THERMO-ELASTODYNAMICS OF NONLINEARLY VISCOUS SOLIDS

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Abstract. In this paper, we study the thermo-elastodynamics of nonlinearly viscous solids in the Kelvin-Voigt rheology where both the elastic and the viscous stress tensors comply with the frame-indifference principle. The system features a force balance including inertia in the frame of nonsimple materials and a heat-transfer equation which is governed by the Fourier law in the deformed configuration. Combining a staggered minimizing movement scheme for quasi-static thermoviscoelasticity [35, 2] with a variational approach to hyperbolic PDEs developed in [5], our main result consists in establishing the existence of weak solutions in the dynamic case. This is first achieved by including an additional higher-order regularization for the dissipation. Afterwards, this regularization can be removed by passing to a weaker formulation of the heat-transfer equation which complies with a total energy balance. The latter description hinges on regularity theory for the fourth order p-Laplacian which induces regularity estimates of the deformation beyond the standard estimates available from energy bounds. Besides being crucial for the proof, these extra regularity properties might be of independent interest and seem to be new in the setting of nonlinear viscoelasticity, also in the static or quasi-static case.

1. Introduction

Understanding the coupling between mechanical and thermal phenomena in viscoelastic solids has been a mainstay in the mathematical and physical literature over the last decades. Even at small strains, the problem is notoriously difficult since the heat-transfer equation has no obvious variational structure due to the low regularity of data. In fact, after the pioneering work of Dafermos [15, 16, 17, 18] in one space dimension, new fundamental ideas related to the existence theory for parabolic equations with measure-valued data developed in [9, 10] were needed to obtain results in three dimensions [8, 11, 44]. At large strains, the problem is still considered to be extremely difficult even in the isothermal case, due to the highly nonlinear nature of models respecting material frame indifference [1]. For some results without temperature coupling, we refer to [37, 38] for existence of global-in-time weak solutions for initial data sufficiently close to a smooth equilibrium and to a local-in-time existence result [33]. By now, more general settings can only be treated by passing to weaker solution concepts such as measure-valued solutions [19, 20, 29]. Resorting to energy densities with higher-order spatial gradients, i.e., to so-called nonsimple materials [46, 47], existence of weak solutions has been shown in [24, 35] for the quasi-static case (without inertia) and in [5] for the dynamic case (with inertia). The variational approach adopted in these papers is quite flexible and has led to various extensions in the last years, ranging from models for dimension reduction [25, 26, 27], to problems with self-contact [12, 13, 32], approximability [14], diffusion [48], or homogenization [28], to applications for fluid-structure interactions [5, 6, 7, 31].

Nonlinear frame-indifferent models in thermoviscoelasticity were analyzed only very recently [2, 3, 4, 35], again adopting the concept of nonsimple materials, yet neglecting inertial effects. The goal of this work is to extend this analysis to the setting of thermo-elastodynamics including inertia. While our work follows the Lagrangian perspective, let us mention that in the last years several works appeared in the isothermal and nonisothermal framework which employ the alternative Eulerian approach instead, see [40, 41, 42, 43, 45]. In this context, higher-order gradients are involved rather in the dissipative than in the conservative part, which sometimes is referred to as multipolar viscous solids. Besides adopting the Lagrangian framework, a main motivation of our work is to establish an existence result *without* higher-order regularization of the dissipation.

We now introduce the large-strain model in more detail. In the Kelvin-Voigt rheology, the force balance of a nonlinearly viscoelastic material in a setting of nonsimple materials is given by the system

$$f = \rho \partial_{tt}^2 y - \operatorname{div}\big(\partial_F W(\nabla y, \theta) + \partial_{\dot F} R(\nabla y, \partial_t \nabla y, \theta) - \nabla (DH(\Delta y))\big) \qquad \text{in } [0,T] \times \Omega. \tag{1.1}$$

Here, $[0,T]$ is a process time interval with $T > 0$, $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ denotes the reference configuration, $y\colon [0,T] \times \Omega \to \mathbb{R}^d$ is the time-dependent deformation, $\theta\colon [0,T] \times \Omega \to [0,\infty)$ denotes the temperature, and $f\colon [0,T]\times\Omega \to \mathbb{R}^d$ is a volume density of external forces acting on Ω. The free energy density $W\colon \mathbb{R}^{d\times d}\times[0,\infty) \to \mathbb{R}\cup\{+\infty\}$ depends on the deformation gradient ∇y (with placeholder $F \in \mathbb{R}^{d\times d}$) and respects frame indifference under rotations as well as positivity of the determinant of ∇y. Additionally, adopting the framework of nonsimple

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materials, the stored energy features a contribution depending on the Laplacian Δy given in terms of a convex potential $H\colon \mathbb{R}^d \to \mathbb{R}$ with p-growth for some $p > d$. Finally, $R\colon \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d} \times [0,\infty) \to \mathbb{R}$ denotes a (pseudo)potential of dissipative forces ($\dot F$ is the time derivative of F). As observed by Antman [1], R must comply with a time-continuous frame indifference principle meaning that R can be written in terms of the right Cauchy-Green tensor $C := F^TF$ and its time derivative $\dot C := \dot F^T F + F^T \dot F$, see (D.1) below for details.

The system (1.1) is coupled to a heat-transfer equation of the form

$$c_V(\nabla y,\theta)\,\partial_t\theta = \mathrm{div}(\mathcal{K}(\nabla y,\theta)\nabla\theta) + \partial_{\dot F}R(\nabla y,\nabla\partial_t y,\theta) : \nabla\partial_t y + \theta\partial^2_{F\theta}W(\nabla y,\theta) : \nabla\partial_t y \qquad \text{in } [0,T]\times\Omega, \quad (1.2)$$

where $c_V(F,\theta) = -\theta\partial^2_{\theta\theta}W(F,\theta)$ is the heat capacity, $\mathcal{K}$ denotes the matrix of the heat-conductivity coefficients, and the last term plays the role of an adiabatic heat source. This corresponds to a heat transfer modeled by the Fourier law in the deformed configuration which is pulled back to the reference configurations and thus includes dependence on the deformation gradient. The coupled system (1.1)–(1.2) is complemented with suitable initial and boundary conditions, see (2.18)–(2.19) below.

