TEST FUNCTION APPROACH TO FULLY NONLINEAR EQUATIONS IN THIN DOMAINS

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Abstract. In this note we extend to fully nonlinear operators the well known result on thin domains of Hale and Raugel [7]. The result is more general even in the case of the Laplacian.

Contents

1. Introduction

The classical result of Hale and Raugel [7] in thin domains states that if u_ε are solutions of

$$\begin{cases} -\Delta u_\varepsilon + u_\varepsilon = f(x,y) & \text{in } \Omega_\varepsilon \\ \partial_{\nu_\varepsilon} u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon \end{cases}$$

where $\Omega_\varepsilon = \{(x,y) \in \mathbb{R}^N \times \mathbb{R} \,:\, x \in \Omega,\ 0 < y < \varepsilon g(x)\}$, for some $g \in C^3(\overline{\Omega})$ such that $0 < \inf_\Omega g \le \sup_\Omega g < \infty$ then u_ε converges to u_o solution of

$$\begin{cases} -(\Delta u_o + \frac{Dg\cdot Du_o}{g}) + u_o = f(x,0) & \text{in } \Omega \\ \partial_\nu u_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

This result has been extended in a wide variety of related problems see e.g. the works of Arrieta, Pereira, Raugel [1, 2, 10]. But all the above results concern variational problems,

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where the appearance of the first order seems to come from a typical integration by parts, related to the variational nature of the problem.

In this paper, instead, we treat fully nonlinear equation in thin domains i.e. where the equation is given by

$$F(D^2u, Du, u, (x,y)) = 0 \text{ in } \Omega_\varepsilon$$

where $F : \mathcal{S}(N+1) \times \mathbb{R}^{N+1} \times \mathbb{R} \times \Omega_\varepsilon \to \mathbb{R}$ is a proper functional in the sense of the User's guide [4]. Of course the solutions are viscosity solutions and the proof follows the test function approach of Evans [5] which is somehow more direct and completely different from the papers mentioned above. Furthermore the technique does not require the operator to be uniformly elliptic as it will be evident from the hypotheses below. An example of thin domains for degenerate elliptic operator will be given explicitly below.

Even though the results will be proved for a large class of operators, in this introduction, we will illustrate the special case where the fully non linear operator is one of the extremal Pucci operators e.g. for $0 < \lambda \le \Lambda$

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) := \sup_{\lambda I \le A \le \Lambda I} (\operatorname{tr} A(D^2u)) = \lambda \sum_{e_i \le 0} e_i + \Lambda \sum_{e_i \ge 0} e_i,$$

where $e_i = e_i(D^2u)$ denotes the i-th eigenvalue of the Hessian matrix D^2u.

Under the hypothesis

(H1) $g \in C^1(\overline{\Omega})$ and $0 < \inf_\Omega g \le \sup_\Omega g < \infty$,

we will prove that u_ε, the solutions of

$$\begin{cases} -\mathcal{M}^+_{\lambda,\Lambda}(D^2u_\varepsilon) + u_\varepsilon = f(x,y) & \text{in } \Omega_\varepsilon \\ \partial_{\nu_\varepsilon} u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon \end{cases} \tag{1}$$

converge uniformly to u_o solution of

$$\begin{cases} -\mathcal{M}^+_{\lambda,\Lambda}(D^2u_o(x)) - \Lambda\left(\frac{Dg\cdot Du_o}{g}\right)^+ + \lambda\left(\frac{Dg\cdot Du_o}{g}\right)^- + u_o(x) = f(x,0) & \text{in } \Omega \\ \partial_\nu u_o = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Of course in the first equation $\mathcal{M}^+_{\lambda,\Lambda}$ acts on matrices in $\mathcal{S}(N+1)$ while, in the second equation, it acts on matrices in $\mathcal{S}(N)$.

In the special case $\lambda = \Lambda = 1$, when $\mathcal{M}^+_{\lambda,\Lambda} = \Delta$, we recover Hale and Raugel result, but we improve the condition on g that is only required to be the natural condition C^1 and not C^3.

We wish to explain the **heuristic** behind the formal proof which will be given in this paper, for a much larger class of operators. Let u_ε be a solution of (1) and let

$$v_\varepsilon(x,y) := u_\varepsilon(x, \varepsilon g(x) y)$$

so that we have "flattened" the top boundary. In similarity with the linear variational case we can suppose that there exists a constant C such that

$$|\partial_{yy} v_\varepsilon| \le C\varepsilon^2.$$

This in turn implies that for $\varepsilon \to 0$, $\partial_{yy} v_\varepsilon \to 0$ and then, using the boundary condition, we get

$$v_\varepsilon(x,y) \to v_o(x).$$

On the other hand, the above estimates implies also that, for some function $k(x)$, we get that

$$\frac{\partial_{yy} v_\varepsilon(x,y)}{\varepsilon^2} \to k(x) := g^2(x)h(x).$$

So we may use the following *ansatz*:

$$v_\varepsilon(x,y) = w(x) + \varepsilon^2 k(x)\frac{y^2}{2} + o(\varepsilon^2).$$

Substituting the ansatz in the equation, we let formally ε go to zero; after a tedious but simple computation it is easy to see that we obtain

$$-\mathcal{M}^+_{\lambda,\Lambda}(\begin{pmatrix} D^2w(x) & 0 \\ 0 & h(x) \end{pmatrix}) + w(x) = f(x,0).$$

We use the condition on the "top" boundary, in order to determine $h(x)$:

$$Dg(x)\cdot[Dw(x) + \varepsilon^2 D(g^2h)(x)\frac{1}{2}] = g(x)h(x)[1+\varepsilon^2 Dg(x)].$$

Passing to the limit we find

$$h(x) = \frac{Dg(x)\cdot Dw(x)}{g(x)}$$

i.e. the limit equation becomes:

$$-\mathcal{M}^+_{\lambda,\Lambda}(\begin{pmatrix} D^2w(x) & 0 \\ 0 & \frac{Dg(x)\cdot Dw(x)}{g(x)} \end{pmatrix}) + w(x) = f(x,0).$$

Observe that writing the ansatz directly for u_ε one obtains

$$u_\varepsilon(x,y) = w(x) + \frac{h(x)}{2}y^2 + o(\varepsilon^2). \tag{3}$$

In the rest of the paper we will treat the general case and make rigorous the above idea, in particular using Evans's test approach to the problem at hand.

We will first give some a priori bounds, which allow to prove that the upper and lower relaxed limits u^+ and u^- of $\{u^\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ are respectively sub and super solutions of the limit equation:

$$\begin{cases} F\left(\begin{pmatrix} D^2w & 0 \\ 0 & Dg\cdot Dw/g \end{pmatrix}, (Dw,0), w, (x,0)\right) = 0 \quad \text{in } \Omega, \\ \dfrac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4}$$

Under the further condition that the comparison principle holds, we will prove that the convergence of u^ε to the solution of (4) is uniform.

The scope of this paper is to open the results in thin domains from a perspective that is, to our knowledge completely different from the previous results, see e.g. [1, 2, 7, 10]. So, in order to make the exposition the clearer possible, we have decided to concentrate on the more classical case i.e. domains like Ω_ε that in one direction have a one flat boundary and with a Neumann boundary condition. We plan to investigate in a further research, more generale domains, for examples with jumps or with non flat sides, or thin domains that also have an oscillatory boundary. We hope the reader will appreciate this choice.

