THE NON-CONVEX PLANAR LEAST GRADIENT PROBLEM

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Abstract. We study the least gradient problem in bounded regions with Lipschitz boundary in the plane. We provide a set of conditions for the existence of solutions in non-convex simply connected regions. We assume the boundary data is continuous and in the space of functions of bounded variation, and we are interested in solutions that satisfy the boundary conditions in the trace sense. Our method relies on the equivalence of the least gradient problem and the Beckman problem which allows us to use the tools of the optimal transportation theory.

1. Introduction

In this paper, we study the following least gradient problem in bounded Lipschitz planar domains

$$\min\left\{\int_\Omega |Du| \,:\, u\in BV(\Omega),\ u|_{\partial\Omega}=g\right\}, \tag{1.1}$$

where g is assumed to be continuous on $\partial\Omega$ and the restriction $u|_{\partial\Omega}$ represents the trace of u on $\partial\Omega$. The interest in (1.1) stems from pure mathematics, e.g. as a limiting case of the p-harmonic functions when $p\to 1$, as well as from science and engineering. It is known that (1.1) is equivalent to the following Beckman problem also called Free Material Design problem, where one has to distribute the conductive material in an optimal way (see [2])

$$\min\left\{\int_{\bar\Omega} |v| : \nabla\cdot v=0,\ v\cdot\nu=\partial_\tau g \ \text{ on } \ \partial\Omega\right\}, \tag{1.2}$$

where ν is the unit normal and τ is the unit tangent to $\partial\Omega$ such that the system (ν,τ) is positively oriented. This equivalence valid for two-dimensional problems was noted in [10] for convex domains and later in [5] for simply connected ones. The main idea of the equivalence proof is that u is a solution to the least gradient problem if and only if the rotation of measure Du by $-\pi/2$, is a solution to the Beckman problem. There, the assumption that Ω is simply connected is needed to write every closed form $\omega=v_2\,dx-v_1\,dy$ with $v=(v_1,v_2)\in L^1(\Omega,\mathbb{R}^2)$ satisfying $\nabla\cdot v=0$ as a potential of a function $u\in W^{1,1}(\Omega)$.

In the seminal paper [17] the authors showed the existence of minimizes to (1.1) in case the boundary datum g is continuous and the mean curvature of $\partial\Omega$ is non-negative, no part of $\partial\Omega$ is a minimal surface. In the case when Ω is contained in the plane, these conditions reduce to strict convexity of Ω. This strict convexity assumption was later relaxed in [13] and [14] where the domain is assumed to be a convex polygon. In that case, the polygon Ω was approximated by a sequence of strictly convex domains Ω_n with boundary data g_n to ensure the convergence of solutions to (1.1) with data (Ω_n,g_n) to a solution of (1.1) with data (Ω,g). Using such an argument, sufficient conditions on Ω and g were obtained to show the existence of solutions to (1.1).

In the present paper, we exploit the equivalence of (1.1) and (1.2) which permits us to state the sufficient, cf. Theorem 3.18, and necessary conditions, see Theorem 3.22, for the existence of solutions to (1.1) when Ω is a convex domain not necessarily a polygon. We also provide a number of conditions sufficient for existence for classes of non-convex domains, cf. Theorems 4.5, 4.10, 4.15. We also show the continuity of solutions we constructed, see Theorem 4.17.

Switching to (1.2) changes the perspective. From this point, we are interested in the construction of an optimal transport map from the data $(\Omega,\partial_\tau g)$. In this process, we pay special attention to open arcs, where g is strictly monotone, see Condition (H1). Furthermore, we must ensure that all transport rays lie inside Ω, this is Condition (H2). The optimal transportation theory also provides a geometric condition on g, which is related to the cyclic monotonicity of the optimal transportation plan, see Condition (H3).

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The first of our major results are about necessary and sufficient conditions for the existence of solutions to (1.1), see Theorems 3.18 and 3.22. Since the structure of the set where g is monotone matters, we assume that g is not only continuous but is also in $BV(\partial\Omega)$. We first make a regularity assumption on Ω, namely, we assume that the set of singular points of $\partial\Omega$ understood as points without a tangent line to Ω is negligible with respect to the measure $|f|$ where $f = \partial_\tau g$. This allows us to use the projection map in our approximation. Later in the paper, such a condition is relaxed. The final assumption on the continuous datum g is that it is piecewise monotone, see Definition 3.21, i.e. there is a family of open arcs such that g restricted to each of them is strictly monotone and their complement is a null set with respect to the measure $|f|$. In this way, we excluded from the data continuous functions g, which grow on a Cantor set. We prove in Theorem 3.18 that under these assumptions there exists a unique solution. However, as Example 3.20 attests, the piecewise monotonicity is not necessary for existence.

We also show that the existence of solutions for piecewise monotone g (for which Cantor functions as data is excluded) implies that our geometric conditions are satisfied, see Theorem 3.22. In conclusion, at the expense of loss of generality of continuous data, we were able to provide a characterization of solvability of (1.1) in convex, not necessarily strictly convex regions Ω.

Once we establish our existence results in the convex case, we show sufficient conditions for the solvability of (1.1) in non-convex domains Ω. This is done in multiple stages. Initially, we assume that we can partition Ω into convex sets C_i, on which the set of admissibility conditions (H1-H3) established for convex Ω holds. Then, we can show an existence result for (1.1), this is the content of Theorem 4.5. A drawback of this result is that even if Ω is not convex, then nonetheless the curves $\partial C_i \cap \partial\Omega$ are convex. The main thrust of the proof of existence is to show that the optimal transportation plan does not move any mass across the boundaries of ∂C_i, that is, no mass is transported between two distinct C_i's. Our most general existence result permits curves with negative curvature. Note that, in the non-convex case, we provide only a set of sufficient conditions, see Theorems 4.10 and 4.15.

Let us present our results in a broader context. As we stated earlier, the authors of [17] established in 1992 the basic existence for convex regions satisfying additional conditions in case of continuous data g. The main point is that the boundary data is attained in the trace sense. Their construction is geometric in nature. The authors of [17] also showed that violation of their sufficient conditions may lead to the non-existence of solutions. In 2014 Mazon et al [11] showed the existence of a solution to a relaxed version of (1.1) for arbitrary domains and arbitrary data $g \in L^1(\partial\Omega)$. Their construction is based on the approximation by the p-harmonic variational problems. However, the boundary condition is attained in a very weak sense. The paper [11] stirred renewed research on (1.1). For example, some authors were interested in the characterization of the trace space, i.e. the subset of $L^1(\partial\Omega)$ consisting of traces of solutions to (1.1). It turns out that this set is strictly smaller than $L^1(\partial\Omega)$, see [16] and the discussion in [9, Chapter 5]. It is interesting to note that the conditions of [17] sufficient for existence were relaxed or adjusted in various ways. For example, the case of unbounded Ω was studied in [8]. Another line of research was related to weakening the convexity assumptions. The authors of [10] addressed (1.1) on a rectangular region, but Dirichlet data are specified only on a non-trivial subset. In [13], (1.1) in polygonal regions for continuous data was studied. The data (Ω, g) are assumed to satisfy a set of admissible assumptions which are stronger than the current set (H1-H3). This result was extended in [14], where discontinuous data were studied. A different approach was presented in [5], where the authors studied (1.1) in an annulus using the equivalence of (1.1) and (1.2) and the tools of the optimal transportation theory. We also mention that the least gradient problem was studied in its anisotropic form or non-homogeneous. We refer the reader interested in the state of the art to a recent book by Górny and Mazón, see [9].

Here is the organization of the paper. Section 2 is devoted to introducing the Beckmann problem and the tools of the optimal transportation for the associated Monge-Kantorovich problem. In Section 3, we show that the condition (H1-H3) permits to construct an optimal transportation plan for the Monge-Kantorovich problem associated with (1.2). In particular, we show that conditions (H1-H3) are necessary and sufficient for the existence of solutions to (1.1). Section 4 is devoted to establishing sufficient conditions for the existence of solutions in two types of non-convex regions Ω.

Notation. We consider Ω having Lipschitz continuous boundary with the natural, i.e. positive orientation of its boundary. If $\beta \subsetneq \partial\Omega$ is an arc, then we may compare its points: for $x_1, x_2 \in \beta$ means that x_1 precedes x_2 in the natural orientation and we write $x_1 < x_2$. We will write throughout the paper about the monotonicity of g defined on an arc $\beta \subsetneq \partial\Omega$. In the present context, we mean monotonicity with respect to the natural orientation of $\partial\Omega$. When we pick points $x_1, x_2 \in \partial\Omega$ to form an arc $\overset{\frown}{x_1 x_2}$ we take care to state which of the two possibilities we have in mind unless such a choice does not matter. By default, we mean open arcs unless stated otherwise.

For points $x, y \in \mathbb{R}^2$ we write $]x, y[$ to denote the open line segment with endpoints x and y, by $[x, y]$ we will denote the closed segment.

We denote by $\mathcal{M}(\overline{\Omega}, \mathbb{R}^2)$ the vector space of Borel vector measures with values in $\mathbb{R}^2$ defined on $\overline{\Omega}$ and $\mathcal{M}^+(\overline{\Omega} \times \overline{\Omega})$ the set of nonnegative Borel measures on $\overline{\Omega} \times \overline{\Omega}$.

Π_x and Π_y are the projection maps from $\overline{\Omega} \times \overline{\Omega}$ into the first and second component, respectively. For a measurable map $S : X \mapsto Y$ and a nonnegative Borel measure μ, we write the pushforward measure $S_{\#}\mu(A) = \mu(S^{-1}(A))$ for every Borel set $A \subseteq Y$.

$\mathrm{Lip}_1(\overline{\Omega})$ denotes the space of Lipschitz continuous functions on $\overline{\Omega}$ with Lipschitz constant equal to 1.

For a measure μ, $\mathrm{spt}(\mu)$ will denote the support of μ.

2. Preliminaries on the Beckmann problem and the Optimal Transportation Theory

The present paper benefits from the method developed in [5], where the equivalence between the least gradient problem and the Beckmann problem was shown to hold in simply connected regions. Before entering into the details, we introduce some definitions. First of all, we assume that Ω is an open, bounded, simply connected, and Lipschitz domain of $\mathbb{R}^2$. Let g be a given continuous function on $\partial\Omega$. Since the boundary of our domain Ω may not be smooth, then we need to define the tangential derivative of g. For this purpose, we will denote by α any arc length parametrization of $\partial\Omega$ with positive orientation (i.e., $\alpha : [0, L) \mapsto \partial\Omega$ with $|\alpha'| = 1$ a.e.).

Definition 2.1. Let $h : \partial\Omega \mapsto \mathbb{R}$ be a Lipschitz function. For almost every $s_0 \in [0, L)$, we set

$$\partial_\tau h(\alpha(s_0)) = \frac{\mathrm{d}}{\mathrm{d}s}[h(\alpha(s))]_{|s=s_0}.$$

In particular, $\partial_\tau h$ is well-defined $\mathcal{H}^1$-a.e. on $\partial\Omega$, provided that h is Lipschitz. Now, we will extend this definition to less regular functions on $\partial\Omega$. More precisely, we define the distributional tangential derivative in an obvious way as a functional over Lipschitz continuous functions.

Definition 2.2. We say that $g : \partial\Omega \mapsto \mathbb{R}$ belongs to $BV(\partial\Omega)$ provided that $g \in L^1(\partial\Omega)$ and the distributional tangential derivative of g is a measure with a finite total variation, i.e. there exists a measure denoted by $\partial_\tau g$ with finite total variation ($|\partial_\tau g|(\partial\Omega) < \infty$) such that for all functions $h \in \mathrm{Lip}(\partial\Omega)$, we have

$$\int_{\partial\Omega} h \, \mathrm{d}[\partial_\tau g] = -\int_{\partial\Omega} g \cdot \partial_\tau h \, \mathrm{d}\mathcal{H}^1. \tag{2.1}$$

From now on, we always assume that $g \in BV(\partial\Omega) \cap C(\partial\Omega)$. So, we may introduce more carefully the so-called Beckmann problem, see (1.2):

$$\min\left\{ \int_{\overline{\Omega}} |v| \; : \; v \in \mathcal{M}(\overline{\Omega}, \mathbb{R}^2), \; \nabla \cdot v = f \;\text{ in }\; \overline{\Omega} \right\}. \tag{2.2}$$

Here, $f := \partial_\tau g$ is understood in accordance with Definition 2.2 and the divergence is taken in the distributional sense in $\mathbb{R}^2$, i.e. $-\int_{\overline{\Omega}} \nabla\varphi \cdot \mathrm{d}v = \int_{\partial\Omega} \varphi \, \mathrm{d}f$, for all $\varphi \in C^1(\overline{\Omega})$. We note that the definition of f implies that this measure on $\partial\Omega$ is such that

$$f(\partial\Omega) = 0. \tag{2.3}$$

The equivalence we announced at the beginning of this section reads as follows, see [5, Theorem 3.4] for the proof.

Proposition 2.3. *Suppose that $\Omega \subset \mathbb{R}^2$ is simply connected with Lipschitz boundary. If $g \in C(\partial\Omega) \cap BV(\partial\Omega)$ and $f = \partial_\tau g$, then Problems (1.1) and (2.2) are equivalent in the following sense:*
(1) The values of the infima are equal, i.e. (1.1) = (2.2).
(2) Given a solution $u \in BV(\Omega)$ of (1.1), we can construct $v \in \mathcal{M}(\Omega, \mathbb{R}^2)$ a solution of (2.2) by $v = R_{\frac{\pi}{2}} Du$, where $R_{\frac{\pi}{2}}$ is the rotation of the plane by $\frac{\pi}{2}$.

(3) Given a solution $v \in \mathcal{M}(\overline{\Omega}, \mathbb{R}^2)$ of (2.2) with $|v|(\partial\Omega) = 0$, we can construct $u \in BV(\Omega)$ a solution of (1.1) with $v = R_{\frac{\pi}{2}} Du$.

Hence, in order to prove the existence of a solution to Problem (1.1), we just need to show the existence of a solution v to the Beckmann problem (2.2), which gives zero mass to the boundary.

Let us comment on the solvability of (2.2). It is well known, see e.g. [15, Chapter 4], that for convex domains Ω the Beckmann problem (2.2) is equivalent to the following Monge-Kantorovich problem:

$$\min\left\{ \int_{\overline{\Omega}\times\overline{\Omega}} |x-y| \,\mathrm{d}\gamma \,:\, \gamma \in \mathcal{M}^+(\overline{\Omega}\times\overline{\Omega}),\, (\Pi_x)_\#\gamma = f^+ \text{ and } (\Pi_y)_\#\gamma = f^- \right\}, \tag{2.4}$$

where f^+ and f^- are the positive and negative parts of f. Moreover, it is known, see [15], that for any bounded domain Ω, not necessarily convex, the optimal transport problem (2.4) has a dual formulation:

$$\sup\left\{ \int_{\overline{\Omega}} \phi \,\mathrm{d}(f^+ - f^-) \,:\, \phi \in \mathrm{Lip}_1(\overline{\Omega}) \right\}. \tag{2.5}$$

Due to the duality (2.4) = (2.5), we get that if γ is an optimal transport plan in Problem (2.4) and ϕ is a Kantorovich potential, i.e. a maximizer for Problem (2.5), then from [15, Chapter 3] we have the following equality:

$$\phi(x) - \phi(y) = |x-y|, \quad \text{for all} \quad (x,y) \in \mathrm{spt}(\gamma). \tag{2.6}$$

Any maximal line segment $[x,y]$ that satisfies the equality (2.6) will be called a *transport ray*. In other words, any optimal transport plan γ has to move the mass f^+ to the destination f^- along these transportation rays. Due to (2.6), one can show that if ϕ is a Kantorovich potential then ϕ is differentiable in the interior of any transport ray $]x,y[$ with $\nabla\phi(z) = \frac{x-y}{|x-y|}$, for all $z \in]x,y[$, see [15]. In particular, two different transport rays cannot intersect at an interior point of one of them.

Coming back to Problem (2.2), we see that we always have the following inequality (2.5) $\le$ (2.2) even if the domain Ω is not convex. Indeed, if v is admissible in Problem (2.2) and ϕ is a 1-Lip smooth function on $\overline{\Omega}$, then we must have

$$\int_{\overline{\Omega}} \phi \,\mathrm{d}(f^+ - f^-) = -\int_{\overline{\Omega}} \nabla\phi \cdot \mathrm{d}v \le \int_{\overline{\Omega}} |v|.$$

Now, let us assume that the domain Ω is convex. Then, one can show that (2.2) = (2.4), see for instance, [15, Chapter 4]. In fact, from an optimal transport plan γ of Problem (2.4), one can construct a minimal vector field for Problem (2.2) by considering the vector field v_γ which is defined as follows:

$$\langle v_\gamma, \xi\rangle := \int_{\overline{\Omega}\times\overline{\Omega}} \int_0^1 \xi((1-t)x + ty)\cdot(y-x)\,\mathrm{d}t\,\mathrm{d}\gamma(x,y), \text{ for all } \xi \in C(\overline{\Omega}, \mathbb{R}^2). \tag{2.7}$$

Moreover, we associate a scalar measure σ_γ, called *transport density*, with vector measure v_γ. Measure σ_γ represents the amount of transport taking place in each region of $\overline{\Omega}$. This measure is defined as follows:

$$\langle \sigma_\gamma, \varphi\rangle := \int_{\overline{\Omega}\times\overline{\Omega}} \int_0^1 \varphi((1-t)x + ty)|x-y|\,\mathrm{d}t\,\mathrm{d}\gamma(x,y), \text{ for all } \varphi \in C(\overline{\Omega}). \tag{2.8}$$

It is clear that the convexity of the domain Ω suffices for the transport density σ_γ (or the vector measure v_γ) to be well-defined. In fact, one can see easily that this vector field v_γ is admissible in Problem (2.2). In addition, if ϕ is a Kantorovich potential between f^+ and f^- then we have $v_\gamma = -\sigma_\gamma \nabla\phi$. In particular, we get that $\int_{\overline{\Omega}} |v_\gamma| = \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\,\mathrm{d}\gamma(x,y) =$ (2.4) and so, v_γ minimizes Problem (2.2). Moreover, one can show that any optimal vector field v of Problem (2.2) is of the form $v = v_\gamma$, for some optimal transport plan γ, see [15, Theorem 4.13].

Hence, the question of the existence of a solution u for Problem (1.1) becomes equivalent to whether the transport density σ_γ gives zero mass to the boundary $\partial\Omega$ or not. If Ω is strictly convex, then we can easily deduce from (2.8) that,

$$\sigma_\gamma(\partial\Omega) = \int_{\partial\Omega\times\partial\Omega} \mathcal{H}^1([x,y]\cap\partial\Omega)\, d\gamma(x,y) = 0.$$

Consequently, Problem (1.1) has a solution u as soon as Ω is strictly convex. However, when Ω is only assumed to be convex but not necessarily strictly convex, it is not true in general that

$\sigma_\gamma(\partial\Omega) = 0$. As a result, a solution u for Problem (1.1) may not exist, for example this happens when $\Omega := [0,1]^2$, $f^+ := \chi_{[0,\frac{1}{2}]\times\{0\}} \llcorner \mathcal{H}^1$ and $f^- := \chi_{[\frac{1}{2},1]\times\{0\}} \llcorner \mathcal{H}^1$.

At this point, we state an observation, which we will frequently use in the sequel.

Lemma 2.4. *Let us suppose that $\{f_n^+\}_{n\geq 1}$ and $\{f_n^-\}_{n\geq 1}$ are two sequences of data for the Monge-Kantorovich problem (2.4) and, $\{\gamma_n\}_{n\geq 1}$ (resp. $\{\phi_n\}_{n\geq 1}$) is a sequence of corresponding optimal transportation plans (resp. Kantorovich potentials such that $\phi_n(x_0) = 0$ for a fixed point $x_0 \in \Omega$). Then, we have the following:*
(1) If $f_n^+ \rightharpoonup f^+$ and $f_n^- \rightharpoonup f^-$, then there exists a subsequence such that $\gamma_{n_k} \rightharpoonup \gamma$ weakly and $\phi_{n_k} \to \phi$ uniformly. Moreover, γ is an optimal transportation plan for f^+, f^- and ϕ is the corresponding Kantorovich potential.
(2) If $[x_n, y_n]$ is a transportation ray between f_n^+ and f_n^- such that $x_n \to x$ and $y_n \to y$, then $[x,y]$ is a transportation ray between f^+, f^-.

Proof. Since measures γ_n are uniformly bounded, we can select a subsequence (not relabeled) converging weakly to $\gamma \in \mathcal{M}^+(\overline{\Omega}\times\overline{\Omega})$. Moreover, the marginals are preserved,

$$\langle f^+, \varphi\rangle = \lim_{n\to\infty} \langle f_n^+, \varphi\rangle = \lim_{n\to\infty} \int_{\overline{\Omega}\times\overline{\Omega}} \varphi(x)\, d\gamma_n(x,y) = \int_{\overline{\Omega}\times\overline{\Omega}} \varphi(x)\, d\gamma(x,y)$$

and the same argument is also valid for f^-. Namely, we integrate the map $(x,y) \mapsto \varphi(y)$ with respect to γ.

Since $\phi_n(x_0) = 0$, then ϕ_n, up to a subsequence, converges uniformly to a function $\phi \in \mathrm{Lip}_1(\overline{\Omega})$. Due to the duality (2.4) = (2.5), we have

$$\int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\,\mathrm{d}\gamma_n = \int_{\overline{\Omega}} \phi_n\,\mathrm{d}(f_n^+ - f_n^-).$$

Letting $n \to \infty$, we get that γ is an optimal transport plan between f^+ and f^-. Moreover, ϕ is the corresponding Kantorovich potential, because

$$(2.4) \leq \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\,\mathrm{d}\gamma = \int_{\overline{\Omega}} \phi\,\mathrm{d}(f^+ - f^-) \leq (2.5).$$

This concludes the proof of Part (1). Part (2) follows immediately from Part (1) and (2.6), because of the following

$$\phi(x) - \phi(y) = \lim_{n\to\infty} \phi_n(x_n) - \phi_n(y_n) = \lim_{n\to\infty} |x_n - y_n| = |x-y|. \quad \square$$

Finally, we conclude this section with a few remarks about the endpoints of transportation rays. Of course, they all belong to the support of $f = \partial_\tau g$, which is contained in $\partial\Omega$. It may happen that $x \in \partial\Omega$ is an endpoint of more than one transportation ray. We shall call such points the *multiple points*. So, we have the following:

Lemma 2.5. *Let us suppose that Ω is convex, then the set of multiple points $\mathcal{N}$ is at most countable. In particular, this set $\mathcal{N}$ is f^+-negligible (i.e. $f^+(\mathcal{N}) = 0$) as soon as f^+ is nonatomic.*

Proof. Suppose that x_0 is a multiple point, then there are two distinct points x_1 and x_2 in $\partial\Omega$ such that $[x_0, x_1]$ and $[x_0, x_2]$ are transportation rays. In particular, the arc $\widehat{x_1x_2}$ has positive $\mathcal{H}^1$ measure (here, we assume that $x_0 \notin \widehat{x_1x_2}$). Moreover, if $x_0 \neq \tilde{x}_0$ are multiple points, then their corresponding arcs $\widehat{x_1x_2}$ and $\widehat{\tilde{x}_1\tilde{x}_2}$ must be disjoint. But on $\partial\Omega$, we can fit at most a countable number of such disjoint open arcs corresponding to multiple points. Hence, the set of multiple points $\mathcal{N}$ is at most countable. $\square$

Throughout the paper, we will assume that f^+ is nonatomic, in fact, this follows immediately from the continuity of g.

