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Functions of bounded variation on RCD spaces

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Abstract

In the Euclidean framework, a (vector valued) function of bounded variation is a function $f \in L^1(\mathbb{R}^n)^m$ whose distributional differential is a matrix valued Radon measure, meaning that there exist nm Radon measures, $(\mathrm{D}_i f_j)_{i=1,\dots,n, j=1,\dots,m}$, satisfying

$$\int_{\mathbb{R}^n} f_j \partial_{x_i} g \mathrm{d}\mathcal{L}^n = -\int_{\mathbb{R}^n} g \mathrm{d}\mathrm{D}_i f_j \qquad \text{for every } g \in C^1_{\mathrm{c}}(\mathbb{R}^n),$$

for every $i = 1, \dots, n$, and for every $j = 1, \dots, m$. An important class of functions of bounded variation is the one of sets of finite perimeter, i.e. those sets whose characteristic function $f = \chi_E$ has bounded variation.

Functions of bounded variation have proven to be an essential tool in many situations, both from the theoretical perspective and from the applied perspective, for instance in the fields of minimal surfaces, image segmentation, and denoising. This led to the study of many properties of functions of bounded variation (in particular, of sets of finite perimeter), such as their fine properties and calculus rules. For example, a fundamental theorem of De Giorgi characterizes a suitable measure theoretic boundary for sets of finite perimeter and shows properties similar to those of sets with smooth boundary, whereas the Vol'pert chain rule describes the distributional differential of $\varphi \circ f$ in terms of the distributional differential of f, for a (vector valued) function of bounded variation f and a C^1 Lipschitz function φ.

Recently, there has been growing interest in the class of $\mathrm{RCD}(K, N)$ spaces. $\mathrm{RCD}(K, N)$ spaces are those metric measure spaces whose Sobolev space is a Hilbert space and having, in a synthetic sense, Ricci curvature bounded from below by K and dimension bounded from above by N. A metric measure space consists in a triplet $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ given by a set, a complete and separable distance and a non-negative Borel measure that is finite on bounded sets. In this non-smooth framework, it is possible to give a meaning to various function spaces. The $\mathrm{RCD}(K, N)$ condition can be enforced through several (equivalent) methods, such as requiring the convexity of certain entropy functionals defined on the space of probability measures or a version of the Bochner inequality, coupled with the request that the Sobolev space is a Hilbert space. One of the motivations of the interest in $\mathrm{RCD}(K, N)$ spaces is the following: any sequence of Riemannian manifolds with Ricci curvature bounded from below by K and dimension bounded from above by N, has a subsequence that converges to an $\mathrm{RCD}(K, N)$ space (which, in general, is not a Riemannian manifold). For this reason, $\mathrm{RCD}(K, N)$ spaces have been proven to be useful also when one starts from an investigation in the smooth context of Riemannian manifolds.

The goal of this manuscript is to provide a comprehensive study of (vector valued) functions of bounded variation defined on RCD spaces, describing their distributional differential and investigating their fine properties and calculus rules. More precisely, this note begins with a recollection of results already present in the literature, some of which are proved in a slightly different way in comparison to the original references: the ideas and techniques used are still the same, but a careful reordering and slight modifications bring to an improvement of the presentation. Then, the bulk of this note contains a collection of results obtained by the author together with coauthors, see [32, 33, 46, 45, 44, 42, 43]. Improvements about the organization of the results and their proofs, in comparison to the original references, are obtained also in this part.

Contents

Chapter 1

Introduction

This manuscript is about functions of bounded variation on RCD spaces. We start this introduction by recalling the most classical results about functions of bounded variation on Euclidean spaces, with the goal of generalizing them to the non-smooth framework. Then, we recall the definition of RCD space along with some properties. Finally, we discuss how the results mentioned at the beginning of this introduction can be proved in the context of RCD spaces.

1.1 Functions of bounded variation on Euclidean spaces

As stated above, we begin with a recollection of results that are nowadays part of the classical literature of geometric measure theory, e.g. [18, 77, 76, 79, 91], and are fundamental tools in various areas of both applied and theoretical analysis.

1.1.1 Definitions

Functions of bounded variation are those functions whose distributional derivatives can be represented by a Radon measure and arise naturally, for example, taking weak limits of sequences of smooth functions with uniformly bounded 1-Sobolev norm. Namely, for $f \in L^1(\mathbb{R}^n)$, we say that f is a function of bounded variation, or $f \in \mathrm{BV}(\mathbb{R}^n)$, if there exist n Radon measures $\mathrm{D}_1 f, \dots, \mathrm{D}_n f$ satisfying

$$\int_{\mathbb{R}^n} f \partial_{x_i} g \, \mathrm{d}\mathcal{L}^n = -\int_{\mathbb{R}^n} g \, \mathrm{d}\mathrm{D}_i f \qquad \text{for every } g \in C^1_{\mathrm{c}}(\mathbb{R}^n),$$

for every $i = 1, \dots, n$. We also have a local version of this definition: $f \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^n)$ if for every open and bounded set B, we can find $f_B \in \mathrm{BV}(\mathbb{R}^n)$ such that $f_B = f$ on B. This induces a well-defined notion of distributional differential also for functions belonging to $\mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^n)$.

Finally, applications often require the study of vector valued functions of bounded variation, i.e. m-tuples of functions of bounded variation. To a map of bounded variation $f = (f_1, \dots, f_m) \in \mathrm{BV}(\mathbb{R}^n)^m$ (or in $\mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^n)^m$) one associates in a natural way the matrix-valued Radon measure $\mathrm{D}f := (\mathrm{D}_i f_j)_{i=1,\dots,n, j=1,\dots,m}$, the distributional differential of f. Then, the total variation of a map $f \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^n)^m$ is, by definition, the total variation (as a matrix-valued Radon measure, with respect to the Hilbert–Schmidt norm) of $\mathrm{D}f$ and is denoted by $|\mathrm{D}f|$. It is also customary to write the polar decomposition

$$\mathrm{D}f = \frac{\mathrm{d}\mathrm{D}f}{\mathrm{d}|\mathrm{D}f|} |\mathrm{D}f|,$$

where $\frac{\mathrm{d}\mathrm{D}f}{\mathrm{d}|\mathrm{D}f|}$ is the polar matrix which has (Hilbert–Schmidt) norm equal to 1 $|\mathrm{D}f|$-a.e.

As a particularly relevant example, we have sets of finite perimeter, also known as Caccioppoli sets, after [54, 55]. A set of finite perimeter is a Borel set E such that $\chi_E \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^n)$ with $|\mathrm{D}\chi_E|(\mathbb{R}^n) < \infty$. The total variation of the indicator function of a set of finite perimeter is also called perimeter and is also denoted by $\mathrm{Per}(E, \cdot) := |\mathrm{D}\chi_E|$. In this case, one denotes the polar vector as ν_E, so that $\mathrm{D}\chi_E = \nu_E|\mathrm{D}\chi_E| = \nu_E\mathrm{Per}(E, \cdot)$. A similar notation is used for sets of locally finite perimeter, i.e. those Borel sets E satisfying $\chi_E \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^n)$. The perimeter measure can be seen as a generalization of the concept of $(n-1)$-dimensional surface area.

Of great importance, and a hint towards the strong link between sets of finite perimeter and functions of bounded variation in general, we have the coarea formula [78],

$$|\mathrm{D}f| = \int_{\mathbb{R}} |\mathrm{D}\chi_{\{f>t\}}|\mathrm{d}t, \tag{1.1.1}$$

which holds for any $f \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^n)$.

1.1.2 De Giorgi's Theorem

A deep result establishes that sets of (locally) finite perimeter behave infinitesimally as sets with regular boundary do. Specifically, as proved in [65], given a set of (locally) finite perimeter E, there exists an $(n-1)$-rectifiable set $\mathcal{F}E$ such that the perimeter measure is concentrated on $\mathcal{F}E$, more precisely

$$|\mathrm{D}\chi_E| = \mathcal{H}^{n-1} \llcorner \mathcal{F}E$$

(here and after, $\mathcal{H}$ denotes the Hausdorff measure), and satisfying the following blow-up conditions. For every $x_0 \in \mathcal{F}E$, the blow-ups $\iota_{x_0,r}(E)$ converge in measure to the half-space

$$\{z \in \mathbb{R}^n : z \cdot \nu_E(x_0) \geq 0\}$$

and the blow-ups $r^{-(n-1)}(\iota_{x_0,r})_*|\mathrm{D}\chi_E|$ weakly converge to

$$\mathcal{H}^{n-1} \llcorner \{z \in \mathbb{R}^n : z \cdot \nu_E(x_0) = 0\},$$

as $r \searrow 0$. Here $\iota_{x_0,r} : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $y \mapsto r^{-1}(y - x_0)$ and $(\iota_{x_0,r})_*|\mathrm{D}\chi_E|$ is the push-forward of $|\mathrm{D}\chi_E|$ through $\iota_{x_0,r}$.

1.1.3 Fine properties

Given $f \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^n)^m$, one considers its singular set S_f, i.e. the complement of the set of approximate continuity of f, the latter defined as the set of those $x \in \mathbb{R}^n$ such that there exists (a unique) $\bar{f}(x) \in \mathbb{R}^m$ satisfying

$$\lim_{r\searrow 0} \fint_{B_r(x)} |f - \bar{f}(x)|\mathrm{d}\mathcal{L}^n = 0. \tag{1.1.2}$$

Here and after, the dashed integral sign denotes the averaged integral. Moreover, there exists a $\mathcal{H}^{n-1}$-rectifiable set J_f, the jump set of f, such that $\mathcal{H}^{n-1}(S_f \setminus J_f) = 0$ (this is meaningful, as $|\mathrm{D}f| \ll \mathcal{H}^{n-1}$), that is characterized as follows. If $\nu_f^J : J_f \to \mathbb{R}^n$ denotes a Borel choice of the unit normal to J_f, every point $x \in J_f$ is a jump point, i.e. there exist $f^r(x), f^l(x) \in \mathbb{R}^m$, satisfying

$$\lim_{r\searrow 0} \fint_{B_r^+(x,\nu_f^J(x))} |f - f^r(x)|\mathrm{d}\mathcal{L}^n = \lim_{r\searrow 0} \fint_{B_r^-(x,\nu_f^J(x))} |f - f^l(x)|\mathrm{d}\mathcal{L}^n = 0, \tag{1.1.3}$$

where $B_r^{\pm}(x,\nu_f^J(x)) := \{y \in B_r(x) : \pm(y-x)\cdot\nu_f^J(x) \geq 0\}$. On J_f, we set

$$\bar{f}(x) := \frac{f^r(x)+f^l(x)}{2}.$$

Recall (1.1.2) and notice that $\bar{f}$ is a Borel representative of f, called the precise representative.

It is customary to split the total variation into absolutely continuous part and singular part as

$$|\mathrm{D}f| = |\mathrm{D}f|^a + |\mathrm{D}f|^s \qquad \text{where } |\mathrm{D}f|^a \ll \mathcal{L}^n \text{ and } |\mathrm{D}f|^s \perp \mathcal{L}^n,$$

and further split the singular part into jump part and Cantor part,

$$|\mathrm{D}f|^s = |\mathrm{D}f|^j + |\mathrm{D}f|^c \qquad \text{where } |\mathrm{D}f|^j = |\mathrm{D}f| \llcorner J_f.$$

1.1.4 Calculus rules

As soon as one has at disposal the precise representative of a map of bounded variation, it is possible to investigate various calculus rules. The most powerful is the general chain rule of [12]: if $f \in \mathrm{BV}(\mathbb{R}^n)^m$ and $\varphi \in \mathrm{LIP}(\mathbb{R}^m)$ with $\varphi(0)=0$, then $\varphi \circ f \in \mathrm{BV}(\mathbb{R}^m)$ and it holds

$$\mathrm{D}(\varphi\circ f)\llcorner J_f = \big(\varphi(f^r)-\varphi(f^l)\big)\nu_f^J|\mathrm{D}f|\llcorner J_f,$$
$$\mathrm{D}(\varphi\circ f)\llcorner (\mathsf{X}\setminus J_f) = \left(\nabla\varphi(\bar{f})\cdot\frac{\mathrm{d}\mathrm{D}f}{\mathrm{d}|\mathrm{D}f|}\right)|\mathrm{D}f|\llcorner(\mathsf{X}\setminus J_f).$$

Here, it is part of the claim that for $|\mathrm{D}f|$-a.e. $x \notin J_f$, φ is differentiable at $\bar{f}(x)$ in the directions of the image of $\frac{\mathrm{d}\mathrm{D}f}{\mathrm{d}|\mathrm{D}f|}$. Hence, the term in brackets in the second line is meaningful (this is not due to Rademacher Theorem, notice indeed that it is possible that the image of $\bar{f}$ is contained in a negligible set, see the discussion after (4.3.10)).

Among the consequences of the general chain rule, we find the Vol'pert chain rule (which has been proved, previously and independently, in [125, 126]), that is the specialization to the case $\varphi \in C^1(\mathbb{R}^m)\cap\mathrm{LIP}(\mathbb{R}^m)$: in this case, we can even combine the two equations above using the so-called "Vol'pert averaged superposition" as

$$\mathrm{D}(\varphi\circ f) = \left(\int_0^1 \nabla\varphi(tf^r+(1-t)f^l)\cdot\frac{\mathrm{d}\mathrm{D}f}{\mathrm{d}|\mathrm{D}f|}\,\mathrm{d}t\right)|\mathrm{D}f|.$$

Finally, if $f,g \in \mathrm{BV}(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$, it holds that $fg \in \mathrm{BV}(\mathbb{R}^n)$ with

$$\mathrm{D}(fg) = \bar{f}\mathrm{D}g + \bar{g}\mathrm{D}f,$$

which can be seen as a straightforward consequence of the above chain rule applied to $\varphi(t)=t^2$ (by polarization) or to $\varphi(u,v)=uv$ (applied to (f,g)).

1.1.5 Subgraphs

The results of this section provide another inspiring link between sets of finite perimeter and general functions of bounded variation (recall the coarea formula (1.1.1)). Let $f \in \mathrm{BV}(\mathbb{R}^n)$. Then, as investigated in [107], see also [79], setting

$$\mathcal{G}_f := \{(x,t)\in\mathbb{R}^n\times\mathbb{R} : t < f(x)\},$$

it turns out that $\mathcal{G}_f$ is a set of locally finite perimeter. Furthermore, if we denote $\pi^1 : \mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^n$ the map defined by $(x,t)\mapsto x$, the expressions concerning the push-forwards $\pi^1_*\mathrm{D}_i\chi_{\mathcal{G}_f}$, $\pi^1_*|\mathrm{D}_i\chi_{\mathcal{G}_f}|$, and $\pi^1_*|\mathrm{D}\chi_{\mathcal{G}_f}|$ can be computed in terms of $\mathrm{D}f$ (such rules are obviously compatible with the usual ones, recall e.g. the area formula, in the case in which f is smooth).

1.1.6 Alberti's Rank one Theorem

It is not hard to prove that on J_f, $\frac{\mathrm{D}f}{\mathrm{d}|\mathrm{D}f|} = (f^r - f^l) \otimes \nu_f^J$, in particular, $\frac{\mathrm{D}f}{\mathrm{d}|\mathrm{D}f|}$ is a rank one matrix $|\mathrm{D}f| \llcorner J_f$-a.e. Alberti's Rank one Theorem, conjectured in [14] and proved in [1] (see also [106]), shows that this behaviour also holds for the Cantor part. In other words, for any $f \in \mathrm{BV}(\mathbb{R}^n)^m$, $\frac{\mathrm{D}f}{\mathrm{d}|\mathrm{D}f|}$ is a rank one matrix $|\mathrm{D}f|^s$-a.e. Notice that the same claim for the absolutely continuous part of $|\mathrm{D}f|$ is obviously false.

The Rank one Theorem has found application, for example, in the field of partial differential equations, as in [8, 38] (it has to be said that the Rank one Theorem is not strictly necessary for [8], see [8, Remark 3.7]), and in the field of relaxation of integral functionals, see [13].

1.2 RCD spaces

We introduce RCD spaces and some of their key properties that will be needed in the rest of this introduction, with the aim of motivating the fact that, despite being non-smooth spaces, RCD spaces have a regular enough structure to allow the generalization of the results recalled in Section 1.1. We do not provide a systematic introduction to this deep subject, as it is not the focus of the manuscript and also as we are going to use the fine properties of RCD spaces as a "black box" rather than a comprehensive understanding of the subject. We refer to the surveys [124, 9, 85], the references therein, and the textbook [123], for an account on this subject.

1.2.1 Definitions

Now we briefly introduce the RCD condition, which stands for Riemannian Curvature-Dimension condition. In essence, $\mathrm{RCD}(K, N)$ spaces are those metric measure spaces whose Ricci curvature is bounded from below by K and whose dimension is bounded from above by N, in a synthetic sense. Here and after, a metric measure space is a triplet $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ where X is a set, d is a complete and separable distance and m is a (non-negative) Borel measure giving finite mass to bounded sets. On metric measure spaces one can define the 2-Sobolev space $W^{1,2}(\mathsf{X})$. For example, the approach of [21], after [57], is as follows. Given a function $g \in \mathrm{LIP}(\mathsf{X})$, one defines

$$\mathrm{lip}(g)(x) := \begin{cases} \limsup\limits_{y \to x} \dfrac{|g(y) - g(x)|}{\mathsf{d}(y, x)} & \text{if } x \text{ is not isolated,} \\ 0 & \text{if } x \text{ is isolated} \end{cases}$$

and then the Cheeger energy of $f \in L^2(\mathsf{m})$ is defined by

$$\mathrm{Ch}(f) := \inf \left\{ \liminf_k \int_{\mathsf{X}} \mathrm{lip}(f_k)^2 \mathrm{d}\mathsf{m} : \{f_k\}_k \subseteq L^2(\mathsf{m}) \cap \mathrm{LIP}(\mathsf{X}), f_k \to f \text{ in } L^2(\mathsf{m}) \right\}.$$

Then the 2-Sobolev space $W^{1,2}(\mathsf{X})$ is defined as

$$W^{1,2}(\mathsf{X}) := \{ f \in L^2(\mathsf{m}) : \mathrm{Ch}(f) < \infty \}$$

and it is endowed with the $\| \cdot \|_{W^{1,2}(\mathsf{X})}$ norm,

$$\|f\|^2_{W^{1,2}(\mathsf{X})} := \|f\|^2_{L^2(\mathsf{m})} + \mathrm{Ch}(f) \qquad \text{for every } f \in W^{1,2}(\mathsf{X}).$$

It turns out that $(W^{1,2}(\mathsf{X}), \| \cdot \|_{W^{1,2}(\mathsf{X})})$ is always a Banach space, but, in general, is not a Hilbert space. Moreover, to $f \in W^{1,2}(\mathsf{X})$, one associates $|\nabla f| \in L^2(\mathsf{m})$, called the minimal asymptotic relaxed slope, satisfying $\mathrm{Ch}(f) = \int_{\mathsf{X}} |\nabla f|^2 \mathrm{d}\mathsf{m}$.

By definition, $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ is an $\mathrm{RCD}(K, N)$ space, where $K \in \mathbb{R}$ and $N \in [1, \infty]$, if it satisfies the Curvature-Dimension condition $\mathrm{CD}(K, N)$ ([120, 121], [104], a suitable convexity property of certain entropy functionals in terms of Wasserstein geodesics) and its 2-Sobolev space is a Hilbert space ([82], [22]). It is possible to give an idea towards the rigorous definition of the RCD condition, considering an equivalent formulation with respect to the one mentioned above. Namely, a metric measure space $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ is an $\mathrm{RCD}(K, N)$ space if, in addition to certain technical assumptions (a bound for the growth of volumes and the so-called Sobolev to Lipschitz property), the associated 2-Sobolev space is a Hilbert space, and moreover the "Bochner inequality"

$$\Delta \frac{|\nabla f|^2}{2} \geq \frac{(\Delta f)}{N} + \nabla f \cdot \nabla \Delta f + K |\nabla f|^2 \tag{1.2.1}$$

holds in a weak sense for any $f \in D(\Delta)$ such that $\Delta f \in H^{1,2}(\mathsf{X})$.

Let us motivate this definition involving the weak Bochner inequality. For an n-dimensional Riemannian manifold (M, g), the well known Bochner equality states that, for a C^3 function f,

$$\Delta \frac{|\nabla f|^2}{2} = \nabla \Delta f \cdot \nabla f + |\mathrm{Hess} f|^2_{\mathrm{HS}} + \mathrm{Ric}(\nabla f, \nabla f). \tag{1.2.2}$$

If the Ricci curvature is bounded from below by K, in the sense that $\mathrm{Ric} \geq Kg$, and $N \geq n$, we can use the bound from below on the Ricci tensor and the inequality $N|\mathrm{Hess} f|^2_{\mathrm{HS}} \geq n|\mathrm{Hess} f|^2_{\mathrm{HS}} \geq (\Delta f)^2$ to prove that (1.2.1) is satisfied in this setting. Conversely, if (1.2.1) above holds for any C^3 function f at any point, then indeed $\mathrm{Ric} \geq Kg$ and $N \geq n$. This is seen by plugging into (1.2.1) suitable chosen test functions, namely, for every $p \in M$, $v \in T_pM$, and $\lambda > 0$, it is possible to choose a C^3 function $f = f_{p,v,\lambda}$ satisfying $\nabla f(p) = v$ and $\nabla^2 f(p) = \lambda \mathrm{Id}$, see [123, Proof of Theorem 14.8] for details in a more general situation (i.e. the weighted case). Thus, we have seen that the inequality (1.2.1) is strong enough to entail bounds on the Ricci tensor and on the dimension, but, with respect to (1.2.2), the term $|\mathrm{Hess} f|$ (which involves second order derivatives and that cannot be directly understood in the sense of distributions) disappeared.

In this Introduction, we will only consider finite dimensional RCD spaces, i.e. the case $N \in [1, \infty)$.

RCD spaces enjoy many interesting properties, for example, the class of RCD spaces (with a bound on the volume of unit balls) is compact with respect to the pointed-measured-Gromov–Hausdorff (pmGH) topology (at the end of this section, we give a brief description of such convergence). Moreover, as discussed above, the $\mathrm{RCD}(K, N)$ condition is compatible with the smooth case: the metric measure space $(M, \mathsf{d}_g, \mathrm{Vol}_g)$ naturally induced by an n-dimensional Riemannian manifold (M, g) is $\mathrm{RCD}(K, N)$ if and only if $\mathrm{Ric}_g \geq K$ and $n \leq N$. As a consequence, the following important fact is usually used to motivate the interest in the class of RCD space:

> given any sequence of Riemannian manifolds with Ricci curvature uniformly bounded from below and dimension uniformly bounded from above, there exists a subsequence converging (in the pmGH topology) to an RCD space.

It is worth mentioning that RCD spaces obtained with such procedure are usually called Ricci-limits, after [59, 60, 61], and are not, in general, Riemannian manifolds.

1.2.2 Properties

This short section recollects results about the structure theory of finite dimensional RCD spaces, obtained in several papers [67, 98, 111, 52, 90, 49]. Given an $\mathrm{RCD}(K, N)$ space $(\mathsf{X}, \mathsf{d}, \mathsf{m})$, there exists $n \in \mathbb{N}$, $n \leq N$, such that $\mathsf{m} = \Theta_n \mathcal{H}^n$, where Θ_n is a Borel function. The integer n is called the essential dimension of the space. Moreover, an RCD space of essential dimension n is n-rectifiable, in the sense that it can be covered, up to an m-negligible set, by countably many Lipschitz images of Borel subsets of $\mathbb{R}^n$. A notable property of RCD spaces is that at m-a.e. $x \in \mathsf{X}$, the blow-up of $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ at x is $(\mathbb{R}^n, \mathsf{d}_e, c_n \mathcal{L}^n)$, where the blow-up is the suitable generalization of the tangent space, and c_n is a renormalization constant that depends only on n. Rigorously, a blow-up is the pmGH limit of the sequence

$$\left(\mathsf{X}, r^{-1}\mathsf{d}, \frac{\mathsf{m}}{\mathsf{m}(B_r(x))}, x\right)$$

as $r \searrow 0$ (later on, we are going to take a different renormalization factor, but this plays no role). Informally speaking, a sequence of pointed $\mathrm{RCD}(K, N)$ spaces $(\mathsf{X}_i, \mathsf{d}_i, \mathsf{m}_i, x_i)$ converges to a pointed metric measure space $(\mathsf{X}_\infty, \mathsf{d}_\infty, \mathsf{m}_\infty, x_\infty)$ in the pmGH topology if there exists a complete and separable metric space $(\mathsf{Z}, \mathsf{d}_\mathsf{Z})$ such that, for every $i \in \mathbb{N} \cup \{\infty\}$, we can isometrically embed $(\mathsf{X}_i, \mathsf{d}_i)$ in $(\mathsf{Z}, \mathsf{d}_\mathsf{Z})$ and the image measures of m_i weakly converge to the image measure of m_∞ and the image points corresponding to x_i converge to the image of x_∞. It is worth mentioning that, after [37], rectifiability and characterization of blow-ups are morally equivalent.

1.2.3 Functions of bounded variation on metric measure spaces

In the seminal paper [108] (and in [6, 7]), a definition of function of bounded variation on general metric measure spaces has been given (see [71] and references therein), inspired by [35] (recall also the definition of the 2-Sobolev space above, [57]). In particular, this definition suits the context of RCD spaces, but, nevertheless, in this section we will consider also less regular spaces. More precisely, given a metric measure space $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ and $f \in L^1(\mathsf{m})$, define

$$|\mathrm{D}f|(U) := \inf\left\{\liminf_k \int_\mathsf{X} \mathrm{lip}(f_k) \mathrm{d}\mathsf{m} : \{f_k\}_k \subseteq L^1(\mathsf{m}) \cap \mathrm{LIP}(\mathsf{X}), f_k \to f \text{ in } L^1(\mathsf{m})\right\} \tag{1.2.3}$$

for any $U \subseteq \mathsf{X}$ open. Then, $f \in \mathrm{BV}(\mathsf{X})$ if $|\mathrm{D}f|(\mathsf{X}) < \infty$ and, in such case, it is possible to prove that $|\mathrm{D}f|$ is induced by a Radon measure, still denoted by $|\mathrm{D}f|$. As in the Euclidean framework, one can define the space $\mathrm{BV}_{\mathrm{loc}}(\mathsf{X})$ and sets of (locally) finite perimeter.

1.2.4 Functions of bounded variation on PI spaces

An intermediate (thanks to [113], [121]) setting between general metric measure spaces and RCD spaces is given by the one of PI spaces. PI spaces are those metric measure spaces $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ such that there exists $\lambda \in \mathbb{R}$ and, for every $R > 0$, there exist $C_D(R), C_R(P) > 0$ satisfying

$$\mathsf{m}(B_{2r}(x)) \leq C_D(R) \mathsf{m}(B_r(x)) \qquad \text{for every } x \in \mathsf{X},\ r \in (0, R),$$

and, for every $f \in \mathrm{LIP}(\mathsf{X})$,

$$\fint_{B_r(x)} \left|f - \fint_{B_r(x)} f \mathrm{d}\mathsf{m}\right| \mathrm{d}\mathsf{m} \leq C_P(R) r \fint_{B_{\lambda R}(x)} \mathrm{lip}(f) \mathrm{d}\mathsf{m} \qquad \text{for every } x \in \mathsf{X},\ r \in (0, R).$$

BV functions on metric measure spaces start behaving like the ones on Euclidean spaces already in the PI framework. For example, given a set of (locally) finite perimeter E, it holds that ([6, 7])

$$|\mathrm{D}\chi_E| = \Theta_E \mathcal{H}^h \llcorner \partial^* E,$$

where Θ_E is a suitable Borel function, $\mathcal{H}^h$ is the one-codimensional spherical Hausdorff measure and is a replacement for $\mathcal{H}^{n-1}$, and $\partial^* E$, the essential boundary of E, is the replacement for $\mathcal{F}E$ and is defined as

$$\partial^* E := \left\{ x \in \mathsf{X} : \limsup_{r \searrow 0} \frac{\mathsf{m}(B_r(x) \cap E)}{\mathsf{m}(B_r(x))} > 0 \text{ and } \limsup_{r \searrow 0} \frac{\mathsf{m}(B_r(x) \setminus E)}{\mathsf{m}(B_r(x))} > 0 \right\}.$$

Then, one is tempted to study further fine properties of functions of bounded variation and hence, for a (scalar valued function) of bounded variation f, one defines $f^\vee, f^\wedge$ (which play the role of f^r, f^l) using approximate limits:

$$f^\wedge(x) := \operatorname{ap} \liminf_{y \to x} f(y) := \sup \left\{ t \in \bar{\mathbb{R}} : \lim_{r \searrow 0} \frac{\mathsf{m}(B_r(x) \cap \{f < t\})}{\mathsf{m}(B_r(x))} = 0 \right\},$$
$$f^\vee(x) := \operatorname{ap} \limsup_{y \to x} f(y) := \inf \left\{ t \in \bar{\mathbb{R}} : \lim_{r \searrow 0} \frac{\mathsf{m}(B_r(x) \cap \{f > t\})}{\mathsf{m}(B_r(x))} = 0 \right\},$$

and finally defines $\bar{f} := \frac{f^\vee + f^\wedge}{2}$. However, it turns out that the usual Leibniz rule does not hold even if one only looks at total variations: the seemingly natural inequality

$$|\mathrm{D}(fg)| \le |\bar{f}||\mathrm{D}g| + |\bar{g}||\mathrm{D}f| \qquad \text{for } f, g \in \mathrm{BV}(\mathsf{X}) \cap L^\infty(\mathsf{X})$$

does not hold in general, see e.g. [102, Example 4.8]. For various reasons, including this one, further investigation is needed in the more regular setting of RCD spaces.

1.3 Contributions

Now we want to discuss how the results of Section 1.1 can be generalized to the setting of RCD spaces as in Section 1.2.

1.3.1 Structure of the thesis

We first draw a parallel between the results recalled in Section 1.1 and their generalization contained in this manuscript.

The first section of this manuscript aims at generalizing the results mentioned in Section 1.1.2. This manuscript begins by recalling the main results about the study of sets of finite perimeter on RCD spaces obtained in [10, 51, 50]. At the time when [10, 51] were written, the powerful result of [70] was still not available. Later on, the result of [70] was used in [50] to sharpen one of the conclusions of [51]. We can then leverage the findings from [70] right from the beginning to introduce the results of the articles mentioned above. This not only significantly shortens the proofs but also makes some of them simpler and better suited for the subsequent development of the theory. The main ideas and techniques used in this section, even after the modifications described above, are the ones contained in [10, 51, 50]. As we do not use the remarkable "splitting via rigidity in the Bakry–Émery inequality in BV" of [10] in this manuscript, we add a brief discussion about the topic in Section 3.6.

The second section of the manuscript deals with the results of Section 1.1.1, Section 1.1.3, and Section 1.1.4. In particular, we define the distributional differential of (vector valued) functions of bounded variation and study some related properties. The material is mostly taken from [42, 43], but simplifications have been made especially for what concerns the proofs. It is worth noticing that in the particular case of sets of finite perimeter, a definition of the distributional differential has already been proposed in [51]. Here, the result has been improved to hold for vector valued functions of bounded variation, but the technique is still borrowed from [51]. With respect to the definition of the distributional differential, a completely different technique can be used, see [44].

The third section deals with the interplay between functions of bounded variations and the sets of locally finite perimeter represented by their subgraphs, see the results of Section 1.1.5. The material is taken from [32, 33] and proofs have been rewritten to become more readable.

The fourth section deals with the bulk of the proof of the Rank one Theorem, [32], and is a generalization of Section 1.1.6. Thanks to the adaptations of some key lemmas about the rectifiability of the reduced boundaries of sets of finite perimeter, the proof contained in this manuscript is shorter than the original one.

Finally, the fifth section is extracted from [46] and deals with the nonlocal characterization of the total variation, but it will not be discussed in this introduction.

1.3.2 Contents

Having thus outlined the structure of the manuscript, we pass to a more detailed discussion about the mathematical content. The statements are not always completely precise, and are sometimes informal: this is done not to overwhelm this introduction with the necessary technicalities, which, given the technical nature of this work, would make this introduction unreasonably long. Also, as most of the results of the first section were already present in the literature, the discussion is limited to some properties of functions of bounded variation, taken from the second, third, and fourth sections (we do not discuss here the fifth section). The purpose of this introduction is to highlight the main challenges faced in obtaining the results presented in this thesis as well as to give a taste of the techniques of geometric measure theory on non-smooth spaces we used.

For what remains of the introduction, fix an $\mathrm{RCD}(K,N)$ space $(\mathsf{X},\mathsf{d},\mathsf{m})$. First, one has to define the objects that are at the centre of the investigation, namely the distributional differentials of functions of bounded variation. There are (at least) two equivalent approaches: the polar decomposition as in [51] and the abstract one as in [44], here we consider the first one. In order to define this object, one needs the machinery of normed modules on metric measure spaces ([84], [69]). Without entering into details, a normed module is the suitable generalization of a vector bundle. For example, the notion of tangent module, read in the smooth (Riemannian) framework, is the algebraic object whose elements are the sections of the tangent bundle. In the non-smooth world, elements in normed modules are defined up to equivalence with respect to a suitable outer/Radon measure; even though the possibility of defining these elements almost everywhere with respect to total variations is a key point, let us avoid this discussion. To sum up, it is possible to show that given a function of bounded variation f, we have the "polar decomposition" $\mathrm{D}f := \nu_f|\mathrm{D}f|$, where

- ν_f is an element of the tangent module, defined up to $|\mathrm{D}f|$-a.e. equivalence,
- $|\mathrm{D}f|$ is the total variation of f, as defined in (1.2.3).

Generalizing this notion to vector valued functions of bounded variation does not take much effort. Indeed, if $f = (f_1, \dots, f_m) \in \mathrm{BV}(\mathsf{X})^m$, one defines $\mathrm{D}f := \nu_f|\mathrm{D}f|$:

- ν_f is an element of the Cartesian product of m copies of the tangent module, defined up to $|\mathrm{D}f|$-a.e. equivalence,

- $|\mathrm{D}f|$ is the total variation of f, seen as a vector valued function of bounded variation. We omit the adaptation of (1.2.3) to the vector valued setting, but it is important to mention that this definition is compatible with the standard one in the Euclidean context.

As soon as that one has the object $\mathrm{D}f$, the first step is the investigation of its fine properties. Namely, a generalization of the coarea formula as in Section 1.1.1 ([108]) and rather soft arguments building upon the theory of sets of finite perimeter, allow us to prove what follows. For a vector valued function of bounded variation $f \in \mathrm{BV}(\mathsf{X})^m$, there exists a vector field ν_f^J ("normal to J_f") such that, for every $i = 1, \dots, m$,

$$\nu_{f_i} \text{ is parallel to } \nu_f^J \qquad \text{on } J_f,$$

where J_f is a suitable generalization of the jump set of f. Also, we obtain the analogue of (1.1.3). Moreover, adopting the notation as in Section 1.1.1, we prove that

$$|\mathrm{D}f| \llcorner J_f = d_n |f^r - f^l| \mathrm{d}\mathcal{H}^h \llcorner J_f,$$

where d_n is a constant that depends only on the essential dimension of $(\mathsf{X}, \mathsf{d}, \mathsf{m})$. The just mentioned properties will play a role in the derivation of the calculus rules, as they imply that it is morally enough to establish the calculus properties outside the jump set.

For what concerns the calculus properties, we generalize the results of Section 1.1.4. In the Euclidean framework, one can follow this scheme:

1. General chain rule: compute $\mathrm{D}(\varphi \circ f)$, where $f \in \mathrm{BV}(\mathsf{X})^m$ and $\varphi \in \mathrm{LIP}(\mathbb{R}^m)$,
2. Vol'pert chain rule: compute $\mathrm{D}(\varphi \circ f)$, where $f \in \mathrm{BV}(\mathsf{X})^m$ and $\varphi \in \mathrm{LIP}(\mathbb{R}^m) \cap C^1(\mathbb{R}^m)$,
3. Leibniz rule: compute $\mathrm{D}(fg)$, where $f, g \in \mathrm{BV}(\mathsf{X}) \cap L^\infty(\mathsf{X})$,
4. Chain rule: compute $\mathrm{D}(\varphi \circ f)$, where $f \in \mathrm{BV}(\mathsf{X})$ and $\varphi \in \mathrm{LIP}(\mathbb{R}) \cap C^1(\mathbb{R})$.

Indeed, item (1) can be proved by slicing, relying on elementary results about functions of bounded variation of real variable, [12]. Then item (2) and item (4) follow as particular cases whereas item (3) follows taking as $\varphi \in \mathrm{LIP}(\mathbb{R}^2)$ a function that coincides with $(u, v) \mapsto uv$ on a sufficiently large neighbourhood of $0 \in \mathbb{R}^2$.

In the non-smooth setting the slicing technique used to prove item (1) in the Euclidean framework is not available so we have to resort to other ideas. The technique is then to reverse the procedure, i.e. starting from item (4) and obtain, as a chain of consequences, items (3), (2), and finally (1). In particular, item (4) is obtained via a suitable modification of the coarea formula, carefully exploiting fine properties of functions of bounded variation. Then item (3) is obtained by polarization, from item (4). Item (2) is then proved in the case in which φ is a polynomial, iterating item (3), and then is proved by approximation, in the general case. Finally, (1) is obtained by approximation from (2), exploiting a link between "closability of certain differentiation operators" and "differentiability of Lipschitz functions in related directions", [2]. Just to explain the difficulties encountered in proving the implication from item (2) to item (1), notice the following fact: it is false, in general, that φ is differentiable at $\bar{f}(x)$ for $|\mathrm{D}f|$-a.e. x, hence a statement of the kind $\mathrm{D}(\varphi \circ f) = \nabla\varphi(f)\mathrm{D}f$ has to be suitably interpreted.

In order to build the tools for the proof of the Rank one Theorem, we need to study the relations between functions of bounded variation and the sets of locally finite perimeter given by their subgraphs. Namely, let $f \in \mathrm{BV}(\mathsf{X})$, and consider, in the product space,

$$\mathcal{G}_f := \{(x, t) \in \mathsf{X} \times \mathbb{R} : t < f(x)\}.$$

It is not hard to prove that $\mathcal{G}_f$ is a set of locally finite perimeter. Let $\pi : \mathsf{X} \times \mathbb{R} \to \mathsf{X}$ be defined by $(x,t) \mapsto x$. Then we establish the link between the push-forward $\pi_*|\mathrm{D}\chi_{\mathcal{G}_f}|$ and $|\mathrm{D}f|$. Moreover, we study important relations between $\nu_{\mathcal{G}_f}(x,t)$ and $\nu_f(x)$. As a result, it turns out that the singular behaviour of f is presented at the projections of the points (x,t) at which the vector $\nu_{\mathcal{G}_f}$ is horizontal. In particular, we have a description of the set C_f, i.e. the set on which the Cantor part of $|\mathrm{D}f|$ is concentrated, as a projection of a suitable set in $\mathsf{X} \times \mathbb{R}$.

With the tools mentioned above at our disposal, it is possible to prove the Rank one Theorem, which is the most technical result of the manuscript. The motivation behind the interest in this theorem is to showcase the possibility of obtaining very refined results in the non-smooth setting of RCD spaces. The proof of this result builds upon all the theory developed in this thesis and its bulk aims at showing what follows. Given $f \in \mathrm{BV}(\mathsf{X})^m$, the polar vector ν_f has rank one $|\mathrm{D}f|^c$-a.e., where this means that $\nu_{f_1}, \dots, \nu_{f_m}$ are parallel $|\mathrm{D}f|^c$-a.e. It is not hard to realize that it is enough to prove this statement in the case $m = 2$.

The discussion above implies that it is enough to show that ν_{f_1} and ν_{f_2} are parallel a.e. with respect to $|\mathrm{D}f_1|^c \wedge |\mathrm{D}f_2|^c$, where the wedge denotes the minimum of two measures, i.e. the biggest measure μ satisfying $\mu \le |\mathrm{D}f_1|^c$ and $\mu \le |\mathrm{D}f_2|^c$. Then, one relies on fine properties of functions of bounded variation to reduce the claim to a statement about the reduced boundaries of subgraphs of functions of bounded variation. Then, we prove a transversality lemma in the spirit of [106], exploiting heavily the rectifiability of reduced boundaries of sets of finite perimeter, which ultimately implies the thesis.

Chapter 2

Preliminaries

The aim of this chapter is to give the preliminary notions that are necessary for the understanding of the thesis. For most of the results, we will not give a proof but rather some references, and in some cases, we refer to the references also for the definition of objects involved. Most of the material contained in Section 2.1 and Section 2.2 (and much more) can be found in [89]. We assume that the reader has familiarity with the content of [89], nevertheless, we recall below the notions that we are going to use most frequently.

2.1 Metric measure spaces

Here we deal with metric measure spaces in general, which will be the framework in this manuscript. Later on, we are going to focus on RCD spaces, as in Section 2.2.

2.1.1 Definitions

In this manuscript, we consider only complete and separable metric spaces. A *metric measure space* is a triplet $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ where X is a set, d is a (complete and separable) distance on X and m is a non-negative Borel measure that is finite on balls. We adopt the convention that metric measure spaces have full topological support, that is to say that for any $x \in \mathsf{X}$ and $r > 0$, we have $\mathsf{m}(B_r(x)) > 0$. Also, to avoid pathological situations, we assume that metric measure spaces are not single points.

A *pointed metric measure space* is a quadruplet $(\mathsf{X}, \mathsf{d}, \mathsf{m}, x)$ where $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ is a metric measure space and $x \in \mathsf{X}$. We consider two pointed metric measure spaces $(\mathsf{X}', \mathsf{d}', \mathsf{m}', x')$ and $(\mathsf{X}'', \mathsf{d}'', \mathsf{m}'', x'')$ to be isomorphic if there exists an isometry $\Psi : \mathsf{X}' \to \mathsf{X}''$ such that $\Psi(x') = x''$ and $\Psi_* \mathsf{m}' = \mathsf{m}''$, where the notation $f_* \nu$ denotes the push-forward of the measure ν through the measurable map f.

We denote the Borel σ-algebra of X by $\mathcal{B}(\mathsf{X})$. For B subset of X and A open subset of X, we write $B \Subset A$ if B is a bounded subset of A with $\mathsf{d}(B, \mathsf{X} \setminus A) > 0$. Clearly, if the space is proper (i.e. bounded sets are relatively compact), $B \Subset A$ if and only if $\bar{B}$ is a compact subset of A.

Given $A \subseteq \mathsf{X}$ open, we denote with $\mathrm{LIP}_{\mathrm{loc}}(A)$ the space of Borel functions that are Lipschitz in a neighbourhood of x, for any $x \in A$. If the space is locally compact, $\mathrm{LIP}_{\mathrm{loc}}(A)$ coincides with the space of functions that are Lipschitz on compact subsets of A. $\mathrm{LIP}_{\mathrm{bs}}(\mathsf{X})$ denotes the space of Lipschitz functions that have bounded support. We adopt the usual notation for the various Lebesgue spaces and we extend the meaning of the subscripts loc and bs in the natural way to other function spaces.

2.1.2 Sobolev functions

The *Cheeger energy* ([57], see also [116, 21, 20]) associated to a metric measure space $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ is the convex and lower semicontinuous functional defined on $L^2(\mathsf{m})$ as

$$\mathrm{Ch}(f) := \frac{1}{2} \inf \left\{ \liminf_{k} \int \mathrm{lip}(f_k)^2 \, \mathrm{d}\mathsf{m} : \{f_k\}_k \subseteq \mathrm{LIP_b}(\mathsf{X}) \cap L^2(\mathsf{m}), f_k \to f \text{ in } L^2(\mathsf{m}) \right\}, \tag{2.1.1}$$

where $\mathrm{lip}(f)$ is defined as

$$\mathrm{lip}(f)(x) := \limsup_{y \to x} \frac{|f(y) - f(x)|}{\mathsf{d}(y, x)},$$

which has to be understood to be 0 if x is an isolated point. The finiteness domain of the Cheeger energy is denoted by $H^{1,2}(\mathsf{X})$ and is endowed with the complete norm $\|f\|^2_{H^{1,2}(\mathsf{X})} := \|f\|^2_{L^2(\mathsf{m})} + 2\mathrm{Ch}(f)$. It is possible to identify a canonical and local object $|\mathrm{d}f| \in L^2(\mathsf{m})$, called minimal relaxed slope, providing the integral representation

$$\mathrm{Ch}(f) = \frac{1}{2} \int |\mathrm{d}f|^2 \, \mathrm{d}\mathsf{m} \qquad \text{for every } f \in H^{1,2}(\mathsf{X}).$$

We denote with $H^{1,2}_{\mathrm{loc}}(\mathsf{X})$ the space of measurable functions $f : \mathsf{X} \to \mathbb{R}$ such that for every point $x \in \mathsf{X}$, there exist a neighbourhood of x, $B = B_x$, and a function $f_B \in H^{1,2}(\mathsf{X})$, such that $f = f_B$ m-a.e. on B, and we define $|\mathrm{d}f|$ exploiting locality. We define $S^2(\mathsf{X})$ as the space of functions f such that, for every $n \in \mathbb{N}$, $(f \wedge n) \vee -n \in H^{1,2}_{\mathrm{loc}}(\mathsf{X})$ and such that $|\mathrm{d}f|$ (which is well defined by locality), belongs to $L^2(\mathsf{m})$. Sometimes, with a slight abuse, we write $|\nabla f|$ in place of $|\mathrm{d}f|$. In the setting of infinitesimal Hilbertian spaces, however, it is possible to prove that this is not an abuse at all.

It is important to mention that most of the usual calculus rules hold also in this context.

2.1.3 Infinitesimal Hilbertianity

Any metric measure space on which Ch is a quadratic form is said to be *infinitesimally Hilbertian*, [82]. Under this assumption, (see [82]) it is possible to define a symmetric bilinear form

$$H^{1,2}(\mathsf{X}) \times H^{1,2}(\mathsf{X}) \ni (g, f) \to \nabla f \cdot \nabla g \in L^1(\mathsf{m})$$

such that

$$\nabla f \cdot \nabla f = |\mathrm{d}f|^2 =: |\nabla f|^2 \qquad \mathsf{m}\text{-a.e. for every } f \in H^{1,2}(\mathsf{X}).$$

On infinitesimally Hilbertian metric measure spaces it is possible to define a linear Laplacian operator $\Delta : D(\Delta) \subseteq H^{1,2}(\mathsf{X}) \to L^2(\mathsf{m})$ in the following way: we let $D(\Delta)$ to be the set of those $f \in H^{1,2}(\mathsf{X})$ such that, for some $h \in L^2(\mathsf{m})$, one has

$$\int \nabla f \cdot \nabla g \, \mathrm{d}\mathsf{m} = -\int h g \, \mathrm{d}\mathsf{m} \qquad \text{for every } g \in H^{1,2}(\mathsf{X}), \tag{2.1.2}$$

and, if this is the case, we put $\Delta f = h$, which is uniquely determined by the equation above, as $\mathrm{LIP_{bs}}(\mathsf{X}) \subseteq H^{1,2}(\mathsf{X})$ is dense in $L^2(\mathsf{m})$.

We are going also to use the Laplacian for functions defined on balls, see [25], or [51, Section 1.2.2]. In particular, one fist defines the space $H^{1,2}_{\mathrm{loc}}(B)$, where B is any (open) ball, similarly to what done at the end of Section 2.1.2. Then, $H^{1,2}(B)$ are those functions $f \in H^{1,2}_{\mathrm{loc}}(B)$ with $f, |\nabla f| \in L^2(B, \mathsf{m})$. Then, for $f \in H^{1,2}(B)$, one defines Δf as the unique (if exists) function $h \in L^2(B, \mathsf{m})$ satisfying (2.1.2), tested against Lipschitz functions whose support is contained in B.

2.1.4 Heat flow on infinitesimally Hilbertian spaces

We can define the *heat flow* h_t as the L^2 gradient flow of Ch, whose existence and uniqueness follow from the Komura-Brezis theory. On infinitesimally Hilbertian spaces, we can characterize the heat flow by requiring that for any $u \in L^2(\mathsf{m})$, the curve $[0,\infty) \ni t \mapsto \mathrm{h}_t u \in L^2(\mathsf{m})$ is continuous in $[0,\infty)$, locally absolutely continuous in $(0,\infty)$ and satisfies

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{h}_t u = \Delta \mathrm{h}_t u & \text{for every } t \in (0,\infty), \\ \mathrm{h}_0 u = u, \end{cases}$$

where we implicitly state that if $t > 0$, $\mathrm{h}_t u \in D(\Delta)$. It is possible to prove that on infinitesimally Hilbertian spaces, the heat flow provides a linear, continuous and self-adjoint contraction semigroup in $L^2(\mathsf{m})$, which extends to a linear and continuous contraction semigroup, that we still denote with h_t, in all spaces $L^p(\mathsf{m})$, for $p \in [1,\infty)$. We define h_t on $L^\infty(\mathsf{m})$ in duality with $L^1(\mathsf{m})$, i.e. if $f \in L^\infty(\mathsf{m})$,

$$\int g \mathrm{h}_t f \,\mathrm{d}\mathsf{m} = \int f \mathrm{h}_t g \,\mathrm{d}\mathsf{m} \qquad \text{for every } g \in L^1(\mathsf{m}),$$

and, with this extension, h_t turns out to be a linear and continuous contraction semigroup in all spaces $L^p(\mathsf{m})$ with $p \in [1,\infty]$. Moreover, Δ and h_t, when well defined, commute.

2.1.5 Normed modules

We assume that the reader is familiar with the notion of normed module, introduced in [84], inspired by the theory developed in [127]. To briefly introduce this concept, take a metric measure space $(\mathsf{X},\mathsf{d},\mathsf{m})$. Let R be either $L^\infty(\mathsf{m})$ or $L^0(\mathsf{m})$ and let $\mathcal{M}$ be an algebraic module of the commutative ring R. A L^p*-pointwise norm*, for $p \in \{0\} \cup [1,\infty)$, is a map $|\cdot| : \mathcal{M} \to L^p(\mathsf{m})$ such that

- $|v| \geq 0$ m-a.e. for every $v \in \mathcal{M}$, and $|v| = 0$ m-a.e. if and only if $v = 0$,
- $|v + w| \leq |v| + |w|$ m-a.e. for every $v, w \in \mathcal{M}$
- $|fv| = |f||v|$ m-a.e. for every $f \in R$ and $v \in \mathcal{M}$.

One can then consider

- $L^p(\mathsf{m})$*-normed* $L^\infty(\mathsf{m})$*-modules*, for $p \in [1,\infty)$, i.e. the case of a module $\mathcal{M}$ over $R = L^\infty(\mathsf{m})$, endowed with an $L^p(\mathsf{m})$-pointwise norm $|\cdot|$ such that $\|v\|_{\mathcal{M}} = \||v|\|_{L^p(\mathsf{m})}$ is a complete norm on $\mathcal{M}$,
- $L^0(\mathsf{m})$*-normed* $L^0(\mathsf{m})$*-modules*, i.e. the case of a module $\mathcal{M}$ over $R = L^0(\mathsf{m})$, endowed with an $L^0(\mathsf{m})$-pointwise norm $|\cdot|$ inducing a complete distance on $\mathcal{M}$.

It is possible to prove that there exists a unique couple $(L^2(T^*\mathsf{X}), \mathrm{d})$ where $L^2(T^*\mathsf{X})$ is a L^2-normed L^∞-module (the *cotangent module*) and $\mathrm{d} : H^{1,2}(\mathsf{X}) \to L^2(T^*\mathsf{X})$ a linear operator (the *differential*), such that

i) $|\mathrm{d}f|$ is equal to the one introduced in Section 2.1.2, for every $f \in H^{1,2}(\mathsf{X})$,

ii) $L^2(T^*\mathsf{X})$ is generated (in the sense of modules) by $\{\mathrm{d}f : f \in H^{1,2}(\mathsf{X})\}$.

We define the *tangent module* $L^2(T\mathsf{X})$ as the dual (in the sense of modules) of $L^2(T^*\mathsf{X})$. We define $L^0(T^*\mathsf{X})$ as the L^0-completion of the cotangent module $L^2(T^*\mathsf{X})$ and also (this definition coincides with the previous one if $p=2$)

$$L^p(T^*\mathsf{X}) := \{v \in L^0(T^*\mathsf{X}) : |v| \in L^p(\mathsf{m})\} \qquad \text{for } p \in [1,\infty].$$

Similarly, we define $L^0(T\mathsf{X})$ as the L^0-completion of $L^2(T\mathsf{X})$ and

$$L^p(T\mathsf{X}) := \{v \in L^0(T\mathsf{X}) : |v| \in L^p(\mathsf{m})\} \qquad \text{for } p \in [1,\infty].$$

We also remark that our definition of the tangent and cotangent modules is, in general, different from the one given in [53] for $p \neq 2$. If the space is infinitesimally Hilbertian, it turns out that $L^2(T^*\mathsf{X})$ is a Hilbert module so that we can, and will, identify $L^2(T^*\mathsf{X})$ with its dual $L^2(T\mathsf{X})$, via a map that sends $\mathrm{d}f$ to ∇f (the latter vector field being given by Riesz Theorem).

Definition 2.1.1. Let $p \in \{2,\infty\}$. For $v \in L^p(T\mathsf{X})$ we say that $v \in D(\mathrm{div}^p)$ if there exists a function $g \in L^p(\mathsf{m})$ such that

$$\int \mathrm{d}f(v)\,\mathrm{d}\mathsf{m} = -\int fg\,\mathrm{d}\mathsf{m} \qquad \text{for every } f \in H^{1,2}(\mathsf{X}) \text{ with bounded support,} \tag{2.1.3}$$

and such g, which is uniquely determined, is denoted by $\mathrm{div}\,v$.

Notice that if $v \in D(\mathrm{div}^2) \cap D(\mathrm{div}^\infty)$, then the two objects $\mathrm{div}\,v$ as above coincide, in particular, $\mathrm{div}\,v \in L^2(\mathsf{m}) \cap L^\infty(\mathsf{m})$. From (2.1.3) it follows that $\mathrm{supp}\,(\mathrm{div}\,v) \subseteq \mathrm{supp}\,v$ and also notice that, if the space is infinitesimally Hilbertian and $p=2$ (then $\mathrm{LIP_{bs}}(\mathsf{X}) \subseteq H^{1,2}(\mathsf{X})$ is dense, as a consequence of the result in [20, Section 8.3] or [11]), (2.1.3) reads

$$\int \nabla f \cdot v\,\mathrm{d}\mathsf{m} = \int fg\,\mathrm{d}\mathsf{m} \qquad \text{for every } f \in \mathrm{LIP_{bs}}(\mathsf{X}).$$

Also, the classical calculus rule holds: if $v \in D(\mathrm{div}^\infty)$ and $f \in \mathrm{LIP_b}(\mathsf{X})$, then $fv \in D(\mathrm{div}^\infty)$ and

$$\mathrm{div}(fv) = \mathrm{d}f(v) + f\,\mathrm{div}\,v. \tag{2.1.4}$$

In the case $p=2$, again from the algebra properties of bounded Sobolev functions together with an easy approximation argument, we have that if $v \in D(\mathrm{div}^2) \cap L^\infty(T\mathsf{X})$ and $f \in \mathrm{S}^2(\mathsf{X}) \cap L^\infty(\mathsf{m})$, then $fv \in D(\mathrm{div}^2)$ and the calculus rule above holds.

In the case $p=2$, we often omit to write the superscript 2 for what concerns the divergence.

2.2 RCD spaces

Now we turn to the particular framework of RCD metric measure spaces.

2.2.1 Definitions

The main setting for our investigation is the one of $\mathrm{RCD}(K,N)$ metric measure spaces (for $K \in \mathbb{R}$ and $N \in [1,\infty]$), that are infinitesimally Hilbertian spaces ([82]) satisfying a lower Ricci curvature bound and an upper dimension bound (meaningful if $N < \infty$) in synthetic sense according to [120, 121], [104]. General references on this topic are [19, 21, 22, 23, 29, 83, 84, 74, 89] and we assume the reader to be familiar with part of this material. See also [124, 9, 85] and references therein.

Our focus be mainly on *finite dimensional* RCD spaces, so that in the sequel when we write $\mathrm{RCD}(K,N)$ we will assume $1 \leq N < \infty$.

2.2.2 Doubling and Poincaré

Recall that $\mathrm{RCD}(K,N)$ spaces are *locally uniformly doubling* ([104, 121]), i.e. for every $R>0$ there exists $C_D=C_D(R)>0$ such that

$$\mathsf{m}(B_{2r}(x))\le C_D\mathsf{m}(B_r(x))\qquad\text{for every } x\in\mathsf{X}\text{ and } 0<r<R,\tag{2.2.1}$$

and support a *weak local* $(1,1)$*-Poincaré inequality* ([113]), i.e. there exists $\lambda\ge 1$ and for every $R>0$, there exists $C_P=C_P(R)>0$ such that, for every $f\in\mathrm{LIP}(\mathsf{X})$,

$$\fint_{B_r(x)}\Big|f-\fint_{B_r(x)}f\,\mathrm{d}\mathsf{m}\Big|\,\mathrm{d}\mathsf{m}\le C_P r\fint_{B_{\lambda r}(x)}\mathrm{lip}(f)\,\mathrm{d}\mathsf{m}\qquad\text{for every } x\in\mathsf{X}\text{ and }\ 0<r<R.\tag{2.2.2}$$

By [93, Theorem 5.1], the Poincaré inequality improves to the following form (see also [57] for what concerns this formulation): there exists $\lambda\ge 1$ and for every $R>0$, there exist $C_P'=C_P'(R)>0$ and $Q=Q(R)>1$ such that, for every $f\in\mathrm{LIP}(\mathsf{X})$,

$$\left(\fint_{B_r(x)}\Big|f-\fint_{B_r(x)}f\,\mathrm{d}\mathsf{m}\Big|^{\frac{Q}{Q-1}}\mathrm{d}\mathsf{m}\right)^{\frac{Q-1}{Q}}\le C_P' r\fint_{B_{2\lambda r}(x)}\mathrm{lip}(f)\,\mathrm{d}\mathsf{m}\qquad\text{for every } x\in\mathsf{X}\text{ and } 0<r<R.\tag{2.2.3}$$

Recall that locally uniformly doubling spaces are proper. We call locally uniformly doubling spaces supporting a weak local $(1,1)$-Poincaré inequality PI spaces. We can, and will, assume that $R\mapsto C_D(R), R\mapsto C_P(R), R\mapsto C_P'(R)$ are non-decreasing functions.

2.2.3 Test functions and test vector fields

Following [84, 114] (with the additional request of a L^∞ bound on the Laplacian), we define the vector space of *test functions* on an $\mathrm{RCD}(K,\infty)$ space as

$$\mathrm{TestF}(\mathsf{X}):=\{f\in\mathrm{LIP}(\mathbb{R})\cap L^\infty(\mathsf{m})\cap D(\Delta):\Delta f\in H^{1,2}(\mathsf{X})\cap L^\infty(\mathsf{m})\},$$

and the vector space of *test vector fields* as

$$\mathrm{TestV}(\mathsf{X}):=\left\{\sum_{i=1}^n f_i\nabla g_i: f_i\in\mathrm{S}^2(\mathsf{X})\cap L^\infty(\mathsf{m}), g_i\in\mathrm{TestF}(\mathsf{X})\right\}.$$

To be precise, the original definition of $\mathrm{TestV}(\mathsf{X})$ was slightly different. However, when using test vector fields to define regular subsets of vector fields such as $H^{1,2}_{\mathrm{H}}(T\mathsf{X})$ and $H^{1,2}_{\mathrm{C}}(T\mathsf{X})$ (see Section 2.2.4), the two definitions produce the same subspaces, as one may readily check inspecting the proofs of Lemma 2.2.2 and Lemma 2.2.3 below.

It is possible to see that $\mathrm{TestF}(\mathsf{X})\subseteq H^{1,2}(\mathsf{X})$ is dense. Also, if $f\in H^{1,2}(\mathsf{X})\cap L^\infty(\mathsf{m})$, we can find a sequence $\{f_n\}_n\subseteq\mathrm{TestF}(\mathsf{X})$ with $f_n\to f$ in $H^{1,2}(\mathsf{X})$ and $\|f_n\|_{L^\infty(\mathsf{m})}\le\|f\|_{L^\infty(\mathsf{m})}$. Using [89, Theorem 6.1.11] (extracted from [114]), one proves that $\mathrm{TestF}(\mathsf{X})$ is an algebra. Clearly, if $f\in\mathrm{S}^2(\mathsf{X})\cap L^\infty(\mathsf{m})\supseteq\mathrm{TestF}(\mathsf{X})$ and $v\in\mathrm{TestV}(\mathsf{X})$, then $fv\in\mathrm{TestV}(\mathsf{X})$.

2.2.4 Second order calculus

By [84], we know that RCD spaces admit second order calculus. In particular, it has been defined the object of covariant derivative for vector fields in $L^2(T\mathsf{X})$, in particular, of Hessian of functions, denoted by $\mathrm{Hess}f$. These definitions build upon integration by parts and the use of test vector fields as in Section 2.2.3. Vector fields admitting a covariant derivative in L^2 are said to belong to $W^{1,2}_{\mathrm{C}}(T\mathsf{X})$, which is proved to be a normed space, subspace of a tensor product of the tangent/cotangent module. The closure of $\mathrm{TestV}(\mathsf{X})$ in the $W^{1,2}_{\mathrm{C}}(T\mathsf{X})$ topology is denoted by $H^{1,2}_{\mathrm{C}}(T\mathsf{X})$. A remarkable fact is that for any $f \in D(\Delta)$, it holds that $\nabla f \in H^{1,2}_{\mathrm{C}}(T\mathsf{X})$, in other words, f admits a Hessian in L^2 (see [84], but in this direction, also [36, 114, 122]). The spaces $W^{1,2}_{\mathrm{H}}(T\mathsf{X})$ and $H^{1,2}_{\mathrm{H}}(T\mathsf{X})$ have been defined in [84]. The former is the space of L^2 vector fields with divergence in L^2 and exterior derivative in L^2, whereas the latter is the closure of test vector fields in the topology of $W^{1,2}_{\mathrm{H}}(T\mathsf{X})$.

For future reference, we record a consequence of the improved Bochner inequality of [94], stated in [51, (1.22)]. For every $f \in D(\Delta)$ with $\Delta f = 0$,

$$r^2 \int_{B_r(x)} |\mathrm{Hess} f|^2 \mathrm{d}\mathsf{m} \le C_{K,N} \inf_{m\in\mathbb{R}} \int_{B_{2r}(x)} \big||\nabla f|^2 - m\big| \mathrm{d}\mathsf{m} - r^2 K \int_{B_{2r}(x)} |\nabla f|^2 \mathrm{d}\mathsf{m}. \tag{2.2.4}$$

We conclude this subsection with a couple of simple lemmas.

Lemma 2.2.1. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K,\infty)$ *space and let* $v = (v_1, \dots, v_m) \in H^{1,2}_{\mathrm{H}}(T\mathsf{X})^m$ *with* $|v| \le 1$ m*-a.e. Then there exists a sequence* $\{v^k = (v^k_1, \dots, v^k_m)\}_k \subseteq \mathrm{TestV}(\mathsf{X})^m$ *such that* $|v^k| \le 1$ m*-a.e. for every* k *and* $v^k_i \to v_i$ *in* $H^{1,2}_{\mathrm{H}}(T\mathsf{X})$ *for every* $i = 1, \dots, m$.

Proof. By the very definition of $H^{1,2}_{\mathrm{H}}(T\mathsf{X})$, for every $i = 1, \dots, m$, we have a sequence $\{w^k_i\} \subseteq \mathrm{TestV}(\mathsf{X})^m$ with $w^k_i \to v_i$ in $H^{1,2}_{\mathrm{H}}(T\mathsf{X})$. Set then, for $\varepsilon > 0$,

$$v^{k,\varepsilon}_i := \frac{1}{(1+\varepsilon) \vee \sqrt{\sum_j |w^k_j|^2}} w^k_i$$

and finally $v^{k,\varepsilon} := (v^{k,\varepsilon}_1, \dots, v^{k,\varepsilon}_m)$. It is clear that $|v^{k,\varepsilon}| \le 1$ m-a.e. so that, using also a diagonal argument, it suffices to show that for every $i = 1, \dots, m$,

$$v^{k,\varepsilon}_i \to \frac{1}{1+\varepsilon} v_i \qquad \text{in } H^{1,2}_{\mathrm{H}}(T\mathsf{X}) \text{ as } k \to \infty.$$

Fix then $i = 1, \dots, m$ and $\varepsilon > 0$. It is clear that

$$v^{k,\varepsilon}_i \to \frac{1}{1+\varepsilon} v_i \qquad \text{in } L^2(T\mathsf{X}) \text{ as } k \to \infty.$$

By the calculus rules in Lemma 2.2.2 below, we just have to show that

$$\left| \nabla \frac{1}{(1+\varepsilon) \vee \sqrt{\sum_j |w^k_j|^2}} \right| |w^k_i| \to 0 \qquad \text{in } L^2(\mathsf{m}) \text{ as } k \to \infty.$$

Set now $A^{k,\varepsilon} := \left\{ \sqrt{\sum_j |w^k_j|^2} > 1 + \varepsilon \right\}$ and notice that $A^{k,\varepsilon} \to \emptyset$ in $L^0(\mathsf{m})$ as $k \to \infty$. Using the calculus rules, we can estimate

$$
\begin{aligned}
&\left|\nabla \frac{1}{(1+\varepsilon)\vee\sqrt{\sum_j |w_j^k|^2}}\right||w_i^k| \\
&\quad\le \frac{1}{2}\chi_{A^{k,\varepsilon}}\left(\frac{1}{\sum_j |w_j^k|^2}\right)^{3/2}\left|\nabla\sum_j |w_j^k|^2\right||w_i^k| \\
&\quad\le \chi_{A^{k,\varepsilon}}\left(\frac{1}{\sum_j |w_j^k|^2}\right)^{3/2}\left(\sum_j |w_j^k|^2\right)^{1/2}\left(\sum_j |\nabla w_j^k|^2\right)^{1/2}|w_i^k| \\
&\quad\le \chi_{A^{k,\varepsilon}}\left(\frac{1}{1+\varepsilon}\right)^{2}\sum_j |\nabla w_j^k||w_i^k|,
\end{aligned}
$$

where in the second inequality we used the Cauchy-Schwarz inequality, and then we see that the last term converges to 0 in $L^2(\mathsf{m})$ as $k\to\infty$. □

In the previous proof we used the following calculus rules, which are an immediate consequence of the already known ones proved in [84]. We add also another lemma, again based on [84], which is not explicitly used in this work but whose proof grants coincidence between the definitions of $H^{1,2}_{\mathrm{C}}(T\mathsf{X})$ via the usual definition of test vector fields and our definition of test vector field.

Lemma 2.2.2. *Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,\infty)$ space, $X\in W^{1,2}_{\mathrm{H}}(T\mathsf{X})\cap L^\infty(T\mathsf{X})$ and $f\in \mathrm{S}^2(\mathsf{X})\cap L^\infty(\mathsf{m})$. Then $fX\in W^{1,2}_{\mathrm{H}}(T\mathsf{X})$ and*

$$
\begin{aligned}
\mathrm{div}(fX) &= \nabla f\cdot X + f\,\mathrm{div}\, X,\\
\mathrm{d}(fX) &= \nabla f\wedge X + f\mathrm{d}X.
\end{aligned}
$$

If moreover $X\in H^{1,2}_{\mathrm{H}}(T\mathsf{X})$, then $fX\in H^{1,2}_{\mathrm{H}}(T\mathsf{X})$.

Proof. Recall that $W^{1,2}_{\mathrm{H}}(T\mathsf{X})$ and $H^{1,2}_{\mathrm{H}}(T\mathsf{X})$ are Hilbert spaces. We prove the claim with an approximation argument. Also, as in the discussion after [84, Definition 3.5.11], if $\omega\in L^2(T\mathsf{X})$, then $\omega\in D(\delta)$ if and only if $\omega\in D(\mathrm{div})$ and, if this is the case, $\delta\omega=-\mathrm{div}(\omega)$.

If $f\in\mathrm{TestF}(\mathsf{X})$, the claim is a consequence of [84, Proposition 3.5.4] (which is stated with a slightly different definition of $\mathrm{TestV}(\mathsf{X})$) and the calculus rules for the divergence and the following approximation argument. If $\{X_n\}_n\subseteq\mathrm{TestV}(\mathsf{X})$ with $X_n\to X$ in $W^{1,2}_{\mathrm{H}}(T\mathsf{X})$, then $fX_n\in\mathrm{TestV}(\mathsf{X})$ and $fX_n\to fX$ in $W^{1,2}_{\mathrm{H}}(T\mathsf{X})$ (see the next paragraph for more details).

If $f\in H^{1,2}(\mathsf{X})\cap L^\infty(\mathsf{m})$, take $\{f_n\}_n\subseteq\mathrm{TestF}(\mathsf{X})$ with $\|f_n\|_{L^\infty(\mathsf{m})}$ uniformly bounded and $f_n\to f$ in $H^{1,2}(\mathsf{X})$. Now we can use the calculus rules for $f_n\in\mathrm{TestF}(\mathsf{X})$ and easily prove, using also dominated convergence,

$$
\begin{aligned}
\mathrm{div}(f_nX) &= \nabla f_n\cdot X + f_n\mathrm{div}\,X \to \nabla f\cdot X + f\mathrm{div}\,X &&\text{in } L^2(\mathsf{m})\\
\mathrm{d}(f_nX) &= \nabla f_n\wedge X + f_n\mathrm{d}X \to \nabla f\wedge X + f\mathrm{d}X &&\text{in } L^2(\Lambda^2 T\mathsf{X}).
\end{aligned}
$$

This shows that $fX\in W^{1,2}_{\mathrm{H}}(T\mathsf{X})$, that the calculus rules hold and finally that $f_nX\to fX$ in $W^{1,2}_{\mathrm{H}}(T\mathsf{X})$, so that if $X\in H^{1,2}_{\mathrm{H}}(T\mathsf{X})$, then $fX\in H^{1,2}_{\mathrm{H}}(T\mathsf{X})$.

If $f\in\mathrm{S}^2(\mathsf{X})\cap L^\infty(\mathsf{m})$ we fix $\bar x\in\mathsf{X}$ and we take $\{\varphi_n\}_n\subseteq\mathrm{LIP_{bs}}(\mathsf{X})$ with $\varphi_n(x):=((n-\mathsf{d}(x,\bar x))\wedge 1)^+$. Similar computations to the ones of the previous paragraph with $f\varphi_n$ in place of f_n show that

$fX \in W^{1,2}_{\rm H}(T\mathsf{X})$, that the calculus rules hold and finally that $(f\varphi_n)X \to fX$ in $W^{1,2}_{\rm H}(T\mathsf{X})$, so that if $X \in H^{1,2}_{\rm H}(T\mathsf{X})$, then $fX \in H^{1,2}_{\rm H}(T\mathsf{X})$. □

Lemma 2.2.3. *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *be an* $\mathrm{RCD}(K,\infty)$ *space,* $X \in W^{1,2}_{\rm C}(T\mathsf{X}) \cap L^\infty(T\mathsf{X})$ *and* $f \in \mathrm{S}^2(\mathsf{X}) \cap L^\infty(\mathsf{m})$. *Then* $fX \in W^{1,2}_{\rm C}(T\mathsf{X})$ *and*

$$\nabla(fX) = \nabla f \otimes X + f\nabla X.$$

If moreover $X \in H^{1,2}_{\rm C}(T\mathsf{X})$, *then* $fX \in H^{1,2}_{\rm C}(T\mathsf{X})$.

Proof. Recall that $W^{1,2}_{\rm C}(T\mathsf{X})$ and $H^{1,2}_{\rm C}(T\mathsf{X})$ are Hilbert spaces. We prove the claim with an approximation argument.

Assume first $f \in H^{1,2}(\mathsf{X}) \cap L^\infty(\mathsf{m})$. Then the first part of the statement has been proved in [84, Proposition 3.4.5]. The second part follows approximating X with a sequence of test vector fields.