The goal of this article is to establish an existence result for weak solutions to the nonlinear thermo-elastodynamic system (1.1)–(1.2), see Theorem 2.5. Our proof strategy heavily hinges on two recent advances in the variational analysis of nonlinearly elastic solids: we combine the staggered minimizing movement scheme for proving existence results in quasi-static thermoviscoelasticity [35, 2] with a variational approach to hyperbolic PDEs [5] which allows to include inertia.

In the following, we describe the main ingredients for the proof in more detail. The fundamental idea in [5] consists in replacing the acceleration term $\rho\partial^2_{tt}y$ by a discrete difference $\rho\frac{\partial_t y - \partial_t y(\cdot-h)}{h}$ which allows to turn the hyperbolic problem (1.1) into a parabolic one. The latter time-delayed problem can be approximated by a time-discretized scheme as in [35, 2] with time step $\tau > 0$. Then, given solutions to the discretized problems with two different length scales τ and h (called the velocity and the acceleration time scale, respectively), one first passes to $\tau \to 0$ and afterwards to $h \to 0$ to obtain a weak solution for (1.1). As in [35], a generalized version of Korn's inequality [36] relying on the second-order regularization is essential in order to tame the nonlinearity arising from the frame indifference of the dissipation term. Concerning the coupling to the heat-transfer equation, the approach in [35, 2] crucially relies on the theory of parabolic equations with measure-valued right-hand side [10]. A delicate part of the proof lies in the passage to the limit $\tau \to 0$ in the dissipation term $\partial_{\dot F}R(\nabla y,\nabla\partial_t y,\theta) : \nabla\partial_t y$, see (1.2). For this, strong convergence of the time-discrete approximations $\nabla\partial_t y_\tau$ is indispensable which is guaranteed by exploiting the convergence of a mechanical energy balance, cf. [35, Proposition 5.1] for details.

Although all techniques mentioned above are crucial ingredients in our work, it turns out that they do not suffice in the setting with heat coupling *and* inertia. The main reason lies in missing regularity which impedes the derivation of a mechanical energy balance. To explain this issue, let as consider the simplified problem

$$\rho\partial^2_{tt}y - \Delta\partial_t y + \Delta(|\Delta y|^{p-2}\Delta y) = f, \tag{1.3}$$

which arises from (1.1) by neglecting the first Piola-Kirchhoff stress tensor $\partial_F W$, and considering a linear variant of $\partial_{\dot F}R$ as well as a p-homogeneous variant of H. In the quasi-static case $\rho = 0$ or in the time-delayed problem where $\rho\partial^2_{tt}y$ is replaced by the discrete difference $\rho\frac{\partial_t y-\partial_t y(\cdot-h)}{h}$, a test of the time-discretized problem with $\partial_t y$ and an integration by parts (neglecting boundary terms) leads to the natural energy bounds $\Delta y \in L^\infty([0,T];L^p(\Omega))$ and $\nabla\partial_t y \in L^2([0,T];L^2(\Omega))$. Then, in the case $\rho = 0$, a mechanical energy balance is achieved by testing (1.3) with $\partial_t y$, cf. [2, Equation (4.11)]. In this context, the term $\Delta(|\Delta y|^{p-2}\Delta y)$ might in principle not have the correct duality coupling to apply the chain rule. However, since the other two terms f and $\Delta\partial_t y$ are in duality, also the delicate fourth-order term can be handled by comparison. In contrast, for $\rho > 0$, the two terms $\rho\partial^2_{tt}y$ and $\Delta(|\Delta y|^{p-2}\Delta y)$ are not in duality and the chain rule (and thus the mechanical energy balance) may fail.

This fundamental issue has already been observed in [35, Remark 6.6]. A possible workaround lies in adding an additional regularization for the dissipation, see (2.17a), which in the simplified setting reads as

$$\rho\partial^2_{tt}y - \Delta\partial_t y + \Delta(|\Delta y|^{p-2}\Delta y) - \varepsilon\partial_t\Delta^3 y = f. \tag{1.4}$$

With the test $\partial_t y$, this induces the energy bound $\nabla\Delta\partial_t y \in L^2([0,T];L^2(\Omega))$ which is strong enough to recover the chain rule. In this case, a mechanical energy balance can be guaranteed and we can follow the strategy devised in [35, 2] and [5], see Theorem 2.2. (Note that we choose a simple higher-order regularization which does not comply with the principle of dynamical frame indifference. A frame-indifferent regularization would necessarily be very nonlinear.) This regularized setting is related to [43] where existence results under higher-order regularizations of the dissipation have been derived in a Eulerian settting. Yet, a main motivation of our work is to derive an existence result *without* such regularization.

Our strategy relies on passing to a weaker formulation of the heat-transfer equation (1.2) which is inspired by the derivation of a total energy balance (see [35, Equation (2.21)] or (2.28) below) and does not feature the delicate dissipation term $\partial_{\dot F} R(\nabla y, \nabla \partial_t y, \theta) : \nabla \partial_t y$, see (2.27) for details. On a formal level, the idea is to test (1.1) with $\partial_t y$ which allows to replace the dissipation term in (1.2). As discussed above, however, this test is actually not allowed in (1.3). Therefore, we perform this replacement first on the regularized level (1.4), and afterwards we pass to the limit $\varepsilon \to 0$. This procedure leads to a modified weak formulation of the system which does not guarantee a mechanical energy balance but has the essential feature that the total energy is in equilibrium with the work by external body forces and heat sources, see (2.28). Moreover, the solution concept introduced here becomes a standard weak solution or a strong solution once the necessary regularity properties for the deformation and the temperature are available.

