A NOTE ON RICCI–PINCHED THREE–MANIFOLDS

LUCA BENATTI, CARLO MANTEGAZZA, FRANCESCA ORONZIO, AND ALESSANDRA PLUDA

ABSTRACT. Let (M,g) be a complete, connected, non–compact Riemannian 3–manifold. Suppose that (M,g) satisfies the *Ricci–pinching condition* $\mathrm{Ric} \geqslant \varepsilon \mathrm{R} g$ for some $\varepsilon > 0$, where Ric and R are the Ricci tensor and the scalar curvature, respectively. In this short note, we give an alternative proof based on potential theory of the fact that if (M,g) has Euclidean volume growth, then it is flat. This result was previously shown by Deruelle–Schulze–Simon [8] and Huisken–Körber [14] and together with the contributions of Lott [17] and Lee–Topping [15], it led to a proof of the so–called *Hamilton's pinching conjecture*.

1. INTRODUCTION

Let (M,g) be a complete and connected Riemannian 3–manifold. We denote by Ric and R the Ricci and scalar curvature, respectively.

Definition 1.1. A Riemannian manifold (M,g) is *Ricci–pinched* if $\mathrm{Ric} \geqslant 0$ and there exists a constant $\varepsilon > 0$ such that $\mathrm{Ric} \geqslant \varepsilon \mathrm{R} g$.

The following theorem was known as *Hamilton's pinching conjecture* and its proof required the joint efforts of Lott [17], Deruelle–Schulze–Simon [8] and Lee–Topping [15].

Theorem 1.2. *Let (M,g) be a complete, connected Riemannian 3–manifold. Suppose that (M,g) is Ricci–pinched, then it is flat or compact.*

Notice that being flat or compact is not mutually exclusive, consider for instance a flat 3–torus. This result is a generalization of the well–known *Myers's diameter estimate*: if (M,g) is a complete and connected n–dimensional Riemannian manifold such that $\mathrm{Ric} \geqslant (n-1)kg$, for some constant $k > 0$, then M is compact and $\mathrm{diam}(M,g) \leqslant \pi/k^2$. Richard Hamilton conjectured Theorem 1.2, possibly taking inspiration from its extrinsic counterpart that he proved for hypersurfaces of the Euclidean space [11].

Theorem 1.3. *Let M be a smooth, strictly convex, complete hypersurface in $\mathbb{R}^n$. If the second fundamental form of M is* pinched*, in the sense that there exists $\varepsilon > 0$ such that*

$$h_{ij} \geqslant \varepsilon \, \mathrm{H} \, g_{ij},$$

where g_{ij} is the induced Riemannian metric, then M is compact.

A first step towards the proof of Theorem 1.2 was done by Chen and Zhu [6] who proved, employing the Ricci flow, that a 3–dimensional, complete and non–compact Riemannian manifold, with bounded and nonnegative sectional curvature, which is Ricci–pinched is flat. Then, Lott [17] improved their result, requiring milder assumptions on the sectional curvature, and Deruelle–Schulze–Simon [8] showed that the conjecture is true if the curvature is

Date: February 19, 2025.
2020 *Mathematics Subject Classification.* 53C21, 53E10, 83C99, 49J45.
Key words and phrases. Riemannian 3–manifold, Ricci–pinching, potential theory.

bounded. Finally, Lee and Topping [15] removed the bounded curvature assumption. All these results employ the Ricci flow. We mention that higher–dimensional versions of Hamilton's conjecture were proven by Ma and Cheng in [18] and by Deruelle–Schulze–Simon [9] (see also [5]).

In this short note, we give an alternative, direct, and mostly self–contained proof of a weaker version of Theorem 1.2.

The *asymptotic volume ratio* of (M,g) is defined as

$$\mathrm{AVR} = \frac{3}{4\pi} \lim_{r\to+\infty} \frac{\mathrm{Vol}(B_r(p))}{r^3},$$

for any point $p\in M$. When $\mathrm{Ric}\geqslant 0$, thanks to the Bishop–Gromov theorem, the quantity AVR is well–defined and independent of the point $p\in M$. Moreover, $\mathrm{AVR}\in[0,1]$ and $\mathrm{AVR}=1$ if and only if the manifold is $\mathbb{R}^3$ endowed with the Euclidean metric.