If $f \in \mathrm{S}^2(\mathsf{X}) \cap L^\infty(\mathsf{m})$ we fix $\bar{x} \in \mathsf{X}$ and we take $\{\varphi_n\}_n \subseteq \mathrm{LIP}_{\rm bs}(\mathsf{X})$ with $\varphi_n(x) := ((n - \mathsf{d}(x,\bar{x})) \wedge 1)^+$ and we set $f_n := f\varphi_n$. We can use the calculus rule for $f_n \in H^{1,2}(\mathsf{X}) \cap L^\infty(\mathsf{m})$ and easily prove, using also dominated convergence,

$$\nabla(f_nX) = \nabla f_n \otimes X + f_n\nabla X \to \nabla f \otimes X + f\nabla X \qquad \text{in } L^2(T^{\otimes 2}\mathsf{X}).$$

This shows that $fX \in W^{1,2}_{\rm C}(T\mathsf{X})$, that the calculus rule holds and finally that $f_nX \to fX$ in $W^{1,2}_{\rm C}(T\mathsf{X})$, so that if $X \in H^{1,2}_{\rm C}(T\mathsf{X})$, then $fX \in H^{1,2}_{\rm C}(T\mathsf{X})$. □

2.2.5 Heat flow on RCD spaces

The RCD condition entails good properties at the level of the heat flow semigroup. A first important feature is the *L^∞-Lipschitz regularization property*: given any function $f \in L^\infty(\mathsf{m})$ in an $\mathrm{RCD}(K,\infty)$ space and $t \in (0,1]$, it holds $|\nabla \mathrm{h}_t f| \in L^\infty(\mathsf{m})$ and

$$\||\nabla \mathrm{h}_t f|\|_{L^\infty(\mathsf{m})} \le \frac{\|f\|_{L^\infty(\mathsf{m})}}{e^K\sqrt{2t}}. \tag{2.2.5}$$

In particular, $\mathrm{h}_t f$ admits a Lipschitz representative (which we still denote by $\mathrm{h}_t f$) having Lipschitz constant at most $\frac{\|f\|_{L^\infty(\mathsf{m})}}{e^K\sqrt{2t}}$; this is a consequence of the so-called *Sobolev-to-Lipschitz property* of RCD spaces, which states that every Sobolev function $f \in W^{1,2}(\mathsf{X})$ satisfying $|\nabla f| \le 1$ m-a.e. has a 1-Lipschitz representative. Notice also that the maximum principle ensures that for any $f \in L^\infty(\mathsf{m})$

$$|\mathrm{h}_t f| \le \|f\|_{L^\infty(\mathsf{m})} \qquad \text{everywhere on } \mathsf{X}.$$

Given any $\mu \in \mathscr{P}_2(\mathsf{X})$ (i.e. μ is a Borel probability measure on X with finite second moment), it makes sense to define $\mathrm{h}_t\mu$ for any $t > 0$ as the unique element of $\mathscr{P}_2(\mathsf{X})$ satisfying the identity

$$\int f \mathrm{d}\mathrm{h}_t\mu = \int \mathrm{h}_t f \mathrm{d}\mu \qquad \text{for every } f \in C_{\rm b}(\mathsf{X}), \tag{2.2.6}$$

where we took the Lipschitz representative of f at the right hand side. Equivalently ([19, 22, 21, 80]), h_t is the EVI_K gradient flow of the entropy, and it turns out that the existence of this gradient flow for any initial datum can be used to characterize $\mathrm{RCD}(K,\infty)$ spaces among length spaces with a growth condition on the reference measure. Moreover, the heat flow of measures is K-contractive with respect to the Wasserstein W_2 distance and, for $t > 0$, maps probability measures into probability measures that are absolutely continuous with respect to m (the latter assertion is an immediate consequence of fact that $t \mapsto \mathrm{h}_t\mu$ is the gradient flow of the entropy). Then, we can define the *heat kernel* $(0,+\infty) \times \mathsf{X} \times \mathsf{X} \ni (t,x,y) \mapsto p_t(x,y)$ as

$$p_t(x,\cdot) := \frac{\mathrm{d}\mathrm{h}_t\delta_x}{\mathrm{d}\mathsf{m}},$$

where δ_x stands for the Dirac measure at x.

We now focus on $\mathrm{RCD}(K,N)$ spaces $(\mathsf{X},\mathsf{d},\mathsf{m})$ with $N < \infty$. Recall from Section 2.2.2 that these spaces $(\mathsf{X},\mathsf{d},\mathsf{m})$ are PI, thus accordingly $(t,x,y) \mapsto p_t(x,y)$ admits a locally Hölder continuous representative by [118, 119]. In [97] it has been proved that, for any $\varepsilon > 0$, there exist positive constants $C_1 = C_1(\varepsilon,K,N) > 0$ and $C_2 = C_2(\varepsilon,K,N) > 0$ such that for every $t > 0$, $x,y \in \mathsf{X}$, the following estimate holds

$$\frac{1}{C_1\mathsf{m}(B_{\sqrt{t}}(y))}\exp\left\{-\frac{\mathsf{d}(x,y)^2}{(4-\varepsilon)t} - C_2t\right\} \le p_t(x,y) \le \frac{C_1}{\mathsf{m}(B_{\sqrt{t}}(y))}\exp\left\{-\frac{\mathsf{d}(x,y)^2}{(4+\varepsilon)t} + C_2t\right\}. \tag{2.2.7}$$

For any finite Borel measure $\mu \ge 0$ on X we can define

$$\mathrm{h}_t\mu(x) := \int p_t(x,y)\mathrm{d}\mu(y) \qquad \text{for every } x \in \mathsf{X}.$$

Using Fubini's theorem, one can check that this definition is consistent with the one in (2.2.6) when $\mu \in \mathscr{P}_2(\mathsf{X})$, and that $\mathrm{h}_t(f\mathsf{m}) = \mathrm{h}_t f$ for every $f \in L^1(\mathsf{m})$ non-negative.

It is worth mentioning that the regularizing properties of the heat flow on (finite dimensional) RCD spaces have been used to prove the existence of "good" cut-off functions, [111, Lemma 3.1] and [29, Lemma 6.7].

Lemma 2.2.4. *Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,N)$ space. For any $0 < 2r < R$ and $x \in \mathsf{X}$, there exists $\eta \in \mathrm{TestF}(\mathsf{X})$ satisfying*

- $0 \le \eta \le 1$, $\eta = 1$ *on* $B_r(x)$, $\eta = 0$ *on* $\mathsf{X} \setminus B_{2r}(x)$,
- $r^2|\Delta\eta| + r|\nabla\eta| \le C$, *where* $C = C_{K,N,R}$ *is a constant that depends only on* K, N *and* R.

Heat flow and vector fields

On an $\mathrm{RCD}(K,\infty)$ space $(\mathsf{X},\mathsf{d},\mathsf{m})$, following [84], one can consider $\mathrm{h}_{\mathrm{H},t}$, the gradient flow relative to the augmented Hodge energy functional in $L^2(T\mathsf{X})$, which is defined as

$$\mathcal{E}_{\mathrm{H}}(\omega) = \begin{cases} \frac{1}{2}\displaystyle\int |\mathrm{d}\omega|^2 + \mathrm{div}(\omega)^2\mathrm{d}\mathsf{m} & \text{if } \omega \in W^{1,2}_{\mathrm{H}}(TX), \\ +\infty & \text{otherwise.} \end{cases}$$

This means that for every $v \in L^2(T\mathsf{X})$ the curve $t \mapsto \mathrm{h}_{\mathrm{H},t}v \in L^2(T\mathsf{X})$ is the unique curve that is continuous in $[0,\infty)$, locally absolutely continuous in $(0,\infty)$ and satisfies

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{h}_{\mathrm{H},t}v = -\Delta_{\mathrm{H}}\mathrm{h}_{\mathrm{H},t}(v) & \text{for every } t \in (0,\infty), \\ \mathrm{h}_{\mathrm{H},0}v = v, \end{cases}$$

where we implicitly state that if $t > 0$, $\mathrm{h}_{\mathrm{H},t}v \in D(\Delta_{\mathrm{H}}) \subseteq H^{1,2}_{\mathrm{C}}(T\mathsf{X})$.

In [84] and [51, Section 1.4] there are proved several properties of the heat flow $\mathrm{h}_{\mathrm{H},t}$, we recall here some of them. The first is the pointwise estimate for $v \in L^2(T\mathsf{X})$

$$|\mathrm{h}_{\mathrm{H},t}v|^2 \le e^{-2Kt}\mathrm{h}_t(|v|^2) \qquad \mathsf{m}\text{-a.e. for every } t \ge 0.$$

Then we recall that $\mathrm{h}_{\mathrm{H},t}$ is self-adjoint, meaning that for every $v, w \in L^2(T\mathsf{X})$,

$$\int \mathrm{h}_{\mathrm{H},t}v \cdot w \, \mathrm{d}\mathsf{m} = \int v \cdot \mathrm{h}_{\mathrm{H},t}w \, \mathrm{d}\mathsf{m} \qquad \text{for every } t \ge 0.$$

Also, we recall the commutation, for $v \in D(\mathrm{div})$,

$$\mathrm{div}(\mathrm{h}_{\mathrm{H},t}v) = \mathrm{h}_t(\mathrm{div}\, v) \qquad \mathsf{m}\text{-a.e. for every } t \ge 0,$$

where we recall $\mathrm{h}_{\mathrm{H},t}v \in D(\Delta_{\mathrm{H}}) \subseteq D(\mathrm{div})$. Finally we state that if $f \in H^{1,2}(\mathsf{X})$, then

$$\mathrm{h}_{\mathrm{H},t}(\nabla f) = \nabla \mathrm{h}_t f \qquad \text{for every } t \ge 0.$$

2.2.6 Convergence of spaces

We assume the reader to be familiar with the notion of *pointed-measured-Gromov–Hausdorff topology* (*pmGH* for short), see [92] and [120, 88], and we recall that we are assuming that metric measure spaces have full topological support.

We only need to consider the pmGH topology on a collection $\mathcal{X}$ whose elements are pointed $\mathrm{RCD}(K,N)$ spaces with

$$\sup_{(\mathsf{X},\mathsf{d},\mathsf{m},x)\in\mathcal{X}} \mathsf{m}(B_1(x)) < \infty,$$

and we give a bit more detail in this specific context. We have the following properties. First, the pmGH topology on $\mathcal{X}$ is metrizable, say by $\mathsf{d}_{\mathrm{pmGH}}$. This is due to the fact that elements in $\mathcal{X}$ are PI with uniform parameters as in (2.2.1). Then, $\mathcal{X}$ is relatively compact with respect to the pmGH topology, this is to say that for any sequence of elements of $\mathcal{X}$, there exists a subsequence converging to a pointed $\mathrm{RCD}(K,N)$ space. We add a bit more detail to this statement, as there is a slight abuse of notation in it. First, by Gromov compactness Theorem and weak compactness in the space of measures, we have a subsequence converging in the pmGH topology to a metric measure space. We are committing a slight abuse, as it may very well happen that for this limit metric measure space, the measure does not have full support. Then, the deep result of stability of the RCD condition under pmGH convergence implies that the support of the measure is an $\mathrm{RCD}(K,N)$ space. Finally, we have this equivalent characterization of pmGH convergence on $\mathcal{X}$. A sequence $\{(\mathsf{X}_k, \mathsf{d}_k\, \mathsf{m}_k, x_k)\}_{k\in\mathbb{N}} \subseteq \mathcal{X}$ converges in the pmGH topology to $(\mathsf{X}_\infty, \mathsf{d}_\infty, \mathsf{m}_\infty, x_\infty)$, or

$$\mathsf{d}_{\mathrm{pmGH}}\big((\mathsf{X}_k, \mathsf{d}_k\, \mathsf{m}_k, x_k), (\mathsf{X}_\infty, \mathsf{d}_\infty\, \mathsf{m}_\infty, x_\infty)\big) \to 0, \tag{2.2.8}$$

if and only if we have a realization of the convergence as follows: we have a proper metric space $(\mathsf{Z}, \mathsf{d}_\mathsf{Z})$ together with a sequence of isometric embeddings $\{\iota_k\}_{k\in\mathbb{N}\cup\{\infty\}}$, where for every k, $\iota_k(\mathsf{X}_k, \mathsf{d}_k) \to (\mathsf{Z}, \mathsf{d}_\mathsf{Z})$ and it holds that

$$\begin{aligned} \iota_k(x_k) &\to \iota_\infty(x_\infty) &&\text{in } (\mathsf{Z}, \mathsf{d}_\mathsf{Z}), \\ (\iota_k)_*\mathsf{m}_k &\to (\iota_\infty)_*\mathsf{m}_\infty &&\text{in duality with } C_{\mathrm{bs}}(\mathsf{Z}). \end{aligned} \tag{2.2.9}$$

We assume familiarity with convergences of function spaces along convergent spaces, as in [24, 25]. See, for instance, the preliminaries of [10, 51].

Fix a sequence as in (2.2.8), together with a realization $(\mathsf{Z}, \mathsf{d}_\mathsf{Z})$. We remark that the following results may depend on the choice of the particular realization, i.e. the notions are *not* intrinsic.

The following is [26, Theorem 3.3], building upon [80, 88].

Theorem 2.2.5 (Pointwise convergence of heat kernels)**.** *For every sequence* $u_k, v_k, t_k \in \mathsf{X}_k \times \mathsf{X}_k \times (0,\infty)$*, for* $k \in \mathbb{N} \cup \{\infty\}$ *with* $\iota_k(u_k) \to \iota_\infty(u_\infty)$ *and* $\iota_k(v_k) \to \iota_\infty(v_\infty)$ *in* $(\mathsf{Z}, \mathsf{d}_\mathsf{Z})$ *and* $t_k \to t_\infty \in (0,\infty)$*, it holds that*

$$p^{\mathsf{X}_k}(u_k, v_k, t_k) \to p^{\mathsf{X}_\infty}(u_\infty, v_\infty, t_\infty).$$

Definition 2.2.6. Let $f_k : \mathsf{X}_k \to \mathbb{R}$ for $k \in \mathbb{N}$ and let $f_\infty : \mathsf{X}_\infty \to \mathbb{R}$.

- We say that f_k *locally uniformly converge* to f_∞ if for every $R > 0$ and for every $\varepsilon \in (0,1)$, there exist $k_0 \in \mathbb{N}$ and $\delta \in (0,1)$ such that for every $k \geq k_0$, $z_k \in \mathsf{X}_k$ and $z_\infty \in B_R(x_\infty)$ with $\mathsf{d}_\mathsf{Z}(\iota_k(z_k), \iota_\infty(z_\infty)) < \delta$, it holds $|f_k(z_k) - f_\infty(z_\infty)| < \varepsilon$. This convergence is also sometimes called *pointwise convergence*.
- We say that $\{f_k\}_{k\in\mathbb{N}}$ is *locally equi-uniformly continuous* if for every $R > 0$ there exists a function $\omega_R : (0,+\infty) \to (0,+\infty)$ with $\omega_R(s) \searrow 0$ as $s \searrow 0$ such that for every $k \in \mathbb{N}$, for every $u_k, v_k \in B_R(x_k)$, it holds that $|f_k(u_k) - f_k(v_k)| \leq \omega_R(\mathsf{d}(u_k, v_k))$.

The previous definitions can be localized to the case of functions defined on balls, as well as the following following proposition, which is proved as the classical Arzelà–Ascoli Theorem.

Proposition 2.2.7. *Let* $\{f_k\}_{k\in\mathbb{N}}$ *be a sequence of locally equi-uniformly continuous functions such that* $\{f_k(x_k)\}_{k\in\mathbb{N}}$ *is bounded. Then there exist a subsequence* $\{n_k\}_k$ *and function* $f_\infty : \mathsf{X}_\infty \to \mathbb{R}$ *such that* f_{n_k} *locally uniformly converges to* f_∞*. Moreover, if* $\omega_R : (0,\infty) \to (0,\infty)$ *for* $R > 0$ *are the function as in the definition of locally equi-uniform continuity for* $\{f_k\}_k$*, then for every* $u_\infty, v_\infty \in B_R(z_\infty)$*, it holds that* $|f_\infty(u_\infty) - f_\infty(v_\infty)| \leq \omega_R(\mathsf{d}(u_\infty, v_\infty))$*.*

Now we deal with L^p or Sobolev convergence.

Definition 2.2.8. Let $f_k \in L^2(\mathsf{X}_k)$ for $k \in \mathbb{N}$ and let $f_\infty \in L^2(\mathsf{X}_\infty)$.

- We say that f_k *weakly converge in* L^2 to f_∞ if $f_k \mathsf{m}_k \to f_\infty \mathsf{m}_\infty$ in duality with $C_{\mathrm{bs}}(\mathsf{Z})$ and $\sup_k \|f_k\|_{L^2(\mathsf{m}_k)} < \infty$.
- We say that f_k *strongly converge in* L^2 to f_∞ if f_k weakly converge in L^2 to f_∞ and $\|f_k\|_{L^2(\mathsf{m}_k)} \to \|f_\infty\|_{L^2(\mathsf{m}_\infty)}$.

Definition 2.2.9. Let $f_k \in L^1(\mathsf{X}_k)$ for $k \in \mathbb{N}$ and let $f_\infty \in L^1(\mathsf{X}_\infty)$.

- We say that f_k *strongly converge in* L^1 to f_∞ if $\sigma \circ f_k$ strongly converge in L^2 to $\sigma \circ f_\infty$, where $\sigma(t) := \mathrm{sign}(t)\sqrt{|t|}$.

We also recall the following definition.

Definition 2.2.10. Let $E_k \subseteq \mathsf{X}_k$ and $E_\infty \subseteq \mathsf{X}_\infty$ be Borel sets, with $\mathsf{m}_k(E_k) < \infty$ and $\mathsf{m}(E_\infty) < \infty$.

- We say that E_k *strongly converge in* L^1 to E_∞ if $\mathsf{m}_k(E_k) \to \mathsf{m}_\infty(E)$ and $\mathsf{m}_k \llcorner E_k \to \mathsf{m}_\infty \llcorner E_\infty$ in duality with $C_{\mathrm{bs}}(\mathsf{Z})$.
- We say that E_k *strongly converge in* L^1_{loc} to E_∞ if $E_k \cap B_R(x_k) \to E_\infty \cap B_R(x_\infty)$ in L^1 for every $R > 0$.

We have the following properties.

Proposition 2.2.11. *Let* $p = 1, 2$*, let* $f_k, g_k \in L^p(\mathsf{X}_k)$ *for* $k \in \mathbb{N}$ *and let* $f_\infty, g_\infty \in L^p(\mathsf{X}_\infty)$*.*

- *If $f_k \to f_\infty$ strongly in L^p, and $g_k \to g_\infty$ strongly in L^p, then $f_k + g_k \to f_\infty + g_\infty$ strongly in L^p.*
- *In the case $p = 2$, if $f_k \to f_\infty$ strongly in L^2, and $g_k \to g_\infty$ strongly in L^2, then $f_k g_k \to f_\infty g_\infty$ strongly in L^1.*
- *In the case $p = 1$, if $f_k \to f_\infty$ strongly in L^1 and $\sup_k \|f_k\|_{L^\infty(\mathsf{m}_k)} < \infty$, then $f_k \to f_\infty$ strongly in L^2.*

For what concerns (local) Sobolev spaces, we have the following definitions.

Definition 2.2.12. Let $f_k \in H^{1,2}(\mathsf{X}_k)$ for $k \in \mathbb{N}$ and let $f_\infty \in H^{1,2}(\mathsf{X}_\infty)$.

- We say that f_k *weakly converge in $H^{1,2}$* to f_∞ if f_k weakly converge in L^2 to f_∞ and $\sup_k \mathrm{Ch}(f_k) < \infty$.
- We say that f_k *strongly converge in $H^{1,2}$* to f_∞ if f_k weakly converge in L^2 to f_∞ and $\mathrm{Ch}(f_k) \to \mathrm{Ch}(f_\infty)$.

The local counterpart is defined similarly.

2.2.7 Tangents

We recall now the definition of tangent cone to an $\mathrm{RCD}(K,N)$ space. First, given a pointed metric measure space $(\mathsf{X},\mathsf{d},\mathsf{m},x)$ and $r \in (0,1)$ we define the rescaled space $(\mathsf{X}, r^{-1}\mathsf{d}, \mathsf{m}_r^x, x)$ where

$$\mathsf{m}_r^x := (C_x^r)^{-1}\mathsf{m},$$

for

$$C_x^r := \int_{B_r(x)} \big(1 - r^{-1}\mathsf{d}(x,z)\big)\mathrm{d}\mathsf{m}(z). \tag{2.2.10}$$

The transformation from m to m_r^x is performed in order to have the space *normalized*, i.e.

$$\int_{B_1^{r^{-1}\mathsf{d}}(x)} \big(1 - r^{-1}\mathsf{d}(x,z)\big)\, \mathrm{d}\mathsf{m}_r^x(z) = 1.$$

As a notation, we set

$$\underline{\mathcal{L}}^k := (\mathcal{L}^k)_1^0 = \frac{k+1}{\omega_k}\mathcal{L}^k, \tag{2.2.11}$$

where $\mathcal{L}^k$ denotes the k dimensional Lebesgue measure and

$$\omega_k := \mathcal{L}^k(B_1^{\mathbb{R}^k}(0)).$$

Definition 2.2.13. Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,N)$ space and $x \in \mathsf{X}$. We say that a pointed metric measure space $(\mathsf{X}',\mathsf{d}',\mathsf{m}',x')$ is *tangent* to $(\mathsf{X},\mathsf{d},\mathsf{m})$ at x if there exists a sequence of radii $r_j \searrow 0$ such that $(\mathsf{X}, r_j^{-1}\mathsf{d}, \mathsf{m}_{r_j}^x, x) \to (\mathsf{X}',\mathsf{d}',\mathsf{m}',x')$ in the pointed-measured-Gromov–Hausdorff topology. We denote the collection of all tangent spaces to $(\mathsf{X},\mathsf{d},\mathsf{m})$ at x as $\mathrm{Tan}_x(\mathsf{X},\mathsf{d},\mathsf{m})$.

If $(\mathsf{X},\mathsf{d},\mathsf{m})$ is an $\mathrm{RCD}(K,N)$ space, Gromov compactness theorem shows that for every $x \in \mathsf{X}$, $\mathrm{Tan}_x(\mathsf{X},\mathsf{d},\mathsf{m})$ is not empty. Moreover, by the stability and rescaling property of the $\mathrm{RCD}(K,N)$ condition, we see that elements of $\mathrm{Tan}_x(\mathsf{X},\mathsf{d},\mathsf{m})$ are $\mathrm{RCD}(0,N)$ spaces.

2.2.8 Structure theory

The known results of structure theory for $\mathrm{RCD}(K,N)$ spaces can be summed up in the following theorem, which, in particular, states that $\mathrm{RCD}(K,N)$ spaces are rectifiable as metric measure spaces (see [49, 52, 87, 98, 111, 90]).

Theorem 2.2.14. *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *be an* $\mathrm{RCD}(K,N)$ *space. Then there exists a unique* $n\in\mathbb{N}$*, called the* essential dimension *of* X*, with* $1\le n\le N$*, such that:*

i) for m*-a.e.* $x\in\mathsf{X}$*,*

$$\mathrm{Tan}_x(\mathsf{X},\mathsf{d},\mathsf{m})=\{(\mathbb{R}^n,\mathsf{d}_e,\underline{\mathcal{L}}^n,0)\}$$

and we call the collection of points $x\in\mathsf{X}$ *satisfying the equation above* $\mathcal{R}_n(\mathsf{X})$*.*

ii) $(\mathsf{X},\mathsf{d},\mathsf{m})$ *is countably* n*-rectifiable. More precisely, given any* $\varepsilon>0$*, we can cover* $(\mathsf{X},\mathsf{d},\mathsf{m})$*, up to an* m*-negligible subset, by a countable union of subsets that are* $(1+\varepsilon)$*-bilipschitz equivalent to measurable subsets of* $\mathbb{R}^n$*.*

iii) There exists a non-negative density $\theta\in L^1_{\mathrm{loc}}(\mathsf{X},\mathcal{H}^n\llcorner\mathcal{R}_n(\mathsf{X}))$ *such that*

$$\mathsf{m}=\theta\mathcal{H}^n\llcorner\mathcal{R}_n(\mathsf{X}). \tag{2.2.12}$$

Remark 2.2.15. We point out that the set $\mathcal{R}_n(\mathsf{X})$ of n-regular points is Borel. To check it, define $\varphi:\mathsf{X}\to[0,\infty)$ as $\varphi(x):=\limsup_{r\searrow 0}\mathsf{d}_{\mathrm{pmGH}}\big((\mathsf{X},r^{-1}\mathsf{d},\mathsf{m}_x^r,x),(\mathbb{R}^n,\mathsf{d}_e,\underline{\mathcal{L}}^n,0)\big)$. One can readily verify that $(0,1)\ni r\mapsto(\mathsf{X},r^{-1}\mathsf{d},\mathsf{m}_x^r,x)$ is $\mathsf{d}_{\mathrm{pmGH}}$-continuous for any given $x\in\mathsf{X}$, whence it follows that

$$\varphi(x)=\inf_{k\in\mathbb{N}}\sup_{q\in\mathbb{Q}\cap(0,1/k)}\mathsf{d}_{\mathrm{pmGH}}\big((\mathsf{X},q^{-1}\mathsf{d},\mathsf{m}_x^q,x),(\mathbb{R}^n,\mathsf{d}_e,\underline{\mathcal{L}}^n,0)\big)\qquad\text{for every }x\in\mathsf{X}. \tag{2.2.13}$$

Since $\mathsf{X}\ni x\mapsto(\mathsf{X},r^{-1}\mathsf{d},\mathsf{m}_x^r,x)$ is $\mathsf{d}_{\mathrm{pmGH}}$-continuous for any given $r\in(0,1)$, we deduce that $\mathsf{X}\ni x\mapsto\mathsf{d}_{\mathrm{pmGH}}\big((\mathsf{X},q^{-1}\mathsf{d},\mathsf{m}_x^q,x),(\mathbb{R}^n,\mathsf{d}_e,\underline{\mathcal{L}}^n,0)\big)$ is a continuous function for any $q\in\mathbb{Q}\cap(0,1)$. Consequently, (2.2.13) ensures that $\mathcal{R}_n(\mathsf{X})=\{x\in\mathsf{X}:\varphi(x)=0\}$ is a Borel set (in fact, a countable intersection of F_σ sets), as we claimed. ■

If $(\mathsf{X},\mathsf{d},\mathsf{m})$ is an $\mathrm{RCD}(K,N)$ space of essential dimension n, it holds that for every $x\in\mathsf{X}$,

$$(\mathbb{R}^k,\mathsf{d}_e,\underline{\mathcal{L}}^k,0)\notin\mathrm{Tan}_x(\mathsf{X},\mathsf{d},\mathsf{m})\qquad\text{if }k>n,$$

this is due to the lower semicontinuity of the essential dimension, [101, 49].

Now, we introduce the set $\mathcal{R}_n^*(\mathsf{X})$. As customary, given a metric measure space $(\mathsf{X},\mathsf{d},\mu)$ and a real number $k\ge 0$, we define the upper and lower k-dimensional densities of μ as

$$\overline{\Theta}_k(\mu,x):=\limsup_{r\searrow 0}\frac{\mu(B_r(x))}{\omega_k r^k},\qquad\underline{\Theta}_k(\mu,x):=\liminf_{r\searrow 0}\frac{\mu(B_r(x))}{\omega_k r^k}\qquad\text{for every }x\in\mathsf{X},$$

respectively. In the case where $\overline{\Theta}_k(\mu,x)$ and $\underline{\Theta}_k(\mu,x)$ coincide, we denote their common value by $\Theta_k(\mu,x)\in[0,\infty]$ and we call it the *k-dimensional density* of μ at x.

Definition 2.2.16. Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,N)$ space having essential dimension n. Then we define the Borel set $\mathcal{R}_n^*(\mathsf{X})$ as

$$\mathcal{R}_n^*(\mathsf{X}):=\big\{x\in\mathcal{R}_n(\mathsf{X}):\exists\Theta_n(\mathsf{m},x)\in(0,\infty)\big\}.$$

We have that, cf. (2.2.12),

$$\mathsf{m} = \Theta_n(\mathsf{m}, x) \llcorner \mathcal{R}_n^*(\mathsf{X}). \tag{2.2.14}$$

Take an $\mathrm{RCD}(K, N)$ space $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ of essential dimension n and consider the product $(\mathsf{X} \times \mathbb{R}, \mathsf{d} \otimes \mathsf{d}_e, \mathsf{m} \otimes \mathcal{L}^1)$. We will often use, without mention, this fact: $x \in \mathcal{R}_n^*(\mathsf{X})$ if and only if $(x, t) \in \mathcal{R}_{n+1}^*(\mathsf{X} \times \mathbb{R})$ for some (hence all) $t \in \mathbb{R}$ and, if this is the case,

$$\Theta_{n+1}(\mathsf{m} \otimes \mathcal{L}^1, x) = \Theta_n(x, \mathsf{m}).$$

2.2.9 Fine modules

We assume familiarity with the definition of capacitary modules, quasi-continuous functions and vector fields and related material in [69]. A summary of the material we use can be found in [51, Section 1.3].

We start by recalling the notion of capacity.

Definition 2.2.17. Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be a metric measure space. For any set $A \subseteq \mathsf{X}$ we define the 2-*capacity*, (or simply *capacity*) as

$$\mathrm{Cap}(A) := \inf \left\{ \|f\|^2_{H^{1,2}(\mathsf{X})} : f \in H^{1,2}(\mathsf{X}), f \geq 1 \ \mathsf{m}\text{-a.e. on some neighbourhood of } A \right\}.$$

It turns out that Cap is a submodular outer measure, finite on bounded sets, and, obviously, $\mathsf{m} \leq \mathrm{Cap}$. Moreover, a function $f : \mathsf{X} \to \mathbb{R}$ is *quasi-continuous* if for every $\varepsilon > 0$, there exists $E_\varepsilon \subseteq \mathsf{X}$ with $\mathrm{Cap}(E_\varepsilon) < \varepsilon$ such that the restriction of f to $\mathsf{X} \setminus E_\varepsilon$ is continuous. We denote by QCR the quasi-continuous representative.

The following theorem states that, on RCD spaces, gradients of test functions have a better representative than gradients of general Sobolev functions. These representatives enjoy the property (actually, define the property) of being quasi-continuous.

Theorem 2.2.18 ([69, Theorem 2.6])**.** *Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be an $\mathrm{RCD}(K, \infty)$ space. Then there exists a unique couple $(L^0_{\mathrm{Cap}}(T\mathsf{X}), \bar{\nabla})$, where $L^0_{\mathrm{Cap}}(T\mathsf{X})$ is a $L^0(\mathrm{Cap})$-normed $L^0(\mathrm{Cap})$-module and*

$$\bar{\nabla} : \mathrm{TestF}(\mathsf{X}) \to L^0_{\mathrm{Cap}}(T\mathsf{X})$$

is a linear operator such that:

i) $|\bar{\nabla} f| = \mathrm{QCR}(|\nabla f|)$ Cap-*a.e. for every* $f \in \mathrm{TestF}(\mathsf{X})$,

ii) the set $\{\sum_n \chi_{E_n} \bar{\nabla} f_n\}$*, where* $\{f_n\}_n \subseteq \mathrm{TestF}(\mathsf{X})$ *and* $\{E_n\}_n$ *is a Borel partition of* X *is dense in* $L^0_{\mathrm{Cap}}(T\mathsf{X})$*.*

Uniqueness is up to unique isomorphism, in the sense that, if another couple $(L^0_{\mathrm{Cap}}(T\mathsf{X})', \bar{\nabla}')$ satisfies the same properties, then there exists a unique module isomorphism $\Phi : L^0_{\mathrm{Cap}}(T\mathsf{X}) \to L^0_{\mathrm{Cap}}(T\mathsf{X})'$ such that $\Phi \circ \bar{\nabla} = \bar{\nabla}'$. Moreover, $L^0_{\mathrm{Cap}}(T\mathsf{X})$ is a Hilbert module that we call capacitary tangent module.

It is worth spending a few words on $L^0(\mathrm{Cap})$-normed $L^0(\mathrm{Cap})$-modules and, in particular, on $L^0_{\mathrm{Cap}}(T\mathsf{X})$, as the $L^0(\mathrm{Cap})$ and $L^0(\mathsf{m})$ topologies may behave quite differently. First, $L^0(\mathrm{Cap})$-normed $L^0(\mathrm{Cap})$-modules enjoy the following important properties (cf. [84, Definition 1.2.1]):

i) *locality*: for every $v \in L^0_{\mathrm{Cap}}(T\mathsf{X})$ and $\{A_i\}_i$ sequence of Borel subsets of X such that $\chi_{A_i} v = 0$ for every $i \in \mathbb{N}$, then $\chi_{\bigcup_i A_i} v = 0$,

ii) *gluing*: if $\{v_i\}_i \subseteq L^0_{\mathrm{Cap}}(T\mathsf{X})$ and $\{A_i\}_i$ is a sequence of pairwise disjoint Borel subsets of X, there exists $v \in L^0_{\mathrm{Cap}}(T\mathsf{X})$ such that $\chi_{A_i} v = \chi_{A_i} v_i$ for every $i \in \mathbb{N}$.

Indeed, the first property follows trivially from the existence of the pointwise norm. For what concerns the second property, notice first that, partitioning the sets A_i and using locality, we can with no loss of generality assume that $|v_i| \in L^\infty(\mathrm{Cap})$ for every i. We can then set $a_i := 2^{-i}\|1 + |v_i|\|_{L^\infty(\mathrm{Cap})}$ and consider the Cauchy sequence $n \mapsto \sum_{i=1}^n a_i^{-1}\chi_{A_i} v_i$ and then multiply its limit by $f := \sum_{i=1}^\infty a_i \chi_{A_i}$ so that we can conclude by locality. However, the gluing property for $L^0_{\mathrm{Cap}}(T\mathsf{X})$ follows directly from its construction, starting from the set of infinite linear combinations as in item *ii)* of Theorem 2.2.18. Notice that one needs the gluing property for $L^0_{\mathrm{Cap}}(T\mathsf{X})$ to define the multiplication by functions in $L^0(\mathrm{Cap})$ so that we cannot use the argument above to prove the gluing property for $L^0_{\mathrm{Cap}}(T\mathsf{X})$. This discussion is relevant because the map

$$L^0(\mathrm{Cap}) \times L^0_{\mathrm{Cap}}(T\mathsf{X}) \ni (f, v) \mapsto fv \in L^0_{\mathrm{Cap}}(T\mathsf{X})$$

is not continuous in general. For example, set $(\mathsf{X}, \mathsf{d}, \mathsf{m}) = ([0,1], \mathsf{d}_e, \mathcal{L}^1)$, recall [69, Example 2.17], and notice that $L^\infty(\mathrm{Cap})$ is a closed (non-trivial) subspace of $L^0(\mathrm{Cap})$. Take $v_n := (1 + n^{-1})\chi_{(0,1)}$ and $f(x) := 1/x$. Clearly $v_n \to v := \chi_{(0,1)} \in L^0_{\mathrm{Cap}}(T\mathsf{X})$, however $\{f v_n\} \in L^0_{\mathrm{Cap}}(T\mathsf{X})$ is not even a Cauchy sequence.

Notice that we can, and will, extend the map QCR from $H^{1,2}(\mathsf{X})$ to $\mathrm{S}^2(\mathsf{X}) \cap L^\infty(\mathsf{m})$ by a locality argument. We define

$$\mathrm{Test}\bar{\mathrm{V}}(\mathsf{X}) := \left\{ \sum_{i=1}^n \mathrm{QCR}(f_i)\bar{\nabla} g_i : f_i \in \mathrm{S}^2(\mathsf{X}) \cap L^\infty(\mathsf{m}), g_i \in \mathrm{TestF}(\mathsf{X}) \right\}.$$

We define also the vector subspace of *quasi-continuous vector fields*, $\mathcal{QC}(T\mathsf{X})$, as the closure of $\mathrm{Test}\bar{\mathrm{V}}(\mathsf{X})$ in $L^0_{\mathrm{Cap}}(T\mathsf{X})$ and finally,

$$\mathcal{QC}^\infty(T\mathsf{X}) := \{v \in \mathcal{QC}(T\mathsf{X}) : |v| \text{ is Cap-essentially bounded}\}.$$

Recall now that as $\mathsf{m} \ll \mathrm{Cap}$, we have a natural projection map

$$\mathrm{Pr} : L^0(\mathrm{Cap}) \to L^0(\mathsf{m}) \qquad \text{defined as} \qquad [f]_{L^0(\mathrm{Cap})} \mapsto [f]_{L^0(\mathsf{m})}$$

where $[f]_{L^0(\mathrm{Cap})}$ (resp. $[f]_{L^0(\mathsf{m})}$) denotes the Cap (resp. m) equivalence class of f. It turns out that Pr, restricted to the set of quasi-continuous functions, is injective ([69, Proposition 1.18]). We have the following projection map $\bar{\mathrm{Pr}}$, given by [69, Proposition 2.9 and Proposition 2.13], which plays the role of Pr on vector fields.

Proposition 2.2.19. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, \infty)$ *space. There exists a unique linear continuous map*

$$\bar{\mathrm{Pr}} : L^0_{\mathrm{Cap}}(T\mathsf{X}) \to L^0(T\mathsf{X})$$

that satisfies

i) $\bar{\mathrm{Pr}}(\bar{\nabla} f) = \nabla f$ *for every* $f \in \mathrm{TestF}(\mathsf{X})$,

ii) $\bar{\mathrm{Pr}}(gv) = \mathrm{Pr}(g)\bar{\mathrm{Pr}}(v)$ *for every* $g \in L^0(\mathrm{Cap})$ *and* $v \in L^0_{\mathrm{Cap}}(T\mathsf{X})$.

Moreover, for every $v \in L^0_{\mathrm{Cap}}(T\mathsf{X})$,

$$\left|\bar{\mathrm{Pr}}(v)\right| = \mathrm{Pr}(|v|) \qquad \mathsf{m}\text{-}a.e.$$

and $\bar{\mathrm{Pr}}$, *when restricted to the set of quasi-continuous vector fields, is injective.*

We point out that if $v \in \mathcal{QC}(T\mathsf{X})$, [69, Proposition 2.12] shows that $|v| \in L^0(\mathrm{Cap})$ is quasi-continuous, in particular, $v \in \mathcal{QC}^\infty(T\mathsf{X})$ if and only if $\bar{\mathrm{Pr}}(v) \in L^\infty(T\mathsf{X})$.

In what follows, with a little abuse, we often write, for $v \in L^0_{\mathrm{Cap}}(T\mathsf{X})$, $v \in D(\mathrm{div})$ if and only if $\bar{\mathrm{Pr}}(v) \in D(\mathrm{div})$ and, if this is the case, $\mathrm{div}\, v = \mathrm{div}(\bar{\mathrm{Pr}}(v))$. Similar notation will be used for other operators acting on subspaces of $L^0(T\mathsf{X})$.

Theorem 2.2.20 ([69, Theorem 2.14 and Proposition 2.13])**.** *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, \infty)$ *space. Then there exists a unique map* $\mathrm{Q\bar{C}R} : H^{1,2}_{\mathrm{C}}(T\mathsf{X}) \to L^0_{\mathrm{Cap}}(T\mathsf{X})$ *such that*

i) $\mathrm{Q\bar{C}R}(v) \in \mathcal{QC}(T\mathsf{X})$ *for every* $v \in H^{1,2}_{\mathrm{C}}(T\mathsf{X})$,

ii) $\bar{\mathrm{Pr}} \circ \mathrm{Q\bar{C}R}(v) = v$ *for every* $v \in H^{1,2}_{\mathrm{C}}(T\mathsf{X})$.

Moreover, $\mathrm{Q\bar{C}R}$ *is linear and satisfies*

$$\left|\mathrm{Q\bar{C}R}(v)\right| = \mathrm{QCR}(|v|) \qquad \mathrm{Cap}\text{-}a.e. \text{ for every } v \in H^{1,2}_{\mathrm{C}}(T\mathsf{X}),$$

so that $\mathrm{Q\bar{C}R}$ *is continuous.*

We often omit to write the $\mathrm{Q\bar{C}R}$ (which, as before, stands for quasi-continuous) operator for simplicity of notation. This should cause no ambiguity thanks to the fact that

$$\mathrm{Q\bar{C}R}(gv) = \mathrm{QCR}(g)\mathrm{Q\bar{C}R}(v) \qquad \text{for every } g \in H^{1,2}(\mathsf{X}) \cap L^\infty(\mathsf{m}) \text{ and } v \in H^{1,2}_{\mathrm{C}}(T\mathsf{X}) \cap L^\infty(T\mathsf{X}). \tag{2.2.15}$$

This can be proved easily as the continuity of the map QCR implies that $\mathrm{QCR}(g)\mathrm{Q\bar{C}R}(v)$ as above is quasi-continuous and the injectivity of the map $\bar{\mathrm{Pr}}$ restricted the set of quasi-continuous vector fields yields the conclusion. Again by locality, we have that (2.2.15) holds even for $g \in \mathrm{S}^2(\mathsf{X}) \cap L^\infty(\mathsf{m})$.

We have the following dimensional decomposition of the capacitary tangent module, along with the existence of a basis on the sets of a suitable partition. This is proved in [43]. Notice that to every element of the partition A_k is associated an integer $n(k) \le n$, and the inequality may be strict, see e.g. [69, Example 3.17]

Theorem 2.2.21. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space of essential dimension* n*. Then there exists a partition of* X *made of countably many bounded Borel sets* $\{A_k\}_k$ *such that for every* k *there exist* $n(k)$ *with* $0 \le n(k) \le n$ *and* $\left\{v_1^k, \dots, v_{n(k)}^k\right\} \subseteq \mathrm{TestV}(\mathsf{X})$ *with bounded support which is an orthonormal basis of* $L^0_{\mathrm{Cap}}(T\mathsf{X})$ *on* A_k*, in the sense that*

$$v_i \cdot v_j = \delta_i^j \qquad \mathrm{Cap}\text{-}a.e. \text{ on } A_k$$

and for every $v \in L^0_{\mathrm{Cap}}(T\mathsf{X})$ *there exist* $g_1, \dots, g_{n(k)} \in L^0(\mathrm{Cap})$ *such that*

$$v = \sum_{i=1}^{n(k)} g_i v_i^k \qquad \mathrm{Cap}\text{-}a.e. \text{ on } A_k,$$

where, in particular,

$$g_i = v \cdot v_i^k \qquad \mathrm{Cap}\text{-}a.e. \text{ on } A_k.$$

Here we implicitly state that if $n(k) = 0$ *then for every* $v \in L^0_{\mathrm{Cap}}(T\mathsf{X})$ *we have* $v = 0$ Cap-*a.e. on* A_k*.*

Proof. First, we remark that that it is easy to show that in item *ii*) of Theorem 2.2.18 we can replace $\mathrm{TestF}(\mathsf{X})$ with a countable subset, say $\{f_k\}_{k\in\mathbb{N}}$. This is due to the fact that $H^{1,2}_C(T\mathsf{X})$ is separable, so that we can take a countable dense subset of $\mathrm{TestV}(\mathsf{X})$ in the $W^{1,2}_C(T\mathsf{X})$ topology. Set now $v_k := \tilde{\nabla} f_k$ and $D := \{v_k\}_k$.

Consider now the (countable) sequence of Cap-a.e. defined functions

$$F_I := \det\left(v_i \,\cdot\, v_j\right)_{i,j\in I}$$

where I ranges over all finite subsets of $\mathbb{N}$. Notice that if $|I| > n$ then the fact that $L^2(T\mathsf{X})$ has dimension n and basic linear algebra yield that $F_I = 0$ m-a.e. hence $F_I = 0$ Cap-a.e. because F_I is quasi-continuous. We set then for $i \in \mathbb{N}$, $i \geq 1$

$$A_i := \bigcup_{|I|=i} \{F_I \neq 0\} \bigcap_{|J|\geq i+1} \{F_J = 0\}$$

and $A_0 := \mathsf{X} \setminus \cup_{i\geq 1} A_i$. Notice $\mathsf{X} = A_0 \cup \cdots \cup A_n$ as a disjoint union.

Notice now that we can show, by density, that for every $v \in L^0_{\mathrm{Cap}}(T\mathsf{X})$ we have $v = 0$ Cap-a.e. on A_0. Then, by definition, we can write $A_i = \bigcup_I A_i^I$ countable (disjoint) union, where on A_i^I we have that $F_I \neq 0$ Cap-a.e. and $F_J = 0$ Cap-a.e. if $|J| > |I|$. We claim now that $\{v_k\}_{k\in I}$ is a basis of $L^0_{\mathrm{Cap}}(T\mathsf{X})$ on A_i^I, in the sense that for every $v \in L^0_{\mathrm{Cap}}(T\mathsf{X})$ there exists $\{g_k\}_{k\in I}$ such that

$$v = \sum_{k\in I} g_k v_k \quad \text{Cap-a.e. on } A_i^I$$

and that if for some $\{g_k\}_{k\in I} \subseteq L^0(\mathrm{Cap})$ we have that $\sum_{k\in I} g_k v_k = 0$ Cap-a.e. then $g_k = 0$ Cap-a.e. on A_i^I for every $k \in I$.

The second claim follows by basic linear algebra: indeed if $\sum_{k\in I} g_k v_k = 0$ Cap-a.e. then we have in particular

$$\sum_{k,h\in I} g_h g_k v_h \,\cdot\, v_k = 0 \quad \text{Cap-a.e.}$$

and this implies $g_k = 0$ Cap-a.e. on A_i^I for every $k \in I$ as $F_I \neq 0$ Cap-a.e. on A_i^I.

We show now the first claim. This is again basic linear algebra together with a simple density argument. Take any w in $L^0_{\mathrm{Cap}}(T\mathsf{X})$, then a density-continuity argument and the fact that $F_{I\cup\{k\}} = 0$ Cap-a.e. on A_i^I for every k show that if we set (with an abuse) $v_{\bar{k}} := w$ we have that

$$F_{I\cup\{\bar{k}\}} = 0 \quad \text{Cap-a.e. on } A_i^I.$$

In particular, as $F_I \neq 0$ Cap-a.e. on A_i^I, we have that $v_{\bar{k}} \,\cdot\, v_l = \sum_{j\in I} g_j v_j \,\cdot\, v_l$ Cap-a.e. on A_i^I for $l \in\; I \cup \{\bar{k}\}$ where $\{g_j\}_j \subseteq L^0(\mathrm{Cap})$ (as they are the unique solution of a linear system with coefficients in $L^0(\mathrm{Cap})$). This immediately implies

$$\left| v_{\bar{k}} - \sum_{j\in I} g_j v_j \right|^2 = 0 \quad \text{Cap-a.e. on } A_i^I.$$

We do now a further decomposition of the sets A_i^I. First, we orthogonalize the basis $\{v_k\}_{k\in I}$ by means of a Gram-Schmidt procedure as follows. Assume for simplicity $I = \{1, 2, \dots, m\}$. We set recursively

$$v_k' := c_k^k v_k + \sum_{l=1}^{k-1} c_l^k v_l' \quad \text{for } k = 1, \dots, m,$$

where $\{c_l^k\}_{1\le l\le k\le m}$ are defined as

$$c_l^k := \begin{cases} \prod_{j=1}^{k-1} v_j' \cdot v_j' & \text{if } l = k, \\ -\frac{v_k \cdot v_l'}{v_l' \cdot v_l'} \prod_{j=1}^{k-1} v_j' \cdot v_j' & \text{if } l < k. \end{cases}$$

Notice that $\{c_l^k\} \subseteq S^2(\mathsf{X}) \cap L^\infty(\mathsf{m})$ and that $\{v_k'\}_{k\in I}$ is still a basis on A_i^I in the sense described above. Also, $v_h' \cdot v_k' = 0$ Cap-a.e. on A_i^I if $h \ne k, h, k \in I$.

If $\varepsilon > 0$, we set then $(A_i^I)_\varepsilon := A_i^I \cap \{|v_k'| > \varepsilon$ for every $k \in I\}$, notice that $A_i^I = \bigcup_{\varepsilon>0}(A_i^I)_\varepsilon$ and we can write such union as a countable union. We rescale now the basis writing

$$v_k'' := \frac{1}{\varepsilon \vee |v_k'|} v_k'$$

and this allows us to conclude the proof. □

The following theorem, which is [51, Section 1.3] (see also [42]), is crucial in the construction of modules tailored to particular measures.

Theorem 2.2.22. *Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be a metric measure space and let μ be a Borel measure finite on balls such that $\mu \ll$ Cap. Let also $\mathcal{M}$ be a $L^0(\mathrm{Cap})$-normed $L^0(\mathrm{Cap})$-module. Define the natural (continuous) projection*

$$\pi_\mu : L^0(\mathrm{Cap}) \to L^0(\mu).$$

We define an equivalence relation $\sim_\mu$ on $\mathcal{M}$ as

$$v \sim_\mu w \text{ if and only if } |v - w| = 0 \qquad \mu\text{-a.e.}$$

Define the quotient module $\mathcal{M}_\mu^0 := \mathcal{M}/\sim_\mu$ with the natural (continuous) projection

$$\bar\pi_\mu : \mathcal{M} \to \mathcal{M}_\mu^0.$$

Then $\mathcal{M}_\mu^0$ is a $L^0(\mu)$-normed $L^0(\mu)$-module, with the pointwise norm and product induced by the ones of $\mathcal{M}$: more precisely, for every $v \in \mathcal{M}$ and $g \in L^0(\mathrm{Cap})$,

$$\begin{cases} |\bar\pi_\mu(v)| := \pi_\mu(|v|), \\ \pi_\mu(g)\bar\pi_\mu(v) := \bar\pi_\mu(gv). \end{cases} \tag{2.2.16}$$

If $p \in [1, \infty]$, we set

$$\mathcal{M}_\mu^p := \left\{v \in \mathcal{M}_\mu^0 : |v| \in L^p(\mu)\right\},$$

which is a $L^p(\mu)$-normed $L^\infty(\mu)$-module. Moreover, if $\mathcal{M}$ is a Hilbert module, also $\mathcal{M}_\mu^0$ and $\mathcal{M}_\mu^2$ are Hilbert modules.

In the particular case in which $\mathcal{M} = L^0_{\mathrm{Cap}}(T\mathsf{X})$ and μ is a Borel measure finite on balls such that $\mu \ll$ Cap, we set

$$L_\mu^p(T\mathsf{X}) := (L^0_{\mathrm{Cap}}(T\mathsf{X}))_\mu^p \qquad \text{for } p \in \{0\} \cup [1, \infty].$$

In the case $\mu = \mathsf{m}$ notice that considering the map

$$\dot{\nabla} : \mathrm{TestF}(\mathsf{X}) \xrightarrow{\bar{\nabla}} L^0_{\mathrm{Cap}}(T\mathsf{X}) \xrightarrow{\bar{\pi}_{\mathsf{m}}} (L^0_{\mathrm{Cap}}(T\mathsf{X}))^0_{\mathsf{m}}$$

we can show that $(L^0_{\mathrm{Cap}}(T\mathsf{X}))^0_{\mathsf{m}}$ is isomorphic to the usual L^0 tangent module via a map that sends ∇f to $\dot{\nabla} f$ so that we have no ambiguity of notation and, by construction, the map $\bar{\pi}_{\mathsf{m}}$ coincides with $\bar{\mathrm{Pr}}$ defined in Proposition 2.2.19. We define the traces

$$\begin{aligned} \mathrm{tr}_\mu &: H^{1,2}_{\mathrm{loc}}(\mathsf{X}) \to L^0(\mu) \qquad &\text{as} \qquad \mathrm{tr}_\mu := \pi_\mu \circ \mathrm{QCR},\\ \bar{\mathrm{tr}}_\mu &: H^{1,2}_{\mathrm{C}}(T\mathsf{X}) \to L^0_\mu(T\mathsf{X}) \qquad &\text{as} \qquad \bar{\mathrm{tr}}_\mu := \bar{\pi}_\mu \circ \mathrm{Q\bar{C}R}. \end{aligned}$$

To simplify the notation, we often omit to write the trace operators. This should cause no ambiguity because from (2.2.15) and (2.2.16) it follows that

$$\bar{\mathrm{tr}}_\mu(gv) = \mathrm{tr}_\mu(g)\bar{\mathrm{tr}}_\mu(v) \qquad \text{for every } g \in H^{1,2}_{\mathrm{loc}}(\mathsf{X}) \cap L^\infty(\mathsf{m}) \text{ and } v \in H^{1,2}_{\mathrm{C}}(T\mathsf{X}) \cap L^\infty(T\mathsf{X}).$$

We define

$$\mathrm{TestV}_\mu(\mathsf{X}) := \bar{\mathrm{tr}}_\mu(\mathrm{TestV}(\mathsf{X})) \subseteq L^\infty_\mu(T\mathsf{X})$$

and the proof of [51, Lemma 2.7] gives the following result.

Lemma 2.2.23. *Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be an $\mathrm{RCD}(K, \infty)$ space and let μ be a finite Borel measure such that $\mu \ll \mathrm{Cap}$. Then $\mathrm{TestV}_\mu(\mathsf{X})$ is dense in $L^p_\mu(T\mathsf{X})$ for every $p \in [1, \infty)$.*

We also need Cartesian products of normed modules. Fix $n \in \mathbb{N}$, $n \geq 1$ and denote by $\| \cdot \|_e$ the Euclidean norm of $\mathbb{R}^n$. Given a $L^0(\mathsf{m})$-normed $L^0(\mathsf{m})$-module $\mathcal{N}$, we can consider its Cartesian product $\mathcal{N}^n$ and endow it with the natural module structure and with the pointwise norm

$$|(v_1, \dots, v_n)| := \|(|v_1|, \dots, |v_n|)\|_e$$

which is induced by a scalar product if and only if the one of $\mathcal{N}$ is, and if this is the case, we still denote the pointwise scalar product on $\mathcal{N}^n$ by the dot $\,\cdot\,$. We endow $\mathcal{N}^n$ with the norm induced by the Lebesgue norm of the relevant exponent of the pointwise norm with respect to m. Also, $\mathcal{N}^n$ is a $L^2(\mathsf{m})$-normed $L^\infty(\mathsf{m})$-module if and only if $\mathcal{N}$ is, and, if this is the case, a subspace $\mathcal{N}_1$ of $\mathcal{N}$ is dense if and only if $(\mathcal{N}_1)^n$ is dense in $\mathcal{N}^n$. Similar considerations hold if m is replaced by a Borel measure, finite on balls and (with the suitable interpretation) in the case of $L^0(\mathrm{Cap})$-normed $L^0(\mathrm{Cap})$-modules or if we alter the integrability exponent. It is clear that if $\mathcal{M}$ is a $L^0(\mathrm{Cap})$-normed $L^0(\mathrm{Cap})$-module and μ is a Borel measure finite on balls such that $\mu \ll \mathrm{Cap}$, then also

$$(\mathcal{M}^p_\mu)^n = (\mathcal{M}^n)^p_\mu \qquad \text{for } p \in \{0\} \cup [1, \infty].$$

We adopt the natural notation

$$\mathrm{L}^p_\mu(T^n\mathsf{X}) := L^p_\mu(T\mathsf{X})^n$$

and, when possible, we endow $\mathrm{L}^p_\mu(T^n\mathsf{X})$ with the norm induced by the $\mathrm{L}^p(\mu)$ norm of the (Euclidean) pointwise norm $|\cdot|$.

The following remark will be used in the sequel without further notice: if $v = (v_1, \dots, v_n) \in \mathcal{N}^n$ is such that for every $i = 1, \dots, n$, $v_i \in H^{1,2}(\mathsf{X})$, then $|v| \in H^{1,2}(\mathsf{X})$. This follows from the fact that if $f_1, \dots, f_n \in H^{1,2}(\mathsf{X})$ and $\varphi \in \mathrm{LIP}(\mathbb{R}^n; \mathbb{R})$ is such that $\varphi(0) = 0$, then $\varphi(f_1, \dots, f_n) \in H^{1,2}(\mathsf{X})$.

2.3 Functions of bounded variation

Now we turn to the study of functions of bounded variation on metric measure spaces. Most of the material can be found in the thesis [71] and the references therein.

2.3.1 Definitions and basic properties

We assume that the reader is familiar with the theory of functions of bounded variation and sets of (locally) finite perimeter in metric measure spaces developed in [6, 7, 108] and in the more recent [10, 51] for what concerns the $\mathrm{RCD}(K,N)$ setting. We recall now the main notions.

Fix a metric measure space $(\mathsf{X},\mathsf{d},\mathsf{m})$. Given $f\in L^1_{\mathrm{loc}}(\mathsf{m})$, we define, for any $A\subseteq\mathsf{X}$ open, the *total variation*

$$|\mathrm{D}f|(A):=\inf\left\{\liminf_k\int\operatorname{lip}(f_k)\,\mathrm{d}\mathsf{m}:\{f_k\}_k\subseteq\mathrm{LIP}_{\mathrm{loc}}(A),f_k\to f\text{ in }L^1_{\mathrm{loc}}(A,\mathsf{m})\right\},\tag{2.3.1}$$

where $f_k\to f$ in $L^1_{\mathrm{loc}}(A,\mathsf{m})$ if for every $x\in A$ there exists a neighbourhood $U=U_x$ of x such that $f_k\to f$ in $L^1(U,\mathsf{m})$.

For $\Omega\subseteq\mathsf{X}$ open, we say that f has *locally bounded variation on* Ω provided that for every $x\in\Omega$, there exists a neighbourhood $A=A_x$ of x with $|\mathrm{D}f|(A)<\infty$. In this case we write $f\in\mathrm{BV}_{\mathrm{loc}}(A)$. If moreover $f\in L^1(\Omega)$ and $|\mathrm{D}f|(\Omega)<\infty$, we say that f *has bounded variation on* Ω and we write $f\in\mathrm{BV}(\Omega)$. In particular, we say that f is a *function of bounded variation*, and we write $f\in\mathrm{BV}(\mathsf{X})$, if $f\in L^1(\mathsf{m})$ and $|\mathrm{D}f|(\mathsf{X})<\infty$. In this case it is easy to show that in (2.3.1), L^1 convergence can be equivalently taken instead of L^1_{loc} convergence.

If $f=\chi_E$, we say that E is a *set of locally finite perimeter* if $\chi_E\in\mathrm{BV}_{\mathrm{loc}}(\mathsf{X})$ and we say that E is a *set of finite perimeter* if $|\mathrm{D}\chi_E|(\mathsf{X})<\infty$.

If $f\in\mathrm{BV}_{\mathrm{loc}}(\Omega)$, then $|\mathrm{D}f|(\,\cdot\,)$ turns out to be the restriction to open sets of a Borel measure (finite or locally finite) that we denote with the same symbol and that we still call *total variation*. If $f=\chi_E$, we denote $|\mathrm{D}f|(\,\cdot\,)$ also with $\mathrm{Per}(E,\,\cdot\,)$ and we call it *perimeter*.

Notice that, by its very definition, the total variation is lower semicontinuous with respect to L^1_{loc} convergence, is subadditive and $|\mathrm{D}(\varphi\circ f)|\le L|\mathrm{D}f|$ whenever $f\in\mathrm{BV}(\mathsf{X})$ and φ is L-Lipschitz. Finally, (2.2.3) and, in particular, (2.2.2) extend immediately to the case $f\in\mathrm{BV}(\mathsf{X})$, with the term $\frac{1}{\mathsf{m}(B_{cr}(x))}|\mathrm{D}f|(B_{cr}(x))$ in place of $⨍_{B_{cr}(x)}\operatorname{lip}(f)\,\mathrm{d}\mathsf{m}$. Therefore,

$$\min\{\mathsf{m}(B_r(x)\cap E),\mathsf{m}(B_r(x)\setminus E)\}\le 2C_P r|\mathrm{D}\chi_E|(B_{\lambda r}(x))\qquad\text{for every }x\in\mathsf{X}\text{ and }r\in(0,R).\tag{2.3.2}$$

The following remark will be used with no further reference.

Remark 2.3.1. Let $f\in\mathrm{BV}(\mathsf{X})$. Whenever we have an optimal sequence $\{f_k\}_k\subseteq\mathrm{LIP}_{\mathrm{loc}}(\mathsf{X})\cap L^1(\mathsf{m})$ for the computation of $|\mathrm{D}f|(\mathsf{X})$ as in (2.3.1), i.e. $f_k\to f$ in $L^1(\mathsf{m})$ and $\int_{\mathsf{X}}\operatorname{lip}(f_k)\mathrm{d}\mathsf{m}\to|\mathrm{D}f|(\mathsf{X})$, then it holds that $\operatorname{lip}(f_k)\mathsf{m}\to|\mathrm{D}f|$ in duality with $C_{\mathrm{b}}(\mathsf{X})$. Moreover, by the results in [71], there exists at least one such sequence, which moreover satisfies $\{f_k\}_k\subseteq\mathrm{LIP}_{\mathrm{bs}}(\mathsf{X})$. ■

Several classical results have been generalized to the abstract framework of metric measure spaces. Among them, the Fleming–Rishel *coarea* formula, which we now state.

Proposition 2.3.2 (Coarea)**.** *Let $A\subseteq\mathsf{X}$ open and let $f\in L^1_{\mathrm{loc}}(A)$. Then*

$$|\mathrm{D}f|(A)=\int_{\mathbb{R}}\mathrm{Per}(\{f>r\},A)\,\mathrm{d}r\,.$$

In particular, for $f\in L^1(A)$, it holds that $f\in\mathrm{BV}(A)$ if and only if $\int_{\mathbb{R}}\mathrm{Per}(\{f>r\},A)\,\mathrm{d}r<\infty$.

In the proposition above, it is part of the statement that if $|\mathrm{D}f|(A) < \infty$, then the set $\{f > r\}$ has finite perimeter for $\mathcal{L}^1$-a.e. $r \in \mathbb{R}$.

We also have the following consequence of the coarea formula above. If $f \in \mathrm{BV}(\mathsf{X})$, then

$$\int h \,\mathrm{d}|\mathrm{D}f| = \int_{\mathbb{R}} \int h \,\mathrm{d}\mathrm{Per}(\{f > r\}, \,\cdot\,) \,\mathrm{d}r \qquad \text{for any Borel function } h : \mathsf{X} \to [0, \infty]. \tag{2.3.3}$$

A standard consequence of the coarea formula is that given $x \in \mathsf{X}$, then for $\mathcal{L}^1$-a.e $r \in (0, \infty)$ the ball $B_r(x)$ has finite perimeter. In the framework of $\mathrm{RCD}(K, N)$ spaces this conclusion holds for every $r \in (0, \infty)$ and the Bishop–Gromov inequality ([121]) provides sharp upper bounds for perimeters of balls. We also recall that sets of finite perimeters are an algebra, more precisely, if E and F are sets of finite perimeter, then

$$\mathrm{Per}(E, \,\cdot\,) = \mathrm{Per}(\mathsf{X} \setminus E, \,\cdot\,) \qquad \text{and} \qquad \mathrm{Per}(E \cap F, \,\cdot\,) + \mathrm{Per}(E \cup F, \,\cdot\,) \le \mathrm{Per}(E, \,\cdot\,) + \mathrm{Per}(F, \,\cdot\,).$$

2.3.2 Fine properties on PI spaces

We now fix a PI space $(\mathsf{X}, \mathsf{d}, \mathsf{m})$. The following definitions make sense on arbitrary metric measure spaces, but most of the results of this section hold for PI spaces.

Given a measurable set E, we define its *essential boundary* as

$$\partial^* E := \left\{ x \in \mathsf{X} : \limsup_{r \searrow 0} \frac{\mathsf{m}(B_r(x) \cap E)}{\mathsf{m}(B_r(x))} > 0 \text{ and } \limsup_{r \searrow 0} \frac{\mathsf{m}(B_r(x) \setminus E)}{\mathsf{m}(B_r(x))} > 0 \right\},$$

and given a measurable function $f : \mathsf{X} \to \mathbb{R}$, we define the *approximate lower and upper limits* as

$$\begin{aligned} f^\wedge(x) &:= \operatorname{ap} \liminf_{y \to x} f(y) \;:= \sup \left\{ t \in \bar{\mathbb{R}} : \lim_{r \searrow 0} \frac{\mathsf{m}(B_r(x) \cap \{f < t\})}{\mathsf{m}(B_r(x))} = 0 \right\}, \\ f^\vee(x) &:= \operatorname{ap} \limsup_{y \to x} f(y) := \inf \left\{ t \in \bar{\mathbb{R}} : \lim_{r \searrow 0} \frac{\mathsf{m}(B_r(x) \cap \{f > t\})}{\mathsf{m}(B_r(x))} = 0 \right\}. \end{aligned}$$

Notice that if E is a measurable subset of X, then

$$\partial^* E = \{x : (\chi_E^\wedge(x), \chi_E^\vee(x)) = (0, 1)\}.$$

We define

$$S_f := \left\{ x \,:\, f^\wedge(x) < f^\vee(x) \right\}. \tag{2.3.4}$$

If $x \in \mathsf{X} \setminus S_f$, then $f^\wedge(x) = f^\vee(x)$ and we denote their common value by $\bar{f}(x)$. If $x \in S_f$ we define

$$\bar{f}(x) := \frac{f^\wedge(x) + f^\vee(x)}{2},$$

adopting the convention $+\infty + (-\infty) = 0$. We call $\bar{f}$ the *precise representative* of f.

We also need the definition of *codimension one spherical Hausdorff measure*, defined as

$$\mathcal{H}^h(A) := \lim_{\delta \searrow 0} \mathcal{H}^h_\delta(A)$$

where

$$\mathcal{H}^h_\delta(A) := \inf \left\{ \sum_{i \in \mathbb{N}} \frac{\mathsf{m}(B_{r_i}(x_i))}{r_i} : A \subseteq \bigcup_{i \in \mathbb{N}} B_{r_i}(x_i), \; r_i \le \delta \right\}.$$

Notice that, for any $\delta > 0$,

$$\mathcal{H}^h_\delta(A) = 0 \qquad \Rightarrow \qquad \mathcal{H}^h(A) = 0.$$

It is possible to prove (see [99, Lemma 3.2]) that for $f \in \mathrm{BV}(\mathsf{X})$,

$$-\infty < f^\vee(x) \le f^\wedge(x) < +\infty \qquad \text{for } \mathcal{H}^h\text{-a.e. } x \in \mathsf{X}. \tag{2.3.5}$$

Moreover, the following relations can be easily proved with standard measure theoretic arguments (see e.g. [28, Proposition 5.2]):

$$\begin{aligned} &\text{if } x \in S_f \text{ and } t \in (f^\wedge(x), f^\vee(x)) && \text{then } x \in \partial^*\{f > t\},\\ &\text{if } x \in \partial^*\{f > t\} && \text{then } t \in [f^\wedge(x), f^\vee(x)],\\ &\qquad \text{in particular} && \\ &\text{if } x \notin S_f \text{ and } x \in \partial^*\{f > t\} && \text{then } \bar{f}(x) = t. \end{aligned} \tag{2.3.6}$$

We recall that [7, Lemma 5.2] and [51, Theorem 1.12], together with the coarea formula show that, in the framework of PI spaces (in particular, in the framework of $\mathrm{RCD}(K, N)$ spaces),

$$|\mathrm{D}f| \ll \mathcal{H}^h \ll \mathrm{Cap} \qquad \text{for every } f \in \mathrm{BV}_{\mathrm{loc}}(\mathsf{X}). \tag{2.3.7}$$

We also have the following more precise version of the first absolute continuity in (2.3.7), by [6, 7] and [28]: for any set of finite perimeter E, there exists a Borel function Θ_E, which is bounded uniformly from below and from above by strictly positive constants that depend only on the parameters in (2.2.1) and (2.2.2), such that

$$|\mathrm{D}\chi_E| = \Theta_E \mathcal{H}^h \llcorner \partial^* E. \tag{2.3.8}$$

However, we will prove in Theorem 2.3.7 that

$$|\mathrm{D}f| \ll \mathrm{Cap} \tag{2.3.9}$$

holds for any m.m.s. $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ and $f \in \mathrm{BV}(\mathsf{X})$.

Moreover, [28], it holds that the coarea formula implies that

$$\mathcal{H}^h(B) < \infty \qquad \Rightarrow \qquad |\mathrm{D}f|(B \setminus S_f) = 0.$$

The following proposition summarizes results about sets of finite perimeter that are now well-known in the context of PI spaces and are proved in [7, 75], see also [6].

Proposition 2.3.3. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be a* PI *space and let* $E \subseteq \mathsf{X}$ *be a set of locally finite perimeter. Then, for* $|\mathrm{D}\chi_E|$*-a.e.* $x \in \mathsf{X}$ *the following hold:*

i) E *is* asymptotically minimal *at* x*, i.e., there exist* $r_x > 0$ *and a function* $\omega_x : (0, r_x) \to (0, \infty)$ *with* $\lim_{r \searrow 0} \omega_x(r) = 0$ *satisfying*

$$|\mathrm{D}\chi_E|(B_r(x)) \le (1 + \omega_x(r))|\mathrm{D}\chi_{E'}|(B_r(x)) \qquad \text{if } r \in (0, r_x) \text{ and } E' \Delta E \Subset B_r(x),$$

ii) $|\mathrm{D}\chi_E|$ *is* asymptotically doubling *at* x*, i.e.*

$$\limsup_{r \searrow 0} \frac{|\mathrm{D}\chi_E|(B_{2r}(x))}{|\mathrm{D}\chi_E|(B_r(x))} < \infty,$$

iii) we have the following estimates:

$$0<\liminf_{r\searrow 0}\frac{r|\mathrm{D}\chi_E|(B_r(x))}{\mathsf{m}(B_r(x))}\le\limsup_{r\searrow 0}\frac{r|\mathrm{D}\chi_E|(B_r(x))}{\mathsf{m}(B_r(x))}<\infty,$$

iv) the following holds:

$$\liminf_{r\searrow 0}\min\left\{\frac{\mathsf{m}(B_r(x)\cap E)}{\mathsf{m}(B_r(x))},\frac{\mathsf{m}(B_r(x)\setminus E)}{\mathsf{m}(B_r(x))}\right\}>0.$$

As customary, we split the total variation of a function of bounded variation in absolutely continuous, jump, and Cantor part.

Definition 2.3.4 (Decomposition of the total variation)**.** Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be a PI space and let $f\in \mathrm{BV}_{\mathrm{loc}}(\mathsf{X})$. We split the total variation in *absolutely continuous part* and *singular part*,

$$|\mathrm{D}f|=|\mathrm{D}f|^a+|\mathrm{D}f|^s\qquad\text{where } |\mathrm{D}f|^a=g_f\mathsf{m}\ll\mathsf{m}\text{ and }|\mathrm{D}f|^s\perp\mathsf{m},$$

and further split the singular part into *jump part* and *Cantor part*,

$$|\mathrm{D}f|^s=|\mathrm{D}f|^j+|\mathrm{D}f|^c\qquad\text{where } |\mathrm{D}f|^j=|\mathrm{D}f|\llcorner S_f,$$

so that we can write

$$|\mathrm{D}f|^c=|\mathrm{D}f|\llcorner C_f\qquad\text{with }\mathsf{m}(C_f)=0\text{ and }C_f\cap S_f=\emptyset.$$

2.3.3 Integration by parts

The following representation formula for the total variation is based on a result proved in [72] and then modified in [53] (see [53, Remark 3.18]). In the particular setting of $\mathrm{RCD}(K,\infty)$ spaces, it is possible to use an approximation argument to provide a direct proof (cf. Proposition 2.3.18). See [42] for the following formulation.

Proposition 2.3.5 (Representation formula)**.** *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *be a metric measure space and let* $f\in\mathrm{BV}(\mathsf{X})$*. Then, for every A open subset of* X*, it holds that*

$$|\mathrm{D}f|(A)=\sup\left\{\int_A f\,\mathrm{div}\,v\,\mathrm{d}\mathsf{m}\right\},\tag{2.3.10}$$

where the supremum is taken among all $v\in\mathcal{W}_A$, where

$$\mathcal{W}_A:=\{v\in D(\mathrm{div}^\infty):|v|\le 1\ \mathsf{m}\text{-}a.e.\ \operatorname{supp}v\Subset A\}.$$

Finally, the supremum can be equivalently taken among all $v\in\tilde{\mathcal{W}}_A$, where

$$\tilde{\mathcal{W}}_A:=\{v\in D(\mathrm{div}^\infty):|v|\le 1\ \mathsf{m}\text{-}a.e.\ \operatorname{supp}v\subseteq A\}.$$

Proof. Fix $A\subseteq\mathsf{X}$ open. If $v\in\mathcal{W}_A$, as $\operatorname{supp}v\Subset A$, we can find B open with $v\in\mathcal{W}_B$ and $\bar B\subseteq A$. Take now a sequence $\{f_n\}_n\subseteq\mathrm{LIP}_{\mathrm{bs}}(\mathsf{X})$ with $f_n\to f$ in $L^1(\mathsf{m})$ and $\int_{\mathsf{X}}\mathrm{lip}(f_n)\mathrm{d}\mathsf{m}\to|\mathrm{D}f|(\mathsf{X})$ (hence $\mathrm{lip}(f_n)\mathsf{m}\to|\mathrm{D}f|$ in duality with $C_{\mathrm{b}}(\mathsf{X})$). Then

$$\int f\,\mathrm{div}\,v\,\mathrm{d}\mathsf{m}=\lim_n\int f_n\mathrm{div}\,v\,\mathrm{d}\mathsf{m}=-\lim_n\int\mathrm{d}f_n(v)\,\mathrm{d}\mathsf{m}\,.$$

We have that for every n (recall the bound $|\mathrm{d}f_n| \le \mathrm{lip}(f_n)$ m-a.e.),

$$\left|\int \mathrm{d}f_n(v)\,\mathrm{d}\mathsf{m}\right| \le \int_B \mathrm{lip}(f_n)\,\mathrm{d}\mathsf{m}\,.$$

Exploiting the weak convergence of $\mathrm{lip}(f_n)\mathsf{m}$ to $|\mathrm{D}f|$ we have

$$\limsup_n \int_B \mathrm{lip}(f_n)\,\mathrm{d}\mathsf{m} \le |\mathrm{D}f|(\bar{B}) \le |\mathrm{D}f|(A)$$

and this proves that the quantity defined by the supremum in (2.3.10) is bounded by $|\mathrm{D}f|(A)$.