2. Preliminaries

Let $F : \mathcal{S}(N+1) \times \mathbb{R}^{N+1} \times \mathbb{R} \times \overline{\Omega_\varepsilon} \to \mathbb{R}$ be a proper functional in the sense of the User's guide [4], i.e.

$$\text{(H2)} \qquad \begin{cases} F \in C(\mathcal{S}(N+1) \times \mathbb{R}^{N+1} \times \mathbb{R} \times \overline{\Omega_\varepsilon}, \mathbb{R}), \\ F(X, p, r, (x, y)) \le F(Y, p, s, (x, y)) \text{ whenever } r \le s, \text{ and } Y \le X. \end{cases}$$

Furthermore, for simplicity of the presentation, we strengthen the monotonicity condition on F in the above as follows.

(H3) There exists $\alpha > 0$ such that

$$\alpha(r - s) \le F(X, p, r, (x, y)) - F(X, p, s, (x, y))$$

for $r \ge s$ and $(X, p, (x, y)) \in \mathcal{S}(N+1) \times \mathbb{R}^{N+1} \times \Omega_\varepsilon$.

Our PDE problem is:

$$F(D^2u^\varepsilon, Du^\varepsilon, u^\varepsilon, (x, y)) = 0 \quad \text{in } \Omega_\varepsilon \quad \text{and} \quad \frac{\partial u^\varepsilon}{\partial \nu_\varepsilon} = 0 \quad \text{on } \partial\Omega_\varepsilon, \tag{5}$$

where ν_ε denotes the outward (unit) normal to Ω_ε.

Since our concern is the asymptotic behavior of solutions u^ε to (5), we will restrict ourself to the parameter ε in the range $(0, \varepsilon_0]$, where $\varepsilon_0 > 0$ is a number fixed throughout. Let us note that we shall use ν to indicate the normal to Ω and ν_ε for Ω_ε. Obviously, the assumptions above are not enough to ensure the existence of viscosity solutions in the sense of the User's guide [1] to (5). To keep the generality of the assumptions made above, we consider the notion of viscosity solutions to (5) which eliminates the continuity requirement. That is, we call a bounded function u on $\overline{\Omega_\varepsilon}$ a (viscosity) solution of (5) if its upper and lower semicontinuous envelopes are viscosity sub and super solutions, in the sense of [1], to (5), respectively.

We assume throughout that

(H4) Ω is a bounded C^1 domain of $\mathbb{R}^N$.

Accordingly, we may choose a function $\rho \in C^1(\mathbb{R}^N)$ so that

$$\rho(x) < 0 \quad \text{for } x \in \Omega, \quad D\rho(x) \ne 0, \quad \text{and} \quad \rho(x) > 0 \quad \text{for } x \in \mathbb{R}^N \setminus \overline{\Omega}. \tag{6}$$

Note that the outward unit normal ν to Ω at $x \in \partial\Omega$ is given by $\nu = |D\rho(x)|^{-1} D\rho(x)$. The domain Ω_ε has corners, where the N–dimensional hypersurface $\partial\Omega \times \mathbb{R}$ intersects either the hypersurfaces $y = g(x)$ or $y = 0$, respectively.

We denote by $\partial_{\mathrm{L}}\Omega_\varepsilon$, $\partial_{\mathrm{B}}\Omega_\varepsilon$, and $\partial_{\mathrm{T}}\Omega_\varepsilon$ the lateral, bottom, and top portions of the boundary $\partial\Omega_\varepsilon$, which are described respectively as

$$\{(x, y) \in \partial\Omega_\varepsilon \,:\, x \in \partial\Omega\}, \quad \{(x, y) \in \partial\Omega_\varepsilon \,:\, y = 0\}, \quad \text{and}$$
$$\{(x, y) \in \partial\Omega_\varepsilon \,:\, y = \varepsilon g(x)\}.$$

Furthermore, the outward unit normal to Ω_ε at the lateral boundary, at the bottom, $y = 0$, and at the top boundary, $y = \varepsilon g(x)$, is given, respectively, by $\nu_{\mathrm{L}} = (|D\rho(x)|^{-1}D\rho(x), 0)$, $\nu_{\mathrm{B}} = -e_{N+1} = -(0, \ldots, 0, 1)$, and

$$\nu_{\mathrm{T}} = \frac{(-\varepsilon Dg(x), 1)}{\sqrt{1 + \varepsilon^2|Dg(x)|^2}}.$$

The appearance of corners of the domain Ω_ε requires a little care in the definition of sub and super solutions to (5). For instance, when u is a bounded upper semicontinuous function on $\overline{\Omega_\varepsilon}$, we call u a viscosity subsolution of (5) if the following condition holds: whenever $\phi \in C^2(\overline{\Omega_\varepsilon})$, $\hat{z} = (\hat{x}, \hat{y}) \in \overline{\Omega_\varepsilon}$ and $\max_{\overline{\Omega_\varepsilon}}(u - \phi) = (u - \phi)(\hat{z})$, we must have

$$F(D^2\phi(\hat{z}), D\phi(\hat{z}), u(\hat{z}), \hat{z}) \leq 0 \tag{7}$$

if $\hat{z} \in \Omega_\varepsilon$, we have either (7) or

$$\nu_{\mathrm{L}} \cdot D\phi(\hat{z}) \leq 0 \tag{8}$$

if $\hat{z} \in \partial_{\mathrm{L}}\Omega_\varepsilon \setminus (\partial_{\mathrm{B}}\Omega_\varepsilon \cup \partial_{\mathrm{T}}\Omega_\varepsilon)$, we have either (7) or

$$\nu_{\mathrm{B}} \cdot D\phi(\hat{z}) \leq 0 \tag{9}$$

if $\hat{z} \in \partial_{\mathrm{B}}\Omega_\varepsilon \setminus \partial_{\mathrm{L}}\Omega_\varepsilon$, we have either (7) or

$$\nu_{\mathrm{T}} \cdot D\phi(\hat{z}) \leq 0 \tag{10}$$

if $\hat{z} \in \partial_{\mathrm{T}}\Omega_\varepsilon \setminus \partial_{\mathrm{L}}\Omega_\varepsilon$, we have either (7), (8), or (9) if $\hat{z} \in \partial_{\mathrm{L}}\Omega_\varepsilon \cap \partial_{\mathrm{B}}\Omega_\varepsilon$, and we have either (7), (8), or (10) if $\hat{z} \in \partial_{\mathrm{L}}\Omega_\varepsilon \cap \partial_{\mathrm{T}}\Omega_\varepsilon$. Replacing "max" and "$\leq$" with "min" and "$\geq$", respectively, in the above condition yields the right definition of viscosity supersolution.