Theorem 1.4. *Let (M,g) be a complete, connected, non–compact, Ricci–pinched Riemannian 3–manifold. Suppose that $\mathrm{AVR}>0$, then (M,g) is flat.*

We will get Theorem 1.4 as a consequence of a slightly more general result, where the assumption $\mathrm{AVR}>0$ is replaced by another condition on the asymptotic volume growth. We say that (M,g) has *superquadratic volume growth* if there exist a point $p\in M$ and two constants $\mathrm{C}_{\mathrm{vol}}>0$ and $\alpha\in(1,2]$ such that, for sufficiently large r,

$$\mathrm{C}_{\mathrm{vol}}^{-1}\, r^{1+\alpha} \leqslant \mathrm{Vol}(B_r(q)) \leqslant \mathrm{C}_{\mathrm{vol}}\, r^{1+\alpha}. \tag{1.1}$$

Theorem 1.5. *Let (M,g) be a complete, connected, non–compact, Ricci–pinched Riemannian 3–manifold. Suppose that (M,g) has* superquadratic volume growth *with $\alpha>4/3$ in (1.1), then (M,g) is flat.*

Condition (1.1) holding with $\alpha=2$ is equivalent to $\mathrm{AVR}>0$, hence Theorem 1.4 is a special case of Theorem 1.5. We mention that Theorem 1.5 is contained in the paper of Deruelle–Schulze–Simon [8, Theorem 1.3] and has been proved also by Huisken–Köerber using the *inverse mean curvature flow* [14]. Our proof in the next sections avoids the existence and regularity theory for the inverse mean curvature flow [12, 13], being replaced with the more widely known potential theory. At the end of the paper, we also show an application to manifolds with boundary.

2. Proof of Theorem 1.5

Let (M,g) be a complete, connected, non–compact, Ricci–pinched Riemannian 3–manifold. We suppose by contradiction that (M,g) is not flat, then there must exist a point $o\in M$ with $\mathrm{R}(o)>0$. As a consequence, by considering the asymptotic expansion of the surface area and the mean curvature H of the small spheres $\partial B_r(o)$, as $r\to 0$, there exists a radius $r\ll 1$ such that $\partial B_r(o)$ is a smooth surface and

$$\int_{\partial B_r(o)} \mathrm{H}^2\, \mathrm{d}\mu < 16\pi, \tag{2.1}$$

see for instance [10, Theorem 3.2].

We then set $\Omega=\overline{B}_r(o)$ and we define the function w as the solution of the elliptic problem

$$\begin{cases} \Delta w = |\nabla w|^2 & \text{on } M\setminus\Omega \\ w = 0 & \text{on } \partial\Omega \\ w \to +\infty & \text{as } d(x,o)\to+\infty \end{cases} \tag{2.2}$$

The existence and regularity of such a solution are granted by the classical theory of harmonic functions. Consider indeed the following problem

$$\begin{cases} \Delta u = 0 & \text{on } M \setminus \Omega \\ u = 1 & \text{on } \partial\Omega \\ u \to 0 & \text{as } d(x,o) \to +\infty \end{cases} \tag{2.3}$$

and assume that (M,g) has superquadratic volume growth, that is condition (1.1) holds. Then, if $\Omega \subseteq M$ is a regular domain, problem (2.3) admits a unique solution $u \in C^\infty(M \setminus \mathring{\Omega})$ which takes values in $(0,1]$ and it is smooth till the boundary (see the papers by Varopoulos [19], Li–Yau [16] and Agostiniani–Fogagnolo–Mazzieri [1]). Then, $w = -\log u$ is a smooth solution of problem (2.2).