Now, (with the notation of [72, 53]), let $\delta \in \mathrm{Der}^{\infty,\infty}(\mathsf{X})$ be with $|\delta| \le 1$ m-a.e. and $\operatorname{supp}\delta \Subset A$. Then $\delta \in \mathrm{Der}^{2,2}(\mathsf{X})$ so that, using [53, Lemma 3.12], we can find a vector field $v_\delta \in D(\mathrm{div})$ such that $|v_\delta| \le |\delta|$ m-a.e. and $\operatorname{div} v_\delta = \operatorname{div}\delta$ m-a.e. and then also the opposite inequality in (2.3.10) is proved, in virtue of [72, Theorem 3.4].

In order to conclude, we just have to show that if $A \subseteq \mathsf{X}$ is open and $v \in \tilde{\mathcal{W}}_A$, then

$$\int f \operatorname{div} v\,\mathrm{d}\mathsf{m} \le |\mathrm{D}f|(A).$$

By an immediate approximation argument, there is no loss of generality in assuming that v has bounded support. Let $\varepsilon > 0$. By regularity, let $K \subseteq \mathsf{X}$ be a compact set with $K \subseteq \mathsf{X} \setminus A$ and $|\mathrm{D}f|((\mathsf{X} \setminus A) \setminus K) < \varepsilon$. It is clear that $\operatorname{supp} v \Subset \mathsf{X} \setminus K$, so that

$$\int f \operatorname{div} v\,\mathrm{d}\mathsf{m} \le |\mathrm{D}f|(\mathsf{X} \setminus K) \le |\mathrm{D}f|(A) + \varepsilon,$$

so that the proof is concluded being $\varepsilon > 0$ arbitrary. □

Remark 2.3.6. If $f \in \mathrm{BV}(\mathsf{X})$, $v \in D(\mathrm{div}) \cap L^\infty(\mathsf{m})$ and $\{n_k\}_k \subseteq (0,\infty)$, $\{m_k\}_k \subseteq (0,\infty)$ are two sequences with $\lim_k n_k = \lim_k m_k = +\infty$, then the limit

$$\lim_k \int (f \vee -m_k) \wedge n_k \operatorname{div} v\,\mathrm{d}\mathsf{m} \tag{2.3.11}$$

exists finite and does not depend on the particular choice of the sequences $\{n_k\}_k$ and $\{m_k\}_k$. Indeed, a cut-off argument and an approximation argument as the one in the proof of Proposition 2.3.5 yields that, if $g \in \mathrm{BV}(\mathsf{X}) \cap L^\infty(\mathsf{m})$ and v is as above, then

$$\left|\int g \operatorname{div} v\,\mathrm{d}\mathsf{m}\right| \le |\mathrm{D}g|(\mathsf{X})\|v\|_{L^\infty(T\mathsf{X})},$$

so that, using also coarea, we get the claim.

Therefore, if $f \in \mathrm{BV}(\mathsf{X})$ and $v \in D(\mathrm{div}) \cap L^\infty(\mathsf{m})$, we can write

$$\int f \operatorname{div} v\,\mathrm{d}\mathsf{m}\,,$$

with the convention that it has to be interpreted as the limit in (2.3.11). ■

2.3.4 Total variation and capacity

We recall Definition 2.2.17. For a function of bounded variation f, on a PI space, it holds that $|\mathrm{D}f| \ll \mathrm{Cap}$, by (2.3.7). For general metric measure space, there is little hope to recover (2.3.7). However, the fact that $|\mathrm{D}f| \ll \mathrm{Cap}$ still holds, as the following theorem shows. Notice that this fact is of crucial importance, as it allows to exploit the results of Section 2.2.9.

Theorem 2.3.7. *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *be a metric measure space and let* $f \in \mathrm{BV}(\mathsf{X})$*. Then*

$$|\mathrm{D}f| \ll \mathrm{Cap}.$$

We prove Theorem 2.3.7 after Lemma 2.3.8, which states that we can compute the capacity of compact sets using Lipschitz functions, instead of Sobolev ones.

Lemma 2.3.8. *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *be a metric measure space and let* $K \subseteq \mathsf{X}$ *be a compact set. Then*

$$\mathrm{Cap}(K) = \inf \|f\|^2_{H^{1,2}(\mathsf{X})} = \inf \int f^2 + \mathrm{lip}(f)^2 \,\mathrm{d}\mathsf{m} \tag{2.3.12}$$

where both the infima are taken among all functions $f \in \mathrm{LIP_{bs}}(\mathsf{X})$ *such that* $f \ge 1$ *on a neighbourhood of* K*.*

Proof. Recalling that if $f \in \mathrm{LIP_{bs}}(\mathsf{X})$, then $f \in H^{1,2}(\mathsf{X})$ and

$$\|f\|^2_{H^{1,2}(\mathsf{X})} \le \int f^2 + \mathrm{lip}(f)^2 \,\mathrm{d}\mathsf{m}\,,$$

we immediately obtain the two inequalities ($\le$) in (2.3.12).

To conclude, we can assume with no loss of generality that $\mathrm{Cap}(K) < \infty$. If $\varepsilon > 0$, fix $g \in H^{1,2}(\mathsf{X})$ with $g \ge 1$ m-a.e. on a neighbourhood of K such that $\|g\|^2_{H^{1,2}(\mathsf{X})} \le \mathrm{Cap}(K) + \varepsilon$. Up to replacing g with $0 \vee g \wedge 1$, there is no loss of generality in assuming that g takes values in $[0,1]$ and that $g = 1$ m-a.e. on a neighbourhood of K, call this neighbourhood A. Let also $\{g_n\} \subseteq \mathrm{LIP_{bs}}(\mathsf{X})$ be such that $g_n \to g$ in $L^2(\mathsf{m})$ and $\int \mathrm{lip}(g_n)^2 \,\mathrm{d}\mathsf{m} \to \mathrm{Ch}(g)$ (using an immediate cut-off argument we can replace $\mathrm{LIP_b}(\mathsf{X}) \cap L^2(\mathsf{m})$ with $\mathrm{LIP_{bs}}(\mathsf{X})$ in (2.1.1)). Take $\eta \in \mathrm{LIP_{bs}}(\mathsf{X})$ such that $\eta = 1$ on a neighbourhood of K, $\eta(x) \in [0,1]$ for every $x \in \mathsf{X}$ and $\operatorname{supp} \eta \subseteq A$ (here we use the compactness of K). Set now $f_n := (1-\eta)g_n + \eta \in \mathrm{LIP_{bs}}(\mathsf{X})$ and notice that $f_n \ge 1$ on a neighbourhood of K. Exploiting the fact that $g_n \to g$ in $L^2(\mathsf{m})$ and $g = 1$ m-a.e. on A,

$$\limsup_n \int f_n^2 \,\mathrm{d}\mathsf{m} = \int g^2 \,\mathrm{d}\mathsf{m}\,.$$

Using the convexity inequality for the slope (e.g. [71, Lemma 1.3.2]) and arguing as above, we have that

$$\mathrm{lip}(f_n) \le (1-\eta)\mathrm{lip}(g_n) + \mathrm{lip}(\eta)|g_n - 1|$$

so that

$$\limsup_n \int \mathrm{lip}(f_n)^2 \,\mathrm{d}\mathsf{m} \le \limsup_n \int \mathrm{lip}(g_n)^2 \,\mathrm{d}\mathsf{m}\,.$$

All in all, we conclude as $\varepsilon > 0$ was arbitrary and

$$\limsup_n \int f_n^2 + \mathrm{lip}(f_n)^2 \,\mathrm{d}\mathsf{m} \le \|g\|^2_{H^{1,2}(\mathsf{X})} \le \mathrm{Cap}(K) + \varepsilon. \qquad \square$$

Remark 2.3.9. It is worth pointing out that Lemma 2.3.8 holds also replacing lip with the bigger lip_a in (2.3.12), which is defined by

$$\mathrm{lip}_a(f)(x) := \limsup_{y,z\to x} \frac{|f(y)-f(z)|}{\mathsf{d}(y,z)},$$

which has to be understood to be 0 if x is an isolated point, for any f locally Lipschitz. The proof is exactly the same, if one takes into account the main result of [20]. ■

Proof of Theorem 2.3.7. First notice that thanks to coarea and the regularity of $|\mathrm{D}f|$, we can reduce ourselves to prove that $|\mathrm{D}f|(K)=0$ whenever $K\subseteq \mathsf{X}$ is a compact set with $\mathrm{Cap}(K)=0$ and assuming also $f\in \mathrm{BV}(\mathsf{X})\cap L^\infty(\mathsf{m})$. Thanks to Lemma 2.3.8, we can take a sequence $\{\varphi_n\}_n\subseteq \mathrm{LIP_{bs}}(\mathsf{X})$ such that $\varphi_n(x)\in[0,1]$ for every $x\in\mathsf{X}$, $\varphi_n(x)=1$ on a neighbourhood of K (this neighbourhood depends on n) and $\|\varphi_n\|_{H^{1,2}(\mathsf{X})}\to 0$.

Take $v\in D(\mathrm{div}^\infty)$ with $|v|\le 1$ m-a.e. and $\mathrm{supp}\, v$ bounded. Consider now

$$\int f \mathrm{div}\, v \,\mathrm{d}\mathsf{m} = \int f \mathrm{div}(\varphi_n v)\,\mathrm{d}\mathsf{m} + \int f\mathrm{div}((1-\varphi_n)v)\,\mathrm{d}\mathsf{m}$$

and notice that, by the calculus rules for the divergence in (2.1.4) (recall that we are assuming $f\in L^2(\mathsf{m})$),

$$\int f\mathrm{div}(\varphi_n v)\,\mathrm{d}\mathsf{m}\to 0 \qquad \text{as } n\to\infty$$

and also that, by Proposition 2.3.5,

$$\left|\int f\mathrm{div}((1-\varphi_n)v)\,\mathrm{d}\mathsf{m}\right| \le |\mathrm{D}f|(\mathsf{X}\setminus K)$$

as $\mathrm{supp}\,((1-\varphi_n)v)\Subset \mathsf{X}\setminus K$. If we let $n\to\infty$ and then take the supremum among all v as above, we have, by Proposition 2.3.5,

$$|\mathrm{D}f|(\mathsf{X})\le |\mathrm{D}f|(\mathsf{X}\setminus K),$$

which proves our claim. □

2.3.5 Cartesian surfaces

Here and after, when we deal with a Cartesian product of sets of the kind $\mathsf{X}_1\times\cdots\times\mathsf{X}_k$, we denote by

$$\pi^j:\mathsf{X}_1\times\cdots\times\mathsf{X}_k\to\mathsf{X}_j \qquad (x_1,\dots,x_k)\mapsto x_j$$

the projection onto the corresponding factor, for $j=1,\dots,k$.

Definition 2.3.10. Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be a metric measure space and let $f:\mathsf{X}\to\mathbb{R}$ be Borel. Then we define the *subgraph* of f as the Borel set $\mathcal{G}_f\subseteq\mathsf{X}\times\mathbb{R}$ given by

$$\mathcal{G}_f := \{(x,t)\in\mathsf{X}\times\mathbb{R}: t<f(x)\}.$$

Lemma 2.3.11. *Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be a locally uniformly doubling metric measure space and let $f:\mathsf{X}\to\mathbb{R}$ be a Borel function. Then it holds that*

$$\begin{aligned}(x,t)\in\partial^*\mathcal{G}_f \qquad &\Rightarrow \qquad t\in[f^\wedge(x),f^\vee(x)],\\ t\in(f^\wedge(x),f^\vee(x)) \qquad &\Rightarrow \qquad (x,t)\in\partial^*\mathcal{G}_f.\end{aligned}$$

In particular, if $x\in\mathsf{X}\setminus S_f$, then it holds that $\partial^*\mathcal{G}_f\cap(\{x\}\times\mathbb{R})$ is either empty or coincides with $\{(x,\bar f(x))\}$.

Proof. In the proof the constant C may change from line to line and it only depends on the doubling constant at scale $R = 1$. We can compute, for $r \in (0, \varepsilon)$, using Fubini's Theorem,

$$\begin{aligned}\frac{(\mathsf{m} \otimes \mathcal{L}^1)(B_r(x,t) \cap \mathcal{G}_f)}{(\mathsf{m} \otimes \mathcal{L}^1)(B_r(x,t))} &\le \frac{(\mathsf{m} \otimes \mathcal{L}^1)\big((B_r(x) \times B_r(t)) \cap \mathcal{G}_f\big)}{(\mathsf{m} \otimes \mathcal{L}^1)(B_{r/2}(x) \times B_{r/2}(t))} \\ &\le C\frac{(\mathsf{m} \otimes \mathcal{L}^1)(\{(y,t) \in B_r(x) \times B_r(t) : t < f(y)\})}{r\mathsf{m}(B_r(x))} \\ &\le C\frac{\fint_{t-r}^{t+r} \mathsf{m}(\{y \in B_r(x) : s < f(y)\})\,\mathrm{d}s}{\mathsf{m}(B_r(x))} \\ &\le C\frac{\mathsf{m}(B_r(x) \cap \{f > t - \varepsilon\})}{\mathsf{m}(B_r(x))}.\end{aligned}$$

Therefore, if $(x,t) \in \partial^*\mathcal{G}_f$, then $t \le f^\vee(x)$. Similarly, we can show that if $r \in (0, \varepsilon)$,

$$\frac{(\mathsf{m} \otimes \mathcal{L}^1)(B_r(x,t) \setminus \mathcal{G}_f)}{(\mathsf{m} \otimes \mathcal{L}^1)(B_r(x,t))} \le C\frac{\mathsf{m}(B_r(x) \cap \{f < t + \varepsilon\})}{\mathsf{m}(B_r(x))},$$

which in turn shows that if $(x,t) \in \partial^*\mathcal{G}_f$, then $t \ge f^\wedge(x)$. Conversely, arguing as above, we can show that if $r \in (0, \varepsilon)$,

$$\frac{(\mathsf{m} \otimes \mathcal{L}^1)(B_{2r}(x,t) \cap \mathcal{G}_f)}{(\mathsf{m} \otimes \mathcal{L}^1)(B_{2r}(x,t))} \ge C\frac{\mathsf{m}(B_r(x) \cap \{f > t + \varepsilon\})}{\mathsf{m}(B_r(x))}$$

and that

$$\frac{(\mathsf{m} \otimes \mathcal{L}^1)(B_r(x,t) \setminus \mathcal{G}_f)}{(\mathsf{m} \otimes \mathcal{L}^1)(B_r(x,t))} \ge C\frac{\mathsf{m}(B_r(x) \cap \{f < t - \varepsilon\})}{\mathsf{m}(B_r(x))}$$

which yield the second claim. □

By [30, Theorem 5.1] and its proof, taking into account the elementary inequality

$$a \le \sqrt{1 + a^2} \le 1 + a, \qquad \text{for every } a > 0,$$

(or see [28, Proposition 4.2]) we obtain the following proposition.

Proposition 2.3.12. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be a* PI *space and* $f \in \mathrm{BV}(\mathsf{X})$. *Then* $\mathcal{G}_f$ *is a set of locally finite perimeter in* $\mathsf{X} \times \mathbb{R}$ *and*

$$|\mathrm{D}f| \le \pi^1_*|\mathrm{D}\chi_{\mathcal{G}_f}| \le |\mathrm{D}f| + \mathsf{m}. \tag{2.3.13}$$

In particular, if C_f *and* S_f *are the* m*-negligible sets as in Definition 2.3.4,*

$$(\pi^1_*|\mathrm{D}\chi_{\mathcal{G}_f}|) \llcorner (C_f \cup J_f) = |\mathrm{D}f| \llcorner (C_f \cup J_f).$$

2.3.6 Vector valued functions of bounded variation

In what follows we fix $m \in \mathbb{N}$, $m \ge 1$. We treat now the case of vector valued BV functions, i.e. functions of bounded variation taking values in $\mathbb{R}^m$, or equivalently, collections of m real valued functions of bounded variation. As the case $m = 1$ has already been treated, we focus on $m \in \mathbb{N}$, $m \ge 2$.

Definition 2.3.13. Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be a metric measure space and $f \in L^1(\mathsf{m})^m$. We define, for any A open subset of X,

$$|\mathrm{D}f|(A) := \inf\left\{\liminf_k \int_A \|(\mathrm{lip}(f_{i,k}))_{i=1,\dots,m}\|_e \,\mathrm{d}\mathsf{m}\right\} \tag{2.3.14}$$

where the infimum is taken among all sequences $\{f_{i,k}\}_k \subseteq \mathrm{LIP}_{\mathrm{loc}}(A)$ such that $f_{i,k} \to f_i$ in $L^1_{\mathrm{loc}}(A, \mathsf{m})$ for every $i = 1, \dots, m$.

Remark 2.3.14. Notice that we are taking the relaxation of the integral of the Euclidean norm of the vector whose components are the local Lipschitz constants of the various coordinates, not the local Lipschitz constant of a vector valued function. The former approach follows [18], while the latter (a slight variant of the one in) [108]. For open subsets of $\mathbb{R}^n$ the former approach corresponds to the relaxation of the integral of the Hilbert-Schmidt norm of the Jacobian matrix of a sequence of approximating functions, while the latter employs the operator norm instead, and is seen to be equivalent to the one proposed in [4]. Also, it is straightforward to show that $f \in \mathrm{BV}(\mathsf{X})^m$ if and only if the quantity defined in (2.3.14) for $|\mathrm{D}f|(\mathsf{X})$ is finite, and similarly for $\mathrm{BV}_{\mathrm{loc}}(\mathsf{X})$. ■

Proposition 2.3.15. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be a metric measure space and* $f \in \mathrm{BV}(\mathsf{X})^m$. *Then* $|\mathrm{D}f|(\,\cdot\,)$ *as defined in* (2.3.14) *is the restriction to open sets of a finite non-negative Borel measure that we call* total variation *of* f *and still denote with the same symbol.*

Proof. The proof of [15, Lemma 5.2] can be easily adapted with no substantial changes. Indeed, one has only to notice that the convexity inequality for the slope used in [15, Lemma 5.4] and the properties of the Euclidean norm imply a suitable version of the convexity inequality for the slope in our situation. □

Definition 2.3.16. Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be a metric measure space and $f = (f_1, \dots, f_m) \in \mathrm{BV}(\mathsf{X})^m$. We define the Borel set

$$S_f := \bigcup_{i=1}^m S_{f_i},$$

where S_{f_i} is defined in (2.3.4).

Also in this case we adopt the terminology (and notation) of Definition 2.3.4 for what concerns the splitting of the total variation of a vector valued function of bounded variation in *absolutely continuous*, *jump* and *Cantor* part.

2.3.7 Functions of bounded variation on RCD spaces

The proof of the following result can be found in [86, Remark 3.5], we briefly sketch it here for the sake of completeness. Notice that there is a slight abuse in the statements of (2.3.15) and (2.3.16), as we identified measures with their densities freely.

Proposition 2.3.17 (Bakry–Émery estimate in BV)**.** *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, \infty)$ *space and* $f \in \mathrm{BV}(\mathsf{X})$. *Then, if* $t > 0$, $\mathrm{h}_t f \in \mathrm{BV}(\mathsf{X})$ *and it holds*

$$|\mathrm{D}\mathrm{h}_t f| \le e^{-Kt}\mathrm{h}_t|\mathrm{D}f|. \tag{2.3.15}$$

If moreover $f \in \mathrm{BV}(\mathsf{X}) \cap L^\infty(\mathsf{m})$, *then* $\mathrm{h}_t f \in \mathrm{BV}(\mathsf{X}) \cap H^{1,2}(\mathsf{X})$ *and*

$$|\nabla \mathrm{h}_t f| \le e^{-Kt}\mathrm{h}_t|\mathrm{D}f| \qquad \mathsf{m}\text{-}a.e. \tag{2.3.16}$$

Proof. First notice that by the general theory of Sobolev spaces, we easily obtain that $|\mathrm{Dh}_t f| \le |\nabla \mathrm{h}_t f|\mathsf{m}$ if $f \in \mathrm{BV}(\mathsf{X}) \cap L^\infty(\mathsf{m})$. Then, thanks to the lower semicontinuity of the total variation and a truncation argument, the first statement follows from the second.

In order to conclude the proof, take a sequence $\{f_k\}_k \subseteq \mathrm{LIP_{bs}}(\mathsf{X})$ with $f_k \to f$ in $L^1(\mathsf{m})$ and $\int_\mathsf{X} \mathrm{lip}(f_k)\mathrm{d}\mathsf{m} \to |\mathrm{D}f|(\mathsf{X})$ (hence $\mathrm{lip}(f_k)\mathsf{m} \to |\mathrm{D}f|$ in duality with $C_\mathrm{b}(\mathsf{X})$). Clearly, we can assume that $\|f_k\|_{L^\infty(\mathsf{m})} \le \|f\|_{L^\infty(\mathsf{m})}$, so that $\mathrm{h}_t f_k \in H^{1,2}(\mathsf{X}) \cap \mathrm{LIP_b}(\mathsf{X})$ for every k, with uniformly bounded Lipschitz constants, by the L^∞-LIP regularization property. Also, by [114, Corollary 4.3], we have that for every k,

$$|\nabla \mathrm{h}_t f_k| \le e^{-Kt}\mathrm{h}_t|\nabla f_k| \le e^{-Kt}\mathrm{h}_t \mathrm{lip}(f_k) \qquad \mathsf{m}\text{-a.e.}$$

Then, $\{|\nabla \mathrm{h}_t f_k|\}_k \subseteq L^2(\mathsf{m})$ is bounded and, as $|\nabla \mathrm{h}_t f|$ is bounded from above by any L^2 weak limit of $\{|\nabla \mathrm{h}_t f_k|\}_k$, we can conclude easily, recalling that the heat flow on finite measures preserves the weak convergence in duality with $C_\mathrm{b}(\mathsf{X})$. □

For the rest of this subsection, we fix $m \in \mathbb{N}$, $m \ge 1$. The following proposition provides us with a generalization of Proposition 2.3.5 (actually, also with a different proof, but only for the special context of RCD spaces) to the multi dimensional case, in the context of $\mathrm{RCD}(K,\infty)$ spaces.

In view of the following proposition, recall that the interpretation of the integral in (2.3.17) is given by Remark 2.3.6.

Proposition 2.3.18. *Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,\infty)$ space and let $f \in \mathrm{BV}(\mathsf{X})^m$. Then, for every A open subset of X, it holds that*

$$|\mathrm{D}f|(A) = \sup\left\{\sum_{i=1}^m \int_A f_i \mathrm{div}\, v_i \,\mathrm{d}\mathsf{m}\right\}, \tag{2.3.17}$$

where the supremum is taken among all $v = (v_1, \dots, v_m) \in \mathcal{W}_A^m$, where

$$\mathcal{W}_A^m := \Big\{v = (v_1, \dots v_m) \in \mathrm{TestV}(\mathsf{X})^m : |v| \le 1 \ \mathsf{m}\text{-}a.e.\ \mathrm{supp}\,|v| \Subset A\Big\}. \tag{2.3.18}$$

Finally, the supremum can be equivalently taken among all $v \in \tilde{\mathcal{W}}_A^m$, where

$$\tilde{\mathcal{W}}_A^m := \Big\{v = (v_1, \dots v_m) \in \mathrm{TestV}(\mathsf{X})^m : |v| \le 1 \ \mathsf{m}\text{-}a.e.\ \mathrm{supp}\,|v| \subseteq A\Big\}.$$

Proof. Call $|\mathrm{D}f|^*$ the quantity defined by the right hand side of (2.3.17), we show now that $|\mathrm{D}f|^*$ is the restriction to open sets of a finite Borel measure, that we still denote with $|\mathrm{D}f|^*$ and that $|\mathrm{D}f|^* = |\mathrm{D}f|$ as measures.

Step 1. We show that $|\mathrm{D}f|^*(A) \le |\mathrm{D}f|(A)$ for every open set A. Fix then $A \subseteq \mathsf{X}$ open. Assume for the moment that also $f_i \in L^\infty(A,\mathsf{m}) \cap \mathrm{LIP_{loc}}(A)$ is such that $\int_A \mathrm{lip}(f_i)\,\mathrm{d}\mathsf{m} < \infty$ for every $i = 1, \dots, m$ and take any $v = (v_1, \dots, v_m) \in \mathcal{W}_A^m$. Set now

$$C := \mathrm{supp}\,|v|$$

Notice $C \Subset A$ and take a cut-off function $\psi \in \mathrm{LIP_{bs}}(\mathsf{X})$ with $\psi(x) \in [0,1]$ for every $x \in \mathsf{X}$, $\psi = 1$ on a neighbourhood of C and $\mathrm{supp}\,\psi \Subset A$. Therefore, for every $i = 1, \dots, m$, $\psi f_i \in L^\infty(\mathsf{m}) \cap \mathrm{LIP_{loc}}(\mathsf{X})$

is such that $\int \operatorname{lip}(\psi f_i)\,\mathrm{d}\mathsf{m} < \infty$. We can now estimate, for $t > 0$, using the Cauchy-Schwarz inequality and (2.3.16),

$$-\sum_{i=1}^{m}\int \mathrm{h}_t(\psi f_i)\mathrm{div}\, v_i\,\mathrm{d}\mathsf{m} = \sum_{i=1}^{m}\int \nabla \mathrm{h}_t(\psi f_i)\,\cdot\, v_i\,\mathrm{d}\mathsf{m} \le \int_C \|(|\nabla \mathrm{h}_t(\psi f_i)|)_i\|_e\,\mathrm{d}\mathsf{m}$$
$$\le e^{-Kt}\int_C \|(\mathrm{h}_t|\mathrm{D}(\psi f_i)|)_i\|_e\,\mathrm{d}\mathsf{m} \le e^{-Kt}\int_C \|(\mathrm{h}_t\operatorname{lip}(\psi f_i))_i\|_e\,\mathrm{d}\mathsf{m}$$

so that, letting $t \searrow 0$,

$$-\sum_{i=1}^{m}\int f_i\mathrm{div}\, v_i\,\mathrm{d}\mathsf{m} \le \int_C \|(\operatorname{lip}(\psi f_i))_i\|_e\mathrm{d}\mathsf{m} \le \int_A \|\operatorname{lip}(f_i)\|_e\mathrm{d}\mathsf{m}.$$

Back to the general case $f \in \mathrm{BV}(\mathsf{X})^m$, we notice that we have to show the claim in the case $f_i \in L^\infty(A,\mathsf{m})$ for every $i = 1,\dots,m$. Then, we can conclude by the very definition of $|\mathrm{D}f|$ and what said above, noticing that approximating sequences can be taken made of functions uniformly bounded in $L^\infty(\mathsf{m})$ with no loss of generality.

Step 2. We show that $|\mathrm{D}f|^*$ is the restriction to open sets of a finite Borel measure (that we still call $|\mathrm{D}f|^*$). To this aim, we can use Carathéodory criterion ([18], cf. [15, Proof of Lemma 5.2]) and is then enough to verify (all the sets in consideration are assumed to be open):

1. $|\mathrm{D}f|^*(A) \le |\mathrm{D}f|^*(B)$ if $A \subseteq B$,
2. $|\mathrm{D}f|^*(A\cup B) \ge |\mathrm{D}f|^*(A) + |\mathrm{D}f|^*(B)$ if $\mathsf{d}(A,B) > 0$,
3. $|\mathrm{D}f|^*(A) = \lim_k |\mathrm{D}f|^*(A_k)$ if $A_k \nearrow A$,
4. $|\mathrm{D}f|^*(A\cup B) \le |\mathrm{D}f|^*(A) + |\mathrm{D}f|^*(B)$.

We notice that (1) and (2) follow trivially from the definition of $|\mathrm{D}f|^*$ and that (2) does not even need the sets to be well separated. We prove now property (3). Fix $\varepsilon > 0$ and take a compact subset K with $K \subseteq A$ and $|\mathrm{D}f|(A\setminus K) \le \varepsilon$. Then there exists $\bar k$ such that $K \subseteq A_{\bar k}$, in particular we can find $\psi \in \mathrm{LIP_{bs}}(\mathsf{X})$ with $\psi(x) \in [0,1]$ for every $x \in \mathsf{X}$, $\psi = 1$ on a neighbourhood of K and $\operatorname{supp}\psi \Subset A_{\bar k}$. If we take $v = (v_1,\dots,v_m) \in \mathcal{W}^m_A$, we can write $v_i = \psi v_i + (1-\psi)v_i$ for $i = 1,\dots,m$, notice $(\psi v_i)_i \in \mathcal{W}^m_{A_{\bar k}}$ and $((1-\psi)v)_i \in \mathcal{W}^m_{A\setminus K}$. Then we can compute, using that $|\mathrm{D}f|^*(A\setminus K) \le |\mathrm{D}f|(A\setminus K) \le \varepsilon$,

$$\sum_{i=1}^{m}\int_A f_i\mathrm{div}\, v_i\,\mathrm{d}\mathsf{m} = \sum_{i=1}^{m}\int_{A_{\bar k}} f_i\mathrm{div}(\psi v_i)\,\mathrm{d}\mathsf{m} + \sum_{i=1}^{m}\int_{A\setminus K} f_i\mathrm{div}((1-\psi)v_i)\,\mathrm{d}\mathsf{m} \le |\mathrm{D}f|^*(A_{\bar k}) + \varepsilon$$

so that (3) follows as $v \in \mathcal{W}^m_A$ and $\varepsilon > 0$ are arbitrary. We prove now (4). Take a sequence of bounded open sets $\{A_k\}_k$ with $A_k \nearrow A$ and $A_k \subseteq \left\{x \in A : \mathsf{d}(x,\mathsf{X}\setminus A) > k^{-1}\right\}$; take similarly $\{B_k\}_k$. Fix k and take $\tilde\psi_A \in \mathrm{LIP_{bs}}(\mathsf{X})$ with $\tilde\psi_A(x) \in [0,1]$ for every $x \in \mathsf{X}$, $\tilde\psi_A = 1$ on a neighbourhood of A_k and $\operatorname{supp}\tilde\psi_A \Subset A$; define similarly $\tilde\psi_B$. Define also $\psi_A := \tilde\psi_A$ and $\psi_B := \tilde\psi_B(1-\tilde\psi_A)$. Take then $v = (v_1,\dots,v_m) \in \mathcal{W}^m_{A_k\cup B_k}$. Writing $v_i = \psi_A v_i + \psi_B v_i$ for $i = 1,\dots,m$ we can argue similarly as above to verify that

$$|\mathrm{D}f|^*(A_k\cup B_k) \le |\mathrm{D}f|^*(A) + |\mathrm{D}f|^*(B)$$

so that (4) follows letting $k \to \infty$, taking into account (3).

Step 3. We conclude that $|\mathrm{D}f|^* = |\mathrm{D}f|$. By the previous steps, it is enough to show $|\mathrm{D}f|^*(\mathsf{X}) \geq |\mathrm{D}f|(A)$ if $A \subseteq \mathsf{X}$ is open and bounded. Assume for the moment that also $f \in L^\infty(\mathsf{m})^m$. Let $t_k \searrow 0$ and consider $f_{i,k} := \mathrm{h}_{t_k} f_i$. By lower semicontinuity of the total variation,

$$|\mathrm{D}f|(A) \leq \liminf_k |\mathrm{D}(f_{1,k}, \dots, f_{n,k})|(A)$$

and then, taking into account that $\mathsf{m}(A) < \infty$ and the general theory of Sobolev spaces,

$$|\mathrm{D}f|(A) \leq \liminf_k \int_A \|(|\nabla f_{i,k}|)_{i=1,\dots,m}\|_e \, \mathrm{d}\mathsf{m} \leq \liminf_k \int \|(|\nabla f_{i,k}|)_{i=1,\dots,m}\|_e \, \mathrm{d}\mathsf{m} \,. \tag{2.3.19}$$

By density, we take $f_{i,k,l} \subseteq \mathrm{TestF}(\mathsf{X})$ such that $f_{i,k,l} \to f_{i,k}$ in $H^{1,2}(\mathsf{X})$ as $l \to \infty$ for every i. We can write the right hand side of (2.3.19) as

$$\liminf_k \lim_{\varepsilon \searrow 0} \lim_l \lim_{\delta \searrow 0} \sum_i \int \nabla f_{i,k} \cdot \frac{\nabla \mathrm{h}_\delta f_{i,k,l}}{\sqrt{\sum_j \mathrm{h}_\delta(|\nabla f_{j,k,l}|^2) + \varepsilon}} \, \mathrm{d}\mathsf{m} \,,$$

that is,

$$\liminf_k \lim_{\varepsilon \searrow 0} \lim_l \lim_{\delta \searrow 0} \sum_i \int f_{i,k} \mathrm{div} \left(\frac{\nabla \mathrm{h}_\delta f_{i,k,l}}{\sqrt{\sum_j \mathrm{h}_\delta(|\nabla f_{j,k,l}|^2) + \varepsilon}} \right) \mathrm{d}\mathsf{m} \,. \tag{2.3.20}$$

Recalling the properties of the heat flow $\mathrm{h}_{\mathrm{H},t}$, we can rewrite the quantity in (2.3.20) as

$$\liminf_k \lim_{\varepsilon \searrow 0} \lim_l \lim_{\delta \searrow 0} \sum_i \int f_i \mathrm{div} \left(\mathrm{h}_{\mathrm{H},\mathrm{t_k}} \left(\frac{\nabla \mathrm{h}_\delta f_{i,k,l}}{\sqrt{\sum_j \mathrm{h}_\delta(|\nabla f_{j,k,l}|^2) + \varepsilon}} \right) \right) \mathrm{d}\mathsf{m}$$

and see that it is bounded by $|\mathrm{D}f|^*(\mathsf{X})$, by an approximation argument that relies on Lemma 2.2.1; here we used that an immediate approximation argument yields that if $A = \mathsf{X}$ the request that $\mathrm{supp}\, v_i$ is compact in (2.3.18) is irrelevant. We have therefore proved $|\mathrm{D}f|^*(\mathsf{X}) = |\mathrm{D}f|(\mathsf{X})$ in the case f bounded.

We treat the general case. We write

$$f^l := ((f_1 \vee -l) \wedge l, \dots, (f_m \vee -l) \wedge l). \tag{2.3.21}$$

Now we can conclude easily, as, by lower semicontinuity, what we just proved, and coarea

$$\begin{aligned} |\mathrm{D}f|(\mathsf{X}) &\leq \liminf_l |\mathrm{D}f^l|(\mathsf{X}) = \liminf_l |\mathrm{D}f^l|^*(\mathsf{X}) \leq |\mathrm{D}f|^*(\mathsf{X}) + \liminf_l |\mathrm{D}(f^l - f)|^*(\mathsf{X}) \\ &\leq |\mathrm{D}f|^*(\mathsf{X}) + \sum_i \limsup_l |\mathrm{D}(f_i^l - f_i)|(\mathsf{X}) = |\mathrm{D}f|^*(\mathsf{X}). \end{aligned}$$

The last claim can be proved as for Proposition 2.3.5. □

Remark 2.3.19. One may wonder whether Proposition 2.3.18 holds also in the more general setting of (infinitesimally Hilbertian) metric measure spaces, with the obvious modifications (i.e. whether we can extend Proposition 2.3.5 to functions taking values in $\mathbb{R}^m$ instead of $\mathbb{R}$). It seems anything but straightforward to adapt the argument used in [71] (extracted from [72, 15]) as here we face a difficulty generalizing the approach via test plans. For this reason we had to provide a completely different proof, obtained via approximation arguments, at the price of working in more regular spaces. We give here an example of this issue, using the notation of the articles just cited. We point out that the difference $|\mathrm{D}f| \neq |\mathrm{D}f|_w$ that we are going to see is what we expect, given the choice of the relaxation made to define the total variation, cf. Remark 2.3.14. Nevertheless, being all the norms on finite dimensional spaces equivalent, the two objects are equivalent, in the sense that one bounds the other, up to a multiplicative constant.

Consider $\mathsf{X} := [0,1]^2 \subseteq \mathbb{R}^2$ endowed with the Euclidean distance and the Lebesgue measure. Let $f : \mathsf{X} \to \mathbb{R}^2$ be the identity. It is clear that $f \in \mathrm{BV}(\mathsf{X})^2$ and $|\mathrm{D}f|(\mathsf{X}) = \sqrt{2}$. However, computing the total variation defined via test plans, $|\mathrm{D}f|_w = 1$. Indeed, if $B \subseteq \mathsf{X}$ is a Borel set and $\boldsymbol{\pi}$ is a test plan, we obtain, using Fubini's theorem,

$$\begin{aligned}\int \gamma_\# |\mathrm{D}(f \circ \gamma)|(B)\, \mathrm{d}\boldsymbol{\pi}(\gamma) &\le \int \mathcal{L}^1(\{t : \gamma_t \in B\}) \mathrm{Lip}(\gamma)\, \mathrm{d}\boldsymbol{\pi}(\gamma) \\ &\le \|\mathrm{Lip}(\gamma)\|_{L^\infty(\boldsymbol{\pi})} (\mathcal{L}^1 \otimes \boldsymbol{\pi})(\{(t,\gamma) : \gamma_t \in B\}) \\ &\le \|\mathrm{Lip}(\gamma)\|_{L^\infty(\boldsymbol{\pi})} C(\boldsymbol{\pi}) \mathcal{L}^2(B),\end{aligned}$$

so that $|\mathrm{D}f|_w \le \mathcal{L}^2$. ■

Chapter 3

Sets of finite perimeter

This chapter is about the fine properties of sets of finite perimeter on finite dimensional RCD spaces. Namely, blow-ups of sets of finite perimeter are studied, along with the rectifiability of reduced boundaries and representation formulae for perimeter measures.

3.1 Regular behaviour

The main result of this section states that total variations of functions of bounded variation are concentrated on the regular set $\mathcal{R}_n^*(\mathsf{X})$. We state it for functions of bounded variation, nevertheless notice that by coarea the same result for characteristic function of sets of finite perimeter would imply the result as stated below.

Theorem 3.1.1. *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *be an* $\mathrm{RCD}(K,N)$ *of essential dimension* n*. Then*

$$|\mathrm{D}f|(\mathsf{X}\setminus\mathcal{R}_n^*(\mathsf{X}))=0\qquad \textit{for every } f\in\mathrm{BV}(\mathsf{X}).$$

Proof. The statement can be achieved by repeating *verbatim* the proof of [50, Theorem 3.1], using $\mathcal{R}_n^*(\mathsf{X})$ instead of $\mathcal{R}_n(\mathsf{X})$, and Lemma 3.1.2 below instead of [50, Proposition 2.14]. □

The proof of Theorem 3.1.1 builds upon the characterization of total variation along curves, as well as the following crucial result, which is based on [70] (which, in turn, is the generalization to RCD spaces of [62], obtained in the context of Ricci limits).

Lemma 3.1.2. *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *be an* $\mathrm{RCD}(K,N)$ *space of essential dimension* n*. Suppose that* $\gamma:[0,1]\to\mathsf{X}$ *is a geodesic satisfying* $\gamma_t\in\mathcal{R}_n^*(\mathsf{X})$ *for a dense family of* $t\in(0,1)$*. Then it holds that* $\gamma_t\in\mathcal{R}_n^*(\mathsf{X})$ *for every* $t\in(0,1)$*.*

Proof. Let $\delta\in(0,1/20)$ be fixed. The non-branching property of X, proved in [70, Theorem 1.3], ensures that the constant-speed reparametrisation of $\gamma|_{[\delta/2,1-\delta/2]}$ on $[0,1]$ is the unique geodesic between its endpoints.

We first prove that $\gamma_t\in\mathcal{R}_n(\mathsf{X})$ for every $t\in[\delta,1-\delta]$. Now by [70, Theorem 1.1] (see also [70, Theorem 1.2]), we have $\bar r$ and C such that

$$\mathsf{d}_{\mathrm{pGH}}((B_r(\gamma_s),\gamma_s),(B_r(\gamma_{s'}),\gamma_{s'}))\le Cr|s-s'|\qquad\text{for every } r\in(0,\bar r)\text{ and } s,s'\in[\delta,1-\delta].$$

Fix for the moment $s'\in[\delta,1-\delta]$ and take $(\mathsf{X}',\mathsf{d}',\mathsf{m}',x')\in\mathrm{Tan}_{\gamma_{s'}}(\mathsf{X},\mathsf{d},\mathsf{m})$. Notice that if $\gamma_s\in\mathcal{R}_n(\mathsf{X})$, then, by the equation above, for any $R>0$,

$$\mathsf{d}_{\mathrm{pGH}}((B_R^{\mathsf{X}'}(x'),x'),(B_R^{\mathbb{R}^n}(0),0)<CR|s-s'|$$

and since $\gamma_s \in \mathcal{R}_n(\mathsf{X})$ for a dense set of s, we infer that $(B_R^{\mathsf{X}'}(x'), \mathsf{d}', x') = (B_R^{\mathbb{R}^n}(0), \mathsf{d}_e, 0)$ so that, being R arbitrary, $(\mathsf{X}', \mathsf{d}', x') = (\mathbb{R}^n, \mathsf{d}_e, 0)$. As $(\mathsf{X}', \mathsf{d}', \mathsf{m}', x')$ is RCD$(0, N)$, an iterative application of the splitting Theorem ([81]) yields that $(\mathsf{X}', \mathsf{d}', \mathsf{m}', x') = (\mathbb{R}^n, \mathsf{d}_e, \underline{\mathcal{L}}^n, 0)$, whence $\gamma_{s'} \in \mathcal{R}_n(\mathsf{X})$.

Now we prove that $\gamma_t \in \mathcal{R}_n^*(\mathsf{X})$ for every $t \in [\delta, 1-\delta]$. Recall that [70, Eq. (166)] gives constants ε, $\bar{r}$ and C such that

$$\left|\frac{\mathsf{m}(B_r(\gamma_s))}{\mathsf{m}(B_r(\gamma_{s'}))} - 1\right| \leq C|s-s'|^{\frac{1}{2(1+2N)}} \qquad \text{for every } r \in (0, \bar{r}) \text{ and } s, s' \in [\delta, 1-\delta] \text{ with } |s-s'| < \varepsilon.$$

In particular, for any $s, s' \in [\delta, 1-\delta]$ with $|s-s'| < \varepsilon$ we have that

$$\left|\frac{\mathsf{m}(B_r(\gamma_s))}{\omega_n r^n}\left(\frac{\mathsf{m}(B_r(\gamma_{s'}))}{\omega_n r^n}\right)^{-1} - 1\right| \leq C|s-s'|^{\frac{1}{2(1+2N)}} \qquad \text{for every } r \in (0, \bar{r}). \tag{3.1.1}$$

Now let $t \in [\delta, 1-\delta]$ be fixed and choose a sequence $\{t_i\}_{i\in\mathbb{N}} \subseteq \gamma^{-1}(\mathcal{R}_n^*(\mathsf{X})) \cap [\delta, 1-\delta] \cap (t-\varepsilon, t+\varepsilon)$ such that $t_i \to t$. Up to a not relabelled subsequence, we can assume that $\Theta_n(\mathsf{m}, \gamma_{t_i}) \to \lambda$ for some $\lambda \in [0, \infty]$. Pick sequences $\{r_j\}_{j\in\mathbb{N}}, \{\tilde{r}_j\}_{j\in\mathbb{N}} \subseteq (0, \bar{r})$ such that

$$\frac{\mathsf{m}(B_{r_j}(\gamma_t))}{\omega_n r_j^n} \to \overline{\Theta}_n(\mathsf{m}, \gamma_t) \qquad \text{and} \qquad \frac{\mathsf{m}(B_{\tilde{r}_j}(\gamma_t))}{\omega_n \tilde{r}_j^n} \to \underline{\Theta}_n(\mathsf{m}, \gamma_t).$$

Plugging $(s, s', r) = (t, t_i, r_j)$ or $(s, s', r) = (t, t_i, \tilde{r}_j)$ in (3.1.1), and letting $j \to \infty$, we deduce that $\overline{\Theta}_n(\mathsf{m}, \gamma_t) < \infty$ and

$$\left|\frac{\overline{\Theta}_n(\mathsf{m}, \gamma_t)}{\Theta_n(\mathsf{m}, \gamma_{t_i})} - 1\right|, \left|\frac{\underline{\Theta}_n(\mathsf{m}, \gamma_t)}{\Theta_n(\mathsf{m}, \gamma_{t_i})} - 1\right| \leq C|t-t_i|^{\frac{1}{2(1+2N)}} \qquad \text{for every } i \in \mathbb{N}. \tag{3.1.2}$$

Similarly, plugging $(s, s', r) = (t_i, t, r_j)$ or $(s, s', r) = (t_i, t, \tilde{r}_j)$ in (3.1.1), and letting $j \to \infty$, we deduce that $\underline{\Theta}_n(\mathsf{m}, \gamma_t) > 0$ and

$$\left|\frac{\Theta_n(\mathsf{m}, \gamma_{t_i})}{\overline{\Theta}_n(\mathsf{m}, \gamma_t)} - 1\right|, \left|\frac{\Theta_n(\mathsf{m}, \gamma_{t_i})}{\underline{\Theta}_n(\mathsf{m}, \gamma_t)} - 1\right| \leq C|t-t_i|^{\frac{1}{2(1+2N)}} \qquad \text{for every } i \in \mathbb{N}. \tag{3.1.3}$$

Observe that (3.1.2) and (3.1.3) imply, respectively, that for every $i \in \mathbb{N}$ it holds that

$$\left|\overline{\Theta}_n(\mathsf{m}, \gamma_t) - \underline{\Theta}_n(\mathsf{m}, \gamma_t)\right| \leq 2C|t-t_i|^{\frac{1}{2(1+2N)}}\, \Theta_n(\mathsf{m}, \gamma_{t_i}), \tag{3.1.4}$$

$$\left|\overline{\Theta}_n(\mathsf{m}, \gamma_t) - \underline{\Theta}_n(\mathsf{m}, \gamma_t)\right| \leq 2C|t-t_i|^{\frac{1}{2(1+2N)}}\, \frac{\overline{\Theta}_n(\mathsf{m}, \gamma_t)\underline{\Theta}_n(\mathsf{m}, \gamma_t)}{\Theta_n(\mathsf{m}, \gamma_{t_i})}. \tag{3.1.5}$$

Hence, we can conclude that $\overline{\Theta}_n(\mathsf{m}, \gamma_t) = \underline{\Theta}_n(\mathsf{m}, \gamma_t) \in (0, +\infty)$ by letting $i \to \infty$ in (3.1.4) if $\lambda < \infty$, or in (3.1.5) if $\lambda = +\infty$. This shows that $\gamma_t \in \mathcal{R}_n^*(\mathsf{X})$ for every $t \in [\delta, 1-\delta]$.

Thanks to the arbitrariness of δ, we proved that $\gamma_t \in \mathcal{R}_n^*(\mathsf{X})$ for every $t \in (0, 1)$, as desired. □

3.2 Blow-ups

The goal of this section is to study how sets of finite perimeter behave after a blow-up of the space. We start introducing the definition of blow-up of a set of finite perimeter and a technical/compactness tool. Then we discuss splitting maps, good coordinates, and their interplay. Finally, we prove the main result of the section, i.e. the characterization of blow-ups of sets of finite perimeter.

3.2.1 Tools

The following is one of the most important definitions of this manuscript.

Definition 3.2.1. Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,N)$ space, let $x\in\mathsf{X}$ and let E be a measurable subset of X. We say that the quintuple $(\mathsf{X}_\infty,\mathsf{d}_\infty,\mathsf{m}_\infty,x_\infty,E_\infty)$ *is tangent to* $(\mathsf{X},\mathsf{d},\mathsf{m},E)$ at x if there exists a sequence of radii $r_k\searrow 0$, such that

i) $(\mathsf{X},r_k^{-1}\mathsf{d},\mathsf{m}^x_{r_k},x)\to(\mathsf{X}_\infty,\mathsf{d}_\infty,\mathsf{m}_\infty,x_\infty)$ in the pointed measured Gromov-Hausdorff topology, say in a realization (proper metric space) $(\mathsf{Z},\mathsf{d}_\mathsf{Z})$ with respect to isometric embeddings $\iota_k:(\mathsf{X},r_k^{-1}\mathsf{d})\to(\mathsf{Z},\mathsf{d}_\mathsf{Z})$ and $\iota_\infty:(\mathsf{X}_\infty,\mathsf{d}_\infty)\to(\mathsf{Z},\mathsf{d}_\mathsf{Z})$.

ii) E_∞ is measurable subset of X_∞ such that $\iota_k(E)$ converge in L^1_{loc} to $\iota_\infty(E_\infty)$ with respect to the realization as above.

iii) E_∞ is a set of locally finite perimeter and *weak convergence of rescaled perimeters* holds, i.e.

$$(\iota_k)_*|\mathrm{D}^k\chi_E|\to(\iota_\infty)_*|\mathrm{D}\chi_{E_\infty}|\qquad\text{in duality with } C_{\mathrm{bs}}(\mathsf{Z})\tag{3.2.1}$$

with respect to the realization as above, where $|\mathrm{D}^k\chi_E|$ is the perimeter measure relative to the rescaled space $(\mathsf{X},r_k^{-1}\mathsf{d},\mathsf{m}^x_{r_k})$.

We denote the collection of all tangent spaces to $(\mathsf{X},\mathsf{d},\mathsf{m},E)$ at x as $\mathrm{Tan}_x(\mathsf{X},\mathsf{d},\mathsf{m},E)$ and we will write a converging sequence as above by

$$(\mathsf{X},r_k^{-1}\mathsf{d},\mathsf{m}^x_{r_k},x,E)\to(\mathsf{X}_\infty,\mathsf{d}_\infty,\mathsf{m}_\infty,x_\infty,E_\infty).$$

We consider two elements $(\mathsf{X}_\infty,\mathsf{d}_\infty,\mathsf{m}_\infty,x_\infty,E_\infty)$, $(\mathsf{X}'_\infty,\mathsf{d}'_\infty,\mathsf{m}'_\infty,x'_\infty,E'_\infty)\in\mathrm{Tan}_x(\mathsf{X},\mathsf{d},\mathsf{m},E)$ to be isomorphic if there exists an isometry $\Psi:\mathsf{X}_\infty\to\mathsf{X}'_\infty$ such that $\Psi(x_\infty)=x'_\infty$, $\Psi_*\mathsf{m}_\infty=\mathsf{m}'_\infty$ (i.e. $(\mathsf{X}_\infty,\mathsf{d}_\infty,\mathsf{m}_\infty,x_\infty)$ and $(\mathsf{X}'_\infty,\mathsf{d}'_\infty,\mathsf{m}'_\infty,x'_\infty)$ are isomorphic as pointed metric measure spaces through Ψ) and moreover $\mathsf{m}'_\infty(\Psi(E_\infty)\Delta E'_\infty)=0$.

Now we show a result which implies, in a sense, compactness of sets of (locally) finite perimeter during a blow-up procedure.

Theorem 3.2.2. *Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,N)$ space and let $E\subseteq\mathsf{X}$ be a set of locally finite perimeter. Let $x\in\operatorname{supp}|\mathrm{D}\chi_E|$ be satisfying the conclusions of Proposition 2.3.3.*

Let $r_k\searrow 0$ be such that $(\mathsf{X},r_k^{-1}\mathsf{d},\mathsf{m}^x_{r_k},x)\to(\mathsf{X}_\infty,\mathsf{d}_\infty,\mathsf{m}_\infty,x_\infty)$ in the pmGH topology, for some realization. Then we can extract a not relabelled subsequence such that there exists a set of locally finite perimeter $E_\infty\subseteq\mathsf{X}_\infty$ and, with respect to the same realization as above, $(\mathsf{X},r_k^{-1}\mathsf{d},\mathsf{m}^x_{r_k},x,E)\to(\mathsf{X}_\infty,\mathsf{d}_\infty,\mathsf{m}_\infty,x_\infty,E_\infty)$ according to Definition 3.2.1. In particular, we have that $\mathrm{Tan}_x(\mathsf{X},\mathsf{d},\mathsf{m},E)$ is not empty.

Proof. This is [10, Corollary 4.10] with its proof, we give some of the details, we use the notation of Definition 3.2.1. By item *iii)* of Proposition 2.3.3, it holds that, for any $R>0$,

$$\begin{aligned}\limsup_k|\mathrm{D}^k\chi_E|(B^k_R(x))&=\limsup_k\frac{r_k|\mathrm{D}\chi_E|(B_{Rr_k}(x))}{\mathsf{m}(B_{r_k}(x))}\\&\le\limsup_k\frac{r_k|\mathrm{D}\chi_E|(B_{Rr_k}(x))}{\mathsf{m}(B_{Rr_k}(x))}\mathsf{m}^x_{r_k}(B^k_R(x))\\&\le\limsup_k\frac{Rr_k|\mathrm{D}\chi_E|(B_{Rr_k}(x))}{\mathsf{m}(B_{Rr_k}(x))}\frac{\mathsf{m}_\infty(B_R(x_\infty))}{R}<\infty,\end{aligned}\tag{3.2.2}$$

so that we can apply [10, Corollary 3.4] and extract a (not relabelled) subsequence such that also item ii) of Definition 3.2.1 holds, for some Borel set $E_\infty \subseteq \mathsf{X}_\infty$ (notice that the realization $(\mathsf{Z}, \mathsf{d}_\mathsf{Z})$ is fixed in [10, Section 3], so that such convergence happens in the realization $(\mathsf{Z}, \mathsf{d}_\mathsf{Z})$ that we have in the statement). It remains to show that, up to passing to a subsequence, also item item iii) of Definition 3.2.1 is satisfied, as the last conclusion follows from Gromov compactness Theorem for $\mathrm{RCD}(K, N)$ spaces.

Item iii) of Definition 3.2.1 follows from [10, Proposition 3.9], we repeat the argument. We know that $\iota_k(E) \to \iota_\infty(E_\infty)$ in L^1_{loc} and, up to passing to a (not relabelled) subsequence, that $(\iota_k)_*|\mathrm{D}^k\chi_E| \to (\iota_\infty)_*\nu$ in duality with $C_{\mathrm{bs}}(\mathsf{Z})$, for some Radon measure ν on $(\mathsf{X}_\infty, \mathsf{d}_\infty)$. The second conclusion is due to weak compactness in the space of measures, taking into account (3.2.2) again and the fact that any weak limit of $(\iota_k)_*|\mathrm{D}^k\chi_E|$ is supported on $\iota_\infty(\mathsf{X}_\infty)$. In other words, we only have to prove that $\nu = |\mathrm{D}\chi_{E_\infty}|$.

Now we argue as in the proof of [10, Corollary 3.4] (recall (3.2.2)), and we see that, up to extracting a not relabelled subsequence, for a sequence of radii $R_l \nearrow \infty$, for every l,

$$\sup_k |\mathrm{D}^k\chi_{B^k_{R_l}(x)}|(\mathsf{X}) < \infty \qquad \text{and} \qquad \sup_k |\mathrm{D}^k\chi_{E\cap B^k_{R_l}(x)}|(\mathsf{X}) < \infty. \tag{3.2.3}$$

Notice that, by the very definition of L^1_{loc} convergence, we have that $\iota_k(\chi_E \cap B^k_{R_l}(x)) = \iota_k(\chi_E) \cap B_{R_l}(\iota_k(x))$ converge in L^1-strong to $\iota_\infty(\chi_{E_\infty} \cap B_{R_l}(x_\infty)) = \iota_\infty(\chi_{E_\infty}) \cap B_{R_l}(\iota_\infty(x_\infty))$, for every l. This observation allows us to use [10, Proposition 3.6], in particular, [10, Equation (3.9)], to conclude that, for every l, $E_\infty \cap B_{R_l}(x_\infty)$ has finite perimeter and

$$\int_{\mathsf{X}_\infty} g\mathrm{d}|\mathrm{D}\chi_{E_\infty\cap B_{R_l}(x_\infty)}| \le \liminf_k \int_{\mathsf{X}} g\mathrm{d}|\mathrm{D}^k\chi_{E\cap B^k_{R_l}(x)}| \qquad \text{for every } g \in \mathrm{LIP_{bs}}(\mathsf{Z}, \mathsf{d}_\mathsf{Z}) \text{ non-negative.}$$

In particular, by locality, taking into account that $R_l \nearrow \infty$, we deduce that E_∞ has locally finite perimeter and

$$|\mathrm{D}\chi_{E_\infty}| \le \nu. \tag{3.2.4}$$

Now, for every l we apply [10, Proposition 3.8] to obtain a sequence $\{E^l_k\}_k$ of sets of finite perimeter, where $E^l_k \subseteq \mathsf{X}$ satisfy

$$E^l_k \to E_\infty \cap B_{R_l}(x_\infty) \quad \text{in } L^1\text{-strong} \qquad \text{and} \qquad |\mathrm{D}^k\chi_{E^l_k}|(\mathsf{X}) \to |\mathrm{D}\chi_{E_\infty\cap B_{R_l}(x_\infty)}|(\mathsf{X}^\infty).$$

In particular, using as before [10, Proposition 3.6] (but this time, we can exploit the second conclusion above), we deduce that

$$(\iota_k)_*|\mathrm{D}^k\chi_{E^l_k}| \to (\iota_\infty)_*|\mathrm{D}\chi_{E_\infty\cap B_{R_l}(x_\infty)}| \qquad \text{in duality with } C_{\mathrm{bs}}(\mathsf{Z}). \tag{3.2.5}$$

Again, we have taken a not relabelled subsequence of $\{r_k\}_k$ (with a diagonal argument).

Now fix l and let $s \in (R_l/2, R_l)$ be such that

$$\nu(\partial B_s(x_\infty)) = |\mathrm{D}\chi_{E_\infty}|(\partial B_s(x_\infty)) = |\mathrm{D}^k\chi_E(\partial B^k_s(x))| = 0 \qquad \text{for every } k$$

(notice that a.e. $s \in (R_l/2, R_l)$ is suitable) and define

$$\tilde{E}^l_k := (E^l_k \cap B^k_s(x)) \cup (E \setminus B^k_s(x)) = (E^l_k \cap B_{sr_k}(x)) \cup (E \setminus B_{sr_k}(x)).$$

By item *ii*) of Proposition 2.3.3, we have that, for k big enough (namely, such that $R_l r_k < r_x$),

$$\begin{aligned}|\mathrm{D}^k\chi_E|(B^k_{R_l}(x)) = \frac{r_k|\mathrm{D}\chi_E|(B_{R_l r_k}(x))}{\mathsf{m}(B_{r_k}(x))} &\le (1+\omega_x(R_l r_k))\frac{r_k|\mathrm{D}\chi_{\tilde{E}^l_k}|(B_{R_l r_k}(x))}{\mathsf{m}(B_{r_k}(x))}\\ &= (1+\omega_x(R_l r_k))|\mathrm{D}^k\chi_{\tilde{E}^l_k}|(B^k_{R_l}(x)) = \lambda_k|\mathrm{D}^k\chi_{\tilde{E}^l_k}|(B^k_{R_l}(x)),\end{aligned}$$

where $\lambda_k := (1+\omega_x(R_l r_k)) \to 1$ as $k \nearrow \infty$. We can then compute

$$\begin{aligned}|\mathrm{D}^k\chi_E|(\bar B^k_s(x)) &= |\mathrm{D}^k\chi_E|(B^k_{R_l}(x)) - |\mathrm{D}^k\chi_E|(B^k_{R_l}(x)\setminus \bar B^k_s(x))\\ &\le \lambda_k|\mathrm{D}^k\chi_{\tilde E^l_k}|(B^k_{R_l}(x)) - |\mathrm{D}^k\chi_E|(B^k_{R_l}(x)\setminus \bar B^k_s(x))\\ &= \lambda_k|\mathrm{D}^k\chi_{\tilde E^l_k}|(B^k_s(x)) + \lambda_k|\mathrm{D}^k\chi_{\tilde E^l_k}|(\partial B^k_s(x))\\ &\qquad + \lambda_k|\mathrm{D}^k\chi_{\tilde E^l_k}|(B^k_{R_l}(x)\setminus \bar B^k_s(x)) - |\mathrm{D}^k\chi_E|(B^k_{R_l}(x)\setminus \bar B^k_s(x))\\ &= \lambda_k|\mathrm{D}^k\chi_{E^l_k}|(B^k_s(x)) + \lambda_k|\mathrm{D}^k\chi_{\tilde E^l_k}|(\partial B^k_s(x))\\ &\qquad + (\lambda_k-1)|\mathrm{D}^k\chi_E|(B^k_{R_l}(x)\setminus \bar B^k_s(x)).\end{aligned}$$

Hence, using the second conclusion of (3.2.3) and the fact that $\lambda_k \to 1$, and then (3.2.5) for the second inequality,

$$\begin{aligned}\nu(B_s(x_\infty)) &\le \liminf_k |\mathrm{D}^k\chi_E|(\bar B^k_s(x)) \le \limsup_k |\mathrm{D}^k\chi_{E^l_k}|(B^k_s(x)) + \liminf_k |\mathrm{D}^k\chi_{\tilde E^l_k}|(\partial B^k_s(x))\\ &\le |\mathrm{D}\chi_{E_\infty\cap B_{R_l}(x_\infty)}|(\bar B_s(x_\infty)) + \liminf_k |\mathrm{D}^k\chi_{\tilde E^l_k}|(\partial B^k_s(x))\\ &= |\mathrm{D}\chi_{E_\infty}|(B_s(x_\infty)) + \liminf_k |\mathrm{D}^k\chi_{\tilde E^l_k}|(\partial B^k_s(x)),\end{aligned}$$

where the last equality is due to the constraints on s.

Therefore, recalling (3.2.4), it remains to prove that

$$\liminf_k |\mathrm{D}^k\chi_{\tilde E^l_k}|(\partial B^k_s(x)) = 0 \qquad \text{for a.e. } s\in(R_l/2,R_l). \tag{3.2.6}$$

We start by noticing that

$$\chi_{\tilde E^l_k} = \chi_{E^l_k}\chi_{B_{sr_k}(x)} + \chi_E\chi_{\mathsf{X}\setminus B_{sr_k}(x)} = (\chi_{E^l_k}-\chi_E)\chi_{B_{sr_k}}(x) + \chi_E,$$

so that

$$|\mathrm{D}\chi_{\tilde E^l_k}| \le \big|\mathrm{D}\big((\chi_{E^l_k}-\chi_E)\chi_{B_{sr_k}}(x)\big)\big| + |\mathrm{D}\chi_E|. \tag{3.2.7}$$

Now we apply (with an immediate localization argument) [10, Lemma 3.10] with $\chi_{E^l_k}-\chi_E$ in place of f to deduce that

$$\big|\mathrm{D}\big((\chi_{E^l_k}-\chi_E)\chi_{B_{sr_k}}(x)\big)\big|(\partial B_{sr_k(x)}) \le \int_{\mathsf{X}} |\chi_{E^l_k}-\chi_E|\mathrm{d}|\mathrm{D}\chi_{B_{sr_k}}(x)| \qquad \text{for a.e. } s\in(R_l/2,R_l). \tag{3.2.8}$$

Hence, recalling the constraint on s, reading (3.2.7) and (3.2.8) in the rescaled spaces,

$$|\mathrm{D}^k\chi_{\tilde E^l_k}|(\partial B^k_s(x)) \le \int_{\mathsf{X}} |\chi_{E^l_k}-\chi_E|\mathrm{d}|\mathrm{D}^k\chi_{B^k_s}(x)| \qquad \text{for a.e. } s\in(R_l/2,R_l).$$

Then, using also Fatou's Lemma and the coarea formula,

$$\begin{aligned}\int_{R_l/2}^{R_l}\liminf_k |\mathrm{D}^k\chi_{\tilde E^l_k}|(\partial B^k_s(x))\mathrm{d}s &\le \liminf_k \int_{R_l/2}^{R_l} |\mathrm{D}^k\chi_{\tilde E^l_k}|(\partial B^k_s(x))\mathrm{d}s\\ &\le \liminf_k \int_{R_l/2}^{R_l}\int_{\mathsf{X}} |\chi_{E^l_k}-\chi_E|\mathrm{d}|\mathrm{D}^k\chi_{B^k_s}(x)|\mathrm{d}s\\ &\le \liminf_k \int_{B^k_{R_l}(x)} |\chi_{E^l_k}-\chi_E|\mathrm{d}\mathsf{m}^x_{r_k}.\end{aligned} \tag{3.2.9}$$

Now notice that

$$\int_{B^k_{R_l}(x)} |\chi_{E^l_k} - \chi_{E\cap B_{R_l}(x)}| \mathsf{dm}^x_{r_k} = \int_{B^k_{R_l}(x)} \chi_{E^l_k} + \chi_{E\cap B_{R_l}(x)} - 2\chi_{E^l_k\cap(E\cap B_{R_l}(x))} \mathsf{dm}^x_{r_k} \to 0,$$

where we used that $E^l_k \to E_\infty \cap B_{R_l}(x_\infty)$ and $E\cap B_{R_l}(x) \to E_\infty \cap B_{R_l}(x_\infty)$ in L^1-strong and hence also $E^l_k \cap (E\cap B_{R_l}(x)) \to E_\infty \cap B_{R_l}(x_\infty)$ in L^1-strong by [10, Lemma 3.5]. This implies that the last term of (3.2.9) is 0, so that (3.2.6) follows. □

3.2.2 Splitting maps

We introduce splitting maps, after [58]. In the definition below, $\hat{C}_{K,N}$ is the constant appearing in the definition of δ splitting map in [51], which is linked to the (local) Lipschitz constant of harmonic functions [96].

A splitting map is an object that plays the role of some coordinate functions $(u^1,\dots,u^k) : \mathbb{R}^n \to \mathbb{R}^k$. Notice that these standard coordinate functions are Lipschitz, harmonic, have vanishing Hessian and have orthonormal gradients. Moreover, for any $\mathrm{RCD}(0,N)$ space, the presence of a map satisfying these properties implies that the space splits off a factor $\mathbb{R}^k$, isometrically (this follows from an iterated application of the splitting Theorem [81] as in [34, Lemma 1.21], see also [10, Section 2]).

Definition 3.2.3 (Splitting map)**.** Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,N)$ space. Let $x\in\mathsf{X}$, let $k\in\mathbb{N}$, and let $r,\delta>0$ be given. We say that a map

$$u = (u^1,\dots,u^k) : B_r(x) \to \mathbb{R}^k$$

is a δ-*splitting map* if the following three properties hold:

i) for every $i=1,\dots,k$, u^i is harmonic and $\hat{C}_{K,N}$-Lipschitz,

ii) for every $i=1,\dots,k$,
$$r^2 \fint_{B_r(x)} |\mathrm{Hess}\, u^i|^2 \mathsf{dm} < \delta,$$

iii) for every $i,j=1,\dots,k$,
$$\fint_{B_r(x)} |\nabla u^i \cdot \nabla u^j - \delta_{i,j}| \mathsf{dm} < \delta.$$

Remark 3.2.4. Let $u,v : B_r(x) \to \mathbb{R}$ be harmonic maps on an RCD space (hence u and v are locally Lipschitz, [96]). Then $\nabla u \cdot \nabla v$ has Lebesgue points with respect to m everywhere in $B_r(x)$. This is [50, Remark 2.10] together with a polarization argument. ■

Remark 3.2.5. Let $u,v : B_r(x) \to \mathbb{R}$ be harmonic maps on an RCD space. Then ∇u admits a quasi-continuous representative on $B_r(x)$ as well as $\nabla u \cdot \nabla v$ admits a quasi-continuous representative on $B_r(x)$. In particular, for any Borel measure, finite on balls, $\mu \ll \mathrm{Cap}$, ∇u admits a well defined trace on $L^0_\mu(T\mathsf{X})$ on $B_r(x)$ (see Theorem 2.2.22) as well as $\nabla u \cdot \nabla v$ admits a well defined trace on $L^0(\mu)$ on $B_r(x)$. We are going to exploit this traces throughout with no further notice.

Moreover, the Lebesgue representative (with respect to m) for $\nabla u \cdot \nabla v$, say f, is quasi-continuous on $B_r(x)$ (see [69] and the references therein for the relevant notions). In particular, f coincides with the trace of $\nabla u \cdot \nabla v$ on $L^0(\mu)$. We prove this fact. By a cut-off argument and the local nature of this claim, we can assume that f has compact support in $B_r(x)$, hence $f \in H^{1,2}(\mathsf{X})$. We denote with $\bar{f}$ the quasi-continuous representative for f, our claim is then $\bar{f} = f$ in $L^0(\mathrm{Cap})$. Take also $\{f_n\}_n \subseteq \mathrm{LIP_{bs}}(\mathsf{X})$ with $f_n \to f$ in $H^{1,2}(\mathsf{X})$. Notice that (taking the continuous representative of f_n) $f_n \to \bar{f}$ in $L^0(\mathrm{Cap})$ by the continuity of the quasi-continuous representative map $H^{1,2}(\mathsf{X}) \to L^0(\mathrm{Cap})$. Now the conclusion comes from the argument in the first part of the proof of [39, Theorem 5.62], see in particular [39, Equation (5.15)], with $\bar{f}$ in place of u and f_k in place of f. ■

Remark 3.2.6. The condition *ii*) of Definition 3.2.3 is easily seen to be unnecessary, in applications, but we preferred to keep it to follow the original definition. For example, inspecting the proof of Proposition 3.2.9 from [51, Proposition 3.7], we see that this condition does not enter into play, and the smallness of the Hessian is ultimately implied by condition *iii*), thanks to the local Bochner inequality (2.2.4), together with the doubling inequality (cf. Remark 3.2.13). ■

Now we state four key propositions [51] which show the interplay between the existence of splitting maps and the fact that a suitable rescaling of the space is close to a space that splits isometrically an Euclidean factor. The first shows existence of a splitting map, provided that the space is close enough to a product space.

Proposition 3.2.7. *Let $N > 1$ and let $k \in \mathbb{N}$ with $k \leq N$. For every $\delta \in (0,1)$, there exists $\varepsilon = \varepsilon(N,\delta) \in (0,1)$ such that for every $\mathrm{RCD}(-\varepsilon, N)$ space $(\mathsf{X},\mathsf{d},\mathsf{m})$ and $x \in \mathsf{X}$, if there exists a pointed $\mathrm{RCD}(0, N-k)$ space $(\mathsf{Z},\mathsf{d_Z},\mathsf{m_Z},z)$ satisfying*

$$\mathsf{d}_{\mathrm{pmGH}}\big((\mathsf{X},\mathsf{d},\mathsf{m},x),(\mathbb{R}^k \times \mathsf{Z},\mathsf{d}_{\mathbb{R}^k\times\mathsf{Z}},\mathcal{L}^k \otimes \mathsf{m_Z},(0,z))\big) < \varepsilon,$$

then there exists a δ-splitting map $u : B_5(x) \to \mathbb{R}^k$.

Then we have the scale-invariant version of Proposition 3.2.7.

Proposition 3.2.8. *Let $N > 1$ and let $k \in \mathbb{N}$ with $k \leq N$. For every $\delta \in (0,1)$, there exists $\varepsilon = \varepsilon(N,\delta) \in (0,1)$ such that for every $\mathrm{RCD}(K, N)$ space $(\mathsf{X},\mathsf{d},\mathsf{m})$ and $x \in \mathsf{X}$, if $r \in (0,1)$ is such that $r^2 \max\{-K, 0\} < \varepsilon$ and there exists a pointed $\mathrm{RCD}(0, N-k)$ space $(\mathsf{Z},\mathsf{d_Z},\mathsf{m_Z},z)$ satisfying*

$$\mathsf{d}_{\mathrm{pmGH}}\big((\mathsf{X},r^{-1}\mathsf{d},\mathsf{m}_x^r,x),(\mathbb{R}^k \times \mathsf{Z},\mathsf{d}_{\mathbb{R}^k\times\mathsf{Z}},\mathcal{L}^k \otimes \mathsf{m_Z},(0,z))\big) < \varepsilon,$$

then there exists a δ-splitting map $u : B_{5r}(x) \to \mathbb{R}^k$.

The third proposition is a sort of converse of the two above: if we have a map that is a splitting map at sufficiency many scales, then the space is close to a product space. Notice that while the assumption (the existence of a splitting map defined on $B_{\delta^{-1}}(x)$) is local, the conclusion ($\mathsf{d}_{\mathrm{pmGH}}(\mathsf{X},\mathbb{R}^k \times \mathsf{Z}) < \varepsilon$) has more a "global taste". The possibility of having such a statement comes from the very definition of the pmGH distance, through the usual use of cut-off functions.