Thanks to (H2), we can fix a constant $C_0 > 0$ so that

$$|F(0, 0, 0, (x, y))| \leq C_0 \quad \text{for } (x, y) \in \overline{\Omega_{\varepsilon_0}}.$$

Under the assumptions (H1) and (H4), we define $\kappa \in [0, +\infty)$ and $\varepsilon_1 \in (0, +\infty]$ by

$$\kappa = \max_{x \in \partial\Omega}(\nu(x) \cdot Dg(x))^+ \quad \text{and} \quad \varepsilon_1 = \frac{1}{\kappa}, \tag{11}$$

where $\frac{1}{\kappa} = +\infty$ if $\kappa = 0$. Define $\varepsilon_* = \min\{\varepsilon_0, \varepsilon_1\}$.

Proposition 1. *Assume that (H1)–(H4) hold. Let $\varepsilon \in (0, \varepsilon_*)$.*

1. *The constant functions $\alpha^{-1}C_0$ and $-\alpha^{-1}C_0$ are classical super and sub solutions to (5), respectively.*
2. *There is a viscosity solution to (5).*
3. *Any viscosity solution u to (5) satisfies $\sup_{\overline{\Omega_\varepsilon}} |u| \leq \alpha^{-1}C_0$.*

The boundary of the domain Ω_ε has corners, $\partial_{\mathrm{T}}\Omega_\varepsilon \cap \partial_{\mathrm{L}}\Omega_\varepsilon$ and $\partial_{\mathrm{B}}\Omega_\varepsilon \cap \partial_{\mathrm{L}}\Omega_\varepsilon$. The following two lemmas take care of the main difficulties arising from the corners.

Lemma 2. *Assume (H1) and (H4). Let $\varepsilon \in (0, \varepsilon_1)$. Let $c \in \partial_{\mathrm{T}}\Omega_\varepsilon \cap \partial_{\mathrm{L}}\Omega_\varepsilon$. Let $\psi \in C^2(\overline{\Omega}_\varepsilon)$ take a maximum at c. Then, either*

$$\nu_{\mathrm{T}} \cdot D\psi(c) \geq 0 \quad \text{or} \quad \nu_{\mathrm{L}} \cdot D\psi(c) \geq 0.$$

Proof. We write (c', c_{N+1}) for c, where $c_{N+1} = \varepsilon g(c')$. Set $\theta = \sqrt{\varepsilon^2|Dg(c')|^2 + 1}$ and $\zeta = \nu_{\rm L}(c) + \theta\nu_{\rm T}(c)$. First, we show that for some $t_0 > 0$,

$$c - t\zeta \in \Omega_\varepsilon \quad \text{for all } t \in (0, t_0). \tag{12}$$

Let $\rho \in C^1(\mathbb{R}^N)$ be a function satisfying (6). Writing $\zeta = (\zeta', \zeta_{N+1})$, we have

$$\zeta' = \nu(c') - \varepsilon Dg(c') \quad \text{and} \quad \zeta_{N+1} = 1,$$

and wish to show that for some $t_0 > 0$,

$$\rho(c' - t\zeta') < 0 \quad \text{and} \quad c_{N+1} - t\zeta_{N+1} < \varepsilon g(c' - t\zeta') \quad \text{for all } t \in (0, t_0). \tag{13}$$

Since

$$\varepsilon g(c' - t\zeta') = c_{N+1} - t\varepsilon\zeta' \cdot Dg(c') + o(t) \quad \text{as } t \to 0^+,$$

it is obvious that if

$$\zeta' \cdot D\rho(c') > 0 \quad \text{and} \quad \zeta_{N+1} - \varepsilon\zeta' \cdot Dg(c') > 0, \tag{14}$$

then (13) holds.

We compute that

$$\zeta' \cdot D\rho(c') = |D\rho(c')|(\nu(c') - \varepsilon Dg(c')) \cdot \nu(c') \geq |D\rho(c')|(1 - \varepsilon\kappa),$$

and

$$\zeta_{N+1} - \varepsilon\zeta' \cdot Dg(c') = 1 - \varepsilon(\nu(c') - \varepsilon Dg(c')) \cdot Dg(c') \geq 1 - \varepsilon\kappa.$$

Since $\varepsilon\kappa < 1$, we find that (14) is valid. In view of (12), we see that

$$0 \geq \frac{d}{dt}\psi(c - t\zeta))\Big|_{t=0} = -\zeta \cdot D\psi(c),$$

which can be stated as

$$(\nu_{\rm L} + \theta\nu_{\rm B}) \cdot D\psi(c) \geq 0.$$

Hence, we have either $\nu_{\rm L} \cdot D\psi(c) \geq 0$ or $\nu_{\rm B} \cdot D\psi(c) \geq 0$. □

Remark 3. A claim similar to the above lemma holds for other boundary points. Indeed, assume (H1) and (H4), and let $\varepsilon > 0$, $c \in \partial\Omega_\varepsilon \setminus (\partial_{\rm T}\Omega_\varepsilon \cap \partial_{\rm L}\Omega_\varepsilon)$, $\psi \in C^2(\overline{\Omega}_\varepsilon)$, and $\psi(c) = \max_{\overline{\Omega}_\varepsilon} \psi$. In addition, if $c \in \partial_{\rm L}\Omega_\varepsilon$, then $c - t\nu_{\rm L}(c) \in \overline{\Omega_\varepsilon}$ for all $t \in (0, t_0)$ and some $t_0 > 0$, which implies that $\nu_{\rm L} \cdot D\psi(c) \geq 0$. If $c \in \partial_{\rm T}\Omega_\varepsilon$, then $c - t\nu_{\rm T}(c) \in \Omega_\varepsilon$ for $t \in (0, t_0)$ and some $t_0 > 0$ and, therefore, $\nu_{\rm T} \cdot D\psi(c) \geq 0$. If $c \in \partial_{\rm B}\Omega_\varepsilon$, then $c - t\nu_{\rm B}(c) \in \overline{\Omega_\varepsilon}$ for $t \in (0, t_0)$ and some $t_0 > 0$, which yields that $\nu_{\rm B} \cdot D\psi(c) \geq 0$.

Remark 4. An important consequence of Lemma 2 and Remark 3 is this. Assume (H1), (H2), and (H4) and let $\varepsilon \in (0, \varepsilon_*)$. If $u \in C^2(\overline{\Omega_\varepsilon})$ is a classical subsolution to (5), then it is a subsolution to (5) in the viscosity sense. Let us start by remarking that a classical subsolution $u \in C^2(\overline{\Omega_\varepsilon})$ to (5) means that all the following conditions hold pointwise; $F(D^2u, Du, u, (x, y)) \leq 0$ on $\overline{\Omega_\varepsilon}$, $\nu_{\rm T} \cdot Du \leq 0$ on $\partial_{\rm T}\Omega_\varepsilon$, $\nu_{\rm B} \cdot Du \leq 0$ on $\partial_{\rm B}\Omega_\varepsilon$, and $\nu_{\rm L} \cdot Du \leq 0$ on $\partial_{\rm L}\Omega_\varepsilon$. For instance, on $\partial_{\rm T}\Omega_\varepsilon \cap \partial_{\rm L}\Omega_\varepsilon$, classical subsolution u satisfies the pointwise inequality $\max\{F(D^2u, Du, u, x)), \nu_{\rm T} \cdot Du, \nu_{\rm L} \cdot Du\} \leq 0$. Similarly, classical supersolutions to (5) are defined just by reversing the inequalities. Now suppose that $\phi \in C^2(\overline{\Omega_\varepsilon})$, $c \in \overline{\Omega_\varepsilon}$, and $(u - \phi)(c) = \max_{\overline{\Omega_\varepsilon}}(u - \phi)$. The function $\psi := u - \phi$ takes a maximum at c. If $c \in \Omega_\varepsilon$, then $0 \geq D^2\psi(c) = D^2u(c) - D^2\phi(c)$, $0 = D\psi(c) = Du(c) - D\phi(c)$, which, together with (H2), yields