Although curing the issue with the dissipation, the passage to the weaker modified setting causes a new problem: the resulting weak formulation features a third-order term $\nabla(DH(\Delta y))$ which is not compatible with the available energy bound $\Delta y \in L^\infty([0,T]; L^p(\Omega))$, see below (1.3). Therefore, it is necessary to improve the regularity of the deformation. Loosely speaking, this is achieved by testing (1.3) with $-\Delta y$ which after integration by parts (omitting any boundary terms) leads to an *elliptic estimate.* In fact, using $\nabla \partial_t y \in L^2([0,T]; L^2(\Omega))$, the first term $|\int \partial^2_{tt} y \Delta y \,\mathrm{d}t\,\mathrm{d}x| \le C\|\nabla \partial_t y\|^2_{L^2([0,T];L^2(\Omega))} \le C$ is controlled. Assuming $p = 2$ for simplicity here, the second and third term can be controlled as

$$\Big|\int_0^T \int_\Omega \Delta \partial_t y \cdot \Delta y \,\mathrm{d}t\,\mathrm{d}x\Big| \le C\|\nabla \partial_t y\|_{\in L^2([0,T];L^2(\Omega))} \|\nabla \Delta y\|_{L^2([0,T];L^2(\Omega))} \le C\|\nabla \Delta y\|_{L^2([0,T];L^2(\Omega))},$$

$$\Big|\int_0^T \int_\Omega \Delta(\Delta y) \cdot \Delta y \,\mathrm{d}t\,\mathrm{d}x\Big| \ge \tfrac{1}{C}\|\nabla \Delta y\|^2_{L^2([0,T];L^2(\Omega))}.$$

This allows to obtain the control $\nabla \Delta y \in L^2([0,T]; L^2(\Omega))$ which suffices to give sense to the term $\nabla(DH(\Delta y))$ in the weak formulation. Again, on a rigorous level, this test is performed for the regularized problem (1.4) with ε-independent bounds, and then the regularity for y is obtained in the limit of vanishing regularization $\varepsilon \to 0$, see Proposition 3.11 and Lemma 3.12 for details. More precisely, for given $p > d$, the additional regularity reads as $(1+|\Delta y|)^{\frac{p-2}{2}} |\nabla \Delta y|^2 \in L^2([0,T]; L^2(\Omega))$, see Theorem 2.5. Even in the nonlinear case of $p \neq 2$, the regularity estimates introduced here rely on the theory for the Laplace operator only and as such are *independent of nonlinear regularity techniques.* This seems to be a special feature of the fourth order p-Laplacian, which was observed by the authors very much to their surprise. Up to their knowledge, it has not been used before.

Besides being crucial for our proof, the result might be of independent interest and improves the known regularity properties also for results in the quasi-static case ($\rho = 0$) [35, 2] or in the isothermal case [24]. It seems that even in the static case of elastic minimizers this extra regularity property has not been shown previously. Let us mention, however, that compared to [2, 24, 35] the regularity issues force us to impose Dirichlet conditions on the *entire* boundary $\partial\Omega$.

The plan of the paper is as follows. Section 2 introduces the nonlinear model and states our main results. Then, the results are proved in Sections 3–5. We start by considering the ε-regularized problem and introduce a discretized solution with time stepping τ of the parabolic approximation with time delay $h > 0$. This introduces a three layer approximation, and we successively pass to the limits in the layers, namely first in τ (Section 3), then in h (Section 4), and eventually in the regularization ε (Section 5). It is important to mention that all essential a priori bounds are already established on the τ-level in Section 3 and transfer over to the limits $\tau \to 0$, $h \to 0$, and $\varepsilon \to 0$. Moreover, some additional higher-order bounds based on elliptic regularity theory are provided (see Lemma 3.12) that blow up in the limit $\varepsilon \to 0$. Still, they are crucial to perform the final limiting passage in the heat-transfer equation to control some ε-dependent terms resulting from the regularization, see Proposition 5.4 for details.

2. The model and main results

2.1. **Notation.** Denoting by $d \in \{2,3\}$ the dimension, we indicate by $\Omega \subset \mathbb{R}^d$ an open bounded set with C^5-boundary and fix $p \in (d, 2^*)$, where $2^* = \infty$ for $d = 2$ and $2^* = 6$ for $d = 3$. In what follows, we use standard notation for Lebesgue, Sobolev, and Bochner spaces. By $\mathbb{1}_J$ we denote the indicator function of a set $J \subset \mathbb{R}$ or $J \subset \Omega$. The lower index $_+$ means nonnegative elements, i.e., $L^2_+(\Omega)$ denotes the convex cone of nonnegative functions belonging to $L^2(\Omega)$ and a similar definition is used for $H^1_+(\Omega)$. Mean integrals are denoted by $⨍$. We also set $\mathbb{R}_+ := [0, +\infty)$. Let $a \wedge b := \min\{a, b\}$ for $a, b \in \mathbb{R}$. Moreover, we let $\mathbf{Id} \in \mathbb{R}^{d\times d}$ be the identity matrix, and $\mathbf{id}(x) := x$ stands for the identity map on $\mathbb{R}^d$. We define the subsets $SO(d) := \{A \in \mathbb{R}^{d\times d} \colon A^T A = \mathbf{Id}, \det A = 1\}$, $GL^+(d) := \{F \in \mathbb{R}^{d\times d} \colon \det(F) > 0\}$, and $\mathbb{R}^{d\times d}_{\mathrm{sym}} := \{A \in \mathbb{R}^{d\times d} \colon A^T = A\}$. Furthermore, for a matrix $F \in \mathbb{R}^{d\times d}$ we write $F^{-T} := (F^{-1})^T = (F^T)^{-1}$, and given a tensor G (of arbitrary dimension and order), $|G|$ will denote its Frobenius norm. We write the scalar product between vectors and matrices as $\cdot$ and $:$, respectively. The tensor product of two vectors $v_1, v_2 \in \mathbb{R}^d$ is denoted by $v_1 \otimes v_2 \in \mathbb{R}^{d\times d}$. As usual, in the proofs generic constants C are strictly positive and may vary from line to line. If not stated otherwise, all constants only depend on d, p, Ω, and the potentials and data defined in Subsection 2.2 below. We frequently use a scaled version of Young's inequality with constant $\lambda \in (0,1)$ by which we mean $ab \le \lambda a^q + C_\lambda b^{q'}$ for $a, b \ge 0$, exponents q, $q' > 1$ with $1/q + 1/q' = 1$, and a suitable constant $C_\lambda > 0$.