Definition 2.6. Let γ be an optimal transport plan in Problem (2.4), we associate to γ the multivalued map $R_\mathcal{N} : \mathrm{spt}\, f^+ \mapsto \mathcal{P}(\partial\Omega)$ as follows

$$R_\mathcal{N}(x) := \{y \in \partial\Omega : [x,y] \text{ is a transportation ray}\}.$$

In virtue of Lemma 2.4 the graph of $R_\mathcal{N}$ is closed, and so from [1, Chapter 18], $R_\mathcal{N}$ admits a Borel selector which we will denote by R and call it a *transportation map*.

We make the following observation about R.

Lemma 2.7. *Suppose that an open arc $G \subset \operatorname{spt} f^+$ does not intersect $\mathcal{N}$. Then, R is continuous on G and $R(G)$ is an arc. Moreover, $R(G) \cap \mathcal{N} = \emptyset$.*

Proof. Let us suppose that $x_1, x_2 \in G$ with $x_1 < x_2$, then $R(x_1) > R(x_2)$, otherwise the transportation rays $[x_1, R(x_1)]$ and $[x_2, R(x_2)]$ would intersect, thanks to the convexity of Ω. Since R is monotone, then it is continuous except at a countable set of jump points. Suppose that $x_0 \in G$ is a point, where R has a jump. In this case, Lemma 2.4 implies that x_0 is a multiple point, contrary to our assumption. Hence, continuity follows and $R(G)$ is an arc.

Suppose now that $y \in R(G) \cap \mathcal{N}$, then there exist w_1, w_2 such that $[w_1, y]$, $[w_2, y]$ are transportation rays and $w_1 \in G$. This means that $[w_2, y]$ must intersect some transportation rays with endpoints near y. Thus, we reached a contradiction, again we use the convexity of Ω. □

We end this section with the following Lemma.

Lemma 2.8. *Let us suppose that Ω is strictly convex, $f^\pm$ are as defined above, and γ is an optimal transportation plan, then γ is concentrated on the graph of R. Moreover, γ is unique.*

Proof. Thanks to Lemma 2.5, we see that for $\gamma-$almost every couple $(x, y) \in (\partial\Omega)^2$, the point x does not belong to $\mathcal{N}$. As a result, there is a unique transport ray starting at x and going to the support of f^-. Since $\Omega \subset \mathbb{R}^2$ is strictly convex, then this ray must intersect $\operatorname{spt}(f^-)$ at exactly one point $R(x)$. This yields that γ is concentrated on the graph of the map R. In particular, we have

$$R_\# f^+ = f^- \tag{2.9}$$

because for all $\varphi \in C(\overline{\Omega})$, one has

$$\int_{\overline{\Omega}} \varphi(y) \, \mathrm{d}f^-(y) = \int_{\overline{\Omega} \times \overline{\Omega}} \varphi(y) \, \mathrm{d}\gamma(x, y) = \int_{\operatorname{spt}\gamma} \varphi(R(x)) \, \mathrm{d}\gamma(x, y) = \int_{\overline{\Omega}} \varphi(R(x)) \, \mathrm{d}f^+(x).$$

At the same time, we infer that the optimal transport plan γ is unique because it is determined by the map R which depends only on the Kantorovich potential ϕ and it is a Borel selector of $R_{\mathcal{N}}$ which is unique up to the negligible set of multiple points, $\mathcal{N}$. □

3. Necessary and sufficient conditions for existence and uniqueness in the convex case

In this section, we assume that Ω is an open bounded and convex (but not necessarily strictly convex) domain in $\mathbb{R}^2$ and the trace g is in $C(\partial\Omega) \cap BV(\partial\Omega)$. Under these conditions, we aim to find necessary and sufficient conditions for the existence of a solution u to the least gradient problem (1.1). We also show that this solution u is unique.

Our construction of solutions to (1.1) is based on the equivalence of the least gradient problem (1.1) and the Beckmann problem (2.2), which in turn is equivalent in convex domains with the Monge-Kantorovich optimal transportation problem (2.4). Thus, our effort is concentrated on the construction of optimal transportation plans. We first construct them for strictly convex domains for relatively simple data. The general convex domains and general data are dealt with in a series of approximations. Once we characterize the optimal transportation plans we present quite general sufficient conditions for the existence of solutions to (1.1), see Theorem 3.18. It turns out that to obtain the necessary conditions we have to restrict our attention to a class of absolutely continuous g, see Theorem 3.22. We achieve these goals by proper approximation of Ω by strictly convex sets and g by more regular data. Our first task, however, is to state the admissibility conditions.

3.1. Admissibility conditions.

First of all, we introduce our admissibility conditions (H1), (H2), and (H3) on the boundary datum g. Let us start with the first condition:

• **Condition (H1).** Suppose that there are three (possibly infinite) index sets I_Γ, I_χ, $I_F \subseteq \mathbb{N}$ such that the boundary $\partial\Omega$ can be decomposed, up to a $|f|-$negligible set, into disjoint open arcs $\Gamma_i^\pm = \overset{\frown}{a_i^\pm b_i^\pm}$ $(i \in I_\Gamma)$, $\chi_i^\pm = \overset{\frown}{c_i c_i^\pm}$ $(i \in I_\chi)$ and F_i $(i \in I_F)$ such that:

(1) For every $i \in I_\Gamma$, we have $\operatorname{dist}(\Gamma_i^+, \Gamma_i^-) := \inf\{|x - y| : \ x \in \Gamma_i^+, \ y \in \Gamma_i^-\} > 0$, g is strictly increasing on Γ_i^+ and strictly decreasing on Γ_i^- with

$$TV(g|_{\Gamma_i^+}) = TV(g|_{\Gamma_i^-}).$$

For the sake of convenience, we assume that $g(a_i^+) = g(a_i^-) < g(b_i^+) = g(b_i^-)$.

(2) For every $i \in I_\chi$, we have $\overline{\chi_i^+} \cap \overline{\chi_i^-} = \{c_i\}$, g is strictly increasing on χ_i^+ and strictly decreasing on χ_i^- with
$$TV(g|_{\chi_i^+}) = TV(g|_{\chi_i^-}).$$

(3) For every $i \in I_F$, the boundary datum g is constant on F_i. Moreover, each F_i is maximal with this property, i.e. if an open arc $\alpha \supset F_i$ is such that $g|_\alpha$ is constant, then $\alpha = F_i$. We shall say that F_i is a flat part.

(4) For every $i \in I_\Gamma \cup I_\chi$, we denote by T_i the convex hull of Γ_i^+ and Γ_i^- and D_i the convex hull of χ_i^+ and χ_i^-. Then, we assume that the sets $\{T_i, D_i : i \in I_\Gamma \cup I_\chi\}$ are mutually disjoint.

Keeping (H1) in mind, we introduce the following notation: $\Gamma_i = \Gamma_i^+ \cup \Gamma_i^-$ $(i \in I_\Gamma)$, $\chi_i = \chi_i^+ \cup \{c_i\} \cup \chi_i^-$ $(i \in I_\chi)$, $\Gamma^\pm = \bigcup_{i \in I_\Gamma} \Gamma_i^\pm$, $\chi^\pm = \bigcup_{i \in I_\chi} \chi_i^\pm$, $\Gamma = \Gamma^+ \cup \Gamma^-$ and $\chi = \chi^+ \cup \chi^-$.

In order to introduce the other assumptions (H2) and (H3) on the boundary datum g, we need to define a transport map $\mathbf{T} : \Gamma^+ \cup \chi^+ \mapsto \Gamma^- \cup \chi^-$ which will be a good candidate to be the optimal transport map between f^+ and f^-. So, we proceed as follows:

$$\mathbf{T}(x^+) = \begin{cases} x^- \in \chi_i^- & \text{if } x^+ \in \chi_i^+ \text{ and } TV(g|_{\widehat{c_i x^+}}) = TV(g|_{\widehat{c_i x^-}}), \quad i \in I_\chi, \\ x^- \in \Gamma_i^- & \text{if } x^+ \in \Gamma_i^+ \text{ and } TV(g|_{\widehat{a_i^+ x^+}}) = TV(g|_{\widehat{a_i^- x^-}}), \quad i \in I_\Gamma. \end{cases} \tag{3.1}$$

Here, we take the arcs $\widehat{c_i x^\pm} \subset \chi_i^\pm$ and $\widehat{a_i^\pm x^\pm} \subset \Gamma_i^\pm$. Due to the fact that g is strictly increasing on χ_i^+ (resp. Γ_i^+) and strictly decreasing on χ_i^- (resp. Γ_i^-) and $TV(g|_{\chi_i^+}) = TV(g|_{\chi_i^-})$ (resp. $TV(g|_{\Gamma_i^+}) = TV(g|_{\Gamma_i^-})$), one can see that the map $\mathbf{T}$ is well defined and it is also one-to-one (see Lemma 3.1 below). Hence, the inverse map of $\mathbf{T}$ is well defined and we will denote it by $\mathbf{T}^{[-1]}$. On the other hand, it is clear that by construction, $\mathbf{T}$ is a transport map from f^+ to f^-, i.e. $\mathbf{T}_\# f^+ = f^-$.

Lemma 3.1. *For every $i \in I_\chi$ (resp. $i \in I_\Gamma$), the restriction of the transportation map $\mathbf{T}$ to each arc χ_i^+ (resp. Γ_i^+) is a homeomorphism onto χ_i^- (resp. Γ_i^-).*

Proof. The restriction of $\mathbf{T}$ to χ_i^+ is strictly monotone. In fact, if $x_1, x_2 \in \chi_i^+$ and $x_1 < x_2$ with respect to the positive orientation of $\partial\Omega$, then $\mathbf{T}(x_1) > \mathbf{T}(x_2)$ (if $\mathbf{T}(x_1) = \mathbf{T}(x_2)$ then this would mean that g is constant on $\widehat{x_1 x_2}$, which is impossible). Hence, $\mathbf{T}$ is continuous except for at most countably many points. However, if $\mathbf{T}$ has a jump at $x_0 \in \chi_i^+$, then g would be constant on the arc $\widehat{\mathbf{T}(x_0^+)\mathbf{T}(x_0^-)} \subset \chi_i^-$, where $\mathbf{T}(x_0^+) := \lim_{x \to x_0, x > x_0} \mathbf{T}(x)$ and $\mathbf{T}(x_0^-) := \lim_{x \to x_0, x < x_0} \mathbf{T}(x)$. But, this is impossible due to the strict monotonicity of g on χ_i^-.

Since $\mathbf{T}$ is continuous and strictly monotone, then we deduce that the inverse is also continuous and strictly monotone as desired.

The same argument applies to Γ_i^+. □

Now, we continue stating our assumptions. Since the domain Ω is not necessarily strictly convex, we want to prohibit any transport ray from gliding along the boundary $\partial\Omega$. So, we impose the following condition:

- **Condition (H2).** For every $x^+ \in \Gamma^+ \cup \chi^+$, the open segment $]x^+, \mathbf{T}(x^+)[$ is contained in Ω.

Finally, since we need $\mathbf{T}$ to be an optimal transport map between f^+ and f^-, where $f = \partial_\tau g$, so we make the following assumption that will be crucial in the course of proving this fact:

- **Condition (H3).** Let us consider any finite sequence of points $\{e_i^+\}_{1 \le i \le m}$ (where $m \in \mathbb{N}$) in $\Gamma^+ \cup \chi^+$, then we assume the following inequality:

$$\sum_{i=1}^{m} |e_i^+ - \mathbf{T}(e_i^+)| < \sum_{i=1}^{m-1} |e_i^+ - \mathbf{T}(e_{i+1}^+)| + |e_m^+ - \mathbf{T}(e_1^+)|. \tag{3.2}$$

We note that this condition is related to the cyclical monotonicity property which is satisfied by any optimal transport plan γ of Problem (2.4) (here, $\gamma = (Id, \mathbf{T})_\# f^+$), see [15, Theorem 1.38]. We also remark that this condition is weaker than [13, Condition (C2)] and so, this allows us to extend the class of boundary data for which we can deduce the existence of solutions to (1.1), see Example 3.19 below.

Before starting, we note that existence of a unique solution to Problem (1.1) for strictly convex domains Ω and continuous data g is well-know, see [17]. However, our idea here is to characterize this solution in order to be able to pass to the limit in the convex case.

3.2. **The case of strictly convex domain.** In this section, we will show that thanks to conditions (H1) and (H3), ((H2) is trivial here), we can construct solutions to Problem (2.4). Since we know that the transport rays are the boundaries of the level sets of the solutions to Problem (1.1), we would like to identify these rays or equivalently, to characterize the optimal transport map R. More precisely, our bet is that if $x \in \Gamma^+ \cup \chi^+$, then the segment $[x, \mathbf{T}(x)]$ is a transport ray and so, $R(x) = \mathbf{T}(x)$ for f^+-a.e. x. Establishing this fact is the main result of this section.

Proposition 3.2. *Assume that Ω is bounded strictly convex and $g \in BV(\partial\Omega) \cap C(\partial\Omega)$ satisfies conditions (H1) & (H3), where the sets I_χ & I_Γ are finite. We set $f = \partial_\tau g$. Let γ be the optimal transport plan for Problem (2.4) between f^+ and f^-. For $\gamma-$a.e. $(x^+, x^-) \in \partial\Omega \times \partial\Omega$, we either have $x^+ \in \chi_i^+$ and $x^- = \mathbf{T}(x^+) \in \chi_i^-$ $(i \in I_\chi)$ or $x^+ \in \Gamma_i^+$ and $x^- = \mathbf{T}(x^+) \in \Gamma_i^-$ $(i \in I_\Gamma)$. In other words, we have $\gamma = (Id, \mathbf{T})_\# f^+$.*

Before we embark on proving this proposition, we lay out our tools. An important part of our argument is monitoring how the boundary $\partial\Omega$ is divided by a transportation ray. Namely, we notice that any point e^+ and its image $R(e^+)$ defined in Lemma 2.8) separate $\partial\Omega$ into two parts of zero measure f. More precisely, we have the following.

Lemma 3.3. *If $e^+ \in \chi_i^+$, $i \in I_\chi$ (resp. $e^+ \in \Gamma_i^+$, $i \in I_\Gamma$), then $f(\overset{\frown}{e^+ R(e^+)}) = 0$.*

Proof. Let us note that the definition of the arc $\overset{\frown}{e^+ R(e^+)}$ is ambiguous, however, the result holds independently of the chosen orientation.

We consider the transportation ray $[e^+, R(e^+)]$. Its endpoints separate $\partial\Omega$ into two open sets E_1, $E_2 \subset \partial\Omega$. We claim that $R(E_i) \subset E_i$, for $i = 1, 2$. Indeed, if there is a point $x \in E_1$ such that $R(x) \in E_2$ then, thanks to the convexity of Ω, the transportation rays $[e^+, R(e^+)]$ and $[x, R(x)]$ must intersect, but this is impossible.

Since $R^{-1}(E_i) \subset E_i$ $(i = 1, 2)$ and due to Lemma 2.8 we have $f^- = R_\# f^+$, then we see $f^-(E_i) = f^+(R^{-1}(E_i)) \le f^+(E_i)$, $i = 1, 2$. In particular, we get that $f^-(\partial\Omega) = f^-(E_1) + f^-(E_2) \le f^+(E_1) + f^+(E_2) = f^+(\partial\Omega)$. But, we know that $f^+(\partial\Omega) = f^-(\partial\Omega)$. Hence, $f^+(E_i) = f^-(E_i)$, $i = 1, 2$. □

Keeping in mind the setting of Proposition 3.2 we also make the following observation.

Lemma 3.4. *Under the conditions of Proposition 3.2, the set of multiple points $\mathcal{N}$ is finite.*

Proof. In Lemma 2.5, we already showed that $\mathcal{N}$ is at most countable. Let $x_0 \in \mathcal{N}$ and $x_1, x_2 \in R_\mathcal{N}(x_0)$. We claim that x_1 and x_2 do not belong to the closure of the same arc from $\Gamma \cup \chi$. Indeed, if the claim is not true, then $\overset{\frown}{x_1 x_2} \subset \Gamma_i^-$ or $\overset{\frown}{x_1 x_2} \subset \chi_i^-$ and this arc must be contained in $R_\mathcal{N}(x_0)$. As a result,

$$0 < f^-(\overset{\frown}{x_1 x_2}) = R_\# f^+(\overset{\frown}{x_1 x_2}) = f^+(R^{-1}(\overset{\frown}{x_1 x_2})) = f^+(\{x_0\}) = 0,$$

which is a contradiction.

Let us suppose now that $x_0' \neq x_0$ is another multiple point and $x_1' \neq x_2'$ are in $R_\mathcal{N}(x_0')$. Since transport rays cannot intersect we see that at least one of the x_1', x_2' belongs to the closure of an arc not containing neither x_1 nor x_2. We can iterate this process to infer that the number of elements in $\mathcal{N}$ must be finite. □

Now, we can state another important tool.

Lemma 3.5. *If $e^+ \in \chi_i^+$, $i \in I_\chi$ (resp. $e^+ \in \Gamma_i^+$, $i \in I_\Gamma$) and $\mathbf{T}^{[-1]}(R(e^+)) \in \chi_i^+$ (resp. $\mathbf{T}^{[-1]}(R(e^+)) \in \Gamma_i^+$), then $e^+ = \mathbf{T}^{[-1]}(R(e^+))$.*

Proof. By the definition of $\mathbf{T}$, we note that if $\mathbf{T}^{[-1]}(R(e^+)) \in \chi_i^+$ (resp. $\mathbf{T}^{[-1]}(R(e^+)) \in \Gamma_i^+$), then we have $R(e^+) \in \chi_i^-$ (resp. $R(e^+) \in \Gamma_i^-$). Thanks to Lemma 3.3, we know that $f(\overset{\frown}{e^+R(e^+)}) = 0$. Moreover, for every $x^+ \in \chi_i^+$, $i \in I_\chi$ (resp. $x^+ \in \Gamma_i^+$, $i \in I_\Gamma$), we show that

$$f(\overset{\frown}{x^+\mathbf{T}}(x^+)) = 0. \tag{3.3}$$

In fact, if $x^+ \in \chi_i^+$ then (3.3) immediately follows from the definition of $\mathbf{T}$ (one can assume that the arc is so chosen that $c_i \in \overset{\frown}{x^+\mathbf{T}(x^+)})$.

If $x^+ \in \Gamma_i^+$ then, again due to the definition of $\mathbf{T}$, we have $f^+(\overset{\frown}{x^+b_i^+}) = f^-(\overset{\frown}{\mathbf{T}(x^+)\,b_i^-})$ and the arcs are chosen so that $\overset{\frown}{x^+b_i^+}\subset \Gamma_i^+$ and $\overset{\frown}{\mathbf{T}(x^+)\,b_i^-} = \mathbf{T}(\overset{\frown}{x^+b_i^+}) \subset \Gamma_i^-$. We also see that $\partial\Omega \setminus (\overset{\frown}{x^+b_i^+} \cup \overset{\frown}{\mathbf{T}(x^+)\,b_i^-})$ has exactly two connected components E_1 and E_2. For the sake of definitness, we assume that $b_i^+, b_i^- \in E_1$. Notice that if there is $j \in I_\Gamma$ $(i \neq j)$ and such that $E_1 \cap \Gamma_j^\pm \neq \emptyset$, then $\Gamma_j^\pm \subset E_1$. Indeed, if $\Gamma_j^\pm \setminus E_1 \neq \emptyset$, then $\Gamma_j^\pm$ must intersect $\Gamma_i^+ \cup \Gamma_i^-$, but this is impossible. The same conclusion, i.e. $\chi_j^\pm \subset E_1$ holds when $E_1 \cap \chi_j^\pm \neq \emptyset$ $(j \in I_\chi)$.

Let us suppose now that $\Gamma_j^+ \subset E_1$, $j \in I_\Gamma$. Then, we also have $\Gamma_j^- \subset E_1$. Indeed, if $\Gamma_j^- \subset E_2$ then the geometry would imply that $T_i \cap T_j \neq \emptyset$ which contradicts (H1). By the same argument, if $\chi_j^+ \subset E_1$ $(j \in I_\chi)$, then $\chi_j^- \subset E_1$. The observations we made imply that

$$E_1 = \bigcup_{\alpha\in A} \chi_\alpha \cup \bigcup_{\beta\in B} \Gamma_\beta \cup \bigcup_{\gamma\in C} F_\gamma \cup N,$$

for appropriate sets of indices A, B, C, where $|f|(N) = 0$. Hence, we have $f(E_1) = 0$. As a result, from the definition of $\mathbf{T}$ we get that

$$f(\overset{\frown}{x^+\mathbf{T}(x^+)}) = f(\overset{\frown}{x^+b_i^+}) + f(E_1) + f(\overset{\frown}{\mathbf{T}(x^+)\,b_i^-}) = f^+(\overset{\frown}{x^+b_i^+}) - f^-(\overset{\frown}{\mathbf{T}(x^+)\,b_i^-}) = 0.$$

This observation completes the proof of (3.3).

Finally, take $x^+ = \mathbf{T}^{[-1]}(R(e^+))$, then we get that $f(\overset{\frown}{\mathbf{T}^{[-1]}(R(e^+))R(e^+)}) = 0$. After combining this with $f(\overset{\frown}{e^+R(e^+)}) = 0$, we have

$$\begin{aligned} 0 = f(\partial\Omega) &= f(\overset{\frown}{e^+\,\mathbf{T}^{[-1]}(R(e^+))}) + f(\overset{\frown}{e^+\,R(e^+)}) + f(\overset{\frown}{\mathbf{T}^{[-1]}(R(e^+))R(e^+)}) \\ &= f(\overset{\frown}{e^+\,\mathbf{T}^{[-1]}(R(e^+))}). \end{aligned}$$

Since $f(\overset{\frown}{e^+\,\mathbf{T}^{[-1]}(R(e^+))}) = f^+(\overset{\frown}{e^+\,\mathbf{T}^{[-1]}(R(e^+))})$, we reach $e^+ = \mathbf{T}^{[-1]}(R(e^+))$. $\square$

Now, we are ready to carry out the *proof of Proposition 3.2*. We are going to show that for f^+−a.e. $x \in \chi_i^+$, $i \in I_\chi$ (resp. $x \in \Gamma_i^+$, $i \in I_\Gamma$), we have $R(x) = \mathbf{T}(x)$. Assume that our claim does not hold. Hence, due to piecewise continuity of R implied by Lemma 2.7, there is an arc $E_1^+ \subset \chi_{i_1}^+$ (resp. $E_1^+ \subset \Gamma_{i_1}^+$) that is transported outside $\chi_{i_1}^-$ (resp. $\Gamma_{i_1}^-$) to an arc $E_2^- := R(E_1^+)$ such that $E_2^- \subset \chi_{i_2}^-$ or $E_2^- \subset \Gamma_{i_2}^-$. Without loss of generality, we may assume that $E_1^+ \subset \chi_{i_1}^+$ and $E_2^- \subset \chi_{i_2}^-$. The argument when $E_2^- \subset \Gamma_{i_2}^-$ is the same and it will be omitted. Set $E_2^+ := \mathbf{T}^{[-1]}(E_2^-) \subset \chi_{i_2}^+$. Due to Lemma 3.4, we may make the arcs E_2^+ and E_2^- disjoint from $\mathcal{N}$. We claim that $R(E_2^+) \cap \chi_{i_2}^- = \emptyset$. Let us suppose the contrary, i.e. there is an arc E^+ of E_2^+ such that $R(E^+) \subset \chi_{i_2}^-$. Then, Lemma 3.5 implies that $R(E^+) = \mathbf{T}(E^+) \subset E_2^-$. However, this is impossible due to the strict convexity of Ω and the fact that two different transport rays cannot intersect at an interior point and that $E_2^- = R(E_1^+)$. Our claim follows.