$$0 \geq F(D^2u(c), Du(c), u(c), c) \geq F(D^2\phi(c), D\phi(c), u(c), c).$$

If $c \in \partial_{\mathrm{T}}\Omega_\varepsilon \cap \partial_{\mathrm{L}}\Omega_\varepsilon$, then, by Lemma 2, either

$$0 \leq \nu_{\mathrm{T}} \cdot D\psi(c) = \nu_{\mathrm{T}} \cdot Du(x) - \nu_{\mathrm{T}} \cdot D\phi(c) \leq -\nu_{\mathrm{T}} \cdot D\phi(c)$$

or

$$0 \leq \nu_{\mathrm{L}} \cdot D\psi(c) = \nu_{\mathrm{L}} \cdot Du(c) - \nu_{\mathrm{L}} \cdot D\phi(c) \leq -\nu_{\mathrm{L}} \cdot D\phi(c),$$

that is, either $\nu_{\mathrm{T}} \cdot D\phi(c) \leq 0$ or $\nu_{\mathrm{L}} \cdot D\phi(c) \leq 0$. Finally, consider the case $c \in \partial\Omega_\varepsilon \setminus (\partial_T\Omega_\varepsilon \cap \partial_{\mathrm{L}}\Omega_\varepsilon)$. By Remark 3, if $c \in \partial_{\mathrm{T}}\Omega_\varepsilon$, then $\nu_{\mathrm{T}} \cdot D\psi(c) \geq 0$ and, hence, $\nu_{\mathrm{T}} \cdot D\phi(c) \leq 0$. If $c \in \partial_{\mathrm{L}}\Omega_\varepsilon$, then $\nu_{\mathrm{L}} \cdot D\psi(c) \geq 0$ and $\nu_{\mathrm{L}} \cdot D\phi(c) \leq 0$. If $c \in \partial_{\mathrm{B}}\Omega_\varepsilon$, then $\nu_{\mathrm{B}} \cdot \psi(c) \geq 0$ and $\nu_{\mathrm{B}} \cdot D\phi(c) \leq 0$. Thus, u is a subsolution to (5) in the viscosity sense. Similarly, we deduce that any classical supersolution of (5) is a viscosity supersolution of (5).

Lemma 5. *Assume (H1) and (H4). Let $\varepsilon \in (0, \varepsilon_1)$. Then, there exists a function $\psi \in C^2(\overline{\Omega}_\varepsilon, \mathbb{R})$ such that*

$$\nu \cdot D\psi(z) > 0 \quad \textit{for} \begin{cases} \nu = \nu_{\mathrm{L}} \ \textit{and} \ z \in \partial_{\mathrm{L}}\Omega_\varepsilon, \\ \nu = \nu_{\mathrm{B}} \ \textit{and} \ z \in \partial_{\mathrm{B}}\Omega_\varepsilon, \\ \nu = \nu_{\mathrm{T}} \ \textit{and} \ z \in \partial_{\mathrm{T}}\Omega_\varepsilon. \end{cases} \tag{15}$$

Proof. Let $\rho \in C^1(\mathbb{R}^N)$ be a function satisfying (6). We note here (see Remark 6) that, for each $\gamma > 0$, it is possible to choose ρ such that $||D\rho(x)| - 1| < \gamma$ for $x \in \partial\Omega$.

Choose a function $\eta \in C^1(\mathbb{R})$ such that

$$\eta(r) = 0 \quad \text{for } r \leq -g_0, \quad 0 \leq \eta'(r) \leq 1 \quad \text{for } r \in \mathbb{R}, \quad \text{and} \quad \eta'(0) = 1,$$

where $g_0 := \inf_\Omega g > 0$. We define $\psi = \psi_\varepsilon$ on $\mathbb{R}^{N+1}$ by setting

$$\psi(x, y) = \rho(x) + \varepsilon\Big(\eta\Big(-\frac{y}{\varepsilon}\Big) + \eta\Big(\frac{y - \varepsilon g(x)}{\varepsilon}\Big)\Big) \quad \text{for } (x, y) \in \mathbb{R}^N \times \mathbb{R}.$$

If $z = (x, y) \in \partial_{\mathrm{L}}\Omega_\varepsilon$, then

$$\begin{aligned} \nu_{\mathrm{L}} \cdot D\psi(z) &= (\nu(x), 0) \cdot \Big((D\rho, 0) - \eta'\Big(-\frac{y}{\varepsilon}\Big)e_{N+1} \\ &\qquad + \eta'\Big(\frac{y - \varepsilon g(x)}{\varepsilon}\Big)(-\varepsilon Dg(x), 1)\Big) \\ &\geq |D\rho(x)| - \eta'\Big(\frac{y - \varepsilon g(x)}{\varepsilon}\Big)\varepsilon\kappa \geq |D\rho(x)| - \varepsilon\kappa. \end{aligned}$$

Similarly, if $z = (x, 0) \in \partial_{\mathrm{B}}\Omega_\varepsilon$, then

$$\nu_{\mathrm{B}} \cdot D\psi(z) = -e_{N+1} \cdot D\psi(z) = \eta'(0) - \eta'(-g(x)) = 1,$$

and if $z = (x, y) \in \partial_{\mathrm{T}}\Omega_\varepsilon$, then

$$\begin{aligned}\nu_{\mathrm{T}} \cdot D\psi(z) &\geq \frac{1}{\sqrt{\varepsilon^2|Dg|^2+1}}\Big(- \varepsilon|D\rho|\nu \cdot Dg(x) - \eta'(-g(x)) + \eta'(0)(\varepsilon^2|Dg(x)|^2+1)\Big)\Big) \\ &\geq \frac{1}{\sqrt{\varepsilon^2|Dg|^2+1}}\Big(- \varepsilon\kappa|D\rho| + (\varepsilon^2|Dg(x)|^2+1)\Big)\Big) \\ &\geq \frac{1}{\sqrt{\varepsilon^2|Dg|^2+1}}\Big(1 - \varepsilon\kappa|D\rho|\Big).\end{aligned}$$

By th choice of ε, we have $\varepsilon\kappa < 1$ and, as noted above, we may assume that

$$\varepsilon\kappa < |D\rho(x)| \quad \text{and} \quad \varepsilon\kappa|D\rho(x)| < 1 \quad \text{for } x \in \partial\Omega.$$

The function ψ satisfies the property (15). □

Remark 6. For any $\gamma > 0$, there exists a $\rho \in C^1(\mathbb{R}^N)$ which satisfies (6) and $||D\rho(x)| - 1| < \gamma$ for $x \in \partial\Omega$. To see this, fix any $\rho \in C^1(\mathbb{R}^N)$ having the property (6). Let $\gamma > 0$. Since $D\rho(x) \neq 0$ for $x \in \partial\Omega$, we may choose a function $\lambda \in C^1(\mathbb{R}^N)$ such that $\lambda > 0$ in $\mathbb{R}^N$ and $|\lambda(x)|D\rho(x)| - 1| < \gamma$ for $x \in \partial\Omega$. Then, noting that $D(\lambda\rho)(x) = \lambda(x)D\rho(x)$ for $x \in \partial\Omega$, we find that the function $\lambda\rho$ has the required properties.