Let $\Omega_t = \{w \leqslant t\} \cup \Omega$. We define the following function $\mathscr{F}$ at every regular value $t \in [0,+\infty)$ of w solution of problem (2.2), as

$$\mathscr{F}(t) = \int_{\partial\Omega_t} \mathrm{H}\,|\nabla w| - |\nabla w|^2 \,\mathrm{d}\mu,$$

where H denotes the mean curvature with respect to the outward pointing unit normal $\nu = \nabla w/|\nabla w|$ and μ is the surface measure of the level set $\partial\Omega_t = \{w = t\}$. By Sard theorem the set of critical values of w has zero Lebesgue measure, hence the function $\mathscr{F}$ is then well defined almost everywhere in $[0,+\infty)$.

Notice that, by simply expanding the square in $(\mathrm{H}/2 - |\nabla w|)^2 \geqslant 0$, we have

$$\mathscr{F}(t) = \int_{\partial\Omega_t} \mathrm{H}\,|\nabla w| - |\nabla w|^2 \,\mathrm{d}\mu \leqslant \int_{\partial\Omega_t} \mathrm{H}^2/4 \,\mathrm{d}\mu. \tag{2.4}$$

In particular, being $\partial\Omega_0 = \partial B_r(o)$ a regular level set of w, we have $\mathscr{F}(0) < 4\pi$, by equation (2.1).

The following lemma is in the spirit of similar results in [1, 3].

Lemma 2.1. *The function $\mathscr{F}$ admits a locally absolutely continuous, nonincreasing extension (still denoted by $\mathscr{F}$) to the whole $[0,+\infty)$. Moreover, at the regular values of w, there holds*

$$\mathscr{F}'(t) = -\int_{\partial\Omega_t} \left[\frac{|\nabla^\top |\nabla w||^2}{|\nabla w|^2} + \mathrm{Ric}(\nu,\nu) + |\mathring{\mathrm{h}}|^2 + \frac{1}{2}\,(\mathrm{H} - 2|\nabla w|)^2 \right] \mathrm{d}\mu \leqslant 0, \tag{2.5}$$

where $\nu = \nabla w/|\nabla w|$ and h *are the outward pointing unit normal and the second fundamental form of $\partial\Omega_t$, $\mathring{\mathrm{h}}$ the traceless part of* h *and $\nabla^\top$ denotes the tangential part of the gradient (with respect to $\partial\Omega_t$).*

Proof. At every regular value $t \in [0,+\infty)$ of w, it is straightforward to see that

$$\mathscr{F}(t) = -\int_{\partial\Omega_t} \left\langle \nabla|\nabla w|, \frac{\nabla w}{|\nabla w|} \right\rangle \mathrm{d}\mu, \qquad \text{hence} \qquad \mathscr{F}(t) - \mathscr{F}(s) = -\int_{\{s<w<t\}} \mathrm{div}\,(\nabla|\nabla w|)\,\mathrm{d}\mu,$$

(by the divergence theorem) for every pair of regular values $s < t$ of w in $[0,+\infty)$ such that the open set $\{s < w < t\}$ has no critical points.

The vector field $\nabla|\nabla w|$ is well defined and smooth outside the set of the critical points of w and by direct computation, we get

$$\mathrm{div}\,(\nabla|\nabla w|) = |\nabla w| \left[\frac{|\nabla^\top |\nabla w||^2}{|\nabla w|^2} + \mathrm{Ric}(\nu,\nu) + |\mathring{\mathrm{h}}|^2 + \frac{1}{2}\,(\mathrm{H} - 2|\nabla w|)^2 \right].$$

If the open set $\{s < w < t\}$ does not contain critical points of w, then the inequality $\mathscr{F}(s) - \mathscr{F}(t) \geqslant 0$ follows and equation (2.5) is immediate. If instead the open set $\{s < w < t\}$ contains some critical points, to obtain the same conclusion, one can use appropriate approximating vector fields $\eta(|\nabla w|)\nabla|\nabla w|$, smooth on all $M \setminus \Omega$ and with nonnegative divergence, as in [1, 3]. Following such argument, one also gets that $\mathscr{F} \in W^{1,1}_{\mathrm{loc}}(0,+\infty)$, with a weak derivative given almost everywhere by formula (2.5). □

Lemma 2.2. *There exists $\widetilde{t} \in [0,+\infty)$ such that for all $t \geqslant \widetilde{t}$, there holds $\mathscr{F}(t) \leqslant \mathrm{C}\,\mathrm{e}^{-2t}$, for a positive constant* C.