We presented above a general procedure: given an arc E_1^+ such that $E_1^+ \cap \mathcal{N} = \emptyset$ we constructed two arcs E_2^- and E_2^+ by setting $R(E_1^+) = E_2^-$ and $\mathbf{T}(E_2^+) = E_2^-$. Let us suppose that arcs E_1^+, E_2^-, $E_2^+, \ldots, E_n^+$ (where $n \geq 1$) have been already constructed by the above algorithm, where $R(E_n^+) \cap E_n^- = \emptyset$. Then, we define E_{n+1}^- as follows: if we have $R(E_n^+) \subset \chi_{i_1}^-$, then we set $E_{n+1}^- := R(E_n^+)$ and the construction terminates. If $R(E_n^+) \cap \chi_{i_1}^- = \emptyset$, then we set

$$E_{n+1}^- := R(E_n^+) \qquad \text{and} \qquad E_{n+1}^+ := \mathbf{T}^{[-1]}(E_{n+1}^-)$$

and due to Lemma 3.4 we may guarantee, after possible restriction of the initial set E_1^+, that $E_{n+1}^- \cap \mathcal{N} = \emptyset$. Hence, we know that $R(E_{n+1}^+) \cap E_{n+1}^- = \emptyset$. We note that this sequence of arcs $\{E_k^\pm\}$ may be either: (1) finite with elements

$$E_1^+,\, E_2^-,\, E_2^+, \ldots,\, E_m^-,\, E_m^+,\, E_{m+1}^-. \tag{3.4}$$

(2) or infinite

$$E_1^+,\, E_2^-,\, E_2^+, \ldots,\, E_n^+,\, E_n^-, \ldots \tag{3.5}$$

In addition, thanks to Lemma 3.4 we choose these arcs $E_k^\pm$ such that they do not contain any multiple point.

Assume that the case (3.4) holds. Let e_{m+1}^- be any point in E_{m+1}^- and set $e_m^+ = R^{-1}(e_{m+1}^-) \in E_m^+$. Then, we define $e_m^- = \mathbf{T}(e_m^+)$ and $e_{m-1}^+ = R^{-1}(e_m^-) \in E_{m-1}^+$. In this way, we get a sequence of points $\{e_k^\pm : 1 \le k \le m+1\}$ such that $e_k^\pm \in E_k^\pm$,

$$R(e_k^+) = e_{k+1}^- \quad \text{and} \quad \mathbf{T}(e_k^+) = e_k^-. \tag{3.6}$$

Therefore, Lemma 3.6 below tells us that not only $e_{m+1}^- \in \chi_{i_1}^-$ but also $e_{m+1}^- = e_1^-$. The final observation is made in Lemma 3.7, where we claim that in fact the sequence (3.4) consists of two elements E_1^+, $E_{m+1}^- = E_1^-$.

At last, we address the case of seemingly infinite sequence (3.5). We notice that since the number of arcs Γ_i and χ_i is finite, then some of these arcs must be visited by the sequence more than once. In other words, a loop forms. As a result we are back to (3.4), hence the seemingly infinite sequence consists only of E_1^+ and E_1^-. This is the content of Lemma 3.8 below. Consequently, our proposition follows. □

Here are the Lemmas we referred to.

Lemma 3.6. *Assume that* (3.4) *holds. Let us consider the finite sequence of points* $e_1^+, e_2^-, e_2^+, \ldots, e_m^-, e_m^+, e_{m+1}^-$ *described above. Then, we have*

$$e_{m+1}^- = e_1^- := \mathbf{T}(e_1^+).$$

Proof. First, we claim that $f(\widehat{e_1^+ e_k^-}) = 0$ for all $k \ge 2$ (we note that this property holds even if the sequence (3.5) is infinite). We will prove this by induction. Thanks to Lemma 3.3, we see that $f(\widehat{e_1^+ e_2^-}) = 0$, because by definition $e_2^- = R(e_1^+)$. We also recall from (3.3) that $f(\widehat{e_k^+ e_k^-}) = 0$, for all $2 \le k \le m$. Recursively, assume that $f(\widehat{e_1^+ e_k^-}) = 0$, for some k. We shall prove that we also have $f(\widehat{e_1^+ e_{k+1}^-}) = 0$. For this aim, we have to consider the following three possible configurations, while taking the arc $\widehat{e_1^+ e_{k+1}^-}$ containing e_k^- (the other cases when $e_k^- \notin \widehat{e_1^+ e_{k+1}^-}$ can be treated similarly):
(i) $e_k^+ \in \widehat{e_k^- e_{k+1}^-} \subset \widehat{e_1^+ e_{k+1}^-}$, (ii) $e_k^+ \in \widehat{e_1^+ e_k^-} \subset \widehat{e_1^+ e_{k+1}^-}$, (iii) $e_k^+ \in \partial\Omega \setminus \widehat{e_1^+ e_{k+1}^-}$.
Let us start by considering case (i). Then, we see that the arcs $\widehat{e_1^+ e_k^-}$, $\widehat{e_k^+ e_k^-}$, $\widehat{e_k^+ e_{k+1}^-}$ are disjoint. Hence,

$$f(\widehat{e_1^+ e_{k+1}^-}) = f(\widehat{e_1^+ e_k^-} \cup \widehat{e_k^+ e_k^-} \cup \widehat{e_k^+ e_{k+1}^-}) = f(\widehat{e_1^+ e_k^-}) + f(\widehat{e_k^+ e_k^-}) + f(\widehat{e_k^+ e_{k+1}^-}) = 0.$$

Indeed, here we used the inductive assumption $f(\widehat{e_1^+ e_k^-}) = 0$, property (3.3) as well as Lemma 3.3, $f(\widehat{e_k^+ e_{k+1}^-}) = f(\widehat{e_k^+ R(e_k^+)}) = 0$. Now, we take care of case (ii). We have

$$f(\widehat{e_1^+ e_{k+1}^-}) = f(\widehat{e_1^+ e_k^-} \cup \widehat{e_k^+ e_{k+1}^-}).$$

However, the arcs $\widehat{e_k^+ e_{k+1}^-}$ and $\widehat{e_1^+ e_k^-}$ have a nontrivial intersection which is the arc $\widehat{e_k^+ e_k^-}$. Thus, one has

$$f(\widehat{e_1^+ e_{k+1}^-}) = f(\widehat{e_1^+ e_k^-}) - f(\widehat{e_k^+ e_k^-}) + f(\widehat{e_k^+ e_{k+1}^-}) = 0,$$

because each term in the sum above is zero. Finally, we consider the case (iii). We apply the same argument we used in (ii) and obtain

$$f(\overset{\frown}{e_1^+ e_{k+1}^-}) = f(\overset{\frown}{e_1^+ e_k^-}) + f(\overset{\frown}{e_k^+ e_k^-}) - f(\overset{\frown}{e_k^+ e_{k+1}^-}) = 0.$$

In particular, induction yields $f(\overset{\frown}{e_1^+ e_{m+1}^-}) = 0$. We also know that $e_{m+1}^- \in \chi_{i_1}^-$ and $f(\overset{\frown}{e_1^+ e_1^-}) = 0$. Hence, we infer that $f^-(\overset{\frown}{e_1^- e_{m+1}^-}) = 0$ and so, $e_{m+1}^- = e_1^-$. □

In addition, we have the following:

Lemma 3.7. *The sequence $\{e_k^\pm : 1 \leq k \leq m+1\}$ consists of the couple $\{e_1^+, e_1^-\}$.*

Proof. Let us suppose the contrary and $m > 1$. Thanks to Lemma 3.6, we have that $(e_m^+, e_1^-) \in \mathrm{spt}(\gamma)$. Yet, $(e_k^+, e_{k+1}^-) \in \mathrm{spt}(\gamma)$, for all $1 \leq k \leq m-1$. Let ϕ be a Kantorovich potential for measures f^+ and f^-. Hence, by the equality (2.6), we must have the following inequality:

$$\begin{aligned}\sum_{k=1}^{m-1} |e_k^+ - e_{k+1}^-| + |e_m^+ - e_1^-| &= \sum_{k=1}^{m-1} [\phi(e_k^+) - \phi(e_{k+1}^-)] + [\phi(e_m^+) - \phi(e_1^-)] = \sum_{k=1}^{m} [\phi(e_k^+) - \phi(e_k^-)] \\ &\leq \sum_{k=1}^{m} |e_k^+ - e_k^-|,\end{aligned}$$

where the last inequality follows from the fact that ϕ is 1−Lipschitz. Considering the sequence's definition in (3.6), it follows that the above inequality contradicts our condition (H3). □

Finally, it remains to show that the sequence (3.5) cannot be infinite.

Lemma 3.8. *The apparently infinite sequence (3.5) is in fact finite and it consists of the arcs E_1^+, E_1^-.*

Proof. Let us consider the infinite sequence (3.5). Since the number of arcs Γ_i and χ_i is by assumption finite, then there exist $l \geq 1$ and $k \geq 2$ such that $E_l^-, E_{l+k}^- \subset \chi_{i_l}^-$ (or $E_l^-, E_{l+k}^- \subset \Gamma_{i_l}^-$). Let us denote again by $\{e_i^\pm\}_i$ the sequence of points on these arcs $E_i^\pm$ defined above in the course of proof of Proposition 3.2. Thanks to Lemma 3.6, one can show similarly that $e_{l+k}^- = e_l^-$. Now, we have a finite sequence starting at $e_l^+ \in \chi_{i_l}^+$ (or $e_l^+ \in \Gamma_{i_l}^+$) (this point will take the role of a new e_1^+) and terminating at $e_{l+k}^- = e_l^- \in \chi_{i_l}^-$ (or $e_{l+k}^- = e_l^- \in \Gamma_{i_l}^-$). Hence, we are in a situation, when we deal with a finite sequence, we invoke Lemma 3.7 to deduce that the loop consists of a pair $\{e_l^+, e_l^-\}$. But, this is a contradiction. □

In this way, we have provided all the details for the proof of Proposition 3.2, which will play an important role in the next section to show that even if Ω is convex and not necessarily strictly convex, the transportation map $\mathbf{T}$ will be also the optimal transport map R between f^+ and f^-. We conclude this section with the following observation.

Remark 3.9. The scrutiny of the proof of Lemma 3.7 shows that in the case when Ω is strictly convex, one can relax the condition (H3) by assuming that inequality (3.2) holds only for every sequence of points $\{e_i^+\}_{1\leq i\leq m}$ that belong to different arcs Γ_i^+ $(i \in I_\Gamma)$ or χ_i^+ $(i \in I_\chi)$. This remark will be used in Step 3 of Proposition 3.12.

3.3. The case of convex, but not strictly convex domain.

In this section, we will extend the result of Proposition 3.2 to the case of convex domain Ω, without assuming its strict convexity. We will proceed in a natural manner by finding a sequence of strictly convex domains Ω_n whose closures converge to $\overline{\Omega}$ in the Hausdorff metric. At the same time, we have to come up with a good choice of the boundary data g_n defined on $\partial\Omega_n$ and approximating g, to keep the assumptions (H1) and (H3) in force. Since our approximation is based on the orthogonal projection onto $\overline{\Omega}$, then the singular points of the support of f will play a role. We recall their definition

$$\mathcal{S} := \Big\{x \in \Gamma \cup \chi \,:\, \text{there is no tangent line to } \partial\Omega \text{ at } x\Big\}.$$

We give our first observation about the projection map onto a convex set.

Lemma 3.10. *Suppose that $\Omega \subset \mathbb{R}^2$ is an open bounded convex set and $\overline{\Omega} \subset \Omega'$, where Ω' is open bounded and strictly convex. We assume that $\tilde{P} : \mathbb{R}^2 \mapsto \overline{\Omega}$ is the orthogonal projection, its restriction to $\partial\Omega'$ will be denoted by P. For every $x_0 \in \partial\Omega$, we have the following:*
(1) if $x_0 \notin \mathcal{S}$, then the preimage $P^{-1}(x_0)$ is a singleton,
(2) if $x_0 \in \mathcal{S}$, then $P^{-1}(x_0) \subset \partial\Omega'$ is an arc of positive Hausdorff measure.

Proof. (1) In this case, the normal cone $N(x_0)$ reduces to a ray, which has a unique intersection with $\partial\Omega'$.

(2) If x_0 is a singular point, then the normal cone has a positive opening. Its intersection with $\partial\Omega'$ has a positive Hausdorff measure. □

The conclusion from this lemma is as follows. If we approximate Ω by a sequence of strictly convex sets Ω_n and we try to partition the boundaries of these sets into arcs of types $\chi^{\pm}_{i,n}$ and $\Gamma^{\pm}_{i,n}$, then we will meet some difficulty since Lemma 3.10 tells us that $(P_{|\partial\Omega_n})^{-1}$ will not preserve the structure of arcs $\chi^{\pm}_i$ and $\Gamma^{\pm}_i$ due to possible presence of singular points on these arcs. As a result, we are forced to rearrange our partition into arcs $\chi^{\pm}_i$ and $\Gamma^{\pm}_i$ in such a way that the new partition avoids singular points. We will then make the following assumption about the set of singular points

$$|f|(\overline{\mathcal{S}}) = 0. \tag{S}$$

We note that the assumption (S) is satisfied as soon as the closure of $\mathcal{S}$ is countable; in particular, this will cover the case of polygonal Ω.

Lemma 3.11. *Let us suppose that Ω is convex and conditions (H1), (H2), (H3) and (S) are satisfied. Then, there exists another partition of $\partial\Omega$ into smooth arcs $\tilde{\chi}^{\pm}_i$ ($i \in I_{\tilde{\chi}} \equiv I_\chi$), $\tilde{\Gamma}^{\pm}_i$ ($i \in I_{\tilde{\Gamma}}$ where $I_\Gamma \subset I_{\tilde{\Gamma}}$) and $\tilde{F}_i$ ($i \in I_{\tilde{F}}$ where $I_F \subset I_{\tilde{F}}$), satisfying the same conditions (H1), (H2) and (H3).*

Proof. Since $\partial\Omega \setminus \overline{\mathcal{S}}$ is open, then there are open arcs U_k ($k \in I \subset \mathbb{N}$) such that

$$\partial\Omega \setminus \overline{\mathcal{S}} = \bigcup_{k\in I} U_k.$$

Consider the following families of open arcs

$$\tilde{\Gamma}^+_{k,i,j} := \Gamma^+_k \cap U_i \cap \mathbf{T}^{[-1]}(U_j) \qquad \text{and} \qquad \tilde{\Gamma}^-_{k,i,j} := \Gamma^-_k \cap \mathbf{T}(U_i) \cap U_j,$$

with $k \in I_\Gamma$ and $i, j \in I$. In fact, these arcs $\tilde{\Gamma}^{\pm}_{k,i,j}$ are in a one to one correspondence, because we have

$$\mathbf{T}(\tilde{\Gamma}^+_{k,i,j}) = \mathbf{T}(\Gamma^+_k \cap U_i \cap \mathbf{T}^{[-1]}(U_j)) = \Gamma^-_k \cap \mathbf{T}(U_i) \cap U_j = \tilde{\Gamma}^-_{k,i,j}.$$

Moreover, it is clear that $\tilde{\Gamma}^+_{k,i,j}$ and $\tilde{\Gamma}^-_{k,i,j}$ inherit the properties of Γ^+_k, Γ^-_k.

When we subdivide arcs $\chi^{\pm}_k$ we proceed slightly differently. We first construct similar families

$$\chi^+_k \cap U_i \cap \mathbf{T}^{[-1]}(U_j) \qquad \text{and} \qquad \chi^-_k \cap \mathbf{T}(U_i) \cap U_j, \qquad k \in I_\chi,\ i,\, j \in I.$$

We consider two cases: $c_k \in \overline{\mathcal{S}}$ and $c_k \notin \overline{\mathcal{S}}$. In the first case, we set

$$\hat{\Gamma}^+_{k,i,j} := \chi^+_k \cap U_i \cap \mathbf{T}^{[-1]}(U_j) \qquad \text{and} \qquad \hat{\Gamma}^-_{k,i,j} := \chi^-_k \cap \mathbf{T}(U_i) \cap U_j, \qquad k \in I_\Gamma,\ i,\, j \in I.$$

Since $\mathbf{T}(\hat{\Gamma}^+_{k,i,j}) = \hat{\Gamma}^-_{k,i,j}$, we can see that these arcs satisfy the conditions required for arcs of type Γ.

In the case when $c_k \notin \overline{\mathcal{S}}$, there exist i_0, $j_0 \in I$ such that if we set

$$\hat{\chi}^+_{k,i_0,j_0} := \chi^+_k \cap U_{i_0} \cap \mathbf{T}^{[-1]}(U_{j_0}), \qquad \hat{\chi}^-_{k,i_0,j_0} := \chi^-_k \cap \mathbf{T}(U_{i_0}) \cap U_{j_0},$$

then $\overline{\chi^+_{k,i_0,j_0}} \cap \overline{\chi^-_{k,i_0,j_0}} = \{c_k\}$ and χ^+_{k,i_0,j_0}, χ^-_{k,i_0,j_0} inherit the properties of χ^+_k, χ^-_k. This concludes the proof. □

Thanks to Lemma 3.11, one can assume that the decomposition of $\partial\Omega$ is such that all arcs $\chi^{\pm}_i$ ($i \in I_\chi$) and $\Gamma^{\pm}_i$ ($i \in I_\Gamma$) are smooth. However, it is possible that the point $c_i = \overline{\chi^+_i} \cap \overline{\chi^-_i}$ is a singular point. Moreover, we note that we do not care about singular points on the flat parts F_i ($i \in I_F$).

Now, we are ready to show existence of a solution to Problem (1.1) in the convex case. For this purpose, we divide our task into several parts. First, we assume that I_Γ and I_χ are finite, and we approximate Ω by a sequence of strictly convex domains. Finally, by approximating the boundary data, we complete the analysis considering the case when the number of arcs is infinite.

Let us assume that the set of singular points $\mathcal{S}$ is finite. In addition, we strengthen our condition (H3) by assuming that for every $m \in \mathbb{N}$, there is a $\delta_m > 0$ such that for any sequence of points $\{e_k^+\}_{1\le k\le m}$ described in (H3), the following inequality holds:

$$\text{(H3)}' \qquad \sum_{k=1}^{m} |e_k^+ - \mathbf{T}(e_k^+)| \le \sum_{k=1}^{m-1} |e_k^+ - \mathbf{T}(e_{k+1}^+)| + |e_m^+ - \mathbf{T}(e_1^+)| - \delta_m.$$

Proposition 3.12. *Let us assume that Ω is convex, conditions (H1), (H2) & (H3)$'$ are satisfied, the set $\mathcal{S}$ is finite and the number of arcs $\chi_i^\pm$ ($i \in I_\chi$) as well as $\Gamma_i^\pm$ ($i \in I_\Gamma$) is finite. Then, $\mathbf{T}$ is an optimal transport map from f^+ to f^-.*

Proof. Our task will be to find a sequence of decreasing strictly convex regions Ω_n such that $\overline{\Omega}_n$ converges to $\overline{\Omega}$ in the Hausdorff distance. At the same time, we need to approximate the boundary datum g by a sequence of functions g_n defined on $\partial\Omega_n$. Our choice of (Ω_n, g_n) must be such that for every $n \in \mathbb{N}$, the boundary $\partial\Omega_n$ can be decomposed into arcs $\chi_{i,n}$, $\Gamma_{i,n}$ and $F_{i,n}$ for appropriate sets of indices.

We divide the proof into several steps:

Step 1. Since Ω is convex, then due to [12, Theorem 20.4] one can always find a sequence of decreasing closed polygons Δ_n containing Ω and converging to $\overline{\Omega}$ in the Hausdorff distance. After that, we use the argument in [13] to construct a strictly convex region $\tilde{\Omega}_n$ containing Δ_n such that the Hausdorff distance between them does not exceed $\frac{1}{n}$. Thus, we obtain a decreasing sequence of strictly convex domains Ω_n such that $\overline{\Omega}_n \to \overline{\Omega}$ in the Hausdorff metric. Indeed, if we set $\Omega_n = \bigcap_{i=1}^n \tilde{\Omega}_i$, then

$$\Omega \subset \Omega_{n+1} = \tilde{\Omega}_{n+1} \cap \Omega_n \subset \Omega_n.$$

Step 2. Due to Lemma 3.11 we may assume that all arcs $\chi_i^\pm$ and $\Gamma_j^\pm$ are smooth because $\mathcal{S}$ is finite.

We recall that $\tilde{P}$, the orthogonal projection onto $\overline{\Omega}$, is Lipschitz continuous on $\mathbb{R}^2$. Let us fix $n \in \mathbb{N}$ and set $\tilde{P}_n := \tilde{P}|_{\partial\Omega_n}$, $\mathcal{C}_{i,n}^\pm := \tilde{P}_n^{-1}(\chi_i^\pm)$ for $i \in I_\chi$, and $\mathcal{C'}_{i,n}^\pm := \tilde{P}_n^{-1}(\Gamma_i^\pm)$ for $i \in I_\Gamma$, $\mathcal{C}_n := \bigcup_{i\in I_\chi \cup I_\Gamma}(\mathcal{C}_{i,n}^\pm \cup \mathcal{C'}_{i,n}^\pm)$. Let $P_n : \mathcal{C}_n \mapsto \Gamma \cup \chi$ be a further restriction of $\tilde{P}_n$ to $\mathcal{C}_n$. Thanks to Lemma 3.10, we see that P_n is one-to-one, hence by the open mapping theorem the inverse of P_n is continuous too. In the sequel, we denote by $P_n^{-1} : \Gamma \cup \chi \mapsto \mathcal{C}_n$ the inverse map of P_n. Then, we define a measure $\tilde{f}_n$ on $\mathcal{C}_n$ as follows:

$$\tilde{f}_n := (P_n^{-1})_\# f.$$

Since f is concentrated on $\Gamma \cup \chi$ and P_n^{-1} is continuous on $\Gamma \cup \chi$, then the measure $\tilde{f}_n$ is well defined. Now, we extend it to a Borel measure on $\partial\Omega_n$ by setting

$$f_n(B) := \tilde{f}_n(B \cap \mathcal{C}_n),$$

for any Borel set $B \subset \partial\Omega_n$. By the definition of f_n we have that $|f_n|(\partial\Omega_n \backslash \mathcal{C}_n) = 0$. It is also clear that $f_n(\partial\Omega_n) = f(\partial\Omega) = 0$ and $|f_n|(P^{-1}(\mathcal{S}) \cap \partial\Omega_n) = 0$.

Moreover, if $x_1, x_2 \in \mathcal{C}_n$ then we have

$$f_n(\widehat{x_1 x_2}) = f(\widehat{P_n(x_1) P_n(x_2)})$$

because $\widehat{P_n(x_1) P_n(x_2)} = P_n(\widehat{x_1 x_2})$.

After these preparations, we define the trace function g_n on $\partial\Omega_n$. Since it has to satisfy $\partial_\tau g_n = f_n$, then we proceed as follows. For a fixed $x_0 \in \partial\Omega \backslash \mathcal{S}$, we define $x_n := P_n^{-1}(x_0) \in \partial\Omega_n$. Then, we set

$$g_n(x) := f_n(\widehat{x_n x}), \text{ for every } \ x \in \partial\Omega_n.$$

Since f is atomless, so is f_n. Hence, $g_n \in C(\partial\Omega_n)$.