Proof of Proposition 1. (1) Set $u(z) = \alpha^{-1}C_0$ for $z \in \overline{\Omega_\varepsilon}$. It is clear that $\nu_{\mathrm{L}} \cdot Du(z) = 0$ for $z \in \partial_{\mathrm{L}}\Omega_\varepsilon$, $\nu_{\mathrm{B}} \cdot Du(z) = 0$ for $z \in \partial_{\mathrm{B}}\Omega_\varepsilon$, and $\nu_{\mathrm{T}} \cdot Du(z) = 0$ for $z \in \partial_{\mathrm{T}}\Omega_\varepsilon$. It follows that

$$\begin{aligned}F(D^2u(z), Du(z), u(z), z) &= F(0, 0, u(z), z) \geq F(0, 0, 0, z) + \alpha u(z) \\ &\geq -C_0 + C_0 = 0 \quad \text{for } z \in \overline{\Omega_\varepsilon}.\end{aligned}$$

Thus, the constant function $\alpha^{-1}C_0$ is a classical supersolution of (5). Similarly, the constant function $-\alpha^{-1}C_0$ is a classical subsolution of (5).

(2) Thanks to Remark 4, the constant functions $\alpha^{-1}C_0$ and $-\alpha^{-1}C_0$ are viscosity sub and super solutions of (5), respectively. Due to Remark 3, we infer that the Perron method works for the boundary value problem (5). Indeed, if we set

$$\begin{aligned}u(z) = \sup\{v(z)\ : &v \text{ is a viscosity subsolution of (5)}, \\ &-\alpha^{-1}C_0 \leq v \leq \alpha^{-1}C_0 \text{ on } \overline{\Omega_\varepsilon}\} \quad \text{for } z \in \overline{\Omega_\varepsilon},\end{aligned}$$

then the function u is a viscosity solution to (5). (Remark 3 is critical when one checks that the lower semicontinuous envelope of u is a supersolution to (5). See a related remark [4, Remark 4.5].)

(3) According to Lemma 5, there is a function $\psi \in C^2(\overline{\Omega_\varepsilon})$ satisfying (15). Let u be any viscosity solution of (5). Let v and w be the upper and lower semicontinuous envelopes of u on $\overline{\Omega_\varepsilon}$, respectively. Fix any $\delta > 0$. We prove by contradiction that $v \leq \alpha^{-1}C_0$ on $\overline{\Omega_\varepsilon}$. Thus, we suppose that $\max_{\overline{\Omega_\varepsilon}} v > \alpha^{-1}C_0$. Choosing positive constants δ and γ small enough, we have $\max_{\overline{\Omega_\varepsilon}}(v - \gamma\psi) > \delta + \alpha^{-1}C_0$. Set $\phi = \gamma\psi + \delta + \alpha^{-1}C_0$ on $\overline{\Omega_\varepsilon}$. Let $\hat{z} \in \overline{\Omega_\varepsilon}$ be a maximum point of the function $v - \phi$. Noting that (15) holds with ϕ in place of ψ, we find by the subsolution property of v that

$$\begin{aligned}0 &\geq F(D^2\phi(\hat{z}), D\phi(\hat{z}), v(\hat{z}), \hat{z}) \\ &\geq F(\gamma D^2\psi(\hat{z}), \gamma D\psi(\hat{z}), \gamma\psi(\hat{z}) + \alpha^{-1}C_0, \hat{z}) + \alpha\delta.\end{aligned}$$

Sending $\gamma \to 0^+$, we obtain $F(0,0,\alpha^{-1}C_0,\tilde{z})+\alpha\delta \le 0$ for some $\tilde{z} \in \overline{\Omega_\varepsilon}$, which contradicts that $\alpha^{-1}C_0$ is a classical supersolution of (5). Hence, we conclude that $u \le v \le \alpha^{-1}C_0$ on $\overline{\Omega_\varepsilon}$. A parallel argument ensures that $u \ge w \ge -\alpha^{-1}C_0$ on $\overline{\Omega_\varepsilon}$. □

3. Convergence results

Henceforth, unless otherwise stated, we always assume that $\varepsilon \in (0,\varepsilon_*)$.

3.1. **Relaxed limits.** Let u^ε be a solution of (5). By Proposition 1, we have

$$\|u^\varepsilon\|_\infty \le \frac{C_0}{\alpha}.$$

This allows us to define the upper and lower relaxed limits u^+ and u^- of $\{u^\varepsilon\}_{\varepsilon\in(0,\varepsilon_*)}$:

$$\begin{cases} u^+(x) = \lim_{r\to 0^+} \sup\{u^\varepsilon(\xi,\eta) \,:\, (\xi,\eta) \in \overline{\Omega}_\varepsilon, |\xi - x| < r, 0 < \varepsilon < r\}, \\ u^-(x) = \lim_{r\to 0^+} \inf\{u^\varepsilon(\xi,\eta) \,:\, (\xi,\eta) \in \overline{\Omega}_\varepsilon, |\xi - x| < r, 0 < \varepsilon < r\}. \end{cases} \tag{16}$$

It follows that $u^+ \ge u^-$ on $\overline{\Omega}$ and $u^+, -u^- \in \mathrm{USC}(\overline{\Omega})$. The limit equation will be

$$\begin{cases} F\left(\begin{pmatrix} D^2 w & 0 \\ 0 & Dg\cdot Dw/g \end{pmatrix}, (Dw,0), w, (x,0)\right) = 0 \quad \text{in } \Omega, \\ \dfrac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{17}$$

Theorem 7. *Suppose that (H1)–(H4) hold. The functions u^+ and u^- are, respectively, sub and super solutions of* (17)*.*

Proof. We treat only the subsolution property. By replacing u^ε by its upper semicontinuous envelope, we may assume that u^ε is upper semicontinuous on $\overline{\Omega_\varepsilon}$. Let $\phi \in C^2(\overline{\Omega})$ and assume that for some $\hat{x} \in \overline{\Omega}$,

$$(u^+ - \phi)(x) < (u^+ - \phi)(\hat{x}) \quad \text{if} \quad x \ne \hat{x}.$$

In the following computation, we fix $\delta > 0$ arbitrarily. We choose $h_\delta \in C^2(\overline{\Omega})$ so that

$$\left|\left(\frac{Dg\cdot D\phi}{g}\right)(x) - h_\delta(x)\right| < \delta \quad \text{for } x \in \overline{\Omega}.$$

We set

$$\psi_\delta^\pm(x,y) = \frac{1}{2}y^2\left(\pm 2\delta + h_\delta(x)\right),$$

and in view of (3), explained in the heuristic of the introduction, we consider the function

$$\Phi(x,y) = \phi(x) + \psi_\delta^+(x,y) + \gamma\varepsilon^2\zeta(y/\varepsilon),$$

where $\zeta \in C^2(\mathbb{R})$ is a bounded function on $\mathbb{R}$ having the properties

$$-1 < \zeta'(0) < 0 < \zeta'(y) < 1 \;\text{ for } y \ge \min g \quad \text{and} \quad |\zeta''(y)| < 1 \;\text{ for } y \in [0,\max g],$$

and $\gamma > 0$.