Proof. If Σ is a closed, connected surface in (M,g) with $\mathrm{Ric} \geqslant \varepsilon \mathrm{R} g$, we have

$$2\int_\Sigma \mathrm{Ric}(\nu,\nu)\,\mathrm{d}\mu \geqslant \varepsilon\left(16\pi - \int_\Sigma \mathrm{H}^2\,\mathrm{d}\mu\right) \qquad \text{if genus}(\Sigma) = 0, \tag{2.6}$$

$$2\int_\Sigma \mathrm{Ric}(\nu,\nu) + \big|\mathring{\mathrm{h}}\big|^2\,\mathrm{d}\mu \geqslant \int_\Sigma \mathrm{H}^2\,\mathrm{d}\mu \qquad \text{if genus}(\Sigma) \geqslant 1. \tag{2.7}$$

These two inequalities follow from the Gauss–Bonnet theorem and the Gauss–Codazzi equations, taking into account the pinching condition in the first one (see [14, Lemma 8]).
Suppose that $t \geqslant 0$ is a regular value of w, then the number of the connected components of $\partial\Omega_t$ is finite, by its compactness. If all of them have genus greater or equal to one, by inequality (2.5) and using estimate (2.7) for every single connected component, after adding we obtain

$$-2\mathscr{F}'(t) \geqslant \int_{\partial\Omega_t} 2\,\mathrm{Ric}(\nu,\nu) + 2|\mathring{\mathrm{h}}|^2\,\mathrm{d}\mu \geqslant \int_{\partial\Omega_t} \mathrm{H}^2\,\mathrm{d}\mu \geqslant 4\mathscr{F}(t),$$

where the last inequality is given by formula (2.4). If there exists at least one connected component with genus zero, letting $\Sigma_t^1 \neq \varnothing$ be the union of the $n \in \mathbb{N}$ connected components of genus zero and Σ_t^2 the union of the connected components of genus greater than one, by inequalities (2.5) and (2.6), we have

$$\begin{aligned}
-2\mathscr{F}'(t) &\geqslant \int_{\partial\Omega_t} 2\,\mathrm{Ric}(\nu,\nu) + (\mathrm{H} - 2|\nabla w|)^2\,\mathrm{d}\mu \\
&\geqslant \int_{\Sigma_t^1} 2\,\mathrm{Ric}(\nu,\nu) + \varepsilon\,(\mathrm{H} - 2|\nabla w|)^2\,\mathrm{d}\mu + \varepsilon\int_{\Sigma_t^2} (\mathrm{H} - 2|\nabla w|)^2\,\mathrm{d}\mu \\
&\geqslant \varepsilon\left(16n\pi - 4\int_{\Sigma_t^1} \mathrm{H}\,|\nabla w| - |\nabla w|^2\,\mathrm{d}\mu\right) - 4\varepsilon\int_{\Sigma_t^2} \mathrm{H}\,|\nabla w| - |\nabla w|^2\,\mathrm{d}\mu \\
&= \varepsilon\left(16n\pi - 4\int_{\partial\Omega_t} \mathrm{H}\,|\nabla w| - |\nabla w|^2\,\mathrm{d}\mu\right) \\
&\geqslant \varepsilon\,(16\pi - 4\mathscr{F}(t)),
\end{aligned}$$

where we used the fact that $\varepsilon \leqslant 1/3$ (this follows by tracing the Ricci–pinching condition).
Hence, we can conclude that for almost every $t \in [0,+\infty)$, there holds

$$\mathscr{F}'(t) \leqslant \max\{-2\mathscr{F}(t), \varepsilon\,(2\mathscr{F}(t) - 8\pi)\}.$$