Let us discuss the forms of the sets $\mathcal{C}_{i,n}^\pm$ and $\mathcal{C'}_{i,n}^\pm$. We notice that $\mathcal{C'}_{i,n}^\pm = P_n^{-1}(\Gamma_i^\pm)$ are two arcs of the form $\Gamma_{i,n}^\pm$, because we have $\overline{\mathcal{C'}_{i,n}^+} \cap \overline{\mathcal{C'}_{i,n}^-} = \emptyset$ and

$$f_n(\mathcal{C'}_{i,n}^+) = f^+(\Gamma_i^+) \quad \text{and} \quad f_n(\mathcal{C'}_{i,n}^-) = f^-(\Gamma_i^-).$$

Moreover, g_n is strictly increasing on ${\mathcal{C}'}^{+}_{i,n}$ (resp. decreasing on ${\mathcal{C}'}^{-}_{i,n}$), because if $x_1,\, x_2 \in {\mathcal{C}'}^{\pm}_{i,n}$ are such that $x_1 < x_2$, then $P_n(x_1),\, P_n(x_2) \in \Gamma_i^{\pm}$ with $P_n(x_1) < P_n(x_2)$ and, we have

$$g_n(x_2) - g_n(x_1) = f_n(\overset{\frown}{x_1 x_2}) = f(P_n(x_1)\overset{\frown}{\,} P_n(x_2)),$$

where $\overset{\frown}{x_1 x_2} \subset {\mathcal{C}'}^{\pm}_{i,n}$ and $P_n(x_1)\overset{\frown}{\,}P_n(x_2) = P_n(\overset{\frown}{x_1 x_2}) \subset \Gamma_i^{\pm}$.

The above argument applies too when the point c_i (the common endpoint of $\chi_i^{\pm}$) is a singular point. Hence, $\mathcal{C}^{\pm}_{i,n} = P_n^{-1}(\chi_i^{\pm})$ are two arcs of the form $\Gamma^{\pm}_{i,n}$, because by Lemma 3.10, $P_n^{-1}(c_i)$ is an arc of positive $\mathcal{H}^1$ measure (see Figure 1).

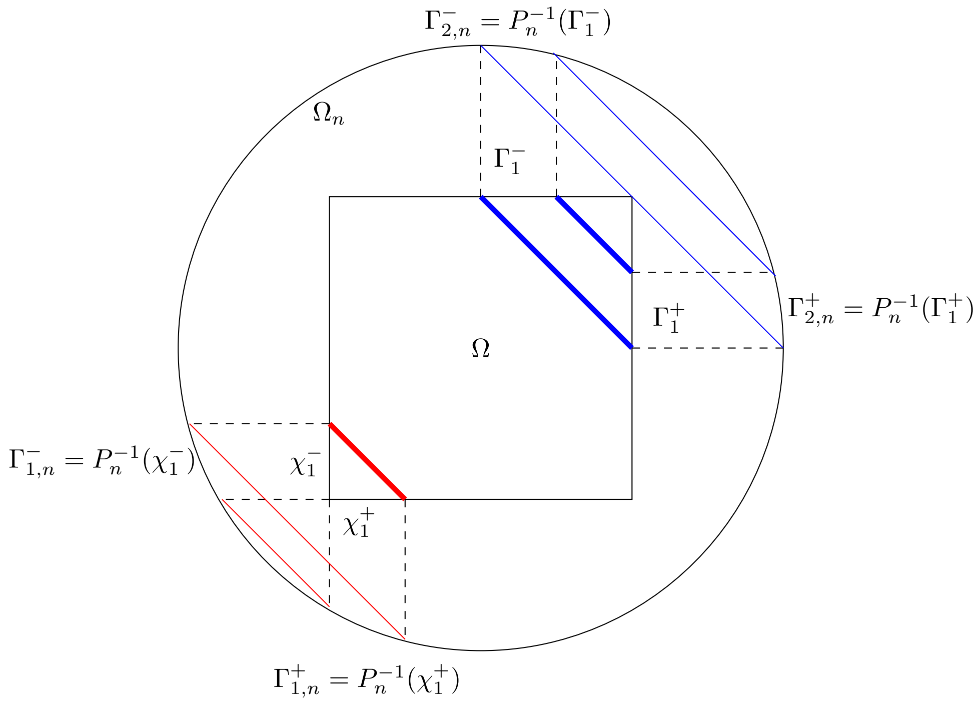

FIGURE 1

Finally, when $c_i \notin \mathcal{S}$, then by Lemma 3.10, $P_n^{-1}(c_i)$ is a singleton and $\overline{\mathcal{C}^{+}_{i,n}} \cap \overline{\mathcal{C}^{-}_{i,n}} = \{P_n^{-1}(c_i)\}$. Hence, $\mathcal{C}^{\pm}_{i,n} = P_n^{-1}(\chi_i^{\pm})$ are two arcs of the form $\chi^{\pm}_{i,n}$. Indeed, we have

$$f_n(\mathcal{C}^{+}_{i,n} \cup \mathcal{C}^{-}_{i,n}) = f(\chi_i) = 0.$$

We also see that g_n is strictly increasing on $\mathcal{C}^{+}_{i,n}$ (resp. decreasing on $\mathcal{C}^{-}_{i,n}$), because if $x_1,\, x_2 \in \mathcal{C}^{\pm}_{i,n}$ are such that $x_1 < x_2$, then $P_n(x_1),\, P_n(x_2) \in \chi_i^{\pm}$ with $P_n(x_1) < P_n(x_2)$. Moreover,

$$g_n(x_2) - g_n(x_1) = f_n(\overset{\frown}{x_1 x_2}) = f(P_n(x_1)\overset{\frown}{\,} P_n(x_2)),$$

where $\overset{\frown}{x_1 x_2} \subset \mathcal{C}^{\pm}_{i,n}$ and $P_n(x_1)\overset{\frown}{\,}P_n(x_2) = P_n(\overset{\frown}{x_1 x_2}) \subset \chi_i^{\pm}$.

Step 3. Now, let us check that the assumption (H3) is also satisfied by g_n. Let $\mathbf{T}_n$ be the transportation map defined on $\Gamma_n^+ \cup \chi_n^+$ (see (3.1)). First, we see that if $e^+ \in \Gamma_n^+ \cup \chi_n^+$, then we have

$$e^- := \mathbf{T}_n(e^+) = P_n^{-1}(\mathbf{T}(P_n(e^+))). \tag{3.7}$$

Indeed, if $e^+ \in \Gamma^+_{i,n}$, $i \in I_{\Gamma_n}$, (the argument when $e^+ \in \chi^+_{i,n}$, $i \in I_{\chi_n}$, is the same and it will be omitted), then $P_n(e^+) \in \Gamma_j^+$ for some $j \in I_\Gamma$ or $P_n(e^+) \in \chi_j^+$ for some $j \in I_\chi$, in case c_j is a singular point. As a result, $\mathbf{T}(P_n(e^+)) \in \Gamma_j^-$ or $\mathbf{T}(P_n(e^+)) \in \chi_j^-$, hence $P_n^{-1}(\mathbf{T}(P_n(e^+))) \in \Gamma^-_{i,n}$. Due to (3.3) we also notice that

$$f_n(e^+ \overset{\frown}{P_n^{-1}(\mathbf{T}}(P_n(e^+)))) = f(P_n(e^+)\overset{\frown}{\mathbf{T}}(P_n(e^+))) = 0.$$

We can use (3.7) again to deduce that for all $x \in \partial\Omega_n$ we have

$$|\mathbf{T}_n(x) - \mathbf{T}(P_n(x))| = |P^{-1}(\mathbf{T}(P_n(x))) - \mathbf{T}(P_n(x))| \leq d_H(\overline{\Omega}_n, \overline{\Omega}) \to 0, \quad \text{when } n \to \infty. \tag{3.8}$$

Let $\{e_i^+\}_{1\leq i\leq m}$ be any finite sequence of points such that $e_i^+ \in \chi_{i,n}^+ \cup \Gamma_{i,n}^+$. We recall that since Ω_n is strictly convex, then by Remark 3.9, it is sufficient to consider points belonging to different arcs $\Gamma_{i,n}^+$ or $\chi_{i,n}^+$. In this way, the index m appearing in (H3) will be at most the number of arcs $\chi_i^\pm$ and $\Gamma_i^\pm$, which is finite and does not depend on n. Now, applying (3.8) we see that

$$\begin{aligned}\sum_{i=1}^m |e_i^+ - \mathbf{T}_n(e_i^+)| &\leq \sum_{i=1}^m |e_i^+ - P_n(e_i^+)| + |P_n(e_i^+) - \mathbf{T}(P_n(e_i^+))| + |\mathbf{T}(P_n(e_i^+)) - \mathbf{T}_n(e_i^+)| \\ &\leq \sum_{i=1}^m |P_n(e_i^+) - \mathbf{T}(P_n(e_i^+))| + 2m\,\varepsilon_n,\end{aligned}$$

where $\varepsilon_n := d_H(\overline{\Omega}_n, \overline{\Omega})$. Moreover, we have the following inequality:

$$\begin{aligned}&\sum_{i=1}^{m-1} |P_n(e_i^+) - \mathbf{T}(P_n(e_{i+1}^+))| + |P_n(e_m^+) - \mathbf{T}(P_n(e_1^+))| \\ &\qquad\leq \sum_{i=1}^{m-1} \big(|P_n(e_i^+) - e_i^+| + |e_i^+ - \mathbf{T}_n(e_{i+1}^+)| + |\mathbf{T}_n(e_{i+1}^+) - \mathbf{T}(P_n(e_{i+1}^+))|\big) \\ &\qquad\qquad + |P_n(e_m^+) - e_m^+| + |e_m^+ - \mathbf{T}_n(e_1^+)| + |\mathbf{T}_n(e_1^+) - \mathbf{T}(P_n(e_1^+))| \\ &\qquad\leq \sum_{i=1}^{m-1} |e_i^+ - \mathbf{T}_n(e_{i+1}^+)| + |e_m^+ - \mathbf{T}_n(e_1^+)| + 2m\,\varepsilon_n.\end{aligned}$$

Since by assumption g satisfies (H3)$'$, there exists a $\delta_m > 0$ (independent of n) such that

$$\sum_{i=1}^m |P_n(e_i^+) - \mathbf{T}(P_n(e_i^+))| \leq \sum_{i=1}^{m-1} |P_n(e_i^+) - \mathbf{T}(P_n(e_{i+1}^+))| + |P_n(e_m^+) - \mathbf{T}(P_n(e_1^+))| - \delta_m.$$

These inequalities imply that g_n satisfies the assumption (H3) because for n large enough, we get the following inequality:

$$\begin{aligned}\sum_{i=1}^m |e_i^+ - \mathbf{T}_n(e_i^+)| &\leq \sum_{i=1}^{m-1} |e_i^+ - \mathbf{T}_n(e_{i+1}^+)| + |e_m^+ - \mathbf{T}_n(e_1^+)| - (\delta_m - 4m\,\varepsilon_n) \\ &< \sum_{i=1}^{m-1} |e_i^+ - \mathbf{T}_n(e_{i+1}^+)| + |e_m^+ - \mathbf{T}_n(e_1^+)|.\end{aligned}$$

Step 4. For the sake of studying the convergence of the sequence of measures $f_n \in \mathcal{M}(\partial\Omega_n)$, we need first to extend them all to a common domain. For this purpose, we fix any $n_0 \in \mathbb{N}$. For $n \geq n_0$, we define a Borel measure $\tilde{f}_n$ on $\overline{\Omega}_{n_0}$ by setting $\tilde{f}_n(A) := f_n(A \cap \partial\Omega_n)$, for every Borel set A. For the sake of a convenient notation, we will drop the tildes. Let f_n^+ and f_n^- be the positive and negative parts of f_n. Hence, we claim that $f_n^\pm \rightharpoonup f^\pm$. Indeed, for any continuous function φ on $\overline{\Omega}_{n_0}$, one has

$$\langle f_n^\pm, \varphi\rangle = \langle (P_n^{-1})_\# f^\pm, \varphi\rangle = \int_{\partial\Omega} \varphi(P_n^{-1}(x))\,\mathrm{d}f^\pm(x) \to \int_{\partial\Omega} \varphi(x)\,\mathrm{d}f^\pm(x) = \langle f^\pm, \varphi\rangle$$

because due to (3.8), $P_n^{-1}(x)$ converges to x, for all $x \in \Gamma \cup \chi$.

Let γ_n be an optimal transport plan in Problem (2.4) between f_n^+ and f_n^-, where $\overline{\Omega}_{n_0}$ plays now the role of $\overline{\Omega}$. By Lemma 2.4, we infer that up to a subsequence, γ_n weakly converges to a measure γ in $\mathcal{M}^+(\overline{\Omega}_{n_0} \times \overline{\Omega}_{n_0})$ with $(\Pi_x)_\#\gamma = f^+$, $(\Pi_y)_\#\gamma = f^-$. Moreover, γ is an optimal transportation plan between f^+ and f^-.

Yet, due to Proposition 3.2, we have that $\gamma_n = (Id, \mathbf{T}_n)_\# f_n^+$. We will show that $\gamma = (Id, \mathbf{T})_\# f^+$. Take $\xi \in C(\overline{\Omega} \times \overline{\Omega})$, then we have

$$\begin{aligned}\int_{\overline{\Omega}\times\overline{\Omega}} \xi(x,y)\,\mathrm{d}\gamma_n(x,y) &= \int_{\overline{\Omega}} \xi(x, \mathbf{T}_n(x))\,\mathrm{d}f_n^+(x) = \int_{\overline{\Omega}} \xi(x, \mathbf{T}_n(P_n^{-1}(x)))\,\mathrm{d}f^+(x) \\ &\to \int_{\overline{\Omega}} \xi(x, \mathbf{T}(x))\,\mathrm{d}f^+(x)\end{aligned}$$

where the convergence of $\mathbf{T}_n(P_n^{-1}(x))$ to $\mathbf{T}(x)$ follows from (3.8). This implies that $\gamma = (Id, \mathbf{T})_\# f^+$. □

We will see in the next proposition that one can relax (H3)′ in Proposition 3.12 by showing that (H3) is in fact sufficient to get that $\mathbf{T}$ is an optimal transport map.

Proposition 3.13. *Assume Ω is convex, (H1), (H2) & (H3) hold, $\mathcal{S}$ is finite and the number of arcs $\chi_i^\pm$ ($i \in I_\chi$) and $\Gamma_i^\pm$ ($i \in I_\Gamma$) is finite. Then, $\mathbf{T}$ is an optimal transport map from f^+ to f^-.*

Proof. We will construct a sequence $g_n \in C(\partial\Omega) \cap BV(\partial\Omega)$ satisfying (H3)′ and converging uniformly to g.

We set
$$t^\star := \min_{i \in I_\Gamma \cup I_\chi} \{TV(g \llcorner \chi_i^\pm), TV(g \llcorner \Gamma_i^\pm)\} > 0.$$
We fix $n \in \mathbb{N}^\star$ large enough so that $\frac{1}{n} < t^\star$. For every $i \in I_\chi$, there exists $c_{i,n}^\pm \in \chi_i^\pm$ such that $|g(c_{i,n}^\pm) - g(c_i^\pm)| = \frac{1}{n}$. For every $i \in I_\Gamma$, there are $a_{i,n}^\pm, b_{i,n}^\pm \in \Gamma_i^\pm$ such that $g(a_{i,n}^\pm) = g(a_i^\pm) + \frac{1}{n}$ and $g(b_{i,n}^\pm) = g(b_i^\pm) - \frac{1}{n}$. Set $I_\chi^n = I_\chi$, $I_\Gamma^n = I_\Gamma$ and $I_F^n = I_F \cup I_\chi \cup I_\Gamma$. So, we define the arcs
$$\chi_{i,n}^\pm = \overset{\frown}{c_i\, c_{i,n}^\pm} \;\; (i \in I_\chi^n), \qquad \Gamma_{i,n}^\pm = \overset{\frown}{a_{i,n}^\pm b_{i,n}^\pm} \;\; (i \in I_\Gamma^n),$$
and
$$F_{i,n} = \begin{cases} F_i & \text{if} \quad i \in I_F \\ \Gamma_i^\pm \setminus \Gamma_{i,n}^\pm & \text{if} \quad i \in I_\Gamma^n \\ \chi_i^\pm \setminus \chi_{i,n}^\pm & \text{if} \quad i \in I_\chi^n \end{cases}.$$

We define measure f_n on $\partial\Omega$ as follows:
$$f_n := \sum_{i \in I_\chi^n} f \llcorner \chi_{i,n} + \sum_{i \in I_\Gamma^n} f \llcorner \Gamma_{i,n}.$$
Then, we have $f_n \rightharpoonup f$. Now, set
$$g_n(x) = g(x_0) + f_n(\overset{\frown}{x_0 x}).$$

We construct the corresponding $\mathbf{T}_n$ as in (3.1). Since g satisfies (H3) and the arcs $\chi_{i,n}$ (resp. $\Gamma_{i,n}$) are compactly contained in the open arcs χ_i (resp. Γ_i) and $\mathbf{T}_n$ is continuous on the compact set $\overline{\Gamma_n^+} \cup \overline{\chi_n^+} \subset \Gamma^+ \cup \chi^+$ (see Lemma 3.1), then we deduce that (H3)′ is satisfied with this choice of boundary datum g_n. Thus, by Proposition 3.12, we get that $\mathbf{T}_n$ is an optimal transport map, i.e. $\gamma_n = (Id, \mathbf{T}_n)_\# f_n^+$ solves (2.4) between f_n^+ and f_n^-. We notice that the construction of new arcs yields that for every $x \in \Gamma_n^+ \cup \chi_n^+$, we have
$$\mathbf{T}_n(x) = \mathbf{T}(x). \tag{3.9}$$
Thus, one can show again that $\gamma_n \rightharpoonup \gamma = (Id, \mathbf{T})_\# f^+$ and γ is an optimal transport plan between f^+ and f^-. □

Now, we take another approximation in order to cover the case of convex domain with infinite number of arcs $\chi_i^\pm$ ($i \in I_\chi$) and $\Gamma_i^\pm$ ($i \in I_\Gamma$) and, possibly an infinite number of singular points. More precisely, we have the following:

Proposition 3.14. *Assume that Ω is convex and conditions (H1), (H2), (H3) and (S) hold. Then, $\mathbf{T}$ is an optimal transport map from f^+ to f^-.*

Proof. In the present case, we have a priori an infinite number of arcs $\chi_i^\pm$ and $\Gamma_i^\pm$ with an infinite number of singular points. However, due to Lemma 3.11, we may assume after a repartition of $\partial\Omega$, that all the arcs $\chi_i^\pm$ and $\Gamma_i^\pm$ are smooth. We then proceed again by approximation, but this time we approximate the trace functions g by g_n defined on $\partial\Omega$ and such that g_n has a finite number of smooth arcs $\chi_{i,n}^\pm$ and $\Gamma_{i,n}^\pm$. Namely, for a fixed $n \in \mathbb{N}^\star$, we set
$$I_\chi^n = \left\{ i \in I_\chi : \mathcal{H}^1(\chi_i) \geq \frac{1}{n} \right\}, \qquad I_\Gamma^n = \left\{ i \in I_\Gamma : \mathcal{H}^1(\Gamma_i^+ \cup \Gamma_i^-) \geq \frac{1}{n} \right\}$$
and
$$\chi_{i,n} = \chi_i \;\; (\text{for all } \; i \in I_\chi^n), \qquad \Gamma_{i,n}^\pm = \Gamma_i^\pm \;\; (\text{for all } \; i \in I_\Gamma^n),$$

$F_{i,n} = \chi_j$ (for some $j \in I_\chi \backslash I_\chi^n$), $F_{i,n} = \Gamma_j$ (for some $j \in I_\Gamma \backslash I_\Gamma^n$) or $F_{i,n} = F_j$ (for some $j \in I_F$). Now, we define measure f_n on $\partial\Omega$ as follows:

$$f_n := \sum_{i \in I_\chi^n} f \llcorner \chi_i + \sum_{i \in I_\Gamma^n} f \llcorner \Gamma_i.$$

We shall see that $f_n \rightharpoonup f$. Indeed, for any continuous function φ on $\partial\Omega$, we have that

$$\langle f_n, \varphi \rangle = \sum_{i \in I_\chi^n} \langle f \llcorner \chi_i, \varphi \rangle + \sum_{i \in I_\Gamma^n} \langle f \llcorner \Gamma_i, \varphi \rangle \longrightarrow \sum_{i \in I_\chi} \langle f \llcorner \chi_i, \varphi \rangle + \sum_{i \in I_\Gamma} \langle f \llcorner \Gamma_i, \varphi \rangle = \langle f, \varphi \rangle.$$

We note that $f_n(\partial\Omega) = 0$, for all $n \in \mathbb{N}^\star$. Fix $x_0 \in \partial\Omega$, so we define the trace function g_n on $\partial\Omega$ as follows:

$$g_n(x) := f_n(\widehat{x_0 x})$$

where $\widehat{x_0 x}$ is the arc from x_0 to x going in the positive orientation. So, we clearly have $g_n \in C(\partial\Omega) \cap BV(\partial\Omega)$ with $\partial_\tau g_n = f_n$. Let $\mathbf{T}_n$ be the transport map from f_n^+ to f_n^- defined as in (3.1). It is convenient to extend it to $\Gamma^+ \cup \chi^+$ by setting $\mathbf{T}_n(x) = x$, for every $x \in (\bigcup_{i \in I_\chi \backslash I_\chi^n} \chi_i^+) \cup (\bigcup_{i \in I_\Gamma \backslash I_\Gamma^n} \Gamma_i^+)$. It is obvious that $\mathbf{T}_n(x) = \mathbf{T}(x)$, for every $x \in \Gamma_n^+ \cup \chi_n^+$. It is also easy to see that the assumption (H3) is satisfied by g_n, because for every $i \in I_\chi^n$ (resp. $i \in I_\Gamma^n$), we have $\chi_{i,n} = \chi_i$ (resp. $\Gamma_{i,n} = \Gamma_i$). In particular, Proposition 3.13 yields that $\mathbf{T}_n$ is an optimal transport map from f_n^+ to f_n^-.

Let $\gamma_n = (Id, \mathbf{T}_n)_\# f_n^+$ be the optimal transport plan between f_n^+ and f_n^-. Now, we invoke Lemma 2.4 to deduce that up to a subsequence, $\gamma_n \rightharpoonup \gamma$, where $\gamma \in \mathcal{M}^+(\overline{\Omega} \times \overline{\Omega})$ is an optimal transportation plan between f^+ and f^-. We notice that for any $\xi \in C(\overline{\Omega} \times \overline{\Omega})$, we have

$$\int_{\overline{\Omega} \times \overline{\Omega}} \xi(x,y) \,\mathrm{d}\gamma_n(x,y) = \int_{\overline{\Omega}} \xi(x, \mathbf{T}_n(x)) \,\mathrm{d}f_n^+(x) = \int_{\overline{\Omega}} \xi(x, \mathbf{T}(x)) \,\mathrm{d}f_n^+(x) \to \int_{\overline{\Omega}} \xi(x, \mathbf{T}(x)) \,\mathrm{d}f^+(x).$$

Hence, $\gamma = (Id, \mathbf{T})_\# f^+$. □

Remark 3.15. We have seen that the projection map P is easy to visualize geometrically, but the set $\mathcal{S}$ leads to technical difficulties when we deal with P. However, we could introduce a more complicated map that does not take into account the presence of singular points on $\partial\Omega$ and, we will do so later in the proof of Proposition 4.12.