We choose a maximum point $(\bar{x},\bar{y}) = (\bar{x}(\varepsilon,\gamma),\bar{y}(\varepsilon,\gamma))$ of the function $u^\varepsilon - \Phi$ on $\overline{\Omega}_\varepsilon$. We are to take the limit $\varepsilon \to 0^+$. In our limit process as $\varepsilon \to 0^+$, the choice of γ

depends on ε in such a way that $\lim \gamma/\varepsilon = 0$. A possible choice is $\gamma = \varepsilon^2$. It is a standard observation (see Remark 8 below) that as $\varepsilon \to 0^+$,

$$(\bar{x}, \bar{y}) \to (\hat{x}, 0) \quad \text{and} \quad u^\varepsilon(\bar{x}, \bar{y}) \to u^+(\hat{x}). \tag{18}$$

Since u^ε is a subsolution of (5), if

$$(\bar{x}, \bar{y}) \in \Omega_\varepsilon, \tag{i}$$

then we have

$$F(D^2\Phi(\bar{x}, \bar{y}), D\Phi(\bar{x}, \bar{y}), u^\varepsilon(\bar{x}, \bar{y}), (\bar{x}, \bar{y})) \le 0; \tag{19}$$

if

$$(\bar{x}, \bar{y}) \in \partial_{\mathrm{T}}\Omega_\varepsilon \setminus \partial_{\mathrm{L}}\Omega_\varepsilon, \tag{ii}$$

then we have either (19) or

$$-\varepsilon Dg(\bar{x}) \cdot D_x\Phi(\bar{x}, \varepsilon g(\bar{x})) + \Phi_y(\bar{x}, \varepsilon g(\bar{x})) \le 0; \tag{20}$$

if

$$(\bar{x}, \bar{y}) \in \partial_{\mathrm{B}}\Omega_\varepsilon \setminus \partial_{\mathrm{L}}\Omega_\varepsilon, \tag{iii}$$

then we have either (19) or

$$-\Phi_y(\bar{x}, 0) \le 0; \tag{21}$$

if

$$(\bar{x}, \bar{y}) \in \partial_{\mathrm{L}}\Omega_\varepsilon \setminus (\partial_{\mathrm{T}}\Omega_\varepsilon \cup \partial_{\mathrm{B}}\Omega_\varepsilon), \tag{iv}$$

then we have either (19) or

$$\frac{\partial\Phi(\bar{x}, \bar{y})}{\partial\nu_\varepsilon} = \nu_{\mathrm{L}} \cdot D\Phi(\bar{x}, \bar{y}) \le 0; \tag{22}$$

if

$$(\bar{x}, \bar{y}) \in \partial_{\mathrm{T}}\Omega_\varepsilon \cap \partial_{\mathrm{L}}\Omega_\varepsilon, \tag{v}$$

then we have either (19), (20), or (22); if

$$(\bar{x}, \bar{y}) \in \partial_{\mathrm{B}}\Omega_\varepsilon \cap \partial_{\mathrm{L}}\Omega_\varepsilon, \tag{vi}$$

then we have either (19), (21), or (22).

Observe that

$$\begin{aligned}
&D_x\Phi = D\phi(x) + \frac{y^2}{2}Dh_\delta(x), &&\Phi_y = y\,(2\delta + h_\delta(x)) + \varepsilon\gamma\zeta'\left(\frac{y}{\varepsilon}\right),\\
&D_x^2\Phi = D^2\phi(x) + \frac{y^2}{2}D^2h_\delta(x), &&\Phi_{yy} = 2\delta + h_\delta(x) + \gamma\zeta''\left(\frac{y}{\varepsilon}\right),\\
&\Phi_{x_i y} = \Phi_{y x_i} = y(h_\delta)_{x_i}(x).
\end{aligned}$$

Inequalities (19), (20), (21), and (22), can be written, respectively, as

$$F(\bar{X}, \bar{p}, u^\varepsilon(\bar{x}, \bar{y}), (\bar{x}, \bar{y})) \le 0, \tag{23}$$

where

$$\bar{X} = \begin{pmatrix} D^2\phi(\bar{x}) + \dfrac{\bar{y}^2}{2}D^2h_\delta(\bar{x}) & \bar{y}Dh_\delta(\bar{x}) \\ \bar{y}Dh_\delta(\bar{x})^T & 2\delta + h_\delta(\bar{x}) + \gamma\zeta''\left(\dfrac{\bar{y}}{\varepsilon}\right) \end{pmatrix},$$

and $\bar{p} = (D\phi(\bar{x}) + \dfrac{\bar{y}^2}{2}Dh_\delta(\bar{x}), \bar{y}\,(2\delta + h_\delta(\bar{x})) + \varepsilon\gamma\zeta'\left(\dfrac{\bar{y}}{\varepsilon}\right)),$

$$\begin{aligned} -Dg(\bar{x})\cdot &\left(D\phi(\bar{x}) + \frac{\varepsilon^2 g(\bar{x})^2}{2}Dh_\delta(\bar{x})\right) \\ &+ g(\bar{x})\,(2\delta + h_\delta(\bar{x})) + \gamma\zeta'(g(\bar{x})) \le 0, \end{aligned} \tag{24}$$

$$\zeta'(0) \ge 0, \tag{25}$$

$$\frac{\partial\Phi(\bar{x},\bar{y})}{\partial\nu_{\mathrm{L}}} = (|D\rho|^{-1}D\rho(\bar{x}), 0)\cdot D\Phi(\bar{x},\bar{y}) \le 0. \tag{26}$$

Choosing $\varepsilon > 0$ small enough, we may assume that

$$\delta \ge \frac{\varepsilon^2 g(\bar{x})}{2}Dg(\bar{x})\cdot Dh_\delta(\bar{x}).$$

If (24) holds, then we have

$$\begin{aligned} 0 \ge &-Dg(\bar{x})\cdot\left(D\phi(\bar{x}) + \frac{\varepsilon^2 g(\bar{x})^2}{2}Dh_\delta(\bar{x})\right) \\ &+ g(\bar{x})\left(\delta + \left(\frac{Dg\cdot D\phi}{g}\right)(\bar{x})\right) + \gamma\zeta'(g(\bar{x})) \\ \ge &\,\gamma\zeta'(g(\bar{x})). \end{aligned}$$

This contradicts our choice of ζ, and also (25) is a contradiction.

Thus, we have (23) in the case when either (i), (ii), or (iii) is valid, and we have either (23) or (26) in the cases when either (iv), (v), or (vi) holds.