Proposition 3.2.9. *Let $N > 1$ and let $k \in \mathbb{N}$ with $k \leq N$. For every $\varepsilon \in (0,1)$ there exists $\delta = \delta(N,\varepsilon) \in (0,1)$ such that for any $\mathrm{RCD}(-\delta, N)$ space $(\mathsf{X},\mathsf{d},\mathsf{m})$ and $x \in \mathsf{X}$, if there exists a map $u : B_{\delta^{-1}}(x) \to \mathbb{R}^k$ such that for every $s \in (0,\delta^{-1})$, $u : B_s(x) \to \mathbb{R}^k$ is a δ-splitting map, then there exists a pointed $\mathrm{RCD}(0, N-k)$ metric measure space $(\mathsf{Z},\mathsf{d_Z},\mathsf{m_Z},z)$ such that*

$$\mathsf{d}_{\mathrm{pmGH}}\big((\mathsf{X},\mathsf{d},\mathsf{m},x),(\mathbb{R}^k \times \mathsf{Z},\mathsf{d}_{\mathbb{R}^k\times\mathsf{Z}},\mathcal{L}^k \otimes \mathsf{m_Z},(0,z))\big) < \varepsilon.$$

Then we have the scale-invariant version of Proposition 3.2.9.

Proposition 3.2.10. *Let $N > 1$ and let $k \in \mathbb{N}$ with $k \leq N$. For every $\varepsilon \in (0,1)$ there exists $\delta = \delta(N,\varepsilon) \in (0,1)$ such that for every $r \in (0,1)$, for every $\mathrm{RCD}(K, N)$ space $(\mathsf{X},\mathsf{d},\mathsf{m})$ with $x \in \mathsf{X}$,* *if there exists a map* $u : B_r(x) \to \mathbb{R}^k$ *such that for every* $s \in (0, r)$*,* $u : B_s(x) \to \mathbb{R}^k$ *is a* δ*-splitting map, then for every* $(\mathsf{X}_\infty, \mathsf{d}_\infty, \mathsf{m}_\infty, x_\infty) \in \mathrm{Tan}_x(\mathsf{X}, \mathsf{d}, \mathsf{m})$*, there exists a pointed* $\mathrm{RCD}(0, N-k)$ *space* $(\mathsf{Z}, \mathsf{d}_\mathsf{Z}, \mathsf{m}_\mathsf{Z}, z)$ *such that*

$$\mathsf{d}_{\mathrm{pmGH}}\big((\mathsf{X}_\infty, \mathsf{d}_\infty, \mathsf{m}_\infty, x_\infty), (\mathbb{R}^k \times \mathsf{Z}, \mathsf{d}_{\mathbb{R}^k \times \mathsf{Z}}, \mathcal{L}^k \otimes \mathsf{m}_\mathsf{Z}, (0, z))\big) < \varepsilon.$$

Finally, for any splitting map, there exists a “bad set” with small Hausdorff content outside which the map is still a splitting map at any smaller scale. We omit its proof ([50]), which is based on a maximal function argument.

Proposition 3.2.11. *Let* $N > 1$ *and let* $k \in \mathbb{N}$ *with* $k \le N$*. Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space, let* $x \in \mathsf{X}$ *and let* $u : B_{4r}(x) \to \mathbb{R}^k$ *be a* δ*-splitting map for some* $r \in (0, 1/2)$ *such that* $r^2 \max\{-K, 0\} \le 4$*. Then there exists a Borel set* $G \subseteq B_{2r}(x)$ *with*

$$\mathcal{H}^h_5(B_{2r}(x) \setminus G) \le C_N \sqrt{\delta} \frac{\mathsf{m}(B_{2r}(x))}{2r},$$

where $C_N \in (0, \infty)$ *is a constant that depends only on* N*, such that for every* $y \in G$ *and* $s \in (0, r)$*,* $u : B_s(y) \to \mathbb{R}^k$ *is a* $C_N \delta^{1,4}$*-splitting map.*

The proposition above motivates the following definition.

Definition 3.2.12 (Good splitting map)**.** Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be an $\mathrm{RCD}(K, N)$ space of essential dimension n. Let $x \in \mathsf{X}$, let $r > 0$ and let $\eta \in (0, n^{-1})$. We say that a map

$$u = (u^1, \ldots, u^n) : B_{2r}(x) \to \mathbb{R}^n$$

is a *good* η*-splitting map on* $D \subseteq B_r(x)$ if the following two properties hold:

i) for every $i = 1, \ldots, n$, u^i is harmonic and $\hat{C}_{K,N}$-Lipschitz,

ii) for every $i, j = 1, \ldots, n$, for every $y \in D$ and $s \in (0, r)$,

$$\fint_{B_s(y)} |\nabla u^i \cdot \nabla u^j - \delta_{i,j}| \mathrm{d}\mathsf{m} < \eta$$

(i.e. u satisfies items i) and satisfies ii) of Definition 3.2.3 for every centre $y \in D$ and scale $s \in (0, r)$). We simply write good splitting map if the value of $\eta \in (0, n^{-1})$ is not important.

Remark 3.2.13. It may seem strange that in the definition of good splitting map we have dropped the scale-invariant averaged control on the squared Hilbert–Schmidt norm of the Hessian, i.e. the bound on $r^2 \fint_{B_r(y)} |\mathrm{Hess}\, u^i|^2 \mathrm{d}\mathsf{m}$ as in item iii) of Definition 3.2.3. This, however, is coherent with the motivation for good splitting maps, i.e. being able to operate at arbitrarily small scales. Indeed, $|\nabla u^i|^2$ has Lebesgue points everywhere in $B_r(x)$ (Remark 3.2.4) hence, by (2.2.4) and the doubling inequality, for every $y \in B_r(x)$,

$$r^2 \fint_{B_r(y)} |\mathrm{Hess}\, u^i|^2 \mathrm{d}\mathsf{m} \to 0,$$

so that the bound mentioned above would have been redundant. ■

Given a good splitting map, we can rotate its components in order to have them asymptotically form an orthonormal basis.

Lemma 3.2.14. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K,N)$ *space of essential dimension* n. *Let* $\eta \in (0, n^{-1})$ *and let* $u : B_{2r}(x) \to \mathbb{R}^n$ *be a good* η*-splitting map on* $D \subseteq B_r(x)$. *We define the matrix valued Borel map* $M = \{M_{i,j}\}_{i,j} : B_{2r}(x) \to \mathbb{R}^{n\times n}$ *as the Gram matrix*

$$M_{i,j} := \nabla u^i \cdot \nabla u^j.$$

Then there exists a matrix valued Borel map $A = \{A_{i,j}\}_{i,j} : D \to \mathbb{R}^{n\times n}$ *satisfying*

$$AMA^T = \mathrm{Id} \text{ and } |A| \le C_{\eta,n} \qquad \text{on } D,$$

where $C_{\eta,n} \in (0,\infty)$ *is a constant depending only on* η *and* n.

In particular, if $y \in D$ *and we set* $v_y := A(y)u$, *then*

$$(\nabla v_y^i \cdot \nabla v_y^j)(y) = \delta_{i,j} \qquad \text{for } i,j = 1,\dots,n.$$

Proof. Notice first that being u a good η-splitting map (recall Remark 3.2.4), for every $i,j = 1,\dots,n$,

$$|M_{i,j} - \delta_{ij}| \le \eta < n^{-1} \qquad \text{on } D. \tag{3.2.10}$$

Now fix for the moment $y \in D$. Let $\lambda \in \mathbb{R}$ be an eigenvalue of $M(y)$ and let $w = (w^1,\dots,w^n)$ be a corresponding eigenvector, assume that $|w| = 1$. Then

$$\lambda = w^T M(x) w = w^T (M(y) - \mathrm{Id}) w + |w|^2$$

so that by (3.2.10) and Holder's inequality,

$$|\lambda - 1| = |w^T(M(y)-\mathrm{Id})w| = \left|\textstyle\sum_{i,j} w^i (M(y)_{i,j} - \delta_{i,j}) w^j\right| \le \eta \left|\textstyle\sum_{i,j} w^i w^j\right| \le \eta |w|^2_{\ell^1} \le \eta n < 1.$$

In particular, $M(y)$ is invertible.

Fix for the moment $y \in D$. We perform a formal Gram–Schmidt orthogonalization algorithm. Namely, we set

$$\tilde v_y^1 := u^1, \qquad v_y^1 := \frac{\tilde v_y^1}{\sqrt{(\tilde\nabla v_y^1 \cdot \nabla \tilde v_y^1)(y)}}$$

and for $i > 1$,

$$\tilde v_y^i := u^i - (\nabla u^i \cdot \nabla v_y^1)(y) v_y^1 - \dots - (\nabla u^i \cdot \nabla v_y^{i-1})(y) v_y^{i-1}, \qquad v_y^i := \frac{\tilde v_y^i}{\sqrt{(\nabla \tilde v_y^i \cdot \nabla \tilde v_y^i)(y)}},$$

notice that the denominators are strictly positive by the invertibility of $M(y)$. Then we set $A(y)$ to be the natural matrix satisfying $v_y = (v_y^1,\dots,v_y^n) = A(y)u$ according to the procedure above, namely

$$A(y)_{i,j} = \begin{cases} 0 & \text{if } i < j, \\ \frac{1}{\sqrt{(\nabla \tilde v_y^i \cdot, \nabla \tilde v_y^i)(y)}} & \text{if } i = j, \\ -\frac{(\nabla u^i \cdot \nabla v_y^j)(y)}{\sqrt{(\nabla \tilde v_y^i \cdot, \nabla \tilde v_y^i)(y)}} & \text{if } i > j. \end{cases}$$

Now we compute

$$(A(y)M(y)A(y)^T)_{i,j} = \textstyle\sum_{k,l} A(y)_{i,k} A(y)_{j,l} (\nabla u^k \cdot \nabla u^l)(y) = (\nabla v_y^i \cdot \nabla v_y^j)(y) = \delta_{i,j},$$

where the last equality is due to the Gram–Schmidt-type construction. Let now $w \in \mathbb{R}^n$. We compute

$$|A(y)^T w|^2 = w^T A(y)A(y)^T w = w^T A(y)M(y)A(y)^T w + w^T A(y)(\mathrm{Id} - M(y))A(y)^T w \\ \leq |w|^2 + (\eta n)|A(y)^T w|^2$$

where we used that the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the symmetric matrix $M(y)$ satisfy $|\lambda_i - 1| \leq \eta n < 1$. Therefore we can bound $|A(y)^T|$ by a constant that depends only on η and n. □

Now we need to introduce a notion of unit normal, in the sense of calculus, to a set of finite perimeter. We need this object in order to keep track of the set of finite perimeter in a blow-up procedure. We only state the result, revealing in advance a particular version of Theorem 4.1.1 of Chapter 4 (actually, the bulk of the proof of Theorem 4.1.1 is the same as the one of Theorem 3.2.15).

Theorem 3.2.15 (Gauss–Green formula)**.** *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space and let* $E \subseteq \mathsf{X}$ *be a set of finite perimeter and finite measure. Then there exists a unique vector field* $\nu_E \in L^\infty_{|\mathrm{D}\chi_E|}(T\mathsf{X})$ *such that it holds*

$$\int_E \operatorname{div} v \mathrm{d}\mathsf{m} = -\int v \cdot \nu_E \, \mathrm{d}|\mathrm{D}\chi_E| \qquad \textit{for every } v \in H^{1,2}_{\mathrm{C}}(T\mathsf{X}) \cap D(\operatorname{div}) \cap L^\infty(T\mathsf{X}).$$

Moreover, $|\nu_E| = 1$ $|\mathrm{D}\chi_E|$*-a.e.*

Now we want to isolate a condition for which, given a harmonic Lipschitz vector valued map, this map is suitable to recognize the blow-up of a particular set of finite perimeter at a point.

Definition 3.2.16 (Good coordinates)**.** Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be an $\mathrm{RCD}(K, N)$ space of essential dimension n. Let $E \subseteq \mathsf{X}$ be a set of finite perimeter and let $x \in \operatorname{supp} |\mathrm{D}\chi_E|$ be given. Then we say that an n-tuple of harmonic Lipschitz functions

$$u = (u^1, \ldots, u^n) : B_r(x) \to \mathbb{R}^n$$

is a *system of good coordinates* for E at x provided the following properties are satisfied.

i) For any $i, j = 1, \ldots, n$, it holds that

$$\lim_{s \searrow 0} \fint_{B_s(x)} |\nabla u^i \cdot \nabla u^j - \delta_{ij}| \mathrm{d}\mathsf{m} = \lim_{s \searrow 0} \fint_{B_s(x)} |\nabla u^i \cdot \nabla u^j - \delta_{ij}| \mathrm{d}|\mathrm{D}\chi_E| = 0. \tag{3.2.11}$$

ii) For any $i = 1, \ldots, n$, there exists $\bar{\nu}^u_i(x) \in \mathbb{R}$ such that

$$\lim_{s \searrow 0} \fint_{B_s(x)} |\nu_E \cdot \nabla u^i - \bar{\nu}^u_i(x)| \mathrm{d}|\mathrm{D}\chi_E| = 0.$$

iii) The resulting vector $\bar{\nu}^u(x) := (\bar{\nu}^u_1(x), \ldots, \bar{\nu}^u_n(x)) \in \mathbb{R}^n$ satisfies $|\bar{\nu}^u(x)| = 1$.

Remark 3.2.17. Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be an $\mathrm{RCD}(K, N)$ space of essential dimension n, let $x \in \mathsf{X}$ and let $u = (u^1, \ldots, u^n)$ be an n-tuple of harmonic Lipschitz functions satisfying for every $i, j = 1, \ldots, n$ (cf. (3.2.11))

$$\lim_{r \searrow 0} \fint_{B_r(x)} |\nabla u^i \cdot \nabla u^j - \delta_{ij}| \mathrm{d}\mathsf{m} = 0.$$

Given a sequence of radii $r_i \searrow 0$ such that

$$(\mathsf{X}, r_i^{-1}\mathsf{d}, \mathsf{m}_x^{r_i}, x) \to (\mathbb{R}^n, \mathsf{d}_e, \underline{\mathcal{L}}^n, 0)$$

and fixed a realization of such convergence, it follows from the results of [24, 25] (recalled in [51, Section 1.2.3]) and (2.2.4) that, up to extracting a not relabelled subsequence, the functions in

$$\{u_i^j := r_i^{-1}(u^j - u^j(x))\}_i \qquad \text{for } j = 1, \dots, n$$

converge locally uniformly and locally in $H^{1,2}$-strong on $B_R(0)$ (for every $R > 0$) to orthogonal coordinate functions of $\mathbb{R}^n$. ■

The fact that systems of good coordinates have components that are asymptotically orthonormal grants that, for sufficiently small scales, these maps are splitting maps.

Lemma 3.2.18. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space of essential dimension* n*. Let* $u : B_r(x) \to \mathbb{R}^n$ *a* n*-tuple of harmonic Lipschitz functions satisfying for every* $i, j = 1, \dots, n$ *(cf.* (3.2.11)*)*

$$\lim_{s \searrow 0} \fint_{B_s(x)} |\nabla u^i \cdot \nabla u^j - \delta_{i,j}| \mathrm{d}\mathsf{m} = 0.$$

Then, for every $\delta \in (0, 1)$*, there exists* $r_\delta \in (0, 1)$ *such that* u *is a* $(1 + \delta)$*-Lipschitz* δ*-splitting map on* $B_s(x)$ *for every* $s \in (0, r_\delta)$*.*

Proof. As in Remark 3.2.13, we have that for every i,

$$\lim_{r \searrow 0} r^2 \fint_{B_r(x)} |\mathrm{Hess}\, u^i|^2 \mathrm{d}\mathsf{m} = 0.$$

Finally, considering the scale-invariant version of [48, Remark 3.3] we have that for every $\delta \in (0, 1)$, there exists $r_\delta \in (0, 1)$ such that u is $(1 + C_N \delta^{1/2})$-Lipschitz on $B_{r_\delta}(x)$. □

Now we decompose an RCD space as a countable union of Borel sets $\{D_k\}_k$, up to a set that is not seen by total variations, in such a way that every D_k is a "good set" of a good splitting map. It is worth noticing that this decomposition depends only on the RCD space and not on a particular function of bounded variation.

Lemma 3.2.19. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space of essential dimension* n *and let* $\eta \in (0, n^{-1})$*. Then there exists a sequence of maps* $\{u_k : B_{2r_k}(x_k) \to \mathbb{R}^n\}_k$ *and a sequence of pairwise disjoint Borel sets* $\{D_k \subseteq B_{r_k}(x_k)\}_k$ *such that*

i) for every $f \in \mathrm{BV}(\mathsf{X})$*,*

$$|\mathrm{D}f| \left(\mathsf{X} \setminus \bigcup_k D_k \right) = 0, \tag{3.2.12}$$

ii) for every k*,* u_k *is a good* η*-splitting map on* D_k*.*

To any such collection of η-splitting maps, we can therefore associate a natural map

$$\bigcup_k D_k \to \mathbb{N} \qquad x \mapsto k(x).$$

Proof. The proof follows the arguments given in the proof of [51, Theorem 3.2], but the outcome is stronger, thanks to a suitable improvement of the proof.

Fix a countable dense set $S \subseteq \mathcal{R}_n^*(\mathsf{X})$. Let $y \in S$ be given. If $\varepsilon > 0$ is small enough and $r \in (0, \sqrt{\varepsilon/|K|}) \cap \mathbb{Q}$ is such that

$$\mathsf{d}_{\mathrm{pmGH}}\big((\mathsf{X}, r^{-1}\mathsf{d}, \mathsf{m}_y^r, y), (\mathbb{R}^n, \mathsf{d}_e, \underline{\mathcal{L}}^n, 0)\big) < \varepsilon,$$

then, by Proposition 3.2.8 we obtain a δ-splitting map $u_{y,r} : B_{5r}(y) \to \mathbb{R}^n$ for some δ (which can be made arbitrarily small, taking ε small enough). Let

$$D_{y,r} := \big\{x \in B_{5/2r}(y) \big| u_{y,r} \text{ is an } \eta\text{-splitting map on } B_s(x) \text{ for every } s \in (0, 5/4r)\big\}.$$

The claim of the lemma will be proved with the sequence of sets $\{D_{y,r}\}_{y,r}$ and maps $\{u_{y,r}\}_{y,r}$, after having made the sets $D_{y,r}$ disjoint.

Assume now, by contradiction, that the claim is false. Then, by coarea, we find a set of finite perimeter $E \subseteq \mathsf{X}$ such that

$$|\mathrm{D}\chi_E|\left(\mathsf{X} \setminus \bigcup_{y,r} D_{y,r}\right) > 0.$$

In particular, by Proposition 2.3.3 and Theorem 3.1.1,

$$|\mathrm{D}\chi_E|\left(B \setminus \bigcup_{y,r} D_{y,r}\right) > 0, \tag{3.2.13}$$

where (notice $B \subseteq \mathcal{R}_n^*(\mathsf{X}) \cap \partial^* E$), for some j that will be assumed fixed,

$$B := \left\{x \in \mathcal{R}_n^*(\mathsf{X}) \cap \partial^* E : \frac{r|\mathrm{D}\chi_E|(B_r(x))}{\mathsf{m}(B_r(x))} > j^{-1} \text{ for every } r \in (0, j^{-1})\right\}.$$

Fix $\varepsilon > 0$ to be determined later. If $x \in B$, then there exists $r = r(x) \in \mathbb{Q} \cap (0,1)$ such that $|K|r^2 < \varepsilon < 4$ and

$$\mathsf{d}_{\mathrm{pmGH}}\big((\mathsf{X}, r^{-1}\mathsf{d}, \mathsf{m}_x^r, x), (\mathbb{R}^n, \mathsf{d}_e, \underline{\mathcal{L}}^n, 0)\big) < \varepsilon \qquad \text{and} \qquad \frac{r|\mathrm{D}\chi_E|(B_{r/4}(x))}{\mathsf{m}(B_{r/4}(x))} > 4j^{-1}.$$

By density of S and thanks to an easy continuity argument, we deduce that for some point $y = y(x) \in S \cap B_{r/2}(x)$,

$$\mathsf{d}_{\mathrm{pmGH}}\big((\mathsf{X}, r^{-1}\mathsf{d}, \mathsf{m}_y^r, y), (\mathbb{R}^n, \mathsf{d}_e, \underline{\mathcal{L}}^n, 0)\big) < \varepsilon, \qquad \text{and} \qquad \frac{r|\mathrm{D}\chi_E|(B_{r/4}(y))}{\mathsf{m}(B_{r/4}(y))} > 4j^{-1}. \tag{3.2.14}$$

By the discussion above, we obtain a δ-splitting map $u_{y,r} : B_{5r}(y) \to \mathbb{R}^n$ for some $\delta = \delta(\varepsilon)$ (which can be made arbitrarily small, taking ε small enough). By Proposition 3.2.11, $u_{y,r}$ is a $C_N\delta^{1/4}$-splitting map on $B_s(x)$ for any $x \in D_{y,r}^\varepsilon \subseteq B_{5/2r}(y)$ and $s \in (0, 5/4r)$, where

$$\mathcal{H}_5^h(B_{5/2r}(y) \setminus D_{y,r}^\varepsilon) \le C_N\delta^{1/2}\frac{\mathsf{m}(B_{5/2r}(x))}{5/2r}.$$

Therefore, $D_{y,r}^\varepsilon \subseteq D_{y,r}$ if $C_N\delta^{1/4} < \eta$.

We apply Vitali covering lemma to the family $\{B_{r(x)/2}(y(x))\}_{x\in B}$ constructed arguing as above and we obtain a sequence of disjoint balls $\{B_{r(x_i)/2}(y(x_i))\}_i$ such that

$$B \subseteq \bigcup_i B_{5/2r(x_i)}(y(x_i)).$$

Set

$$D^\varepsilon := \bigcup_i D^\varepsilon_{y(x_i),r(x_i)}.$$

Then we can compute

$$\begin{aligned}\mathcal{H}^h_5(B\setminus D^\varepsilon) &\le \sum_{i\in\mathbb{N}} \mathcal{H}^h_5\left(B_{5/2r(x_i)}(y(x_i))\setminus D^\varepsilon_{y(x_i),r(x_i)}\right) \le C_N\delta^{1/2}\sum_{i\in\mathbb{N}}\frac{\mathsf{m}(B_{5/2r(x_i)}(y(x_i)))}{5/2r(x_i)}\\ &\le C_N\delta^{1/2}\sum_{i\in\mathbb{N}}\frac{\mathsf{m}(B_{r(x_i)/4}(y(x_i)))}{r(x_i)/4} \le C_N j\delta^{1/2}|\mathrm{D}\chi_E|(\mathsf{X}),\end{aligned}$$

where the constants C_N may change from line to line, in the third inequality we are using the doubling property together with the fact that $r(x_i)$ is sufficiently small, and in the last inequality we are using (3.2.14) together with the fact that $\{B_{r(x_i)/2}(y(x_i))\}$ are disjoint. Let now $\{\varepsilon_i\}_i$ with $\varepsilon_i \searrow 0$ be such that the corresponding $\{\delta_i\}_i$ satisfy both $\delta_i^{1/2}\le 2^{-i}$ and $C_N j\delta_i^{1/4}<\eta$, and set

$$G := \bigcup_i D^{\varepsilon_i} \subseteq D_{y,r}.$$

Then $\mathcal{H}^h_5(B\setminus G)=0$, which contradicts (3.2.13). □

In view of the following proposition, recall Definition 3.2.16, in particular the definition of $\bar\nu$. We are going to prove that, given a good splitting map, we can rotate it in order to have a system of good coordinates.

Proposition 3.2.20. *Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,N)$ space of essential dimension n and let $E\subseteq\mathsf{X}$ be a set of finite perimeter. Let $\eta\in(0,n^{-1})$ and let $u:2B\to\mathbb{R}^n$ be a good η-splitting map on $D\subseteq B$. Let $A:D\to\mathbb{R}^{n\times n}$ be the matrix valued Borel map given by Lemma 3.2.14 and set, for $y\in D$, $v_y:=A(y)u$. Then, for $|\mathrm{D}\chi_E|$-a.e. $y\in D$, the map v_y is a system of good coordinates for E at y and*

$$\bar\nu_E^{v_y}(y) = A(y)\big((\nu_E\cdot\nabla u^i)(y)\big)_i\in\mathbb{R}^n \qquad \text{for } |\mathrm{D}\chi_E|\text{-a.e. } x\in D. \tag{3.2.15}$$

In particular, for $|\mathrm{D}\chi_E|$-a.e. y, there exists a system of good coordinates for E at y.

Proof. Up to removing a $|\mathrm{D}\chi_E|$-negligible set, we can assume that every point of D is a Lebesgue point of $\nabla u^i\cdot\nabla u^j$, $\nu_E\cdot\nabla u^i$, with respect to $|\mathrm{D}\chi_E|$ for every $i,j=1,\dots,n$. This is due to the asymptotically doubling property of $|\mathrm{D}\chi_E|$. Also, up to removing another $|\mathrm{D}\chi_E|$-negligible set, we can assume that at every point of D, the Lebesgue value for $\nabla u^i\cdot\nabla u^j$ with respect to m coincides with the Lebesgue value for (the trace of the quasi-continuous representative of) $\nabla u^i\cdot\nabla u^j$ with respect to $|\mathrm{D}\chi_E|$, for every $i,j=1,\dots,n$. This is due to Remark 3.2.5 together with the fact that $|\mathrm{D}\chi_E|$ is asymptotically doubling. Notice that this last property implies, in particular, that (taking the traces on $L^2_{|\mathrm{D}\chi_E|}(T\mathsf{X})$)

$$A\{\nabla u^i\cdot\nabla u^j\}_{i,j}A^T = \mathrm{Id} \qquad |\mathrm{D}\chi_E|\text{-a.e. on } D, \tag{3.2.16}$$

where we also used Lemma 3.2.14.

By the paragraph above, for every $y \in D$, the map v_y satisfies (3.2.11). We compute, exploiting the fact that every $y \in D$ is a Lebesgue point for $\nu_E \cdot \nabla u^i$,

$$\begin{aligned}\limsup_{r\searrow 0} \fint_{B_r(y)} &\left|\textstyle\sum_k A(y)_{i,k}(\nu_E \cdot \nabla u^k)(z) - \sum_k A(y)_{i,k}(\nu_E \cdot \nabla u^k)(y)\right| \mathrm{d}|\mathrm{D}\chi_E|(z)\\ &\le \limsup_{r\searrow 0} C_{\eta,n} \textstyle\sum_k \fint_{B_r(y)} |(\nu_E \cdot \nabla u^k)(z) - (\nu_E \cdot \nabla u^k)(y)| \mathrm{d}|\mathrm{D}\chi_E|(z) = 0,\end{aligned}$$

where the inequality is due to Lemma 3.2.14. Hence (3.2.15) is satisfied.

It remains to prove only that $|\nu_E^{v_y}(y)| = 1$ for $|\mathrm{D}\chi_E|$-a.e. $y \in D$, notice that $y \mapsto \nu_E^{v_y}(y) \in \mathbb{R}^n$ is $|\mathrm{D}\chi_E|$-measurable thanks to (3.2.15). First, by (3.2.16),

$$\left(\textstyle\sum_k A_{i,k} \nabla u^k\right) \cdot \left(\textstyle\sum_k A_{j,k} \nabla u^k\right) = \delta_{i,j} \qquad |\mathrm{D}\chi_E|\text{-a.e. on } D. \tag{3.2.17}$$

Now, as $L^2_{|\mathrm{D}\chi_E|}(T\mathsf{X})$ has dimension at most n (Theorem 2.2.21) we deduce that the n vector fields $\{\sum_k A(y)_{i,k} \nabla u^k\}_{i=1,\dots,n}$ form an orthonormal basis for $L^2_{|\mathrm{D}\chi_E|}(T\mathsf{X})$ on D. Therefore,

$$\nu_E = \sum_i \Big(\left(\textstyle\sum_k \nu_E \cdot A_{i,k} \nabla u^k\right) \left(\textstyle\sum_k A_{i,k} \nabla u^k\right) \Big) \qquad \text{on } D,$$

as element of $L^2_{|\mathrm{D}\chi_E|}(T\mathsf{X})$. Now, as $|\nu_E| = 1$ $|\mathrm{D}\chi_E|$-a.e. and by (3.2.17),

$$\begin{aligned}1 &= \sum_{i,j} \Big(\left(\textstyle\sum_k \nu_E \cdot A_{i,k} \nabla u^k\right) \left(\textstyle\sum_k \nu_E \cdot A_{j,k} \nabla u^k\right) \left(\textstyle\sum_k A_{i,k} \nabla u^k\right) \cdot \left(\textstyle\sum_k A_{j,k} \nabla u^k\right) \Big)\\ &= \sum_i \left(\textstyle\sum_k \nu_E \cdot A_{i,k} \nabla u^k\right)^2\end{aligned}$$

$|\mathrm{D}\chi_E|$-a.e. on D, whence the claim follows from (3.2.15).

The last conclusion is due to Lemma 3.2.19. □

3.2.3 Main result

The following important theorem completely characterizes blow-ups of sets of finite perimeter, and how we can recognize the blow-up, in the limit space, using a system of good coordinates.

Theorem 3.2.21. *Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be an $\mathrm{RCD}(K,N)$ space of essential dimension n and let $E \subseteq \mathsf{X}$ be a set of locally finite perimeter. Let $x \in \operatorname{supp} |\mathrm{D}\chi_E|$ such that the conclusions of Proposition 2.3.3 hold at x. Assume that there exists a system of good coordinates for E at x, $u : B_r(x) \to \mathbb{R}^n$. Then*

$$\mathrm{Tan}_x(\mathsf{X}, \mathsf{d}, \mathsf{m}, E) = \big\{ (\mathbb{R}^n, \mathsf{d}_e, \underline{\mathcal{L}}^n, 0, \{x_n > 0\}) \big\}. \tag{3.2.18}$$

More precisely, if $H \subseteq \mathbb{R}^n$ is a half-space such that

$$(\mathbb{R}^n, \mathsf{d}_e, \underline{\mathcal{L}}^n, 0, H) \in \mathrm{Tan}_x(\mathsf{X}, \mathsf{d}, \mathsf{m}, E),$$

where the coordinates in $\mathbb{R}^n$ are chosen as limits of rescalings of u as in Remark 3.2.17, then

$$H = \{ y : y \cdot \bar{\nu}_E^u(x) \ge 0 \}.$$

In particular, (3.2.18) holds for $|\mathrm{D}\chi_E|$-a.e. $x \in \mathsf{X}$.

Proof. By a standard reduction argument, we can assume that E has finite perimeter and finite measure. Fix $x \in \operatorname{supp}|D\chi_E|$ and fix $u : B_{r_x}(x) \to \mathbb{R}^n$, system of good coordinates for E at x. Notice that if O is a rotation matrix, then $v := Ou : B_{r_x}(x) \to \mathbb{R}^n$ is still a system of good coordinates for E at x with $\bar{\nu}^v_E(x) = O\bar{\nu}^u_E(x)$ (here we are making the dependence on the system of good coordinates explicit). Hence, for **Step 2**, up to changing also the coordinates on $\mathbb{R}^n$, we see that there is no loss of generality in assuming that $\bar{\nu}^u_E(x) = (0, \dots, 0, 1)$. This has the only effect to simplify some computations.

Step 1. By Lemma 3.2.18, for every $\delta \in (0,1)$, there exists $r_\delta \in (0,1)$ such that u is a δ-splitting map on $B_s(x)$ for every $s \in (0, r_\delta)$. Now, take $(\mathsf{X}', \mathsf{d}', \mathsf{m}', x') \in \operatorname{Tan}_x(\mathsf{X}, \mathsf{d}, \mathsf{m})$ (which is not empty, by Gromov compactness Theorem). Now we can apply Proposition 3.2.10 for any $\delta \in (0,1)$. Take indeed $\varepsilon = j^{-1}$ and $\delta = \delta(\varepsilon)$ given by Proposition 3.2.10, then, by the map u, we know that there exists a pointed $\mathrm{RCD}(0, N-n)$ space $(\mathsf{Z}^j, \mathsf{d}^j, \mathsf{m}^j, z^j)$ such that

$$\mathsf{d}_{\mathrm{pmGH}}((\mathsf{X}', \mathsf{d}', \mathsf{m}', x'), (\mathbb{R}^n \times \mathsf{Z}^j, \mathsf{d}_e \otimes \mathsf{d}^j, \mathcal{L}^n \otimes \mathsf{m}^j, (0, z^j))) < j^{-1}.$$

Applying Gromov compactness Theorem, we have that for some pointed $\mathrm{RCD}(0, N-n)$ space $(\mathsf{Z}^\infty, \mathsf{d}^\infty, \mathsf{m}^\infty, z^\infty)$ we have that

$$\mathsf{d}_{\mathrm{pmGH}}((\mathsf{Z}^\infty, \mathsf{d}^\infty, \mathsf{m}^\infty, z^\infty), (\mathsf{Z}^j, \mathsf{d}^j, \mathsf{m}^j, z^j)) \to 0$$

so that we infer

$$\mathsf{d}_{\mathrm{pmGH}}((\mathsf{X}', \mathsf{d}', \mathsf{m}', x'), (\mathbb{R}^n \times \mathsf{Z}^\infty, \mathsf{d}_e \otimes \mathsf{d}^\infty, \mathcal{L}^n \otimes \mathsf{m}^\infty, (0, z^\infty))) = 0.$$

The lower semicontinuity and additivity of the essential dimension forces Z^∞ to be a point, so that $(\mathsf{X}', \mathsf{d}', \mathsf{m}', x')$ must be $\mathbb{R}^n$. This shows that

$$\operatorname{Tan}_x(\mathsf{X}, \mathsf{d}, \mathsf{m}) = \left\{(\mathbb{R}^n, \mathsf{d}_e, \underline{\mathcal{L}}^n, 0)\right\}.$$

Step 2. Now take a sequence $r_j \searrow 0$ such that $(\mathsf{X}, r_j^{-1}\mathsf{d}, \mathsf{m}^x_{r_j}, x) \to (\mathbb{R}^n, \mathsf{d}_e, \underline{\mathcal{L}}^n, 0)$ in the pmGH topology and let $(\mathsf{Z}, \mathsf{d}_\mathsf{Z})$ be a realization of such convergence, with isometric embeddings $\iota_j : (\mathsf{X}, r_j^{-1}\mathsf{d}) \to (\mathsf{Z}, \mathsf{d}_\mathsf{Z})$ and $\iota_j : (\mathbb{R}^n, \mathsf{d}_e) \to (\mathsf{Z}, \mathsf{d}_\mathsf{Z})$. We can assume (possibly taking a not relabelled subsequence, see Remark 3.2.17) that the maps $\{u^i_j := r_j^{-1}(u^i - u^i(x))\}_j$ converge locally uniformly and in $H^{1,2}$-strong on $B_R(0)$ (for every $R > 0$) to coordinate functions on $\mathbb{R}^n$, say $e^1, \dots, e^n$. We will use these coordinates on $\mathbb{R}^n$. We can assume (possibly taking a not relabelled subsequence, see Theorem 3.2.2) that $(\mathsf{X}, r_j^{-1}\mathsf{d}, \mathsf{m}^x_{r_j}, x, E) \to (\mathbb{R}^n, \mathsf{d}_e, \underline{\mathcal{L}}^n, 0, E')$, i.e. $\iota_j(E)$ converge in L^1_{loc} to $\iota_\infty(E')$ and we have weak convergence of rescaled perimeters, see (3.2.1), for some set of locally finite perimeter $E' \subseteq \mathbb{R}^n$. We have to show that $E' = \{y : y \cdot \bar{\nu}^u_E(x) \ge 0\}$, recall that we are assuming

$$\bar{\nu}^u_E(x) = (0, \dots, 0, 1).$$

We prove this by a limiting procedure involving the Gauss–Green formula.

For this part of the proof we are going to follow the arguments of the proofs of [51, Proposition 4.7] and [50, Proposition 4.8], building heavily on the convergence results of [24, 25], recalled also in [51, Section 1.2.3]. As these convergence results are by now standard, we are going to use them freely, referring the reader to the references above for details. Let $\varphi_\infty \in \mathrm{LIP}_{\mathrm{bs}}(\mathbb{R}^n)$. We take $\varphi_j \in \mathrm{LIP}_{\mathrm{bs}}(\mathsf{X}, r_j^{-1}\mathsf{d})$, uniformly Lipschitz, with uniformly bounded support and strongly converging to φ_∞ in $H^{1,2}$ and locally uniformly converging to φ_∞. By Theorem 3.2.15, we have that for $i = 1, \dots, n$, if $\varphi \in \mathrm{LIP_{bs}}(\mathsf{X})$ has support contained in $B_{r_x}(x)$,

$$\int_E \nabla\varphi \cdot \nabla u^i \mathrm{d}\mathsf{m} = -\int \varphi \nabla u^i \cdot \nu_E \mathrm{d}|\mathrm{D}\chi_E|,$$

where we used that as u^i is harmonic on $B_{r_x}(x)$, then $\varphi\nabla u^i \in H^{1,2}_{\mathrm{C}}(T\mathsf{X}) \cap D(\mathrm{div})$ and $\mathrm{div}(\varphi\nabla u^i) = \varphi\nabla u^i$. We can multiply both sides by $\frac{r_j}{C_x^{r_j}}$ and use the scaling properties of the gradients and the measures to infer that

$$\int_{E_j} \nabla^j\varphi_j \cdot \nabla^j u^i_j \mathrm{d}\mathsf{m}^x_{r_j} = -\frac{r_j}{C_x^{r_j}} \int \varphi_j \nabla u^i \cdot \nu_E \mathrm{d}|\mathrm{D}\chi_E|, \tag{3.2.19}$$

where this equality makes sense for j large enough as for j large enough, with a little abuse, $\mathrm{supp}\,\varphi_j \subseteq B_{r_x}(x)$. Concerning the left hand side of (3.2.19), we have the convergence

$$\int_{E_j} \nabla^j\varphi_j \cdot \nabla^j u^i_j \mathrm{d}\mathsf{m}^x_{r_j} \to \int_{E'} \nabla\varphi_\infty \cdot \nabla e^i \mathrm{d}\underline{\mathcal{L}}^n. \tag{3.2.20}$$

Using twice the rescaling property $|\mathrm{D}^j\chi_E| = \frac{r_j}{C_x^{r_j}}|\mathrm{D}\chi_E|$, if R is big enough,

$$\begin{aligned}
&\left|\frac{r_j}{C_x^{r_j}} \int \varphi_j \nabla u^i \cdot \nu_E \mathrm{d}|\mathrm{D}\chi_E| - \delta_{i,n}\int \varphi_j \mathrm{d}|\mathrm{D}^j\chi_E|\right| \\
&\qquad \le \|\varphi_j\|_\infty \frac{r_j}{C_x^{r_j}} \int_{B_{Rr_j}(x)} |\nabla u^i \cdot \nu_E - \delta_{i,n}| \mathrm{d}|\mathrm{D}\chi_E| \\
&\qquad = \|\varphi_j\|_\infty |\mathrm{D}\chi_{E_j}|(B_R(x)) \fint_{B_{Rr_j}(x)} |\nabla u^i \cdot \nu_E - \delta_{i,n}| \mathrm{d}|\mathrm{D}\chi_E| \to 0.
\end{aligned}$$

Therefore, concerning the right hand side of (3.2.19), we have the convergence

$$\frac{r_j}{C_x^{r_j}} \int \varphi_j \nabla u^i \cdot \nu_E \mathrm{d}|\mathrm{D}\chi_E| \to \delta_{i,n} \int \varphi_\infty \mathrm{d}|\mathrm{D}\chi_{E'}|, \tag{3.2.21}$$

where we used again the weak convergence of rescaled perimeters together with the local uniform convergence $\varphi_j \to \varphi_\infty$. All in all by (3.2.19), (3.2.20) and (3.2.21), we obtain that in the space $(\mathbb{R}^n, \mathsf{d}_e, \mathcal{L}^n)$ (we are not rescaling the measure now), we have that for every $\varphi \in \mathrm{LIP_{bs}}(\mathbb{R}^n)$,

$$\int_{E'} \partial_i\varphi \mathrm{d}\mathcal{L}^n = -\delta_{i,n} \int \varphi \mathrm{d}\mathcal{H}^{n-1} \llcorner \partial^* E'. \tag{3.2.22}$$

Now, for any $i = 1, \dots, n-1$, by (3.2.22),

$$\int_{E'} \partial_i\varphi \mathrm{d}\mathcal{L}^n = 0,$$

which easily implies that $\chi_{E'}(y) = \chi_{H'}(y_n)$ for some $H' \subseteq \mathbb{R}$. Taking product test functions, one verifies that (3.2.22) with $i = n$ implies that H' is a set of locally finite perimeter and that for every $\psi \in \mathrm{LIP_{bs}}(\mathbb{R})$,

$$\int_{H'} \psi' \mathrm{d}t = -\int \psi \mathrm{d}\mathcal{H}^0 \llcorner \partial^* H'.$$

This shows that

$$|\mathrm{D}\chi_{H'}| = \mathrm{D}\chi_{H'},$$

so that we have that $H' = \chi_{[a,\infty)}$ for some $a \in \mathbb{R}$. Now we can use the weak convergence of rescaled perimeters together with item *iii)* of Proposition 2.3.3 to infer that $a = 0$, whence the claim.

Step 3. The last conclusion is due to Proposition 3.2.20 and Proposition 2.3.3. □

We briefly comment on **Step 1** of the proof of Theorem 3.2.21. Notice that the conclusion entails a "constancy of the dimension" statement (which is not obtained for free but hidden in the proof of Lemma 3.2.19, which exploits Theorem 3.1.1). Also, if one is only interested on the conclusion of Theorem 3.2.21 at almost every point with respect to the perimeter measure, the argument of **Step 1** is unnecessary and can be replaced directly by Theorem 3.1.1.

3.3 Reduced boundary

In view of the following definition, recall the definition of tangent to a set given in Definition 3.2.1. We want to have a subset of the essential boundary that is big enough to carry all the perimeter measure, but made of points that are regular enough to have nice properties.

Definition 3.3.1 (Reduced boundary)**.** Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be an $\mathrm{RCD}(K, N)$ space of essential dimension n. Let $E \subseteq \mathsf{X}$ be a set of locally finite perimeter. Then we define the *reduced boundary* $\mathcal{F}E \subseteq \partial^* E$ of E as the set of all those points $x \in \mathcal{R}_n^*(\mathsf{X})$ satisfying all the four conclusions of Proposition 2.3.3 and such that

$$\mathrm{Tan}_x(\mathsf{X}, \mathsf{d}, \mathsf{m}, E) = \big\{(\mathbb{R}^n, \mathsf{d}_e, \underline{\mathcal{L}}^n, 0, \{x_n > 0\})\big\}.$$

By Theorem 3.2.21, Theorem 3.1.1 and Proposition 2.3.3 we obtain the following corollary, which confirms that Definition 3.3.1 is meaningful.

Corollary 3.3.2. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space of essential dimension* n *and let* $E \subseteq \mathsf{X}$ *be a set of locally finite perimeter. Then*

$$|\mathrm{D}\chi_E|(\mathsf{X} \setminus \mathcal{F}E) = 0.$$

Remark 3.3.3. For any $x \in \mathcal{F}E$ the following hold (recall the definition of C_x^r in (2.2.10)).

i) If $r_i \searrow 0$ is such that

$$(\mathsf{X}, r_i^{-1}\mathsf{d}, \mathsf{m}_x^{r_i}, x) \to (\mathbb{R}^n, \mathsf{d}_e, \underline{\mathcal{L}}^n, 0) \tag{3.3.1}$$

in a realization $(\mathsf{Z}, \mathsf{d}_\mathsf{Z})$, then, up to not relabelled subsequences and a change of coordinates in $\mathbb{R}^n$,

$$(\mathsf{X}, r_i^{-1}\mathsf{d}, \mathsf{m}_x^{r_i}, x, E) \to (\mathbb{R}, \mathsf{d}_e, \underline{\mathcal{L}}^n, 0, \{x_n > 0\}),$$

in the same realization $(\mathsf{Z}, \mathsf{d}_\mathsf{Z})$. Notice that, given a sequence $r_i \searrow 0$, it is always possible to find a subsequence satisfying (3.3.1) by Gromov compactness Theorem.

ii) We have

$$\begin{aligned}
&\lim_{r\searrow 0} \frac{\mathsf{m}(B_r(x))}{r^n} = \omega_n \Theta_n(\mathsf{m}, x) \in (0,\infty),\\
&\lim_{r\searrow 0} \frac{C_x^r}{r^n} = \frac{\omega_n}{n+1}\Theta_n(\mathsf{m}, x),\\
&\lim_{r\searrow 0} \frac{|\mathrm{D}\chi_E|(B_r(x))}{r^{n-1}} = \omega_{n-1}\Theta_n(\mathsf{m}, x).
\end{aligned}$$

iii) x is a point of density $1/2$ for E, i.e.

$$\lim_{r\searrow 0}\frac{\mathsf{m}(B_r(x)\cap E)}{\mathsf{m}(B_r(x))}=1/2.$$

This follows indeed by Theorem 3.2.2 and by the membership to $\mathcal{R}^*_n(\mathsf{X})$. ■

3.4 Rectifiability

In view of the following theorem, recall that $|\mathrm{D}\chi_E|$ is concentrated on $\mathcal{F}E$, by Corollary 3.3.2, for any set of locally finite perimeter E. We are then going to prove the rectifiability of reduced boundaries of sets of finite perimeter, and give an explicit description of the maps that we can use to have rectifiability: indeed, we are going to use good splitting maps.

Theorem 3.4.1. *Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,N)$ space of essential dimension n and let $E\subseteq\mathsf{X}$ be a set of locally finite perimeter. Then $\mathcal{F}E$ is countably $(n-1)$-rectifiable. More precisely, for every $\varepsilon>0$, we can write*

$$\mathcal{F}E=\bigcup_{i\in\mathbb{N}}B_i\cup N,$$

where, for every i, B_i is $(1+\varepsilon)$-bilipschitz to a Borel subset of $\mathbb{R}^{n-1}$ and $|\mathrm{D}\chi_E|(N)=0$.

Theorem 3.4.1 is an easy consequence of the following lemma, and we postpone its proof after the proof of Lemma 3.4.2. Lemma 3.4.2 is proved by blow-up.

Lemma 3.4.2. *Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,N)$ space of essential dimension n and let $E\subseteq\mathsf{X}$ be a set of locally finite perimeter. Let u be a good splitting map on D and let $A:D\to\mathbb{R}^{n\times n}$ be the matrix valued Borel map given by Lemma 3.2.14. Let ε small enough so that $\varepsilon^2(\hat{C}_{K,N}+1)<\varepsilon$, where $\hat{C}_{K,N}$ is the constant appearing in Definition 3.2.12.*

Let $G\subseteq\mathcal{F}E\cap D$ be a set of points with the following properties:

i) for some $l\in\mathbb{N}$, $l\geq 1$,

$$\frac{r|\mathrm{D}\chi_E|(B_r(y))}{\mathsf{m}(B_r(y))}>l^{-1}\qquad\text{for every } y\in G \text{ and } r\in(0,l^{-1}),$$

ii) for every $y\in G$, the map $v_y:=A(y)u$ is a system of good coordinates for E at y,

iii) there exist a matrix $\bar{A}\in\mathbb{Q}^{n\times n}$ and a a vector $\bar{\nu}\in\mathbb{Q}^n$ such that

$$|A(y)-\bar{A}|<\varepsilon^2\qquad\text{and}\qquad|\bar{\nu}_E^{v_y}(y)-\bar{\nu}|<\varepsilon^2\qquad\text{for every } y\in G.$$

Then there exists a projection map $\bar{\pi}:\mathbb{R}^n\to\mathbb{R}^n$ onto a hyperplane and a sequence of sets $\{G_k\}_k$ with $G_k\subseteq G$ and

$$|\mathrm{D}\chi_E|\left(G\setminus\bigcup_k G_k\right)=0,$$

such that, for every k,

$$\bar{\pi}\bar{A}u:G_k\to\mathbb{R}^n$$

is $(1+2\varepsilon)$-bilipschitz onto its $(n-1)$-dimensional image.

Proof. Define $\bar\pi:\mathbb{R}^n\to\mathbb{R}^n$ to be the projection onto the subspace orthogonal to $\bar\nu$, i.e. $\bar\pi(z)=z-(z\cdot\bar\nu)\bar\nu$ and set $\bar u:=\bar A u$.

Take any $y\in G$. We claim that there exists $r_y\in(0,1]$ satisfying

$$\big||\bar\pi\bar u(y)-\bar\pi\bar u(z)|-\mathsf{d}(y,z)\big|\le 2\varepsilon\mathsf{d}(y,z)\qquad\text{for every } z\in G \text{ such that } \mathsf{d}(y,z)<r_y. \tag{3.4.1}$$

Assume the contrary and take a sequence $\{z_k\}_k\subseteq G$ with $r_k:=\mathsf{d}(y,z_k)\to 0$ satisfying

$$\big||\bar\pi\bar u(y)-\bar\pi\bar u(z_k)|-\mathsf{d}(y,z_k)\big|>2\varepsilon\mathsf{d}(y,z_k).$$

Now we consider the rescaled spaces $(\mathsf{X},r_k^{-1}\mathsf{d},\mathsf{m}_y^{r_k},y)$. Recall that v_y is a system of good coordinates for E at y, hence, up to taking not relabelled subsequences we can exploit the membership $y\in\mathcal{F}E$ (in particular, Theorem 3.2.21) and Remark 3.2.17 to have that $\{v^j_{y,k}:=r_k^{-1}(v_y^j-v_y^j(y))\}_k$ converge locally uniformly to orthogonal coordinates on $\mathbb{R}^n$, say $v^j_{y,\infty}$, for $j=1,\dots,n$, where $(\mathsf{X},r_k^{-1}\mathsf{d},\mathsf{m}_y^{r_k},y,E)\to(\mathbb{R}^n,\mathsf{d}_e,\underline{\mathcal{L}}^n,0,H)$ in the pmGH topology, for the half-space

$$H=\{z:v_{y,\infty}(z)\,\cdot\,\bar\nu_E^{v_y}(y)\ge 0\}.$$

Also, up taking again subsequences, we can assume that $z_k\to z_\infty$ for some $z_\infty\in\mathbb{R}^n$, notice that $\mathsf{d}_e(0,z_\infty)=1=|v_{y,\infty}(0)-v_{y,\infty}(z_\infty)|$. Now we claim that $z_\infty\in\partial H$. Assume not, say $z_\infty\notin\partial H$, then for some $\sigma\in(0,1)$, taking into account weak convergence of measures involved, and that $\{z_k\}_k\subseteq G$,

$$\begin{aligned}0<\underline{\mathcal{L}}^n(B_\sigma^{\mathbb{R}^n}(z_\infty))=\lim_k \mathsf{m}_y^{r_k}(B^k_\sigma(z_k))=\lim_k\frac{\mathsf{m}(B_{\sigma r_k}(z_k))}{C_y^{r_k}}\le\limsup_k l\frac{r_k|\mathrm{D}\chi_E|(B_{\sigma r_k}(z_k))}{C_y^{r_k}}\\ =l\limsup_k|\mathrm{D}\chi_{E_k}|(B^k_\sigma(z_k))=l|\mathrm{D}\chi_H|(B_\sigma^{\mathbb{R}^n}(z_\infty))=0\end{aligned}$$

which is a contradiction. Therefore $z_\infty\in\partial H$ so that, by the description of H,

$$(v_{y,\infty}(0)-v_{y,\infty}(z_\infty))\perp\bar\nu_E^{v_y}(y),$$

and hence

$$\big||\bar\pi v_{y,\infty}(0)-\bar\pi v_{y,\infty}(z_\infty)|-|v_{y,\infty}(0)-v_{y,\infty}(z_\infty)|\big|\le|\bar\nu-\bar\nu_E^{v_y}(y)||v_{y,\infty}(0)-v_{y,\infty}(z_\infty)|<\varepsilon^2.$$

Therefore, by local uniform convergence,

$$\begin{aligned}\lim_k r_k^{-1}\big||\bar\pi v_y(y)-\bar\pi v_y(z_k)|-\mathsf{d}(y,z_k)\big|&=\lim_k\big||\bar\pi v_{y,k}(y)-\bar\pi v_{y,k}(z_k)|-r_k^{-1}\mathsf{d}(y,z_k)\big|\\&=\big||\bar\pi v_{y,\infty}(0)-\bar\pi v_{y,\infty}(z_\infty)|-\mathsf{d}_e(0,z_\infty)\big|<\varepsilon^2.\end{aligned}$$

Hence, if k is big enough,

$$\big||\bar\pi v_y(y)-\bar\pi v_y(z_k)|-\mathsf{d}(y,z_k)\big|<(\varepsilon+\varepsilon^2)r_k.$$

Now we compute, if k is big enough, as ε is small enough,

$$\begin{aligned}\big||\bar\pi\bar u(y)-\bar\pi\bar u(z_k)|-\mathsf{d}(y,z_k)\big|&\le|\bar A-A(y)||u(y)-u(z_k)|+\big||\bar\pi v_y(y)-\bar\pi v_y(z_k)|-\mathsf{d}(y,z_k)\big|\\&\le\varepsilon^2\hat C_{K,N}\mathsf{d}(y,z_k)+(\varepsilon+\varepsilon^2)\mathsf{d}(y,z_k)<2\varepsilon\mathsf{d}(y,z_k),\end{aligned}$$

which is a contradiction. Therefore (3.4.1) is proved.

By an exhaustion argument, we can assume that for every $y \in G_k$, $r_y > m^{-1}$, for some $m \in \mathbb{N}$, where we are taking the maximal $r_y \in [0,1]$, which is not zero by (3.4.1). Notice indeed that, taking the maximal r_y, $y \mapsto r_y$ is upper semicontinuous. By a partitioning argument we assume moreover that the diameter of G_k is smaller than m^{-1}: if $y, z \in G_k$, then

$$\big| |\bar{\pi}\bar{u}(y) - \bar{\pi}\bar{u}(z)| - \mathsf{d}(y,z) \big| \le 2\varepsilon \mathsf{d}(y,z),$$

as $\mathsf{d}(y,z) \le m^{-1} < r_y$. Hence the map $\bar{\pi}\bar{u}$ provides the suitable bilipschitz map. □

Proof of Theorem 3.4.1. Fix $\varepsilon > 0$ small enough (this smallness depending on $\hat{C}_{K,N}$, which appears in Lemma 3.4.2).

Now we use Lemma 3.2.19 for some $\eta \in (0, n^{-1})$ and, keeping the same notation, we concentrate on one fixed k. Dropping the subscript k, we have to prove rectifiability of $|\mathrm{D}\chi_E| \llcorner D$, where there exists a good splitting map $u : 2B \to \mathbb{R}^n$ on $D \subseteq B$, for some ball $B \subseteq \mathsf{X}$. Now we apply Proposition 3.2.20 (the matrix valued Borel map $D \ni y \mapsto A(y) \in \mathbb{R}^{n\times n}$ given by Lemma 3.2.14) and, up to discarding a $|\mathrm{D}\chi_E|$-negligible subset from D, $v_y := A(y)u$ is a system of good coordinates for E at y, for any $y \in D$. Up to a partitioning and exhaustion argument we see that we can assume that assumptions of Lemma 3.4.2 are in place (recall Remark 3.3.3). Hence we can use a further partitioning argument and exploit the maps given by Lemma 3.4.2 (as their image lies in a hyperplane, we can choose coordinates of such hyperplane and assume that these maps take values in $\mathbb{R}^{n-1}$). □

We add now a technical result which will be useful later on. Namely, fixed a BV function f we can cover an RCD space, up to a $|\mathrm{D}f|$-null set, by a countable union of sets that are bi-Borel to Borel subsets of the Euclidean space.

Proposition 3.4.3. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K,N)$ *space of essential dimension* n *and let* $f \in \mathrm{BV}(\mathsf{X})^m$. *Then there exists a countable collection* $\{(G_i, \Psi_i, B_i)\}_i$ *such that*

i) $\{G_i\}_i$ *is a collection of pairwise disjoint Borel subset of* X *satisfying*

$$|\mathrm{D}f|\Big(\mathsf{X} \setminus \bigcup_i G_i\Big) = 0,$$

ii) $\{B_i\}_i$ *is a collection of Borel subsets of* $\mathbb{R}^n$,

iii) for every i,

$$\Psi_i : G_i \to B_i$$

is an invertible Borel map with Borel inverse

$$\Psi_i^{-1} : B_i \to G_i,$$

iv) for every i, Ψ_i *is the restriction of some* $\tilde{\Psi}_i \in \mathrm{BV}(\mathsf{X})^n$ *to* G_i *with* $J_{\tilde{\Psi}_i} \cap G_i = \emptyset$.

Proof. As $|\mathrm{D}f| \le |\mathrm{D}f_1| + \cdots + |\mathrm{D}f_m|$, there is no loss of generality in assuming $m = 1$, so write $f = f_1$.

Split X in the disjoint union $A_f \cup C_f \cup S_f$, according to the decomposition

$$|\mathrm{D}f| = |\mathrm{D}f|^a + |\mathrm{D}f|^c + |\mathrm{D}f|^j$$

in absolutely continuous, Cantor and jump part (Definition 2.3.4). Then the claim on the portion S_f follows from the rectifiability result of Theorem 3.4.1. The claim on the portion A_f follows from the rectifiability of $(\mathsf{X},\mathsf{d},\mathsf{m})$, e.g. [111] or [49]. As the maps providing rectifiability are bilipschitz, an application of McShane extension Theorem shows that item *iv)* can be satisfied.

We treat now the part C_f. First, we define the subgraph of f as in Section 2.3.5, i.e.

$$\mathcal{G}_f := \{(x,t) \in \mathsf{X}\times\mathbb{R} : t < f(x)\}.$$

By Proposition 2.3.12, $\mathcal{G}_f$ is a set of locally finite perimeter and that, if $\pi : \mathsf{X}\times\mathbb{R} \to \mathsf{X}$ denotes the projection onto the first factor, it holds that

$$|\mathrm{D}f| \le \pi_*|\mathrm{D}\chi_{\mathcal{G}_f}|.$$

By the rectifiability result of Theorem 3.4.1 again, there exists a countable collection $\{\hat{C}_i\}_i$ of pairwise disjoint Borel subsets of $\partial^*\mathcal{G}_f \subseteq \mathsf{X}\times\mathbb{R}$ such that

$$|\mathrm{D}\chi_{\mathcal{G}_f}|\Big((\mathsf{X}\times\mathbb{R})\setminus\bigcup_i \hat{C}_i\Big) = 0$$

and for every i there exists a map

$$\Phi_i : \hat{C}_i \to \hat{B}_i \subseteq \mathbb{R}^n$$

which is bilipschitz onto its image. To our aim, there is no loss of generality in assuming that for every i, $\hat{C}_i \subseteq (C_f \cap \{f \in \mathbb{R}\})\times\mathbb{R}$ (we use also (2.3.5)). We set for every i, $C_i := \pi(\hat{C}_i)$ and $\Psi_i := \Phi_i \circ (\mathrm{Id}, f)_{|C_i}$, and it is easy to show that this assignment satisfies the request in item *iii)*, recalling (2.3.6).

We now show item *iv)*, up to removing from C_i the $|\mathrm{D}f|$-negligible subset $J_{\tilde{\Psi}_i}\cap C_i$ (the fact that $J_{\tilde{\Psi}_i}\cap C_i$ is $|\mathrm{D}f|$-negligible follows from (3.4.2) below and the fact that $|\mathrm{D}f|\llcorner(\mathsf{X}\setminus S_f)$ does not charge jump sets of functions of bounded variation). We first use McShane extension Theorem for Φ_i to obtain a L-Lipschitz function $\tilde{\Phi}_i$ and set $\tilde{\Psi}_i := \tilde{\Phi}_i \circ (\mathrm{Id}, f)$. Notice that if $g \in \mathrm{LIP_{loc}}(\mathsf{X})$ it holds that $\mathrm{lip}(\tilde{\Phi}_i \circ (\mathrm{Id}, g)) \le L(\mathrm{lip}(g)+1)$, therefore an approximation argument yields that $\mathrm{D}\tilde{\Psi}_i \in \mathrm{BV_{loc}}(\mathsf{X})^n$ with

$$|\mathrm{D}\tilde{\Psi}_i| \le L(|\mathrm{D}f| + \mathsf{m}), \tag{3.4.2}$$

and then the conclusion follows. □

3.5 Representation formula

Now we want to state formulae which give the representation of the perimeter measure. The first one, (3.5.1), is in terms of the 1-codimensional Hausdorff measure, whereas the second one, (3.5.2), is a purely metric formula (and answers in the affirmative to [115, Conjecture 5.32]). It is worth comparing (3.5.1) to (2.3.8), which holds on any PI space.

Theorem 3.5.1 (Representation formula for the perimeter)**.** *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *be an* $\mathrm{RCD}(K,N)$ *space of essential dimension* n. *Let* $E \subseteq \mathsf{X}$ *be a set of locally finite perimeter. Then,*

$$|\mathrm{D}\chi_E| = \frac{\omega_{n-1}}{\omega_n}\mathcal{H}^h \llcorner \mathcal{F}E \tag{3.5.1}$$

and

$$|\mathrm{D}\chi_E| = \Theta_n(\mathsf{m},\cdot)\mathcal{H}^{n-1}\llcorner\mathcal{F}E. \tag{3.5.2}$$

In particular, it holds that $\Theta_{n-1}(|\mathrm{D}\chi_E|, x) = \Theta_n(\mathsf{m}, x)$ *for* $\mathcal{H}^{n-1}$*-a.e.* $x \in \mathcal{F}E$.

Proof. Formula (3.5.1) is proved in [51, Corollary 3.15], taking into account Corollary 3.3.2.

Now we prove (3.5.2). For any $x \in \mathcal{F}E$ we compute, by Remark 3.3.3,

$$\exists \Theta_{n-1}(|D\chi_E|, x) = \lim_{r \searrow 0} \frac{|D\chi_E|(B_r(x))}{\omega_{n-1} r^{n-1}} = \Theta_n(\mathsf{m}, x) \in (0, \infty). \tag{3.5.3}$$

Notice that $|D\chi_E|$ is concentrated on $\mathcal{F}E$ (Corollary 3.3.2) and that we are heavily exploiting the fact that, by definition, $\mathcal{F}E \subseteq \mathcal{R}_n^*(\mathsf{X})$, which is meaningful thanks to Theorem 3.1.1. we deduce that $\mu_E := \mathcal{H}^{n-1} \llcorner \mathcal{F}E$ is a σ-finite. Now, by [31, Theorem 2.4.3] we have that $\mathcal{H}^{n-1} \llcorner \mathcal{F}E$ is a σ-finite measure and moreover that

$$\mathcal{H}^{n-1} \llcorner \mathcal{F}E \ll |D\chi_E| \llcorner \mathcal{F}E \ll \mathcal{H}^{n-1} \llcorner \mathcal{F}E.$$

Also, from Theorem 3.4.1 (and Remark 3.5.2), we see that we can apply [27, Theorem 5.4] (which is based on [100]) and the computation in (3.5.3) and deduce that

$$|D\chi_E| = \lim_{r \searrow 0} \frac{|D\chi_E|(B_r(x))}{\omega_{n-1} r^{n-1}} \mathcal{H}^{n-1} \llcorner \mathcal{F}E = \Theta_n(\mathsf{m}, x) \mathcal{H}^{n-1} \llcorner \mathcal{F}E,$$

whence the conclusion. □

Remark 3.5.2. Consider an RCD(K, N) space $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ of essential dimension n and a set of locally finite perimeter $E \subseteq \mathsf{X}$. Notice that

$$\mathcal{H}^h(B) = 0 \text{ if and only if } |D\chi_E|(B) = 0 \qquad \text{for every } B \subseteq \partial^* E \text{ Borel} \tag{3.5.4}$$

and that

$$\mathcal{H}^h(B) = 0 \text{ if and only if } \mathcal{H}^{n-1}(B) = 0 \qquad \text{for every } B \subseteq \mathcal{R}_n^*(\mathsf{X}) \text{ Borel.} \tag{3.5.5}$$

Indeed, (3.5.4) follows from [7, Theorem 5.3], whereas (3.5.5) is proved via standard arguments, starting from the claim on

$$B_j := \left\{ x \in B : \frac{\mathsf{m}(B_r(x))}{r^n} \in (j^{-1}, j) \text{ for every } r \in (0, j^{-1}) \right\}$$

and using σ-sub additivity as $j \to \infty$.

In particular, we have

$$\mathcal{H}^h \llcorner \partial^* E \ll |D\chi_E| \ll \mathcal{H}^h \llcorner \mathcal{F}E, \tag{3.5.6}$$

where we also used Corollary 3.3.2. ■

3.6 Bibliographical notes

The theory of sets of finite perimeter in Euclidean spaces is nowadays well understood. The results stated in this chapter, in the smooth framework, are by now classical, see for example the references already recalled in this manuscript, e.g. [18, 77, 76, 79, 91] as well as [64, 65].

The systematic study of fine properties of sets of finite perimeter in RCD spaces was initiated in [10] (for what concerns blow-ups), and then continued in [51, 50] (for what concerns rectifiability and finer results). These papers are the main references for this chapter. However, at the time when [10, 51] were written, the powerful result of [70] was not available. Later on, [70] was used in [50] to sharpen the result of [51]. We are then in a position to leverage the findings from [70] already from the beginning of this note. This not only significantly shortens the proofs but also makes some of them simpler and better suited for the subsequent development of the theory. In particular, this allows us to bypass the technique of splitting via rigidity in the Bakry–Émery inequality of [10] and prove directly that the blow-up of an RCD space, at almost every point with respect to the perimeter measure, is Euclidean. Also, the part of the technique of [51] borrowed from [111] is now unnecessary. The main ideas and techniques used in this section, even after the modifications described above, are the ones contained in [10, 51, 50].

The statement of Theorem 3.1.1 is taken from [32], however, its proof is from [50] (and relies on [112, 70]), and uses Proposition 3.1.2 (proved in [32]), which is a sharpening of [50, Proposition 2.14].

Definition 3.2.1 is taken from [10], up to the fact that here we add the requirement of item iii). i.e. the weak convergence of (3.2.1), however, this will make basically no difference, by Theorem 3.2.2. Theorem 3.2.2 is proved in [10], here we give a statement that, under some aspects, is slightly sharper. See also [10] for what concerns references for the argument used.

Definition 3.2.3 is [51, Definition 3.4], which, in turn, recalls [58]. Proposition 3.2.7, Proposition 3.2.8, Proposition 3.2.9, Proposition 3.2.10, and Proposition 3.2.11 are taken from [51]. Definition 3.2.12 is clearly inspired by the ideas just mentioned and is motivated by [32, Definition 2.28]. Notice that, with respect to the literature, we do not include any Hessian bound (cf. item ii) of Definition 3.2.3) in the definition of good splitting map, see Remark 3.2.13. Notice also that good η-splitting maps for $\eta = 1/(2n)$ are enough for the material of this manuscript. Lemma 3.2.14 is standard and takes its roots from [32, Lemma 2.27 and Definition 2.28].

Theorem 3.2.15 is [51, Theorem 2.4]. We are going to slightly improve this result in Chapter 4, however, such result is needed also in this chapter and hence we have decided to move up its statement.

Definition 3.2.16 is [50, Definition 4.6]. We have added item iii), with an immaterial difference with respect to the reference, thanks to [50, Proposition 3.6].

Lemma 3.2.19 is extracted from [32, Lemma 2.27], which, in turn, builds upon the argument of [51, Theorem 3.2] and uses, as main ingredients, [51, Corollary 3.10 and Corollary 3.12]. One of the contributions of Lemma 3.2.19 is to "throw away" the bad set of small Hausdorff content of Proposition 3.2.11 inside its proof, avoiding the necessity to treat this set afterwards. The main part of Proposition 3.2.20 is that $\nu_E^{v_y}(y)$ has norm 1. This is done in [50, Proposition 3.6].

Theorem 3.2.21 sums up achievements of [10, 51, 50]. Its proof, however, is slightly modified, as we prove that the blow-up of the set of finite perimeter is a half-space without relying on [10]. This, as explained in the introduction, is due to the fact that we can exploit the result of [70] right from the beginning of this manuscript. The major drawback is that, in this note, we take [70] as a black box, and, without doing so, it would have been possible to prove a slightly weaker (in the sense that at $|\mathrm{D}\chi_E|$-a.e. point the tangent is Euclidean, but possibly of not constant dimension) version of Theorem 3.2.21, in a self-contained way. Nevertheless, to obtain the constancy of dimension of the tangents as in Theorem 3.2.21, [70] seems necessary ([50]), and this is the reason why we decided to follow this approach. It is worth spending some lines about the technique of [10]. The authors proved that, for an $\mathrm{RCD}(0, N)$ space $(\mathsf{X}, \mathsf{d}, \mathsf{m})$, if there exists a Lipschitz and bounded function f satisfying equality in (2.3.16) for some $s > 0$, i.e.

$$|\nabla \mathrm{h}_s f| = \mathrm{h}_s |\nabla f| \qquad \mathsf{m}\text{-a.e.} \tag{3.6.1}$$

then the space splits a line, isometrically, and f, read in the split space, depends only on the real variable, in a monotone way. It was indeed shown that the flow trajectories of the vector field $\frac{\nabla \mathrm{h}_s f}{|\nabla \mathrm{h}_s f|}$ are lines and provide a splitting of the space, [81]. It was then shown that, if $E \subseteq \mathsf{X}$ is a set of finite perimeter, then, for $|D\chi_E|$-a.e. $x \in \mathsf{X}$, for any $(\mathsf{X}_\infty, \mathsf{d}_\infty, \mathsf{m}_\infty, x_\infty, E_\infty) \in \mathrm{Tan}_x(\mathsf{X}, \mathsf{d}, \mathsf{m}, E)$, for any $t > 0$, $f := \mathrm{h}_t^{\mathsf{X}_\infty}(\chi_{E_\infty})$ satisfies (3.6.1). Hence, tangents at such points split a line, and an iterated application of Preiss' iterated tangents principle provides us the with existence, almost everywhere with respect to the perimeter measure, of Euclidean tangents. Later on, in [51], this conclusion has been sharpened, but still taking the above discussion as starting point, using techniques of geometric analysis: almost everywhere with respect to the perimeter measure, the tangent to a set of finite perimeter is unique and Euclidean of a certain (possibly not constant) dimension.

Definition 3.3.1 is taken from [10], here we introduce also the requirement of the conclusions of Proposition 2.3.3 (which is equivalent, up to a set which is negligible with respect to the perimeter, by Proposition 2.3.3).

Theorem 3.4.1 was first proved in [51]. We remark (see [32, Appendix A]) that Theorem 3.2.21 is already enough to establish rectifiability, by [37]. However, we give a more direct proof, following [51], as we will also need more information about the maps providing rectifiability. Compared to [51], we give a slightly different proof of the rectifiability statement, as we use Lemma 3.4.2 instead of [51, Proposition 4.7]. The argument used is the same, i.e. blow-up, but here we manage to avoid the introduction of "bad sets" with small Hausdorff content. The precise statement of Lemma 3.4.2 appears here for the first time and simplifies the proof of Theorem 6.1.2, with respect to [32]. Proposition 3.4.3 is taken from [43] and will be used in the proof of Theorem 4.3.6.

For what concerns Theorem 3.5.1, equation (3.5.1) is proved in [10], whereas (3.5.2) is proved in [32] (in [10], for what concerns non-collapsed RCD spaces).

Chapter 4

Distributional differential of BV functions

In this chapter we give a definition of "distributional differential" for vector valued functions of bounded variation on RCD spaces, and we study the properties that this newly defined object satisfies.

4.1 Existence and basic notions

Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,\infty)$ space and let $f\in\mathrm{BV}(\mathsf{X})^m$. Recall that (2.3.9) states that $|\mathrm{D}f|\ll\mathrm{Cap}$, so that we have, by Theorem 2.2.22, the module $L^p_{|\mathrm{D}f|}(T\mathsf{X})$. In view of the following theorem, recall also that the interpretation of the integral in (4.1.1) is given by Remark 2.3.6. The proof of Theorem 4.1.1 relies on the strategy developed in [51] in the case in which f is a simple function, and on suitable arguments to extend the result to the general case.

Theorem 4.1.1. *Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,\infty)$ space and let $f=(f_1,\dots,f_m)\in\mathrm{BV}(\mathsf{X})^m$. Then there exists a unique vector field $\nu_f\in L^\infty_{|\mathrm{D}f|}(T^m\mathsf{X})$ such that it holds*

$$\sum_{i=1}^m\int f_i\,\mathrm{div}\,v_i=-\int v\cdot\nu_f\,\mathrm{d}|\mathrm{D}f|\qquad\text{for every } v=(v_1,\dots,v_m)\in(\mathcal{QC}^\infty(T\mathsf{X})\cap D(\mathrm{div}))^m. \tag{4.1.1}$$

Moreover, $|\nu_f|=1$ $|\mathrm{D}f|$-a.e.