The thesis then follows from this differential inequality, keeping into account that $\mathscr{F}$ is locally absolutely continuous, by Lemma 2.1. Indeed, by the monotonicity of $\mathscr{F}$, either $\mathscr{F}(t) \geqslant 8\pi\varepsilon/(2+2\varepsilon)$ for every $t \geqslant 0$, or there exists $\widetilde{t} \geqslant 0$ such that $\mathscr{F}(t) \leqslant 8\pi\varepsilon/(2+2\varepsilon)$ for every $t \geqslant \widetilde{t}$. In the first case, $\mathscr{F}'(t) \leqslant \varepsilon(2\mathscr{F}(t) - 8\pi)$, for every $t \geqslant 0$ and $\mathscr{F}(t) \leqslant \mathscr{F}(0) < 4\pi$. Hence, there must exist some $t \geqslant 0$ such that $\mathscr{F}(t) < 8\pi\varepsilon/(2+2\varepsilon)$, which is a contradiction.

In the second case, $\mathscr{F}'(t) \leqslant -2\mathscr{F}(t)$ for all $t \geqslant \widetilde{t}$, which implies $\mathscr{F}(t) \leqslant 4\pi\,\mathrm{e}^{-2(t-\widetilde{t})}$, hence the thesis. ☐

Now we introduce another function $\mathscr{G}$, defined at every regular value $t \in [0,+\infty)$ of w as

$$\mathscr{G}(t) = \int_{\partial\Omega_t} |\nabla w|^2 \,\mathrm{d}\mu.$$

Lemma 2.3. *For almost every $t \in [0,+\infty)$, there holds $0 \leqslant \mathscr{G}(t) \leqslant \mathscr{F}(t)$. In particular,*

$$\lim_{t\to+\infty} \mathscr{F}(t) = \lim_{t\to+\infty} \mathscr{G}(t) = 0.$$

Proof. As a consequence of [4, Theorem 3.1] the function $\mathscr{G}$ admits a nonincreasing C^1–extension on all $[0,+\infty)$ (indeed, $\mathscr{G}(t) = F_2^1(e^t)$, where F_p^β are the monotone quantities introduced in [4]). One can readily check that at every regular value $t \in [0,+\infty)$ of w (almost all, by Sard theorem), we have

$$0 \geqslant \mathscr{G}'(t) = \mathscr{G}(t) - \mathscr{F}(t),$$

which gives the thesis. ☐

We then need the notion of *normalized capacity* of a bounded closed set $D \subseteq M$:

$$\mathfrak{c}_2(\partial D) = \inf\left\{ \frac{1}{4\pi}\int_{M\setminus D} |\nabla\psi|^2 \,\mathrm{dVol} \;\middle|\; \psi \in C_c^\infty(M), \psi \geqslant \chi_D \right\}.$$

The relation of such capacity with the function w is given by the fact that (recalling that $w = -\log u$ with u the harmonic function solving problem (2.3))

$$\mathfrak{c}_2(\partial\Omega) = \frac{1}{4\pi}\int_{M\setminus\Omega} |\nabla u|^2 \,\mathrm{dVol} = \frac{1}{4\pi}\int_{\partial\Omega} |\nabla u| \,\mathrm{d}\mu = \frac{1}{4\pi}\int_{\partial\Omega} |\nabla w| \,\mathrm{d}\mu,$$

where we kept into account that $|\nabla w| = |\nabla u|$ on $\partial\Omega$, as $u = 1$ (see [4, Proposition 2.8] for a detailed justification of the first two equalities). Moreover, with the same argument, at every regular value $t \in [0,+\infty)$ of w, we have ([4, Proposition 2.9])

$$\mathfrak{c}_2(\partial\Omega_t) = \frac{1}{4\pi}\int_{\partial\Omega_t} |\nabla w| \,\mathrm{d}\mu = \frac{e^t}{4\pi}\int_{\partial\Omega_t} |\nabla u| \,\mathrm{d}\mu = \frac{e^t}{4\pi}\int_{\partial\Omega} |\nabla u| \,\mathrm{d}\mu = \mathrm{e}^t\,\mathfrak{c}_2(\partial\Omega), \tag{2.8}$$

where we used again the divergence theorem in the domain $\Omega_t \setminus \Omega$.