Sending $\varepsilon \to 0^+$, we have

$$\bar{X} \to \begin{pmatrix} D^2\phi(\hat{x}) & 0 \\ 0 & 2\delta + h_\delta(\hat{x}) \end{pmatrix} \le \begin{pmatrix} D^2\phi(\hat{x}) & 0 \\ 0 & 3\delta + \dfrac{Dg(\hat{x})\cdot D\phi(\hat{x})}{g(\hat{x})} \end{pmatrix}$$

and

$$\bar{p} \to (D\phi(\hat{x}), 0).$$

Therefore, we see that if $\hat{x} \in \Omega$, then we have

$$F\left(\begin{pmatrix} D^2\phi(\hat{x}) & 0 \\ 0 & 3\delta + \dfrac{Dg(\hat{x})D\phi(\hat{x})}{g(\hat{x})} \end{pmatrix}, (D\phi(\hat{x}), 0), u^+(\hat{x}), (\hat{x}, 0)\right) \le 0, \tag{27}$$

if $\hat{x} \in \partial\Omega$, then we have either (27) or

$$\nu_{\mathrm{L}}\cdot(D\phi(\hat{x}), 0) = \frac{\partial\phi(\hat{x})}{\partial\nu} \le 0.$$

This guarantees that u^+ is a subsolution of (17).

A remark on the proof of the supersolution property of u^- is that, in this case, one should use the perturbed test function

$$\Phi(x,y) = \phi(x) + \psi_\delta^-(x,y) - \gamma\varepsilon^2\zeta\left(\frac{y}{\varepsilon}\right). \qquad \square$$

Remark 8. For a general approach to the proof of (18), we may refer to the User's guide [4]. Here, for the reader's convenience, we give a straightforward proof of (18). By the definition of $u^+(\hat{x})$, we may choose $\{(\varepsilon_j, x_j, y_j)\}_{j\in\mathbb{N}}$ so that

$$\varepsilon_j \to 0^+, \quad (x_j, y_j) \in \overline{\Omega_{\varepsilon_j}}, \quad |x_j - \hat{x}| < \frac{1}{j}, \quad u^+(\hat{x}) < \frac{1}{j} + u^{\varepsilon_j}(x_j, y_j).$$

Since Φ depends on ε, we write Φ_ε for Φ. Also, we write $(\bar{x}_j, \bar{y}_j)$ for $(\bar{x}, \bar{y})$ with $\varepsilon = \varepsilon_j$. Thus, $(\bar{x}_j, \bar{y}_j)$ is a maximum point of $u^{\varepsilon_j} - \Phi_{\varepsilon_j}$, and we have

$$(u^{\varepsilon_j} - \Phi_{\varepsilon_j})(\bar{x}_j, \bar{y}_j) \geq (u^{\varepsilon_j} - \Phi_{\varepsilon_j})(x_j, y_j) > -\frac{1}{j} + u^+(\hat{x}) - \Phi_{\varepsilon_j}(x_j, y_j).$$

We may assume after passing to a subsequence that for some $\tilde{x} \in \overline{\Omega}$ and $\tilde{u} \in \mathbb{R}$,

$$\lim(\bar{x}_j, \bar{y}_j) = (\tilde{x}, 0) \quad \text{and} \quad \lim u^{\varepsilon_j}(\bar{x}_j, \bar{y}_j) = \tilde{u}.$$

Since $(\varepsilon_j, \bar{x}_j, \bar{y}_j) \to (0, \tilde{x}, 0)$, we see, by the definition of $u^+(\tilde{x})$, that

$$u^+(\tilde{x}) \geq \lim u^{\varepsilon_j}(\bar{x}_j, \bar{y}_j) = \tilde{u}.$$

All the above together, we see in the limit as $j \to \infty$ that

$$u^+(\tilde{x}) - \Phi_0(\tilde{x}, 0) \geq \tilde{u} - \Phi_0(\tilde{x}, 0) \geq u^+(\hat{x}) - \Phi_0(\hat{x}, 0),$$

where $\Phi_0(x,y) := \lim_{\varepsilon\to 0^+} \Phi_\varepsilon(x,y) = \phi(x) + \psi_\delta^-(x,y)$, that is,

$$(u^+ - \phi)(\tilde{x}) \geq \tilde{u} - \phi(\tilde{x}) \geq (u^+ - \phi)(\hat{x}),$$

which shows that $\tilde{x} = \hat{x}$ and $\tilde{u} = u^+(\hat{x})$. $\square$

3.2. **Uniform convergence.** Let F, Ω, and g be as in the previous section. Define the function $G \in C(\mathcal{S}(N) \times \mathbb{R}^N \times \mathbb{R} \times \overline{\Omega}, \mathbb{R})$ by

$$G(X,p,r,x) = F\Big(\begin{pmatrix} X & 0 \\ 0 & Dg(x)\cdot p/g(x) \end{pmatrix}, (p,0), r, (x,0)\Big). \tag{28}$$

Recall that the limit equation (17) for u is stated as

$$G(D^2u, Du, u, x) = 0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{29}$$

A convenient assumption for Theorem 9 to draw a uniform convergence result is the validity of the comparison principle for (29):

(H5) If v and w are viscosity sub and super solutions to (29), respectively, then $v \leq w$ on $\overline{\Omega}$.

Indeed, we have

Theorem 9. *Assume (H1)–(H5). Let u^ε be a viscosity solution to* (5) *for $\varepsilon \in (0, \varepsilon_0]$. Then, for a unique continuous viscosity solution u^0 of* (29)*, we have*

$$\lim_{\varepsilon\to 0^+} \max_{(x,y)\in\overline{\Omega_\varepsilon}} |u^\varepsilon(x,y) - u^0(x)| = 0. \tag{30}$$

Proof. The following argument is standard in the asymptotic analysis based on the half-relaxed limits, but we here present it for the reader's convenience. Let u^+ and u^- be the functions defined by (16). By the definition, we have $u^- \le u^+$ on $\overline{\Omega}$ and $u^+, -u^- \in \mathrm{USC}(\overline{\Omega})$. Theorem 7 ensures that u^+ and u^- are viscosity sub and super solutions to (29), respectively. Furthermore, (H5) assures that $u^+ \le u^-$ on $\overline{\Omega}$. Hence, we see that $u^+ = u^-$ on $\overline{\Omega}$, which readily shows that $u^+ = u^-$ is continuous on $\overline{\Omega}$. Writing u^0 for $u^+ = u^-$, we find that u^0 is a continuous viscosity solution to (29).