From Proposition 3.14, we know that $\gamma = (Id, \mathbf{T})_\# f^+$ is an optimal transport plan between f^+ and f^-. In fact, one can show that this is a unique optimal transport plan in (2.4). More precisely, we have the following:

Proposition 3.16. *Under the assumptions that Ω is convex and (H1), (H2), (H3) and (S) hold, Problem (2.4) has a unique optimal transport plan $\gamma = (Id, \mathbf{T})_\# f^+$.*

Proof. Let γ be an optimal transport plan in Problem (2.4) and let us assume that $\gamma \neq (Id, \mathbf{T})_\# f^+$, i.e. the set

$$A = \left\{ x \in \partial\Omega : \exists\, y \in \partial\Omega,\ y \neq \mathbf{T}(x),\ (x,y) \in \operatorname{spt}(\gamma) \right\}$$

has a positive measure, i.e., $f^+(A) > 0$. At the same time, by Proposition 3.14, $]x, \mathbf{T}(x)[\subset \Omega$ is a transport ray for f^+-a.e. x. We know that two transport rays cannot intersect at an interior point of one of them. As a result, we get that $A \subset \mathcal{N}$. Yet, thanks to the convexity of Ω (see Lemma 2.5), $\mathcal{N}$ is at most countable. Hence, this yields that $f^+(A) = 0$, which is a contradiction. We conclude that $\gamma = (Id, \mathbf{T})_\# f^+$ is the unique optimal transport plan in Problem (2.4). □

In the sequel, we set $\gamma := (Id, \mathbf{T})_\# f^+$. Thanks to the convexity of Ω, the vector measure v_γ given by (2.7) (resp. the transport density σ_γ (2.8)) is well defined. Moreover, due to condition (H2), one has

$$|v_\gamma|(\partial\Omega) = \sigma_\gamma(\partial\Omega) = \int_{\partial\Omega} \mathcal{H}^1([x, \mathbf{T}(x)] \cap \partial\Omega) \,\mathrm{d}f^+(x) = 0. \tag{3.10}$$

Hence, we get the following:

Proposition 3.17. *Assume that Ω is convex and (H1), (H2), (H3) & (S) hold. Then, v_γ is the unique solution for Problem (2.2).*

Proof. Thanks to [15, Chapter 4] and the fact that Ω is assumed to be convex, we have that (2.2) = (2.4) and, v_γ is a minimizer for Problem (2.2). Moreover, we recall that if v is another minimizer in (2.2), then there will be an optimal transport plan γ' in (2.4) such that $v = v_{\gamma'}$ (we refer the reader to [15, Theorem 4.13] for more details). Yet, by Proposition 3.16, we know that the optimal transport plan γ is unique. Hence, Problem (2.2) has a unique solution as well. □

Finally, we are ready to state the first of our main theorems in this section:

Theorem 3.18. *Assume that Ω is convex, $g \in BV(\partial\Omega) \cap C(\partial\Omega)$ and conditions (H1), (H2), (H3), and (S) are satisfied. Then, the least gradient problem* (1.1) *has a unique solution.*

Proof. Due to Proposition 2.3, Problems (1.1) and (2.2) are equivalent. As a result, our claim follows immediately from Proposition 3.17 and the fact that $|v_\gamma|(\partial\Omega) = 0$, see (3.10). □

Now, we present an example showing that violation of (H3) might lead to the non-existence of solutions to Problem (1.1).

Example 3.19. Let $\Omega := (0,1)^2$ be the unit square and define a function $g \in BV(\partial\Omega) \cap C(\partial\Omega)$ as follows

$$g(x_1, x_2) := \begin{cases} \min\{x_1,\, \delta,\, 1 - x_1\} & \text{if} \quad x_1 \in [0,1], \;\; x_2 = 0 \;\; \text{or} \;\; x_2 = 1, \\ \min\{x_2,\, \delta,\, 1 - x_2\} & \text{if} \quad x_2 \in [0,1], \;\; x_1 = 0 \;\; \text{or} \;\; x_1 = 1, \end{cases}$$

where

$$\delta \in \left] \frac{1}{2+\sqrt{2}}, \frac{1}{2} \right[.$$

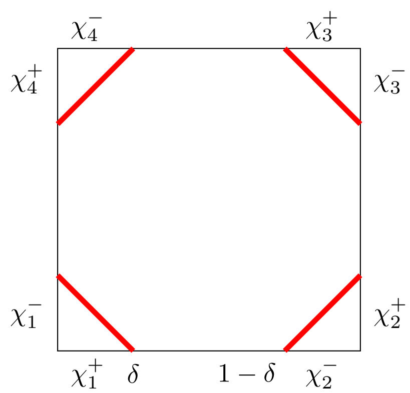

FIGURE 2

We notice that in this case, the boundary of Ω can be divided into arcs $\chi_i^\pm$ with $i = 1, ..., 4$ (as shown in Figure 1):

$$\begin{aligned} \chi_1^+ &:= \{(x_1, 0) : 0 \le x_1 \le \delta\}, & \chi_1^- &:= \{(0, x_2) : 0 \le x_2 \le \delta\}, \\ \chi_2^+ &:= \{(1, x_2) : 0 \le x_2 \le \delta\}, & \chi_2^- &:= \{(x_1, 0) : 1-\delta \le x_1 \le 1\}, \\ \chi_3^+ &:= \{(x_1, 1) : 1-\delta \le x_1 \le 1\}, & \chi_3^- &:= \{(1, x_2) : 1-\delta \le x_2 \le 1\}, \\ \chi_4^+ &:= \{(0, x_2) : 1-\delta \le x_2 \le 1\}, & \chi_4^- &:= \{(x_1, 1) : 0 \le x_1 \le \delta\}. \end{aligned}$$

Due to the definition of g, the measures f^+, f^- are given by $f^+ = \mathcal{H}^1 \llcorner (\chi_1^+ \cup \chi_2^+ \cup \chi_3^+ \cup \chi_4^+)$ and $f^- = \mathcal{H}^1 \llcorner (\chi_1^- \cup \chi_2^- \cup \chi_3^- \cup \chi_4^-)$.

Now, we claim that there are transport rays between f^+ and f^- which are contained in the boundary of Ω. Indeed, if this was not the case then it would not be difficult to see that the line segments $[(\delta,0),(0,\delta)]$, $[(1-\delta,0),(1,\delta)]$, $[(1,1-\delta),(1-\delta,1)]$ and $[(\delta,1),(0,1-\delta)]$ must be transport rays. Hence, we should have the following inequality:

$$\begin{aligned} &|(\delta,0)-(0,\delta)| + |(1-\delta,0)-(1,\delta)| + |(1,1-\delta)-(1-\delta,1)| + |(\delta,1)-(0,1-\delta)| \\ &\qquad \le |(\delta,0)-(1-\delta,0)| + |(1,\delta)-(1,1-\delta)| + |(1-\delta,1)-(\delta,1)| + |(0,1-\delta)-(0,\delta)|, \end{aligned}$$

which is not possible since $\delta > \frac{1}{2+\sqrt{2}}$. In particular, the assumption (H3) is not satisfied and, a solution u does not exist. On the other hand, we see that (H3) is satisfied as soon as $\delta \le \frac{1}{2+\sqrt{2}}$. Thus, we infer that Problem (1.1) has a unique solution, provided that this inequality holds.

From Theorem 3.18, we have already seen that if (H1), (H2), (H3) and (S) are satisfied, then Problem (1.1) attains a minimum. In other words, these conditions are sufficient to get existence of a solution to Problem (1.1).

In the following example the data violates condition (H1) but (1.1) has a solution, hence (H1) is not necessary for existence.

Example 3.20. Let $\Omega = [0,1]^2$ and $C : [0,1] \mapsto [0,1]$ be the Cantor function. Then, we define the boundary datum g on $\partial\Omega$ as follows:

$$g(x_1,x_2) = \begin{cases} C(x_1) & \text{if} \quad 0 \le x_1 \le 1 \quad \text{and} \quad x_2 = 0 \quad \text{or} \quad x_2 = 1, \\ 0 & \text{if} \quad 0 \le x_2 \le 1 \quad \text{and} \quad x_1 = 0, \\ 1 & \text{if} \quad 0 \le x_2 \le 1 \quad \text{and} \quad x_1 = 1. \end{cases}$$

With this choice of g, it is clear that the least gradient problem (1.1) admits a solution $u(x_1,x_2) = C(x_1)$. Indeed, let us define the transport map

$$T(x_1,0) = (x_1,1).$$

The Kantorovitch potential is given by $\phi(x_1,x_2) = -x_2$, because

$$1 = \int_\Omega \phi \,\mathrm{d}(f^+ - f^-) \le \sup_{\mathrm{Lip}(\psi)\le 1} \int_\Omega \psi \,\mathrm{d}(f^+ - f^-) = \int_0^1 (\psi(x_1,0) - \psi(x_1,1)) \,\mathrm{d}f^+ \le 1.$$

Moreover,

$$\int_\Omega \phi \,\mathrm{d}(f^+ - f^-) = \int_\Omega |x - T(x)| \,\mathrm{d}f^+.$$

Notice that for such a function g, (H1) is not satisfied.

In the example above, sets I_χ and I_Γ are empty; but g is monotone on each side of the square and $\partial_\tau g$ is a singular measure with $\int_{\partial\Omega} |\partial_\tau g| = 2$. These observations suggest the following definition.

Definition 3.21. We say that a function $g \in W^{1,1}(\partial\Omega)$ is *piecewise monotone* provided that $\partial\Omega$ may be decomposed into disjoint sets such that $\partial\Omega = U^+ \cup U^- \cup U_0$, where $U^\pm$ are open and $\partial_\tau g > 0$ a.e. on U^+, $\partial_\tau g < 0$ a.e. on U^- and $\int_{U_0} |\partial_\tau g| \, ds = 0$.

If we keep this definition in mind, then one can show that (H1), (H2), and (H3) are all at the same time necessary conditions for the solvability of the least gradient problem for piecewise monotone data. This is the content of the second of our main theorems:

Theorem 3.22. *Let Ω be a convex domain and g in $W^{1,1}(\partial\Omega)$ be piecewise monotone. Assume that Problem (1.1) has a solution u. Then, the boundary datum g must satisfy conditions (H1), (H2) and (H3).*

Proof. Step 1. First, we define $v := R_{\frac{\pi}{2}} Du$. Due to Proposition 2.3 the vector field v turns out to be a solution for problem (2.2). At the same time, by [15, Theorem 4.13], there will be an optimal transport plan γ for Problem (2.4) such that $v = v_\gamma$. Since u is a solution to Problem (1.1), then no level set $\partial\{u \ge t\}$ of u is contained in the boundary $\partial\Omega$. In optimal transport terms, this means that there are no transport rays between f^+ and f^- which are contained in $\partial\Omega$ (we recall that f^+ and f^- are the positive and negative parts of f, the tangential derivative of g).

Let Δ be the interior of the set where u is locally constant or equivalently, where the transport density $\sigma = |v|$ vanishes. This set has at most a countable number of disjoint connected components denoted by Δ_i, $i \in I_\Delta$, with positive Lebesgue measures. We shall introduce $I_F = \{i \in I_\Delta : \mathcal{H}^1(\partial\Delta_i \cap \partial\Omega) > 0\}$. Then, we define the flat parts as follows:

$$F_i = \partial\Delta_i \cap \partial\Omega, \qquad i \in I_F.$$

We shall see that for any i, the set $\Omega \cap \partial\Delta_i$ is composed of transport rays. Indeed, if $z \in \Omega \cap \partial\Delta_i$ then there will be a sequence $z_n \in \Omega \setminus \overline{\Delta}$ such that $z_n \to z$. Since $z_n \notin \overline{\Delta}$ then there will be a transport ray $\mathcal{R}_n = [x_n, y_n]$ such that $z_n \in \mathcal{R}_n$. Yet, after extracting a subsequence (not relabeled), these transport rays $\mathcal{R}_n$ converge to a line segment $\mathcal{R} = [x,y]$, where $x_n \to x$ and $y_n \to y$. Moreover, $z \in \mathcal{R}$ and due to Lemma 2.4 (2) applied to constant sequences $f_n^\pm = f^\pm$, we infer that $\mathcal{R}$ is a transport ray. In particular, $\partial\Delta_i$ intersects $\partial\Omega$. At the same time, we see that the set $\partial\Omega\setminus\overline{\bigcup_{i\in I_\Delta}\Delta_i}$ is a sum of open arcs, let us call it $\bigcup_{j\in J}\alpha_j$, where $\alpha_j\subset\partial\Omega$ and J is at most countable index set.

Step 2. Let us suppose that g has at least one strict local minimum or maximum. If this is not the case we will proceed to Step 3. We take c_k a strict local minimum or maximum of g. Since $c_k\in\partial\Omega\setminus\overline{\bigcup_{i\in I_\Delta}\Delta_i}$, then there is $j\in J$ so that $c_k\in\alpha_j$. Moreover, there will be an open arc $\tilde{\chi}_k:=\overset{\frown}{c_k^+c_k^-}$ around c_k, where g is strictly increasing on $\chi_k^+:=\overset{\frown}{c_k\,c_k^+}$ and strictly decreasing on $\chi_k^-:=\overset{\frown}{c_k\,c_k^-}$. We notice that $[c_k^+,c_k^-]$ is a transport ray. This follows from the fact that any level set of a solution to (1.1) is a transportation ray. Let us denote by $D_k\subset\Omega$ the convex hull of χ_k, we notice that $\partial D_k\cap\Omega=[c_k^+,c_k^-]$.

Step 3. We define an open set $T:=\Omega\setminus\left(\overline{\bigcup_{i\in I_\chi}D_i}\cup\overline{\bigcup_{i\in I_\Delta}\Delta_i}\right)$. We claim that any open connected component of T is of the form T_i (see condition (H1)), i.e. it is the convex hull of two open arcs Γ_i^+ and Γ_i^-, where g is strictly increasing on Γ_i^+ and strictly decreasing on Γ_i^- with $TV(g_{|\Gamma_i^+})=TV(g_{|\Gamma_i^-})$ and $\operatorname{dist}(\Gamma_i^+,\Gamma_i^-)>0$.

Let C be an open connected component of T. We claim that $\partial C\cap\partial\Omega$ consists of a sum of two disjoint closed arcs. First, it is easy to see that $\partial C\cap\Omega$ consists of transportation rays, hence C is convex.

Since $C\cap\overline{\Delta}=\emptyset$, then the interior relative to $\partial\Omega$ of any arc of $\partial C\cap\partial\Omega$ does not contain any multiple point, otherwise there will be an $i\in I_\Delta$ such that Δ_i divides C into two parts but this is a contradiction because C is connected. By the same argument, one can see that the interior of any arc of $\partial C\cap\partial\Omega$ does not also intersect the flat part F of g. Moreover, it is clear that g does not attain a strict local minimum/maximum in the interior of $\partial C\cap\partial\Omega$. Hence, on any arc of $\partial C\cap\partial\Omega$, the boundary datum g is either strictly increasing or strictly decreasing. Hence, any point $x^+\in\partial C\cap\partial\Omega$ is an endpoint of a transportation ray $[x^+,x^-]$, where $x^-\in\partial C\cap\partial\Omega$.

Since the set $U=\partial\Omega\setminus\left(\overline{\bigcup_{i\in I_\chi}D_i}\cup\overline{\bigcup_{i\in I_\Delta}\Delta_i}\right)$ is open it is a sum of open disjoint arcs. We set

$$A_C=\{\alpha\subset U:\alpha\text{ is an open connected component and }\alpha\cap\partial C\neq\emptyset\}.$$

It is clear that if $\alpha\in A_C$ then $\overline{\alpha}\subset\partial C$. Now, we claim that if $\alpha\neq\beta$ are both in A_C, then $\overline{\alpha}\cap\overline{\beta}=\emptyset$. Let us suppose otherwise, i.e. $\overline{\alpha}\cap\overline{\beta}=\{p\}$. Hence, g is strictly monotone on $\theta:=\alpha\cup\{p\}\cup\beta$, for otherwise we have an arc of type χ. Then, we have two possibilities: (i) either p is a multiple point, or (ii) p is not a multiple point.

In the first case, we see that there will be two transportation rays $\mathcal{R}_1=[p,q_1]$ and $\mathcal{R}_2=[p,q_2]$, where q_1, $q_2\in\partial C\cap\partial\Omega$ and $q_1\neq q_2$. Yet, these rays $\mathcal{R}_1$ and $\mathcal{R}_2$ separate two components of C, which was assumed to be connected, a contradiction.

If (ii) holds, then there will be a unique transportation ray $\mathcal{R}=[p,q]$ starting at p such that q belongs to $\partial C\cap\partial\Omega$. Since g is strictly monotone on θ, this ray $]p,q[$ must be contained in C. Suppose that $p_n^\alpha\in\alpha$ (resp. $p_n^\beta\in\beta$) is a sequence of points converging to p. We take the rays emanating from these points, $[p_n^\alpha,q_n^\alpha]$, $[p_n^\beta,q_n^\beta]$. Moreover, by strict monotonicity of g, these rays are contained in C and we have

$$\lim_{n\to\infty}q_n^\alpha=q=\lim_{n\to\infty}q_n^\beta,$$

since otherwise we would reach a contradiction with the fact that p is not a multiple point. Thus, there is a ball $B(p,r)$, which does not contain any point from $\overline{\bigcup_{i\in I_\chi}D_i}\cup\overline{\bigcup_{i\in I_\Delta}\Delta_i}$, but this yields again a contradiction with the definition of p. Hence, we conclude that $\overline{\alpha}\cap\overline{\beta}=\emptyset$.

Now, take any arc $\alpha_1\subset\partial C\cap\partial\Omega$ and any point x^+ in the interior of α_1. Assume that $[x^+,x^-]$ is a transport ray. As we noted above g is strictly monotone on α_1 so that x^- cannot belong to α_1. Thus, $x^-\in\alpha_2\subset\partial C\cap\partial\Omega$, where $\overline{\alpha_1}\cap\overline{\alpha_2}=\emptyset$. Then, all the other points in the interior of α_1 must be transported to the same arc α_2 because otherwise there would be a multiple point inside α_1, which is a contradiction as we already showed that there are no multiple points inside any arc of $\partial C\cap\partial\Omega$. Consequently, $\partial C\cap\partial\Omega$ can be decomposed into two arcs Γ^+ and Γ^-, where g is strictly increasing on Γ^+ and strictly decreasing on Γ^- with $TV(g_{|\Gamma^+})=TV(g_{|\Gamma^-})$ and $\operatorname{dist}(\Gamma^+,\Gamma^-)>0$.

Step 4. Set

$$I_\Gamma=\{i\in I_\Delta:T_i\ \text{ is a connected component of }T\}.$$

Since sets T_i, $i\in I_\Gamma$, are disjoint and open, then set I_Γ is at most countable. It is also clear that the sets T_i ($i\in I_\Gamma$) and D_j ($j\in I_\chi$) are mutually disjoint. Hence, the condition (H1) is satisfied.

Moreover, we obviously have that for every $x^+ \in \chi_i^+$, $i \in I_\chi$ (resp. $x^+ \in \Gamma_i^+$, $i \in I_\Gamma$), the line segment $[x^+, \mathbf{T}(x^+)]$ is a transport ray (see Lemma 3.5), which is contained in Ω. Hence, (H2) holds as well.

Finally, (H3) is also satisfied. Consider any finite sequence of points $\{e_k^+\}_{1 \le k \le m}$ (where $m \in \mathbb{N}$) such that $e_k^+ \in \chi_{i_k}^+ \cup \Gamma_{i_k}^+$, for some $i_k \in I_\chi \cup I_\Gamma$. Then, we have $(e_k^+, \mathbf{T}(e_k^+)) \in \mathrm{spt}(\gamma)$, for every $1 \le k \le m$ and so, thanks to the equality (2.6) which is satisfied by the Kantorovich potential ϕ, we get the following identity:

$$\sum_{k=1}^{m} |e_k^+ - \mathbf{T}(e_k^+)| = \sum_{k=1}^{m} \phi(e_k^+) - \phi(\mathbf{T}(e_k^+)) \;=\; \sum_{k=1}^{m-1} [\phi(e_k^+) - \phi(\mathbf{T}(e_{k+1}^+))] + \phi(e_m^+) - \phi(\mathbf{T}(e_1^+)).$$

Recalling the proof of Proposition 3.16, we see that $[e_k^+, \mathbf{T}(e_{k+1}^+)]$ is not a transport ray and then, we have $\phi(e_k^+) - \phi(\mathbf{T}(e_{k+1}^+)) < |e_k^+ - \mathbf{T}(e_{k+1}^+)|$. Consequently, we get

$$\sum_{k=1}^{m} |e_k^+ - \mathbf{T}(e_k^+)| < \sum_{k=1}^{m-1} |e_k^+ - \mathbf{T}(e_{k+1}^+)| + |e_m^+ - \mathbf{T}(e_1^+)|.$$

This concludes the proof. $\square$

4. Sufficient conditions for existence and uniqueness in the non-convex case

In this section, we will extend the equivalence between Problems (2.2) and (2.4) to the general case of a bounded, simply connected, not necessarily convex domain, $\Omega \subset \mathbb{R}^2$, however, some admissibility conditions on the Dirichlet datum g are imposed. We stress that the assumption of the piecewise monotonicity of $g \in C(\partial\Omega)$ is a standing assumption. Then, we will be able to show, under these conditions, the existence and uniqueness of a solution to the least gradient problem (1.1). We stress that the condition (S), which is a regularity assumption on $\partial\Omega$ is in force (but, we will see in Proposition 4.12 that one can relax this condition).

To begin with, we introduce the following assumptions:

• **Condition (L1)**. The domain Ω can be decomposed into convex disjoint sets $\tilde{C}_i$ ($i \in I_C$) and disjoint open sets X_i ($i \in I_X$) such that g is constant on $\partial X_i \cap \partial\Omega$. In order to use the setting of Section 3 we need data (C_i, g_i), $i \in I_C$, where C_i is open and convex, $g_i \in C(\partial C_i)$. Namely, we set $C_i := \tilde{C}_i^{\circ}$ and $g_i(x) = g(x_i) + (f \llcorner (\partial C_i \cap \partial\Omega))(\widehat{x_i x})$ and $x_i \in \partial C_i \cap \partial\Omega$ is fixed. Of course, sets C_i are disjoint. Moreover, we have the following:

(1) For every $i \in I_C$, we assume that $g_i \in C(\partial C_i)$ satisfies the condition (H1), i.e. $\partial C_i \cap \partial\Omega$ can be decomposed into open arcs ${\Gamma_j^i}^{\pm}$ ($j \in I_\Gamma^i$), ${\chi_j^i}^{\pm}$ ($j \in I_\chi^i$) and F_j^i ($j \in I_F^i$), satisfying all the points of (H1) (see Section 3). We note that g_i is constant on each component of $\partial C_i \cap \Omega$.

In the sequel, we also use the following notations: $\Gamma_j^i = {\Gamma_j^i}^+ \cup {\Gamma_j^i}^-$ ($i \in I_C$, $j \in I_\Gamma^i$), $\chi_j^i = {\chi_j^i}^+ \cup {\chi_j^i}^-$ ($i \in I_C$, $j \in I_\chi^i$), ${\Gamma^i}^{\pm} = \bigcup_{j \in I_\Gamma} {\Gamma_j^i}^{\pm}$, ${\chi^i}^{\pm} = \bigcup_{j \in I_\chi} {\chi_j^i}^{\pm}$, $\Gamma^i = {\Gamma^i}^+ \cup {\Gamma^i}^-$, $\chi^i = {\chi^i}^+ \cup {\chi^i}^-$ ($i \in I_C$), $\Gamma^{\pm} = \bigcup_{i \in I_C} {\Gamma^i}^{\pm}$, $\chi^{\pm} = \bigcup_{i \in I_C} {\chi^i}^{\pm}$, $\Gamma = \Gamma^+ \cup \Gamma^-$ and, $\chi = \chi^+ \cup \chi^-$. This will help us express the next conditions.

We will note further simple consequences of the assumption (L1).

