assign $\mathcal{Q}_1^2$ to the bad cubes $\mathfrak{B}_2$ and remove it, together with all the cubes $\mathcal{Q} \in \mathfrak{F}_2$ such that $\frac{1}{\theta}Q$ is at distance less than $D_0\rho^2$ from $\frac{1}{\theta}\mathcal{Q}_1^2$. Continue the procedure until there are no cubes left in $\mathfrak{F}_2$. By property (ii) again, exactly the same argument as in the first part shows that $\mathfrak{F}_2$ contains at most $N_0 = m(2+2D_0+4\sqrt{n}/\theta)^n$ cubes assigned to the bad set, that is $\#\mathfrak{B}_2 \le N_0$. This produces a partition $\mathfrak{F}_2 = \mathfrak{G}_2 \cup \mathfrak{B}_2$, and we get

$$\sum_{\mathcal{Q}\in\mathfrak{B}_1} \mathcal{S}(\mathcal{Q}) \le \sum_{\mathcal{Q}\in\mathfrak{G}_2} \mathcal{S}(\mathcal{Q}) + \sum_{\mathcal{Q}\in\mathfrak{B}_2} \mathcal{S}(\mathcal{Q}) \le N_0M_0\rho^{2\beta-n} + \sum_{\mathcal{Q}\in\mathfrak{B}_2} \mathcal{S}(\mathcal{Q})\,.$$

Iterating this argument, after k steps we have always $\#\mathfrak{B}_k \le N_0$, and in particular by (i) and subadditivity

$$\mathcal{S}(\mathfrak{B}_k) \le \sum_{\mathcal{Q}\in\mathfrak{B}_k} \mathcal{S}(\mathcal{Q}) \to 0\,,$$

since each $\mathcal{Q} \in \mathfrak{B}_k$ has side $\theta\rho^k \to 0$. Thus, the set of the points belonging to infinitely many bad families is $\mathcal{S}$-negligible. Hence

$$\begin{aligned}\mathcal{S}(\mathcal{Q}_\theta(0)) &\leq M_0\rho^{\beta-n} + N_0M_0\rho^{2\beta-n} + N_0M_0\rho^{3\beta-n} + \dots \\ &\leq N_0M_0\rho^{\beta-n}\sum_{j\geq 0}\rho^{j\beta} = \frac{N_0}{\rho^n(\rho^{-\beta}-1)}M_0\,.\end{aligned}$$

Now notice that $\mathcal{Q}_{1/2}(0)$ can be covered, for some $\xi = \xi_n$ dimensional constant, by $\xi_n\theta^{-n}$ many cubes of side $\theta/10$ such that the cube with the same center and side $1/10$ still is contained in $\mathcal{Q}_1(0)$. Since property (ii) is translation invariant, covering $\mathcal{Q}_{1/2}(0)$ in such a way gives

$$\mathcal{S}(\mathcal{Q}_{1/2}(0)) \leq \frac{\xi_n\theta^{-n}N_0}{\rho^n(\rho^{-\beta}-1)}M_0 = \frac{\xi_n\theta^{-n}m(2D_0+3\sqrt{n}/\theta)^n}{\rho^n(\rho^{-\beta}-1)}M_0\,,$$

and as this holds for every $\rho = 2^{-k}$, just choosing any fixed k gives the desired estimate. □

Proof of Theorem 3.6. Consider the subadditive function on cubes

$$\mathcal{S}(\mathcal{Q}_r(x)) := \int_{\mathcal{Q}_r(x)} |\nabla u|\,dx.$$

By Lemma 3.8 and Proposition 3.9 rescaled, there exists a constant $D_0 = D_0(n,s) > 0$ such that (ii) of Lemma 3.10 is satisfied with $\beta = n-1$ and $\theta = 1/4$. Clearly also $\mathcal{S}$ satisfies (i) since u is smooth. Then, Lemma 3.10 concludes the proof. □

4. Appendix

In the proof of the first variation formula, we need the following simple result for the derivatives of integrals over moving sets.

Lemma 4.1. *Let $E \subset \mathbb{R}^n$ be smooth and $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth one-parameter family of diffeomorphisms such that $\phi_0(x) = x$ and with $X := \frac{\partial}{\partial t}\big|_{t=0}\phi_t$ with compact support. Then*

$$\frac{d}{dt}\bigg|_{t=0}\int_{\phi_t(E)} f(t,x)\,dx = \int_E \frac{\partial f}{\partial t}(0,x)\,dx + \int_{\partial E} f(0,x)(N\cdot X)\,d\sigma_x\,,$$

where N is the outer-unit normal to ∂E from E.

Proof. The fact that X has compact support in $\mathbb{R}^n$ implies that all the integrals and derivatives below are finite. We have

$$\int_{\phi_t(E)} f(t,x)\,dx = \int_E f(t,\phi_t(y))|\det(d\phi_t(y))|\,dy\,.$$

Thus

$$\frac{d}{dt}\bigg|_{t=0}\int_{\phi_t(E)} f(t,x)\,dx = \int_E \frac{\partial f}{\partial t}(0,y) + \nabla_x f(0,y)\cdot X + f(0,y)\,\mathrm{div}(X)\,dy\,, \tag{39}$$

where we have used that

$$\frac{d}{dt}\bigg|_{t=0}|\det(d\phi_t)| = \mathrm{tr}(d\phi_t\big|_{t=0}) = \mathrm{tr}(\nabla X) = \mathrm{div}(X).$$

Finally, by the divergence theorem, the right-hand side of (39) equals to

$$\int_E \frac{\partial f}{\partial t}(0,y)\,dy + \int_E \mathrm{div}(f(0,\cdot)X)\,dy = \int_E \frac{\partial f}{\partial t}(0,y)\,dy + \int_{\partial E} f(0,\cdot)(N\cdot X)\,d\sigma\,,$$

and this finishes the proof. □

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