Proof of Theorem 1.5. We need the following "classical" estimates for a solution $u : M \setminus \Omega \to (0,1]$ of problem (2.3) (see for instance [1, 7, 16]): there exist a positive constant $\mathrm{C} = \mathrm{C}(M,\Omega)$ such that for all $x \in M \setminus \Omega$,

$$u(x) \leqslant \mathrm{C}\, d(x,o)^{1-\alpha}, \tag{2.9}$$

where α is the exponent in condition (1.1).
By equation (2.8) and Hölder inequality, at every regular value $t \in [0,+\infty)$ of w, we have

$$\mathrm{e}^{3t}\,\mathfrak{c}_2(\partial\Omega)^3 = \mathfrak{c}_2(\partial\Omega_t)^3 = \left(\frac{1}{4\pi}\int_{\partial\Omega_t} |\nabla w| \,\mathrm{d}\mu\right)^3 \leqslant \frac{1}{(4\pi)^3}\left(\int_{\partial\Omega_t} |\nabla w|^{-1} \,\mathrm{d}\mu\right)\left(\int_{\partial\Omega_t} |\nabla w|^2 \,\mathrm{d}\mu\right)^2$$

and from Lemmas 2.2 and 2.3, we know that there exists $\widetilde{t} \in [0,+\infty)$ such that for all $t \in [\widetilde{t},+\infty)$, there holds

$$\int_{\partial\Omega_t} |\nabla w|^2 \,\mathrm{d}\mu = \mathscr{G}(t) \leqslant \mathrm{C}\,\mathrm{e}^{-2t},$$

for a positive constant C. Thus, using the coarea formula, we obtain

$$\frac{d}{dt}\mathrm{Vol}(\{w \leqslant t\}) = \int_{\partial\Omega_t} |\nabla w|^{-1}\,\mathrm{d}\mu \geqslant \big[4\pi\,\mathfrak{c}_2(\partial\Omega)\big]^3\,\mathrm{e}^{3t}/\mathscr{G}^2(t) \geqslant \big[4\pi\,\mathfrak{c}_2(\partial\Omega)\big]^3\,\mathrm{e}^{7t}/\mathrm{C}^2.$$

for almost every $t \in [0,+\infty)$. Let $R_t = \sup\big\{d(q,o) : q \in \{w \leqslant t\} = \Omega_t\big\}$ for any $t \in [0,+\infty)$ and $t_n \to +\infty$ be an increasing sequence of regular values of w (whose existence is again guaranteed by Sard theorem). Integrating the above inequality on $[0,t_n]$ and using the superquadratic volume growth assumption, we get

$$\frac{1}{7\mathrm{C}^2}\big[4\pi\,\mathfrak{c}_2(\partial\Omega)\big]^3\big(\mathrm{e}^{7t_n}-1\big) \leqslant \mathrm{Vol}(\{w \leqslant t_n\}) \leqslant \mathrm{Vol}(B_{R_{t_n}}(o)) \leqslant \mathrm{C}_{\mathrm{vol}}\,R_{t_n}^{1+\alpha}. \tag{2.10}$$