Proof. We divide the proof in several steps.

Step 1. We show that if $f\in\mathrm{BV}(\mathsf{X})\cap L^\infty(\mathsf{m})$, then there exists a unique $\nu_f\in L^\infty_{|\mathrm{D}f|}(T\mathsf{X})$ such that

$$\int f\,\mathrm{div}\,v=-\int v\cdot\nu_f\,\mathrm{d}|\mathrm{D}f|\qquad\text{for every } v\in H^{1,2}_{\mathrm{C}}(T\mathsf{X})\cap D(\mathrm{div})\cap L^\infty(T\mathsf{X}), \tag{4.1.2}$$

and moreover $|\nu_f|=1$ $|\mathrm{D}f|$-a.e. This can be proved following *verbatim* the proof of [51, Theorem 2.2]. Notice that in [51] the assumption that the dimension was finite could have been dropped taking into account Theorem 2.3.7 and Proposition 2.3.17.

Step 2. Under the same assumptions of **Step 1**, we show that (4.1.2) holds for every $v\in\mathcal{QC}^\infty(T\mathsf{X})\cap D(\mathrm{div})$. Fix then $v\in\mathcal{QC}^\infty(T\mathsf{X})\cap D(\mathrm{div})$. By an easy cut-off argument, there is no loss of generality in assuming that v has bounded support. We take a sequence $t_k\searrow 0$, then we define $v_k := \psi e^{Kt_k} \mathrm{h}_{\mathrm{H},t_k} v$, where $\psi \in \mathrm{LIP_{bs}}(\mathsf{X})$ (say $\operatorname{supp}\psi \Subset B$ for some ball B) is a cut-off function that is identically 1 on a neighbourhood of $\operatorname{supp} v$. Now notice that the equality in (4.1.2) holds for v_k and that, as $\operatorname{div} v_k \to \operatorname{div} v$ in $L^2(\mathsf{m})$, it suffices to check that $v_k \to v$ in $L^0_{\mathrm{Cap}}(T\mathsf{X})$, then dominated convergence together with the theory developed in [69] implies the conclusion.

We thus have to show that $v_k \to v$ in in $L^0_{\mathrm{Cap}}(T\mathsf{X})$. We can assume with no loss of generality that $|v| \le 1$ m-a.e. so that also $|v_k| \le 1$ m-a.e. for every k. As $v \in \mathcal{QC}(T\mathsf{X})$, we can take a sequence $\{w_l\}_l \subseteq \mathrm{TestV}(\mathsf{X})$ such that $w_l \to v$ in $L^0_{\mathrm{Cap}}(T\mathsf{X})$, $\operatorname{supp} w_l \Subset B$ and $|w_l| \le 1$ m-a.e.

We claim that

$$w_{l,k} := \psi e^{Kt_k} \mathrm{h}_{\mathrm{H},t_k} w_l \to \psi e^{Kt_k} \mathrm{h}_{\mathrm{H},t_k} v = v_k \text{ uniformly in } k \qquad \text{in } L^0_{\mathrm{Cap}}(T\mathsf{X}) \text{ as } l \to \infty. \tag{4.1.3}$$

Fix for the moment $\varepsilon > 0$. As $w_l \to v$ in $L^0_{\mathrm{Cap}}(T\mathsf{X})$ and the fact that all these vector fields have uniformly bounded support, we can take functions $\{g_l\}_l \subseteq H^{1,2}(\mathsf{X})$ such that $\|g_l\|_{H^{1,2}(\mathsf{X})} \to 0$, $g_l(x) \in [0,1]$ for m-a.e. x and $\{|w_l - v| > \varepsilon\}$ is contained in the interior of $\{g_l \ge 1\}$. Therefore, taking into account that

$$\begin{aligned}|w_{l,k} - v_k| &\le \chi_B \left(e^{Kt_k} |\mathrm{h}_{\mathrm{H},t_k} \left((w_l - v)\chi_{\{|w_l - v| > \varepsilon\}}\right)| + e^{Kt_k} |\mathrm{h}_{\mathrm{H},t_k} \left((w_l - v)\chi_{\{|w_l - v| \le \varepsilon\}}\right)|\right) \\ &\le \chi_B \left(\mathrm{h}_{t_k}(|(w_l - v)\chi_{\{|w_l - v| > \varepsilon\}}|) + \mathrm{h}_{t_k}(|(w_l - v)\chi_{\{|w_l - v| \le \varepsilon\}}|)\right) \le 2\chi_B \mathrm{h}_{t_k} g_l + 2\varepsilon\end{aligned}$$

and that

$$\|\mathrm{h}_{t_k} g_l\|_{H^{1,2}(\mathsf{X})} \le \|g_l\|_{H^{1,2}(\mathsf{X})} \to 0 \text{ uniformly in } k \qquad \text{as } l \to \infty,$$

it follows that, uniformly in k,

$$\limsup_l \mathrm{Cap}\left\{|w_{l,k} - v^k| > 4\varepsilon\right\} \le \limsup_l \mathrm{Cap}\left(B \cap \{|\mathrm{h}_{t_k} g_l| > \varepsilon\}\right) = 0$$

and hence (4.1.3) follows.

Now we can conclude easily, noticing that

$$\mathsf{d}_{L^0_{\mathrm{Cap}}(T\mathsf{X})}(v_k, v) \le \mathsf{d}_{L^0_{\mathrm{Cap}}(T\mathsf{X})}(v_k, w_{l,k}) + \mathsf{d}_{L^0_{\mathrm{Cap}}(T\mathsf{X})}(w_{l,k}, w_l) + \mathsf{d}_{L^0_{\mathrm{Cap}}(T\mathsf{X})}(w_l, v)$$

as we can first take l large enough to estimate the first and last summand (uniformly in k) and then let $k \to \infty$, recalling that as $w_{l,k} \to w_l$ in $H^{1,2}_{\mathrm{H}}(T\mathsf{X})$, $w_{l,k} \to w_l$ in $L^0_{\mathrm{Cap}}(T\mathsf{X})$.

Step 3. We drop the $L^\infty(\mathsf{m})$ bound assumption on f made in **Step 1**. For $k \in \mathbb{Z}$, define

$$f_k := (f \vee k) \wedge (k+1) - (k + \chi_{\{k<0\}}(k)).$$

Notice that for every $k \in \mathbb{Z}$, $|\mathrm{D} f_k| \le |\mathrm{D} f|$ and, by **Step 1**, there exists $\nu_{f_k} \in L^2_{|\mathrm{D} f_k|}(T\mathsf{X})$ with $|\nu_{f_k}| = 1$ $|\mathrm{D} f_k|$-a.e. and such that

$$\int f_k \operatorname{div} v \, \mathrm{d}\mathsf{m} = -\int v \cdot \nu_{f_k} \, \mathrm{d}|\mathrm{D} f_k| \qquad \text{for every } v \in \mathcal{QC}^\infty(T\mathsf{X}) \cap D(\operatorname{div}). \tag{4.1.4}$$

If we consider ν_{f_k} as an element of $L^0_{\mathrm{Cap}}(T\mathsf{X})$, we have that $\bar{\pi}_f(\nu_{f_k}) \frac{\mathrm{d}|\mathrm{D} f_k|}{\mathrm{d}|\mathrm{D} f|}$ is well defined. Now notice that by coarea it holds $|\mathrm{D} f| = \sum_{k \in \mathbb{Z}} |\mathrm{D} f_k|$. Then, as

$$\left\| \bar{\pi}_f(\nu_{f_k}) \frac{\mathrm{d}|\mathrm{D} f_k|}{\mathrm{d}|\mathrm{D} f|} \right\|_{L^2_{|\mathrm{D} f|}(T\mathsf{X})} \le |\mathrm{D} f_k|(\mathsf{X}),$$

and by the completeness of $L^2_{|\mathrm{D}f|}(T\mathsf{X})$, we see that

$$\nu_f := \sum_{k\in\mathbb{Z}} \bar{\pi}_f(\nu_{f_k}) \frac{\mathrm{d}|\mathrm{D}f_k|}{\mathrm{d}|\mathrm{D}f|}$$

is a well defined element of $L^2_{|\mathrm{D}f|}(T\mathsf{X})$ that satisfies $|\nu_f| \le 1$ $|\mathrm{D}f|$-a.e.

Let now $v \in \mathcal{QC}^\infty(T\mathsf{X}) \cap D(\mathrm{div})$. We prove the integration by parts formula (4.1.2) for f and v. Using (4.1.4) we have,

$$\lim_l \int \sum_{k=-l}^{l-1} f_k \mathrm{div}\, v \,\mathrm{d}\mathsf{m} = -\lim_l \int v \,\cdot\, \sum_{k=-l}^{l-1} \bar{\pi}_f(\nu_{f_k}) \frac{\mathrm{d}|\mathrm{D}f_k|}{\mathrm{d}|\mathrm{D}f|}\,\mathrm{d}|\mathrm{D}f| = -\int v \,\cdot\, \nu_f \,\mathrm{d}|\mathrm{D}f|\,.$$

Now we can show that $|\nu_f| \ge 1$ $|\mathrm{D}f|$-a.e. arguing as in the proof of [51, Theorem 2.2]. Finally, uniqueness of ν_f follows from Lemma 2.2.23. All in all, we have proved the theorem in the case $m = 1$.

Step 4. We treat the case $m > 1$. Notice that $|\mathrm{D}f|$ is a finite measure such that

$$|\mathrm{D}f| \le |\mathrm{D}f_1| + \cdots + |\mathrm{D}f_m| \ll \mathrm{Cap}.$$

Also, taking into account that $|\mathrm{D}f_i| \le |\mathrm{D}f|$, we can write, thanks to Radon–Nikodym Theorem,

$$|\mathrm{D}f_i| = g_i|\mathrm{D}f| \qquad \text{for every } i = 1, \dots, m,$$

where $g_i \in L^\infty(|\mathrm{D}f|)$. Then for every $i = 1, \dots, m$ we can set

$$\nu_i := g_i \bar{\pi}_{|\mathrm{D}f|}(\nu_{f_i}) \in L^2_{|\mathrm{D}f|}(T\mathsf{X}),$$

where we are considering ν_{f_i} as a element of $L^0_{\mathrm{Cap}}(T\mathsf{X})$. Notice that as ν_{f_i} is well defined $|\mathrm{D}f_i|$-a.e. ν_i is well defined and that $|\nu_i| = |g_i|$ $|\mathrm{D}f|$-a.e. We set $\nu_f := (\nu_1, \dots, \nu_m) \in \mathrm{L}^2_{|\mathrm{D}f|}(T^m\mathsf{X})$ and clearly (4.1.1) is satisfied.

As uniqueness of such ν_f follows from uniqueness in the unidimensional case, to conclude it remains only to show that

$$|\nu_f| = 1 \qquad |\mathrm{D}f|\text{-a.e.}$$

By Lemma 2.2.23, take a sequence $\{w^k = (w_1^k, \dots, w_m^k)\} \subseteq \mathrm{TestV}(\mathsf{X})^m$ such that $w_k \to \chi_{\{|\nu_f|>0\}} \frac{\nu_f}{|\nu_f|}$ in $\mathrm{L}^2_{|\mathrm{D}f|}(T^m\mathsf{X})$. Then define $\{v^k\}_k \subseteq \mathrm{TestV}(\mathsf{X})^m$ as

$$v^k := \frac{1}{1 \vee |w^k|} w^k.$$

Notice that still $v^k \to \chi_{\{|\nu_f|>0\}} \frac{\nu_f}{|\nu_f|}$ and moreover $|v^k| \le 1$ m-a.e. Let $A \subseteq \mathsf{X}$ open and take a sequence $\{\psi_k\}_k \subseteq \mathrm{LIP_{bs}}(\mathsf{X})$ such that $\psi_k(x) \in [0,1]$ for every $x \in \mathsf{X}$, $\mathrm{supp}\,\psi_k \subseteq A$ and $\psi_k(x) \nearrow 1$ for every $x \in A$. By Proposition 2.3.18 and (4.1.1) we can compute

$$|\mathrm{D}F|(A) \ge -\sum_{i=1}^m \int f_i \mathrm{div}(\psi_k v_i^k)\,\mathrm{d}\mathsf{m} = \int \psi_k v^k \,\cdot\, \nu_f \,\mathrm{d}|\mathrm{D}f| \to \int_A |\nu_F| \,\mathrm{d}|\mathrm{D}f|\,,$$

and this shows that $|\nu_f| \le 1$ $|\mathrm{D}f|$-a.e. Now notice that Proposition 2.3.18 and (4.1.1) imply that

$$\int |\nu_f| \,\mathrm{d}|\mathrm{D}f| \ge 1$$

so that we conclude. □

It is clear that the map $\mathrm{BV}(\mathsf{X})^m \ni f \mapsto \nu_f$ is local, in the sense that if A is an open set and $f, g \in \mathrm{BV}(\mathsf{X})^m$ are such that $f = g$ m-a.e. on A, then $|\mathrm{D}f| = |\mathrm{D}g|$ on A and moreover $\nu_f = \nu_g$ $|\mathrm{D}f|$-a.e. on A. We can therefore extend the definition of $\nu_\cdot$ to $\mathrm{BV}_{\mathrm{loc}}(\mathsf{X})^m$ and it easily follows that (4.1.1) holds even for $f \in \mathrm{BV}_{\mathrm{loc}}(\mathsf{X})^m$ as soon as we consider vector fields v with compact support. In particular, we can associate to every set of locally finite perimeter E the vector field $\nu_E := \nu_{\chi_E}$. Notice that $\nu_{-f} = -\nu_f$ and that $\nu_{\mathsf{X}\setminus E} = -\nu_E$, again by (4.1.1),

Of course, the i-th element of the polar vector of $(f_1, \dots, f_m)$ is linked to the polar vector of f_i. This is the content of the following trivial remark.

Remark 4.1.2. Let $f = (f_1, \dots, f_m) \in \mathrm{BV}_{\mathrm{loc}}(\mathsf{X})^m$. Then,

$$(\nu_f)_i = \frac{\mathrm{d}|\mathrm{D}f_i|}{\mathrm{d}|\mathrm{D}f|}\nu_{f_i} \qquad |\mathrm{D}f|\text{-a.e. for every } i = 1, \dots, m, \tag{4.1.5}$$

which is an immediate consequence of (4.1.1) (or of the construction of ν_f). ■

We are not going to use the following remark in the sequel, but we believe that it is interesting as it provides us with an optimal density result.

Remark 4.1.3. We show that if $F \in \mathrm{BV}(\mathsf{X})^m$, then there exists a sequence

$$\{v^k = (v_1^k, \dots, v_m^k)\}_k \subseteq \mathcal{W}^m$$

where

$$\mathcal{W}^m := \left\{ v = (v_1, \dots v_m) \in H^{1,2}_{\mathrm{H}}(T\mathsf{X})^m : |v| \le 1 \ \mathsf{m}\text{-a.e.} \ \mathrm{div}\, v_i \in L^\infty(\mathsf{m}) \text{ for every } i = 1, \dots, m \right\}$$

such that $v^k \to \nu_f$ in $L^2_{|\mathrm{D}f|}(T^m\mathsf{X})$.

Indeed, we can modify the proof of Proposition 2.3.18 (see in particular its **Step 3**), replacing $\mathrm{TestV}(\mathsf{X})^m$ with $\mathcal{W}^m$ in (2.3.18) and then it is enough to take a sequence $\{v^k = (v_1^k, \dots, v_m^k)\}_k \subseteq \mathcal{W}^m$ such that (with the usual interpretation for the integral)

$$\sum_{i=1}^m \int f_i \mathrm{div}\, v_i^k \, \mathrm{d}\mathsf{m} \to -|\mathrm{D}f|(\mathsf{X})$$

and compute

$$\begin{aligned} \int |v^k - \nu_f|^2 \, \mathrm{d}|\mathrm{D}f| &= \int |v^k|^2 \, \mathrm{d}|\mathrm{D}f| + \int |\nu_f|^2 \, \mathrm{d}|\mathrm{D}f| - 2 \int v^k \cdot \nu_F \, \mathrm{d}|\mathrm{D}f| \\ &\le 2|\mathrm{D}f|(\mathsf{X}) + 2 \sum_{i=1}^m \int f_i \mathrm{div}\, v_i^k \, \mathrm{d}\mathsf{m} \to 0, \end{aligned}$$

where in the last inequality we used (4.1.1). Notice that this argument works only to approximate ν_f, and the difficulty in approximating other elements of $L^2_{|\mathrm{D}f|}(T^m\mathsf{X})$ lies in the request of essentially bounded divergence coupled with the request $|v| \le 1$ m-a.e. ■

4.1.1 Formal interpretation

In this manuscript, we will need to do some algebraic manipulation of the abstract object $\nu_f|\mathrm{D}f|$, especially when dealing with fine properties and calculus rules for functions of bounded variation. We introduce now a formal framework to denote the distributional differential of functions of bounded variation that will allow us to proceed.

Definition 4.1.4. Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,\infty)$ space. Let $\nu \in L^\infty_{\mathrm{Cap}}(T^m\mathsf{X})$ and let μ be a finite measure with $\mu \ll \mathrm{Cap}$. We write $\nu\mu$ to denote the linear operator that acts on $L^\infty_{\mathrm{Cap}}(T^m\mathsf{X})$ as follows:

$$\nu\mu(B)(v) := \int_B v \,\cdot\, \nu \,\mathrm{d}\mu \qquad \text{for every } v \in L^\infty_{\mathrm{Cap}}(T^m\mathsf{X}) \text{ and } B \subseteq \mathsf{X} \text{ Borel.}$$

We also consider the restriction operator: if $C \subseteq \mathsf{X}$ is Borel,

$$((\nu\mu)\llcorner C)(B)(v) := \int_{B\cap C} v \,\cdot\, \nu \,\mathrm{d}\mu \qquad \text{for every } v \in L^\infty_{\mathrm{Cap}}(T^m\mathsf{X}) \text{ and } B \subseteq \mathsf{X} \text{ Borel,}$$

in other words, $(\nu\mu)\llcorner C = \nu(\mu\llcorner C)$, so that we will not write the unnecessary brackets.

Given $\nu_1\mu_1$ and $\nu_2\mu_2$, we write $\nu_1\mu_1 = \nu_2\mu_2$ if and only if

$$\nu_1\mu_2(\mathsf{X})(v) = \nu_1\mu_2(\mathsf{X})(v) \qquad \text{for every } v \in L^\infty_{\mathrm{Cap}}(T^m\mathsf{X}). \tag{4.1.6}$$

Remark 4.1.5. Notice that the expression $\nu\mu$ makes sense even if the vector field ν is defined only μ-a.e. as well as $\mu\nu\llcorner C$ makes sense even if C is only μ-measurable. We will exploit this fact throughout.

Also, we have that (4.1.6) holds if and only if

$$\nu_1\mu_1(\mathsf{X})(v) = \nu_2\mu_2(\mathsf{X})(v) \qquad \text{for every } v \in \mathrm{TestV}(\mathsf{X})^m.$$

This is a standard consequence of Lemma 2.2.23. If this is the case, then $\mu := |\nu_1|\mu_1 = |\nu_2|\mu_2$ and $\frac{\nu_1}{|\mu_1|} = \frac{\nu_2}{|\mu_2|}$ μ-a.e. ■

We define now some formal algebraic operations for objects of the kind $\nu\mu$.

Definition 4.1.6. Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,\infty)$ space.

i) Let $\nu \in L^\infty_{\mathrm{Cap}}(T^m\mathsf{X})$ and let μ be a finite measure with $\mu \ll \mathrm{Cap}$. Let moreover $\varphi : \mathsf{X} \to \mathbb{R}^{l\times m}$ be a μ-measurable function. We define $\varphi(\nu\mu) := (\varphi\nu)\mu$, where

$$(\varphi\nu)_j := \sum_{i=1}^m \varphi_{j,i}\nu_i \qquad j = 1,\dots,l.$$

Notice that

$$(\varphi\nu\mu)(\mathsf{X})(v) = \int \sum_{j=1}^m \sum_{i=1}^n v_j \,\cdot\, \varphi_{j,i}\nu_i \,\mathrm{d}\mu$$

for every $v = (v_1,\dots,v_m) \in L^\infty_{\mathrm{Cap}}(T^m\mathsf{X})$.

ii) Let $\nu_1,\nu_2 \in L^\infty_{\mathrm{Cap}}(T^m\mathsf{X})$ and let μ_1,μ_2 be two finite measures with $\mu_1 \ll \mathrm{Cap}, \mu_2 \ll \mathrm{Cap}$. We define $\nu_1\mu_1 + \nu_2\mu_2$ as sum of linear operators, in the sense that

$$(\nu_1\mu_1 + \nu_2\mu_2)(\mathsf{X})(v) = \int v \,\cdot\, \nu_1 \,\mathrm{d}\mu_1 + \int v \,\cdot\, \nu_2 \,\mathrm{d}\mu_2 \,.$$

Notice that, if we define $\mu := \mu_1 + \mu_2$, then

$$\nu_1\mu_1 + \nu_2\mu_2 = \left(\nu_1\frac{\mathrm{d}\mu_1}{\mathrm{d}\mu} + \nu_2\frac{\mathrm{d}\mu_2}{\mathrm{d}\mu}\right)\mu.$$

We then specialize this language to distributional differentials of functions of bounded variation, and read the characterization of the distributional differential in these terms.

Definition 4.1.7. Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,\infty)$ space and $f \in \mathrm{BV}(\mathsf{X})^m$. We define

$$\mathrm{D}f := \nu_f |\mathrm{D}f|,$$

according to Definition 4.1.4 and we call this object the *distributional differential* of f.

Remark 4.1.8. Let $f \in \mathrm{BV}(\mathsf{X})^m$. Then (4.1.1) reads

$$\sum_{i=1}^m \int f_i \mathrm{div} v_i = -\mathrm{D}f(\mathsf{X})(v) \qquad \text{for every } v=(v_1,\dots,v_m) \in (\mathcal{QC}^\infty(T\mathsf{X}) \cap D(\mathrm{div}))^m.$$

Also, if $\mu\nu$ is as in Definition 4.1.4 and satisfies

$$\sum_{i=1}^m \int f_i \mathrm{div} v_i = -\mu\nu(\mathsf{X})(v) \qquad \text{for every } v=(v_1,\dots,v_f) \in \mathrm{TestV}(\mathsf{X})^m,$$

then $\mathrm{D}F = \nu\mu$.

Finally, it is clear that the map $\mathrm{BV}(\mathsf{X})^m \ni f \mapsto \mathrm{D}f$ is linear. ∎

4.2 Fine properties

Having at our disposal an object that represents the distributional differential of a function of bounded variation as well as a language to deal with such objects, we start investigating some properties. The following lemma, despite being simple, is of crucial importance for the development of the theory and will be used several times. It relates the normal of the superlevel sets of a BV function with the polar vector of the function, according to the principle that for smooth maps reads as "the gradient of the function is normal to the level sets".

Lemma 4.2.1. *Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,\infty)$ space and let $f \in \mathrm{BV}(\mathsf{X})$. Then*

$$\nu_{\{f>t\}} = \bar{\pi}_{|\mathrm{D}\chi_{\{f>t\}}|}(\nu_f) \in L^2_{|\mathrm{D}\chi_{\{f>t\}}|}(T\mathsf{X}) \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in \mathbb{R},$$

where we took a Cap*-representative of ν_f.*

Proof. Notice first that the coarea formula implies that the claim of this lemma is well posed and that $\left|\bar{\pi}_{|\mathrm{D}\chi_{\{f>t\}}|}(\nu_f)\right| = 1$ $\left|\mathrm{D}\chi_{\{f>t\}}\right|$-a.e. for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$. By Cavalieri's integration formula and using twice (4.1.1),

$$\begin{aligned}\int v \cdot \nu_f \,\mathrm{d}|\mathrm{D}f| &= \int_0^{+\infty} \int v \cdot \nu_{\{f>t\}} \,\mathrm{d}\left|\mathrm{D}\chi_{\{f>t\}}\right| \mathrm{d}t - \int_{-\infty}^0 \int v \cdot \nu_{\{f<t\}} \,\mathrm{d}\left|\mathrm{D}\chi_{\{f<t\}}\right| \mathrm{d}t \\ &= \int_{\mathbb{R}} \int v \cdot \nu_{\{f>t\}} \,\mathrm{d}\left|\mathrm{D}\chi_{\{f>t\}}\right| \mathrm{d}t\end{aligned} \tag{4.2.1}$$

for every $v \in \mathrm{TestV}(\mathsf{X})$ with bounded support.

By Lemma 2.2.23 and a cut-off argument, we have a sequence $\{v_k\}_k \subseteq \mathrm{TestV}(\mathsf{X})$ with bounded support such that $v_k \to \nu_f$ in $L^2_{|\mathrm{D}f|}(T\mathsf{X})$, so that, using the coarea formula in (2.3.3),

$$\left| \int_{\mathbb{R}} \int \nu_f \cdot \nu_{\{f>t\}} \,\mathrm{d}\big|\mathrm{D}\chi_{\{f>t\}}\big| \,\mathrm{d}t - \int_{\mathbb{R}} \int v_k \cdot \nu_{\{f>t\}} \,\mathrm{d}\big|\mathrm{D}\chi_{\{f>t\}}\big| \,\mathrm{d}t \right| \\ \leq \int_{\mathbb{R}} \int |\nu_f - v_k| \,\mathrm{d}\big|\mathrm{D}\chi_{\{f>t\}}\big| \,\mathrm{d}t = \int |\nu_f - v_k| \,\mathrm{d}|\mathrm{D}f| \to 0,$$

where we implicitly took the projection of ν_f on $L^2_{|\mathrm{D}\chi_{\{f>t\}}|}(T\mathsf{X})$. We can then compute, using the coarea formula and (4.2.1),

$$\int_{\mathbb{R}} \int |\nu_{\{f>t\}} - \nu_f|^2 \,\mathrm{d}\big|\mathrm{D}\chi_{\{f>t\}}\big| \,\mathrm{d}t = 2|\mathrm{D}f|(\mathsf{X}) - 2\int_{\mathbb{R}} \int \nu_f \cdot \nu_{\{f>t\}} \,\mathrm{d}\big|\mathrm{D}\chi_{\{f>t\}}\big| \,\mathrm{d}t \\ = 2|\mathrm{D}f|(\mathsf{X}) - 2\lim_k \int_{\mathbb{R}} \int v_k \cdot \nu_{\{f>t\}} \,\mathrm{d}\big|\mathrm{D}\chi_{\{f>t\}}\big| \,\mathrm{d}t = 2|\mathrm{D}f|(\mathsf{X}) - 2\lim_k \int v_k \cdot \nu_f \,\mathrm{d}|\mathrm{D}f| = 0,$$

which yields the conclusion, by the coarea formula again. □

The next lemma is a rigidity property of the triangle inequality for total variations.

Lemma 4.2.2. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, \infty)$ *space and* $f, f_1, f_2 \in \mathrm{BV}(\mathsf{X})$ *such that* $f = f_1 + f_2$ *and* $|\mathrm{D}f| = |\mathrm{D}f_1| + |\mathrm{D}f_2|$. *Then*

$$\nu_f = \nu_{f_i} \qquad |\mathrm{D}f_i|\text{-a.e. for } i = 1, 2,$$
$$\nu_{f_1} = \nu_{f_2} \qquad |\mathrm{D}f_1| \wedge |\mathrm{D}f_2|\text{-a.e.}$$

Proof. By Lemma 2.2.23, we have a sequence $\{v_k\}_k \subseteq \mathrm{TestV}(\mathsf{X})$ such that $v_k \to \nu_f$ in $L^2_{|\mathrm{D}f|}(T\mathsf{X})$. In particular,

$$|\mathrm{D}f|(\mathsf{X}) = \lim_k \int v_k \cdot \nu_f \,\mathrm{d}|\mathrm{D}f|\,.$$

Thanks to the hypothesis $|\mathrm{D}f| = |\mathrm{D}f_1| + |\mathrm{D}f_2|$,

$$\|v_k - \nu_{f_1}\|^2_{L^2_{|\mathrm{D}f_1|}(T\mathsf{X})} + \|v_k - \nu_{f_2}\|^2_{L^2_{|\mathrm{D}f_2|}(T\mathsf{X})} \\ = \|\nu_{f_1}\|^2_{L^2_{|\mathrm{D}f_1|}(T\mathsf{X})} + \|\nu_{f_2}\|^2_{L^2_{|\mathrm{D}f_2|}(T\mathsf{X})} + \|v_k\|^2_{L^2_{|\mathrm{D}f|}(T\mathsf{X})} \\ - 2\int v_k \cdot \nu_{f_1} \,\mathrm{d}|\mathrm{D}f_1| - 2\int v_k \cdot \nu_{f_2} \,\mathrm{d}|\mathrm{D}f_2| \\ = |\mathrm{D}f|(\mathsf{X}) + \|v_k\|^2_{L^2_{|\mathrm{D}f|}(T\mathsf{X})} - 2\int v_k \cdot \nu_f \,\mathrm{d}|\mathrm{D}f|$$

where in the last equality we used also the linearity of the map $f \mapsto \mathrm{D}f$. It follows

$$\lim_k \left(\|v_k - \nu_{f_1}\|^2_{L^2_{|\mathrm{D}f_1|}(T\mathsf{X})} + \|v_k - \nu_{f_2}\|^2_{L^2_{|\mathrm{D}f_2|}(T\mathsf{X})} \right) = 0.$$

We conclude as we have proved $v_k \to \nu_f$ in $L^2_{|\mathrm{D}f|}(T\mathsf{X})$ and $v_k \to \nu_{f_i}$ in $L^2_{|\mathrm{D}f_i|}(T\mathsf{X})$ for $i = 1, 2$. □

Now, we begin by investigating the fine properties (and calculus rules for) BV functions. We have our results only in the setting of *finite dimensional* RCD spaces. The reason is that we need the interplay between total variations and 1-codimensional spherical Hausdorff measure $\mathcal{H}^h$ given by (2.3.7), which is available only in the finite dimensional setting.

Now we state a transversality lemma, which is a consequence of Lemma 4.2.2 in the case of χ_E, χ_F, for $E \subseteq F$. The adjective "transversality" is due to the following reason: it states that it two hypersurfaces meet at many points, then they have the same normal vector at the intersection.

Lemma 4.2.3. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space and let* $E, F \subseteq \mathsf{X}$ *be two sets of locally finite perimeter. Then*

$$\nu_E = \pm\nu_F \qquad \mathcal{H}^h\text{-a.e. in } \partial^* E \cap \partial^* F. \tag{4.2.2}$$

More precisely, there exist two sets $N^+, N^- \subseteq \partial^* E \cap \partial^* F$ *with* $\mathcal{H}^h((\partial^* E \cap \partial^* F) \setminus (N^+ \cup N^-)) = 0$ *such that the following holds: for every* $x \in N^+$ *(resp.* N^-*) the set* $E \Delta F$ *has density* 0 *(resp.* 1*) at* x *and* (4.2.2) *holds* $\mathcal{H}^h$*-a.e. in* N^+ *(resp.* N^-*) with the* $+$ *(resp.* $-$*) sign.*

Notice that (3.5.1) implies that (4.2.2) is well defined.

Proof. By a standard reduction, we can treat only the case in which E and F have finite perimeter and finite measure.

First assume $E \subseteq F$. Notice that coarea shows that $f_1 = \chi_E$ and $f_2 = \chi_F$ satisfy the assumptions of Lemma 4.2.2. Indeed,

$$\begin{aligned} |\mathrm{D}(\chi_E + \chi_F)|(\mathsf{X}) &= \int |\mathrm{D}\chi_{\{\chi_E + \chi_F > t\}}|(\mathsf{X}) \mathrm{d}t = \int_0^1 |\mathrm{D}\chi_F|(\mathsf{X}) \mathrm{d}t + \int_1^2 |\mathrm{D}\chi_E|(\mathsf{X}) \mathrm{d}t \\ &= |\mathrm{D}\chi_E|(\mathsf{X}) + |\mathrm{D}\chi_F)|(\mathsf{X}), \end{aligned}$$

where we used that $E \subseteq F$. Therefore, by Lemma 4.2.2, recalling also (3.5.1),

$$\nu_E = \nu_F \qquad \mathcal{H}^h\text{-a.e. in } \partial^* E \cap \partial^* F.$$

Assume now $E \cap F = \emptyset$. Using the same arguments as above (with $f_1 = \chi_E$ and $f_2 = -\chi_F$, notice that $\nu_{-\chi_F} = -\nu_F$),

$$\nu_E = -\nu_F \qquad \mathcal{H}^h\text{-a.e. in } \partial^* E \cap \partial^* F.$$

Thanks to Remark 3.3.3 (recall (3.5.6)), we may consider only the set of points at which the sets E and F have density $1/2$ and the sets $E \setminus F$, $F \setminus E$ and $E \Delta F$ have density in $\{0, 1/2, 1\}$. We easily show that $E \Delta F$ cannot have density $1/2$ at such points. If $E \Delta F$ has density 0 at x, then $E \cap F$ has density $1/2$ at x. We can use the first case treated above to compare first ν_E with $\nu_{E \cap F}$ and then ν_F with with $\nu_{E \cap F}$. If instead $E \Delta F$ has density 1 at x, both $E \setminus F$ and $F \setminus E$ have density $1/2$ at x. We can use the first case treated above to compare first ν_E with $\nu_{E \setminus F}$, then ν_F with $\nu_{F \setminus E}$ and conclude comparing $\nu_{E \setminus F}$ with $\nu_{F \setminus E}$, using the second case treated above. □

In the following proposition, given a set of locally finite perimeter E, we denote by E^1 (resp. E^0) the set of interior (resp. exterior) points in the sense of geometric measure theory, namely the set of density 1 (resp. 0) points of E with respect to m. Notice that $E^1 = \{\chi_E^\wedge = 1\}$ and $E^0 = \{\chi_E^\vee = 0\}$. Also, if F is another set of locally finite perimeter, we denote by

$$\{\nu_E = \pm\nu_F\} := \{x \in \partial^* E \cap \partial^* F : |\nu_E \mp \nu_F|(x) = 0\}.$$

Notice that (3.5.1) implies that the sets above are well defined. The next result is a consequence of Lemma 4.2.3 and gives a finer description of the behaviour of the unit normals to two sets of locally finite perimeter.

Proposition 4.2.4. *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *be an* $\mathrm{RCD}(K,N)$ *space and let* $E,F\subseteq\mathsf{X}$ *be two sets of locally finite perimeter. Then, up to* $\mathcal{H}^h$*-negligible sets,*

$$\{\nu_E=\nu_F\}\cup\{\nu_E=-\nu_F\}=\partial^*E\cap\partial^*F,$$
$$\{\nu_E=\nu_F\}=\partial^*E\cap\partial^*F\cap(E\Delta F)^0,$$
$$\{\nu_E=-\nu_F\}=\partial^*E\cap\partial^*F\cap(E\Delta F)^1.$$

Moreover,

$$\begin{aligned}
\mathrm{D}\chi_{E\cap F}&=\mathrm{D}\chi_E\llcorner(F^1\cup(\{\nu_E=\nu_F\}))+\mathrm{D}\chi_F\llcorner E^1=\mathrm{D}\chi_E\llcorner F^1+\mathrm{D}\chi_F\llcorner(E^1\cup\{\nu_E=\nu_F\}),\\
\mathrm{D}\chi_{E\cup F}&=\mathrm{D}\chi_E\llcorner(F^0\cup\{\nu_E=\nu_F\})+\mathrm{D}\chi_F\llcorner E^0=\mathrm{D}\chi_E\llcorner F^0+\mathrm{D}\chi_F\llcorner(E^0\cup\{\nu_E=\nu_F\}),\\
\mathrm{D}\chi_{E\setminus F}&=\mathrm{D}\chi_E\llcorner(F^0\cup\{\nu_E=-\nu_F\})-\mathrm{D}\chi_F\llcorner E^1=-\mathrm{D}\chi_E\llcorner F^1+\mathrm{D}\chi_F\llcorner(E^0\cup\{\nu_E=-\nu_F\}),
\end{aligned}$$

and

$$\begin{aligned}
|\mathrm{D}\chi_{E\cap F}|&=|\mathrm{D}\chi_E|\llcorner(F^1\cup\{\nu_E=\nu_F\})+|\mathrm{D}\chi_F|\llcorner E^1=|\mathrm{D}\chi_E|\llcorner F^1+|\mathrm{D}\chi_F|\llcorner(E^1\cup\{\nu_E=\nu_F\}),\\
|\mathrm{D}\chi_{E\cup F}|&=|\mathrm{D}\chi_E|\llcorner(F^0\cup\{\nu_E=\nu_F\})+|\mathrm{D}\chi_F|\llcorner E^0=|\mathrm{D}\chi_E|\llcorner F^0+|\mathrm{D}\chi_F|\llcorner(E^0\cup\{\nu_E=\nu_F\}),\\
|\mathrm{D}\chi_{E\setminus F}|&=|\mathrm{D}\chi_E|\llcorner(F^0\cup\{\nu_E=-\nu_F\})-|\mathrm{D}\chi_F|\llcorner E^1\\
&=-|\mathrm{D}\chi_E|\llcorner F^1+|\mathrm{D}\chi_F|\llcorner(E^0\cup\{\nu_E=-\nu_F\}).
\end{aligned}$$

Proof. The first three equalities are a restatement of Lemma 4.2.3.

Thanks to Remark 3.3.3, we can prove the other equalities on the set of points at which the densities of the sets of locally finite perimeter E, F, $E\cap F$, $E\cup F$, $E\setminus F$ are in $\{0,1/2,1\}$. Also, recall that (3.5.6) holds for all the sets involved.

We first prove

$$\mathrm{D}\chi_{E\cap F}=\mathrm{D}\chi_E\llcorner(F^1\cup\{\nu_E=\nu_F\})+\mathrm{D}\chi_F\llcorner E^1. \tag{4.2.3}$$

Notice that $(E\cap F)^1\subseteq E^1\cap F^1$, hence both sides of (4.2.3) vanish on $(E\cap F)^1$. Also, up to $\mathcal{H}^h$-negligible sets, $(E\cap F)^0\subseteq E^0\cup F^0\cup\{\nu_E=-\nu_F\}$, by Lemma 4.2.3, hence both sides of (4.2.3) vanish on $(E\cap F)^0$. Now,

$$\partial^*(E\cap F)=(\partial^*E\cap F^1)\cup(\partial^*F\cap E^1)\cap\big(\partial^*E\cap\partial^*F\cap\partial^*(E\cap F)\big),$$

up to $\mathcal{H}^h$-negligible sets. We can conclude that (4.2.3) is satisfied on $\partial^*(E\cap F)$, as by Lemma 4.2.3, $\nu_{E\cap F}=\nu_E$ $\mathcal{H}^h$-a.e. on $\partial^*(E\cap F)\cap\partial^*E$, $\nu_{E\cap F}=\nu_F$ $\mathcal{H}^h$-a.e. on $\partial^*(E\cap F)\cap\partial^*F$ and finally $(\partial^*E\cap\partial^*F)\cap\partial^*(E\cap F)=\{\nu_E=\nu_F\}$, up to $\mathcal{H}^h$-negligible sets.

Now, through algebraic manipulations, using (4.2.3),

$$\begin{aligned}
\mathrm{D}\chi_{E\cup F}&=\mathrm{D}\chi_{\mathsf{X}\setminus((\mathsf{X}\setminus E)\cap(\mathsf{X}\setminus F))}=-\mathrm{D}\chi_{(\mathsf{X}\setminus E)\cap(\mathsf{X}\setminus F)}\\
&=-\mathrm{D}\chi_{\mathsf{X}\setminus E}\llcorner((\mathsf{X}\setminus F)^1\cup\{\nu_{\mathsf{X}\setminus E}=\nu_{\mathsf{X}\setminus F}\})-\mathrm{D}\chi_{\mathsf{X}\setminus F}\llcorner(\mathsf{X}\setminus E)^1\\
&=\mathrm{D}\chi_E\llcorner(F^0\cup\{\nu_E=\nu_F\})+\mathrm{D}\chi_F\llcorner E^0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathrm{D}\chi_{E\setminus F}=\mathrm{D}\chi_{E\cap(\mathsf{X}\setminus F)}&=\mathrm{D}\chi_E\llcorner((\mathsf{X}\setminus F)^1\cup\{\nu_E=\nu_{\mathsf{X}\setminus F}\})+\mathrm{D}\chi_{\mathsf{X}\setminus F}\llcorner E^1\\
&=\mathrm{D}\chi_E\llcorner(F^0\cup\{\nu_E=-\nu_F\})-\mathrm{D}\chi_F\llcorner E^1.
\end{aligned}$$

Finally, the remaining equalities in the second set of equalities are obtained by symmetry. Then, the equalities in the third set of equalities follow from Remark 4.1.5, as all the summands considered have pairwise disjoint support. □

We state a technical result which will be extremely useful in approximation arguments. Its proof is based on the Gaussian estimates of the heat kernel and a blow-up argument at jump points.

Lemma 4.2.5. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space and let* $f \in \mathrm{BV}(\mathsf{X}) \cap L^\infty(\mathsf{m})$*. Then*

$$\lim_{s \searrow 0} \mathrm{h}_s f(x) = \bar{f}(x) \qquad \mathcal{H}^h\text{-a.e.}$$

Proof. In the sequel, let C denote a numerical constant depending only on the parameters entering into play (it may vary during the proof). We let n denote the essential dimension of the space. Thanks to (2.3.5), we can restrict ourselves to prove the claim for $x \in \mathsf{X}$ such that $-\infty < f^\wedge(x) \le f^\vee(x) < +\infty$.

Step 1: case $x \in \mathsf{X} \setminus S_f$. We can compute

$$\limsup_{s \searrow 0} \left|\mathrm{h}_{s^2} f(x) - \bar{f}(x)\right| \le \limsup_{s \searrow 0} \int p_{s^2}(x, y)\left|f(y) - \bar{f}(x)\right| \mathrm{d}\mathsf{m}(y)\,.$$

Fix now $a > 1$. Using the Gaussian estimate for the heat kernel in (2.2.7)

$$\int_{B_{as}(x)} p_{s^2}(x, y)\left|f(y) - \bar{f}(x)\right| \mathrm{d}\mathsf{m}(y) \le \frac{C}{\mathsf{m}(B_s(x))} \int_{B_{as}(x)} \left|f(y) - \bar{f}(x)\right| \mathrm{d}\mathsf{m}(y)$$

and the right hand side converges to 0, because of the doubling inequality and the fact that points in $\mathsf{X} \setminus S_f$ are Lebesgue points for f. Also, as $f \in L^\infty(\mathsf{m})$,

$$\int_{\mathsf{X} \setminus B_{as}(x)} p_{s^2}(x, y)\left|f(y) - \bar{f}(x)\right| \mathrm{d}\mathsf{m}(y) \le C \int_{\mathsf{X} \setminus B_{as}(x)} p_{s^2}(x, y)\, \mathrm{d}\mathsf{m}(y)\,.$$

Using again the Gaussian estimates for the heat kernel in (2.2.7),

$$\begin{aligned} p_{s^2}(x, y) &\le \frac{C}{\mathsf{m}(B_s(x))} \exp\left\{-\frac{\mathsf{d}(x, y)^2}{6s^2}\right\} \\ &\le C H(x, y) \frac{1}{\mathsf{m}(B_{2s}(x))} \exp\left\{-\frac{\mathsf{d}(x, y)^2}{8s^2}\right\} \le C H(x, y) p_{4s^2}(x, y), \end{aligned}$$

where, using the doubling inequality,

$$H(x, y) := \frac{\mathsf{m}(B_{2s}(x))}{\mathsf{m}(B_s(x))} \exp\left\{-\frac{\mathsf{d}(x, y)^2}{6s^2} + \frac{\mathsf{d}(x, y)^2}{8s^2}\right\} \le C e^{-a^2/24} \qquad \text{if } \mathsf{d}(x, y) \ge as.$$

Therefore

$$\int_{\mathsf{X} \setminus B_{as}(x)} p_{s^2}(x, y)\, \mathrm{d}\mathsf{m}(y) \le C e^{-a^2/24} \int p_{4s^2}(x, y)\, \mathrm{d}\mathsf{m}(y) \le C e^{-a^2/24}. \tag{4.2.4}$$

Being a arbitrary, we conclude $\lim_{s \searrow 0} \mathrm{h}_{s^2} f(x) = \bar{f}(x)$ if $x \notin S_f$.

Step 2: case $x \in S_f$**.** Let $D \subseteq \mathbb{R}$ a countable dense set such that if $t \in D$, then $E_t := \{f > t\}$ is a set of finite perimeter. This is possible thanks to coarea. Set, for every $t \in D$, N_t as the set of points of $\partial^* E_t$ where (3.2.18) fails. We know that $|\mathrm{D}\chi_{E_t}|(N_t) = 0$, hence $\mathcal{H}^h(N_t) = 0$, by Remark 3.5.2. We set

$$N := \bigcup_{t \in D} N_t,$$

notice $\mathcal{H}^h(N)=0$. We show now $\lim_{s\searrow 0}\mathrm{h}_{s^2}f(x)=\bar f(x)$ for every $x\in S_f\setminus N$. Fix $x\in S_f\setminus N$ and $t\in(f^\wedge(x),f^\vee(x))\cap D$ so that $x\in\partial^*E_t$ be (2.3.6), so that, as $x\notin N_t$, x satisfies (3.2.18). Hence by Theorem 2.2.5 together with a sort of Fatou's Lemma and the convergence in L^1_{loc} of sets of finite perimeter, see e.g. the proof of [10, (4.13)],

$$\lim_{s\searrow 0}\mathrm{h}_{s^2}(\chi_{E_t})(x)=\mathrm{h}_1^{\mathbb{R}^n}(\chi_{\{x_n>0\}})(0)=1/2,$$

so that

$$\lim_{s\searrow 0}\mathrm{h}_{s^2}(\chi_{\mathsf{X}\setminus E_t})(x)=1/2.$$

We can then write

$$\begin{aligned}&\limsup_{s\searrow 0}\left|\mathrm{h}_{s^2}f(x)-\bar f(x)\right|\\ &\qquad\le\limsup_{s\searrow 0}\left|\mathrm{h}_{s^2}(\chi_{\mathsf{X}\setminus E_t}(f-f^\wedge(x)))\right|(x)+\limsup_{s\searrow 0}\left|\mathrm{h}_{s^2}(\chi_{E_t}(f-f^\vee(x)))\right|(x).\end{aligned}$$

It is enough then to show $\limsup_{s\searrow 0}|\mathrm{h}_{s^2}(\chi_{E_t}(f-f^\vee(x)))|(x)=0$ (the other term can be dealt similarly). We fix $a>1$. We can compute

$$\begin{aligned}\left|\mathrm{h}_{s^2}(\chi_{E_t}(f-f^\vee(x)))\right|(x)&\le\int_{E_t}p_{s^2}(x,y)\left|f(y)-f^\vee(x)\right|\mathrm{d}\mathsf{m}(y)\\ &=\int_{E_t\cap B_{as}(x)}p_{s^2}(x,y)\left|f(y)-f^\vee(x)\right|\mathrm{d}\mathsf{m}(y)\\ &\qquad+\int_{\mathsf{X}\setminus B_{as}(x)}p_{s^2}(x,y)\left|f(y)-f^\vee(x)\right|\mathrm{d}\mathsf{m}(y)\,.\end{aligned}$$

The second term on the right hand side is bounded by $Ce^{-a^2/24}$, thanks to $f\in L^\infty(\mathsf{m})$ and (4.2.4). As a is arbitrary, we conclude if we show that the first term converges to 0 as $s\searrow 0$. Using the Gaussian estimates for the heat kernel in (2.2.7) and the doubling inequality we estimate the first term by

$$\begin{aligned}&\limsup_{s\searrow 0}\int_{E_t\cap B_{as}(x)}p_{s^2}(x,y)\left|f(y)-f^\vee(x)\right|\mathrm{d}\mathsf{m}(y)\\ &\qquad\le\limsup_{s\searrow 0}\frac{C}{\mathsf{m}(B_{as}(x))}\int_{E_t\cap B_{as}(x)}\left|f(y)-f^\vee(x)\right|\mathrm{d}\mathsf{m}(y)\,.\end{aligned}$$

Take now any $t_1\in(t,f^\vee(x))\cap D$ and $t_2\in(f^\vee(x),\infty)\cap D$. We can split

$$B_{as}(x)\cap E_t=(B_{as}(x)\cap(E_t\setminus E_{t_1}))\cup(B_{as}(x)\cap(E_{t_1}\setminus E_{t_2}))\cup(B_{as}(x)\cap E_{t_2})\,.$$

Now by the very definition of $f^\vee(x)$, E_{t_2} has density 0 at x. Also, $E_t\setminus E_{t_1}$ has density 0 at x, as $E_{t_1}\subseteq E_t$ and both E_t and E_{t_1} have density 1/2 at x, as a consequence of $x\in\partial^*E_{t_1}\cap\partial^*E_t$ and $x\notin N$. Therefore, taking into account $f\in L^\infty(\mathsf{m})$, we are left with

$$\begin{aligned}&\limsup_{s\searrow 0}\frac{C}{\mathsf{m}(B_{as}(x))}\int_{E_t\cap B_{as}(x)}\left|f(y)-f^\vee(x)\right|\mathrm{d}\mathsf{m}(y)\\ &\quad=\limsup_{s\searrow 0}\frac{C}{\mathsf{m}(B_{as}(x))}\int_{(E_{t_1}\setminus E_{t_2})\cap B_{as}(x)}\left|f(y)-f^\vee(x)\right|\mathrm{d}\mathsf{m}(y)\le C(t_2-t_1).\end{aligned}$$

We conclude as we can take $t_2,t_1\to t$ by density of D in $\mathbb{R}$. □

4.2.1 The jump set

The aim of this section is to investigate in detail the behaviour of a function of bounded variation on the jump set.

For the following definition, notice that the intersection with $\mathcal{R}_n^*(\mathsf{X})$ plays essentially no role, by Theorem 3.1.1.

Definition 4.2.6. Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,N)$ space of essential dimension n and let $f \in \mathrm{BV}(\mathsf{X})^m$. We set $J_f := S_f \cap \mathcal{R}_n^*(\mathsf{X})$, the *jump set*.

The following proposition goes towards the study of jump sets for vector valued functions of bounded variation (recall Definition 2.3.16). In particular, total variation and Hausdorff measures are mutually absolutely continuous on jump sets. Moreover, for a vector valued function of bounded variation, at almost every point in the jump set, every component of f contributing to the jump has the same polar vector, up to the sign. Notice that item *iii*) can be seen as a blow-up result for vector valued functions of bounded variation on the jump set. The main tool to prove Proposition 4.2.7 is the transversality Lemma 4.2.3.

Proposition 4.2.7. *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *be an* $\mathrm{RCD}(K,N)$ *space of essential dimension* n *and let* $f \in \mathrm{BV}(\mathsf{X})^m$*. Then*

$$|\mathrm{D}f| \llcorner J_{f_i} \ll \mathcal{H}^h \llcorner J_{f_i} \ll \mathcal{H}^{n-1} \llcorner J_{f_i} \ll |\mathrm{D}f_i| \llcorner J_{f_i} \qquad \text{for every } i = 1, \ldots, m. \tag{4.2.5}$$

Moreover, there exist a pair of $|\mathrm{D}f|$*-measurable functions* $f^l,\ f^r : \mathsf{X} \to \mathbb{R}^m$ *and a vector field* $\nu_f^J \in L^\infty_{|\mathrm{D}f|}(T\mathsf{X})$ *such that*

i) it holds

$$|\nu_f^J| = 1 \text{ and } f^r \neq f^l \qquad |\mathrm{D}f|\text{-a.e. on } J_f,$$

ii) for every $i = 1, \ldots, m$,

$$(f_i^r - f_i^l)\nu_f^J = (f_i^\vee - f_i^\wedge)\nu_{f_i} \qquad |\mathrm{D}f|\text{-a.e.} \tag{4.2.6}$$

where we considered ν_f^J *and* ν_{f_i} *as elements of* $L^0_{\mathrm{Cap}}(T\mathsf{X})$*, set* $\bar f(x) := \frac{f^r(x)+f^l(x)}{2}$,

iii) for $|\mathrm{D}f|$*-a.e.* $x \in J_f$ *there exists a set of finite perimeter* E *such that* $x \in \mathcal{F}E$ *and*

$$\lim_{r\searrow 0} \fint_{B_r(x)\cap E} |f - f^r(x)| \,\mathrm{d}\mathsf{m} = \lim_{r\searrow 0} \fint_{B_r(x)\cap(\mathsf{X}\setminus E)} |f - f^l(x)| \,\mathrm{d}\mathsf{m} = 0, \tag{4.2.7}$$

iv) for $|\mathrm{D}f|$*-a.e.* $x \in \mathsf{X} \setminus J_f$, $\bar f(x) := f^l(x) = f^r(x)$ *and*

$$\lim_{r\searrow 0} \fint_{B_r(x)} \left|f - \bar f(x)\right| \mathrm{d}\mathsf{m} = 0. \tag{4.2.8}$$

Finally, if $\tilde f^l, \tilde f^r, \tilde\nu_f^J$ *is another triplet as above, then* $J_f = \tilde J_f^+ \cup \tilde J_f^-$*, with*

$$(\tilde f^l, \tilde f^r) = \sigma_\pm(f^l, f^r) \text{ and } \tilde\nu_f^J = \pm\nu_f^J \qquad |\mathrm{D}f|\text{-a.e. on } \tilde J_f^\pm$$

(where $\sigma_+(a,b) = (a,b)$ *and* $\sigma_-(a,b) = (b,a)$*) and*

$$\tilde f^l = \tilde f^r = f^l = f^r \qquad |\mathrm{D}f|\text{-a.e. on } \mathsf{X} \setminus J_f.$$

Proof. First notice that part of (4.2.5) follows from the coarea formula in (2.3.3) (see e.g. [28, Theorem 5.3]), recalling also that $|\mathrm{D}f| \ll \mathcal{H}^h$. The remaining part of (4.2.5) is due to Remark 3.5.2 (as $J_{f_i} \subseteq \mathcal{R}_N^*(\mathsf{X})$ by definition). This shows also that (4.2.6) is well posed.

We recall that thanks to (2.3.5), $f_i^\vee$, $f_i^\wedge$, $\overline{f_i} \in \mathbb{R}$ for $\mathcal{H}^h$-a.e. $x \in \mathsf{X}$ and for every $i = 1, \dots, m$. We set $f_i^l = f_i^r := \overline{f_i}$ if $x \notin J_{f_i}$. We define ν_f^J as an element of $L^0_{\mathrm{Cap}}(T\mathsf{X})$, then it is enough to take the projection $\bar{\pi}_{|\mathrm{D}f|}$ to obtain a vector field of $L^\infty_{|\mathrm{D}f|}(T\mathsf{X})$ as in the statement. We define f_i^l, f_i^r and ν_f^J iteratively. More precisely, we define $(f_1^r, f_1^l) = (f_1^\vee, f_1^\wedge)$ on J_{f_1} and $\nu_f^J = \nu_{f_1}\chi_{J_{f_1}}$. At step k, let $G_k := \bigcup_{i=1}^{k-1} J_{f_i}$. We define $(f_k^r, f_k^l) = (f_k^\vee, f_k^\wedge)$ on $J_{f_k} \setminus G_k$ and we add to ν_f^J the Cap vector field $\nu_{f_k}\chi_{J_{f_k}\setminus G_k}$. Now, Lemma 4.2.3 and the construction above imply that at $\mathcal{H}^h$-a.e. $x \in J_{f_k} \cap G_k$ it holds $\nu_{f_k} = \pm\nu_f^J$, whence (f_k^r, f_k^l) is uniquely defined on $J_{f_k} \cap G_k$ by the request *ii*).

Using Lemma 4.2.1 and standard considerations, we can find a countable dense subset of $\mathbb{R}$, $\{t_j\}_{j\in\mathbb{N}}$, such that for every $i = 1, \dots, m$, we have

i) for every j, for
$$E_{i,j} := \{f_i > t_j\},$$
$E_{i,j}$ is a set of locally finite perimeter,

ii) $J_{f_i} = \bigcup_{j\in\mathbb{N}} \partial^* E_{i,j}$,

iii) for every j
$$\nu_{E_{i,j}} = \bar{\pi}_{|\mathrm{D}\chi_{E_{i,j}}|}(\nu_{f_i}).$$

We prove now (4.2.7). First, arguing as in the proof of [99, Theorem 3.5] and taking into account (2.3.5), we see that we can assume with no loss of generality that $f_i \in L^\infty(\mathsf{m})$ for every $i = 1, \dots, m$. Up to discarding a $\mathcal{H}^h$-negligible set, we may restrict ourselves to the set of points of J_f at which the density of every set $E_{i,j}$ is in $\{0, 1/2, 1\}$, see e.g. Remark 3.3.3, Remark 3.5.2 and Corollary 3.3.2. Notice that if $m = 1$ and $x \in \partial^* E_{i,j}$, then (4.2.7) holds with either $E_{i,j}$ or $\mathsf{X} \setminus E_{i,j}$ in place of E. This follows from a standard argument as the one used at the end of **Step 2** of the proof of Lemma 4.2.5. But then the same conclusion holds also if $m > 1$, up to $\mathcal{H}^h$-negligible subsets, by Lemma 4.2.3 (thanks to our choice of f^l and f^r). Finally, we can assume that $x \in \mathcal{F}E$ as $\mathcal{H}^h(\partial^* E_{i,j} \setminus \mathcal{F}E_{i,j}) = 0$. Also, [99, Theorem 3.5] proves (4.2.8).

We prove now uniqueness, in the sense explained at the end of the statement. On $\mathsf{X} \setminus J_f$, this is clear, so let us focus on J_f. We just have to prove that at $|\mathrm{D}F|$-a.e. $x \in J_f$, $(\tilde{f}^l(x), \tilde{f}^r(x))$ coincides, up to the order, with $(f^l(x), f^r(x))$, then we can use (4.2.6) to conclude. We can assume that at x there exist two sets of finite perimeter $E, \tilde{E}$ with $x \in \mathcal{F}E \cap \mathcal{F}\tilde{E}$ and such that (4.2.7) holds and also the variant of (4.2.7) for $\tilde{f}^l(x), \tilde{f}^r(x), \tilde{E}$ holds. Now, notice that it holds that
$$0 < \limsup_{r\searrow 0} \frac{\mathsf{m}(E \cap B_r(x))}{\mathsf{m}(B_r(x))} \le \limsup_{r\searrow 0} \frac{\mathsf{m}(E \cap \tilde{E} \cap B_r(x))}{\mathsf{m}(B_r(x))} + \limsup_{r\searrow 0} \frac{\mathsf{m}(E \setminus \tilde{E} \cap B_r(x))}{\mathsf{m}(B_r(x))}.$$
Therefore, either $0 < \limsup_{r\searrow 0} \frac{\mathsf{m}(E\cap\tilde{E}\cap B_r(x))}{\mathsf{m}(B_r(x))}$ or $0 < \limsup_{r\searrow 0} \frac{\mathsf{m}(E\setminus\tilde{E}\cap B_r(x))}{\mathsf{m}(B_r(x))}$. In the first case, we infer that $\tilde{f}^r(x) = f^r(x)$, in the second case that $\tilde{f}^l(x) = f^r(x)$. We can deal similarly with $f^l(x)$. □

Remark 4.2.8. A careful inspection of the proof of Proposition 4.2.7 shows that in item *iii*) we can replace the integral $\fint_{B_r(x)\cap E} |f - f^r(x)| \,\mathrm{d}\mathsf{m}$ with $\fint_{B_r(x)\cap E} |f - f^r(x)|^{Q/(Q-1)} \,\mathrm{d}\mathsf{m}$ for any $Q = Q(R)$ given as in (2.2.3) and similarly for the integral involving f^l. A similar consideration holds for item *iv*). ■

Proposition 4.2.7 gives us a "canonical" unit normal to the jump set. In the following lemma, we clarify that such normal depends on the jump set rather than on the vector valued function of bounded variation.

Lemma 4.2.9. *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *be an* $\mathrm{RCD}(K,N)$ *space and let* $f\in\mathrm{BV}(\mathsf{X})^m$ *and let* $g\in\mathrm{BV}(\mathsf{X})^l$. *Then we can choose* $\nu_f^J=\nu_g^J$ *on* $J_f\cap J_g$ *and for* $(|\mathrm{D}f|\wedge|\mathrm{D}g|)$*-a.e.* $x\in J_f\cap J_g$, *there exists a set of finite perimeter* E *such that* $x\in\mathcal{F}E$ *and*

$$\lim_{r\searrow 0}\fint_{B_r(x)\cap E}|(f,g)-(f^r(x),g^r(x))|\,\mathrm{d}\mathsf{m}=\lim_{r\searrow 0}\fint_{B_r(x)\cap(\mathsf{X}\setminus E)}|(f,g)-(f^l(x),g^l(x))|\,\mathrm{d}\mathsf{m}=0.$$

In particular, if $g=\varphi\circ f$ *for some* $\varphi\in C^1(\mathbb{R}^m,\mathbb{R}^l)\cap\mathrm{LIP}(\mathbb{R}^m,\mathbb{R}^l)$ *such that* $\varphi(0)=0$, *then we can choose* $\nu^J_{\varphi\circ f}=\nu_f^J$ *on* $J_{\varphi\circ f}$ *as* $J_{\varphi\circ f}\subseteq J_f$ *and we have*

$$(\varphi\circ f)^r=\varphi(f^r)\ \text{and}\ (\varphi\circ f)^l=\varphi(f^l).$$

Proof. The proof is an application of Proposition 4.2.7 to $(f,g)\in\mathrm{BV}(\mathsf{X})^{m+l}$. □

The results obtained so far allow us to obtain a well-defined fine representative of a vector valued function of bounded variation with meaningful properties.

Definition 4.2.10. Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,N)$ space and let $f\in\mathrm{BV}(\mathsf{X})^m$. Define the functions *left representative, right representative and precise representative* $f^l,f^r,\bar f:\mathsf{X}\to\mathbb{R}^m$, respectively, and the vector field *normal to the singular set* ν_f^J as then ones given by Proposition 4.2.7.

There may be more than one possible choice for the triplet (f^l,f^r,ν_f^J), however, notice that the quantity $(f^l-f^r)\nu_f^J$ is well defined $|\mathrm{D}f|$-a.e. on J_f.

It is classical that whenever reduced boundaries of sets of finite perimeter are rectifiable, the same conclusion holds for jump sets of functions of bonded variation. We record this fact in the following proposition.

Proposition 4.2.11. *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *be an* $\mathrm{RCD}(K,N)$ *space of essential dimension* n *and let* $f\in\mathrm{BV}_{\mathrm{loc}}(\mathsf{X})^m$. *Then* $\mathcal{H}^h\llcorner J_f$ *and* $\mathcal{H}^{n-1}\llcorner J_f$ *are* σ*-finite and mutually absolutely continuous. Moreover,* J_f *is countably* $(n-1)$*-rectifiable. More precisely, for every* $\varepsilon>0$, *we can write*

$$J_f=\bigcup_{i\in\mathbb{N}}B_i\cup N,$$

where for every i, B_i *is* $(1+\varepsilon)$*-bilipschitz to a Borel subset of* $\mathbb{R}^{n-1}$ *and* $|\mathrm{D}f|(N)=0$.

Proof. We can clearly assume that $f\in\mathrm{BV}(\mathsf{X})$. By a classical argument, J_f can be obtained as a countable union of reduced boundaries of sets of finite perimeter. For example, by coarea there exists $S\subseteq\mathbb{R}$ countable and dense such that $E_s:=\{f>s\}$ has locally finite perimeter for any $s\in S$, then, by (2.3.4) and (2.3.6) it holds $J_f=\bigcup_{s\in S}\partial^*E_s$. Then it is enough to recall the representation formulae of Theorem 3.5.1 together with Remark 3.5.2. The same argument also yields rectifiability, using Theorem 3.4.1 and taking into account Remark 3.5.2 and (4.2.5). □

In view of the following result, recall that as an immediate consequence of Proposition 4.2.11, $\mathcal{H}^{n-1}\llcorner J_f\ll\mathrm{Cap}$, so that, if ν is a Cap-vector field, $\nu\mathcal{H}^{n-1}\llcorner J_f$ is well defined. We are going to give a precise description of the behaviour of the distributional differential (and of its total variation) of a function of bounded variation on the jump set. The proof is by coarea and a sort of integration by Cavalieri's formula.

Proposition 4.2.12. *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *be an* $\mathrm{RCD}(K,N)$ *space of essential dimension* n *and let* $f \in \mathrm{BV}(\mathsf{X})^m$. *Then*

$$\mathrm{D}f \llcorner J_f = (f^r - f^l)\nu_f^J \frac{\omega_{n-1}}{\omega_n}\mathcal{H}^h \llcorner J_f \tag{4.2.9}$$

and

$$\mathrm{D}f \llcorner J_f = (f^r - f^l)\nu_f^J \Theta_n(\mathsf{m},\cdot)\mathcal{H}^{n-1} \llcorner J_f. \tag{4.2.10}$$

In particular,

$$|\mathrm{D}f| \llcorner J_f = |f^r - f^l| \frac{\omega_{n-1}}{\omega_n}\mathcal{H}^h \llcorner J_f \tag{4.2.11}$$

and

$$|\mathrm{D}f| \llcorner J_f = |f^r - f^l|\Theta_n(\mathsf{m},\cdot)\mathcal{H}^{n-1} \llcorner J_f. \tag{4.2.12}$$

Proof. First notice that the statement makes sense thanks to (4.2.5). We start from the case $m = 1$. By coarea and (3.5.1) we have that for $v \in \mathrm{TestV}(\mathsf{X})$

$$\int v \cdot \nu_f \,\mathrm{d}|\mathrm{D}f| \llcorner J_f = \int_{\mathbb{R}} \int_{J_f} v \cdot \nu_f \,\mathrm{d}\big|\mathrm{D}\chi_{\{f>t\}}\big| \,\mathrm{d}t = \frac{\omega_{n-1}}{\omega_n} \int_{\mathbb{R}} \int_{J_f} \chi_{\partial^*\{f>t\}} v \cdot \nu_f \mathrm{d}\mathcal{H}^h \,\mathrm{d}t\,.$$

Now notice that the map $(t,x) \mapsto \chi_{\partial^*\{f>t\}}(x)$ is measurable, thanks to standard arguments: just notice that for any r the maps

$$(x,t) \mapsto \frac{\mathsf{m}(B_r(x) \cap \{f>t\})}{\mathsf{m}(B_r(x))} \qquad \text{and} \qquad (x,t) \mapsto \frac{\mathsf{m}(B_r(x) \setminus \{f>t\})}{\mathsf{m}(B_r(x))}$$

are continuous everywhere up to a set of null $(\mathcal{H}^h \llcorner J_f) \otimes \mathcal{L}^1$ measure. We can therefore apply Fubini's Theorem (integrability is given by coarea and σ-finiteness of $\mathcal{H}^h$ by Proposition 4.2.11) and infer that

$$\int v \cdot \nu_f \,\mathrm{d}|\mathrm{D}f| \llcorner J_f = \frac{\omega_{n-1}}{\omega_n} \int_{J_f} \int_{\mathbb{R}} \chi_{\partial^*\{f>t\}} v \cdot \nu_f \,\mathrm{d}t \,\mathrm{d}\mathcal{H}^h = \frac{\omega_{n-1}}{\omega_n} \int_{J_f} (f^\vee - f^\wedge) v \cdot \nu_f \mathrm{d}\mathcal{H}^h,$$

where we used (2.3.6) for the last equality. Now (4.2.9) follows from the equation above and (4.2.6). Now we prove (4.2.10), the proof follows from the same argument as above, relying on (3.5.2) (in particular, Theorem 3.1.1) instead of (3.5.1), hence writing

$$|\mathrm{D}\chi_{\{f>t\}}| = |\mathrm{D}\chi_{\{f>t\}}| \llcorner \mathcal{R}_n^*(\mathsf{X}) = \Theta_n(\mathsf{m},\cdot)\chi_{\partial^*\{f>t\}}\mathcal{H}^{n-1} \llcorner \mathcal{R}_n^*(\mathsf{X}).$$

Now we turn to case $m > 1$. Take $v = (v_1, \ldots, v_m) \in \mathrm{TestF}(\mathsf{X})^m$. Then, using (4.1.5), Proposition 4.2.7 and what proved in the case $m = 1$,

$$\mathrm{D}f \llcorner J_f(\mathsf{X})(v) = \int_{J_f} v \cdot \nu_f \mathrm{d}|\mathrm{D}f| = \sum_{i=1}^m \int_{J_f} v_i \cdot \nu_{f_i} \mathrm{d}|\mathrm{D}f_i| = \sum_{i=1}^m \int_{J_f} (f_i^r - f_i^l) v_i \cdot \nu_f^J \frac{\omega_{n-1}}{\omega_n}\mathcal{H}^h$$

which proves (4.2.9) in the general case. Similarly we obtain (4.2.10) in the general case.

Finally, (4.2.11) (resp. (4.2.12)) follows from (4.2.9) (resp. (4.2.9)) and Remark 4.1.5. □

4.3 Calculus rules

We turn now to study the calculus rules for (vector valued) functions of bounded variation. Again, the study is performed only on finite dimensional RCD spaces, as we need to use the fine properties of the previous section.

4.3.1 Chain rule

The first result that we state is the chain rule, and studies the distributional differential of a BV function after taking the composition with a Lipschitz function. It is proved exploiting the fine properties that we have just studied combined with the coarea formula.

Proposition 4.3.1 (Chain rule)**.** *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *be an* $\mathrm{RCD}(K,N)$ *space,* $f\in\mathrm{BV}(\mathsf{X})$ *and* $\varphi\in\mathrm{LIP}(\mathbb{R})$ *such that* $\varphi(0)=0$*. Then* $\varphi\circ f\in\mathrm{BV}(\mathsf{X})$ *and*

$$\mathrm{D}(\varphi\circ f)=\left(\int_0^1\varphi'(tf^\vee+(1-t)f^\wedge)\,\mathrm{d}t\right)\mathrm{D}f. \tag{4.3.1}$$

We comment briefly on the well-posedness of (4.3.1). Recalling (2.3.5), we see that it suffices to check that

$$|\mathrm{D}f|(A)=0,$$

where

$$A:=\left\{x\in\mathsf{X}\setminus J_f:\varphi \text{ is not differentiable at } \bar f(x)\right\}.$$

We can then use coarea, the relations in (2.3.6), (3.5.1) and finally Rademacher's Theorem to compute

$$\begin{aligned}|\mathrm{D}f|(A)&=\int_{\mathbb{R}}\left|\mathrm{D}\chi_{\{f>t\}}\right|(A)\,\mathrm{d}t=\frac{\omega_{n-1}}{\omega_n}\int_{\mathbb{R}}\mathcal{H}^h(A\cap\partial^*\{f>t\})\,\mathrm{d}t\\&=\frac{\omega_{n-1}}{\omega_n}\int_{\mathbb{R}}\mathcal{H}^h(\{x\in\mathsf{X}\setminus J_f:\varphi\text{ is not differentiable at } t \text{ and } \bar f(x)=t\})\,\mathrm{d}t=0.\end{aligned}$$

Proof. The proof is done by coarea, taking inspiration from [5]. Using linearity, we can assume that φ is also bilipschitz and strictly increasing with no loss of generality. For what concerns the jump part, the claim on $J_{\varphi\circ f}$ follows from Proposition 4.2.12 and Lemma 4.2.9.