Lemma 4.1. *Let us suppose that C_i, $i \in I_C$, is one of the sets defined above, then $f(\partial C_i) = 0$.*

Proof. First, we note that $f(\partial C_i) = f(\partial C_i \cap \partial\Omega)$. By definition of C_i, the set $\partial C_i \cap \partial\Omega$ has the following structure:

$$\partial C_i \cap \partial\Omega = \bigcup_{j \in I_\chi^i} ({\chi_j^i}^+ \cup {\chi_j^i}^-) \cup \bigcup_{j \in I_\Gamma^i} ({\Gamma_j^i}^+ \cup {\Gamma_j^i}^-) \cup \bigcup_{j \in I_F^i} F_j^i \cup N^i,$$

where $N^i = (\partial C_i \cap \partial\Omega) \setminus (\Gamma^i \cup \chi^i \cup \bigcup_{j \in I_F^i} F_j^i)$. But, by the assumption that g is piecewise monotone, we see that this set N^i is a null set with respect to the measure $|f|$. Hence, by (L1), we get that

$$f(\partial C_i \cap \partial\Omega) = \sum_{j \in I_\chi^i} f({\chi_j^i}^+ \cup {\chi_j^i}^-) + \sum_{j \in I_\Gamma^i} f({\Gamma_j^i}^+ \cup {\Gamma_j^i}^-) = 0. \quad \square$$

We follow the same strategy of construction of solutions as in the previous section. Namely, we begin by introducing a transport map $\mathbf{T} : \Gamma^+ \cup \chi^+ \mapsto \Gamma^- \cup \chi^-$. In the first step of the construction, we define $\mathbf{T}^i$ on ∂C_i with the help of the formula (3.1). We can do this due to Lemma 4.1. This definition of $\mathbf{T}^i$ implies in particular that $\mathbf{T}^i$ has all the properties proved for the map $\mathbf{T}$ in Section 3, (as usual, $\mathbf{T}^{i[-1]}$ denotes the inverse of $\mathbf{T}^i$).

After this preparation, we set

$$\mathbf{T}(x^+) = \mathbf{T}^i(x^+), \qquad \text{whenever } x^+ \in (\Gamma^+ \cup \chi^+) \cap C_i. \tag{4.1}$$

Since the sets C_i are disjoint, then we see that this map $\mathbf{T}$ is well-defined. Here, come our next requirements on the boundary datum g:

(2) For every $i \in I_C$, the restriction of g to $\partial C_i \cap \partial\Omega$ satisfies the condition (H2), i.e. for all $x^+ \in \Gamma^+ \cup \chi^+$, we have $]x^+, \mathbf{T}(x^+)[\subset \Omega$.

(3) For every $i \in I_C$, g satisfies the condition (H3) on $\partial C_i \cap \partial\Omega$, i.e. for any sequence of points $\{e_k^+\}_{1\le k\le m}$ (where $m \in \mathbb{N}$) such that $e_k^+ \in {\chi_{j_k}^i}^+ \cup {\Gamma_{j_k}^i}^+$ (for some $j_k \in I_\chi^i \cup I_\Gamma^i$), we have the following inequality

$$\sum_{k=1}^{m} |e_k^+ - \mathbf{T}^i(e_k^+)| \;<\; \sum_{k=1}^{m-1} |e_k^+ - \mathbf{T}^i(e_{k+1}^+)| + |e_m^+ - \mathbf{T}^i(e_1^+)|.$$

Remark 4.2. We note that these conditions encompassed in (L1) impose important restrictions on the geometry of $\partial\Omega$. Namely, even if the domain Ω is not convex, (L1) implies that the arcs of $\partial C_i \cap \partial\Omega$ must have non-negative curvature for all $i \in I_C$. As a result the $D = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in (-1,1),\ x_2 \in (\sqrt{1-x_1^2}, \sqrt{1-x_1^2}+1)\}$ does not satisfy (L1) as soon as $g \in C(\partial D)$ is not constant on the graph of $x_1 \in (-1,1) \mapsto \sqrt{1-x_1^2}$. However, we will also cover in Proposition 4.8 the case when some arcs of $\mathrm{spt}(f)$ have negative curvature.

Since Ω is not convex, then we will also need an additional condition that guarantees that the boundary of each set C_i ($i \in I_C$) is transported to itself. Namely, we assume the following:

- **Condition (L2)**. Let $\{{e_k}^\pm\}_{1\le k\le m}$ (where $m \in \mathbb{N}$) be two finite sequences of points such that ${e_k}^\pm \in {\chi^{i_k}}^\pm \cup {\Gamma^{i_k}}^\pm$, with $i_k \ne i_{k'}$ for all $k \ne k'$. Then, we assume the following additional inequality:

$$\sum_{k=1}^{m} |{e_k}^+ - {e_k}^-| < \sum_{k=1}^{m-1} |{e_k}^+ - {e_{k+1}}^-| + |{e_m}^+ - {e_1}^-|.$$

We stress that here e_k^+ and e_k^- are two arbitrary points on ∂C_{i_k}. In particular, e_k^- needs not to be the image of e_k^+ under the map $\mathbf{T}$.

Proposition 4.3. *Assume that conditions (L1) and (L2) hold. Let γ be an optimal transport plan for Problem (2.4), then for $\gamma-$a.e. $(x^+, x^-) \in \mathrm{spt}(\gamma)$, there exists $i \in I_C$ such that $x^- = \mathbf{T}^i(x^+)$. In particular, the optimal transport plan γ is unique and, we have $\gamma = (Id, \mathbf{T})_\# f^+$.*

Proof. We claim that $\gamma(\bar{C}_i \times \bar{C}_j) = 0$ for all $i \ne j$. The proof is similar to the argument of Proposition 3.2 and it is performed in Steps 1.1 till 1.3.

Step 1.1. Let us assume that there is a couple $(e_1^+, e_2^-) \in \mathrm{spt}(\gamma)$ with $e_1^+ \in {\Gamma^{i_1}}^+ \cup {\chi^{i_1}}^+$ but $e_2^- \in {\Gamma^{i_2}}^- \cup {\chi^{i_2}}^-$, where $i_1 \ne i_2$. Then, we are going to construct two sequences of points in $\Gamma^\pm \cup \chi^\pm$,

$$e_1^+,\, e_2^+,\, \dots \quad \text{and} \quad e_2^-,\, e_3^-,\, \dots$$

such that $(e_k^+, e_{k+1}^-) \in \mathrm{spt}(\gamma)$, $e_k^+ \in {\Gamma^{i_k}}^+ \cup {\chi^{i_k}}^+$ and $e_{k+1}^- \in {\Gamma^{i_{k+1}}}^- \cup {\chi^{i_{k+1}}}^-$, where $i_k \ne i_{k+1}$.

Let us suppose that we have $e_k^- \in {\Gamma^{i_k}}^- \cup {\chi^{i_k}}^-$ and $e_{k-1}^+ \in {\Gamma^{i_{k-1}}}^+ \cup {\chi^{i_{k-1}}}^+$. We will construct (e_k^+, e_{k+1}^-). Since f is atomless, then there is a Borel set $G_k^- \subset {\Gamma^{i_k}}^- \cup {\chi^{i_k}}^-$ containing e_k^- with $f^-(G_k^-) > 0$, which was transported from a set $G_{k-1}^+ \subset {\Gamma^{i_{k-1}}}^+ \cup {\chi^{i_{k-1}}}^+$ containing e_{k-1}^+. Yet, thanks to Lemma 4.1, we know that $f(\partial C_{i_k}) = 0$. Hence, the mass imported into C_{i_k} must be balanced with an equal outflow of the mass. Then, there will be a Borel set $G_k^+ \subset {\Gamma^{i_k}}^+ \cup {\chi^{i_k}}^+$ with $0 < f^+(G_k^+) \le f^-(G_k^-)$, which is transported to a set $G_{k+1}^- \subset {\Gamma^{i_{k+1}}}^- \cup {\chi^{i_{k+1}}}^-$. So, let us just pick any couple $(e_k^+, e_{k+1}^-) \in (G_k^+ \times G_{k+1}^-) \cap \mathrm{spt}(\gamma)$. In this way, we get sequences with the desired properties.

Step 1.2. Let us assume that the index set I_C is finite. Now, we claim that $i_k \ne i_{k'}$, for all $k \ne k'$. Let us suppose that there exist $l, m \ge 1$ such that $e_l^\pm,\, e_{l+m+1}^\pm \in {\Gamma^{i_l}}^\pm \cup {\chi^{i_l}}^\pm$ (i.e., one has

$i_l = i_{l+m+1}$). Let ϕ be again a Kantorovich potential between f^+ and f^-. Then, thanks to the equality (2.6), we get that

$$\sum_{k=l}^{l+m-1} |e_k^+ - e_{k+1}^-| + |e_{l+m}^+ - e_{l+m+1}^-| = \sum_{k=l}^{l+m-1} [\phi(e_k^+) - \phi(e_{k+1}^-)] + [\phi(e_{l+m}^+) - \phi(e_{l+m+1}^-)]$$
$$= [\phi(e_l^+) - \phi(e_{l+m+1}^-)] + \sum_{k=l+1}^{l+m} [\phi(e_k^+) - \phi(e_k^-)]$$
$$\leq |e_l^+ - e_{l+m+1}^-| + \sum_{k=l+1}^{l+m} |e_k^+ - e_k^-|.$$

However, this contradicts the inequality in the assumption (L2).

Step 1.3. Finally, it remains to consider the case when $I_C = \mathbb{N}^\star$. From now on, for the sake of simplicity of notation we assume that $i_k = k$.

Since we assumed that our claim does not hold, there is an arc $E_1^+ \subset \partial C_1$ and another one $E_2^- \subset \partial C_2$ with $\gamma(E_1^+ \times E_2^-) = c > 0$. Set $\gamma_1 = \gamma \llcorner (E_1^+ \times E_2^-)^c$, $f_1^+ = (\Pi_x)_\# \gamma_1$ and $f_1^- = (\Pi_y)_\# \gamma_1$. Then, we have

$$f_1^\pm(\partial\Omega) = f^\pm(\partial\Omega) - c.$$

Since $\partial C_1 \subset \partial\Omega \setminus E_2^-$, then we also have

$$f_1^-(\partial\Omega) \geq f^-(\partial C_1) = f^+(\partial C_1) \geq f^+(E_1^+) \geq c.$$

Yet, $f(\partial C_2) = 0$. Then, there will be a set $E_2^+ \subset \partial C_2$ with another set $E_3^- \subset \cup_{k \in I_3} \partial C_k$ such that $\gamma(E_2^+ \times E_3^-) = c$. Again, we define $\gamma_2 = \gamma_1 \llcorner (E_2^+ \times E_3^-)^c$, $f_2^+ = (\Pi_x)_\# \gamma_2$ and $f_2^- = (\Pi_y)_\# \gamma_2$. Thanks to $\partial C_1 \subset \partial\Omega \setminus (E_2^- \cup E_3^-)$, we have

$$f_2^\pm(\partial\Omega) = f_1^\pm(\partial\Omega) - c = f^\pm(\partial\Omega) - 2c \qquad \text{and} \qquad f_2^-(\partial\Omega) \geq f^-(\partial C_1) \geq c.$$

Fix $n \geq 3$. By induction: since $f(\partial C_k) = 0$ for all $k \in I_n$ then there will be a set $E_n^+ \subset \cup_{k \in I_n} \partial C_k$ and another set $E_{n+1}^- \subset \cup_{k \in I_{n+1}} \partial C_k$ such that $\gamma(E_n^+ \times E_{n+1}^-) = c$. We also define $\gamma_n = \gamma_{n-1} \llcorner (E_n^+ \times E_{n+1}^-)^c$, $f_n^+ = (\Pi_x)_\# \gamma_n$ and $f_n^- = (\Pi_y)_\# \gamma_n$. Using that $\partial C_1 \subset \partial\Omega \setminus \cup_{i \in I_{n+1}} E_i^-$, we have

$$f_n^\pm(\partial\Omega) = f_{n-1}^\pm(\partial\Omega) - c = f^\pm(\partial\Omega) - nc \qquad \text{and} \qquad f_n^-(\partial\Omega) \geq f^-(\partial C_1) \geq c.$$

But, this yields obviously to a contradiction as soon as n is large enough. As a consequence, the claim that $\gamma(\bar{C}_i \times \bar{C}_j) = 0$, for all $i \neq j$, is proved.

Step 2. We claim that the restriction of γ to $C_i \times C_i$ is the unique optimal transport plan between its corresponding marginals. For $i \in I_C$, we introduce

$$\gamma_i := \gamma \llcorner (\bar{C}_i \times \bar{C}_i), \qquad f_i^\pm := f^\pm \llcorner \bar{C}_i.$$

Thanks to our assumption on γ, we see that $(\Pi_x)_\# \gamma_i = f_i^+$ and $(\Pi_y)_\# \gamma_i = f_i^-$. Let us suppose that η_i is a solution to the Monge-Kantorovich problem on C_i with data f_i^+, f_i^-, $i \in I_C$. We notice that since by assumption C_i is convex and $g_i := g \llcorner \partial C_i$ satisfies condition (H1)-(H3), then due to Proposition 3.16, η_i is the unique optimal transportation plan and it is induced by a map T_i, i.e. $\eta_i = (Id, T_i)_\# f_i^+$. The optimality of η_i on $\bar{C}_i \times \bar{C}_i$ implies that

$$\int_{\bar{C}_i \times \bar{C}_i} |x - y| \, d\eta_i \leq \int_{\bar{C}_i \times \bar{C}_i} |x - y| \, d\gamma_i. \tag{4.2}$$

Now, we set $\eta = \sum_{i \in I_C} \eta_i$. The optimality of γ between f^+ and f^- as well as the admissibility of η lead to

$$\int_{\bar{\Omega} \times \bar{\Omega}} |x - y| \, d\gamma \leq \int_{\bar{\Omega} \times \bar{\Omega}} |x - y| \, d\eta.$$

Yet, from (4.2), we also have

$$\int_{\bar{\Omega} \times \bar{\Omega}} |x - y| \, d\eta \leq \int_{\bar{\Omega} \times \bar{\Omega}} |x - y| \, d\gamma.$$

Hence, we get that the inequality in (4.2) is in fact an equality for every $i \in I_C$, i.e. we have

$$\int_{\bar{C}_i \times \bar{C}_i} |x - y| \, d\eta_i = \int_{\bar{C}_i \cap \bar{C}_i} |x - y| \, d\gamma_i.$$

This means that $\gamma_i = \eta_i = (Id, \mathbf{T}_i)_\# f_i^+$ is the unique optimal transport plan between f_i^+ and f_i^-. Our claim follows. Yet, we have

$$\gamma = \sum_{i \in I_C} \gamma_i. \tag{4.3}$$

Thus, we deduce that

$$\gamma = \sum_{i \in I_C} (Id, \mathbf{T}_i)_\# f_i^+ = (Id, \mathbf{T})_\# f. \tag{4.4}$$

Hence, we have proved that if γ is an optimal transport plan between f^+ and f^-, then we have $\gamma = (Id, \mathbf{T})_\# f^+$. Yet, the map $\mathbf{T}$ does not depend on γ as a result, the optimal transport plan γ is unique. This concludes the proof. □

Proposition 4.4. *Assume that (L1) and (L2) hold. Then, we have* (2.2) = (2.4)*. Moreover, v_γ is a unique solution of Problem* (2.2)*, provided that $\gamma = (Id, \mathbf{T})_\# f^+$.*

Proof. We recall that we always have (2.5) $\leq$ (2.2). In addition, we see that v_γ is well defined thanks to the fact that $]x, \mathbf{T}(x)[\subset \Omega$ (see (L1)), for f^+-a.e. x. Moreover, v_γ is clearly admissible in Problem (2.2) and we have

$$\int_\Omega |v_\gamma| = \int_{\overline{\Omega} \times \overline{\Omega}} |x-y| \mathrm{d}\,\gamma = (2.4) = (2.5) \leq (2.2).$$

Hence, v_γ solves Problem (2.2) and, we have (2.2) = (2.4). It is worth noting that although Ω is not convex, we have proved that the values of the infima of Problems (2.2) and (2.4) are exactly the same. Thanks to this fact, following the argument in [7, Proposition 2.6 (3)] for non-convex domains, we can adapt the result in [15, Theorem 4.13] to deduce that if v is an optimal vector field for Problem (2.2), then there will be an optimal transport plan γ' for Problem (2.4) such that $v = v_{\gamma'}$ Yet, by Proposition 4.3, the optimal transport plan γ is unique and so, the solution v_γ of Problem (2.2) is unique as well. □

After these preparations we present our first existence result in case of non-convex domains. It is similar in the spirit to the main result of [4].

Theorem 4.5. *Under the assumptions (L1) and (L2), there exists a unique solution u to Problem* (1.1) *provided that $g \in W^{1,1}(\partial\Omega)$ is piecewise monotone.*

Proof. By Proposition 4.4, v_γ (where $\gamma = (Id, \mathbf{T})_\# f^+$) is the unique optimal flow in Problem (2.2). Yet, thanks to (L1), it is clear that $|v_\gamma|(\partial\Omega) = 0$. Then, Proposition 2.3 concludes the proof. □

In the example below we illustrate how the above theorem works. We stress that the partitioning Ω need not be obvious.

Example 4.6. Set $\Omega = \Omega^+ \setminus \Omega^-$ with $\Omega^+ = [-1,1] \times [0,1]$ and $\Omega^- = [-a,a] \times [0,b]$, with $0 < a$, $b < 1$. We define the boundary data g on $\partial\Omega$ as follows:

$$g(x_1,x_2) = \begin{cases} 0 & ([a,1] \times \{0\}) \cup (\{1\} \times [0,b]) \cup (\{-1\} \times [0,b]) \cup ([-1,-a] \times \{0\}), \\ \dfrac{b-x_2}{1-b} & (\{1\} \times [b,1]) \cup (\{-1\} \times [b,1]), \\ -1 & ([-1,1] \times \{1\}) \cup ([-a,a] \times \{b\}), \\ -\dfrac{x_2}{b} & (\{-a\} \times [0,b]) \cup (\{a\} \times [0,b]). \end{cases}$$

In this case, $f^+ = f_1^+ + f_2^+$ and $f^- = f_1^- + f_2^-$, where

$$f_1^+ = \frac{1}{b}\mathcal{H}^1 \llcorner (\{a\} \times [0,b]), \qquad f_1^- = \frac{1}{1-b}\mathcal{H}^1 \llcorner (\{1\} \times [b,1]),$$

$$f_2^+ = \frac{1}{1-b}\mathcal{H}^1 \llcorner (\{-1\} \times [b,1]), \qquad f_2^- = \frac{1}{b}\mathcal{H}^1 \llcorner (\{-a\} \times [0,b]).$$

In order to prove existence of a solution to Problem (1.1), we subdivide our region Ω into sets satisfying conditions (L1) and (L2). As shown in Figure 3, we set X_1 to be the trapezoid with vertices $(-a,b), (a,b), (-1,1), (1,1)$, X_2 the triangle with vertices $(-a,0), (-1,b), (-1,0)$, and X_3 the triangle with vertices $(a,0)$, $(1,b)$ and $(1,0)$. We want to subdivide $C := \Omega \setminus (\bar{X}_1 \cup \bar{X}_2 \cup \bar{X}_3)$

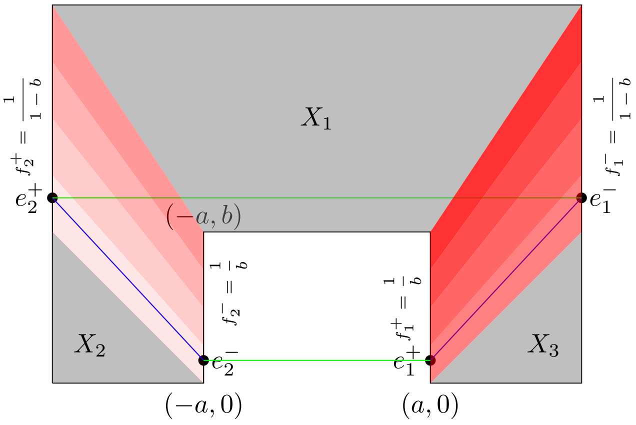

FIGURE 3. Rectilinear C-shape example

into convex sets C_i's (in red in Figure 3) satisfying condition (L2) and such that each C_i satisfies (H1)-(H3). Notice that for $s \in [0,1]$, we have

$$g(a, sb) = g(-a, sb) = g(1, b + (1-b)s) = g(-1, b + (1-b)s) = -s.$$

After taking any partition of $[0,1]$, $0 = s_0 < s_1 < \cdots < s_n = 1$ and $\Delta s_i = s_i - s_{i-1}$, such that $\max_{i\in\{1,\ldots,n\}} \Delta_i \to 0$ as $n \to \infty$, we construct convex domains $C_{i,l/r}$ as follows: $C_{i,l}$ (resp. $C_{i,r}$) is an open quadrilateral in red with vertices $(-a, s_{i-1}b)$, $(-1, b + (1-b)s_{i-1})$, $(-1, b + (1-b)s_i)$, $(-a, s_i b)$ (respectively, the quadrilateral in red with vertices $(a, s_{i-1}b)$, $(1, b + (1-b)s_{i-1})$, $(1, b + (1-b)s_i)$, $(a, s_i b)$). As a result $C = \left((\cup_{i=1}^n \bar{C}_{i,l}) \cup (\cup_{i=1}^n \bar{C}_{i,r})\right)^\circ$.

Notice that $\partial C_{i,r} \cap \partial\Omega$ can be decomposed into two arcs:

$$\Gamma_{i,r}^+ = \overset{\frown}{(a, s_{i-1}b)\,(a, s_i b)}, \quad \Gamma_{i,r}^- = \overset{\frown}{(1, b + (1-b)s_{i-1})\,(1, b + (1-b)s_i)}.$$

By symmetry, we decompose $C_{i,l}$ in similar way. Then, it follows that all convex domains $C_{i,l/r}$ satisfy condition (H1), we conclude that (L1) holds.

Next, we check (L2). Once w set $e_1^+ = (a, sb), e_1^- = (1, b + (1-b)s)$, $e_2^- = (-a, sb)$, $e_2^+ = (-1, b + (1-b)s)$ as in Figure 3, Condition (L2) implies that

$$2|(1-a, b + s(1-2b))| = |e_1^+ - e_1^-| + |e_2^+ - e_2^-| < |e_1^+ - e_2^-| + |e_2^+ - e_1^-| = 2 + 2a.$$

For this inequality to hold we require that

$$|s(2b-1) - b| < 2\sqrt{a},$$

for every $s \in [0,1]$. Notice that the maximum of the left-hand side is $\max(b, 1-b)$. Then, assuming that

$$\max(b, 1-b) < 2\sqrt{a}, \tag{4.5}$$

(L2) follows for $e_i^\pm$ corresponding to transport rays. Since the inequality in (4.5) is strict, then we choose Δs_i small enough so that the convex sets $C_{i,l/r}$ satisfy inequality (L2) for every sequence of points $\{e_k^\pm\}$ from $\partial C_{i_k,l/r}$. Hence, under the assumption (4.5) and thanks to Theorem 4.5, Problem (1.1) has a unique solution.

Here comes the most general instance of data we consider in this paper. Namely, we will extend the result of Proposition 4.3 to the case, when the boundary datum g is monotone on some arcs with negative curvature. On the way, we need to introduce the following assumptions.

- **Condition (A1)**. The domain Ω can be decomposed into disjoint sets C_i $(i \in I_C)$, E_i $(i \in I_E)$ and, X_i $(i \in I_X)$ such that g is constant on $\partial X_i \cap \partial\Omega$ and, we have the following:

(1) For every $i \in I_C$, C_i is convex. In addition, the family of open sets $\{C_i^\circ : i \in I_C\}$ satisfies the assumption (L1).