For $\Omega \subset \mathbb{R}^d$ and p as above, we introduce the set of *admissible deformations* by

$$\mathcal{Y}_{\mathbf{id}} := \big\{ y \in W^{2,p}(\Omega;\mathbb{R}^d) \colon y = \mathbf{id} \text{ on } \partial\Omega,\ \det(\nabla y) > 0 \text{ in } \Omega \big\}, \tag{2.1}$$

and we say that the *absolute temperature* θ is admissible if $\theta \in L^1_+(\Omega)$.

2.2. **Energies and their respective potentials.** The variational setting described in the sequel mostly coincides with the one from [2], up to a more special choice of the strain-gradient energy. In the following, let $C_0 \ge 1$ be some fixed positive constant.

Mechanical energy and coupling energy: The *elastic energy* $\mathcal{W}^{\mathrm{el}} \colon \mathcal{Y}_{\mathbf{id}} \to \mathbb{R}_+$ is given by

$$\mathcal{W}^{\mathrm{el}}(y) := \int_\Omega W^{\mathrm{el}}(\nabla y)\,\mathrm{d}x, \tag{2.2}$$

where $W^{\mathrm{el}} \colon GL^+(d) \to \mathbb{R}_+$ is a frame indifferent elastic energy potential with the usual assumptions in nonlinear elasticity. More precisely, we require:

(W.1) W^{el} is C^2;
(W.2) Frame indifference: $W^{\mathrm{el}}(QF) = W^{\mathrm{el}}(F)$ for all $F \in GL^+(d)$ and $Q \in SO(d)$;
(W.3) Lower bound: $W^{\mathrm{el}}(F) \ge \frac{1}{C_0}\big(|F|^2 + \det(F)^{-q}\big) - C_0$ for all $F \in GL^+(d)$, where $q \ge \frac{pd}{p-d}$.

Adopting the concept of 2nd-grade nonsimple materials, see [46, 47], we also consider a *strain-gradient energy term* $\mathcal{H} \colon \mathcal{Y}_{\mathbf{id}} \to \mathbb{R}_+$, defined as

$$\mathcal{H}(y) := \int_\Omega H(\Delta y)\,\mathrm{d}x. \tag{2.3}$$

Here, $H \colon \mathbb{R}^d \to \mathbb{R}_+$ is of the form

$$H(v) = \mathfrak{h}(|v|) \tag{2.4}$$

for $v \in \mathbb{R}^d$, where $\mathfrak{h} \colon \mathbb{R}_+ \to \mathbb{R}_+$ is defined as

$$\mathfrak{h}(s) := \int_0^s \max\{2\sigma, p\sigma^{p-1}\}\,\mathrm{d}\sigma. \tag{2.5}$$

The definition of $\mathfrak{h}$ ensures that H is uniformly convex and has p-growth. More precisely, we have

(H.1) H is uniformly convex and C^1;
(H.2) Frame indifference: $H(Q\Delta y) = H(\Delta y)$ in Ω for all $y \in \mathcal{Y}_{\mathbf{id}}$ and $Q \in SO(d)$;
(H.3) $|v|^p \le H(v) \le C_0|v|^p$ and $|DH(v)| \le C_0|v|^{p-1}$ for all $v \in \mathbb{R}^d$,

where $DH(v) := (\partial_{v_i} H(v))_{i=1}^d = \max\{2, p|v|^{p-2}\}v$ is the gradient of H with respect to v. The *mechanical energy* $\mathcal{M} \colon \mathcal{Y}_{\mathbf{id}} \to \mathbb{R}_+$ is then defined as

$$\mathcal{M}(y) := \mathcal{W}^{\mathrm{el}}(y) + \mathcal{H}(y). \tag{2.6}$$

Besides the mechanical energy, we introduce a *coupling energy* $\mathcal{W}^{\mathrm{cpl}} \colon \mathcal{Y}_{\mathbf{id}} \times L^1_+(\Omega) \to \mathbb{R}$ given by

$$\mathcal{W}^{\mathrm{cpl}}(y,\theta) := \int_\Omega W^{\mathrm{cpl}}(\nabla y, \theta)\,\mathrm{d}x, \tag{2.7}$$

where $W^{\mathrm{cpl}} \colon GL^+(d) \times \mathbb{R}_+ \to \mathbb{R}$ describes mutual interactions of mechanical and thermal effects, and satisfies

(C.1) W^{cpl} is continuous, and C^2 in $GL^+(d) \times (0,\infty)$;
(C.2) $W^{\mathrm{cpl}}(QF,\theta) = W^{\mathrm{cpl}}(F,\theta)$ for all $F \in GL^+(d)$, $\theta \ge 0$, and $Q \in SO(d)$;
(C.3) $W^{\mathrm{cpl}}(F, 0) = 0$ for all $F \in GL^+(d)$;
(C.4) $|W^{\mathrm{cpl}}(F,\theta) - W^{\mathrm{cpl}}(\tilde F,\theta)| \le C_0(1 + |F| + |\tilde F|)|F - \tilde F|$ for all F, $\tilde F \in GL^+(d)$, and $\theta \ge 0$;
(C.5) For all $F \in GL^+(d)$ and $\theta > 0$ it holds that

$$|\partial^2_{FF} W^{\mathrm{cpl}}(F,\theta)| \le C_0, \qquad |\partial^2_{F\theta} W^{\mathrm{cpl}}(F,\theta)| \le \frac{C_0(1+|F|)}{\max\{\theta, 1\}}, \qquad \frac{1}{C_0} \le -\theta\partial^2_{\theta\theta} W^{\mathrm{cpl}}(F,\theta) \le C_0.$$