It remains to show the claim on $\mathsf{X}\setminus J_f$, as $|\mathrm{D}(\varphi\circ f)|(J_f\setminus J_{\varphi\circ f})=0$. Take any $v\in\mathrm{TestF}(\mathsf{X})$. We compute, using the coarea formula in (2.3.3), Lemma 4.2.1, the change of variables $t=\varphi(s)$ and (2.3.6),

$$\begin{aligned}\mathrm{D}(\varphi\circ f)\llcorner(\mathsf{X}\setminus J_f)(v)&=\int_{\mathsf{X}\setminus J_f}v\,\cdot\,\nu_{\varphi\circ f}\mathrm{d}|\mathrm{D}(\varphi\circ f)|=\int_{\mathbb{R}}\int_{\mathsf{X}\setminus J_f}v\,\cdot\,\nu_{\varphi\circ f}\mathrm{d}|\mathrm{D}\chi_{\{\varphi\circ f>t\}}|\mathrm{d}t\\&=\int_{\mathbb{R}}\int_{\mathsf{X}\setminus J_f}v\,\cdot\,\nu_{\{\varphi\circ f>t\}}\mathrm{d}|\mathrm{D}\chi_{\{\varphi\circ f>t\}}|\mathrm{d}t\\&=\int_{\mathbb{R}}\varphi'(s)\int_{\mathsf{X}\setminus J_f}v\,\cdot\,\nu_{\{f>s\}}\mathrm{d}|\mathrm{D}\chi_{\{f>s\}}|\mathrm{d}s\\&=\int_{\mathbb{R}}\int_{\mathsf{X}\setminus J_f}\varphi'(\bar f)v\,\cdot\,\nu_{\{f>s\}}\mathrm{d}|\mathrm{D}\chi_{\{f>s\}}|\mathrm{d}s.\end{aligned}$$

With the same argument as above, we "reverse"

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathsf{X}\setminus J_f}\varphi'(\bar f)v\,\cdot\,\nu_{\{f>s\}}\mathrm{d}|\mathrm{D}\chi_{\{f>s\}}|\mathrm{d}s&=\int_{\mathbb{R}}\int_{\mathsf{X}\setminus J_f}\varphi'(\bar f)v\,\cdot\,\nu_f\mathrm{d}|\mathrm{D}\chi_{\{f>s\}}|\mathrm{d}s\\&=\int_{\mathsf{X}\setminus J_f}\varphi'(\bar f)v\,\cdot\,\nu_f\mathrm{d}|\mathrm{D}f|=\varphi(\bar f)\mathrm{D}f\llcorner(\mathsf{X}\setminus J_f)(v),\end{aligned}$$

so that the claim is proved. □

4.3.2 Leibniz rule

Now we polarize the chain rule to obtain the Leibniz rule, which describes the distributional differential of the product of two functions of bounded variation. We restrict ourselves to the case f, g bounded functions of bounded variation although the boundedness hypothesis can be slightly weakened using approximation arguments as done in the proof of Proposition 4.3.4 below.

Proposition 4.3.2 (Leibniz rule)**.** *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space and let* $f, g \in \mathrm{BV}(\mathsf{X}) \cap L^\infty(\mathsf{m})$*. Then* $fg \in \mathrm{BV}(\mathsf{X})$ *and*

$$\mathrm{D}(fg) = \bar{f}\mathrm{D}g + \bar{g}\mathrm{D}f. \tag{4.3.2}$$

In particular, $|\mathrm{D}(fg)| \le |\bar{f}||\mathrm{D}g| + |\bar{g}||\mathrm{D}f|$.

Proof. Using the chain rule of Proposition 4.3.1 with $\varphi \in \mathrm{LIP}(\mathbb{R})$ that coincides with $t \mapsto t^2$ on a sufficiently large neighbourhood of 0, we see that

$$\mathrm{D}(f+g)^2 = 2(\overline{f+g})\mathrm{D}(f+g) = 2(\overline{f} + \overline{g})\mathrm{D}(f+g), \tag{4.3.3}$$

$$\mathrm{D}f^2 = 2\overline{f}\mathrm{D}f, \tag{4.3.4}$$

$$\mathrm{D}g^2 = 2\overline{g}\mathrm{D}g. \tag{4.3.5}$$

Here we used that $\overline{f+g} = \overline{f} + \overline{g}$ $\mathcal{H}^h$-a.e. which follows e.g. from Lemma 4.2.5. Using the linearity of the map $f \mapsto \mathrm{D}f$, subtracting (4.3.4) and (4.3.5) from (4.3.3), we obtain (4.3.2). □

As a consequence, we record the following result. Even though we are not going to need it elsewhere, we state it because we believe that it is interesting: it shows that $\nabla \mathrm{h}_t f \mathsf{m}$ weakly converge to $\mathrm{D}f$ in duality with objects of the kind gv, where $g \in \mathrm{BV}(\mathsf{X}) \cap L^\infty(\mathsf{m})$ and $v \in \mathcal{QC}^\infty(T\mathsf{X}) \cap D(\mathrm{div})$.

Proposition 4.3.3. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space and* $f, g \in \mathrm{BV}(\mathsf{X}) \cap L^\infty(\mathsf{m})$*. Then*

$$\lim_{t \searrow 0} \int gv \cdot \nabla \mathrm{h}_t f \, \mathrm{d}\mathsf{m} = \int \bar{g}v \, \cdot \, \nu_f \, \mathrm{d}|\mathrm{D}f| \qquad \text{for every } v \in \mathcal{QC}^\infty(T\mathsf{X}) \cap D(\mathrm{div}).$$

Proof. We can write, thanks to the calculus rules,

$$\int \mathrm{h}_t f \mathrm{h}_s g \mathrm{div}\, v \, \mathrm{d}\mathsf{m} = -\int \mathrm{h}_s g \nabla \mathrm{h}_t f \, \cdot \, v \, \mathrm{d}\mathsf{m} - \int \mathrm{h}_t f \nabla \mathrm{h}_s g \, \cdot \, v \, \mathrm{d}\mathsf{m}\,.$$

We let now first $s \searrow 0$ then $t \searrow 0$, use Lemma 4.2.5 and compare the outcome with the result given by (4.3.2). □

4.3.3 Vol'pert chain rule

The Leibniz rule can be iteratively used to study the distributional differential of the product of several BV functions, hence, by linearity, we know how to compute $\mathrm{D}(\varphi \circ f)$ when φ is a polynomial and f is a vector valued function of bounded variation. By approximation, we can treat the case in which φ is any C^1 function. The result is the following chain rule, also called Vol'pert chain rule.

Theorem 4.3.4 (Vol'pert chain rule)**.** *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space and let* $f \in \mathrm{BV}(\mathsf{X})^m$*. Let* $\varphi \in C^1(\mathbb{R}^m, \mathbb{R}^l) \cap \mathrm{LIP}(\mathbb{R}^m, \mathbb{R}^l)$ *such that* $\varphi(0) = 0$*. Then*

$$\mathrm{D}(\varphi \circ f) = \left(\int_0^1 \nabla \varphi(t f^r + (1-t) f^l) \, \mathrm{d}t \right) \mathrm{D}f. \tag{4.3.6}$$

Proof. By Remark 4.1.2, we see that we can assume $l = 1$ with no loss of generality (see e.g. the second part of the proof of Proposition 4.2.12). The jump part is dealt as in the proof of Proposition 4.3.1, building upon Proposition 4.2.12 and Lemma 4.2.9. Indeed, we know that

$$\int_{J_f} v \cdot \nu_{\varphi\circ f}\mathrm{d}|\mathrm{D}(\varphi\circ f)| = \int_{J_f} (\varphi(f^r) - \varphi(f^l))v \cdot \nu_f^J \mathrm{d}|\mathrm{D}f|. \tag{4.3.7}$$

Now we turn to the proof of (4.3.6) on $\mathsf{X}\setminus J_f$. Assume for the moment also $f \in L^\infty(\mathsf{m})^m$, say $|f_i| \le L$ m-a.e. for $i = 1,\dots,m$. Fix $v \in \mathrm{TestF}(\mathsf{X})$, recalling (4.1.5) we have to show that

$$\int_{\mathsf{X}\setminus J_f} v \cdot \nu_{\varphi\circ f}\mathrm{d}|\mathrm{D}(\varphi\circ f)| = \int_{\mathsf{X}\setminus J_f} \sum_{i=1}^m \partial_i\varphi(\bar f)v \cdot \nu_{f_i}\mathrm{d}|\mathrm{D}f_i|. \tag{4.3.8}$$

Notice that the differential is a closed linear operator, in the sense that if $\{\varphi_k\}_k$ are uniformly Lipschitz functions satisfying the same hypotheses of φ such that $\varphi_k \to \varphi$ pointwise, then $\int \varphi_k \circ f \mathrm{div} v \to \int \varphi\circ f\mathrm{div}v$, so that

$$\mathrm{D}(\varphi_k\circ f)(\mathsf{X})(v) \to \mathrm{D}(\varphi\circ f)(\mathsf{X})(v). \tag{4.3.9}$$

By (4.3.7) and (4.3.9) we infer that if φ_k is as above, then

$$\int_{\mathsf{X}\setminus J_f} v \cdot \nu_{\varphi_k\circ f}\mathrm{d}|\mathrm{D}(\varphi_k\circ f)| \to \int_{\mathsf{X}\setminus J_f} v \cdot \nu_{\varphi\circ f}\mathrm{d}|\mathrm{D}(\varphi\circ f)|.$$

As a consequence of this discussion, we see that if $\{\varphi_k\}_k$ are uniformly Lipschitz functions satisfying the same hypotheses of φ such that (4.3.8) holds for any φ_k and $(\varphi_k, \nabla\varphi_k) \to (\varphi, \nabla\varphi)$ uniformly on $[-L, L]^m$, then (4.3.8) holds also for φ. Also, the left hand side of (4.3.8) is linear in φ.

Let $\varepsilon > 0$. By a mollification and cut-off argument, we find $\tilde\varphi \in C^\infty(\mathbb{R}^m)$ such that $\operatorname{supp}\tilde\varphi \subseteq [-2L, 2L]^m$, $\tilde\varphi(0) = 0$, and

$$\sup_{x\in[-L,L]^m} |\varphi(x) - \tilde\varphi(x)| + |\nabla\varphi(x) - \nabla\tilde\varphi(x)| < \varepsilon.$$

Now, by the Stone–Weierstrass Theorem, we find a polynomial $g : \mathbb{R}^m \to \mathbb{R}$ such that

$$\sup_{x\in[-2L,2L]^m} |\partial_1\cdots\partial_m\tilde\varphi(x) - g(x)| < \varepsilon.$$

Set now

$$\hat\varphi((x_1,\dots,x_m)) := \int_{-2L}^{x_1}\mathrm{d}s_1\cdots\int_{-2L}^{x_m}\mathrm{d}s_m\, g((s_1,\dots,s_m)),$$

it is not hard to verify that still

$$\sup_{x\in[-L,L]^m} |\tilde\varphi(x) - \hat\varphi(x)| + |\nabla\tilde\varphi(x) - \nabla\hat\varphi(x)| < C\varepsilon,$$

where C depends only on L and m. Eventually adding to $\hat\varphi$ a (small) constant, we can assume that $\hat\varphi(0) = 0$. Notice that $\hat\varphi$ is a polynomial.

Then, by discussion above, we see that it is enough to prove (4.3.8) for a polynomial, say φ (notice that the fact that polynomials are not Lipschitz plays no role here, as f is assumed to be bounded). Therefore, using also linearity, we see that we can assume with no loss of generality that φ is a monomial. Also, up to changing m and repeating some function f_i, we can assume that

$$\varphi(x_1,\dots,x_m)=x_1\cdots x_m.$$

All in all, we have reduced the claim to

$$\int_{\mathsf{X}\setminus J_f} v\cdot\nu_{f_1\cdots f_m}\mathrm{d}|\mathrm{D}(f_1\cdots f_m)|=\int_{\mathsf{X}\setminus J_f}\sum_{i=1}^m\Big(\prod_{j\neq i}\bar f_j\Big)v\cdot\nu_{f_i}\mathrm{d}|\mathrm{D}f_i|.$$

The claim is now proved by iteration of the Leibniz rule of Proposition 4.3.2 taking into account that if $x\notin J_f$, then $\overline{\prod_{j\in J}f_j}(x)=\prod_{j\in J}\bar f_j(x)$ for $J\subseteq\{1,\dots,m\}$, by (4.2.8) together with the assumption $f\in L^\infty(\mathsf{m})$. Indeed, restricting all equalities to $\mathsf{X}\setminus J_f$,

$$\begin{aligned}\nu_{f_1\cdots f_m}\mathrm{d}|\mathrm{D}(f_1\cdots f_m)|&=\overline{f_2\cdots f_m}\nu_{f_1}|\mathrm{D}f_1|+\bar f_1\nu_{f_2\cdots f_m}\mathrm{d}|\mathrm{D}(f_2\cdots f_m)|\\&=\bar f_2\cdots\bar f_m\nu_{f_1}|\mathrm{D}f_1|+\bar f_1\nu_{f_2\cdots f_m}\mathrm{d}|\mathrm{D}(f_2\cdots f_m)|\\&=\bar f_2\cdots\bar f_m\nu_{f_1}|\mathrm{D}f_1|+\bar f_1\big(\overline{f_3\cdots f_m}\nu_{f_2}|\mathrm{D}f_2|+\bar f_2\nu_{f_3\cdots f_m}\mathrm{d}|\mathrm{D}(f_3\cdots f_m)|\big)\\&=\cdots=\sum_{i=1}^m\prod_{j\neq i}\bar f_j\nu_{f_i}\mathrm{d}|\mathrm{D}f_i|\end{aligned}$$

so that the proof is concluded under the additional assumption $f\in L^\infty(\mathsf{m})^m$.

Now we get rid of the assumption $f\in L^\infty(\mathsf{m})^m$ using an approximation argument. In this procedure, we consider the approximating sequence $\{f^l\}_l$ as in (2.3.21). Now we can let $l\to\infty$ in (4.3.6) for f^l, recalling Lemma 4.2.2, (2.3.5) and coarea together with the closure property of the differential for what concerns the convergence of the let hand side. □

4.3.4 General chain rule

This section contains the last calculus rule of the chapter. It is the general chain rule, Theorem 4.3.6, which concerns the distributional differential of the composition of a vector valued function of bounded variation and a Lipschitz function. We remark that it is the strongest of our results concerning the calculus rules, in the sense that it immediately implies all the previous results. However, the previous results are needed for its proof.

With start with some preparatory material to state Theorem 4.3.6. For what concerns the notation, if $\nu=(\nu_1,\dots,\nu_m)\in L^0_{\mathrm{Cap}}(T^m\mathsf{X})$ and $v\in L^0_{\mathrm{Cap}}(T\mathsf{X})$, we write

$$\nu\cdot v:=(\nu_1\cdot v),\dots,(\nu_m\cdot v)\in\mathbb{R}^m.$$

The following lemma gives a concept of image of a matrix field in the non-smooth framework, where we do not have pointwise defined objects. $\mathrm{Gr}(\mathbb{R}^m)$ denotes the collection of all the vector subspaces of $\mathbb{R}^m$.

Lemma 4.3.5. *Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,N)$ space, let $\mu\ll\mathrm{Cap}$ be a finite Borel measure and let $\nu\in L^0_{\mathrm{Cap}}(T^m\mathsf{X})$. Then there exists unique (up to μ-a.e. equality) μ-measurable map*

$$G:\mathsf{X}\to\mathrm{Gr}(\mathbb{R}^m)$$

satisfying

i) for every $v \in L^0_{\mathrm{Cap}}(T\mathsf{X})$,

$$\nu \cdot v \in G \qquad \mu\text{-a.e.}$$

ii) if $G' : \mathsf{X} \to \mathrm{Gr}(\mathbb{R}^m)$ *is another map satisfying the requirement i), then*

$$G \subseteq G' \qquad \mu\text{-a.e.}$$

We call the map G given by the lemma above $\mu - \operatorname{ess\,span} \nu$.

Proof. First notice that uniqueness of G trivially follows from item *ii*).

Fix for the moment a set A as in the decomposition given by Theorem 2.2.21 and an orthonormal basis of $L^0_{\mathrm{Cap}}(T\mathsf{X})$ on A, say $\{v_1, \dots, v_k\}$. The map

$$A \ni x \mapsto \mathrm{span}\big(\{\nu \cdot v_i(x)\}_{i=1,\dots,k}\big) \in \mathrm{Gr}(\mathbb{R}^m)$$

is μ-measurable. We then define G equals this map on A and then define G μ-a.e. on X with a gluing argument.

We show now that G satisfies the desired properties. It is sufficient to fix a set A and vector fields $\{v_1, \dots, v_k\}$ as above and prove the claims on A. Item i) follows from the fact that $\{v_1, \dots, v_k\}$ is a basis of $L^0_{\mathrm{Cap}}(T\mathsf{X})$ on A. For what concerns item ii), take G' satisfying item i). In particular, μ-a.e. $\nu \cdot v_i \in G$ for every $i = 1, \dots, k$ so that μ-a.e. $G \subseteq G'$. □

Now we state the main result of this section. We defer the proof after the statement and the proof of Lemma 4.3.7 below. First, we recall a definition about the differentiability of Lipschitz functions, as in Section 8.1, Given $\varphi \in \mathrm{LIP}(\mathbb{R}^m, \mathbb{R}^l)$, we say that φ is *differentiable at x with respect to $V \in \mathrm{Gr}(\mathbb{R}^m)$* if there exists a linear map $\nabla_V \varphi(x) : V \to \mathbb{R}^l$ such that

$$\varphi(x + v) = \varphi(x) + \nabla_V \varphi(x) \cdot v + o(|v|) \qquad \text{for } v \in V.$$

If $v \in \mathbb{R}^m$, we say that φ is *differentiable at x in direction v* if φ is differentiable at x with respect span(v). Notice that every φ is differentiable with respect to $\{0\}$ at any point of $\mathbb{R}^m$.

If we want to generalize the calculus rule of Theorem 4.3.4 to the case in which φ is not C^1, but only Lipschitz, we certainly have to be coherent with (4.3.6). Notice that, outside the jump set, (4.3.6) reads as

$$\mathrm{D}(\varphi \circ f) \llcorner (\mathsf{X} \setminus J_f) = \nabla\varphi(\bar f)\mathrm{D}f \llcorner (\mathsf{X} \setminus J_f). \tag{4.3.10}$$

While Rademacher Theorem states that Lipschitz functions are differentiable almost everywhere with respect to the Lebesgue measure, it is in general false that a Lipschitz function φ is differentiable for $|\mathrm{D}f|$-a.e. $x \in \mathsf{X} \setminus J_f$ at $\bar f(x)$, in the case in which f is a vector valued function of bounded variation (this, however holds if f is scalar valued, see the discussion after the statement of Proposition 4.3.1). Take, for example, $\varphi(u, v) := u \vee v$ and $f(x, y) := (x, x)$. For this reason, Theorem 4.3.6 below can *not* be proved with soft techniques as we did for Theorem 4.3.4 and (4.3.10) has to be suitably interpreted, and this obstacle is not due to the fact that we are working in the RCD realm. The key remark to overcome this difficulty ([12]) is that we do not really need the full differentiability of φ at $\bar f(x)$, but only the differentiability in directions given by the image of the "polar matrix" $\frac{\mathrm{d}\mathrm{D}f}{\mathrm{d}|\mathrm{D}f|}$, as is only against these directions that the differential of φ is tested. It turns out that indeed φ is differentiable a.e. with respect to these directions and Theorem 4.3.4 has a suitable generalization.

Theorem 4.3.6. *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *be an* $\mathrm{RCD}(K,N)$ *space, let* $f\in \mathrm{BV}(\mathsf{X})^m$ *and let* $\varphi\in \mathrm{LIP}(\mathbb{R}^m,\mathbb{R}^l)$ *such that* $\varphi(0)=0$*. Then*

$$\mathrm{D}(\varphi\circ f)\llcorner J_f=\big(\varphi(f^r)-\varphi(f^l)\big)\mathrm{D}f\llcorner J_f.$$

Set now $V:=|\mathrm{D}f|-\operatorname{ess\,span}\nu_f$*. Then for* $|\mathrm{D}f|$*-a.e.* $x\notin J_f$*,* φ *is differentiable at* $\bar f(x)$ *with respect to* V *and it holds*

$$\mathrm{D}(\varphi\circ f)\llcorner(\mathsf{X}\setminus J_f)=\nabla_V\varphi(\bar f)\nu_f|\mathrm{D}f|\llcorner(\mathsf{X}\setminus J_f). \tag{4.3.11}$$

In the theorem above, by $\nabla_V\varphi(\bar f)\nu_f$ we mean the unique, up to $|\mathrm{D}f|$-a.e. equality, vector field in $L^0_{\mathrm{Cap}}(T\mathsf{X})^l$ such that for every $v\in L^0_{\mathrm{Cap}}(T\mathsf{X})$ it holds

$$(\nabla_V\varphi(\bar f)\nu_f)\,\cdot\,v=\nabla_V\varphi(\bar f)(\nu_f\,\cdot\,v)\qquad|\mathrm{D}f|\text{-a.e.}$$

Here we show the main lemma towards the proof of Theorem 4.3.6, whose purpose is to gain directions along which φ is differentiable. Its proof is based on the link between "closability of certain differentiation operators" and "differentiability of Lipschitz functions in related directions" recalled in Section 8.1, coupled with the information that "the map that takes φ and returns the distributional differential of $\varphi\circ f$ is closable".

Lemma 4.3.7. *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *be an* $\mathrm{RCD}(K,N)$ *space, let* $f\in\mathrm{BV}(\mathsf{X})^m$*, let* $\varphi\in\mathrm{LIP}(\mathbb{R}^m)$ *such that* $\varphi(0)=0$*, and let* $v\in\mathrm{TestV}(\mathsf{X})$*. Then for* $|\mathrm{D}f|$*-a.e.* $x\notin J_f$*,* φ *is differentiable at* $\bar f(x)$ *in direction* $(\nu_f\,\cdot\,v)(x)\in\mathbb{R}^m$ *and it holds*

$$\nu_{\varphi\circ f}\,\cdot\,v|\mathrm{D}(\varphi\circ f)|\llcorner(\mathsf{X}\setminus J_f)=\nabla_{\nu_f\,\cdot\,v}\varphi(\bar f)(\nu_f\,\cdot\,v)|\mathrm{D}f|\llcorner(\mathsf{X}\setminus J_f). \tag{4.3.12}$$

Proof. Denote by n the essential dimension of X. By Proposition 3.4.3 and the fact that $|\mathrm{D}(\varphi\circ f)|\le L|\mathrm{D}f|$, where L denotes the Lipschitz constant of φ, it is enough to prove the claim on G, where $G\subseteq\mathsf{X}\setminus J_f$ is a bounded Borel set for which there exists a Borel map $\Psi:G\to B$, where B is a Borel subset of $\mathbb{R}^n$ and Ψ has Borel inverse $\Psi^{-1}:B\to G$ and moreover Ψ is the restriction to G of some $\tilde\Psi\in\mathrm{BV}(\mathsf{X})^n$. Also, $J_{\tilde\Psi}\cap G=\emptyset$ and we can assume that v has compact support. We avoid writing $\bar{\cdot}$ for the precise representative, to simplify the notation.

We set then $f':=(f,\tilde\Psi)$ and $\varphi':=\varphi\circ\pi^1$, where $\pi^1:\mathbb{R}^m\times\mathbb{R}^n\to\mathbb{R}^m$ is the projection onto the first factor. In particular, $|\mathrm{D}f|\le|\mathrm{D}f'|$. Notice that f' and φ' still satisfy the assumptions of the lemma and that still $J_{f'}\cap G=\emptyset$. Notice also that $\varphi'\circ f'=\varphi\circ f$, that ($|\mathrm{D}f|$-a.e.) φ' is differentiable in direction $\nu_{f'}\,\cdot\,v$ if and only if φ is differentiable in direction $\nu_f\,\cdot\,v$ and finally that

$$(\nu_{f'}\,\cdot\,v)_i=(\nu_f\,\cdot\,v)_i\frac{\mathrm{d}|\mathrm{D}f|}{\mathrm{d}|\mathrm{D}f'|}\qquad|\mathrm{D}f'|\text{-a.e. for } i=1,\dots,m,$$

so that it remains to show (4.3.12) on G with φ' in place of φ and f' in place of f.

To simplify the notation, we return to the notation f and φ, keeping in mind that f is injective on G and its inverse is Borel. We set

$$w:=(\nu_f\,\cdot\,v)\circ f^{-1}\qquad\text{and}\qquad\mu:=f_*(|\mathrm{D}f|\llcorner G).$$

Assume for the moment also that $\varphi\in C^1(\mathbb{R}^m)$. Then we know that (4.3.12) holds with this choice of φ by Proposition 4.3.4. We compute now

$$f_*(\nabla\varphi(f)(\nu_f\,\cdot\,v)|\mathrm{D}f|\llcorner G)=\sum_{i=1}^m\partial_i\varphi f_*((\nu_f\,\cdot\,v)_i|\mathrm{D}f|\llcorner G)=\nabla\varphi\,\cdot\,w\mu.$$

We check that the differentiation operator depending on φ defined above is closable in the sense of item i) of Theorem 8.1.1 in the Appendix. We have to check that if $\{\varphi_k\}_k$ is a sequence as in item i) of Theorem 8.1.1 then there exists $\ell \in L^\infty(\mu)$ such that for every $h \in L^1(\mu)$,

$$\int_{\mathbb{R}^m} h\nabla\varphi_k \cdot w \,\mathrm{d}\mu \to \int_{\mathbb{R}^m} h\ell \,\mathrm{d}\mu\,.$$

Clearly, we can assume that $\varphi_k(0) = 0$ for every k. Equivalently, we have to prove that

$$\int_{\mathsf{X}} h \circ f \nabla\varphi_k(f)(\nu_f \cdot v)\,\mathrm{d}|\mathrm{D}f|\llcorner G \to \int_{\mathsf{X}} h\circ f \ell\circ f |\nu_f \cdot v|\,\mathrm{d}|\mathrm{D}f|\llcorner G\,,$$

where $h\circ f \in L^1(|\mathrm{D}f|\llcorner G)$. As also $\varphi_k \in C^1(\mathbb{R}^m)$, by Proposition 4.3.4 we have that

$$\int_{\mathsf{X}} h \circ f \nabla\varphi_k(f)(\nu_f \cdot v)\,\mathrm{d}|\mathrm{D}f|\llcorner G = \int_{\mathsf{X}} h\circ f \nu_{\varphi_k\circ f} \cdot v\,\mathrm{d}|\mathrm{D}(\varphi_k\circ f)|\llcorner G,$$

which is well posed, since $|\mathrm{D}(\varphi_k \circ f)| \le L|\mathrm{D}f|$ for every k, where $L \in (0,\infty)$ denotes the Lipschitz constant of the functions in $\{\varphi_k\}_k$. Also, $|\mathrm{D}(\varphi\circ f)| \le L|\mathrm{D}f|$. For every $\varepsilon > 0$, take $h_\varepsilon \in \mathrm{LIP}_{\mathrm{bs}}(\mathsf{X})$ such that

$$\|h\circ f - h_\varepsilon\|_{L^1(|\mathrm{D}f|)} < \varepsilon,$$

where we understand $h\circ f = 0$ $|\mathrm{D}f|$-a.e. on $\mathsf{X}\setminus G$. By Theorem 4.1.1 (with the usual interpretation of the integrals involving $\mathrm{div}(h_\varepsilon v)$ given by Remark 2.3.6),

$$\begin{aligned}-\int_{\mathsf{X}} h_\varepsilon \nu_{\varphi_k\circ f}\cdot v\,\mathrm{d}|\mathrm{D}(\varphi_k\circ f)| &= \int_{\mathsf{X}} \varphi_k\circ f \mathrm{div}(h_\varepsilon v)\,\mathrm{d}\mathsf{m}\\ \to \int_{\mathsf{X}} \varphi\circ f\mathrm{div}(h_\varepsilon v)\,\mathrm{d}\mathsf{m} &= -\int_{\mathsf{X}} h_\varepsilon\nu_{\varphi\circ f}\cdot v\,\mathrm{d}|\mathrm{D}(\varphi\circ f)|\,.\end{aligned}$$

Now, we have that

$$\left|\int_{\mathsf{X}} (h\circ f - h_\varepsilon)\nu_{\varphi_k\circ f}\cdot v\,\mathrm{d}|\mathrm{D}(\varphi_k\circ f)|\right| \le L\|v\|_{L^\infty(T\mathsf{X})}\|h\circ f - h_\varepsilon\|_{L^1(|\mathrm{D}f|)} \le L\|v\|_{L^\infty(T\mathsf{X})}\varepsilon,$$

and a similar estimate holds for φ in place of φ_k. Then we see that

$$\begin{aligned}\int_{\mathsf{X}} h\circ f\nabla\varphi_k(f)(\nu_f\cdot v)\,\mathrm{d}|\mathrm{D}f|\llcorner G &\to \int_{\mathsf{X}} h\circ f\nu_{\varphi\circ f}\cdot v\,\mathrm{d}|\mathrm{D}(\varphi\circ f)|\llcorner G\\ &= \int_{\mathbb{R}^m} h(\nu_{\varphi\circ f}\cdot v)\circ f^{-1}\frac{\mathrm{d}|\mathrm{D}(\varphi\circ f)|}{\mathrm{d}|\mathrm{D}f|}\circ f^{-1}\,\mathrm{d}\mu\,.\end{aligned}$$

This provides the existence of the sought $\ell \in L^\infty(\mu)$.

Therefore we can apply Theorem 8.1.1. It follows that if φ is as in the statement, then φ is differentiable in direction w μ-a.e. In other words, at $|\mathrm{D}f|\llcorner G$-a.e. x, φ is differentiable at $f(x)$ in direction $(\nu_f\cdot v)(x)$.

Take now $g \in L^1(|\mathrm{D}f|\llcorner G)$. We approximate φ with a sequence $\{\varphi_k\}_k$ as in Lemma 8.1.2. Using Proposition 4.3.4, we see that for every k

$$\int_{\mathsf{X}} g\nu_{\varphi_k\circ f}\cdot v\,\mathrm{d}|\mathrm{D}(\varphi_k\circ f)|\llcorner G = \int_{\mathsf{X}} g\nabla\varphi_k(f)(\nu_f\cdot v)\,\mathrm{d}|\mathrm{D}f|\llcorner G\,.$$

Using dominated convergence to deal with the right hand side and by the very same computations as above to deal with the left hand side, we prove that (4.3.12) holds for φ, as g was arbitrary. □

Having Lemma 4.3.7 at our disposal, the proof of Theorem 4.3.6 is rather classical.

Proof of Theorem 4.3.6. Denote by n the essential dimension of X. We start from a couple of reductions, as in the proof of Proposition 4.3.4. By Remark 4.1.2, we see that we can assume $l = 1$ with no loss of generality (see e.g. the second part of the proof of Proposition 4.2.12). The jump part is dealt as in the proof of Proposition 4.3.1, building upon Proposition 4.2.12 and Lemma 4.2.9.

We prove now (4.3.11) and the differentiability statement. We first show that for $|\mathrm{D}f|$-a.e. $x \notin J_f$, φ is differentiable at $\bar{f}(x)$ with respect to V. Recalling the construction of V in Lemma 4.3.5 (in particular, Theorem 2.2.21), it is enough to show this claim on a Borel subset A on which we have an orthonormal basis of $L^0_{\mathrm{Cap}}(T\mathsf{X})$, say $\{v_1, \dots, v_k\} \subseteq \mathrm{TestV}(\mathsf{X})$: namely, we have to show differentiability at $\bar{f}(x)$ with respect to $\mathrm{span}\big(\{\nu_f \cdot v_i\}_{i=1,\dots,k}\big)$.

By Lemma 4.3.7, if $v \in \mathrm{TestV}(\mathsf{X})$, for $|\mathrm{D}f|$-a.e. $x \in A \setminus J_f$ it holds that φ is differentiable at $\bar{f}(x)$ in direction $\nu_f \cdot v$. Therefore, for $|\mathrm{D}f|$-a.e. $x \in A \setminus J_f$ φ is differentiable at $\bar{f}(x)$ in every direction contained in $\mathrm{span}_{\mathbb{Q}}\big(\{\nu_f \cdot v_i\}_{i=1,\dots,k}\big)$. Lemma 4.3.7 again shows that the differential on $\mathrm{span}_{\mathbb{Q}}\big(\{\nu_f \cdot v_i\}_{i=1,\dots,k}\big)$ is linear, up to discarding a set of null $|\mathrm{D}f| \llcorner (A \setminus J_f)$ measure. It is then classical to infer from this the conclusion. □

4.4 Bibliographical notes

We recall that, as already mentioned in Section 3.6, the topic of this manuscript is to generalize the classical results about functions of bounded variation to the framework of (finite dimensional) RCD spaces. We refer to Section 3.6 for a list of general references on the topic, in the framework of Euclidean spaces. For what concerns the calculus rules, there are two more advanced results: the Vol'pert (who gives the name to the Vol'pert averaged superposition, that is the form in which we state (4.3.1) and (4.3.6)) chain rule formula, proved in the smooth setting in [125] (see also [126]) and the general chain rule formula, proved in the smooth setting in [12].

Most of the material is taken from [42, 43]. Some of the proofs have been revised and became shorter and more transparent, for example, the proof of Theorem 4.3.4 benefits from the separate treatment on the jump part (thanks to Proposition 4.2.12).

As stated during its proof, the bulk of Theorem 4.1.1 is already present in [51]. There, the authors treated only characteristic functions of sets of finite perimeter and finite measure on a finite dimensional RCD space. Our improvement is in the direction of treating more general functions (i.e. vector valued functions of bounded variation) and also by considering possibly infinite dimensional RCD spaces. The first improvement is obtained by a rather soft argument, noticing that in [51] the fact that the function is a characteristic function is not really used. For what concerns the possibility of treating the infinite dimensional case, there were only two ingredients missing in [51]. The first one is the fact that one needs total variations to be absolute continuous with respect to the 2-Sobolev capacity. In [51, Lemma 1.10 and Theorem 1.12], the authors showed that, on finite dimensional spaces,

$$|\mathrm{D}f| \ll \mathcal{H}^h \ll \mathrm{Cap} \qquad \text{for every } f \in \mathrm{BV}(\mathsf{X})$$

(actually, they showed the result for characteristic functions, but, thanks to coarea, there is no difference). This chain of relations is crucial in the subsequent development of the theory (and is one of the major reasons why we need to work on finite dimensional spaces to obtain the most refined results), but really needed the fact that the space is finite dimensional, to use its property of being PI. Hence, to state Theorem 4.1.1 in the infinite dimensional case, we had to give a different proof of the fact that

$$|\mathrm{D}f| \ll \mathrm{Cap} \qquad \text{for every } f \in \mathrm{BV}(\mathsf{X}),$$

see Theorem 2.3.7. The second ingredient is the validity of the Bakry–Émery estimate in BV in the infinite dimensional setting, Proposition 2.3.17. Such result was already stated in [86, Remark 3.5], but only for *proper* $\mathrm{RCD}(K,\infty)$ spaces. The properness assumption can be immediately removed thanks to the approximation result of [71].

It is worth remarking that Theorem 4.1.1 can be proved also with a completely different technique, relying on the language of local vector measures and an abstract Riesz's Theorem, see [44]. A local vector measure is the non-Euclidean analogue of a vector valued measure and is defined as a map from the Borel subsets of a Polish space to the dual of a suitable normed module. This theory experiences an improvement in the setting of RCD spaces (rather than Polish spaces) and allows the abstract construction of a local vector measure $\mathrm{D}f$ satisfying

$$\sum_{i=1}^{m} \int f_i \mathrm{div} v_i = -v \cdot \mathrm{D}f(\mathsf{X}) \qquad \text{for every } v = (v_1, \dots, v_m) \in (\mathcal{QC}^\infty(T\mathsf{X}) \cap D(\mathrm{div}))^m.$$

Also, as soon as one has at disposal the language of local vector measures, Section 4.1.1 becomes unnecessary: the operations described there can be seen as operations among local vector measures. In particular (with the language of [44]) Definition 4.1.4 trivially identifies $\nu\mu$ with the local vector measure whose polar decomposition is $\nu\mu$. We have decided not to include the material of [44] as local vector measures are not strictly necessary for the development of this theory.

Proposition 4.2.4 is not present in [42, 43], but its statement is in [50], with a different proof. Here we exploit our tools to give a more direct proof, that avoids the blow-up argument of [50]. In relation to Proposition 4.2.5, we remark that in [16, 17] the opposite procedure has been used: in these papers, the essential boundary is recognized through the short-time behaviour of heat semigroup. Half of Proposition 4.2.12 is taken from [42] (namely (4.2.9) and (4.2.11)), whereas the other half (namely (4.2.10) and (4.2.12)), follow from the same techniques, taking into account (3.5.2), proved in [32].

Chapter 5

Cartesian Surfaces

In this chapter we extend classical results for subgraphs of functions of bounded variation in $\mathbb{R}^n \times \mathbb{R}$ to the setting of $\mathsf{X} \times \mathbb{R}$, where X is a finite dimensional RCD space. We recall here Definition 2.3.10,

$$\mathcal{G}_f := \{(x,t) \in \mathsf{X} \times \mathbb{R} : t < f(x)\},$$

and that

$$\pi^1 : \mathsf{X} \times \mathbb{R} \to \mathsf{X}, \qquad \pi^2 : \mathsf{X} \times \mathbb{R} \to \mathbb{R}$$

denote the projections onto the corresponding factors.

In particular, we give the precise expression of the push-forward onto X of the perimeter measure of the subgraph in $\mathsf{X} \times \mathbb{R}$ of a BV function on X. Moreover, in properly chosen good coordinates, we write the precise expression of the normal to the boundary of the subgraph of a BV function f with respect to the polar vector of f, and we prove change-of-variable formulae.

5.1 Main results

First, recall the notation of Section 2.3.5 and the decomposition of the total variation as in Definition 2.3.4. The first result of this chapter establishes the equivalence between local finiteness of the total variation of a function and finiteness of the perimeter of the subgraph on cylinders. Notice that it is part of the claim that local integrability is a consequence of finiteness of the perimeter of the subgraph on cylinders. Finally, we have a characterization of the total variation of the function in terms of the push-forward of the perimeter of the subgraph.

Theorem 5.1.1. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K,N)$ *space and let* $f \in L^0(\mathsf{m})$*. Then the following are equivalent:*

i) $f \in \mathrm{BV}_{\mathrm{loc}}(\mathsf{X})$*,*

ii) for every bounded set $B \subseteq \mathsf{X}$*,* $|\mathrm{D}\chi_{\mathcal{G}_f}|(B \times \mathbb{R}) < \infty$*.*

If this is the case, then

$$\pi^1_* |\mathrm{D}\chi_{\mathcal{G}_f}| = \sqrt{g_f^2 + 1}\, \mathsf{m} + |\mathrm{D}f| \llcorner (C_f \cup J_f).$$

We defer the proof of Theorem 5.1.1 to Section 5.1.2 below.

To state the next results we need to read the "components" of polar vector fields, e.g. ν_f or $\nu_{\mathcal{G}_f}$. Even though the following definition is not intrinsic, the quantities introduced will play a key role in the proof of the Rank one Theorem 6.1.2. By tensorization of the energy, the following definition is well posed.

Definition 5.1.2. Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,N)$ space of essential dimension n and let u be a good splitting map on D. Let $f\in \mathrm{BV}_{\mathrm{loc}}(\mathsf{X})$. Then we define

i) the $\mathbb{R}^n$-valued $|\mathrm{D}f|$-measurable map ν_f^u defined for $|\mathrm{D}f|$-a.e. $x\in D$ by

$$\nu_f^u(x) := ((\nu_f\,\cdot\,\nabla u^1)(x),\dots,(\nu_f\,\cdot\,\nabla u^n)(x)),$$

ii) the $\mathbb{R}^{n+1}$-valued $|\mathrm{D}\chi_{\mathcal{G}_f}|$-measurable map $\nu^u_{\mathcal{G}_f}$ defined for $|\mathrm{D}\chi_{\mathcal{G}_f}|$-a.e. $p:=(x,t)\in D\times\mathbb{R}$ by

$$\nu^u_{\mathcal{G}_f}(p) := ((\nu_{\mathcal{G}_f}\,\cdot\,\nabla u^1)(p),\dots,(\nu_{\mathcal{G}_f}\,\cdot\,\nabla u^n)(p),(\nu_{\mathcal{G}_f}\,\cdot\,\nabla \pi^2)(p)).$$

If E is set of locally finite perimeter, we write $\nu_E:=\nu_{\chi_E}$ for simplicity.

Recalling again Lemma 3.2.19, we see that domains of a countable family of good splitting maps cover X up to sets that are negligible with respect to relevant measures. Hence, in practice, there is no loss of generality in working on the domain of a fixed good splitting map. Namely, in Theorem 5.1.3 and Theorem 5.1.4 below, we are going to compare $\nu^u_{\mathcal{G}_f}$ and ν^u_f only for a single good splitting map u, on its domain D. This, however, still allows us to have a complete picture (i.e. the comparison for $|\mathrm{D}\chi_{\mathcal{G}_f}|$-a.e. (x,t)), thanks to Lemma 3.2.19 and the remark that, using the notation of Lemma 3.2.19, we have,

$$|\mathrm{D}\chi_{\mathcal{G}_f}|\left(\left(\mathsf{X}\setminus\bigcup_k D_k\right)\times\mathbb{R}\right)=0\qquad\text{for every } f\in\mathrm{BV}_{\mathrm{loc}}(\mathsf{X}),$$

which is a consequence of (3.2.12) and Proposition 2.3.12. Finally, notice that $\nu^u_{\mathcal{G}_f}$ is well-defined at $(x,\bar f(x))$ for $|\mathrm{D}f|$-a.e. $x\in D\setminus J_f$ and for m-a.e. $x\in D\setminus J_f$. This is due to (2.3.13) and (5.1.1) below, taking into account Lemma 2.3.11.

Now we state the results that link ν^u_f and $\nu^u_{\mathcal{G}_f}$, as in Definition 5.1.2. For what concerns Theorem 5.1.3, we have that, on the regular part (i.e. outside $J_f\cup C_f$) the expression is the same as for smooth maps in Euclidean spaces. Then, on the singular part (i.e. on $J_f\cup C_f$), the last component if $\nu^u_{\mathcal{G}_f}$ vanishes. As a by-product, we obtain that the singular part is identified by the vanishing of the last component of $\nu^u_{\mathcal{G}_f}$.

Theorem 5.1.3. *Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,N)$ space and let $f\in\mathrm{BV}_{\mathrm{loc}}(\mathsf{X})$. Let u be a good splitting map on D. Then, for $|\mathrm{D}\chi_{\mathcal{G}_f}|$-a.e. $(x,t)\in D\times\mathbb{R}$, it holds that*

$$\nu^u_{\mathcal{G}_f}(x,t)=\begin{cases}\left(\sqrt{\frac{1}{1+g_f^2}}\,g_f\nu^u_f,-\sqrt{\frac{1}{1+g_f^2}}\right)(x) & \text{if } x\in D\setminus(J_f\cup C_f),\\ (\nu^u_f,0)(x) & \text{if } x\in D\cap(J_f\cup C_f).\end{cases}$$

With the results above it is not hard to deduce the following "integral version".

Theorem 5.1.4. *Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,N)$ space of essential dimension n and let $f\in\mathrm{BV}_{\mathrm{loc}}(\mathsf{X})$. Let u be a good splitting map on D. Let also $\varphi: D\times\mathbb{R}\to\mathbb{R}$ be a bounded Borel function. Then* *i) for every $i = 1, \dots, n$,*

$$\int_{(D\setminus J_f)\times\mathbb{R}} \varphi(x,t)\big(\nu^u_{\mathcal{G}_f}(x,t)\big)_i \,\mathrm{d}|\mathrm{D}\chi_{\mathcal{G}_f}|(x,t) = \int_{D\setminus J_f} \varphi(x,\bar f(x))\big(\nu^u_f(x)\big)_i \,\mathrm{d}|\mathrm{D}f|(x)\,,$$

ii) it holds

$$\int_{(D\setminus J_f)\times\mathbb{R}} \varphi(x,t)\big(\nu^u_{\mathcal{G}_f}(x,t)\big)_{n+1} \,\mathrm{d}|\mathrm{D}\chi_{\mathcal{G}_f}|(x,t) = -\int_{D\setminus J_f} \varphi(x,\bar f(x))\,\mathrm{d}\mathsf{m}(x)\,,$$

iii) for every $i = 1, \dots, n$,

$$\begin{aligned}&\int_{(D\cap J_f)\times\mathbb{R}} \varphi(x,t)\big(\nu^u_{\mathcal{G}_f}(x,t)\big)_i \,\mathrm{d}|\mathrm{D}\chi_{\mathcal{G}_f}|(x,t)\\ &\qquad= \int_{D\cap J_f} \big(\nu^u_f(x)\big)_i \Theta_n(\mathsf{m},x)\int_{f^\wedge(x)}^{f^\vee(x)} \varphi(x,t)\mathrm{d}t\,\mathrm{d}\mathcal{H}^{n-1}(x)\,,\end{aligned}$$

iv) it holds

$$\int_{(D\cap J_f)\times\mathbb{R}} \varphi(x,t)\big(\nu^u_{\mathcal{G}_f}(x,t)\big)_{n+1} \,\mathrm{d}|\mathrm{D}\chi_{\mathcal{G}_f}|(x,t) = 0.$$

We defer the proof of Theorem 5.1.3 and Theorem 5.1.4 to Section 5.1.2 below.

5.1.1 Auxiliary results

This section contains the auxiliary results that will be needed to prove Theorem 5.1.1, Theorem 5.1.3 and Theorem 5.1.4. The first technical result establishes the absolute continuity $\mathcal{H}^n \llcorner \mathcal{R}^*_n(\mathsf{X}) \ll \pi^1_*|\mathrm{D}\chi_{\mathcal{G}_f}|$. In an imprecise way, this means that every point of X is the projection of some point contained in the reduced boundary of the subgraph.

Lemma 5.1.5. *Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,N)$ space of essential dimension n and let $f \in \mathrm{BV}(\mathsf{X})$. Then*

$$\mathsf{m} \ll \mathcal{H}^n \llcorner \mathcal{R}^*_n(\mathsf{X}) \ll \pi^1_*|\mathrm{D}\chi_{\mathcal{G}_f}|. \tag{5.1.1}$$

Proof. The first absolute continuity of (5.1.1) is due to the structure theory of RCD spaces, see (2.2.14). By Proposition 2.3.12, $\mathcal{G}_f \subseteq \mathsf{X}\times\mathbb{R}$ is a set of locally finite perimeter. By (3.5.2) (see also Remark 3.5.2), we have that

$$|\mathrm{D}\chi_{\mathcal{G}_f}| = \Theta_{n+1}(\mathsf{m}\otimes\mathcal{L}^1,\,\cdot\,)\mathcal{H}^n \llcorner (\partial^*\mathcal{G}_f \cap \mathcal{R}^*_{n+1}(\mathsf{X}\times\mathbb{R})).$$

Now, notice the elementary fact $\mathcal{R}^*_{n+1}(\mathsf{X}\times\mathbb{R}) = \mathcal{R}^*_n(\mathsf{X})\times\mathbb{R}$. Take $B \subseteq \mathcal{R}^*_n(\mathsf{X})$, assume that $\pi^1_*|\mathrm{D}\chi_{\mathcal{G}_f}|(B) = 0$, then $\mathcal{H}^n(\partial^*\mathcal{G}_f \cap (\pi^1)^{-1}(B)) = 0$. Therefore, as the projection is Lipschitz, $\mathcal{H}^n\big(\pi_1(\partial^*\mathcal{G}_f \cap (\pi^1)^{-1}(B))\big) = 0$ so that by Lemma 2.3.11 and the fact that $-\infty < f^\wedge \le f^\vee < +\infty$ for $\mathcal{H}^n$-a.e. $x \in \mathcal{R}^*_n(\mathsf{X})$, we infer that $\mathcal{H}^n\big(\pi_1((\pi^1)^{-1}(B))\big) = 0$, so that $\mathcal{H}^n(B) = 0$. □

Recall that Definition 5.1.2 involves both an object defined on X (ν_f^u) and an object defined on $\mathsf{X}\times\mathbb{R}$ ($\nu_{\mathcal{G}_f}^u$), where the first one is defined in terms of a good splitting map u, and the second one in terms of $(u\circ\pi_1,\pi_2)$. We show that also the second map is a good splitting map, and also record that the relation among the matrices given by Lemma 3.2.14 for u and for $(u\circ\pi_1,\pi_2)$ is the trivial one. Namely, the rotation that we have to apply to u to obtain a system of good coordinates induces naturally a rotation of $(u\circ\pi_1,\pi_2)$ to obtain a system of good coordinates (see also Proposition 3.2.20).

Remark 5.1.6. Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an RCD space of essential dimension n and recall Definition 3.2.12. Let $u: B_{2r}(x)\to\mathbb{R}^n$ be a good η-splitting map on $D\subseteq B_r(x)$. Notice that, by tensorization of the energy, $(u\circ\pi^1,\pi^2): B_{2r}(x,t)\subseteq \mathsf{X}\times\mathbb{R}$ is a good η-splitting map on $\tilde D:=(D\times\mathbb{R})\cap B_r(x,t)$, for any $t\in\mathbb{R}$. Also, if A is the matrix valued Borel map given by Lemma 3.2.14 for u and $\tilde A$ is the matrix valued Borel map given by Lemma 3.2.14 for $(u\circ\pi^1,\pi^2)$, then

$$\tilde A_{i,j}(y,s)=\begin{cases}A_{i,j}(y) & \text{for } i,j\in 1,\dots,n,\\ 0 & \text{for } i=1,\dots,n \text{ and } j=n+1,\\ 0 & \text{for } i=n+1 \text{ and } j=1,\dots,n,\\ 1 & \text{for } i=j=n\end{cases}\tag{5.1.2}$$

for every $(y,s)\in\tilde D$. Hence the map $(y,s)\mapsto\tilde A(y,s)$ is independent of s so that we will assume it to be defined on $D\times\mathbb{R}$. ■

Given a good splitting map on D, the following proposition selects a "nice" subset of D, D_f, that is big enough to describe f, in the sense that the remaining part is seen only by $|\mathrm{D}f|^j$, but satisfies additional convenient properties. We will denote by A the matrix valued Borel map given by Lemma 3.2.14 for a good splitting map u. In item v), we are going to exploit the matrix valued Borel map A and $\tilde A$ as in Remark 5.1.6.

Proposition 5.1.7. *Let $f\in\mathrm{BV}_{\mathrm{loc}}(\mathsf{X})$ and let u be a good splitting map on D. Then there exists a Borel set $D_f\subseteq D$ satisfying the following properties:*

i) $|\mathrm{D}f|^c(D\setminus D_f)=0$ *and* $\mathsf{m}(D\setminus D_f)=0$.

ii) $|\mathrm{D}\chi_{\mathcal{G}_f}|((D\setminus(D_f\cup J_f))\times\mathbb{R})=0$.

iii) $D_f\subseteq\mathcal{R}_n^*(\mathsf{X})\setminus J_f$ *and* $\mathcal{F}\mathcal{G}_f\cap(D_f\times\mathbb{R})=(\mathrm{Id}_{\mathsf{X}},\bar f)(D_f)$. *Hence, for every* $x\in D_f$, $(x,\bar f(x))\in\mathcal{F}\mathcal{G}_f$, *in particular,* $\bar f(x)\in\mathbb{R}$.

iv) For every $x\in D_f$, $(x,\bar f(x))$ *is a Lebesgue point for* $\nu_{\mathcal{G}_f}^u$ *with respect to* $|\mathrm{D}\chi_{\mathcal{G}_f}|$.

v) Given any $x\in D_f$,

$$v_{(x,\bar f(x))}:=(A(x)u\circ\pi^1,\pi^2)=\tilde A(x,\bar f(x))(u\circ\pi^1,\pi^2)$$

is a system of good coordinates for $\mathcal{G}_f$ *at* $(x,\bar f(x))$ *and moreover,*

$$\bar\nu_{\mathcal{G}_f}^{v_{(x,\bar f(x))}}(x,\bar f(x))=\tilde A(x,\bar f(x))\nu_{\mathcal{G}_f}^u(x,\bar f(x)),\tag{5.1.3}$$

where we took the Lebesgue value for $\nu_{\mathcal{G}_f}^u$.

Proof. We will build sets $D \supseteq D_1 \supseteq D_2 \supseteq D_3 \supseteq D_4 =: D_f$. First, we set $D_1 := D \setminus J_f$. Clearly, D_1 still satisfies items *i*) and *ii*).

Take now $N \subseteq \partial^* \mathcal{G}_f \cap ((\mathsf{X} \setminus J_f) \times \mathbb{R})$. Notice that if N is $|\mathrm{D}\chi_{\mathcal{G}_f}|$-negligible, then $\pi^1(N)$ is both $|\mathrm{D}f|^c$-negligible and m-negligible, by Proposition 2.3.12 and Lemma 5.1.5, taking into account also Lemma 2.3.11 that implies $(\pi^1)^{-1}(\pi^1(N)) = N$. Hence removing $\pi^1(N)$ still leaves items *i*) and *ii*) unaffected. We set then $D_2 := D_1 \cap \pi^1(\mathcal{F}\mathcal{G}_f)$. Corollary 3.3.2 ensures that items *i*) and *ii*) are still satisfied whereas Lemma 2.3.11 ensures that item *iii*) holds. We set now

$$D_3 := \{x \in D_2 : (x, \bar f(x)) \text{ is a Lebesgue point for } \nu^u_{\mathcal{G}_f} \text{ with respect to } |\mathrm{D}\chi_{\mathcal{G}_f}|\}.$$

The asymptotic doubling property of $|\mathrm{D}\chi_{\mathcal{G}_f}|$ ensures that items *i*) and *ii*) are still satisfied whereas *iv*) is satisfied thanks to this choice. We set now

$$D_4 := \{x \in D_3 : \text{the conclusions of Proposition 3.2.20 (for } (u \circ \pi^1, \pi^2) \text{ and } \mathcal{G}_f) \text{ hold at } (x, \bar f(x))\},$$

by Proposition 3.2.20 (and a covering argument, as a priori good splitting maps are defined on balls) items *i*) and *ii*) are still satisfied whereas item *v*) is satisfied thanks to this choice. □

The following is a simple consequence of the definition of the set D_f.

Remark 5.1.8. Let D_f be as in Proposition 5.1.7, we keep the same notation. Let $x \in D_f$. By item *iii*) (and the definition of $\mathcal{F}\mathcal{G}_f$), $\mathcal{G}_f$ satisfies the conclusions of Proposition 2.3.3 at $(x, \bar f(x))$. Therefore, also the conclusion of Theorem 3.2.21 is in place, in particular, if

$$(\mathbb{R}^{n+1}, \mathsf{d}_e, \underline{\mathcal{L}}^{n+1}, 0, H) \in \mathrm{Tan}_{(x,\bar f(x))}(\mathsf{X} \times \mathbb{R}, \mathsf{d}_{\mathsf{X}\times\mathbb{R}}, \mathsf{m} \otimes \mathcal{L}^1, \mathcal{G}_f),$$

then

$$H = \{z \in \mathbb{R}^{n+1} : z \cdot \bar\nu^{v_{(x,\bar f(x))}}_{\mathcal{G}_f}(x, \bar f(x)) \geq 0\} = \{z \in \mathbb{R}^{n+1} : z \cdot \tilde A(x, \bar f(x)) \nu^u_{\mathcal{G}_f}(x, \bar f(x)) \geq 0\}, \quad (5.1.4)$$

provided that the coordinates in $\mathbb{R}^{n+1}$ are chosen as limits of appropriate rescalings of the maps $v_{(x,\bar f(x))}$ (Remark 3.2.17) (in a suitable realization). Also, notice that (5.1.3) implies that

$$\big(\nu^u_{\mathcal{G}_f}(x, \bar f(x))\big)_{n+1} = \big(\bar\nu^{v_{(x,\bar f(x))}}_{\mathcal{G}_f}(x, \bar f(x))\big)_{n+1} \in [-1, 1]. \quad (5.1.5)$$

We are going to use these properties throughout. ■

The next proposition is the main technical tool of this section. It shows that on D_f the last component of $\nu^u_{\mathcal{G}_f}$ identifies the Radon–Nikodym derivative of the push forward of the perimeter of the subgraph with respect to the reference measure. This is our first bridge between quantities defined in terms of the subgraph (in $\mathsf{X} \times \mathbb{R}$) and quantities defined in terms of the $\mathrm{BV}_{\mathrm{loc}}$ function (in X). Notice that the claim on the Cantor part is that the normal to the subgraph is horizontal. The proof is a careful blow-up analysis, that uses new techniques as well as very classical techniques of geometric measure theory.

Proposition 5.1.9. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space of essential dimension* n *and let* $f \in \mathrm{BV}_{\mathrm{loc}}(\mathsf{X})$. *Let also* u *be a good splitting map on* D *and let* $D_f \subseteq D \setminus J_f$ *be given by Proposition 5.1.7. Then,*

$$\lim_{r \searrow 0} \frac{\pi^1_* |\mathrm{D}\chi_{\mathcal{G}_f}|(B_r(x))}{\mathsf{m}(B_r(x))} = -\big(\nu^u_{\mathcal{G}_f}(x, \bar f(x))\big)^{-1}_{n+1} \qquad \text{for } |\mathrm{D}\chi_{\mathcal{G}_f}|\text{-a.e. } (x, \bar f(x)) \in D_f \times \mathbb{R}, \quad (5.1.6)$$

where the right hand side has to be understood as $+\infty$ *where* $\big(\nu^u_{\mathcal{G}_f}(x, \bar f(x))\big)_{n+1} = 0$.

In particular, we can compute the Radon–Nikodym derivative as follows:

$$\frac{\mathrm{d}\pi^1_* |\mathrm{D}\chi_{\mathcal{G}_f}|}{\mathrm{d}\mathsf{m}} = -\big(\nu^u_{\mathcal{G}_f}(x, \bar f(x))\big)^{-1}_{n+1} \qquad \text{for } \mathsf{m}\text{-a.e. } x \in D_f. \quad (5.1.7)$$

Proof. First, if $x \in \mathcal{R}_n^*(\mathsf{X})$, we have,

$$\begin{aligned}\Theta_n(\mathsf{m},x) = \lim_{r\searrow 0}\frac{\mathsf{m}(B_r(x))}{\omega_n r^n} &= \lim_{r\searrow 0}\frac{(\mathsf{m}\otimes\mathcal{L}^1)(B_r(x,\bar f(x)))}{\omega_{n+1}r^{n+1}} = \Theta_{n+1}(\mathsf{m}\otimes\mathcal{L}^1,(x,\bar f(x))),\\ \lim_{r\searrow 0}\frac{\mathsf{m}(B_r(x))}{C_x^r} &= \omega_n\frac{(n+1)}{\omega_n},\\ \lim_{r\searrow 0}\frac{r\mathsf{m}(B_r(x))}{C^r_{(x,\bar f(x))}} &= \omega_n\frac{(n+2)}{\omega_{n+1}}.\end{aligned} \tag{5.1.8}$$

Indeed, this can be proved easily taking into account weak convergence of measures and using Fubini's Theorem.

Fix $p := (x,\bar f(x))$ with $x \in D_f$. Let $\{r_i\}_i \subseteq (0,\infty)$ with $r_i \searrow 0$. As $x \in D_f \subseteq \mathcal{R}_n^*(\mathsf{X})$, up to passing to a subsequence that we are still going to call $\{r_i\}_i$, we have the convergence

$$\big(\mathsf{X}, r_i^{-1}\mathsf{d}_{\mathsf{X}\times\mathbb{R}}, \mathsf{m}_x^{r_i}, x\big) \to (\mathbb{R}^n,\mathsf{d}_e,\underline{\mathcal{L}}^n,0),$$

in a realization $(\mathsf{Z},\mathsf{d}_\mathsf{Z})$, where $(\mathsf{Z},\mathsf{d}_\mathsf{Z})$ is a proper metric space. Hence also

$$\big(\mathsf{X}\times\mathbb{R}, r_i^{-1}\mathsf{d}_{\mathsf{X}\times\mathbb{R}}, (\mathsf{m}\otimes\mathcal{L}^1)_p^{r_i}, p\big) \to (\mathbb{R}^{n+1},\mathsf{d}_e,\underline{\mathcal{L}}^{n+1},0)$$

in the realization $(\mathsf{Z}\times\mathbb{R},\mathsf{d}_{\mathsf{Z}\times\mathbb{R}})$. We use Remark 3.3.3 together with Proposition 5.1.7 to obtain that, up to passing to a further subsequence that we are still going to call $\{r_i\}_i$,

$$\big(\mathsf{X}\times\mathbb{R}, r_i^{-1}\mathsf{d}_{\mathsf{X}\times\mathbb{R}}, (\mathsf{m}\otimes\mathcal{L}^1)_p^{r_i}, p, \mathcal{G}_f\big) \to (\mathbb{R}^{n+1},\mathsf{d}_e,\underline{\mathcal{L}}^{n+1},0,H),$$

in the realization $(\mathsf{Z}\times\mathbb{R},\mathsf{d}_{\mathsf{Z}\times\mathbb{R}})$. Passing to a further subsequence that we are still going to call $\{r_i\}_i$, we choose coordinates in $\mathbb{R}^{n+1}$ as limits of rescalings of $v_{(x,\bar f(x))}$ (Remark 3.2.17), hence H is as in (5.1.4).

Then we compute, for any $M \in (0,\infty)$ (notice that $|\mathrm{D}\chi_H|\big(\partial(B_1^{\mathbb{R}^n}(0)\times B_M^{\mathbb{R}}(0))\big) = 0$)

$$\begin{aligned}\lim_i \frac{r_i|\mathrm{D}\chi_{\mathcal{G}_f}|\big(B_{r_i}(x)\times B_{r_iM}(\bar f(x)\big)}{C_p^{r_i}} &= \lim_i |\mathrm{D}\chi_{(\mathcal{G}_f)_i}|\big(B_1^i(x)\times B_M^i(\bar f(x))\big)\\ &= |\mathrm{D}\chi_H|\big(B_1^{\mathbb{R}^n}(0)\times B_M^{\mathbb{R}}(0)\big),\end{aligned}$$

so that, by (5.1.8) and (2.2.11),

$$\lim_i \frac{|\mathrm{D}\chi_{\mathcal{G}_f}|\big(B_{r_i}(x)\times B_{r_iM}(\bar f(x)\big)}{r_i^n} = \Theta_n(\mathsf{m},x)\omega_n^{-1}\mathcal{H}^n\big(\partial H\cap(B_1^{\mathbb{R}^n}(0)\times B_M^{\mathbb{R}}(0))\big).$$

In what follows, we are going to use the Lebesgue value for $\nu^u_{\mathcal{G}_f}(p)$.

Step 1: the case $\big(\nu^u_{\mathcal{G}_f}(p)\big)_{n+1} = 0$. Then $H = H'\times\mathbb{R}$ for some half-space $H'\subseteq\mathbb{R}^n$, so that

$$\liminf_i \frac{\pi^1_*|\mathrm{D}\chi_{\mathcal{G}_f}|(B_{r_i}(x))}{r_i^n} \ge \Theta_n(\mathsf{m},x)\omega_n^{-1}\mathcal{H}^n\big(\partial(H'\times\mathbb{R})\cap(B_1^{\mathbb{R}^n}(0)\times B_M^{\mathbb{R}}(0)\big) = \omega_n^{-1}2M\omega_{n-1}.$$

Being M arbitrary,

$$\liminf_i \frac{\pi^1_*|\mathrm{D}\chi_{\mathcal{G}_f}|(B_{r_i}(x))}{r_i^n} = +\infty$$

and, being the sequence $\{r_i\}_i$ chosen before arbitrary,

$$\lim_{r\searrow 0}\frac{\pi^1_*|\mathrm{D}\chi_{\mathcal{G}_f}|(B_r(x))}{r^n}=+\infty.$$

Step 2: non-positivity of $(\nu^u_{\mathcal{G}_f}(p))_{n+1}$**.** Take $x\in D_f$ such that

$$(\nu^u_{\mathcal{G}_f}(x,\bar f(x)))_{n+1}\neq 0 \tag{5.1.9}$$

and set, as before, $p:=(x,\bar f(x))$. Let

$$\pm B_\varepsilon:=B^{\mathsf{Z}}_\varepsilon(0^{\mathbb{R}^n})\times B^{\mathbb{R}}_\varepsilon(\pm 1^{\mathbb{R}})\subseteq \mathsf{Z}\times\mathbb{R}$$

for $\varepsilon\in(0,1)$ small enough so that $(\pm B_\varepsilon)\cap\partial H\neq\emptyset$, that exists by (5.1.9). Now, by convergence in L^1_{loc} and Fubini's Theorem

$$\begin{aligned}\underline{\mathcal{L}}^{n+1}(H\cap(\pm B_\varepsilon))&=\lim_i(\mathsf{m}\otimes\mathcal{L}^1)^{r_i}_p((\mathcal{G}_f)_i\cap\pm B_\varepsilon)\\&=\lim_i(\mathsf{m}\otimes\mathcal{L}^1)^{r_i}_p\big((\mathcal{G}_f)_i\cap(B^i_\varepsilon(x)\times B^i_\varepsilon(\pm r_i))\big)\\&=\lim_i\frac{(\mathsf{m}\otimes\mathcal{L}^1)\big(\mathcal{G}_f\cap(B_{\varepsilon r_i}(x)\times B_{\varepsilon r_i}(\pm r_i))\big)}{C^{r_i}_p}\\&=\lim_i\frac{1}{C^{r_i}_p}\int_{B_{\varepsilon r_i}(x)}\mathcal{H}^1\big((\{z\}\times B_{\varepsilon r_i}(\pm r_i))\cap\mathcal{G}_f\big)\mathrm{d}\mathsf{m}^i(z).\end{aligned}$$

Therefore, recalling the definition of $\mathcal{G}_f$ we obtain

$$\underline{\mathcal{L}}^{n+1}(H\cap(-B_\varepsilon))-\underline{\mathcal{L}}^{n+1}(H\cap B_\varepsilon)\geq 0,$$

whence the claim follows.

Step 3: the case $\big(\nu^u_{\mathcal{G}_f}(p)\big)_{n+1}\neq 0$**.** We set (by (5.1.5))

$$\alpha:=\big(\nu^u_{\mathcal{G}_f}(p)\big)_{n+1}\in[-1,1]\setminus\{0\}.$$

Set also

$$\beta:=\sqrt{1-\alpha^2}\vee 1/2\in(0,1).$$

An immediate computation yields that $\partial H\cap B^{\mathbb{R}^{n+1}}_1(0)\subseteq B^{\mathbb{R}^n}_1(0)\times B^{\mathbb{R}}_\beta(0)$ whence

$$\lim_i\frac{r_i|\mathrm{D}\chi_{\mathcal{G}_f}|\big(B_{r_i}(p)\setminus(\mathsf{X}\times B_{\beta r_i}(\bar f(x)))\big)}{C^{r_i}_p}=\lim_i|\mathrm{D}\chi_{(\mathcal{G}_f)_i}|\big(B^i_1(p)\setminus(\mathsf{X}\times B^i_\beta(\bar f(x)))\big)=0,$$

hence, by arbitrariness of the sequence $\{r_i\}_i$ chosen before and by (5.1.8),

$$\lim_{r\searrow 0}\frac{|\mathrm{D}\chi_{\mathcal{G}_f}|\big(B_r(p)\setminus(\mathsf{X}\times B_{\beta r}(\bar f(x)))\big)}{r^n}=0. \tag{5.1.10}$$

Now, for $\gamma\in(0,\infty)$ and $(x,t)\in\mathsf{X}\times\mathbb{R}$, we denote the cone

$$C_\gamma(x,t):=\{(y,s)\in\mathsf{X}\times\mathbb{R}:\gamma\mathsf{d}(y,x)\geq|s-t|\}.$$

Set

$$\gamma := \sqrt{\frac{1+\beta}{1-\beta}} \in (1,\infty).$$

Now we claim that

$$\lim_{r\searrow 0} \frac{|\mathrm{D}\chi_{\mathcal{G}_f}|(B_r(p)\setminus C_\gamma(p))}{r^n} = 0. \tag{5.1.11}$$

This will follow from a "cube-density implies cone-density" argument exploiting (5.1.10). In order to prove the claim, fix $\delta \in (0,1)$. By (5.1.10), we can take $\tilde r_0 > 0$ small enough so that

$$\sup_{r\in(0,\tilde r_0)} \frac{|\mathrm{D}\chi_{\mathcal{G}_f}|\big(B_r(p)\setminus(\mathsf{X}\times B_{\beta r}(\bar f(x)))\big)}{r^n} < \delta. \tag{5.1.12}$$

Notice that

$$B_{\tilde r_0}(p)\setminus C_\gamma(p) \subseteq \bigcup_i B_{\tilde r_i}(p)\setminus(\mathsf{X}\times B_{\beta\tilde r_i}(\bar f(x))), \tag{5.1.13}$$

where for any $i\in\mathbb{N}$ with $i\ge 1$ we defined

$$\tilde r_i := \beta\sqrt{\frac{\gamma^2+1}{\gamma^2}}\tilde r_{i-1} = \left(\beta\sqrt{\frac{\gamma^2+1}{\gamma^2}}\right)^i \tilde r_0.$$

Given that by (5.1.12)

$$|\mathrm{D}\chi_{\mathcal{G}_f}|\big(B_{\tilde r_i}(p)\setminus(\mathsf{X}\times B_{\beta\tilde r_i}(p))\big) \le \delta\tilde r_i^n = \delta\left(\beta\sqrt{\frac{\gamma^2+1}{\gamma^2}}\right)^{ni}\tilde r_0^n,$$

it follows from the inclusion in (5.1.13) that

$$\frac{|\mathrm{D}\chi_{\mathcal{G}_f}|(B_{\tilde r_0}(p)\setminus C_\gamma(p))}{\tilde r_0^n} \le \delta\sum_i \left(\beta\sqrt{\frac{\gamma^2+1}{\gamma^2}}\right)^{ni}.$$

Then, (5.1.11) is proved, thanks to the arbitrariness of $\delta>0$ and the finiteness of the sum at the right hand side (by the definition of γ).

Step 4: making the estimate (5.1.11) **of Step 4 "set theoretic".** Let $\varepsilon>0$. We show that there exists a set $\Sigma=\Sigma_\varepsilon\subseteq(D_f\times\mathbb{R})\cap\mathcal{F}\mathcal{G}_f$ with

$$|\mathrm{D}\chi_{\mathcal{G}_f}|\Big(\big\{(x,\bar f(x))\in\mathcal{F}\mathcal{G}_f : x\in D_f, \big(\nu^u_{\mathcal{G}_f}(x,\bar f(x))\big)_{n+1}\ne 0\big\}\setminus\Sigma\Big) < \varepsilon \tag{5.1.14}$$

and such that there exist $\gamma\in(1,\infty)$ and $r_0\in(0,1)$ satisfying

$$(\Sigma\cap B_{r_0}(p))\setminus C_{2\gamma}(p) = \emptyset \qquad \text{for every } p\in\Sigma. \tag{5.1.15}$$

We do it using a standard argument, see e.g. the proof of [117, Theorem 1.6]. Take indeed $\Sigma\subseteq(D_f\times\mathbb{R})\cap\mathcal{F}\mathcal{G}_f$ satisfying (5.1.14) and

- $\bar f$ is continuous on $\pi^1(\Sigma)$

- there exists $\gamma \in (1,\infty)$ such that for any $\delta \in (0,1)$ there exists $\hat r_0 = \hat r_0(\delta) \in (0,1)$ such that, for every $r \in (0, 2\hat r_0)$ and $p \in \Sigma$,

$$\frac{|\mathrm{D}\chi_{\mathcal{G}_f}|(\Sigma \cap B_r(p))}{\Theta_{n+1}(\mathsf{m}\otimes\mathcal{L}^1,p)\omega_n r^n} \geq 1-\delta \qquad \text{and} \qquad \frac{|\mathrm{D}\chi_{\mathcal{G}_f}|\big((\Sigma\cap B_r(p))\setminus C_\gamma(p)\big)}{\Theta_{n+1}(\mathsf{m}\otimes\mathcal{L}^1,p)\omega_n r^n} \leq \delta. \tag{5.1.16}$$

This is possible thanks to Lusin's and Egorov's Theorems, taking into account (5.1.8), Remark 3.3.3, (5.1.11) of **Step 3** and an exhaustion argument, keeping in mind the fact that in (5.1.14) we are estimating the perimeter of a set of points $(x,\bar f(x))$ satisfying $\big(\nu^u_{\mathcal{G}_f}(x,\bar f(x))\big)_{n+1} \neq 0$.

We aim to show that if $\delta \in (0,1)$ is small enough (to be determined later), then this choice of Σ and $\hat r_0$ satisfies also (5.1.15). Assume now that there exist $p \in \Sigma$ and $q \in (\Sigma\cap B_{\hat r_0}(p))\setminus C_{2\gamma}(p)$. Then denoting $\tilde{\mathsf{d}} := \mathsf{d}_{\mathsf{X}\times\mathbb{R}}$ for brevity,

$$B_\rho(q) \subseteq B_{\tilde{\mathsf{d}}(p,q)+\rho}(p)\setminus C_\gamma(p), \qquad \text{where } \rho := \tilde{\mathsf{d}}(p,q)\sin(\arctan(2\gamma)-\arctan(\gamma)) > 0. \tag{5.1.17}$$

Therefore, we can estimate, by (5.1.16) for what concerns the first and last inequalities (notice that $\tilde{\mathsf{d}}(p,q)+\rho \leq 2\tilde{\mathsf{d}}(p,q) < 2\hat r_0$), by (5.1.17) for the central inequality

$$\begin{aligned}\delta &\geq \frac{|\mathrm{D}\chi_{\mathcal{G}_f}|\big((\Sigma\cap B_{\tilde{\mathsf{d}}(p,q)+\rho}(p))\setminus C_\gamma(p)\big)}{\Theta_{n+1}(\mathsf{m}\otimes\mathcal{L}^1,p)\omega_n(\tilde{\mathsf{d}}(p,q)+\rho)^n} \geq \frac{|\mathrm{D}\chi_{\mathcal{G}_f}|(\Sigma\cap B_\rho(q))}{\Theta_{n+1}(\mathsf{m}\otimes\mathcal{L}^1,p)\omega_n(\tilde{\mathsf{d}}(p,q)+\rho)^n}\\ &\geq (1-\delta)\frac{\rho^n}{(\tilde{\mathsf{d}}(p,q)+\rho)^n} = (1-\delta)\frac{\big(\sin(\arctan(2\gamma)-\arctan(\gamma))\big)^n}{\big(1+\sin(\arctan(2\gamma)-\arctan(\gamma))\big)^n},\end{aligned}$$

which leads to a contradiction provided $\delta > 0$ was chosen small enough (depending on γ), proving thus (5.1.15).