To check (30), fix any $\delta > 0$. By the definition of u^+, for any $x \in \overline{\Omega}$, we select $r = r(\delta, x) > 0$ so that

$$u^\varepsilon(\xi,\eta) < u^0(x) + \delta \quad \text{if } 0 < \varepsilon < r,\ (\xi,\eta) \in \overline{\Omega_\varepsilon}, \quad \text{and} \quad |\xi - x| < r.$$

Reselecting $r > 0$ sufficiently smaller, we may assume that $u^0(x) < u^0(\xi) + \delta$ if $\xi \in \overline{\Omega}$ and $|\xi - x| < r$. Now, the above inequality can be stated as

$$u^\varepsilon(\xi,\eta) < u^0(\xi) + 2\delta \quad \text{if } 0 < \varepsilon < r,\ (\xi,\eta) \in \overline{\Omega_\varepsilon}, \quad \text{and} \quad |\xi - x| < r. \tag{31}$$

Since $\overline{\Omega}$ is compact, we can choose a finite number of balls, $B_1, \dots, B_m$, which cover $\overline{\Omega}$, such that for every $j \in \{1, \dots, m\}$, if x_j and r_j denote, respectively, the center and radius of B_j, then (31), with (x_j, r_j) in place of (x, r), holds. Setting $r_0 = \min\{r_j : j = 1, \dots, m\}$, we find that

$$u^\varepsilon(\xi,\eta) < u^0(\xi) + 2\delta \quad \text{for } (\xi,\eta) \in \overline{\Omega_\varepsilon} \ \text{ and } \ 0 < \varepsilon < r_0.$$

An argument parallel to the above yields, after replacing $r_0 > 0$ by a smaller one if necessary,

$$u^\varepsilon(\xi,\eta) > u^0(\xi) - 2\delta \quad \text{for } (\xi,\eta) \in \overline{\Omega_\varepsilon} \ \text{ and } \ 0 < \varepsilon < r_0,$$

which completes the proof of (30). □

Let us recall that there are a number of contests where the comparison principle (H5) holds. In particular, when dealing with Neumann boundary conditions, one can refer to the results of Hitoshi Ishii [8], Guy Barles [3] and Stefania Patrizi [9].

We consider here the general comparison principle given in [4, Theorem 7.5]. This leads us to assume, further hypotheses on the domain Ω and the operator G.

On Ω, in addition to (H4), we need the uniform exterior sphere condition, i.e. that there is a constant $r_0 > 0$ such that

$$B_{r_0}(x + r_0\nu(x)) \cap \Omega = \emptyset \quad \text{for } x \in \partial\Omega, \tag{32}$$

where $B_r(x)$ denotes the open ball $\{y \in \mathbb{R}^N : |y - x| < r\}$. On the function G a crucial and typical hypothesis is the following:

$$\begin{cases} \text{There is a function } \omega : [0,\infty) \to [0,\infty) \text{ that satisfies } \omega(0^+) = 0 \text{ such that} \\ \qquad G(Y,p,r,y) - G(X,p,r,x) \le \omega(\gamma|x-y|^2 + |x-y|(|p|+1)) \\ \text{whenever } \gamma > 0,\ p \in \mathbb{R}^N,\ x, y \in \overline{\Omega},\ r \in \mathbb{R}, \text{ and } X, Y \in \mathcal{S}(N) \text{ satisfy} \\ \qquad -3\gamma I_{2N} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le 3\gamma \begin{pmatrix} I_N & -I_N \\ -I_N & I_N \end{pmatrix}. \end{cases} \tag{33}$$

Here I_m denotes the identity matrix of order m. We impose another continuity condition on G, which states:

$$\begin{cases}\text{There is a neighborhood } V \text{ of } \partial\Omega \text{, relative to } \overline{\Omega}\text{, such that} \\ \qquad G(X,p,r,x) - G(Y,q,r,x) \le \omega(\|X-Y\| + |p-q|) \\ \text{for } X, Y \in \mathcal{S}(N),\ p, q \in \mathbb{R}^N,\ r \in \mathbb{R}, \text{ and } x \in V. \end{cases} \tag{34}$$

Note that if (H3) holds, then

$$\alpha(r-s) \le G(X,p,r,x) - G(X,p,s,x) \tag{35}$$

for $r \ge s$ and $(X,p,x) \in \mathcal{S}(N) \times \mathbb{R}^N \times \overline{\Omega}$.

The next proposition is a direct consequence of [4, Theorem 7.5] and Theorem 9.

Proposition 10. *Assume (H1)–(H4) and* (32)–(34)*. Then (H5) is satisfied and the uniform convergence* (30) *as in Theorem 9 is valid.*

Before concluding our discussion, we present two important examples of equations to which Theorem 9 applies, one is fully nonlinear and the other is linear but degenerate elliptic.

Example 11. We apply Proposition 10 to show the uniform convergence result for the solution of equation (1), involving the extremal Pucci operator as presented in the Introduction. The extremal Pucci operator $-\mathcal{M}^+_{\lambda,\Lambda}(X)$ has the property (33). Indeed, the matrix inequality on the right-hand side of (33) implies that $X \le Y$ and hence, $-\mathcal{M}^+_{\lambda,\Lambda}(Y) + \mathcal{M}^+_{\lambda,\Lambda}(X) \le 0$. If the regularity of g is strengthened so that $g \in C^{1,1}(\overline{\Omega})$, then both the functions

$$H(p,x) = \left(\frac{Dg(x)\cdot p}{g(x)}\right)^{\pm}$$

satisfy

$$|H(p,y) - H(p,x)| \le C|x-y||p|$$

for all $p \in \mathbb{R}^N$, $x, y \in \overline{\Omega}$ and some constant $C > 0$. It is then obvious to see that the operator

$$G(X,p,r,x) = -\mathcal{M}^+_{\lambda,\Lambda}(X) - \Lambda\left(\frac{Dg(x)\cdot p}{g(x)}\right)^+ + \lambda\left(\frac{Dg(x)\cdot p}{g(x)}\right)^- + \alpha r - f(x,0),$$

where $f \in C(\overline{\Omega_{\varepsilon_0}})$, satisfies (33). Thus, thanks to Proposition 10, we find that the uniform convergence (30) for the solution u^ε to (1), as in Theorem 9, holds, provided that $\alpha > 0$, (H1), $g \in C^{1,1}(\overline{\Omega})$, $f \in C(\overline{\Omega_{\varepsilon_0}})$, (H4), and (32) are satisfied. Of course the case of the Laplacian is recovered just by considering $\lambda = \Lambda = 1$.

Example 12. In these examples we concentrate on simple degenerate elliptic equations in order to emphasize how the nature of the limit equation depends on the direction of the diffusion. Let u_ε be the solution of

$$-\partial^2_{yy}(u_\varepsilon) + u_\varepsilon = f(x,y) \text{ in } \Omega_\varepsilon, \qquad \partial_{\nu_\varepsilon} u_\varepsilon = 0 \text{ on } \partial\Omega_\varepsilon.$$

If $g \in C^{1,1}(\overline{\Omega})$ and $f \in C(\overline{\Omega_{\varepsilon_0}})$, then we are under the hypothesis of Proposition 10, therefore u_ε converges uniformly to u_o solution of a first order equation precisely:

$$-\frac{Dg \cdot Du_o}{g} + u_o = f(x,0) \text{ in } \Omega, \qquad \partial_\nu u_o = 0 \text{ on } \partial\Omega.$$

Instead, if u_ε is the solution of

$$-\partial^2_{x_1x_1}(u_\varepsilon) + u_\varepsilon = f(x,y) \text{ in } \Omega_\varepsilon, \qquad \partial_{\nu_\varepsilon} u_\varepsilon = 0 \text{ on } \partial\Omega_\varepsilon$$

it will converge to u_o solution of

$$-(u_o)_{x_1x_1} + u_o = f(x,0) \text{ in } \Omega, \qquad \partial_\nu u_o = 0 \text{ on } \partial\Omega.$$

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