(2) For every $i \in I_E$, $\partial E_i \cap \partial\Omega$ is the sum of two closed arcs E_i^+ and E_i^- such that at least one of them is not convex and, g is strictly increasing on ${E_i}^+$ and it is strictly decreasing on ${E_i}^-$ with $TV(g_{|{E_i}^+}) = TV(g_{|{E_i}^-})$.

Remark 4.7. Now, it is clear that the domain D defined in Remark 4.2 may satisfy (A1) for a proper choice of g, even if it is not constant on the arc $\{(x_1,x_2):\ x_1\in(-1,1),\ x_2=\sqrt{1-x_1^2}\}$.

In order to construct a solution to (1.1), we apply our usual strategy, namely, we introduce a transport map $\mathbf{T}$. First, we define $\bar{\mathbf{T}}:\Gamma^+\cup\chi^+\mapsto\Gamma^-\cup\chi^-$. Since condition (A1) implies that $f(\bigcup_{i\in I_E}E_i^\pm)=0$, the conclusion of Lemma 4.1 is valid. Hence, for any $x^+\in\Gamma^+\cup\chi^+$, we may define $\bar{\mathbf{T}}(x^+)$ by formula (4.1). A new situation arises when we want to define the transportation map on E_i^+, $i\in I_E$. We proceed as follows:

$$\text{If } x^+\in E_i^+, \text{ then we set } \tilde{\mathbf{T}}^i(x^+):=x^-, \text{ where } x^-\in E_i^- \text{ is such that } f(\widehat{x^+x^-})=0.$$

This map is well defined, thanks to the fact that g is strictly monotone on E_i^-. Finally, we define $\mathbf{T}:\Gamma^+\cup\chi^+\cup(\cup_{i\in I_E}E_i^+)\mapsto\Gamma^-\cup\chi^-\cup(\cup_{i\in I_E}E_i^-)$ as follows:

$$\mathbf{T}(x^+)=\begin{cases}\bar{\mathbf{T}}(x^+) & \text{for } x^+\in\Gamma^+\cup\chi^+,\\ \tilde{\mathbf{T}}^i(x^+) & \text{for } x^+\in E_i^+,\ i\in I_E.\end{cases}\tag{4.6}$$

Subsequently, we will denote by $\mathbf{T}^{[-1]}$ the inverse map of $\mathbf{T}$.

Now, we continue stating further conditions on the data.

- **Condition (A2)**. For all points $x^+\in {E_i}^+$ and $x^-\in {E_i}^-$, we have $]x^+,x^-[\subset\Omega$. We note that x^- here is an arbitrary point on E_i^-, so it is not necessarily the image of x^+ by $\tilde{\mathbf{T}}^i$ (see the difference with (H2)).

- **Condition (A3)**. For any two finite sequences of points $\{e_k^\pm\}_{1\le k\le m}$ such that $e_k^\pm\in{\chi^{i_k}}^\pm\cup{\Gamma^{i_k}}^\pm$ (for some $i_k\in I_C$) or $e_k^\pm\in E_{i_k}^\pm$ (for some $i_k\in I_E$), with $i_k\neq i_{k'}$ for all $k\neq k'$ such that $\{i_k,\,i_{k'}\}\subset I_C$ or $\{i_k,\,i_{k'}\}\subset I_E$, we have the following inequality:

$$\sum_{k=1}^m|{e_k}^+-{e_k}^-|<\sum_{k=1}^{m-1}|e_k^+-e_{k+1}^-|+|e_m^+-e_1^-|.$$

Notice that condition (A3) is just a generalization of the assumption (L2), since now we need also to guarantee that every set E_i^+ is transported to E_i^-, for all $i\in I_E$.

Proposition 4.8. *Suppose that conditions (A1), (A2) and (A3) are satisfied. Let γ be an optimal transport plan between f^+ and f^-. Then, for $\gamma-$a.e. $(x^+,x^-)\in\operatorname{spt}(\gamma)$, we have $x^-=\mathbf{T}(x^+)$. In other words, $\gamma=(Id,\mathbf{T})_\#f^+$ is the unique optimal transport plan.*

Proof. Notice that by definition, we have $f(\partial E_i)=0$, for all $i\in I_E$. As a result, the argument of Lemma 4.1 yields that $f(\partial C_i)=0$, for all $i\in I_C$.

The assumption (A3) here corresponds to condition (L2), which played the key role in the proof of Proposition 4.3. Hence, the argument used there shows that $\gamma(C_i\times E_j)=0$, $i\in I_C$ and $j\in I_E$ (resp. $\gamma(C_i\times C_j)=\gamma(E_i\times E_j)=0$, for all $i\neq j\in I_C$ or $i\neq j\in I_E$). Similarly to Proposition 4.3, one can see that $\gamma\llcorner[C_i\times C_i]$ (resp. $\gamma\llcorner[E_i\times E_i]$) is an optimal transport plan between its own marginals $\bar{f}_i^+:=f^+\llcorner\partial C_i$ and $\bar{f}_i^-:=f^-\llcorner\partial C_i$ (resp. $\tilde{f}_i^+:=f^+\llcorner\partial E_i$ and $\tilde{f}_i^-:=f^-\llcorner\partial E_i$), for all $i\in I_C$ (resp. $i\in I_E$). From Proposition 3.14, we infer that

$$\gamma\llcorner[C_i\times C_i]=(Id,\bar{T})_\#\bar{f}_i^+,\quad\text{for every } i\in I_C.\tag{4.7}$$

Now, we claim that for all $(x^+,x^-)\in\operatorname{spt}(\gamma)\cap(E_i^+\times E_i^-)$ we have $x^-=\tilde{T}(x^+)$. Assume that this is not the case. Then, there must be a couple $(x_1^+,x_1^-)\in\operatorname{spt}(\gamma)\cap(E_i^+\times E_i^-)$ such that $x^+<x_1^+$ and $x_1^->x^-$, since otherwise we get a contradiction with the mass balance. Thanks to assumption (A2), we see that the transport rays $]x^+,x^-[$ and $]x_1^+,x_1^-[$ intersect, which is a contradiction. Then, we also have the following:

$$\gamma\llcorner[E_i\times E_i]=(Id,\tilde{\mathbf{T}})_\#\tilde{f}_i^+,\quad\text{for every } i\in I_E.\tag{4.8}$$

Combining (4.7) and (4.8), we get that $\gamma=(Id,\mathbf{T})_\#f^+$. Hence, γ is unique because it is supported on uniquely defined graph of $\mathbf{T}$ and its first marginal equals f^+. This concludes the proof. □

Proposition 4.9. *Assume that (A1), (A2) & (A3) hold. Then, Problems (2.2) and (2.4) have the same minimal value. Moreover, v_γ (where $\gamma=(Id,\mathbf{T})_\#f^+$) is the unique optimal flow in Problem (2.2).*

Proof. The proof is exactly the same as the one for Proposition (4.4). ☐

Finally, we get the following result:

Theorem 4.10. *Assume that (A1), (A2) & (A3) hold and that $g \in W^{1,1}(\partial\Omega)$ is piecewise monotone. Then, Problem (1.1) has a unique solution.*

Proof. This will immediately follow from Propositions 4.9 & 2.3 once we show that $|v_\gamma|(\partial\Omega) = 0$. Indeed, this is implied by the fact that $\gamma = (Id, \mathbf{T})_\# f^+$and by the assumptions (H2), (A2). ☐

Now, we show how the set of assumptions (A1)-(A3) works.

Example 4.11. Here, we present an example of data (Ω, g), where a piece of $\partial\Omega$ has a negative curvature, nonetheless, conditions (A1)-(A3) hold. Set $\Omega := \{(x_1, x_2) : 1 \le x_1^2 + x_2^2 \le R^2,\ x_2 \ge 0\}$. For a fixed $\alpha \in (0, \pi/2)$, we define the boundary data g in polar coordinates as follows:

$$g(r,\theta) = \begin{cases} \dfrac{\theta}{\alpha} & r \in \{1, R\},\ \theta \in]0, \alpha], \\ 1 & r \in \{1, R\},\ \theta \in]\alpha, \pi - \alpha], \\ \dfrac{\pi - \theta}{\alpha} & r \in \{1, R\},\ \theta \in]\pi - \alpha, \pi[, \\ 0 & 1 \le r \le R,\ \theta \in \{0, \pi\}. \end{cases}$$

As a result, $f^+ = f_1^+ + f_2^+$ and $f^- = f_1^- + f_2^-$, where

$$f_1^+ = \frac{1}{\alpha}\mathcal{H}^1 \llcorner \{(1,\theta) : \theta \in [0,\alpha]\}, \qquad f_1^- = \frac{1}{R\alpha}\mathcal{H}^1 \llcorner \{(R,\theta) : \theta \in [0,\alpha]\},$$
$$f_2^+ = \frac{1}{R\alpha}\mathcal{H}^1 \llcorner \{(R,\theta) : \theta \in [\pi-\alpha,\pi]\}, \qquad f_2^- = \frac{1}{\alpha}\mathcal{H}^1 \llcorner \{(1,\theta) : \theta \in [\pi-\alpha,\pi]\}.$$

In this case, a part of $\partial\Omega$ has negative curvature, so in order to prove the existence of a solution

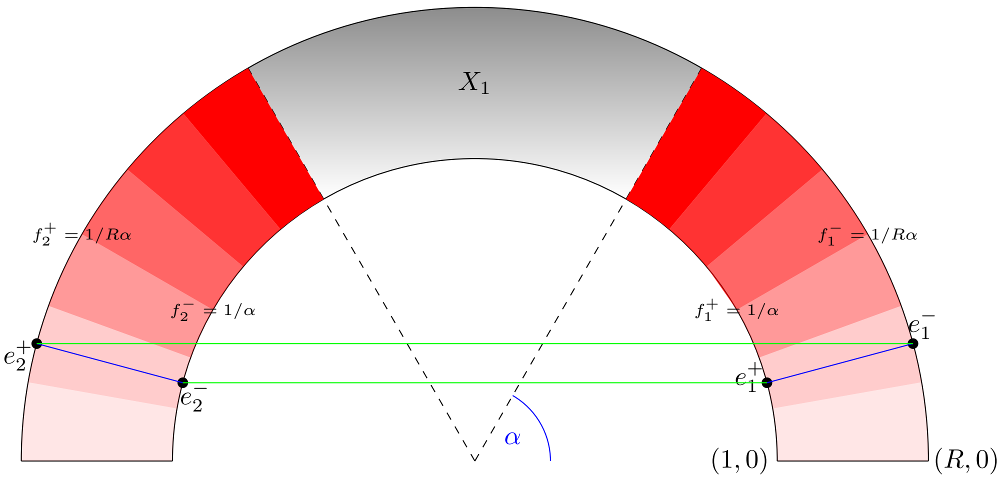

FIGURE 4. Example with negative curvature

to (1.1), we decompose Ω into subsets verifying conditions (A1), (A2), (A3). We refer to Figure 4 for illustration. We let $X_1 = \{(x,y) = (r\cos\theta, r\sin\theta) : (r,\theta) \in (1,R) \times (\alpha, \pi - \alpha)\}$. Notice that for all $s \in [0,1]$, we have

$$g(1, s\alpha) = g(R, s\alpha) = g(1, \pi - s\alpha) = g(R, \pi - s\alpha) = s.$$

Let $0 = s_0 < s_1 < s_2 < \cdots < s_{n-1} < s_n = 1$ be a partition of $[0,1]$. We set $\Delta s_i = s_i - s_{i-1}$. We define $E_{i,r}$ in polar coordinatinates by formula

$$E_{i,r} = \{(r,\theta) : r \in (1,R), \theta \in (s_{i-1}\alpha, s_i\alpha)\}, \qquad i = 1, \dots, n.$$

By definition, $E_{i,l}$ is the symmetric image of $E_{i,r}$ with respect to the vertical coordinate axis. The regions $E_{i,l/r}$ are shaded in red in Figure 4, so $\Omega \backslash [(\cup_{i=1}^n E_{i,r}) \cup (\cup_{i=1}^n E_{i,l})] = X_1$. Notice that $\partial E_{i,r} \cap \partial\Omega$ can be decomposed into two arcs (in polar coordinates):

$$E_{i,r}^+ = (1, s_{i-1}\alpha) \frown (1, s_i\alpha), \quad E_{i,r}^- = (R, s_{i-1}\alpha) \frown (R, s_i\alpha).$$

By symmetry, we decompose $\partial E_{i,l} \cap \partial\Omega$. We choose $\delta_n := \max\{\Delta s_i : \ i = 1, \ldots, n\}$ small enough so that any line segments from $E^+_{i,l/r}$ to $E^-_{i,l/r}$ lies in Ω. We then get that $\partial E_{i,l/r}$ satisfy conditions (A1)-(A2).

Now, we check (A3). We begin with $e^+_1 = (1, s\alpha)$, $e^-_1 = (R, s\alpha)$, $e^-_2 = (1, \pi - s\alpha)$, $e^+_2 = (R, \pi - s\alpha)$. Condition (A3) requires that for every $s \in [0, 1]$, one has

$$2(R-1) = |e^+_1 - e^-_1| + |e^+_2 - e^-_2| < |e^+_1 - e^-_2| + |e^+_2 - e^-_1| = 2\cos(s\alpha) + 2R\cos(s\alpha).$$

This leads to the following relationship

$$\cos\alpha > \frac{R-1}{R+1}. \tag{4.9}$$

For such values of α, (A3) follows for $e^\pm_k$ corresponding to transport rays. We can then further restrict δ_n so that the sets $E_{i,l/r}$ satisfy condition (A3) for every sequence of arbitrary points $\{e^\pm_k\}$ on $\partial E_{i_k,l/r}$.

We conclude that if (4.9) is satisfied, then we may use Theorem 4.10 to deduce that Problem (1.1) has a unique solution.

Finally, we will also cover the case when the inequality in the assumption (A3) becomes the equality. To be more precise, we introduce the following relaxation of (A3):

- **Condition $(\widetilde{A3})$.** Let $\{e^\pm_k\}_{1\le k\le m}$ (where $m \in \mathbb{N}$) be two finite sequences of points such that $e^\pm_k \in \chi^{i_k\pm} \cup \Gamma^{i_k\pm}$ (for some $i_k \in I_C$) or $e^\pm_k \in E^\pm_{i_k}$ (for some $i_k \in I_E$), with $i_k \ne i_{k'}$ for all $k \ne k'$ such that $\{i_k, i_{k'}\} \subset I_C$ or $\{i_k, i_{k'}\} \subset I_E$. Then, we assume that we have the following inequality:

$$\sum_{k=1}^{m} |e^+_k - e^-_k| \le \sum_{k=1}^{m-1} |e^+_k - e^-_{k+1}| + |e^+_m - e^-_1|.$$

We claim that under this weaker assumption $(\widetilde{A3})$, we can establish a version of Proposition 4.8.

Proposition 4.12. *Assume that (A1), (A2) & $(\widetilde{A3})$ hold and g is piecewise monotone. Then, there exists an optimal transport plan $\gamma^\star$ such that $\gamma^\star(C_i \times C_j) = \gamma^\star(E_i \times E_j) = \gamma^\star(C_i \times E_j) = 0$, for all i, j (with $i \ne j$ if $\{i,j\} \subset I_C$ or $\{i,j\} \subset I_E$). Moreover, if (S) is satisfied then $\gamma^\star := (Id, \mathbf{T})_\# f^+$ is an optimal transport plan between f^+ and f^-.*

Proof. We apply the argument used in the proofs of Propositions 3.12 and 3.14. For this purpose we will construct an increasing sequence of domains Ω_n whose closures converge to $\overline{\Omega}$ in the Hausdorff distance as well as a sequence of functions g_n defined on $\partial\Omega_n$ such that for every n, the boundary $\partial\Omega_n$ can be decomposed into sets $\tilde{C}_{i,n}$ and $E_{i,n}$ satisfying condition (A3) and so that we have $\partial\tilde{C}_{i,n} \to \partial C_i$ and $\partial E_{i,n} \to \partial E_i$ in the Hausdorff sense. Here, $\tilde{C}_{i,n}$ need not be convex.

Step 1. Fix $n \in \mathbb{N}^\star$. Let us suppose that α is any of the arcs χ^i_j ($j \in I^i_\chi$, $i \in I_C$), $\Gamma^{i\pm}_j$ ($j \in I^i_\Gamma$, $i \in I_C$) or $E^\pm_i$ ($i \in I_E$). After an appropriate choice of the coordinate system we may assume that α is the graph of a Lipschitz continuous function $h_\alpha : [-r_\alpha, r_\alpha] \mapsto \mathbb{R}$, i.e. $\alpha = G(h_\alpha)$, where $h_\alpha(-r_\alpha) = 0 = h_\alpha(r_\alpha)$ and $G(h_\alpha)$ denotes the graph of h_α. We adopt a convention requiring $h_\alpha \ge 0$ when α is convex (i.e. $\operatorname{conv}\alpha \subset \Omega$), this implies that h_α is concave. Moreover, $h_\alpha \le 0$ when α is concave, so h_α is a convex function. In particular, for convex α there is a $r > 0$ such that $\{(x,y) : \ x \in [-r_\alpha, r_\alpha], y \in [h_\alpha(x), h_\alpha(x) + r]\} \cap \Omega = \emptyset$. When α is concave there is a $r > 0$ such that $\{(x,y) : \ x \in [-r_\alpha, r_\alpha], y \in [h_\alpha(x) - r, h_\alpha(x)]\} \cap \Omega = \emptyset$.

Our construction depends on the curvature of α. First, we consider α which is not a line segment. Then, for any natural $n \ge 1$, we define $\alpha^n := G((1 - \frac{\kappa}{n})h_\alpha)$, where $\kappa = 1$ when α is convex and $\kappa = -1$ when α is concave. When α is convex, then this construction implies that α^n is convex and $\alpha^n \subset \operatorname{conv}(\bar{\alpha}) \subset \bar{\Omega}$. In case $\alpha = E^+_i$ (resp. $\alpha = E^-_i$) is concave, since the distance between E^+_i and E^-_i is positive we conclude that $E^+_{i,n} := \alpha^n$ (resp. $E^-_{i,n} := \alpha^n$) is concave and $\alpha^n \subset E_i$ for sufficiently large n.

When α happens to be a line segment we proceed differently. We take a strictly convex function $\eta_\alpha : [-r_\alpha, r_\alpha] \mapsto (-\infty, 0]$ such that $\eta_\alpha(\pm r_\alpha) = 0$ and $G(\eta_\alpha) \subset \Omega$, then we set $\alpha^n := G(\frac{\eta_\alpha}{n})$, $n \ge 1$. Hence, we conclude that $\alpha^n \subset \Omega$ for large $n \in \mathbb{N}$.

After this preparation we will define the domain Ω_n. More precisely, the boundary $\partial\Omega_n$ is formed by replacing every arc $\alpha = \chi^i_j$ (resp. $\alpha = \Gamma^{i\pm}_j$ or $\alpha = E^\pm_i$) in $\partial\Omega$ by the arc α^n. We note that α^n and α have same endpoints and so $\partial\Omega_n$ is a closed curve (i.e. topologically a circle). In the same way, we define $\partial\tilde{C}_{i,n}$ as ∂C_i but any arc $\alpha = \chi_j^i$ (resp. $\alpha = \Gamma_j^{i\,\pm}$) in ∂C_i is replaced by α^n. Moreover, $E_{i,n}$ is the region bounded by $E_{i,n}^{\pm}$ and $\partial E_i \cap \Omega$. The sets $\Gamma_{j,n}^i$ and $\chi_{j,n}$ are defined in a natural way. It is clear that $\partial\Omega_n \to \partial\Omega$, $\partial\tilde{C}_{i,n} \to \partial C_i$ and $\partial E_{i,n} \to \partial E_i$ in the Hausdorff distance as $n \to \infty$.

Step 2. Now, we define maps $S_n : \partial\Omega \mapsto \partial\Omega_n$ as follows. Since g is piecewise monotone, we first consider points belonging to $U^+ \cup U^-$ (see Definition 3.21). For such points, $S_n(x)$ is set to be the point of intersection of the line segment $[x, \mathbf{T}(x)]$ (resp. $[x, \mathbf{T}^{-1}(x)]$) with the boundary $\partial\Omega_n$, which is the closest to x. We note that S_n is well defined since the line segment $[x, \mathbf{T}(x)]$ (resp. $[x, \mathbf{T}^{-1}(x)]$) intersects Ω transversally and α_n is either strictly convex or strictly concave. Moreover, we see that for each $n \in \mathbb{N}^\star$ the inverse map S_n^{-1} exists.

We extend S_n to $\partial\Omega$ by setting $S_n = Id$ on U_0, (U_0 is from the Definition 3.21). Hence, if $x \in \alpha = \chi_j^i$ (resp. $x \in \alpha = \Gamma_j^{i\,\pm}$ or $x \in \alpha = E_i^{\pm}$), then it is obvious that $S_n(x) \in \alpha^n$ for sufficiently large $n \in \mathbb{N}$. In particular, we have $S_n(\partial C_{i,n}) = \partial\tilde{C}_{i,n}$ and $S_n(E_i^{\pm}) = E_{i,n}^{\pm}$.

Moreover, due to continuity of $\mathbf{T}$ and $\mathbf{T}^{-1}$ it is not difficult to check that the mapping S_n is continuous as well on $\partial\Omega$.

Once we fix any $x_0 \in U_0$ we define the boundary data g_n on $\partial\Omega_n$ by $g_n(x) = f_n(\widehat{x_0\, x})$, where f_n is defined as follows:

$$f_n = S_{n\#} f.$$

It is clear that $f_n(\partial\tilde{C}_{i,n}) = f(S_n^{-1}(\partial\tilde{C}_{i,n})) = f(\partial C_i) = 0$ and $f_n(\partial E_{i,n}) = f(S_n^{-1}(\partial E_{i,n})) = f(\partial E_i) = 0$.

Step 3. Since f_n is defined only on $\partial\Omega_n$, we extend it on $\overline{\Omega}\backslash\partial\Omega_n$, by formula $\bar{f}_n(B) = f_n(B \cap \partial\Omega_n)$ for any Borel set B and for all n. Let $\bar{f}_n^+$ and $\bar{f}_n^-$ be the positive and negative parts of $\bar{f}_n$. Let γ_n be any optimal transport plan between $\bar{f}_n^+$ and $\bar{f}_n^-$ on $\overline{\Omega} \times \overline{\Omega}$. Up to a subsequence, we know that $\gamma_n \rightharpoonup \gamma$ for some $\gamma \in \mathcal{M}^+(\overline{\Omega} \times \overline{\Omega})$. Yet, we see that $\bar{f}_n \rightharpoonup \bar{f}$, where $\bar{f} \llcorner \partial\Omega = f$ and $\operatorname{spt} \bar{f} \subset \partial\Omega$. Indeed, for any $\varphi \in C(\overline{\Omega})$, we have

$$\langle \bar{f}_n, \varphi \rangle = \langle f_n, \varphi \rangle = \langle S_{n\#} f, \varphi \rangle = \int_{\partial\Omega} \varphi(S_n(x))\, \mathrm{d}f(x) \to \int_{\partial\Omega} \varphi(x)\, \mathrm{d}f(x) = \langle f, \varphi \rangle,$$

because $S_n(x)$ converges to x, for all $x \in \partial\Omega$. This follows immediately from the definition of Ω_n which assures us that

$$\lim_{n\to\infty} \max \left\{ |x - S_n(x)| \,:\, x \in \operatorname{spt}(f) \right\} = 0. \tag{4.10}$$

Hence, $(\Pi_x)_\# \gamma = f^+$ and $(\Pi_y)_\# \gamma = f^-$. Similarly, as in the proof of Proposition 3.12, we infer that γ is an optimal transport plan between f^+ and f^-.