Step 5: improved blow-up argument. Let Σ, γ and $\hat r_0$ be given by **Step 4**. We prove that if $x \in \pi^1(\Sigma)$, then

$$\lim_{r\searrow 0}\frac{\pi^1_*(|\mathrm{D}\chi_{\mathcal{G}_f}|\llcorner\Sigma)(B_r(x))}{r^n} = -\Theta_n(\mathsf{m},x)\big(\nu^u_{\mathcal{G}_f}(p)\big)^{-1}_{n+1}. \tag{5.1.18}$$

Fix $x \in \pi^1(\Sigma)$ and set $p := (x,\bar f(x)) \in \Sigma$. Up to removing from Σ a $|\mathrm{D}\chi_{\mathcal{G}_f}|$-negligible subset, we can moreover assume that

$$\lim_{r\searrow 0}\frac{r|\mathrm{D}\chi_{\mathcal{G}_f}|(B_r(p)\setminus\Sigma)}{C^r_p} = 0. \tag{5.1.19}$$

Indeed, this follows from the asymptotically doubling property of $|\mathrm{D}\chi_{\mathcal{G}_f}|$, recalling (5.1.8).

Since $\bar f$ is continuous on $\pi^1(\Sigma)$, there exists $\hat r_1 \in (0,\hat r_0/\sqrt{2})$ such that $|\bar f(y)-\bar f(x)| < \hat r_0/\sqrt{2}$ for all $y \in B_{\hat r_1}(x)\cap\pi^1(\Sigma)$. By $\Sigma \subseteq (D_f\times\mathbb{R})\cap\mathcal{F}\mathcal{G}_f$, $\Sigma \subseteq \{(x,t)\in\mathsf{X}\times\mathbb{R} : t = \bar f(x)\}$, so that

$$\Sigma\cap(B_{\hat r_1}(x)\times\mathbb{R}) \subseteq \Sigma\cap B_{\hat r_0}(p) \subseteq C_{2\gamma}(x,\bar f(x)) \tag{5.1.20}$$

by (5.1.15) of **Step 4**. Now we compute, by (5.1.20),

$$\begin{aligned}\limsup_i \frac{r_i|\mathrm{D}\chi_{\mathcal{G}_f}|\llcorner\Sigma(B_{r_i}(x)\times\mathbb{R})}{C^{r_i}_p} &= \limsup_i \frac{r_i|\mathrm{D}\chi_{\mathcal{G}_f}|\llcorner\Sigma\big(B_{r_i}(x)\times B_{2\gamma r_i}(\bar f(x))\big)}{C^{r_i}_p}\\ &\leq \lim_i |\mathrm{D}\chi_{(\mathcal{G}_f)_i}|\big(B^i_1(x)\times B^i_{2\gamma}(\bar f(x))\big)\\ &= |\mathrm{D}\chi_H|(B^{\mathbb{R}^n}_1(0)\times B^{\mathbb{R}}_{2\gamma}(0)) \leq |\mathrm{D}\chi_H|(B^{\mathbb{R}^n}_1(0)\times\mathbb{R}).\end{aligned}$$

On the other hand, recalling (5.1.19) and the computation right before **Step 1**, for any $M \in (0,\infty)$,

$$\begin{aligned}\liminf_i \frac{r_i|\mathrm{D}\chi_{\mathcal{G}_f}|\llcorner\Sigma(B_{r_i}(x)\times\mathbb{R})}{C_p^{r_i}} &\ge \liminf_i \frac{r_i|\mathrm{D}\chi_{\mathcal{G}_f}|\llcorner\Sigma\big(B_{r_i}(x)\times B_{Mr_i}(\bar f(x))\big)}{C_p^{r_i}}\\ &= \liminf_i \frac{r_i|\mathrm{D}\chi_{\mathcal{G}_f}|\big(B_{r_i}(x)\times B_{Mr_i}(\bar f(x))\big)}{C_p^{r_i}}\\ &= \lim_i |\mathrm{D}\chi_{(\mathcal{G}_f)_i}|\big(B_1^i(x)\times B_M^i(\bar f(x))\big)\\ &= |\mathrm{D}\chi_H|(B_1^{\mathbb{R}^n}(0)\times B_M^{\mathbb{R}}(0)).\end{aligned}$$

Hence, by the arbitrariness of M,

$$\lim_i \frac{r_i|\mathrm{D}\chi_{\mathcal{G}_f}|\llcorner\Sigma(B_{r_i}(x)\times\mathbb{R})}{C_p^{r_i}} = |\mathrm{D}\chi_H|(B_1^{\mathbb{R}^n}(0)\times B_1^{\mathbb{R}}(0)),$$

and, being the sequence $\{r_i\}_i$ chosen before arbitrary, recalling also (5.1.8) and **Step 2**,

$$\begin{aligned}\lim_{r\searrow 0}\frac{\pi^1_*(|\mathrm{D}\chi_{\mathcal{G}_f}|\llcorner\Sigma)(B_r(x))}{r^n} &= \lim_{r\searrow 0}\frac{|\mathrm{D}\chi_{\mathcal{G}_f}|\llcorner\Sigma(B_r(x)\times\mathbb{R})}{r^n}\\ &= \Theta_n(\mathsf{m},x)\left(\omega_n\frac{n+2}{\omega_{n+1}}\right)^{-1}|\mathrm{D}\chi_H|(B_1^{\mathbb{R}^n}(0)\times\mathbb{R})\\ &= -\Theta_n(\mathsf{m},x)\left(\omega_n\frac{n+2}{\omega_{n+1}}\right)^{-1}\frac{n+2}{\omega_{n+1}}\omega_n\big(\nu^u_{\mathcal{G}_f}(p)\big)^{-1}_{n+1}\\ &= -\Theta_n(\mathsf{m},x)\big(\nu^u_{\mathcal{G}_f}(p)\big)^{-1}_{n+1},\end{aligned} \tag{5.1.21}$$

that is (5.1.18).

Step 6: proof of (5.1.6)**.** For $\varepsilon\in(0,1)$, denote by Σ_ε the set given by **Step 4** for this value of ε.

We first prove (5.1.6) for $|\mathrm{D}\chi_{\mathcal{G}_f}|$-a.e. $(x,\bar f(x))\in(D_f\cap C_f)\times\mathbb{R}$. Namely, we prove that for $|\mathrm{D}\chi_{\mathcal{G}_f}|$-a.e. $(x,\bar f(x))\in(D_f\cap C_f)\times\mathbb{R}$, then

$$\big(\nu^u_{\mathcal{G}_f}(x,\bar f(x))\big)_{n+1}=0 \qquad\text{and}\qquad \lim_{r\searrow 0}\frac{\pi^1_*|\mathrm{D}\chi_{\mathcal{G}_f}|(B_r(x))}{\mathsf{m}(B_r(x))}=\infty. \tag{5.1.22}$$

Now, by (5.1.18) of **Step 5** and [31, Theorem 2.4.3] together with an exhaustion argument, we have that

$$\pi^1_*(|\mathrm{D}\chi_{\mathcal{G}_f}|\llcorner\Sigma_\varepsilon)\ll\mathcal{H}^n\llcorner\mathcal{R}^*_n(\mathsf{X})\ll\mathsf{m},$$

hence, letting $\varepsilon\searrow 0$ along a vanishing sequence we obtain by (5.1.14) that (as $\mathsf{m}(C_f)=0$)

$$|\mathrm{D}\chi_{\mathcal{G}_f}|\Big(\big\{(x,\bar f(x))\in\mathcal{F}\mathcal{G}_f : x\in D_f\cap C_f, \big(\nu^u_{\mathcal{G}_f}(x,\bar f(x))\big)_{n+1}\ne 0\big\}\Big)=0,$$

which is the first claim in (5.1.22). Now we show that

$$\lim_{r\searrow 0}\frac{|\mathrm{D}f|(B_r(x))}{\mathsf{m}(B_r(x))}=\infty \qquad \text{for } |\mathrm{D}f|\text{-a.e. } x\in D_f\cap C_f. \tag{5.1.23}$$

This can be easily proved with a classical exhaustion and covering argument. Indeed, assume by contradiction that there exists a compact set $K\subseteq D_f\cap C_f$ with $|\mathrm{D}f|(K)>0$ and

$$\liminf_{r\searrow 0}\frac{|\mathrm{D}f|(B_r(x))}{\mathsf{m}(B_r(x))}<M$$

for some $M \in (0,\infty)$. Let also $\delta \in (0,1)$. For every $x \in K$, choose r_x such that $r_x/5 \in (0,\delta)$ and $|\mathrm{D}f|(B_{r_x}(x)) < M\mathsf{m}(B_{r_x}(x))$. Now the conclusion comes applying Vitali's covering Theorem together with the fact that $\mathsf{m}(B_\delta(K)) \to 0$ as $\delta \searrow 0$ and the local doubling property of m. Then the second claim in (5.1.22) follows, using (2.3.13) twice.

Now we prove (5.1.6) for $|\mathrm{D}\chi_{\mathcal{G}_f}|$-a.e. $(x,\bar f(x)) \in (D_f \setminus C_f)\times\mathbb{R}$. First notice that combining (5.1.1), Proposition 2.3.12 and the fact that $D_f \cap J_f = \emptyset$, we have

$$\mathsf{m}\llcorner D_f \ll (\pi^1_*|\mathrm{D}\chi_{\mathcal{G}_f}|)\llcorner (D_f\setminus C_f) \ll \mathsf{m}. \tag{5.1.24}$$

By (5.1.21) of **Step 5** and differentiation of measures (e.g. combining [95, (3.4.24) and (3.4.32)] with the doubling property of m),

$$\begin{aligned}\lim_{r\searrow 0}\frac{\pi^1_*|\mathrm{D}\chi_{\mathcal{G}_f}|(B_r(x))}{\mathsf{m}(B_r(x))} &= \lim_{r\searrow 0}\frac{\pi^1_*|\mathrm{D}\chi_{\mathcal{G}_f}|(\pi^1(\Sigma_\varepsilon)\cap B_r(x))}{\mathsf{m}(B_r(x))} = \lim_{r\searrow 0}\frac{\pi^1_*(|\mathrm{D}\chi_{\mathcal{G}_f}|\llcorner\Sigma_\varepsilon)(B_r(x))}{\mathsf{m}(B_r(x))}\\ &= -\big(\nu^u_{\mathcal{G}_f}(x,\bar f(x))\big)^{-1}_{n+1},\end{aligned}$$

for m-a.e. $x \in \pi^1(\Sigma_\varepsilon)$, being m locally doubling. By (5.1.14) and (5.1.24), if we let $\varepsilon \searrow 0$ along a sequence, we see that

$$\lim_{r\searrow 0}\frac{\pi^1_*|\mathrm{D}\chi_{\mathcal{G}_f}|(B_r(x))}{\mathsf{m}(B_r(x))} = -\big(\nu^u_{\mathcal{G}_f}(x,\bar f(x))\big)^{-1}_{n+1},$$

for m-a.e. $x \in D_f\setminus C_f$ such that $\big(\nu^u_{\mathcal{G}_f}(x,\bar f(x))\big)^{-1}_{n+1} \neq 0$. Recalling (5.1.24) again, we have proved (5.1.6) for $|\mathrm{D}\chi_{\mathcal{G}_f}|$-a.e. $(x,\bar f(x)) \in (D_f\setminus C_f)\times\mathbb{R}$ such that $\big(\nu^u_{\mathcal{G}_f}(x,\bar f(x))\big)_{n+1} \neq 0$. To conclude, it is enough to notice that by **Step 1**, (5.1.6) is satisfied for $|\mathrm{D}\chi_{\mathcal{G}_f}|$-a.e. $(x,\bar f(x)) \in D_f\times\mathbb{R}$ such that $\big(\nu^u_{\mathcal{G}_f}(x,\bar f(x))\big)_{n+1} = 0$.

Step 7: proof of (5.1.7). First recall that by (2.3.13) it holds

$$(\pi^1_*|\mathrm{D}\chi_{\mathcal{G}_f}|)\llcorner(\mathsf{X}\setminus(C_f\cup J_f)) \ll \mathsf{m} \qquad \text{and} \qquad (\pi^1_*|\mathrm{D}\chi_{\mathcal{G}_f}|)\llcorner(C_f\cup J_f) \perp \mathsf{m},$$

as $|\mathrm{D}f|\llcorner(\mathsf{X}\setminus(C_f\cup J_f)) \ll \mathsf{m}$ and $\mathsf{m}(C_f) = \mathsf{m}(J_f) = 0$. Then (5.1.7) follow from the Radon–Nikodym Theorem, see e.g. [95, Remark 3.4.29] and (5.1.6), taking into account (2.3.13) again. □

The following lemma completes the analysis of Proposition 5.1.9 on D_f: it studies the first n components of ν^u_f and $\nu^u_{\mathcal{G}_f}$ on D_f, which are the ones left out from the previous proposition (notice that the relation is the one satisfied by smooth maps on Euclidean spaces, even though our statement holds also for the Cantor part). In view of it, notice that (5.1.25) below is well defined thanks to (2.3.13).

Lemma 5.1.10. *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *be an* $\mathrm{RCD}(K,N)$ *space of essential dimension* n *and let* $f \in \mathrm{BV}_{\mathrm{loc}}(\mathsf{X})$. *Let also* u *be a good splitting map on* D *and let* $D_f \subseteq D\setminus J_f$ *be given by Proposition 5.1.7. Then,*

$$\sqrt{1-\Big(\nu^u_{\mathcal{G}_f}(x,\bar f(x))\Big)^2_{n+1}}\,\nu^u_f(x) = \big(\nu^u_{\mathcal{G}_f}(x,\bar f(x))\big)_{1,\dots,n} \qquad \text{for } |\mathrm{D}f|\text{-a.e. } x\in D_f. \tag{5.1.25}$$

Proof. By coarea and the representation formula (3.5.2), it is enough to show that for a.e. $t\in\mathbb{R}$ it holds

$$\sqrt{1-\Big(\nu^u_{\mathcal{G}_f}(x,\bar f(x))\Big)^2_{n+1}}\,\nu^u_f(x) = \big(\nu^u_{\mathcal{G}_f}(x,\bar f(x))\big)_{1,\dots,n} \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x\in D_f\cap\mathcal{F}E_t,$$

where, as usual, $E_t := \{f > t\}$. By Lemma 4.2.1, the equality above reads, for a.e. $t \in \mathbb{R}$,

$$\sqrt{1 - \left(\nu^u_{\mathcal{G}_f}(x, \bar f(x))\right)^2_{n+1}} \nu^u_{\chi_{E_t}}(x) = \left(\nu^u_{\mathcal{G}_f}(x, \bar f(x))\right)_{1,\dots,n} \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in D_f \cap \mathcal{F}E_t. \quad (5.1.26)$$

Fix t such that E_t is a set of finite perimeter, it is enough prove (5.1.26) for this value of t at $x \in D_f \cap \mathcal{F}E_t$ satisfying the conclusions of Proposition 3.2.20 (for u and E_t). Set for brevity $p := (x, \bar f(x))$. We are going to use the notation of Proposition 5.1.7 with $v_x := A(x)u$.

By Remark 3.3.3, the assumptions on x and the membership $x \in D_f$, we can find a sequence $r_i \searrow 0$, two half-spaces $H \subseteq \mathbb{R}^{n+1}$ and $H' \subseteq \mathbb{R}^n$, and a proper metric space $(\mathsf{Z}, \mathsf{d}_\mathsf{Z})$ such that

$$\left(\mathsf{X}, r_i^{-1}\mathsf{d}, \mathsf{m}^{r_i}_x, x, E_t\right) \to (\mathbb{R}^n, \mathsf{d}_e, \underline{\mathcal{L}}^n, 0, H'),$$
$$\left(\mathsf{X} \times \mathbb{R}, r_i^{-1}\mathsf{d}_{\mathsf{X}\times\mathbb{R}}, (\mathsf{m} \otimes \mathcal{L}^1)^{r_i}_p, p, \mathcal{G}_f\right) \to (\mathbb{R}^{n+1}, \mathsf{d}_e, \underline{\mathcal{L}}^{n+1}, 0, H),$$

in the realizations Z and $\mathsf{Z} \times \mathbb{R}$, respectively. Also,

$$H' = \{y \in \mathbb{R}^n : y \cdot \bar\nu^{v_x}_{E_t} \ge 0\} \qquad \text{and} \qquad H = \{z \in \mathbb{R}^{n+1} : z \cdot \bar\nu^{v_p}_{\mathcal{G}_f} \ge 0\}, \quad (5.1.27)$$

where we chose the coordinates on $\mathbb{R}^{n+1}$ as limits of appropriate rescalings of v_p (Remark 3.2.17).

Notice also that

$$\{(y, s) \in \mathsf{X} \times \mathbb{R} : s < t\} \to \{(y, s) \in \mathbb{R}^n \times \mathbb{R} : s < 0\} \qquad \text{in } L^1_{\mathrm{loc}},$$

in the realization $\mathsf{Z} \times \mathbb{R}$. Therefore, by stability (e.g. [10, Lemma 3.5]) we deduce that

$$\{(y, s) \in \mathsf{X} \times \mathbb{R} : s < f(y), s < t\} = \mathcal{G}_f \cap (\mathsf{X} \times (-\infty, t)) \to H \cap \{(y, s) \in \mathbb{R}^n \times \mathbb{R} : s < 0\} \qquad \text{in } L^1_{\mathrm{loc}}. \quad (5.1.28)$$

Also, using Fubini's Theorem and dominated convergence, we see that

$$E_t \times (-\infty, t) \to H' \times (-\infty, 0) \qquad \text{in } L^1_{\mathrm{loc}}. \quad (5.1.29)$$

Given that $E_t \times (-\infty, t) = \{(y, s) \in \mathsf{X} \times \mathbb{R} : t < f(y), s < t\} \subseteq \{(y, s) \in \mathsf{X} \times \mathbb{R} : s < f(y), s < t\}$ we obtain from (5.1.28) and (5.1.29) that

$$H' \times (-\infty, 0) \subseteq H \cap \{(y, s) \in \mathbb{R}^n \times \mathbb{R} : s < 0\},$$

so that, recalling (5.1.27),

$$\bar\nu^{v_p}_{\mathcal{G}_f} = \left(\alpha \bar\nu^{v_x}_{E_t}, \left(\bar\nu^{v_p}_{\mathcal{G}_f}\right)_{n+1}\right) \qquad \text{for some } \alpha \in [0, 1].$$

Now, as

$$1 = \left|\bar\nu^{v_p}_{\mathcal{G}_f}\right|^2 = \alpha^2 \left|\bar\nu^{v_x}_{E_t}\right|^2 + \left(\bar\nu^{v_p}_{\mathcal{G}_f}\right)^2_{n+1} = \alpha^2 + \left(\bar\nu^{v_p}_{\mathcal{G}_f}\right)_{n+1},$$

recalling (3.2.15) (for u and χ_{E_t}), (5.1.3) and the fact that $A(x)$ is invertible, (5.1.26) follows. □

The following result is a type of area formula: on the regular part (i.e. outside Cantor and jump part), the area factor (that, for smooth maps, reads $\sqrt{1 + |\nabla f|^2}$) corresponds to the Radon–Nikodym derivative of the perimeter with respect to the reference measure (such object is related to the last component of the normal of the subgraph by Proposition 5.1.9). Our proof is via blow-up, exploiting the results of [57] for what concerns Lipschitz functions, and then by approximation for the general case.

Lemma 5.1.11 (Area formula)**.** *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *be an* $\mathrm{RCD}(K,N)$ *space and let* $f\in\mathrm{BV}_{\mathrm{loc}}(\mathsf{X})$*. Then we have the following expression for the Radon–Nikodym derivative:*

$$\frac{\mathrm{d}\pi^1_*|\mathrm{D}\chi_{\mathcal{G}_f}|}{\mathrm{d}\mathsf{m}}(x)=\sqrt{g_f(x)^2+1}\qquad \textit{for } \mathsf{m}\textit{-a.e. } x\in\mathsf{X}\setminus(C_f\cup J_f).$$

Proof. By Lemma 3.2.19 and Proposition 5.1.7 we can fix u, a good splitting map on D and prove the claim for m-a.e. $x\in D_f$, where the set D_f is the one obtained in Proposition 5.1.7. Also, m-a.e. $x\in D_f$ it holds that $\big(\nu^u_{\mathcal{G}_f}(x,\bar f(x))\big)_{n+1}<0$, by Proposition 5.1.9. Clearly, there is no loss of generality in assuming $f\in\mathrm{BV}(\mathsf{X})$.

Case $f\in\mathrm{BV}(\mathsf{X})\cap\mathrm{LIP}(\mathsf{X})$. First, recall [24, Proposition 6.3] and [57], which imply that $|\mathrm{D}f|=(\mathrm{lip}f)\mathsf{m}$. Take $p:=(x,\bar f(x))$ with $x\in D_f$. We take $\{x_i\}_i\subseteq\mathsf{X}$ with $x_i\to x$ and

$$\lim_i\frac{f(x_i)-f(x)}{\mathsf{d}(x_i,x)}=\pm\mathrm{lip}f(x),$$

where the choice of the sign is any possible choice. Set $r_i:=\mathsf{d}(x,x_i)$, and notice that we can, and will, assume that $r_i\searrow 0$. Therefore (Remark 3.3.3), up to not relabelled subsequences, we have that

$$\big(\mathsf{X}\times\mathbb{R},r_i^{-1}\mathsf{d}_{\mathsf{Z}\times\mathbb{R}},(\mathsf{m}\otimes\mathcal{L}^1)^{r_i}_p,p,\mathcal{G}_f\big)\to(\mathbb{R}^{n+1},\mathsf{d}_e,\underline{\mathcal{L}}^{n+1},0,H),$$

where H is the half-space

$$H:=\big\{y\in\mathbb{R}^{n+1}:y\cdot\bar\nu^{v_p}_{\mathcal{G}_f}(p)\ge 0\big\},$$

(see Proposition 5.1.7 for the notation) and this convergence is realized in a proper metric space $(\mathsf{Z}\times\mathbb{R},\mathsf{d}_{\mathsf{Z}\times\mathbb{R}})$ with respect to isometric embeddings $\iota_i:(\mathsf{X}\times\mathbb{R},{r_i}^{-1}\mathsf{d}_{\mathsf{X}\times\mathbb{R}})\to(\mathsf{Z}\times\mathbb{R},\mathsf{d}_{\mathsf{Z}\times\mathbb{R}})$ and $\iota_\infty:(\mathbb{R}^n\times\mathbb{R},\mathsf{d}_e)\to(\mathsf{Z}\times\mathbb{R},\mathsf{d}_{\mathsf{Z}\times\mathbb{R}})$. Therefore, up to a not relabelled subsequence, $\iota_i(x_i,f(x_i))\to\iota_\infty(\bar z,\pm\mathrm{lip}f(x))$ in $\mathsf{Z}\times\mathbb{R}$, where $|\bar z|=1$.

Now we recall the first equality in (5.1.5), and we set for ease of notation

$$\mathbb{R}^n\times\mathbb{R}\ni\bar\nu^{v_p}_{\mathcal{G}_f}(p)=(\hat\nu,\nu_{n+1})=\big(\hat\nu,\nu^u_{\mathcal{G}_f}(p)_{n+1}\big),$$

for some $\hat\nu\in\mathbb{R}^n$ with $|\hat\nu|^2+\nu_{n+1}^2=1$. Hence, $\partial H=\{(z,t)\in\mathbb{R}^n\times\mathbb{R}:z\cdot\hat\nu+t\nu_{n+1}=0\}$. In particular, if $\nu_{n+1}\neq-1$, for every $(z,t)\in\partial H$,

$$z\cdot\frac{\hat\nu}{|\hat\nu|}\sqrt{(-\nu_{n+1})^{-2}-1}=t,$$

so that

$$(-\nu_{n+1})^{-1}=\sup_{(z,t)\in\partial H:|z|=1}\sqrt{t^2+1}. \tag{5.1.30}$$

Notice that (5.1.30) holds even if $\nu_{n+1}=-1$. Now we claim that

$$\big(-\nu^u_{\mathcal{G}_f}(p)\big)^{-1}_{n+1}=\sqrt{\mathrm{lip}f(x)^2+1}. \tag{5.1.31}$$

Set $\bar q:=(\bar z,\pm\mathrm{lip}f(x))$. Therefore, if we show that $\bar q\in\partial H$, it will follow by (5.1.30) the inequality ($\ge$) in (5.1.31). Take $\bar q'=(\bar z,t)$ such that $\bar q'\in\partial H$, we want to show that $\bar q=\bar q'$. By weak convergence rescaled perimeters and Lemma 2.3.11, we find a sequence of points $\{(x_i',f(x_i'))\}_i\subseteq$

$\mathsf{X}\times\mathbb{R}$ with $\iota_i(x_i',f(x_i'))\to\iota_\infty(\bar q')$ in $\mathsf{Z}\times\mathbb{R}$. Now we compute, if L denotes the global Lipschitz constant of f,

$$
\begin{aligned}
|\pm\operatorname{lip}f(x)-t| &= \lim_i\frac{|f(x_i)-f(x_i')|}{r_i}\le\limsup_i L\frac{\mathsf{d}(x_i,x_i')}{r_i}=\limsup_i L\mathsf{d}_{\mathsf{Z}}(\iota_i(x_i),\iota_i(x_i'))\\
&\le\limsup_i L(\mathsf{d}_{\mathsf{Z}}(\iota_i(x_i),\iota_\infty(\bar z))+\mathsf{d}_{\mathsf{Z}}(\iota_i(x_i'),\iota_\infty(\bar z)))=0.
\end{aligned}
$$

This shows that $t=\pm\operatorname{lip}f(x)$ and hence that $\partial H\ni\bar q'=\bar q$.

Now we show the inequality ($\le$) in (5.1.31), again using (5.1.30). Take any $\bar q:=(\bar z,t)\in\partial H$ with $|\bar z|=1$. As before, we find $\{(x_i,f(x_i))\}_i\subseteq\mathsf{X}\times\mathbb{R}$ with $\iota_i(x_i,f(x_i))\to\iota_\infty(\bar q)$ in $\mathsf{Z}\times\mathbb{R}$. But then

$$
\begin{aligned}
|t| &= \lim_i\frac{|f(x_i)-f(x)|}{r_i}=\lim_i\frac{|f(x_i)-f(x)|}{\mathsf{d}(x_i,x)}\frac{\mathsf{d}(x_i,x)}{r_i}\\
&\le\limsup_i\frac{|f(x_i)-f(x)|}{\mathsf{d}(x_i,x)}\limsup_i\mathsf{d}_{\mathsf{Z}}(\iota_i(x_i),\iota_i(x))\le\operatorname{lip}f(x)|(\bar z,0)|=\operatorname{lip}f(x).
\end{aligned}
$$

Case $f\in\mathrm{BV}(\mathsf{X})$. Let $\varepsilon\in(0,1)$ and, by [99, Proposition 4.3], take $h\in\mathrm{BV}(\mathsf{X})\cap\mathrm{LIP}(\mathsf{X})$ with $\mathsf{m}(\{h\ne f\})<\varepsilon$. Recall Proposition 5.1.7 and call

$$\hat D_\varepsilon:=(D_f\cap D_h\cap\{h=\bar f\})\setminus C_f.$$

It will be enough to prove the claim for m-a.e. $x\in\hat D_\varepsilon$, by the arbitrariness of ε. Notice that by [99, Proposition 3.7], $|\mathrm{D}(f-h)|(\hat D_\varepsilon)=0$, in particular,

$$g_f=\operatorname{lip}h\qquad\mathsf{m}\text{-a.e. on }\hat D_\varepsilon.\tag{5.1.32}$$

Now notice that m-a.e. $x\in\hat D_\varepsilon$ satisfies

$$\lim_{r\searrow0}\frac{(|\mathrm{D}h|+\mathsf{m})(B_r(x)\cap\{h\ne f\})}{r^n}=0.\tag{5.1.33}$$

This follows since $|\mathrm{D}h|\le L\mathsf{m}$, where L is the Lipschitz constant of h, as m is locally doubling and concentrated on $\mathcal{R}_n^*(\mathsf{X})$. Take $x\in\hat D_\varepsilon$ satisfying (5.1.33). We prove that at $p:=(x,h(x))$, $|\mathrm{D}\chi_{\mathcal{G}_h}|$ and $|\mathrm{D}\chi_{\mathcal{G}_h}|\wedge|\mathrm{D}\chi_{\mathcal{G}_f}|$, properly rescaled and in suitable realizations, have the same weak limit. Indeed, we compute, by (3.5.1) together with Remark 3.5.2 for the first inequality, Lemma 2.3.11 for the second inequality and (2.3.13) for the last inequality,

$$
\begin{aligned}
\limsup_{r\searrow0}\frac{\big(|\mathrm{D}\chi_{\mathcal{G}_h}|-|\mathrm{D}\chi_{\mathcal{G}_h}|\wedge|\mathrm{D}\chi_{\mathcal{G}_f}|\big)(B_r(p))}{r^n} &\le\limsup_{r\searrow0}\frac{|\mathrm{D}\chi_{\mathcal{G}_h}|(B_r(p)\setminus\partial^*\mathcal{G}_f)}{r^n}\\
&\le\limsup_{r\searrow0}\frac{|\mathrm{D}\chi_{\mathcal{G}_h}|(\{(y,t)\in B_r(p):h(y)\ne\bar f(y)\})}{r^n}\\
&\le\limsup_{r\searrow0}\frac{\pi^1_*|\mathrm{D}\chi_{\mathcal{G}_h}|(B_r(x)\cap\{h\ne\bar f\})}{r^n}\\
&\le\limsup_{r\searrow0}\frac{(|\mathrm{D}h|+\mathsf{m})(B_r(x)\cap\{h\ne f\})}{r^n},
\end{aligned}
$$

whence, taking into account Remark 3.3.3 and the fact that x satisfies (5.1.33),

$$\lim_{r\searrow 0}\frac{r\big(|\mathrm{D}\chi_{\mathcal{G}_h}|-|\mathrm{D}\chi_{\mathcal{G}_h}|\wedge|\mathrm{D}\chi_{\mathcal{G}_f}|\big)(B_r(p))}{C^r_{p_x}}=0. \tag{5.1.34}$$

Now, as $x\in D_f\cap D_h$, we can find a sequence $r_i\searrow 0$, two half-spaces $H_1,H_2\subseteq\mathbb{R}^{n+1}$ such that

$$\begin{aligned}&\big(\mathsf{X}\times\mathbb{R},r_i^{-1}\mathsf{d}_{\mathsf{X}\times\mathbb{R}},(\mathsf{m}\otimes\mathcal{L}^1)^{r_i}_p,p,\mathcal{G}_h\big)\to(\mathbb{R}^{n+1},\mathsf{d}_e,\underline{\mathcal{L}}^{n+1},0,H_f),\\&\big(\mathsf{X}\times\mathbb{R},r_i^{-1}\mathsf{d}_{\mathsf{X}\times\mathbb{R}},(\mathsf{m}\otimes\mathcal{L}^1)^{r_i}_p,p,\mathcal{G}_f\big)\to(\mathbb{R}^{n+1},\mathsf{d}_e,\underline{\mathcal{L}}^{n+1},0,H_g).\end{aligned}$$

Here (see Proposition 5.1.7 for the notation),

$$H_f=\{y\in\mathbb{R}^{n+1},y\,\cdot\,\bar\nu^{v_p}_{\mathcal{G}_h}\ge 0\},\qquad\text{and}\qquad H_g=\{y\in\mathbb{R}^{n+1},y\,\cdot\,\bar\nu^{v_p}_{\mathcal{G}_f}\ge 0\},$$

where we chose the coordinates on $\mathbb{R}^{n+1}$ as limits of rescalings of v_p (Remark 3.2.17). Now, (5.1.34) implies that $\bar\nu^{v_p}_{\mathcal{G}_h}=\pm\bar\nu^{v_p}_{\mathcal{G}_f}$. With this in mind, Proposition 5.1.9 (together with (5.1.1) and the first equality in (5.1.5) both for $\mathcal{G}_h$ and $\mathcal{G}_f$) implies that

$$\frac{\mathrm{d}\pi^1_*|\mathrm{D}\chi_{\mathcal{G}_f}|}{\mathrm{d}\mathsf{m}}=-\big(\nu^u_{\mathcal{G}_f}(x,\bar f(x))\big)^{-1}_{n+1}=-\big(\nu^u_{\mathcal{G}_h}(x,h(x))\big)^{-1}_{n+1}\qquad\text{for }\mathsf{m}\text{-a.e. }x\in D_\varepsilon,$$

so that the claim follows from what proved in the first **Case** of the proof and (5.1.32). □

With by now standard tools of geometric measure theory, exploiting the rectifiability Theorem 3.4.1 together with the representation formula (3.5.2) of Theorem 3.5.1 (in particular, Theorem 3.1.1), we obtain the following result.

Lemma 5.1.12. *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *be an* $\mathrm{RCD}(K,N)$ *space and let* $E\subseteq\mathsf{X}$ *be a set of finite perimeter. Then*

$$(\mathcal{H}^{n-1}\llcorner\mathcal{F}E)\otimes\mathcal{H}^1=\mathcal{H}^n\llcorner(\mathcal{F}E\times\mathbb{R})$$

as measures on $\mathsf{X}\times\mathbb{R}$.

Proof. Using Theorem 3.4.1 and Corollary 3.3.2 together with Remark 3.5.2, we can use [27, Theorem 5.4] (actually, the result of [100] is enough for this purpose) and see that

$$\Theta_{n-1}(\mathcal{H}^{n-1}\llcorner\mathcal{F}E,x)=\lim_{r\searrow 0}\frac{\mathcal{H}^{n-1}\llcorner\mathcal{F}E(B_r(x))}{\omega_{n-1}r^{n-1}}=1\qquad\text{for }\mathcal{H}^{n-1}\text{-a.e. }x\in\mathcal{F}E.$$

By Fubini's Theorem and what remarked above,

$$\lim_{r\searrow 0}\frac{(\mathcal{H}^{n-1}\otimes\mathcal{H}^1)\llcorner(\mathcal{F}E\times\mathbb{R})(B_r(x,t))}{\omega_n r^n}=1\qquad\text{for }\mathcal{H}^{n-1}\text{-a.e. }x\in\mathcal{F}E,\text{ for every }t\in\mathbb{R}. \tag{5.1.35}$$

In particular, (5.1.35) holds for $(\mathcal{H}^{n-1}\otimes\mathcal{H}^1)$-a.e. $(x,t)\in(\mathcal{F}E\times\mathbb{R})$, so that [31, Theorem 2.4.3] yields that

$$(\mathcal{H}^{n-1}\llcorner\mathcal{F}E)\otimes\mathcal{H}^1\ll\mathcal{H}^n\llcorner(\mathcal{F}E\times\mathbb{R}).$$

Moreover, a simple covering argument shows that if $N\subseteq\mathcal{F}E$ is $\mathcal{H}^{n-1}$-negligible, then $N\times\mathbb{R}\subseteq\mathcal{F}E\times\mathbb{R}$ is $\mathcal{H}^n$-negligible, hence (5.1.35) holds for $\mathcal{H}^n\llcorner(\mathcal{F}E\times\mathbb{R})$-a.e. $(x,t)\in\mathcal{F}E\times\mathbb{R}$. Therefore, with the same arguments as before we can use [27, Theorem 5.4] (which is based on [100]) to conclude. □

Notice that, by (2.3.13), $(\pi^1_*|\mathrm{D}\chi_{\mathcal{G}_f}|)\llcorner J_f = |\mathrm{D}f|\llcorner J_f$, hence (5.1.36) below is well posed. The following lemma concludes the section by studying the relation between $\nu^u_{\mathcal{G}_f}$ and ν^u_f on J_f: the normal to the subgraph is perfectly horizontal and is in the same direction of the polar vector of the function. The proof is done reducing to the case in which f is characteristic function (and in such case, the conclusion is trivial), by comparing the normal of the subgraph to the normal of suitable simple functions.

Lemma 5.1.13. *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *be an* $\mathrm{RCD}(K,N)$ *space and let* $f\in \mathrm{BV}_{\mathrm{loc}}(\mathsf{X})$. *Let also* u *be a good splitting map on* D, *then*

$$\nu^u_{\mathcal{G}_f}(x,t) = (\nu^u_f(x),0) \qquad \text{for } |\mathrm{D}\chi_{\mathcal{G}_f}|\text{-a.e. } (x,t)\in (D\cap J_f)\times\mathbb{R}. \tag{5.1.36}$$

Proof. We can clearly assume $f\in\mathrm{BV}(\mathsf{X})$. As stated right before this lemma,

$$(\pi^1_*|\mathrm{D}\chi_{\mathcal{G}_f}|)\llcorner J_f = |\mathrm{D}f|\llcorner J_f$$

and we are going to exploit throughout this equality. Now, let $S\subseteq\mathbb{R}$ be countable and dense and such that for every $s\in S$, $E_s:=\{f>s\}$ has locally finite perimeter and satisfies the conclusion of Lemma 4.2.1 (this is possible by coarea). As $S\subseteq\mathbb{R}$ is dense, we can use a partitioning argument to see that it is enough to show the claim on $B\times\mathbb{R}$, where $B\subseteq D\cap J_f$ is a $|\mathrm{D}f|$-measurable set satisfying $s\in(f^\vee(x),f^\wedge(x))$, for some $s\in S$. By (2.3.6), $B\subseteq\partial^*E_s$. Then (Remark 3.5.2 and (4.2.11)), if $N\subseteq B$ is with $|\mathrm{D}\chi_{E_s}|(N)=0$, then $|\mathrm{D}\chi_{\mathcal{G}_f}|(N\times\mathbb{R})=0$. In particular, by Corollary 3.3.2, we see that we can assume that $B\subseteq\mathcal{F}E_s$ and, by the assumptions on S, that for every $x\in B$,

$$\nu^u_f(x) = \nu^u_{E_s}(x), \tag{5.1.37}$$

where we fixed Borel representative.

Now, Lemma 5.1.12 together with the fact that graphs have zero (product) measure, implies that

$$\mathcal{H}^n(\{(x,t): x\in B, t=f^\vee(x)\})=0,$$

(here we used also (3.5.1) to deduce that $\mathcal{H}^{n-1}\llcorner\mathcal{F}E_s$ is σ-finite). Then, by (3.5.1),

$$|\mathrm{D}\chi_{\mathcal{G}_f}|(\{(x,t): x\in B, t=f^\vee(x)\})=0,$$

and the same equality holds with $f^\wedge$ in place of $f^\vee$. Therefore, we can use a partitioning argument to reduce ourselves to prove the claim on $B\times I$, where $I\subseteq\mathbb{R}$ is an open interval with $s\in I$ and such that for every $x\in B$, $\bar I\subseteq(f^\wedge(x),f^\vee(x))$. Now recall that for every $(x,t)\in B\times I$, then $x\in\mathcal{F}E_s$, so that $(x,t)\in\mathcal{F}(E_s\times\mathbb{R})$ and that using twice (3.5.1) and Lemma 5.1.12, we have that

$$|\mathrm{D}\chi_{\mathcal{G}_f}|\llcorner(B\times I)\le\frac{\omega_n}{\omega_{n+1}}\mathcal{H}^h\llcorner(B\times I) = |\mathrm{D}\chi_{E_s\times\mathbb{R}}|\llcorner(B\times I) = (|\mathrm{D}\chi_{E_s}|\llcorner B)\otimes(\mathcal{L}^1\llcorner I).$$

Now take $(x,t)\in B\times I$. Then

$$\begin{aligned} E_t\times(-\infty,t) &= \{(y,u)\in\mathsf{X}\times\mathbb{R}: t<f(y), u<t\}\subseteq\{(y,u):\in\mathsf{X}\times\mathbb{R}: u<f(y), u<t\}\\ &=\mathcal{G}_f\cap(\mathsf{X}\times(-\infty,t)) \end{aligned}$$

and

$$\begin{aligned} (\mathsf{X}\setminus E_t)\times[t,\infty) &= \{(y,u)\in\mathsf{X}\times\mathbb{R}: t\ge f(y), u\ge t\}\subseteq\{(y,u)\in\mathsf{X}\times\mathbb{R}: u\ge f(y), u\ge t\}\\ &=((\mathsf{X}\times\mathbb{R})\setminus\mathcal{G}_f)\cap(\mathsf{X}\times[t,\infty)). \end{aligned}$$

Now, if $t \le s$, then $E_s \subseteq E_t$ so that

$$E_s \times (-\infty, t) \subseteq \mathcal{G}_f \qquad \text{and} \qquad E_s \times (-\infty, t) \subseteq E_s \times \mathbb{R}$$

whereas if $t \ge s$, then $E_t \subseteq E_s$ so that

$$(\mathsf{X} \setminus E_s) \times [t, \infty) \subseteq (\mathsf{X} \times \mathbb{R}) \setminus \mathcal{G}_f \qquad \text{and} \qquad (\mathsf{X} \setminus E_s) \times [t, \infty) \subseteq (\mathsf{X} \times \mathbb{R}) \setminus (E_s \times \mathbb{R}).$$

As, for every $x \in B$, E_s has density $1/2$ at x, the inclusions above together with Fubini's Theorem show that $(E_s \times \mathbb{R}) \Delta \mathcal{G}_f$ cannot have density 1 at $(x,t) \in B \times I$. Notice also that for $|\mathrm{D}\chi_{E_s \times \mathbb{R}}|$-a.e. $(x,t) \in B \times I$,

$$\nu^u_{E_s \times \mathbb{R}}(x,t) = (\nu^u_{E_s}(x), 0).$$

The conclusion then comes from Lemma 4.2.3 and (5.1.37). □

5.1.2 Proof of the main results

The main contribution towards the proof of Theorem 5.1.1 is the simple remark that the relative isoperimetric inequality allows us to gain some integrability starting from the finiteness of the total variation.

Proof of Theorem 5.1.1. If $f \in \mathrm{BV}_{\mathrm{loc}}(\mathsf{X})$, then $\mathcal{G}_f$ has perimeter that is finite on cylinders, thanks to the proof of item (a) in [30, Theorem 5.1]. Conversely, assume that $\mathcal{G}_f$ has finite perimeter on cylinders. Then, the argument in the proof of item (b) of [30, Theorem 5.1] yields that for any $x \in \mathsf{X}$ and $r > 0$,

$$\int_{\mathbb{R}} |\mathrm{D}\chi_{\{f>t\}}|(B_r(x))\mathrm{d}t < \infty.$$

Now we take $t_0 \in (0,\infty)$ big enough so that $\mathsf{m}(\{f > t_0\} \cap B_r(x)) \le \min\{1, \mathsf{m}(\{f \le t_0\} \cap B_r(x))\}$ and $\mathsf{m}(\{f < -t_0\} \cap B_r(x)) \le \min\{1, \mathsf{m}(\{f \ge -t_0\} \cap B_r(x))\}$. This is possible as $f \in L^0(\mathsf{m})$. Thus, taking into account that for $\mathcal{L}^1$-a.e. t, $|\mathrm{D}\chi_{\{f>t\}}| = |\mathrm{D}\chi_{\{f<t\}}|$, we obtain from the relative isoperimetric inequality (2.3.2) (that holds with $\lambda = 1$ on finite dimensional RCD spaces, see [93, Section 9] - but this is not important) that

$$\int_{t_0}^{\infty} \mathsf{m}(\{f > t\} \cap B_r(x))\mathrm{d}t < +\infty \qquad \text{and} \qquad \int_{-\infty}^{-t_0} \mathsf{m}(\{f < t\} \cap B_r(x))\mathrm{d}t < +\infty.$$

This implies $f \in L^1_{\mathrm{loc}}(\mathsf{X})$ by Fubini's theorem. By coarea, it also follows that $f \in \mathrm{BV}_{\mathrm{loc}}(\mathsf{X})$.

The last conclusion is an immediate consequence of Lemma 5.1.11 for what concerns the absolutely continuous part and (2.3.13) for what concerns the singular part. □

The proofs of Theorem 5.1.3 and Theorem 5.1.4 follow rather easily from the series of preparatory results of Section 5.1.1.

Proof of Theorem 5.1.3. We first show that

$$\nu^u_{\mathcal{G}_f}(x,t) = \left(\sqrt{\frac{1}{1+g_f^2}} g_f \nu^u_f, -\sqrt{\frac{1}{1+g_f^2}} \right)(x) \qquad \text{for } |\mathrm{D}\chi_{\mathcal{G}_f}|\text{-a.e. } (x,t) \in (D \setminus (J_f \cup C_f)) \times \mathbb{R}.$$

Recall that Proposition 5.1.7 imply that we can reduce ourselves to show this claim for $|\mathrm{D}\chi_{\mathcal{G}_f}|$-a.e. $(x,t) \in (D_f \setminus C_f) \times \mathbb{R}$ (recall that $D_f \cap J_f = \emptyset$).

For $|\mathrm{D}\chi_{\mathcal{G}_f}|$-a.e. $(x,t)\in(\{g_f=0\}\cap D_f\setminus C_f)\times\mathbb{R}$, by Lemma 5.1.11, Proposition 5.1.9 and (2.3.13) it holds that $\big(\nu^u_{\mathcal{G}_f}(x,\bar f(x))\big)_{n+1}=-1$, hence the claim follows by (5.1.3), as $\big|\bar\nu^{(A(x)u\circ\pi^1,\pi^2)}_{\mathcal{G}_f}\big|=1$ (and $A(x)$ is invertible).

Now we show the claim at $|\mathrm{D}\chi_{\mathcal{G}_f}|$-a.e. (x,t) with $x\in(\{g_f>0\}\cap D_f\setminus C_f)\times\mathbb{R}$. Notice that on $\{g_f>0\}\cap D_f\setminus C_f$ it holds that $\mathsf{m}\ll|\mathrm{D}f|\ll\mathsf{m}$. Therefore, by Lemma 5.1.10, taking into account Lemma 5.1.11, Proposition 5.1.9 and (2.3.13), we have the claim.

Now we prove that

$$\nu^u_{\mathcal{G}_f}(x,t)=\big(\nu^u_f(x),0\big)\qquad\text{for }|\mathrm{D}\chi_{\mathcal{G}_f}|\text{-a.e. }(x,t)\in(D\cap C_f)\times\mathbb{R}.$$

By Proposition 5.1.7, we can show the claim for $|\mathrm{D}\chi_{\mathcal{G}_f}|$-a.e. $(x,t)\in(D_f\cap C_f)\times\mathbb{R}$. Then this follows from Proposition 5.1.9 (see also (5.1.23)) together with Lemma 5.1.10 and (2.3.13).

Finally,

$$\nu^u_{\mathcal{G}_f}(x,t)=\big(\nu^u_f(x),0\big)\qquad\text{for }|\mathrm{D}\chi_{\mathcal{G}_f}|\text{-a.e. }(x,t)\in(D\cap J_f)\times\mathbb{R}$$

by Lemma 5.1.13. □

Proof of Theorem 5.1.4. Items *i)* and *ii)* can be proved using Theorem 5.1.3, Theorem 5.1.1, and Lemma 2.3.11. Item *iv)* follows from Lemma 5.1.13.

We show now item *iii)*. By (3.5.2), we write

$$\begin{aligned}\int_{(D\cap J_f)\times\mathbb{R}}&\varphi(x,t)\big(\nu^u_{\mathcal{G}_f}(x,t)\big)_i\,\mathrm{d}|\mathrm{D}\chi_{\mathcal{G}_f}|(x,t)\\&=\int_{(D\cap J_f)\times\mathbb{R}}\varphi(x,t)\big(\nu^u_f(x)\big)_i\chi_{\partial^*\mathcal{G}_f}(x,t)\Theta_n(\mathsf{m},x)\mathrm{d}\mathcal{H}^n(x,t),\end{aligned}$$

where we used the first equality in (5.1.8) and Lemma 5.1.13. Now notice that if $N\subseteq J_f$ is such that $\mathcal{H}^{n-1}(N)=0$, then $\mathcal{H}^n(N\times\mathbb{R})=0$. This can be proved with an easy covering argument.

Therefore, taking into account also Lemma 2.3.11 and coarea, we reduce ourselves to prove that for every $\psi:D\times\mathbb{R}\to[0,1]$ Borel, we have that for $\mathcal{H}^1$-a.e. $s\in\mathbb{R}$

$$\int_{(D\cap\mathcal{F}E_s\cap J_f)\times\mathbb{R}}\psi(x,t)\mathrm{d}\mathcal{H}^n(x,t)=\int_{D\cap\mathcal{F}E_s\cap J_f}\int_{\mathbb{R}}\psi(x,t)\mathrm{d}t\mathrm{d}\mathcal{H}^{n-1}(x),$$

where $E_s:=\{f>s\}$, which follows from Lemma 5.1.12. □

5.2 Bibliographical notes

In the Euclidean setting, the topic exposed in this chapter can be dated back at least to [107, 77]. There, the author was concerned with the study of *Cartesian surfaces*, i.e. subsets of $\mathbb{R}^n\times\mathbb{R}$ that can be written as $\{(x,t)\in\mathbb{R}^n\times\mathbb{R}:x\in\Omega,t=f(x)\}$, where $\Omega\subseteq\mathbb{R}^n$ is open, and $f\in\mathrm{BV}_{\mathrm{loc}}(\Omega)$. A systematic study of Cartesian surfaces and subgraphs of functions of locally bounded variation in Euclidean spaces can be found in [79, Section 4.1.5]. In fact, our results are the generalization of the results contained in [79, Section 4.1.5] to the setting of finite dimensional RCD spaces.

The classical strategy of [79] seems not suitable for our context, as we do not have a canonical way to decompose the distributional derivatives $\mathrm{D}f$ and $\mathrm{D}\chi_{\mathcal{G}_f}$ along different directions. This also causes the need to define the "components" $\big(\nu^u_f\big)_i$ $\big(\nu^u_{\mathcal{G}_f}\big)_i$ exploiting maps that look like charts. The drawback is that these charts are defined only on Borel subsets, hence it is not clear the distributional nature of the objects $\big(\nu^u_f\big)_i$ and $\big(\nu^u_{\mathcal{G}_f}\big)_i$. Nevertheless, in our main result we compare

$\left(\nu_f^u\right)_i$ with $\left(\nu_{\mathcal{G}_f}^u\right)_i$. In order to do so, a new strategy has to be exploited, and we therefore employ a blow-up procedure, which is more compatible with the use of geometric measure theory results and does not need the distributional meaning of such objects.

The results of [79, Section 4.1.5] have already been used in the short proof of the Rank one Theorem in the Euclidean setting of [106], and (after having been generalized to Carnot groups, see e.g. [73, Theorem 4.3]) have been exploited to prove the Rank one Theorem for a subclass of Carnot groups in [73] (see Section 6.2). The generalization of these results to finite dimensional RCD spaces that is contained in this chapter will be used to prove the Rank one Theorem in Chapter 6.

The material of this chapter is taken from [32, 33], after some rewriting. The rewriting is done in such a way to merge smoothly the results contained in [32, 33], but takes also into account some slight differences, for example, the definition of good splitting map that we adopt here is slightly modified with respect to the one used in [32, 33], but this plays no difference, see Remark 3.2.13. Proposition 5.1.7 has to be compared with [32, Proposition 3.6] and [33, Proposition 32]. Notice that what here and in [33, Proposition 32] is called D_f is the analogue of what is called C_f in [32, Proposition 3.6] (the notation of [32] creates a bit of confusion as C_f, here, is any m-negligible set on which $|\mathrm{D}f|^c$ is concentrated). We remark that the first part of Lemma 5.1.11 can be proved also exploiting [30, Theorem 5.1], but the proof given here is tailored to this setting and more in the spirit of this chapter. Also, the claim of Lemma 5.1.12 is contained in [33, Lemma 38], but here we adopt a different proof, which is shorter but less elementary. Part of the proof of Lemma 5.1.13 has been changed with respect to the proof of [33, Lemma 38] and is now simpler.

Chapter 6

Rank one Theorem

The aim of this chapter is to state and prove the Rank one Theorem for vector valued functions of bounded variation on (finite dimensional) RCD spaces. In the Euclidean context such theorem establishes that for a vector valued function of bounded variation f, the polar matrix $\frac{\mathrm{d}\mathrm{D}f}{\mathrm{d}|\mathrm{D}f|}$ has rank one almost everywhere with respect to the singular part of $|\mathrm{D}f|$ (it is clear that this does not hold, in general, with respect to the absolutely continuous part of $|\mathrm{D}f|$). The conclusion on the jump part is not so hard and follows from the material of the previous chapters (i.e. it follows from a sort of transversality condition for rectifiable sets), the bulk of the proof is then the conclusion on the Cantor part.

6.1 Main result

Before stating the main result of this chapter, we define what it means for m-tuple of vector fields to have rank one.

Definition 6.1.1. Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,\infty)$ space. Let $\nu \in L^0_{\mathrm{Cap}}(T^m\mathsf{X})$ and let μ be a Radon measure such that $\mu \ll \mathrm{Cap}$. We say that

$$\mathrm{Rk}(\nu) = 1 \qquad \mu\text{-a.e.}$$

(or that ν *has rank one* μ-a.e.) if there exist $\omega \in L^0_{\mathrm{Cap}}(T\mathsf{X})$ and $\lambda_1,\dots,\lambda_m \in L^0(\mathrm{Cap})$ such that for every $i=1,\dots,m$,

$$\nu_i = \lambda_i\omega \qquad \mu\text{-a.e.}$$

We remark that this is one of the possible definitions we could have given of having rank one. For example, one can give an alternative and equivalent definition exploiting the existence of a local basis (with respect to a decomposition of the space in Borel sets) of $L^0_{\mathrm{Cap}}(T\mathsf{X})$, to recover the language of rank of a matrix, see e.g. Theorem 2.2.21. It is however clear that in Euclidean spaces, the definition given above coincides with the usual one.

We are now ready to state the main theorem of this chapter, which is the generalization of the Rank one Theorem to the setting of $\mathrm{RCD}(K,N)$ spaces.

Theorem 6.1.2 (Rank one Theorem)**.** *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *be an* $\mathrm{RCD}(K,N)$ *space and let* $f \in \mathrm{BV}_{\mathrm{loc}}(\mathsf{X})^m$*. Then*

$$\mathrm{Rk}(\nu_f) = 1 \qquad |\mathrm{D}f|^s\text{-a.e.} \tag{6.1.1}$$

In (6.1.1), we chose a Cap-representative of ν_f. It is clear that its validity does not depend on the choice of this representative.

Now we start proving Theorem 6.1.2. The bulk of the proof is contained in Lemma 6.1.3, whose proof is deferred to Section 6.1.3 below. First notice that if $N \subseteq \mathcal{FG}_f \cap (D_f \times \mathbb{R})$ is $|\mathrm{D}\chi_{\mathcal{G}_f}|$-negligible, then $\pi^1(N)$ is $|\mathrm{D}f|$ negligible, by (2.3.13), as $N \subseteq D_f \times \mathbb{R}$. Hence (6.1.2) below is well posed.

Before stating the lemma, let us shortly explain which is its moral in the Euclidean context ([106]). What one wants to show is that, if Σ_1, Σ_2 are two C^1 hypersurfaces in $\mathbb{R}^{n+1}$ with unit normals $\nu_{\Sigma_1}, \nu_{\Sigma_2}$, the set

$$\{p \in \Sigma_1 : \exists q \in \Sigma_2 : \pi(p) = \pi(q), \big(\nu_{\Sigma_1}(p)\big)_{n+1} = \big(\nu_{\Sigma_2}(q)\big)_{n+1} = 0, \text{ and } \nu_{\Sigma_1}(p) \neq \pm\nu_{\Sigma_2}(p)\}$$

is $\mathcal{H}^n$-negligible, where $\pi : \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ denotes the projection onto the first factor. From this property the Rank one theorem can be easily deduced. It is then clear that the set in (6.1.2) is defined in this spirit, adding one dummy variable to reduce the problem to the study of the intersection of two C^1 hypersurfaces with different normals at the points of intersection.

Lemma 6.1.3. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *and let* $f, g \in \mathrm{BV_{loc}}(\mathsf{X})$. *Let also* u *be a good splitting map on* D *and let* $D_f, D_g \subseteq D$ *be given by Proposition 5.1.7 for* f *and* g *respectively. We define* $\tilde{R} \subseteq \mathsf{X} \times \mathbb{R} \times \mathbb{R}$ *as*

$$\begin{aligned}\tilde{R} := \big\{(x, t, s) \in (D_f \cap D_g) \times \mathbb{R} \times \mathbb{R} :& (x, t) \in \mathcal{FG}_f, (x, s) \in \mathcal{FG}_g, \nu^u_{\mathcal{G}_f}(x, t) \neq \pm\nu^u_{\mathcal{G}_g}(x, s), \\ & (\nu^u_{\mathcal{G}_f}(x, t))_{n+1} = (\nu^u_{\mathcal{G}_g}(x, s))_{n+1} = 0\big\}.\end{aligned} \tag{6.1.2}$$

Then, setting $R := \pi^1(\tilde{R})$, *it holds that*

$$(|\mathrm{D}f| \wedge |\mathrm{D}g|)(R) = 0.$$

The following lemma is basically the Rank one Theorem 6.1.2 in the case $m = 2$ and on the Cantor part. Its proof is based on the study of Cartesian surfaces of the previous chapter and the previous lemma (in which an additional coordinate is added).

Lemma 6.1.4. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space and let* $f, g \in \mathrm{BV_{loc}}(\mathsf{X})$. *Choose two* Cap-*vector fields representatives for* ν_f *and* ν_g. *Then*

$$\nu_f = \pm\nu_g \qquad (|\mathrm{D}f|^c \wedge |\mathrm{D}g|^c)\text{-}a.e.$$

Proof. We denote by n the essential dimension of the space. We use a partitioning argument based on Lemma 3.2.19 together with Proposition 5.1.7 to reduce the claim on $D_f \cap D_g$, where there exists u, a good splitting map on D. Then, from Theorem 5.1.3 and Lemma 6.1.3, taking into account (2.3.13), we deduce that

$$\nu^u_f(x) = \pm\nu^u_g(x) \qquad (|\mathrm{D}f|^c \wedge |\mathrm{D}g|^c)\text{-a.e. on } D_f \cap D_g. \tag{6.1.3}$$

Now, we show that $\{\nabla u^i\}_{i=1,\dots,n}$ generates $L^0_{\mathrm{Cap}}(T\mathsf{X})$ on D. This follows from Remark 3.2.5 together with Theorem 2.2.21. Indeed, set $M_{i,j}$ equal to the Cap-representative of $\nabla u^i \cdot \nabla u^j$. By the fact that u is a good splitting map on D, together with Remark 3.2.5, it holds that $M_{i,j}$ is invertible Cap-a.e. on D. Now, Theorem 2.2.21 shows there can be at most n linearly independent Cap-vector fields. This improves (6.1.3) to

$$\nu_f(x) = \pm\nu_g(x) \qquad (|\mathrm{D}f|^c \wedge |\mathrm{D}g|^c)\text{-a.e. on } D_f \cap D_g,$$

whence the conclusion. □

Proof of Theorem 6.1.2. By Remark 4.1.2, for every $i = 1, \dots, m$,

$$(\nu_f)_i = \frac{\mathrm{d}|\mathrm{D}f_i|}{\mathrm{d}|\mathrm{D}f|}\nu_{f_i} \qquad |\mathrm{D}f|\text{-a.e.}$$

Hence, by classical arguments based on the properties of variation measures, it is enough to consider the case $m = 2$.

Then from Lemma 6.1.4, we have that

$$\nu_{f_1} = \pm\nu_{f_2} \qquad (|\mathrm{D}f_1| \wedge |\mathrm{D}f_2|)\text{-a.e. on } C_{f_1} \cap C_{f_2},$$

whereas, by Lemma 4.2.9 we have that

$$\nu_{f_1} = \pm\nu_{f_2} \qquad (|\mathrm{D}f_1| \wedge |\mathrm{D}f_2|)\text{-a.e. on } J_{f_1} \cap J_{f_2},$$

and thus the conclusion, again by classical properties of variation measures. □

6.1.1 Proof of Lemma 6.1.3

Now we prove Lemma 6.1.3, which is the bulk of the proof of the Rank one Theorem 6.1.2. We remark that one of the difficulties faced in proving Lemma 6.1.3 is the fact that its statement involves a "codimension 2" analysis. Indeed, the set $\tilde{R}$ of (6.1.2) is the intersection of two 1-codimensional objects. There are no known techniques to deal directly with such high codimensional objects on RCD spaces, and for this reason we have to resort to the analysis in the Euclidean space, translating our problem and relying on a well-know transversality result for C^1 hypersurfaces. It is then evident that the problem becomes understanding the structure of "charts" for RCD spaces.

Proof of Lemma 6.1.3. To fix the notation, assume that u is defined on $B_{2\bar{r}}(\bar{x})$, hence $D \subseteq B_{\bar{r}}(x)$, notice that $R \subseteq D_f \cap D_g \subseteq D \subseteq B_{\bar{r}}(\bar{x})$. We let n denote the essential dimension of the space. We start with some exhaustion and partitioning argument that will allow us to gain additional properties on the set R (i.e. for $\tilde{R}$, as $\tilde{R} \subseteq R \times \mathbb{R} \times \mathbb{R}$). This is possible by the nature of the claim, taking advantage of the fact that if we write $R = \bigcup_{k\in\mathbb{N}} R_k \cup N_f \cup N_g$ where $|\mathrm{D}f|(N_f) = |\mathrm{D}g|(N_g) = 0$, then it is enough to show that $(|\mathrm{D}f| \wedge |\mathrm{D}g|)(R_k) = 0$ for every k.

By an exhaustion argument building upon Remark 3.3.3, we can assume that for some $l \in \mathbb{N}$, $l \geq 1$, we have that

$$\frac{r|\mathrm{D}\chi_{\mathcal{G}_f}|(B_r(x, \bar{f}(x)))}{(\mathsf{m} \otimes \mathcal{L}^1)(B_r(x, \bar{f}(x)))} > l^{-1} \qquad \text{for every } x \in R \text{ and } r \in (0, l^{-1}) \tag{6.1.4}$$

and moreover

$$\Theta_n(\mathsf{m}, x) \in (l^{-1}, l) \qquad \text{for every } x \in R. \tag{6.1.5}$$

We assume also that we have the same bounds with g in place of f.

Step 1: rectifiability and surfaces. Up to a partitioning argument, we can assume that $\mathcal{FG}_f \cap (R\times\mathbb{R})$ is contained in $B_{\bar{r}}(\bar{x}, \bar{t})$, for some $\bar{t} \in \mathbb{R}$. Recall (Remark 5.1.6) that $(u\circ\pi^1, \pi^2) : B_{2\bar{r}}(\bar{x}, \bar{t}) \to \mathbb{R}^{n+1}$ is a good splitting map on $(R \times \mathbb{R}) \cap B_{\bar{r}}(\bar{x}, \bar{t})$. Recall also the definitions of the matrix valued Borel maps A and $\tilde{A}$, e.g. Remark 5.1.6 again. Now we are going to use Lemma 3.4.2 as in the proof of Theorem 3.4.1, we give the details. By the definition of D_f, for every $x \in R$, the map

$$v_{(x,\bar{f}(x))} = \tilde{A}(x, \bar{f}(x))(u \circ \pi^1, \pi^2) = (A(x)u \circ \pi^1, \pi^2)$$

is a systems of good coordinates for $\mathcal{F}\mathcal{G}_f$ at $(x, \bar f(x))$. Moreover (we state it for future reference) $(x, \bar f(x))$ is a Lebesgue point for $\nu^u_{\mathcal{G}_f}$ with respect to $|\mathrm{D}\chi_{\mathcal{G}_f}|$ and (5.1.3) is satisfied. Now, take $\varepsilon \in (0,1)$ small enough so that $\varepsilon^2(\hat C_{K,N+1}+1) < \varepsilon$, where $\hat C_{K,N+1}$, the constant appearing in Definition 3.2.12. Then, up to a further partitioning argument, we assume that there exists an invertible (as $A(x)$ is invertible for every $x \in R$) matrix $\bar A \in \mathbb{Q}^{n\times n}$ and a vector $\bar\nu_f \in \mathbb{Q}^{n+1}$ such that for every $x \in R$,

$$|A(x) - \bar A| < \varepsilon^2 \qquad \text{and} \qquad |\bar\nu^u_{\mathcal{G}_f}(x, \bar f(x)) - \bar\nu_f| < \varepsilon^2 \qquad \text{for every } x \in R.$$

In particular, if $\tilde{\bar A}$ denotes the matrix obtained starting from $\bar A$ as in (5.1.2),

$$|\tilde A(x, \bar f(x)) - \tilde{\bar A}| < \varepsilon^2 \qquad \text{for every } x \in R.$$

Therefore, recalling also (6.1.4), the assumptions of Lemma 3.4.2 are in place for $\mathcal{G}_f$ in place of E (with the obvious change of notation). Then we apply Lemma 3.4.2 and a further partitioning argument to assume that the map

$$\bar\pi(\bar A u\circ\pi^1, \pi^2) : (R\times\mathbb{R})\cap\mathcal{F}\mathcal{G}_f \to \mathbb{R}^{n+1}$$

is bilipschitz onto its n-dimensional image, where $\bar\pi$ is a projection onto a hyperplane of $\mathbb{R}^{n+1}$. Hence, also

$$(u\circ\pi^1, \pi^2) : \Gamma_f \to \mathbb{R}^{n+1} \tag{6.1.6}$$

is bilipschitz onto its n-dimensional image, being $\bar A$ invertible and u Lipschitz, where we set, for simplicity of notation,

$$\Gamma_f := (R\times\mathbb{R})\cap\mathcal{F}\mathcal{G}_f.$$

Take now $N \subseteq \mathbb{R}^{n+1}$ with $\mathcal{H}^n(N) = 0$ and set $M := (u\circ\pi^1, \pi^2)^{-1}_{|\Gamma_f}(N) \subseteq \mathsf{X}\times\mathbb{R}$. Being $(u\circ\pi^1,\pi^2)$ bilipschitz, $\mathcal{H}^n(M) = 0$, so that, by the representation formula of Theorem 3.5.1 (in particular, Theorem 3.1.1 and Remark 3.5.2), $|\mathrm{D}\chi_{\mathcal{G}_f}|(M) = 0$, whence $|\mathrm{D}f|(\pi^1(M)) = 0$, by (2.3.13), as $M \subseteq D_f\times\mathbb{R}$. Therefore, by standard results of geometric measure theory in Euclidean spaces and a partitioning argument, we can assume that

$$\Gamma'_f := (u\circ\pi^1, \pi^2)(\Gamma_f) \tag{6.1.7}$$

is contained in a C^1 hypersurface Σ_f. As $\mathcal{H}^n \llcorner \Sigma_f$ is asymptotically doubling, by the discussion above we see that we can assume

$$\lim_{r\searrow 0} \frac{\mathcal{H}^n\llcorner\Sigma_f(B_r(q)\cap\Gamma'_f)}{\omega_n r^n} = 1 \qquad \text{for every } q \in \Gamma'_f. \tag{6.1.8}$$

Finally, taking into account also that $|\mathrm{D}\chi_{\mathcal{G}_f}|$ is asymptotically doubling, we can assume that

$$\lim_{r\searrow 0} \frac{|\mathrm{D}\chi_{\mathcal{G}_f}|(B_r(p)\cap\Gamma_f)}{|\mathrm{D}\chi_{\mathcal{G}_f}|(B_r(p))} = 1 \qquad \text{for every } p \in \Gamma_f. \tag{6.1.9}$$

To sum up, we have reduced ourselves to prove the claim where R is such that the map in (6.1.6) is bilipschitz onto its n-dimensional image which is contained in a C^1 hypersurface Σ_f, and such that (6.1.4), (6.1.5), (6.1.8), (6.1.9) and (5.1.3) hold (taking the Lebesgue representative for $\nu^u_{\mathcal{G}_f}$). Moreover, the similar assertions hold for g in place of f (this is proved exactly with the same argument and a further partitioning and exhaustion).

Step 2: almost one-sided Kuratowski convergence. Let $p \in \Gamma_f$ and let $\rho_k \searrow 0$ such that

$$(\mathsf{X} \times \mathbb{R}, \rho_k^{-1}\mathsf{d}_{\mathsf{X}\times\mathbb{R}}, (\mathsf{m} \otimes \mathcal{L}^1)_p^{\rho_k}, p, \mathcal{G}_f) \to (\mathbb{R}^n \times \mathbb{R}, \mathsf{d}_e, \underline{\mathcal{L}}^{n+1}, 0, H),$$

in a realization $(\mathsf{Z} \times \mathbb{R}, \mathsf{d}_{\mathsf{Z}\times\mathbb{R}})$, where $(\mathsf{Z}, \mathsf{d}_{\mathsf{Z}})$ is a proper metric space, with respect to isometric embeddings $\{\iota_k\}_k$ and ι_∞, where $\iota_k : (\mathsf{X} \times \mathbb{R}, \rho_k^{-1}\mathsf{d}_{\mathsf{X}\times\mathbb{R}}) \to (\mathsf{Z} \times \mathbb{R}, \mathsf{d}_{\mathsf{Z}\times\mathbb{R}})$ and $\iota_\infty : (\mathbb{R}^n \times \mathbb{R}, \mathsf{d}_e) \to (\mathsf{Z} \times \mathbb{R}, \mathsf{d}_{\mathsf{Z}\times\mathbb{R}})$. We claim that for every $\rho, \varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$B_\rho^{\mathsf{Z}\times\mathbb{R}}(\iota_k(p)) \cap \iota_k(\Gamma_f) \subseteq B_\varepsilon^{\mathsf{Z}\times\mathbb{R}}(\iota_\infty(\partial H)) \qquad \text{if } k \ge k_0.$$

We argue by contradiction. Up to taking a not relabelled subsequence, by the contradiction assumption, there exists a sequence $\{q_k\}_k \subseteq \Gamma_f$ such that for every k,

$$\iota_k(q_k) \in B_\rho^{\mathsf{Z}\times\mathbb{R}}(\iota_k(p)) \setminus B_\varepsilon^{\mathsf{Z}\times\mathbb{R}}(\iota_\infty(\partial H)).$$

Up to a not relabelled subsequence, $\iota_k(q_k) \to \iota_\infty(q) \in \mathsf{Z}$, with $\mathsf{d}_{\mathsf{Z}\times\mathbb{R}}(\iota_\infty(q), \iota_\infty(\partial H)) \ge \varepsilon/2$, for some $q \in \mathbb{R}^n \times \mathbb{R}$ (the fact that the limit point of $\{\iota_k(q_k)\}_k$ belongs to $\iota_\infty(\mathbb{R}^n \times \mathbb{R})$ is an easy consequence of the doubling property of the measure). By weak convergence of rescaled perimeters,

$$\lim_k \frac{\rho_k |\mathrm{D}\chi_{\mathcal{G}_f}|(B_{\varepsilon\rho_k/2}(q_k))}{C_p^{\rho_k}} = 0.$$

On the other hand, recalling (6.1.4) and using again the weak convergence of measures,

$$\begin{aligned}\liminf_k \frac{\rho_k |\mathrm{D}\chi_{\mathcal{G}_f}|(B_{\varepsilon\rho_k/2}(q_k))}{C_p^{\rho_k}} &= \liminf_k \frac{\rho_k |\mathrm{D}\chi_{\mathcal{G}_f}|(B_{\varepsilon\rho_k/2}(q_k))}{(\mathsf{m} \otimes \mathcal{L}^1)(B_{\varepsilon\rho_k/2}(q_k))} \frac{(\mathsf{m} \otimes \mathcal{L}^1)(B_{\varepsilon\rho_k/2}(q_k))}{C_p^{\rho_k}} \\ &\ge l^{-1}\underline{\mathcal{L}}^{n+1}(B_{\varepsilon/2}(q)) > 0,\end{aligned}$$

which is a contradiction.

Clearly, the analogue statement holds with g in place of f.