Notice that, by (C.3) and the second bound in (C.5), $\partial_F W^{\rm cpl}$ can be continuously extended to zero temperatures with $\partial_F W^{\rm cpl}(F,0)=0$. For $F\in GL^+(d)$ and $\theta\ge 0$, we define the *total free energy potential*

$$W(F,\theta) := W^{\rm el}(F) + W^{\rm cpl}(F,\theta). \tag{2.8}$$

Dissipation potential: The *dissipation functional* $\mathcal{R}\colon \mathcal{Y}_{\mathbf{id}} \times H^1(\Omega;\mathbb{R}^d)\times L^1_+(\Omega)\to\mathbb{R}_+$ is defined as

$$\mathcal{R}(y,\tilde y,\theta) := \int_\Omega R(\nabla y,\nabla\tilde y,\theta)\,\mathrm{d}x, \tag{2.9}$$

where $R\colon \mathbb{R}^{d\times d}\times\mathbb{R}^{d\times d}\times\mathbb{R}_+\to\mathbb{R}_+$ is the *potential of dissipative forces* satisfying

(D.1) $R(F,\dot F,\theta) := \frac12 D(C,\theta)[\dot C,\dot C] := \frac12 \dot C : D(C,\theta)\dot C$, where $C := F^TF$, $\dot C := \dot F^T F + F^T\dot F$, and $D\in C(\mathbb{R}^{d\times d}_{\rm sym}\times\mathbb{R}_+;\mathbb{R}^{d\times d\times d\times d})$ with $D_{ijkl}=D_{jikl}=D_{klij}$ for $1\le i,j,k,l\le d$;

(D.2) $\frac{1}{C_0}|\dot C|^2 \le \dot C : D(C,\theta)\dot C \le C_0|\dot C|^2$ for all $C,\dot C\in\mathbb{R}^{d\times d}_{\rm sym}$ and $\theta\ge 0$.

Notice that Assumption (D.1) implies that the viscous stress $\partial_{\dot F}R(F,\dot F,\theta)$ is linear in the time derivative $\dot C$ as well as (see e.g. [2, (2.8)])

$$\partial_{\dot F}R(F,\dot F,\theta) = 2F(D(C,\theta)\dot C). \tag{2.10}$$

We also define the associated *dissipation rate* $\xi\colon \mathbb{R}^{d\times d}\times\mathbb{R}^{d\times d}\times\mathbb{R}_+\to\mathbb{R}_+$ as

$$\xi(F,\dot F,\theta) := \partial_{\dot F}R(F,\dot F,\theta) : \dot F = 2R(F,\dot F,\theta), \tag{2.11}$$

where the second identity follows from (2.10) and Assumption (D.1), see also [2, (2.9)].

Below, for technical reasons explained in (1.4), we will also consider a *regularized version of the dissipation* $\mathcal{R}_\varepsilon\colon \mathcal{Y}_{\mathbf{id}}\times H^3(\Omega;\mathbb{R}^d)\times L^1_+(\Omega)\to\mathbb{R}_+$, defined as

$$\mathcal{R}_\varepsilon(y,\tilde y,\theta) := \int_\Omega R(\nabla y,\nabla\tilde y,\theta)\,\mathrm{d}x + \frac{\varepsilon}{2}\int_\Omega |\nabla\Delta\tilde y|^2\,\mathrm{d}x \tag{2.12}$$

for a small regularization parameter $\varepsilon>0$.

Heat conductivity: The map $\mathbb{K}\colon\mathbb{R}_+\to\mathbb{R}^{d\times d}_{\rm sym}$ denotes the *heat conductivity tensor* of the material in the deformed configuration. We require that $\mathbb{K}$ is continuous, symmetric, uniformly positive definite, and bounded. More precisely, for all $\theta\ge0$ it holds that

$$\frac{1}{C_0} \le \mathbb{K}(\theta) \le C_0, \tag{2.13}$$

where the inequalities are meant in the eigenvalue sense. We define the pull-back $\mathcal{K}\colon GL^+(d)\times\mathbb{R}_+\to\mathbb{R}^{d\times d}_{\rm sym}$ of $\mathbb{K}$ into the reference configuration by (see [35, (2.24)])

$$\mathcal{K}(F,\theta) := \det(F)F^{-1}\mathbb{K}(\theta)F^{-T}.$$

Thermal energy and total internal energy: Following [2, 4, 35], the *(thermal part of the) internal energy* $W^{\rm in}\colon GL^+(d)\times(0,\infty)\to\mathbb{R}$ is defined as

$$W^{\rm in}(F,\theta) := W^{\rm cpl}(F,\theta) - \theta\partial_\theta W^{\rm cpl}(F,\theta). \tag{2.14}$$

Using (C.3) and the third bound in (C.5), we see that $W^{\rm in}$ can be continuously extended to zero temperatures by setting $W^{\rm in}(F,0)=0$ for all $F\in GL^+(d)$. Furthermore, by the third bound in (C.5) we have that

$$\partial_\theta W^{\rm in}(F,\theta) = -\theta\partial^2_{\theta\theta}W^{\rm cpl}(F,\theta)\in[C_0^{-1},C_0] \qquad \text{for all } F\in GL^+(d) \text{ and } \theta>0.$$