Step 4. Now, we show that g_n satisfies (A3). Let $\{e_k^{\pm}\}_{1\le k\le m}$ be two finite sequences of points such that $e_k^{\pm} \in \partial\tilde{C}_{i_k,n} \cap \left(\bigcup_{j \in I_\Gamma^{i_k,n}} \Gamma_j^{i_k,n} \cup \bigcup_{j \in I_\chi^{i_k,n}} \chi_j^{i_k,n} \right)$ (for some $i_k \in I_C$) or $e_k^{\pm} \in E_{i_k,n}^{\pm}$ (for some $i_k \in I_E$) with $i_k \neq i_{k'}$ for all $k \neq k'$ such that $\{i_k, i_{k'}\} \subset I_C$ or $\{i_k, i_{k'}\} \subset I_E$.

For every $1 \le k \le m$, let $e_k'^{\,\pm} \in \operatorname{spt}(f^{\pm})$ be such that $[e_k'^{\,+}, e_k'^{\,-}] \cap \partial\Omega_n = \{e_k^+, e_k^-\}$. Thus,

$$\sum_{k=1}^{m} |e_k^+ - e_k^-| = \sum_{k=1}^{m} (|e_k'^{\,+} - e_k'^{\,-}| - |e_k^+ - e_k'^{\,+}| - |e_k^- - e_k'^{\,-}|).$$

At the same time, due to the triangle inequality, we have

$$\begin{aligned} \sum_{k=1}^{m-1} |e_k'^{\,+} - e_{k+1}'^{-}| + |e_m'^{+} - e_1'^{\,-}| \le \sum_{k=1}^{m-1} & |e_k'^{\,+} - e_k^+| + |e_k^+ - e_{k+1}^-| + |e_{k+1}^- - e_{k+1}'^{-}| \\ & + |e_m'^{+} - e_m^+| + |e_m^+ - e_1^-| + |e_1^- - e_1'^{\,-}|. \end{aligned} \tag{4.11}$$

The inequality in (4.11) is strict as soon as there is at least one integer k_0 such that the points $e_{k_0}'^{+}$, $e_{k_0}^+$, $e_{k_0+1}^-$, $e_{k_0+1}'^{-}$ are not co-linear. We proceed while assuming that this is the case. Since $e_k'^{\pm} \in \partial\Omega$, then $(\widetilde{\mathrm{A3}})$ implies that we also have

$$\sum_{k=1}^{m} |e_k'^{\,+} - e_k'^{\,-}| \le \sum_{k=1}^{m-1} |e_k'^{\,+} - e_{k+1}'^{-}| + |e_m'^{+} - e_1'^{\,-}|.$$

Consequently, we infer that

$$\sum_{k=1}^{m} |e_k^+ - e_k^-| < \sum_{k=1}^{m-1} |e_k^+ - e_{k+1}^-| + |e_m^+ - e_1^-|. \tag{4.12}$$

It remains to consider the case when for all $k = 1, \ldots, m$ the points $e_k'^+$, e_k^+, e_{k+1}^-, $e_{k+1}'^-$ are co-linear, (we use the convention that $e_{m+1}^- \equiv e_1^-$ and $e_{m+1}'^- \equiv e'^-_1$). By definition $e_k'^+$, e_k^+, e_k^-, $e_k'^-$ are co-linear, too. Consequently, all points $\{e_k^\pm, e_k'^\pm\}$ are co-linear. We claim that this observation combined with the fact that the sets C_i, $i \in I_C$, E_j, $j \in I_E$ are mutually disjoint, implies then the line segments $[e_k^+, e_k^-]$, $1 \le k \le m$, must be disjoint. Indeed, let us assume that there is a common point $e_{k_1}'^+ = e_{k_2}'^+ =: p \in \partial C_{i_{k_1}} \cap \partial C_{i_{k_2}}$ with $i_{k_1} \neq i_{k_2}$. We notice that $e_{k_1}'^+ = e_{k_2}'^-$ is impossible.

We know that $]e_{k_1}'^+, e_{k_1}'^-[\subset C_{i_{k_1}}$ and $]e_{k_2}'^+, e_{k_2}'^-[\subset C_{i_{k_2}}$ while $C_{i_{k_1}}$ and $C_{i_{k_2}}$ are disjoint. Since point p is by definition in the (relative) interior of both arcs $\partial C_{i_{k_1}}$ and $\partial C_{i_{k_2}}$, then due to the Lipschitz continuity of Ω we see that $\partial C_{i_{k_1}}$ and $\partial C_{i_{k_2}}$ must coincide in a neighborhood of the common point p. As a result, there will be always a triple bifurcation point $a \in \partial C_{i_{k_1}} \cap \partial C_{i_{k_2}}$, which contradicts again the Lipschitz regularity of $\partial\Omega$.

Let us take m_-, m_+ among points $e_k^\pm$, $k = 1, \ldots, m$, such that

$$|m_+ - m_-| = \operatorname{diam}\{e_k^+, e_k^- : 1 \le k \le m\}.$$

Since the intervals $[e_k^+, e_k^-]$, $1 \le k \le m$, are disjoint, then we see that we have the following inequality:

$$\sum_{k=1}^{m} |e_k^+ - e_k^-| < |m_+ - m_-|.$$

At the same time every $x \in]m_+, m_-[$ belongs to at least one interval $[e_k^+, e_{k+1}^-]$. Hence, we get that

$$|m_+ - m_-| \le \sum_{k=1}^{m-1} |e_k^+ - e_{k+1}^-| + |e_m^+ - e_1^-|.$$

As a result (4.12) follows.

Step 5. Thanks to (4.12), one can show exactly as in Step 1 of the proof of Proposition 4.3 that γ_n transports all the mass on $\partial\tilde{C}_{i,n}$ (resp. $\partial E_{i,n}$) to itself. We note that here we are not interested in characterizing the restriction of γ_n to $\partial\tilde{C}_{i,n} \times \partial\tilde{C}_{i,n}$, since we recall that $\tilde{C}_{i,n}$ is now no more convex and thus, (H1) is a priori not satisfied in $\tilde{C}_{i,n}$. From Step 3, we know that $\gamma_n \rightharpoonup \gamma$ and γ is an optimal transport plan between f^+ and f^-. Hence, we infer that γ also transports the mass from ∂C_i (resp. ∂E_i) to itself. To see this, take two continuous functions $\varphi(x)$ and $\psi(y)$ over $\overline{\Omega}$ such that $\operatorname{spt}(\varphi) \subset \overline{C_i}$ and $\operatorname{spt}(\psi) \subset \overline{E_j}$. Yet, we have $\tilde{C}_{i,n} \subset C_i$ and $E_{j,n} \subset E_j$, for n large enough. Hence, we get

$$\int_{\overline{\Omega}\times\overline{\Omega}} \varphi(x)\,\psi(y)\,\mathrm{d}\gamma_n(x,y) = \int_{\overline{C_i}\times\overline{E_j}} \varphi(x)\,\psi(y)\,\mathrm{d}\gamma_n(x,y) = \int_{\overline{\tilde{C}_{i,n}}\times\overline{E_{j,n}}} \varphi(x)\,\psi(y)\,\mathrm{d}\gamma_n(x,y) = 0$$

Consequently, letting $n \to \infty$ we obtain

$$\int_{\overline{\Omega}\times\overline{\Omega}} \varphi(x)\,\psi(y)\,\mathrm{d}\gamma(x,y) = 0.$$

But, φ and ψ are two arbitrary functions supported in $\overline{C_i}$ and $\overline{E_j}$, respectively. Then, we infer that $\gamma(\partial C_i \times \partial E_j) = 0$. In the same way, we show that $\gamma(\partial C_i \times \partial C_j) = 0$ and $\gamma(\partial E_i \times \partial E_j) = 0$, for all $i \neq j$. Recalling Proposition 4.8, we conclude the proof. □

Now, we show that the assumption (S) in Proposition 4.12 can be removed. We recall that we needed this condition in Section 3 because our approach was based on the approximation by strictly convex domains Ω_n, where we used the projection map P_n to the boundary $\partial\Omega$. However, one can use a more suitable map in our approximation, which does not take into account the presence of singular points on $\partial\Omega$ – this is the map S_n that we have just constructed in Proposition 4.12. More precisely, we have:

Proposition 4.13. *Assume that (A1), (A2) and* $(\widetilde{A3})$ *hold and* $g \in W^{1,1}(\partial\Omega)$ *is piecewise monotone. Then,* $\mathbf{T}$ *is an optimal transport map from* f^+ *to* f^-*.*

Proof. From Proposition 4.12, we just need to show that $\gamma^\star \llcorner [C_i \times C_i] = (Id, \mathbf{T})_\#(f^+ \llcorner \partial C_i)$, for all $i \in I_C$. For simplicity of exposition and without loss of generality, let us assume that Ω is convex or equivalently, that I_C is a singleton and $I_E = \emptyset$; so we have $\Omega = C_1$.

Now, we set

$$I_S := \{i \in I_\chi \cup I_\Gamma \,:\, |f|(\chi_i \cap \overline{\mathcal{S}}) > 0 \;\text{ or }\; |f|(\Gamma_i^\pm \cap \overline{\mathcal{S}}) > 0\}.$$

For any fixed $n \in \mathbb{N}$ and every $i \in I_S$, we take n points on χ_i (resp. $\Gamma_i^\pm$), including the endpoints, which are equidistanced in the sense of the arclength. We take the boundary of their convex hull and after removing the open interval connecting endpoints of χ_i (resp. $\Gamma_i^\pm$) we call the resulting polygonal curve $\chi_{i,n}$ (resp. $\Gamma_{i,n}^\pm$). Now, we define the regions Ω_n, $n \in \mathbb{N}$, as the open sets bounded by curves

$$\partial\Omega_n := [\partial\Omega \backslash \bigcup_{i \in I_S} (\chi_i \cup \Gamma_i^\pm)] \cup \bigcup_{i \in I_S} (\chi_{i,n} \cup \Gamma_{i,n}^\pm).$$

It is clear that Ω_n is convex. The map S_n has been already constructed; however, we recall its definition for the sake of clarity of exposition. For points $x \in U^+ \cup U^-$, we define $S_n(x)$ to be the point of intersection of the line segment $[x, \mathbf{T}(x)]$ (resp. $[x, \mathbf{T}^{-1}(x)]$) with the boundary $\partial\Omega_n$, which is the closest to x. On U_0, we set $S_n = Id$. We recall also that the inverse map S_n^{-1} is well defined on $\partial\Omega_n$ and that the map S_n is continuous. Now, we define the Borel measure f_n on $\partial\Omega_n$ as follows:

$$f_n = S_{n\#} f.$$

Let us fix any $x_0 \in U_0$, we note that $x_0 \in \partial\Omega_n$, for all n. Then, we set the boundary data g_n on $\partial\Omega_n$ by $g_n(x) = f_n(\widehat{x_0\, x})$.

We shall prove that $\partial\Omega_n$ can be decomposed into arcs $\chi_{i,n}$ ($i \in I_\chi$) and $\Gamma_{i,n}^\pm$ ($i \in I_\Gamma$) satisfying conditions (H1), (H2) and (H3). For every $i \in I_\chi \backslash I_S$ (resp. $i \in I_\Gamma \backslash I_S$), we have $\chi_{i,n} = \chi_i$ (resp. $\Gamma_{i,n}^\pm = \Gamma_i^\pm$). Since $f_n(\chi_{i,n}) = 0$ for $i \in I_S$, one can see that $\chi_{i,n}$ can be decomposed into two open arcs $\chi_{i,n}^+, \chi_{i,n}^-$ and a singleton $\{c_{i,n}\}$ such that g_n is strictly increasing on $\chi_{i,n}^+$ and strictly decreasing on $\chi_{i,n}^-$ with $TV(g_{|\chi_{i,n}^+}) = TV(g_{|\chi_{i,n}^-})$. In the same way, we can check that g_n is strictly increasing on $\Gamma_{i,n}^+$ (resp. strictly decreasing on $\Gamma_{i,n}^-$) with $TV(g_{|\Gamma_{i,n}^+}) = TV(g_{|\Gamma_i^+})$ (resp. $TV(g_{|\Gamma_{i,n}^-}) = TV(g_{|\Gamma_i^-})$). Indeed, if $x_1, x_2 \in \partial\Omega_n$ and $x_1 < x_2$, then we see that $S_n^{-1}(x_1), S_n^{-1}(x_2) \in \partial\Omega$ satisfy $S_n^{-1}(x_1) < S_n^{-1}(x_2)$ and we have

$$f_n(\widehat{x_1 x_2}) = f(\widehat{S_n^{-1}(x_1)\, S_n^{-1}(x_2)}).$$

Hence,

$$g_n(x_2) - g_n(x_1) = g(S_n^{-1}(x_2)) - g(S_n^{-1}(x_1)).$$

Let $T_{i,n}$ be the convex hull of $\Gamma_{i,n}^+$ and $\Gamma_{i,n}^-$ and $D_{i,n}$ be the convex hull of $\chi_{i,n}$. Then, we have $D_{i,n} \subset D_i$ and $T_{i,n} \subset T_i$, as a result the sets $T_{j,n}$, $j \in I_\Gamma$, $D_{i,n}$, $i \in I_\chi$ are mutually disjoint. In addition, we have $|f_n|(\partial\Omega \backslash (\bigcup_{i \in I_\Gamma} (\Gamma_{i,n}^+ \cup \Gamma_{i,n}^-) \cup \bigcup_{i \in I_\chi} \chi_{i,n})) = 0$. Hence, condition (H1) is satisfied.

Let $\mathbf{T}_n$ be the transport map defined by (4.6) corresponding to f_n and $\partial\Omega_n$. We recall that if e^+ is in the domain of $\mathbf{T}_n$ then one has

$$\mathbf{T}_n(e^+) = S_n(\mathbf{T}(S_n^{-1}(e^+))).$$

Thanks to the fact that $]S_n^{-1}(x), \mathbf{T}(S_n^{-1}(x))[\subset \Omega$ for all $x \in \mathrm{spt}(f_n^+)$, it is clear that $]x, \mathbf{T}_n(x)[\subset \Omega_n$ and so, (H2) is also satisfied.

Now, we need to check that assumption (H3) holds on $\partial\Omega_n$. Take a finite sequence $\{e_k^+\}_{1 \le k \le m}$ ($m \in \mathbb{N}^\star$) such that $e_k^+ \in \chi_{i_k,n}^+ \cup \Gamma_{i_k,n}^+$ for some $i_k \in I_\chi \cup I_\Gamma$. Since g satisfies (H3) on $\partial\Omega$, we know that

$$\sum_{k=1}^{m} |S_n^{-1}(e_k^+) - \mathbf{T}(S_n^{-1}(e_k^+))|$$

$$< \sum_{k=1}^{m-1} |S_n^{-1}(e_k^+) - \mathbf{T}(S_n^{-1}(e_{k+1}^+))| + |S_n^{-1}(e_m^+) - \mathbf{T}(S_n^{-1}(e_1^+))| \tag{4.13}$$

$$\le \sum_{k=1}^{m-1} |S_n^{-1}(e_k^+) - e_k^+| + |e_k^+ - \mathbf{T}_n(e_{k+1}^+)| + |\mathbf{T}_n(e_{k+1}^+) - \mathbf{T}(S_n^{-1}(e_{k+1}^+))| + |S_n^{-1}(e_m^+) - e_m^+|$$
$$+ |e_m^+ - \mathbf{T}_n(e_1^+)| + |\mathbf{T}_n(e_1^+) - \mathbf{T}(S_n^{-1}(e_1^+))|.$$

Due to the definition of S_n, we have the following equality:

$$|e_k^+ - \mathbf{T}_n(e_k^+)| = |S_n^{-1}(e_k^+) - \mathbf{T}(S_n^{-1}(e_k^+))| - |e_k^+ - S_n^{-1}(e_k^+)| - |\mathbf{T}(S_n^{-1}(e_k^+)) - \mathbf{T}_n(e_k^+)|. \tag{4.14}$$

After summing over k in (4.14) and taking into account (4.13), we infer that g_n satisfies (H3) on $\partial\Omega_n$, i.e.

$$\sum_{k=1}^{m} |e_k^+ - \mathbf{T}_n(e_k^+)| < \sum_{k=1}^{m-1} |e_k^+ - \mathbf{T}_n(e_{k+1}^+)| + |e_m^+ - \mathbf{T}_n(e_1^+)|.$$

Condition (S) is satisfied on $\partial\Omega_n$ because it is a polygon with finitely many sides. Hence, thanks to Propositions 3.14 and 3.16, $\gamma_n := (Id, \mathbf{T}_n)_\# f_n^+$ is the unique optimal transport plan between f_n^+ and f_n^-. We know that $\gamma_n \rightharpoonup \gamma$ for some $\gamma \in \mathcal{M}^+(\overline{\Omega} \times \overline{\Omega})$, up to a subsequence. Due to (4.10), we deduce that $f_n \rightharpoonup f$. Thus, we also have $(\Pi_x)_\#\gamma = f^+$ and $(\Pi_y)_\#\gamma = f^-$. By Lemma 2.4, we infer that γ is an optimal transport plan between f^+ and f^-. In addition, for any $x \in U^+ \cup U^-$, we have $S_n(x) \to x$ and $\mathbf{T}_n(S_n(x)) \to \mathbf{T}(x)$, because $\mathbf{T}_n(S_n(x)) = S_n(\mathbf{T}(x))$. Thus, this yields that $\gamma = (Id, \mathbf{T})_\# f^+$ is an optimal transport plan between f^+ and f^-. Since C_1 is convex, then recalling the proof of Proposition 3.16, we infer that $\gamma^\star \llcorner [C_1 \times C_1] = (Id, \mathbf{T})_\# f^+$. □

Finally, we show the existence of a minimal vector field v for Problem (2.2). We note that the optimal transport plan γ in Problem (2.4) is not necessarily unique if condition $(\widetilde{A3})$ holds instead of (A3) (see Example 4.16 below). Despite that, we will be able to prove uniqueness of the minimal vector field v in Problem (2.2) anyway.

Proposition 4.14. *Under the assumptions (A1), (A2) & $(\widetilde{A3})$, Problem (2.2) has a unique minimizer provided that g is piecewise monotone.*

Proof. Due to Proposition 4.13, we know that $\gamma^\star := (Id, \mathbf{T})_\# f^+$ is an optimal transport plan between f^+ and f^-. Since $]x, \mathbf{T}(x)[\subset \Omega$, for all $x \in \operatorname{spt}(f^+)$, then the vector measure $v_{\gamma^\star}$ is well defined and it turns out to be a solution for Problem (2.2). In particular, we have (2.4) = (2.2) and so, we recall, see for instance [15, Chapter 4], that in this case any minimizer v of Problem (2.2) will be of the form $v = v_\gamma$ for some optimal transport plan γ and for all $(x, y) \in \operatorname{spt}(\gamma)$, we must have $[x, y] \subset \overline{\Omega}$. Let $v = v_\gamma$ be such a minimizer. Then, we claim that $\gamma = (Id, \mathbf{T})_\# f^+$. Indeed, let us set

$$A = \Big\{ x \in \partial\Omega : \exists\, y \neq \mathbf{T}(x),\, (x, y) \in \operatorname{spt}(\gamma) \Big\}$$

and assume that $f^+(A) > 0$. Then, one can see exactly as in the proof of Proposition 3.16 that for f^+–a.e. $x \in A$, there are two different transport rays starting at x, $[x, \mathbf{T}(x)]$ and $[x, y]$ which are contained in $\overline{\Omega}$, thus A must be contained in the set of endpoints of arcs of type $\Gamma_i^\pm$, χ_i, or $E_i^\pm$. Hence, A is at most countable and $f^+(A) = 0$. □

Consequently, we get the following extension of Theorem 4.10:

Theorem 4.15. *Under the assumptions (A1), (A2) & $(\widetilde{A3})$, Problem (1.1) has a unique solution provided that g is piecewise monotone.*

Proof. The argument is similar to the proof of Theorem 4.10 and it is left to the interested reader. □

Finally, we present an example where (A3) is violated but $(\widetilde{A3})$ is satisfied and a solution to Problem (1.1) exists.

Example 4.16. Fix $0 < a < b$. Then, the domain Ω as shown in Figure 5 is formed from the vertices of the squares $[-a, a]^2$ and $[-b, b]^2$. We define the boundary data as follows:

$$g(x_1, x_2) = \begin{cases} x_1 & x_2 = \pm x_1, \quad a \le x_1 \le b, \\ -x_1 & x_2 = \pm x_1, \quad -b \le x_1 \le -a, \\ a & x_2 = \pm a, \quad -a \le x_1 \le a, \\ b & x_1 = \pm b, \quad -b \le x_2 \le b. \end{cases}$$

Notice that (A3) is violated in this case and the blue segments and the red ones correspond to all possible transportation rays between f^+ and f^-. In particular, we can construct two optimal

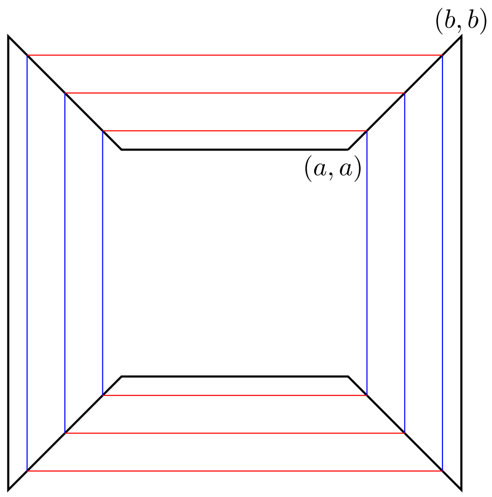

FIGURE 5. Example of non uniquenss

transport plans γ_b and γ_r, where γ_b is supported on the blue segments while γ_r is supported on the red ones. This means that the optimal transport plan is not unique. However, only γ_b corresponds to a solution to the least gradient problems since its transport rays are included in Ω.

We close the paper with an observation on the continuity of the solution u in Problem 1.1.

Theorem 4.17. *Assume that (A1), (A2) & $(\widetilde{A3})$ hold and $g \in C(\partial\Omega)$ is piecewise monotone. Then, the unique solution u of Problem (1.1) is continuous in $\overline{\Omega}$.*

Proof. Assume that there is a point $x_0 \in \overline{\Omega}$ such that u is discontinuous at x_0. Then, there exist two numbers t_1 and t_2 such that

$$\lim_{n\to\infty} \operatorname{ess\,inf}_{B(x_0,\frac{1}{n})} u < t_1 < t_2 < \lim_{n\to\infty} \operatorname{ess\,sup}_{B(x_0,\frac{1}{n})} u.$$

Now, consider the super-level sets $E_{t_1} := \{u \geq t_1\}$ and $E_{t_2} := \{u \geq t_2\}$. Then, we see that $x_0 \in \overline{E_{t_2}} \cap \overline{\mathbb{R}^2 \setminus E_{t_1}}$. However, we have $E_{t_2} \subset E_{t_1}$. Hence, we infer that $x_0 \in \partial E_{t_1} \cap \partial E_{t_2}$. So, the only possibility is to have $x_0 \in \partial\Omega$. Let E be the set bounded by ∂E_{t_1}, ∂E_{t_2} and $\partial\Omega$. It is clear that for $\varepsilon > 0$ small enough, the transport density $\sigma = 0$ on $E \cap B(x_0, \varepsilon)$. Then, u is constant on E and since $t_1 < t_2$ then this means that u has a jump on ∂E_{t_1} or ∂E_{t_2}. But, this yields a contradiction thanks to the fact that f is atomless and so, σ gives zero mass to any transport ray. ☐

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