Step 3: tensorization of inequalities. We start from a couple of useful equalities. First, by (6.1.9), if $p' \in \Gamma_f$,

$$\lim_{r \searrow 0} \frac{|\mathrm{D}\chi_{\mathcal{G}_f}|(B_r(p') \setminus \Gamma_f)}{|\mathrm{D}\chi_{\mathcal{G}_f}|(B_r(p'))} = 0,$$

so that, for every $p = (p', t) \in \Gamma_f \times \mathbb{R}$, as $|\mathrm{D}\chi_{\mathcal{G}_f\times\mathbb{R}}| = |\mathrm{D}\chi_{\mathcal{G}_f}| \otimes \mathcal{L}^1$,

$$\lim_{r \searrow 0} \frac{|\mathrm{D}\chi_{\mathcal{G}_f\times\mathbb{R}}|(B_r(p) \setminus (\Gamma_f \times \mathbb{R}))}{|\mathrm{D}\chi_{\mathcal{G}_f\times\mathbb{R}}|(B_r(p))} = 0. \tag{6.1.10}$$

Also, by (6.1.5) together with Fubini's Theorem (see e.g. the first equality in (5.1.8)),

$$\Theta_{n+2}(\mathsf{m} \otimes \mathcal{L}^1 \otimes \mathcal{L}^1, q) \in (l^{-1}, l) \qquad \text{for every } q \in R \times \mathbb{R} \times \mathbb{R}. \tag{6.1.11}$$

Step 4: blow-up argument. Here and below we denote with $\tau : \mathsf{Y} \times \mathbb{R} \times \mathbb{R}$ the involution given by $(x, t, s) \mapsto (x, s, t)$, for Y any set. Now we set $V := (u \circ \pi^1, \pi^2, \pi^3)$. Clearly,

$$V : \Gamma_f \times \mathbb{R} \to \mathbb{R}^{n+1} \qquad \text{and} \qquad V : \tau(\Gamma_g \times \mathbb{R}) \to \mathbb{R}^{n+1} \tag{6.1.12}$$

are bilipschitz onto their $(n+1)$-dimensional image, by **Step 1**. Set also $V_p := (A(\pi_1(p))u \circ \pi^1, \pi^2, \pi^3)$. Fix $p \in \tilde{R}$, we aim at preparing the setting for the proof of **Step 5** below, i.e. the proof of (6.1.16) and (6.1.17).

By our assumptions and by Remark 3.3.3, we know that there exists a sequence $\rho_k \searrow 0$ and a proper metric space $(\mathsf{Z}, \mathsf{d}_\mathsf{Z})$ that realizes the convergence

$$(\mathsf{X}, \rho_k^{-1}\mathsf{d}, \mathsf{m}_p^{\rho_k}, p) \to (\mathbb{R}^n, \mathsf{d}_e, \underline{\mathcal{L}}^n, 0).$$

Hence, $(\mathsf{Z} \times \mathbb{R} \times \mathbb{R}, \mathsf{d}_{\mathsf{Z}\times\mathbb{R}\times\mathbb{R}})$ realizes the convergence

$$(\mathsf{X} \times \mathbb{R} \times \mathbb{R}, \rho_k^{-1}\mathsf{d}_{\mathsf{X}\times\mathbb{R}\times\mathbb{R}}, (\mathsf{m} \otimes \mathcal{L}^1 \otimes \mathcal{L}^1)_p^{\rho_k}, p) \to (\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}, \mathsf{d}_e, \underline{\mathcal{L}}^{n+2}, 0),$$

for suitable isometric embeddings $\iota_k : (\mathsf{X}\times\mathbb{R}\times\mathbb{R}, \rho_k^{-1}\mathsf{d}_{\mathsf{X}\times\mathbb{R}\times\mathbb{R}}) \to (\mathsf{Z}\times\mathbb{R}\times\mathbb{R}, \mathsf{d}_{\mathsf{Z}\times\mathbb{R}\times\mathbb{R}})$ and $\iota_\infty : (\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}, \mathsf{d}_e) \to (\mathsf{Z} \times \mathbb{R} \times \mathbb{R}, \mathsf{d}_{\mathsf{Z}\times\mathbb{R}\times\mathbb{R}})$. Then, up to taking not relabelled subsequences (Remark 3.3.3) we can assume that, in the same realization, we have

$$(\mathsf{X} \times \mathbb{R} \times \mathbb{R}, \rho_k^{-1}\mathsf{d}_{\mathsf{X}\times\mathbb{R}\times\mathbb{R}}, (\mathsf{m} \otimes \mathcal{L}^1 \otimes \mathcal{L}^1)_p^{\rho_k}, p, \mathcal{G}_f \times \mathbb{R}) \to (\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}, \mathsf{d}_e, \underline{\mathcal{L}}^{n+2}, 0, H'_f) \quad (6.1.13)$$
$$(\mathsf{X} \times \mathbb{R} \times \mathbb{R}, \rho_k^{-1}\mathsf{d}_{\mathsf{X}\times\mathbb{R}\times\mathbb{R}}, (\mathsf{m} \otimes \mathcal{L}^1 \otimes \mathcal{L}^1)_p^{\rho_k}, p, \tau(\mathcal{G}_g \times \mathbb{R})) \to (\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}, \mathsf{d}_e, \underline{\mathcal{L}}^{n+2}, 0, H'_g), \quad (6.1.14)$$

where H'_f and H'_g are half-spaces in $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$. Further, up to passing again to a not relabelled subsequence and changing coordinates in $\mathbb{R}^n$, we will assume that the maps V_p, properly rescaled (i.e. $\{\rho_k^{-1}V_p\}_k$), locally uniformly converge to the coordinate functions of $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ with respect to the convergences above, see Remark 3.2.17 (here and after we assume, to simplify the notation, that $V_p(p) = 0$).

By (5.1.3) and a tensorization argument, it follows that

$$H'_f = \{(z, u, v) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : (z, u) \cdot (\tilde{A}(x, u)\nu^u_{\mathcal{G}_f}(x, u)) \geq 0\},$$

and

$$H'_g = \{(z, u, v) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : (z, v) \cdot (\tilde{A}(x, v)\nu^u_{\mathcal{G}_g}(x, v)) \geq 0\},$$

so that, by $p \in \tilde{R}$,

$$H'_f = H_f \times \mathbb{R} \times \mathbb{R}, \ \tilde{H}'_g = H_g \times \mathbb{R} \times \mathbb{R} \text{ and } H_f \neq \pm H_g$$

for H_f and H_g half-spaces of $\mathbb{R}^n$ (here we also used that $A(x)$ is invertible).

Fix $C \geq 5$ greater than the bilipschitz constants of the maps in (6.1.12) and such that

$$|(A(x), \pi^1, \pi^2)c| \leq (C - 2)|c| \qquad \text{for every } c \in \mathbb{R}^{n+2} \quad (6.1.15)$$

and let $\delta \in (0, C^{-1})$ be sufficiently small so that we find $a \in (\partial H_f \times \mathbb{R} \times \mathbb{R}) \cap B_{1/2}(0) \subseteq \mathbb{R}^{n+2}$ such that

$$B_{2C\delta}(a) \cap (\partial H_g \times \mathbb{R} \times \mathbb{R}) = \emptyset.$$

As a consequence of (6.1.10), we can find a sequence $\{a_k\}_k \subseteq \mathsf{X} \times \mathbb{R} \times \mathbb{R}$ with

$$a_k \in (\Gamma_f \times \mathbb{R}) \cap B_{\rho_k/2}(p) \qquad \text{for } k \text{ large enough,}$$

and $\iota_k(a_k) \to \iota_\infty(a)$ in $\mathsf{Z} \times \mathbb{R} \times \mathbb{R}$. Up to decreasing δ, we assume that

$$B_{C\delta\rho_k}(a_k) \subseteq B_{\rho_k}(p) \qquad \text{for } k \text{ large enough.}$$

Step 5: transversality. We claim that, with the notation introduced in **Step 4**, (recall the definition in (6.1.7))

$$\liminf_k \rho_k^{-n-1}\mathcal{H}^{n+1}\big(B_{\delta\rho_k}(V(a_k))\cap(\Gamma'_f\times\mathbb{R})\big)>0, \tag{6.1.16}$$

$$\liminf_k \rho_k^{-n-1}\mathcal{H}^{n+1}\big(B_{\delta\rho_k}(V(a_k))\cap\tau(\Gamma'_g\times\mathbb{R})\big)=0. \tag{6.1.17}$$

By weak convergence of rescaled perimeters,

$$\liminf_k \frac{\rho_k|\mathrm{D}\chi_{\mathcal{G}_f\times\mathbb{R}}|(B_{C^{-1}\delta\rho_k}(a_k))}{C_p^{\rho_k}}>0.$$

Taking into account (6.1.10) and (6.1.11) together Remark 3.3.3 to deal with the denominators, using (3.5.2), we see that the equation above reads

$$\liminf_k \rho_k^{-n-1}\mathcal{H}^{n+1}\big(B_{C^{-1}\delta\rho_k}(a_k)\cap(\Gamma_f\times\mathbb{R})\big)>0.$$

Therefore, it is easy to verify by contradiction that (6.1.16) follows, by our choice of C.

Now we concentrate on (6.1.17). We claim that there exists $k_0\in\mathbb{N}$ such that

$$V\big(B_{2C^2\rho_k}(p)\cap\tau(\Gamma_g\times\mathbb{R})\big)\cap B_{\delta\rho_k}(V(a_k))=\emptyset.\qquad\text{if } k\ge k_0. \tag{6.1.18}$$

Notice that (6.1.17) would follow from (6.1.18). Indeed, as $\delta<C$ and $a_k\in B_{\rho_k}(p)$,

$$\begin{aligned}B_{\delta\rho_k}(V(a_k))\cap\tau(\Gamma'_g\times\mathbb{R})&\subseteq B_{2C\rho_k}(V(p))\cap B_{\delta\rho_k}(V(a_k))\cap\tau(\Gamma'_g\times\mathbb{R})\\&\subseteq V\big(B_{2C^2\rho_k}(p)\cap\tau(\Gamma_g\times\mathbb{R})\big)\cap B_{\delta\rho_k}(V(a_k)),\end{aligned}$$

so that (6.1.18) implies that the sets considered in (6.1.17) are empty for $k\ge k_0$.

Therefore, we have reduced ourselves to prove (6.1.18). Recall that $\rho_k^{-1}V_p$ locally uniformly converge to the coordinate functions of $\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}$. Hence, there exist $\varepsilon\in(0,\delta)$ and $k_0\in\mathbb{N}$ such that for every $\xi\in B_{3C^2\rho_k}(p)$ and $\xi'\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}$ with $\mathsf{d}_{\mathsf{Z}\times\mathbb{R}\times\mathbb{R}}(\iota_k(\xi),\iota_\infty(\xi'))<\varepsilon$, then $|\rho_k^{-1}V_p(\xi)-\xi'|<\delta$ for every $k\ge k_0$. Up to increasing k_0, we may assume that for every $k\ge k_0$,

$$\mathsf{d}_{\mathsf{Z}\times\mathbb{R}\times\mathbb{R}}(\iota_k(a_k),\iota_\infty(a))<\varepsilon.$$

By (a tensorization of) **Step 2**, we get that, up to increasing k_0, if $k\ge k_0$, then

$$\iota_k(B_{2C^2\rho_k}(p)\cap\tau(\Gamma_g\times\mathbb{R}))\subseteq B_\varepsilon^{\mathsf{Z}\times\mathbb{R}\times\mathbb{R}}(\iota_\infty(\partial H_g\times\mathbb{R}\times\mathbb{R})). \tag{6.1.19}$$

Hence, take $k\ge k_0$ and $b\in B_{2C^2\rho_k}(p)\cap\tau(\Gamma_g\times\mathbb{R})$, to show (6.1.18) we have to prove that

$$|V(b)-V(a_k)|\ge\delta\rho_k. \tag{6.1.20}$$

Notice that (6.1.20) does *not* follow from the fact that the maps in (6.1.12) are C-bilipschitz. By (6.1.19), the exists $b'\in\partial H_g\times\mathbb{R}\times\mathbb{R}$ such that

$$\mathsf{d}_{\mathsf{Z}\times\mathbb{R}\times\mathbb{R}}(\iota_k(b),\iota_\infty(b'))<\varepsilon.$$

Notice that if b' is as above, then, by our choice of δ and a,

$$|b'-a|\ge C\delta.$$

By local uniform convergence,

$$|\rho_k^{-1}V_p(b) - \rho_k^{-1}V_p(a_k)| \geq |b' - a| - 2\delta \qquad \text{if } k \geq k_0$$

so that

$$|V_p(b) - V_p(a_k)| \geq (C-2)\delta\rho_k,$$

which implies, recalling (6.1.15), (6.1.20).

Step 6: proof of σ-finiteness of $\mathcal{H}^n \llcorner \tilde{R}$. We prove that $\mathcal{H}^n \llcorner \tilde{R}$ is σ-finite. Notice that $\tilde{R} \subseteq (\Gamma_f \times \mathbb{R}) \cap \tau(\Gamma_g \times \mathbb{R})$, hence it is enough to show that $V(\tilde{R})$ is σ-finite with respect to $\mathcal{H}^n$. By **Step 1**, $V(\tilde{R}) \subseteq (\Sigma_f \times \mathbb{R}) \cap \tau(\Sigma_g \times \mathbb{R})$, so that, by a standard result of in Geometric Measure Theory on Euclidean spaces, we can simply show that at every $p = (x,t,s) \in \tilde{R}$ it holds that $\Sigma_f \times \mathbb{R} \supseteq \Gamma_f' \times \mathbb{R}$ and $\tau(\Sigma_g \times \mathbb{R}) \supseteq \tau(\Gamma_g' \times \mathbb{R})$ intersect transversally at $V(p)$, or, equivalently, that $\Sigma_f \times \mathbb{R}$ and $\tau(\Sigma_g \times \mathbb{R})$ have different tangent spaces at $V(p)$. Now, by (6.1.16) and (6.1.17) together with (the tensorized version of (6.1.8)) it follows easily that $\Gamma_f' \times \mathbb{R}$ and $\tau(\Gamma_g' \times \mathbb{R})$ have different tangent spaces at 0, whence the conclusion (by the tensorized version of (6.1.8) again). Hence we write

$$\tilde{R} = \bigcup_{i \in \mathbb{N}} \tilde{R}_i,$$

where, for every i, $\tilde{R}_i$ has finite $\mathcal{H}^n$ measure.

Step 7: a technical estimate. Fix $p \in \tilde{R}$. We claim that

$$\lim_{r \searrow 0} \frac{\mathcal{H}^n_5\big((\pi^1,\pi^2)(\tilde{R} \cap B_r(p))\big)}{r^n} = 0.$$

Let us prove the claim. Take a sequence $\rho_k \searrow 0$. We recall, that, with the same notation as in **Step 4**, up to a not relabelled subsequence, we can assume that (6.1.13) and (6.1.14) hold. Let

$$I := I\big(\iota_\infty((\partial H_f \cap \partial H_g) \times \mathbb{R} \times \mathbb{R})\big)$$

be a neighbourhood (in $\mathsf{Z} \times \mathbb{R} \times \mathbb{R}$) of $\iota_\infty(\partial H_f \cap \partial H_g) \times \mathbb{R} \times \mathbb{R}$ that satisfies, for $\varepsilon \in (0,1)$,

$$\mathcal{H}^n_5((\pi^1,\pi^2)(I)) < \varepsilon.$$

This is possible since $\partial H_f \neq \partial H_g$. As a consequence of (a tensorized version of) **Step 2**, there exists $k_0 \in \mathbb{N}$ such that

$$B_1^{\mathsf{Z}\times\mathbb{R}\times\mathbb{R}}(\iota_k(p)) \cap \iota_k(\tilde{R}) \subseteq I \qquad \text{for every } k \geq k_0,$$

from which, taking the projection (π^1,π^2), the claim follows as $\varepsilon \in (0,1)$ was arbitrary.

Step 8: conclusion. By **Step 6**, it is enough to show that

$$(|\mathrm{D}f| \wedge |\mathrm{D}g|)(\pi^1(\tilde{R}_i)) = 0,$$

for every $i \in \mathbb{N}$. We concentrate on a fixed i and, for simplicity, we drop the subscript i from $\tilde{R}_i$. Therefore, $\mathcal{H}^n(\tilde{R}) < \infty$. Fix $\varepsilon > 0$. For every $j \in \mathbb{N},\ j \geq 1$ we consider the sets

$$\tilde{R}_j := \big\{p \in \tilde{R} : r^{-n}\mathcal{H}^n_5\big((\pi^1,\pi^2)(\tilde{R} \cap B_r(p))\big) < \varepsilon \text{ for every } r \in (0, j^{-1})\big\}$$

and
$$\tilde R_j' := \tilde R_j \setminus \bigcup_{i<j} \tilde R_i.$$
Notice that, by **Step 7**,
$$\tilde R = \bigcup_{j\ge 1} \tilde R_j', \tag{6.1.21}$$
and that, by construction, this union is disjoint.

For every $j \ge 1$, we take a countable family of balls $\{B_{r_i^j}(p_i^j)\}_{i\in\mathbb{N}}$ such that, for every $i \in \mathbb{N}$ it holds that $r_i^j < j^{-1}$ and $p_i^j \in \tilde R_j'$, as well as
$$\tilde R_j' \subseteq \bigcup_i B_{r_i^j}(p_i^j) \qquad \text{and} \qquad \sum_i (r_i^j)^n \le 2^n \mathcal{H}^n(\tilde R_j') + 2^{-j}. \tag{6.1.22}$$
We can compute, recalling the definition of $\tilde R_j$ and (6.1.22),
$$\mathcal{H}^n_5((\pi^1,\pi^2)(\tilde R_j')) \le \mathcal{H}^n_5\Big((\pi^1,\pi^2)\Big(\tilde R \cap \bigcup_i B_{r_i^j}(p_i^j)\Big)\Big) \le \sum_i \varepsilon (r_i^j)^n \le \varepsilon(2^n\mathcal{H}^n(\tilde R_j') + 2^{-j}).$$
Therefore, recalling (6.1.21), we see that
$$\mathcal{H}^n_5((\pi^1,\pi^2)(\tilde R)) \le \varepsilon(2^n\mathcal{H}^n(\tilde R) + 1),$$
and, being $\varepsilon > 0$ arbitrary, $|\mathrm{D}\chi_{\mathcal{G}_f}|((\pi^1,\pi^2)(\tilde R)) = 0$, whence the conclusion follows thanks to Proposition 2.3.12 and the fact that $R \subseteq D_f$. □

6.2 Bibliographical notes

In 1988 Ambrosio and De Giorgi [14], motivated by the study of some functionals coming from the Mathematical Physics, conjectured that for every $f \in \mathrm{BV}^m$ the matrix $\frac{\mathrm{d}\mathrm{D}f}{\mathrm{d}|\mathrm{D}f|}$ has rank one $|\mathrm{D}f|^s$-almost everywhere. In 1993 Alberti [1] solved in the affirmative the previous conjecture, see also the account in [66].

As an added value to the theoretical interest of the Rank one Theorem, in 2016 De Philippis and Rindler [68] showed a general structure theorem for $\mathcal{A}$-free vector valued Radon measures on Euclidean spaces, where $\mathcal{A}$ is a linear constant-coefficient differential operator, from which the Rank one Theorem can be derived as a consequence. As a side note, we mention that the main result of [68] is used to prove (2.2.12), namely that the reference measure of an RCD space is absolutely continuous with respect to the Hausdorff measure, see [67, 98, 90]. We also remark that Massaccesi and Vittone recently gave a very short proof of the Rank one Theorem based on the theory of sets of finite perimeter [106], and with Don they used this simplified strategy to prove the analogue of the Rank one Theorem in some Carnot groups [73].

The latter strategy is the starting point of our investigation and it is precisely the strategy of [106, 73] that we have adapted to the non-smooth framework in [32], from which this chapter is taken. Exactly as in the references, the bulk of the proof is the transversality lemma, Lemma 6.1.3. However, the lack of a linear structure forces us to perform a delicate fine analysis, and to resort ultimately to a well-known transversality lemma for C^1 surfaces in the Euclidean space.

In this manuscript, the proof of Lemma 6.1.3 has undergone heavy rewriting (starting from [32]), mostly for what concerns the presentation. Besides some minor improvements, a consistent simplification is obtained by employing Lemma 3.4.2 instead of [51, Proposition 4.7]. There are no deep changes in the techniques, but the possibility of "throwing away" only a set of small perimeter instead of a set of small perimeter together with a set of small content allow us to make the proof less intricate. The possibility of doing so comes from the fact that in this manuscript we have moved up "throwing away" a set of small content to the proof of Lemma 3.2.19, making also the rectifiability lemma, Lemma 3.4.2, more transparent. Moreover, still concerning Lemma 6.1.3, we state and prove the result with respect to a single good splitting map, and not with respect a countable family of good splitting maps, as done in [32]. This improvement is possible thanks to the discussion that we have just made, still thanks to the fact that we use Lemma 3.4.2 instead of [51, Proposition 4.7]. To be more precise, here we also modify the definition of good splitting map, but, taking into account Remark 3.2.13 and Lemma 3.2.14, this does not make a big difference.

Chapter 7

Nonlocal characterization

The aim of this chapter is to show that, even on (finite dimensional) RCD spaces, the total variation of a BV function (as well as the membership to the space BV) can be recognized via a limit of certain nonlocal functionals depending on a parameter, namely, exploiting the short-time behaviour of the heat flow. In particular, we are going to study the limit, as $t \searrow 0$, of the functionals

$$L^1 \ni f \mapsto \frac{1}{\sqrt{t}} \int \int p_t(x,y)|f(x)-f(y)|\mathrm{dm}(x)\mathrm{dm}(y).$$

7.1 Main results

Now we state the main results of this chapter. As discussed above, the first result allows us to compute the total variation of a BV functions in terms of quantities that are nonlocal.

Theorem 7.1.1. *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *be an* $\mathrm{RCD}(K,N)$ *space and let* $f \in L^1(\mathsf{m})$*. Then*

$$\lim_{t\searrow 0} \frac{1}{\sqrt{t}} \int \int p_t(x,y)|f(x)-f(y)|\mathrm{dm}(x)\mathrm{dm}(y) = \frac{2}{\sqrt{\pi}}|\mathrm{D}f|(\mathsf{X}),$$

where the existence of the limit is part of the statement and $|\mathrm{D}f|(\mathsf{X}) = +\infty$ *has to be interpreted as* $f \notin \mathrm{BV}(\mathsf{X})$*.*

The second result studies the first term in the Taylor expansion (with respect to $\sqrt{t}$) of the quantity

$$\int fg\mathrm{dm} - \int \mathrm{h}_t fg\mathrm{dm}.$$

Theorem 7.1.2. *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *be an* $\mathrm{RCD}(K,N)$ *space and let* $f,g \in \mathrm{BV}(\mathsf{X})$ *with* $g \in L^\infty(\mathsf{m})$*. Then*

$$\lim_{t\searrow 0} \frac{1}{\sqrt{t}} \int (f-\mathrm{h}_t f)g\mathrm{dm} = \frac{1}{\sqrt{\pi}}\frac{\omega_{n-1}}{\omega_n}\int_{J_f\cap J_g}(f^\vee - f^\wedge)(g^\vee - g^\wedge)(\nu_f\cdot\nu_g)\mathrm{d}\mathcal{H}^h. \tag{7.1.1}$$

We defer the proof of Theorem 7.1.1 and Theorem 7.1.2 to Section 7.1.2 below.

7.1.1 Auxiliary results

The following lemma is concerned with the part of “recognizing membership in BV” stated in Theorem 7.1.1. It is a straightforward application of a result of [105].

Lemma 7.1.3. *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *be an* $\mathrm{RCD}(K,N)$ *space and let* $f\in L^1(\mathsf{m})$. *Assume that*

$$\liminf_{t\searrow 0}\frac{1}{\sqrt{t}}\int\int p_t(x,y)|f(x)-f(y)|\mathrm{d}\mathsf{m}(x)\mathrm{d}\mathsf{m}(y)<+\infty.$$

Then $f\in\mathrm{BV}(\mathsf{X})$.

Proof. Given any radius $t\in(0,\infty)$, we define the near-diagonal set $\Delta_t\subseteq\mathsf{X}\times\mathsf{X}$ as

$$\Delta_t:=\{(x,y)\in\mathsf{X}\times\mathsf{X}:\mathsf{d}(x,y)<t\}.$$

Since $(\mathsf{X},\mathsf{d},\mathsf{m})$ is a PI space, we know from [105, Theorem 3.1] that for any $f\in L^1(\mathsf{m})$ if

$$\liminf_{t\searrow 0}\frac{1}{t}\int_{\Delta_t}\frac{|f(x)-f(y)|}{\sqrt{\mathsf{m}(B_t(x))}\sqrt{\mathsf{m}(B_t(y))}}\,\mathrm{d}(\mathsf{m}\otimes\mathsf{m})\,(x,y)<+\infty,$$

then $f\in\mathrm{BV}(\mathsf{X})$. Now we conclude as the lower bound in (2.2.7) implies that for some constant $C=C(K,N)\in(0,\infty)$,

$$\frac{\chi_{\Delta_t}(x,y)}{\sqrt{\mathsf{m}(B_t(x))}\sqrt{\mathsf{m}(B_t(y))}}\le Cp_{t^2}(x,y),\qquad\text{for every } t\in(0,1)\text{ and } x,y\in\mathsf{X}.$$ □

Now we state a result which is the main tool to compute total variations starting from the nonlocal quantities as in Theorem 7.1.1 and Theorem 7.1.2. Its proof is based on a delicate blow-up argument, which relies heavily on the fine structure theory of RCD spaces of the previous chapters as well as the special properties of the heat flow on RCD spaces.

Lemma 7.1.4. *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *be an* $\mathrm{RCD}(K,N)$ *space and let* $E\subseteq\mathsf{X}$ *be a set of finite perimeter. Then*

$$\lim_{t\searrow 0}t\mathrm{h}_{t^2}|\nabla\mathrm{h}_{t^2}\chi_E|(x)=\frac{1}{\sqrt{8\pi}}\qquad\text{for every } x\in\mathcal{F}E. \tag{7.1.2}$$

In particular, it holds that

$$\lim_{t\searrow 0}t\int\mathrm{h}_{t^2}|\nabla\mathrm{h}_{t^2}\chi_E|\mathrm{d}|\mathrm{D}\chi_E|=\frac{1}{\sqrt{8\pi}}|\mathrm{D}\chi_E|(\mathsf{X}). \tag{7.1.3}$$

Proof. Fix x as in the statement and take a sequence $t_i\searrow 0$. Up to taking a subsequence, we have that

$$(\mathsf{X},t_i^{-1}\mathsf{d},\mathsf{m}_{t_i}^x,x,E)\to(\mathbb{R}^n,\mathsf{d}_e,\underline{\mathcal{L}}^n,0,H),$$

in a realization $(\mathsf{Z},\mathsf{d}_\mathsf{Z})$, where $(\mathsf{Z},\mathsf{d}_\mathsf{Z})$ is a proper metric space, see Remark 3.3.3. Here $H=\{x_n>0\}$, in a suitable system of coordinates.

We use the superscript i to denote objects relative to the i-th rescaled space, e.g. ∇^i and p^i, and we denote with $\tilde{\ }$ the objects relative to the limit space $(\mathbb{R}^n,\mathsf{d}_e,\underline{\mathcal{L}}^n,0)$. Then we compute for m-a.e. $y\in\mathsf{X}$,

$$\begin{aligned}t_i|\nabla\mathrm{h}_{t_i^2}\chi_E|(y)&=t_i\left|\nabla\int p_{t_i^2}(\cdot,z)\chi_E(z)\mathrm{d}\mathsf{m}(z)\right|(y)=\left|\nabla^i\int p_1^i(\cdot,z)\chi_E(z)\mathrm{d}\mathsf{m}_{t_i}^x(z)\right|(y)\\&=|\nabla^i\mathrm{h}_1^i\chi_E|(y).\end{aligned} \tag{7.1.4}$$

Now observe that for any $R \in (0,\infty)$ we can use (2.2.5) to estimate

$$\begin{aligned}\int_{\mathsf{X}\setminus B^i_R(x)} p^i_1(x,y)|\nabla^i \mathrm{h}^i_1 \chi_E|(y)\mathrm{d}\mathsf{m}^x_{t_i}(y) \leq & C\int_{\mathsf{X}\setminus B^i_R(x)} p^i_1(x,y)\mathrm{d}\mathsf{m}^x_{t_i}(y) \\ =& C\int_{\mathsf{X}\setminus B_{t_i R}(x)} p_{t_i^2}(x,y)\mathrm{d}\mathsf{m}(y) \\ \leq & Ce^{-\frac{R^2}{24}},\end{aligned} \tag{7.1.5}$$

where the last equality is due to (4.2.4) and $C \in (0,\infty)$ denotes a constant that depends only on K and N.

Now we prove that

$$\mathrm{h}^i_1\big(\chi_{B^i_R(x)}|\nabla^i \mathrm{h}^i_1\chi_E|\big)(x) \to \tilde{\mathrm{h}}_1\big(\chi_{\tilde{B}_R(0)}|\tilde{\nabla}\tilde{\mathrm{h}}_1\chi_H|\big)(0).$$

We know from [10, Proposition 4.12] that $|\nabla^i \mathrm{h}^i_1\chi_E|\mathsf{m}^x_{t_i} \to |\tilde{\nabla}\tilde{\mathrm{h}}_1\chi_H|\tilde{\mathcal{L}}^n$ in duality with $C_{\mathrm{bs}}(\mathsf{Z})$, so that, for any $R\in(0,\infty)$, we have that

$$\chi_{B^i_R(x)}|\nabla^i \mathrm{h}^i_1\chi_E|\mathsf{m}^x_{t_i} \to \chi_{\tilde{B}_R(0)}|\tilde{\nabla}\tilde{\mathrm{h}}_1\chi_H|\tilde{\mathcal{L}}^n \qquad \text{in duality with } C_{\mathrm{bs}}(\mathsf{Z}).$$

Now, by [10, Lemma 4.11] we have that

$$\liminf_i \mathrm{h}^i_1\big(\chi_{B^i_R(x)}|\nabla^i \mathrm{h}^i_1\chi_E|\big)(x) \geq \tilde{\mathrm{h}}_1\big(\chi_{\tilde{B}_R(0)}|\tilde{\nabla}\tilde{\mathrm{h}}_1\chi_H|\big)(0). \tag{7.1.6}$$

Also, as $\mathsf{m}^{r_i}_x \to \underline{\mathcal{L}}^n$ in duality with $C_{\mathrm{bs}}(\mathsf{Z})$, we have that

$$\left(\frac{1}{\sqrt{2}e^K} - \chi_{B^i_R(x)}|\nabla^i \mathrm{h}^i_1\chi_E|\right)\mathsf{m}^x_{t_i} \to \left(\frac{1}{\sqrt{2}e^K} - \chi_{\tilde{B}_R(0)}|\tilde{\nabla}\tilde{\mathrm{h}}_1\chi_H|\right)\tilde{\mathcal{L}}^n \qquad \text{in duality with } C_{\mathrm{bs}}(\mathsf{Z}).$$

As by (2.2.5) the measures involved in the convergence above are non-negative, we can use [10, Lemma 4.11] again to deduce that

$$\liminf_i \mathrm{h}^i_1\left(\frac{1}{\sqrt{2}e^K} - \chi_{B^i_R(x)}|\nabla^i \mathrm{h}^i_1\chi_E|\right)(x) \geq \tilde{\mathrm{h}}_1\left(\frac{1}{\sqrt{2}e^K} - \chi_{\tilde{B}_R(0)}|\tilde{\nabla}\tilde{\mathrm{h}}_1\chi_H|\right)(0),$$

i.e.

$$\limsup_i \mathrm{h}^i_1\big(\chi_{B^i_R(x)}|\nabla^i \mathrm{h}^i_1\chi_E|\big)(x) \leq \tilde{\mathrm{h}}_1\big(\chi_{\tilde{B}_R(0)}|\tilde{\nabla}\tilde{\mathrm{h}}_1\chi_H|\big)(0). \tag{7.1.7}$$

Therefore (7.1.3) follows from (7.1.6) and (7.1.7).

Notice that, as $R \nearrow \infty$,

$$\tilde{\mathrm{h}}_1\big(\chi_{\tilde{B}_R(0)}|\tilde{\nabla}\tilde{\mathrm{h}}_1\chi_H|\big)(0) \to \tilde{\mathrm{h}}_1\big(|\tilde{\nabla}\tilde{\mathrm{h}}_1\chi_H|\big)(0) = \frac{1}{\sqrt{8\pi}}, \tag{7.1.8}$$

where the limit is due to monotone convergence and the last equality follows from the direct computation

$$\begin{aligned}|\nabla^{\mathbb{R}^n}\mathrm{h}^{\mathbb{R}^n}_1\chi_H| &= \left|\nabla^{\mathbb{R}^n}\int_{\{z_n>0\}} p^{\mathbb{R}^n}_1(x,z)\mathrm{d}\mathcal{L}^n(z)\right| = \left|\frac{\mathrm{d}}{\mathrm{d}x}\int_{\{z>0\}} p^{\mathbb{R}}_1(x,z)\mathrm{d}\mathcal{L}^1(z)\right| \\ &= \left|\int_{\{z>0\}}\frac{\mathrm{d}}{\mathrm{d}x}p^{\mathbb{R}}_1(x,z)\mathrm{d}\mathcal{L}^1(z)\right| = \left|\int_{\{z>0\}}\frac{\mathrm{d}}{\mathrm{d}z}p^{\mathbb{R}}_1(x,z)\mathrm{d}\mathcal{L}^1(z)\right| = p^{\mathbb{R}}_1(x,0),\end{aligned}$$

which yields

$$\mathrm{h}_1^{\mathbb{R}^n}\big(|\nabla^{\mathbb{R}^n}\mathrm{h}_1^{\mathbb{R}^n}\chi_H|\big)(0)=(\mathrm{h}_1^{\mathbb{R}}p_1^{\mathbb{R}}(\,\cdot\,,0))(0)=p_2^{\mathbb{R}}(0,0)=\frac{1}{\sqrt{8\pi}}.$$

Next, we can estimate, exploiting (7.1.5),

$$\begin{aligned}\left|\int p_1^i(x,y)|\nabla^i\mathrm{h}_1^i\chi_E|(y)\mathrm{d}\mathsf{m}_{t_i}^x(y)-\frac{1}{\sqrt{8\pi}}\right|&\le\left|\int_{B_R^i(x)}p_1^i(x,y)|\nabla^i\mathrm{h}_1^i\chi_E|(y)\mathrm{d}\mathsf{m}_{t_i}^x(y)-\frac{1}{\sqrt{8\pi}}\right|+Ce^{-\frac{R^2}{24}}\\&=\left|\mathrm{h}_1^i\big(\chi_{B_R^i(x)}|\nabla^i\mathrm{h}_1^i\chi_E|\big)(x)-\frac{1}{\sqrt{8\pi}}\right|+Ce^{-\frac{R^2}{24}},\end{aligned}$$

whence if we let first $i\nearrow\infty$ and then $R\nearrow\infty$, keeping in mind (7.1.8), we deduce that

$$\int p_1^i(x,y)|\nabla^i\mathrm{h}_1^i\chi_E|(y)\mathrm{d}\mathsf{m}_{t_i}^x(y)\to\frac{1}{\sqrt{8\pi}},$$

that reads, recalling (7.1.4),

$$t_i\mathrm{h}_{t_i^2}|\nabla\mathrm{h}_{t_i^2}\chi_E|(x)\to\frac{1}{\sqrt{8\pi}}.$$

Since the sequence $t_i\searrow 0$ from which we started was arbitrary, (7.1.2) follows.

Now we show (7.1.3). First notice that by Corollary 3.3.2, (7.1.2) holds for $|\mathrm{D}\chi_E|$-a.e. $x\in\mathsf{X}$. Hence (7.1.3) follows by dominated convergence, where the application of dominated convergence is justified by (2.2.5) and the maximum principle. □

The following lemma is a consequence of Lemma 7.1.4 that we are going to use in the proof of the main results. In the case in which $E=F$, the conclusion comes directly from L'Hôpital's rule and the blow-up computation of Lemma 7.1.4. In the case in which $E\neq F$, we also need a polarization argument.

Lemma 7.1.5. *Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,N)$ space of essential dimension n. Let $E,F\subseteq\mathsf{X}$ be two sets of finite perimeter and finite measure. Then*

$$\lim_{t\searrow 0}\frac{1}{\sqrt{t}}\int(\chi_E-\mathrm{h}_t\chi_E)\chi_F\mathrm{d}\mathsf{m}=\frac{1}{\sqrt{\pi}}\frac{\omega_{n-1}}{\omega_n}\int_{\mathcal{F}E\cap\mathcal{F}F}\nu_E\cdot\nu_F\mathrm{d}\mathcal{H}^h.\tag{7.1.9}$$

In particular, it holds that

$$\lim_{t\searrow 0}\frac{1}{\sqrt{t}}\int(\chi_E-\mathrm{h}_t\chi_E)\chi_E\mathrm{d}\mathsf{m}=\frac{1}{\sqrt{\pi}}|\mathrm{D}\chi_E|(\mathsf{X})=\frac{1}{\sqrt{\pi}}\frac{\omega_{n-1}}{\omega_n}\mathcal{H}^h(\mathcal{F}E).\tag{7.1.10}$$

Proof. We have to compute

$$\lim_{t\searrow 0}\frac{\int(\chi_E-\mathrm{h}_{t^2}\chi_E)\chi_F\mathrm{d}\mathsf{m}}{t}$$

for E and F sets of finite perimeter and finite measure. Notice that as E and F have finite measure, $\int(\chi_E-\mathrm{h}_{t^2}\chi_E)\chi_F\mathrm{d}\mathsf{m}\to 0$ as $t\searrow 0$, so that by L'Hôpital's rule we reduce ourselves to compute

$$\lim_{t\searrow 0}-2t\int\chi_F\Delta\mathrm{h}_{t^2}\chi_E\mathrm{d}\mathsf{m}=\lim_{t\searrow 0}2t\int\nabla\mathrm{h}_{t^2/2}\chi_F\cdot\nabla\mathrm{h}_{t^2/2}\chi_E\mathrm{d}\mathsf{m}.$$

Therefore, we conclude that the left-hand side of (7.1.9) is equal to

$$\lim_{t\searrow 0}\sqrt{8}\,t\int\nabla\mathrm{h}_{t^2}\chi_F\cdot\nabla\mathrm{h}_{t^2}\chi_E\mathrm{d}\mathsf{m}.$$

For $E,F\subseteq \mathsf{X}$ sets of finite perimeter and finite measure, we write

$$g_t(E,F):=\sqrt{8}\,t\int \nabla \mathrm{h}_{t^2}\chi_E\cdot\nabla \mathrm{h}_{t^2}\chi_F \mathrm{d}\mathsf{m}\quad \text{for } t\in(0,\infty).$$

If $E=F$, we simply write $g_t(E)$ instead of $g_t(E,E)$. Our aim is then to study $\lim_{t\searrow 0} g_t(E,F)$, where E and F are sets of finite perimeter and finite measure. We start by studying two particularly simple cases and then we combine the information obtained to treat the general case.

Case $E=F$. First, we compute, using (2.2.5) and Proposition 2.3.16 for the last inequality,

$$\begin{aligned}
&\left|t\int |\nabla \mathrm{h}_{t^2}\chi_E|^2\mathrm{d}\mathsf{m}-te^{-Kt^2}\int|\nabla \mathrm{h}_{t^2}\chi_E|\,\mathrm{h}_{t^2}|\mathrm{D}\chi_E|\mathrm{d}\mathsf{m}\right|\\
&\quad\le \frac{t}{e^{Kt^2}}\int|\nabla \mathrm{h}_{t^2}\chi_E|\left|e^{Kt^2}\frac{|\nabla \mathrm{h}_{t^2}\chi_E|}{\mathrm{h}_{t^2}|\mathrm{D}\chi_E|}-1\right|\mathrm{h}_{t^2}|\mathrm{D}\chi_E|\mathrm{d}\mathsf{m}\\
&\quad\le \frac{1}{\sqrt{2}\,e^{2K}}\int\left(1-e^{Kt^2}\frac{|\nabla \mathrm{h}_{t^2}\chi_E|}{\mathrm{h}_{t^2}|\mathrm{D}\chi_E|}\right)\mathrm{h}_{t^2}|\mathrm{D}\chi_E|\mathrm{d}\mathsf{m}\\
&\quad= \frac{1}{\sqrt{2}\,e^{2K}}\left(|\mathrm{D}\chi_E|-e^{Kt^2}\int|\nabla \mathrm{h}_{t^2}\chi_E|\mathrm{d}\mathsf{m}\right).
\end{aligned}$$

Hence, by lower semicontinuity of the total variation, as $t\searrow 0$,

$$\left|t\int |\nabla \mathrm{h}_{t^2}\chi_E|^2\mathrm{d}\mathsf{m}-te^{-Kt^2}\int|\nabla \mathrm{h}_{t^2}\chi_E|\,\mathrm{h}_{t^2}|\mathrm{D}\chi_E|\mathrm{d}\mathsf{m}\right|\to 0. \tag{7.1.11}$$

Now, Lemma 7.1.4 yields

$$\lim_{t\searrow 0} t\int|\nabla \mathrm{h}_{t^2}\chi_E|\,\mathrm{h}_{t^2}|\mathrm{D}\chi_E|\mathrm{d}\mathsf{m}=\lim_{t\searrow 0} t\int \mathrm{h}_{t^2}|\nabla \mathrm{h}_{t^2}\chi_E|\mathrm{d}|\mathrm{D}\chi_E|=\frac{1}{\sqrt{8\pi}}|\mathrm{D}\chi_E|(\mathsf{X}). \tag{7.1.12}$$

Then, by combining (7.1.11), (7.1.12) and finally (3.5.1), we obtain that

$$\lim_{t\searrow 0} g_t(E)=\sqrt{8}\frac{1}{\sqrt{8\pi}}|\mathrm{D}\chi_E|(\mathsf{X})=\frac{1}{\sqrt{\pi}}\frac{\omega_{n-1}}{\omega_n}\mathcal{H}^h(\mathcal{F}E).$$

Case $E\cap F=\emptyset$. We start noticing that $\chi_{E\cup F}=\chi_E+\chi_F$, so that the linearity of the heat flow and of the gradient yields

$$g_t(E\cup F)=g_t(E)+g_t(F)+2g_t(E,F).$$

Therefore, by the **Case** $E=F$ above,

$$\lim_{t\searrow 0} g_t(E,F)=\frac{1}{\sqrt{\pi}}\frac{|\mathrm{D}\chi_{E\cup F}|(\mathsf{X})-|\mathrm{D}\chi_E|(\mathsf{X})-|\mathrm{D}\chi_F|(\mathsf{X})}{2}.$$

Being E and F disjoint, $\partial^*E\cap F^1=\partial^*F\cap E^1=\{\nu_E=\nu_F\}=\emptyset$, so that, by Proposition 4.2.4,

$$|\mathrm{D}\chi_{E\cup F}|=|\mathrm{D}\chi_E|\llcorner F^0+|\mathrm{D}\chi_F|\llcorner E^0=|\mathrm{D}\chi_E|\llcorner(\mathsf{X}\setminus\partial^*F)+|\mathrm{D}\chi_F|\llcorner(\mathsf{X}\setminus\partial^*E),$$

that implies, by (3.5.1) (see also Remark 3.5.2),

$$\begin{aligned}
|\mathrm{D}\chi_{E\cup F}|(\mathsf{X})&=\frac{\omega_{n-1}}{\omega_n}\big(\mathcal{H}^h(\mathcal{F}E\setminus\mathcal{F}F)+\mathcal{H}^h(\mathcal{F}F\setminus\mathcal{F}E)\big)\\
&=\frac{\omega_{n-1}}{\omega_n}\big(\mathcal{H}^h(\mathcal{F}E)+\mathcal{H}^h(\mathcal{F}F)-2\mathcal{H}^h(\mathcal{F}E\cap\mathcal{F}F)\big).
\end{aligned}$$

Hence, by (3.5.1) again,

$$\lim_{t\searrow 0} g_t(E,F) = -\frac{1}{\sqrt{\pi}}\frac{\omega_{n-1}}{\omega_n}\mathcal{H}^h(\mathcal{F}E\cap\mathcal{F}F).$$

General case. Write $E=(E\cap F)\cup(E\setminus F)$ and $F=(F\cap E)\cup(F\setminus E)$, notice that $E\cap F$, $E\setminus F$ and $F\setminus E$ are pairwise disjoint. Exploiting linearity as before, we write

$$g_t(E,F)=g_t(E\cap F)+g_t(E\cap F,F\setminus E)+g_t(E\setminus F,F\cap E)+g_t(E\setminus F,F\setminus E).$$

Using the two cases treated above, we have that

$$\begin{aligned}\lim_{t\searrow 0} g_t(E,F) =&\frac{1}{\sqrt{\pi}}\frac{\omega_{n-1}}{\omega_n}\big(\mathcal{H}^h(\mathcal{F}(E\cap F))-\mathcal{H}^h(\mathcal{F}(E\cap F)\cap\mathcal{F}(F\setminus E))\\ &-\mathcal{H}^h(\mathcal{F}(E\setminus F)\cap\mathcal{F}(F\cap E))-\mathcal{H}^h(\mathcal{F}(E\setminus F)\cap\mathcal{F}(F\setminus E))\big).\end{aligned} \tag{7.1.13}$$

By a density argument based on Lemma 4.2.3 (see also Proposition 4.2.4), up to $\mathcal{H}^h$-negligible sets, it holds

$$\begin{aligned}\mathcal{F}(E\cap F)&=\big(\mathcal{F}E\cap F^1\big)\cup\big(\mathcal{F}F\cap E^1\big)\cup\big(\mathcal{F}E\cap\mathcal{F}F\cap(E\Delta F)^0\big),\\ \mathcal{F}(F\setminus E)&=\big(\mathcal{F}F\cap E^0\big)\cup\big(\mathcal{F}E\cap F^1\big)\cup\big(\mathcal{F}E\cap\mathcal{F}F\cap(E\Delta F)^1\big),\\ \mathcal{F}(E\setminus F)&=\big(\mathcal{F}E\cap F^0\big)\cup\big(\mathcal{F}F\cap E^1\big)\cup\big(\mathcal{F}E\cap\mathcal{F}F\cap(E\Delta F)^1\big),\end{aligned}$$

where all unions are disjoint. We use these identities to compute, up to $\mathcal{H}^h$-negligible sets,

$$\begin{aligned}\mathcal{F}(E\cap F)\cap\mathcal{F}(F\setminus E)&=\mathcal{F}E\cap F^1,\\ \mathcal{F}(E\setminus F)\cap\mathcal{F}(F\cap E)&=\mathcal{F}F\cap E^1,\\ \mathcal{F}(E\setminus F)\cap\mathcal{F}(F\setminus E)&=\mathcal{F}E\cap\mathcal{F}F\cap(E\Delta F)^1.\end{aligned}$$

We compute then

$$\begin{aligned}&\mathcal{H}^h(\mathcal{F}(E\cap F))-\mathcal{H}^h(\mathcal{F}(E\cap F)\cap\mathcal{F}(F\setminus E))-\mathcal{H}^h(\mathcal{F}(E\setminus F)\cap\mathcal{F}(F\cap E))\\ &\qquad-\mathcal{H}^h(\mathcal{F}(E\setminus F)\cap\mathcal{F}(F\setminus E))\\ &\quad=\mathcal{H}^h\big(\mathcal{F}E\cap F^1\big)+\mathcal{H}^h\big(\mathcal{F}F\cap E^1\big)+\mathcal{H}^h\big(\mathcal{F}E\cap\mathcal{F}F\cap(E\Delta F)^0\big)\\ &\qquad-\mathcal{H}^h\big(\mathcal{F}E\cap F^1\big)-\mathcal{H}^h\big(\mathcal{F}F\cap E^1\big)-\mathcal{H}^h\big(\mathcal{F}E\cap\mathcal{F}F\cap(E\Delta F)^1\big)\\ &\quad=\mathcal{H}^h\big(\mathcal{F}E\cap\mathcal{F}F\cap(E\Delta F)^0\big)-\mathcal{H}^h\big(\mathcal{F}E\cap\mathcal{F}F\cap(E\Delta F)^1\big)\\ &\quad=\int_{\mathcal{F}E\cap\mathcal{F}F}\nu_E\cdot\nu_F\,\mathrm{d}\mathcal{H}^h,\end{aligned}$$

where in the last equality we used Lemma 4.2.3 (or Proposition 4.2.4). Therefore, by recalling (7.1.13) we conclude that the statement holds for any $E,F\subseteq\mathsf{X}$ of finite perimeter and finite measure. □

Having Lemma 7.1.5 at our disposal, there is no effort in obtaining Theorem 7.1.6 below.

Theorem 7.1.6. *Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,N)$ space. Let $E\subseteq\mathsf{X}$ be a set of finite perimeter such that either $\mathsf{m}(E)<\infty$ or $\mathsf{m}(\mathsf{X}\setminus E)<\infty$. Then*

$$\lim_{t\searrow 0}\frac{1}{\sqrt{t}}\int\int p_t(x,y)|\chi_E(x)-\chi_E(y)|\mathrm{d}\mathsf{m}(x)\mathrm{d}\mathsf{m}(y)=\frac{2}{\sqrt{\pi}}|\mathrm{D}\chi_E|(\mathsf{X}). \tag{7.1.14}$$

Proof. We can assume with no loss of generality that $\mathsf{m}(E) < \infty$. We compute (all the integrands are positive)

$$\begin{aligned}
&\int\int p_t(x,y)|\chi_E(y) - \chi_E(x)|\mathrm{d}\mathsf{m}(y)\mathrm{d}\mathsf{m}(x) \\
&\quad= \int (1-\chi_E(x))\int_E p_t(x,y)\mathrm{d}\mathsf{m}(y)\mathrm{d}\mathsf{m}(x) + \int \chi_E(x)\int_{\mathsf{X}\setminus E} p_t(x,y)\mathrm{d}\mathsf{m}(y)\mathrm{d}\mathsf{m}(x) \\
&\quad= \int \chi_{\mathsf{X}\setminus E}\,\mathrm{h}_t\chi_E\mathrm{d}\mathsf{m} + \int \chi_E\,\mathrm{h}_t\chi_{\mathsf{X}\setminus E}\mathrm{d}\mathsf{m} = 2\int_{\mathsf{X}\setminus E}\mathrm{h}_t\chi_E\mathrm{d}\mathsf{m} = 2\int(1-\chi_E)\mathrm{h}_t\chi_E\mathrm{d}\mathsf{m} \\
&\quad= 2\Big(\mathsf{m}(E) - \int\chi_E\,\mathrm{h}_t\chi_E\mathrm{d}\mathsf{m}\Big) = 2\int(\chi_E - \mathrm{h}_t\chi_E)\chi_E\mathrm{d}\mathsf{m}.
\end{aligned} \tag{7.1.15}$$

We obtain (7.1.14) by dividing (7.1.15) by $\sqrt{t}$, letting $t \searrow 0$, and using (7.1.10). □

As a technical tool, we need the following easy computation, obtained via classical techniques. Notice the role played by the regularizing properties of the heat flow on RCD spaces.

Lemma 7.1.7. *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *be an* $\mathrm{RCD}(K,N)$ *space and let* $E,F\subseteq\mathsf{X}$ *two sets of finite perimeter and finite measure. Then*

$$\left|\int(\mathrm{h}_t\chi_E - \chi_E)\chi_F\mathrm{d}\mathsf{m}\right| \le 2e^{K(t/2-1)}|\mathrm{D}\chi_E|(\mathsf{X})\sqrt{t}.$$

Proof. We can compute, using (2.2.5) and (2.3.16) for the second inequality,

$$\begin{aligned}
\left|\int(\mathrm{h}_t\chi_E - \chi_E)\chi_F\mathrm{d}\mathsf{m}\right| &= \left|\int_0^t\frac{\mathrm{d}}{\mathrm{d}s}\int(\mathrm{h}_s\chi_E - \chi_E)\chi_F\mathrm{d}\mathsf{m}\mathrm{d}s\right| = \left|\int_0^t\int\chi_F\Delta\mathrm{h}_s\chi_E\mathrm{d}\mathsf{m}\mathrm{d}s\right| \\
&\le \int_0^t\int|\nabla\mathrm{h}_{s/2}\chi_F||\nabla\mathrm{h}_{s/2}\chi_E|\mathrm{d}\mathsf{m}\mathrm{d}s \le \frac{1}{e^K}\int_0^t\frac{e^{-Ks/2}}{\sqrt{s}}\int\mathrm{h}_{s/2}|\mathrm{D}\chi_E|\mathrm{d}\mathsf{m}\mathrm{d}s \\
&\le 2e^{K(t/2-1)}|\mathrm{D}\chi_E|(\mathsf{X})\sqrt{t},
\end{aligned}$$

which is the claim. □

7.1.2 Proof of the main results

Proof of Theorem 7.1.1. By Lemma 7.1.3, if $f\notin\mathrm{BV}(\mathsf{X})$, then the statement holds. Therefore, in what follows we assume $f\in\mathrm{BV}(\mathsf{X})$. We argue via coarea and integration via Cavalieri's formula, as done in the references [41, Theorem 2.14] and [110, Theorem 4.1], building upon Theorem 7.1.6. Recall that (7.1.15) and Lemma 7.1.7 imply that for every $t\in(0,\infty)$,

$$\frac{1}{\sqrt{t}}\int\int p_t(x,y)|\chi_E(x) - \chi_E(y)|\mathrm{d}\mathsf{m}(x)\mathrm{d}\mathsf{m}(y) \le 4e^{K(\frac{t}{2}-1)}|\mathrm{D}\chi_E|(\mathsf{X}) \tag{7.1.16}$$

for every set of finite perimeter and finite measure $E\subseteq\mathsf{X}$. Notice that (7.1.16) continues to hold as soon at least one between E and $\mathsf{X}\setminus E$ has finite measure.

Take any sequence $t_i \searrow 0$. We denote $E_s := \{f > s\}$ and then we compute, by coarea, (7.1.14), dominated convergence (thanks to (7.1.16)) and Fubini's Theorem,

$$\begin{aligned}
|\mathrm{D}f|(\mathsf{X}) &= \int_{\mathbb{R}} |\mathrm{D}\chi_{E_s}|(\mathsf{X})\mathrm{d}s = \int_{\mathbb{R}} \lim_i \frac{1}{2}\sqrt{\frac{\pi}{t_i}} \int\int p_{t_i}(x,y)|\chi_{E_s}(x) - \chi_{E_s}(y)|\mathrm{d}\mathsf{m}(x)\mathrm{d}\mathsf{m}(y)\mathrm{d}s \\
&= \lim_i \frac{1}{2}\sqrt{\frac{\pi}{t_i}} \int_{\mathbb{R}}\int\int p_{t_i}(x,y)|\chi_{E_s}(x) - \chi_{E_s}(y)|\mathrm{d}\mathsf{m}(x)\mathrm{d}\mathsf{m}(y)\mathrm{d}s \qquad (7.1.17) \\
&= \lim_i \frac{1}{2}\sqrt{\frac{\pi}{t_i}} \int\int p_{t_i}(x,y)\int_{\mathbb{R}}|\chi_{E_s}(x) - \chi_{E_s}(y)|\mathrm{d}s\mathrm{d}\mathsf{m}(x)\mathrm{d}\mathsf{m}(y).
\end{aligned}$$

Now we fix a Borel representative of f and $x, y \in \mathsf{X}$. If $f(y) \le f(x)$,

$$\int_{\mathbb{R}} |\chi_{E_s}(x) - \chi_{E_s}(y)|\mathrm{d}s = \int_{-\infty}^{\infty} \chi_{E_s}(x) - \chi_{E_s}(y)\mathrm{d}s = f(x) - f(y) = |f(x) - f(y)|$$

and a similar conclusion holds if $f(y) \ge f(x)$. Therefore, we can continue (7.1.17) to obtain

$$|\mathrm{D}f|(\mathsf{X}) = \lim_i \frac{1}{2}\sqrt{\frac{\pi}{t_i}} \int\int p_{t_i}(x,y)|f(x) - f(y)|\mathrm{d}\mathsf{m}(x)\mathrm{d}\mathsf{m}(y).$$

As the sequence $\{t_i\}_i$ was arbitrary, this concludes the proof. □

Proof of Theorem 7.1.2. The proof of Theorem 7.1.2 is via coarea and integration via Cavalieri's formula, as done in the reference [110, Theorem 4.3], building upon Lemma 7.1.5. First, we write $f = f^+ - f^-$, where $f^+ := f \vee 0$ and $f^- := (-f) \vee 0$. Thanks to the coarea formula, we can apply Lemma 4.2.2 and infer that $\nu_{f^\pm} = \nu_{\pm f}$ $|\mathrm{D}f^\pm|$-a.e. Also, a direct computation yields that $f^\wedge = (f^+)^\wedge - (f^-)^\vee$ and $f^\vee = (f^+)^\vee - (f^-)^\wedge$. Therefore, by linearity, we can assume that $f \ge 0$ m-a.e. We repeat the same argument for g to see that we can assume that also $g \ge 0$ m-a.e. Up to scaling, we assume that $0 \le g \le 1$ m-a.e.

Let now $E_s := \{f > s\}$ and $F_s := \{g > s\}$ and notice that for $\mathcal{L}^1$-a.e. $\sigma \in \mathbb{R}$, by (7.1.16),

$$\frac{1}{\sqrt{t}}\int\int |p_t(x,y)\chi_{F_\tau}(x)(\chi_{E_\sigma}(x) - \chi_{E_\sigma}(y))|\mathrm{d}\mathsf{m}(x)\mathrm{d}\mathsf{m}(y) \le 4e^{K(t/2-1)}|\mathrm{D}\chi_{E_\sigma}|(\mathsf{X}). \qquad (7.1.18)$$

Notice that we can write for $(\mathsf{m}\otimes\mathsf{m})$-a.e. $(x,y) \in \mathsf{X}\times\mathsf{X}$, as we are assuming $0 \le f$ m-a.e. and $0 \le g \le 1$ m-a.e.

$$p_t(x,y)g(x)(f(x) - f(y)) = \int_0^{\infty}\int_0^1 p_t(x,y)\chi_{F_\tau}(x)(\chi_{E_\sigma}(x) - \chi_{E_\sigma}(y))\mathrm{d}\tau\mathrm{d}\sigma,$$

so that, by Fubini's Theorem (whose application is justified by (7.1.18) and coarea), we obtain

$$\begin{aligned}
\frac{1}{\sqrt{t}}\int (f - \mathrm{h}_t f)g\mathrm{d}\mathsf{m} &= \frac{1}{\sqrt{t}}\int\int p_t(x,y)g(x)(f(x) - f(y))\mathrm{d}\mathsf{m}(y)\mathrm{d}\mathsf{m}(x) \\
&= \frac{1}{\sqrt{t}}\int_0^{\infty}\int_0^1\int\int p_t(x,y)\chi_{F_\tau}(x)(\chi_{E_\sigma}(x) - \chi_{E_\sigma}(y))\mathrm{d}\mathsf{m}(y)\mathrm{d}\mathsf{m}(x)\mathrm{d}\tau\mathrm{d}\sigma \\
&= \frac{1}{\sqrt{t}}\int_0^{\infty}\int_0^1\int \chi_{F_\tau}(\chi_{E_\sigma} - \mathrm{h}_t\chi_{E_\sigma})\mathrm{d}\mathsf{m}\mathrm{d}\tau\mathrm{d}\sigma.
\end{aligned}$$

Lemma 7.1.7 together with coarea again justify an application of the dominated convergence theorem in the limit as $t \searrow 0$ in the equation above, so that, by (7.1.9),

$$\begin{aligned}\lim_{t\searrow 0}\frac{1}{\sqrt{t}}\int (f-\mathrm{h}_t f)g\mathrm{dm} &= \frac{1}{\sqrt{\pi}}\frac{\omega_{n-1}}{\omega_n}\int_0^\infty\int_0^1\int_{\mathcal{F}E_\sigma\cap\mathcal{F}F_\tau}\nu_{E_\sigma}\cdot\nu_{F_\tau}\mathrm{d}\mathcal{H}^h\mathrm{d}\tau\mathrm{d}\sigma\\ &= \frac{1}{\sqrt{\pi}}\frac{\omega_{n-1}}{\omega_n}\int_0^\infty\int_0^1\int_{\mathcal{F}E_\sigma\cap\mathcal{F}F_\tau}\nu_f\cdot\nu_g\mathrm{d}\mathcal{H}^h\mathrm{d}\tau\mathrm{d}\sigma\\ &= \frac{1}{\sqrt{\pi}}\frac{\omega_{n-1}}{\omega_n}\int_0^\infty\int_0^1\int\chi_{\partial^* F_\tau}(\nu_f\cdot\nu_g)\,\mathrm{d}(\mathcal{H}^h\llcorner(\mathcal{F}E_\sigma))\,\mathrm{d}\tau\mathrm{d}\sigma,\end{aligned}$$

where the second equality is due to Lemma 4.2.1.

Now recall that the map $(\tau,x)\mapsto\chi_{\partial^* F_\tau}(x)$ is measurable with respect to $\mathcal{L}^1\otimes(\mathcal{H}^h\llcorner(\mathcal{F}E_\sigma))$ (cf. the proof of Proposition 4.2.12), so that by Fubini's theorem we can write

$$\begin{aligned}&\frac{1}{\sqrt{\pi}}\frac{\omega_{n-1}}{\omega_n}\int_0^\infty\int_0^1\int\chi_{\partial^* F_\tau}(\nu_f\cdot\nu_g)\,\mathrm{d}(\mathcal{H}^h\llcorner(\mathcal{F}E_\sigma))\,\mathrm{d}\tau\mathrm{d}\sigma\\ &\quad=\frac{1}{\sqrt{\pi}}\frac{\omega_{n-1}}{\omega_n}\int_0^\infty\int(g^\vee-g^\wedge)(\nu_f\cdot\nu_g)\,\mathrm{d}(\mathcal{H}^h\llcorner(\mathcal{F}E_\sigma))\,\mathrm{d}\sigma\\ &\quad=\frac{1}{\sqrt{\pi}}\int_0^\infty\int(g^\vee-g^\wedge)(\nu_f\cdot\nu_g)\mathrm{d}|\mathrm{D}\chi_{E_\sigma}|\mathrm{d}\sigma,\end{aligned}$$

where we took into account (2.3.6) and finally (3.5.1) for the last equality. Notice that the integration over X is only on J_g, which is a σ-finite set with respect to $\mathcal{H}^h$ (Proposition 4.2.11), so that $|\mathrm{D}f|\llcorner J_g=|\mathrm{D}f|\llcorner(J_f\cap J_g)$. Hence, by coarea we can continue the computation as

$$\begin{aligned}\frac{1}{\sqrt{\pi}}\int_0^\infty\int(g^\vee-g^\wedge)(\nu_f\cdot\nu_g)\mathrm{d}|\mathrm{D}\chi_{E_\sigma}|\mathrm{d}\sigma &=\frac{1}{\sqrt{\pi}}\int_{J_g}(g^\vee-g^\wedge)(\nu_f\cdot\nu_g)\,\mathrm{d}(|\mathrm{D}f|\llcorner J_f)\\ &=\frac{1}{\sqrt{\pi}}\frac{\omega_{n-1}}{\omega_n}\int_{J_g}(g^\vee-g^\wedge)(f^\vee-f^\wedge)(\nu_f\cdot\nu_g)\,\mathrm{d}(\mathcal{H}^h\llcorner J_f)\,,\end{aligned}$$

where we used (4.2.11). All in all, we have proved (7.1.1). □

7.2 Bibliographical notes

We briefly motivate the investigation contained in this chapter along with some bibliographical notes. In their seminal work [40], Bourgain–Brezis–Mironescu showed that if $\Omega\subseteq\mathbb{R}^n$ is a smooth bounded domain and $p\in(1,\infty)$, then the p-Sobolev seminorm of a function $f\in L^p(\Omega)$ coincides (up to a multiplicative factor, depending only on p and n) with the limit

$$\lim_{i\to\infty}\left(\int_\Omega\int_\Omega\frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_i(|x-y|)\mathrm{d}x\mathrm{d}y\right)^{1/p},$$

where $\{\rho_i\}_i$ are suitably chosen kernels of mollification. The result was then generalized to BV functions by Dávila [63] and Ambrosio independently. In the BV case, it holds that if $\Omega\subseteq\mathbb{R}^n$ is a bounded domain with Lipschitz boundary, then the total variation $|\mathrm{D}f|(\Omega)$ of any function $f\in L^1(\Omega)$ coincides (up to a multiplicative factor, depending only on n) with

$$\lim_{i\to\infty}\int_\Omega\int_\Omega\frac{|f(x)-f(y)|}{|x-y|}\rho_i(|x-y|)\mathrm{d}x\mathrm{d}y.$$

Brezis suggested in [47, Remark 6] that it might be interesting to generalize the theory to more general metric measure spaces $(\mathsf{X}, \mathsf{d}, \mathsf{m})$. Recently, in [103] the authors studied a characterization of the BV space in metric measure spaces supporting a doubling measure and a Poincaré inequality for a large class of mollifiers. In the same paper, the authors also provided a counterexample in the case $p = 1$, demonstrating that (unlike in Euclidean spaces) in metric measure spaces the limit of suitable nonlocal functionals is only comparable, but not necessarily equal, to the total variation measure of the function f. Hence, it is clear that, in order to obtain results like Theorem 7.1.1 and Theorem 7.1.2, additional assumptions on the space are in order.

In many cases of interest, a good choice of mollifiers is given by the (suitably rescaled) heat kernels. In the case of functions of bounded variation, this has been done by several authors: on Euclidean spaces or Riemannian manifolds in [110, 109] and [56], on Carnot groups in [41], and on PI spaces in [105].

Chapter 7 is entirely taken from [46], with no significant modifications, where parts of the arguments are borrowed from from [41, 110]. In particular, for what concerns the proofs of Theorem 7.1.1 and Theorem 7.1.2, we follow the argument as in the proofs of [110, Theorem 4.1 and Theorem 4.3], respectively, (however, to deduce the membership $f \in \mathrm{BV}(\mathsf{X})$ of Theorem 7.1.1, we exploit directly [105]). Our main contributions are then in the proofs of the intermediate results that are combined to prove Theorem 7.1.1 and Theorem 7.1.2. In particular, Lemma 7.1.4 is inspired by the proof of [41, Theorem 2.13], obtained with a blow-up argument, but we have to face the additional difficulty of the lack of the linear structure of Carnot groups. Also, the computations with L'Hôpital's rule used to prove Lemma 7.1.5 have already been used to prove [41, Theorem 2.13], but in our case, again, we cannot exploit the linear structure, so that we have to rely on the regularizing properties of the heat flow on RCD spaces. The case of possibly different E and F is obtained with a polarization argument and a careful computation: due to the lack of a linear structure, the strategy used in the references seems not suitable to our case.

Chapter 8

Appendix

8.1 Differentiability of Lipschitz functions

The aim of this section is to introduce a powerful result that relates "closability of certain differentiation operators" and "differentiability of Lipschitz functions in related directions", and which is an indispensable tool to prove Theorem 4.3.6. We decided to move Theorem 8.1.1 to the appendix, as it is not related with the main framework of this manuscript, which is the one of non-smooth spaces.

Before stating the main result of this section, we start recalling some terminology. Given $\varphi \in \mathrm{LIP}(\mathbb{R}^m, \mathbb{R}^l)$, we say that φ is *differentiable at x with respect to $V \in \mathrm{Gr}(\mathbb{R}^m)$* if there exists a linear map $\nabla_V \varphi(x) : V \to \mathbb{R}^l$ such that

$$\varphi(x+v) = \varphi(x) + \nabla_V \varphi(x) \cdot v + o(|v|) \qquad \text{for } v \in V.$$

If $v \in \mathbb{R}^m$, we say that φ is *differentiable at x in direction v* if φ is differentiable at x with respect to $\mathrm{span}(v)$.

We are going to exploit in a crucial way the following result, which is a restatement of results contained in [3, 2] (see in particular, [3, Theorem 1.1] and [2, Theorem 1.1]). We refer the reader to these references for the definition of $V(\mu, \cdot)$, the *decomposability bundle* associated to the Radon measure μ, as we are not going to use this notion elsewhere (indeed, we are going to exploit the equivalence between items ii) and iii) below, which can be understood without knowing the definition of the decomposability bundle).

Theorem 8.1.1. *Let $v\mu$ be a m-vector valued measure on $\mathbb{R}^m$, where $v \in L^\infty(\mu)^m$ and μ is finite. Then the following assertions are equivalent.*

- *i) $v(x) \in V(\mu, x)$ for μ-a.e. x.*
- *ii) Every Lipschitz function is differentiable in direction $v(x)$ for μ-a.e. x.*
- *iii) The operator*

$$D : C^1(\mathbb{R}^m) \cap \mathrm{LIP}_b(\mathbb{R}^m) \to L^\infty(\mu) \qquad \varphi \mapsto \nabla\varphi \cdot v$$

  is closable, in the sense that if $\{\varphi_k\}_k \subseteq C^1(\mathbb{R}^m) \cap \mathrm{LIP}_b(\mathbb{R}^m)$ is a sequence of uniformly bounded and uniformly Lipschitz functions converging pointwise to $\varphi \in \mathrm{LIP}_b(\mathbb{R}^m)$, then $D(\varphi_k) \to \ell$ in the weak topology of $L^\infty(\mu)$, for some $\ell \in L^\infty(\mu)$.*

If any (hence all) of the items above holds, if ℓ is as in item iii) for φ, then

$$\ell(x) = \nabla_{v(x)}\varphi(x) \cdot v(x) \qquad \text{for } \mu\text{-a.e. } x.$$

In our approximation arguments, we are going to need the following result, which is extracted from [3, Corollary 8.3]. Below, the *global Lipschitz constant* of $\varphi \in \mathrm{LIP}(\mathbb{R}^m)$ is the least number $L \in \mathbb{R}$ such that $|\varphi(x) - \varphi(y)| \leq L|x-y|$ for every $x, y \in \mathbb{R}^m$. Notice that the claim of the following lemma is stronger than item *iii*) of Theorem 8.1.1, as the consequence is pointwise and not only in the weak sense.

Lemma 8.1.2. *Let $\varphi \in \mathrm{LIP}(\mathbb{R}^m)$ and let μ be a finite measure on $\mathbb{R}^m$. Assume also that $x \mapsto v(x)$ is a bounded Borel map such that for μ-a.e. x, φ is differentiable in direction $v(x)$ at x. Then there exists a sequence $\{\varphi_k\}_k \subseteq C^1(\mathbb{R}^m) \cap \mathrm{LIP}(\mathbb{R}^m)$ such that $\varphi_k \to \varphi$ uniformly, the global Lipschitz constant of φ_k converges to the global Lipschitz constant of φ, and finally*

$$\nabla\varphi_k(x) \cdot v(x) \to \nabla_{v(x)}\varphi(x) \cdot v(x) \qquad \text{for } \mu\text{-a.e. } x.$$

8.2 Bibliographical notes

The results of the short Section 8.1 are taken from [3, 2].

In an imprecise way (see [3] for the rigorous definition), the *decomposability bundle* of μ is the "minimal" map $V_\mu : \mathbb{R}^n \to \mathrm{Gr}(\mathbb{R}^n)$ such that, whenever we have a measure space $(I, \mathrm{d}t)$ and a family of measures $(\mu_t)_{t\in I}$ such that

- μ_t is the restriction of $\mathcal{H}^1$ to an 1-rectifiable set E_t, for every $t \in I$,
- $t \mapsto \mu_t$ enjoys suitable regularity assumptions,
- $\int_I \mu_t \mathrm{d}t \ll \mu$,

then

$$\mathrm{Tan}(E_t, x) \subseteq V_\mu(x) \qquad \text{for } \mu_t\text{-a.e. } x \in \mathbb{R}^n \text{ and a.e. } t \in I.$$

The main result of [3] is then to characterize the directions along which Lipschitz functions are differentiable μ-a.e. in terms of V_μ. In particular:

- for every $f \in \mathrm{LIP}(\mathbb{R}^n)$, f is differentiable at μ-a.e. x with respect to the subspace $V_\mu(x)$;
- there exists $\bar{f} \in \mathrm{LIP}(\mathbb{R}^n)$ such that for μ-a.e. x and $v \notin V_\mu(x)$, $\bar{f}$ is not differentiable at x in direction v.

This implies the equivalence between items i) and ii) of Theorem 8.1.1. The equivalence between items i) and iii) of Theorem 8.1.1 is still a consequence of the results [3] and has been proved in [2]. More precisely, the implication iii) $\Rightarrow$ i) (which, together with the implication i) $\Rightarrow$ ii) is what we really need in the proof of Theorem 4.3.6) is a consequence of the construction of "width functions", see (the discussion above) [3, Lemma 4.12].

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