Along with (C.3) this shows that the internal energy is controlled by the temperature in the sense that

$$\frac{1}{C_0}\theta \le W^{\rm in}(F,\theta)\le C_0\theta. \tag{2.15}$$

Finally, we define *total internal energy functional* $\mathcal{E}\colon \mathcal{Y}_{\mathbf{id}}\times L^1_+(\Omega)\to\mathbb{R}_+$ by

$$\mathcal{E}(y,\theta) := \mathcal{M}(y) + \mathcal{W}^{\rm in}(y,\theta) \qquad \text{with } \mathcal{W}^{\rm in}(y,\theta) := \int_\Omega W^{\rm in}(\nabla y,\theta)\,\mathrm{d}x. \tag{2.16}$$

We remark that the above assumptions on the potentials coincide with the ones in [2, Section 2.1], up to the fact that, differently to [2, (2.4)], we only allow the potential of the strain-gradient energy to depend on the norm of the diagonal Δy of $\nabla^2 y$. Moreover, for $d=3$, the range of $p\in(3,6)$ is restricted as we need the Sobolev embedding $H^3(\Omega;\mathbb{R}^d)\subset\subset W^{2,p}(\Omega;\mathbb{R}^d)$. Eventually, in contrast to [2], in the definition of admissible deformations, see (2.1), we need to impose Dirichlet conditions on the *entire* boundary $\partial\Omega$ as this allows us to apply elliptic regularity results. We refer to [35, Examples 2.4 and 2.5] for a class of potentials satisfying all assumptions above.

2.3. **Equations of nonlinear thermoviscoelasticity with inertia: Existence of weak solutions.** Let $I := [0,T]$ where $T > 0$ denotes a time horizon, let $\rho > 0$ be a constant *mass density* in the reference configuration, let $\kappa \geq 0$ be a constant *heat-transfer coefficient*, let $f \in W^{1,1}(I; L^2(\Omega;\mathbb{R}^d))$ be a time-dependent *dead force*, and let $\theta_\flat \in W^{1,1}(I; L^2_+(\partial\Omega))$ be an *external temperature*. Moreover, let $\varepsilon \geq 0$ be a regularization parameter, where $\varepsilon = 0$ corresponds to the setting without regularization. In the strong form, we study the system

$$f = \rho\partial^2_{tt}y - \operatorname{div}\big(\partial_F W(\nabla y,\theta) + \partial_{\dot F}R(\nabla y,\partial_t\nabla y,\theta) - \nabla(DH(\Delta y)) + \varepsilon\partial_t\nabla\Delta^2 y\big)\,, \tag{2.17a}$$

$$-\theta\partial^2_{\theta\theta}W^{\mathrm{cpl}}(\nabla y,\theta)\partial_t\theta = \operatorname{div}\big(\mathcal{K}(\nabla y,\theta)\nabla\theta\big) + \xi(\nabla y,\partial_t\nabla y,\theta) + \theta\partial^2_{F\theta}W^{\mathrm{cpl}}(\nabla y,\theta) : \partial_t\nabla y + \varepsilon|\partial_t\nabla\Delta y|^2\,, \tag{2.17b}$$

coupled with the boundary conditions

$$y = \mathbf{id} \qquad \text{in } I\times\partial\Omega, \tag{2.18a}$$

$$DH(\Delta y) = 0 \qquad \text{in } I\times\partial\Omega, \tag{2.18b}$$

$$\varepsilon\partial_\nu\Delta y = \varepsilon\Delta^2 y = 0 \qquad \text{in } I\times\partial\Omega, \tag{2.18c}$$

$$\mathcal{K}(\nabla y,\theta)\nabla\theta\cdot\nu + \kappa\theta = \kappa\theta_\flat \qquad \text{in } I\times\partial\Omega \tag{2.18d}$$

and subject to the initial conditions

$$y(0) = y_0, \qquad \partial_t y(0) = y_0', \qquad \theta(0) = \theta_0, \tag{2.19}$$

for initial values $y_0 \in \mathcal{Y}_{\mathbf{id}}$, $y_0' \in H^1_0(\Omega;\mathbb{R}^d)$, and $\theta_0 \in L^2_+(\Omega)$. We refer to [35, Section 2] for a thorough explanation of this model. We highlight that, compared to [35], we include inertial effects, i.e., the mechanical equation features the term $\rho\partial^2_{tt}y$. Moreover, for $\varepsilon > 0$ there are regularizing terms both in (2.17a) and (2.17b), complemented with the additional natural boundary condition (2.18c). In the regularized setting, we will assume stronger initial conditions for the deformations, namely $y_{0,\varepsilon} \in \mathcal{Y}^{\mathrm{reg}}_{\mathbf{id}}$ and $y'_{0,\varepsilon} \in H^3(\Omega;\mathbb{R}^d)\cap H^1_0(\Omega;\mathbb{R}^d)$, where

$$\mathcal{Y}^{\mathrm{reg}}_{\mathbf{id}} := \big\{y \in \mathcal{Y}_{\mathbf{id}} \cap H^4(\Omega;\mathbb{R}^d)\colon \partial_\nu\Delta y(t) = \Delta y(t) = 0 \ \mathcal{H}^{d-1}\text{-a.e. in } \partial\Omega\big\}. \tag{2.20}$$

We now first treat the case $\varepsilon > 0$ and afterwards we address the system without regularization.

Existence of weak solutions for the regularized system. We introduce the notion of weak solutions related to (2.17a)–(2.18d) for $\varepsilon > 0$.