Being $w = -\log u$, by estimate (2.9), we have $w(x) \geqslant -\log(\mathrm{C}d(x,o)^{1-\alpha})$, then if $d(q,o) = R_{t_n}$, it must be $q \in \partial\Omega_{t_n}$, that is, $w(q) = t_n$ and we have

$$t_n = w(q) \geqslant -\log(\mathrm{C}d(q,o)^{1-\alpha}) = -\log(\mathrm{C}R_{t_n}^{1-\alpha}),$$

hence, $R_{t_n}^{\alpha-1} \leqslant \mathrm{C}\,\mathrm{e}^{t_n}$, which implies $R_{t_n}^{\alpha+1} \leqslant \mathrm{C}\,\mathrm{e}^{\frac{\alpha+1}{\alpha-1}t_n}$ for a positive constant $\mathrm{C} = \mathrm{C}(M,\Omega)$. Then, by inequality (2.10), we conclude that

$$\mathrm{e}^{7t_n} - 1 \leqslant \mathrm{C}R_{t_n}^{\alpha+1} \leqslant \mathrm{C}e^{\frac{\alpha+1}{\alpha-1}t_n},$$

which is clearly a contradiction if $\alpha > 4/3$, as t_n can be chosen arbitrarily large. □

Replacing the "starting subset" $\overline{B}_r(o)$ with a different regular subset Ω with a compact boundary, such that

$$\int_{\partial\Omega} \mathrm{H}^2\,\mathrm{d}\mu < 16\pi,$$

and repeating the above argument, one obtains the same conclusion. It is then straightforward to obtain also the following result when M has a boundary.

Theorem 2.4. *There not exist a complete, connected, non–compact, Ricci–pinched Riemannian 3–manifold (M,g) that has superquadratic volume growth with $\alpha > 4/3$ in (1.1) and a compact smooth boundary ∂M satisfying*

$$\int_{\partial M} \mathrm{H}^2\,\mathrm{d}\mu < 16\pi. \tag{2.11}$$

Remark 2.5. Clearly, as before, the case AVR > 0 correspond to the case $\alpha = 2$ in assumption (1.1), hence, in particular, there not exist a complete, connected, non–compact, Ricci–pinched Riemannian 3–manifold (M,g) with AVR > 0 and a compact smooth boundary ∂M satisfying condition (2.11).

Remark 2.6. If one is interested in proving only Theorem 1.5, it is known that M must be diffeomorphic to $\mathbb{R}^3$ (see [20]), then, thanks to the strong maximum principle, one can show that the regular level sets of w are connected (see [2, 3] for more detail). This observation simplifies a little bit the proof of Lemma 2.2 in such a case.

Acknowledgements. *All the authors are members of the INDAM–GNAMPA. Luca Benatti and Alessandra Pluda are partially supported by the BIHO Project "NEWS – NEtWorks and Surfaces evolving by curvature" and by the MUR Excellence Department Project awarded to the Department of Mathematics of the University of Pisa. Carlo Mantegazza is partially supported by the PRIN Project 2022E9CF89 "GEPSO – Geometric Evolution Problems and Shape Optimization". Luca Benatti is partially supported by PRIN Project 2022PJ9EFL "Geometric Measure Theory: Structure of*

Singular Measures, Regularity Theory and Applications in the Calculus of Variations". Alessandra Pluda is partially supported by the PRIN Project 2022R537CS "NO^3 *– Nodal Optimization, NOnlinear elliptic equations, NOnlocal geometric problems, with a focus on regularity" and PRA2022 Project "GEODOM".*

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(Luca Benatti) DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI PISA, ITALY
Email address, L. Benatti: luca.benatti@unipi.it

(Carlo Mantegazza) DIPARTIMENTO DI MATEMATICA E APPLICAZIONI "RENATO CACCIOPPOLI", UNIVERSITÀ DI NAPOLI FEDERICO II & SCUOLA SUPERIORE MERIDIONALE, NAPOLI, ITALY
Email address, C. Mantegazza: carlo.mantegazza@unina.it

(Francesca Oronzio) INSTITUTIONEN FÖR MATEMATIK, KUNGLIGA TEKNISKA HÖGSKOLAN, STOCKHOLM, SWEDEN
Email address, F. Oronzio: oronzio@kth.se

(Alessandra Pluda) DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI PISA, ITALY
Email address, A. Pluda: alessandra.pluda@unipi.it