Definition 2.1 (Weak solutions to the regularized thermo-elastodynamic system for viscous solids)**.** Let $y_{0,\varepsilon} \in \mathcal{Y}^{\mathrm{reg}}_{\mathbf{id}}$, $y'_{0,\varepsilon} \in H^3(\Omega;\mathbb{R}^d)\cap H^1_0(\Omega;\mathbb{R}^d)$, $\theta_0 \in L^2_+(\Omega)$, $f \in W^{1,1}(I;L^2(\Omega;\mathbb{R}^d))$, and $\theta_\flat \in W^{1,1}(I;L^2_+(\partial\Omega))$. We say that a pair $(y_\varepsilon,\theta_\varepsilon)$ with

$$y_\varepsilon \in L^\infty(I;\mathcal{Y}_{\mathbf{id}}) \cap H^1\big(I;H^3(\Omega;\mathbb{R}^d)\big) \cap H^2\big(I;(H^3(\Omega;\mathbb{R}^d)\cap H^1_0(\Omega;\mathbb{R}^d))^*\big),$$
$$\theta_\varepsilon \in L^2(I;H^1_+(\Omega))$$

is a solution to the regularized thermo-elastodynamic system with initial conditions $(y_{0,\varepsilon}, y'_{0,\varepsilon},\theta_0)$ if $y_\varepsilon(0) = y_{0,\varepsilon}$, $\partial_t y_\varepsilon(0) = y'_{0,\varepsilon}$, the internal energy $w_\varepsilon := W^{\mathrm{in}}(\nabla y_\varepsilon,\theta_\varepsilon)$ lies in $L^2(I;H^1(\Omega))\cap H^1(I;(H^1(\Omega))^*)$ and satisfies $w_\varepsilon(0) = w_{0,\varepsilon} := W^{\mathrm{in}}(\nabla y_{0,\varepsilon},\theta_0)$, and the following equations are satisfied for every $z \in C^\infty(I\times\overline{\Omega};\mathbb{R}^d)$ with $z = 0$ on $I\times\partial\Omega$ and for every $\varphi \in C^\infty(I\times\overline{\Omega})$:

$$\begin{aligned} 0 &= \int_I\int_\Omega \partial_F W(\nabla y_\varepsilon,\theta_\varepsilon) : \nabla z \,\mathrm{d}x\,\mathrm{d}t + \int_I\int_\Omega DH(\Delta y_\varepsilon)\cdot\Delta z\,\mathrm{d}x\,\mathrm{d}t + \varepsilon\int_I\int_\Omega \partial_t\nabla\Delta y_\varepsilon : \nabla\Delta z\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_I\int_\Omega \partial_{\dot F}R(\nabla y_\varepsilon,\partial_t\nabla y_\varepsilon,\theta_\varepsilon) : \nabla z\,\mathrm{d}x\,\mathrm{d}t + \rho\int_I \langle\partial^2_{tt}y_\varepsilon, z\rangle\,\mathrm{d}t - \int_I\int_\Omega f\cdot z\,\mathrm{d}x\,\mathrm{d}t\,, \end{aligned} \tag{2.21}$$

$$\begin{aligned} 0 &= \int_I\int_\Omega \mathcal{K}(\nabla y_\varepsilon,\theta_\varepsilon)\nabla\theta_\varepsilon\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_I\langle\partial_t w_\varepsilon,\varphi\rangle\,\mathrm{d}t - \kappa\int_I\int_{\partial\Omega}(\theta_\flat - \theta_\varepsilon)\varphi\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\ &\quad - \int_I\int_\Omega \Big(\xi(\nabla y_\varepsilon,\partial_t\nabla y_\varepsilon,\theta_\varepsilon)) + \partial_F W^{\mathrm{cpl}}(\nabla y_\varepsilon,\theta_\varepsilon) : \partial_t\nabla y_\varepsilon + \varepsilon|\partial_t\nabla\Delta y_\varepsilon|^2\Big)\varphi\,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{2.22}$$

where $\langle\cdot,\cdot\rangle$ denotes the dual pairing of $H^3(\Omega;\mathbb{R}^d)\cap H^1_0(\Omega;\mathbb{R}^d)$ and its dual or of $H^1(\Omega)$ and its dual, respectively.

Following the lines of [35, (2.28)–(2.29)], one can show that (2.17a) together with (2.18a)–(2.18c) is equivalent to (2.21). Besides the regularizing term, the only difference in (2.17a) compared to [35] is the presence of the inertial term. Arguing as in [35, (2.16)–(2.17)], we can rewrite the heat-transfer equation (2.17b) in terms of the internal energy w_ε as

$$\partial_t w_\varepsilon = \operatorname{div}\big(\mathcal{K}(\nabla y_\varepsilon,\theta_\varepsilon)\nabla\theta_\varepsilon\big) + \xi(\nabla y_\varepsilon,\partial_t\nabla y_\varepsilon,\theta_\varepsilon) + \varepsilon|\partial_t\nabla\Delta y_\varepsilon|^2 + \partial_F W^{\mathrm{cpl}}(\nabla y_\varepsilon,\theta_\varepsilon) : \partial_t\nabla y_\varepsilon,$$

where we have used (2.14) and the identity

$$\partial_t w_\varepsilon = \partial_F W^{\rm cpl}(\nabla y_\varepsilon, \theta_\varepsilon) : \partial_t \nabla y_\varepsilon - \theta_\varepsilon \partial^2_{F\theta} W^{\rm cpl}(\nabla y_\varepsilon, \theta_\varepsilon) : \partial_t \nabla y_\varepsilon - \theta_\varepsilon \partial^2_{\theta\theta} W^{\rm cpl}(\nabla y_\varepsilon, \theta_\varepsilon)\partial_t \theta_\varepsilon\,.$$

Taking also (2.18d) into account, this yields the weak formulation (2.22).

The first main results of the paper read as follows.

Theorem 2.2 (Existence of weak solutions to the regularized system)**.** *Let $p \in (2, +\infty)$ if $d = 2$ or $p \in (3, 6)$ for $d = 3$. Assume that (W.1)–(W.3), (H.1)–(H.3), (C.1)–(C.5), (D.1)–(D.2), and* (2.13) *hold true. Let $\varepsilon > 0$. Let $y_{0,\varepsilon} \in \mathcal{Y}^{\rm reg}_{\mathbf{id}}$, $y'_{0,\varepsilon} \in H^3(\Omega;\mathbb{R}^d) \cap H^1_0(\Omega;\mathbb{R}^d)$, $\theta_0 \in L^2_+(\Omega)$, $f \in W^{1,1}(I; L^2(\Omega;\mathbb{R}^d))$, and $\theta_\flat \in W^{1,1}(I; L^2_+(\partial\Omega))$. Then, there exists a weak solution $(y_\varepsilon, \theta_\varepsilon)$ to the regularized thermo-elastodynamic system with initial data $(y_{0,\varepsilon}, y'_{0,\varepsilon}, \theta_0)$ in the sense of Definition 2.1.*

For weak solutions, we can derive energy balances and some regularity properties. Recall (2.6) and (2.16).

Theorem 2.3 (Regularity of solutions and total energy balance)**.** *In the setting of Theorem 2.2, we find weak solutions $(y_\varepsilon, \theta_\varepsilon)$ satisfying $y_\varepsilon \in L^\infty(I; \mathcal{Y}^{\rm reg}_{\mathbf{id}})$. Moreover, for each $t \in I$, the system satisfies the mechanical energy balance*

$$\begin{aligned}\mathcal{M}(y_\varepsilon(t)) &+ \frac{\rho}{2}\|\partial_t y_\varepsilon(t)\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega \Big(\xi(\nabla y_\varepsilon, \partial_t \nabla y_\varepsilon, \theta_\varepsilon)) + \varepsilon|\partial_t \nabla \Delta y_\varepsilon|^2 \,\mathrm{d}x + \partial_F W^{\rm cpl}(\nabla y_\varepsilon, \theta_\varepsilon) : \partial_t \nabla y_\varepsilon\Big) \,\mathrm{d}s \\ &= \mathcal{M}(y_{0,\varepsilon}) + \frac{\rho}{2}\|y'_{0,\varepsilon}\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega f \cdot \partial_t y_\varepsilon \,\mathrm{d}x \,\mathrm{d}s, \end{aligned} \tag{2.23}$$

the thermal energy balance

$$\begin{aligned}\int_\Omega w_\varepsilon(t) \,\mathrm{d}x = \int_\Omega w_{0,\varepsilon} \,\mathrm{d}x &+ \int_0^t \int_\Omega \Big(\xi(\nabla y_\varepsilon, \partial_t \nabla y_\varepsilon, \theta_\varepsilon) + \varepsilon|\partial_t \nabla \Delta y_\varepsilon|^2 + \partial_F W^{\rm cpl}(\nabla y_\varepsilon, \theta_\varepsilon) : \partial_t \nabla y_\varepsilon\Big) \,\mathrm{d}x \,\mathrm{d}s \\ &+ \kappa \int_0^t \int_{\partial\Omega} (\theta_\flat - \theta_\varepsilon) \,\mathrm{d}\mathcal{H}^{d-1} \,\mathrm{d}s\,,\end{aligned} \tag{2.24}$$

and the total energy balance

$$\begin{aligned}&\mathcal{E}(y_\varepsilon(t), \theta_\varepsilon(t)) + \frac{\rho}{2}\|\partial_t y_\varepsilon(t)\|^2_{L^2(\Omega)} \\ &\quad= \mathcal{E}(y_{0,\varepsilon}, \theta_0) + \frac{\rho}{2}\|y'_{0,\varepsilon}\|^2_{L^2(\Omega)} + \int_0^t \int_{\partial\Omega} \kappa(\theta_\flat - \theta_\varepsilon) \,\mathrm{d}\mathcal{H}^{d-1} \,\mathrm{d}s + \int_0^t \int_\Omega f \cdot \partial_t y_\varepsilon \,\mathrm{d}x \,\mathrm{d}s.\end{aligned} \tag{2.25}$$

We emphasize that the energy balances are well-defined pointwise for each $t \in I$ since the regularity of the deformation and the temperature imply $y_\varepsilon \in C(I; W^{2,p}(\Omega;\mathbb{R}^d))$, $\partial_t y_\varepsilon \in C(I; L^2(\Omega;\mathbb{R}^d))$, and $w_\varepsilon \in C(I; L^2(\Omega))$, where we use that $p < 2^*$ and [39, Lemma 7.3]. The total energy balance (2.25) arises by summing (2.23) and (2.24). In particular, we observe that the system is closed for $\kappa = 0$ and $f = 0$. Still, an exchange of mechanical energy and thermal energy is possible due to the (regularized) *dissipation rate* $\xi(\nabla y_\varepsilon, \partial_t \nabla y_\varepsilon, \theta_\varepsilon)) + \varepsilon|\partial_t \nabla \Delta y_\varepsilon|^2$ and the *adiabatic heat source* $\partial_F W^{\rm cpl}(\nabla y_\varepsilon, \theta_\varepsilon) : \partial_t \nabla y_\varepsilon$ which cancel out in the summation of (2.23) and (2.24).

Existence of weak solutions for the system without regularization. Our goal is to remove the regularization by passing to the limit $\varepsilon \to 0$ for weak solutions $(y_\varepsilon, \theta_\varepsilon)$ in the sense of Definition 2.1. Unfortunately, the available a priori bounds and compactness results yielding a limit (y, θ), see Lemma 5.1 below, are not strong enough as they guarantee convergence of all terms in (2.21)–(2.22) except for the acceleration $\partial^2_{tt} y_\varepsilon$ in (2.21) and the dissipation rate $\xi(\nabla y_\varepsilon, \partial_t \nabla y_\varepsilon, \theta_\varepsilon)$ in (2.22). Accordingly, also the validity of the mechanical and thermal energy balances (2.23)–(2.24) cannot be expected in the limit $\varepsilon \to 0$ as

$$\liminf_{\varepsilon \to 0} \int_0^t \int_\Omega \xi(\nabla y_\varepsilon, \partial_t \nabla y_\varepsilon, \theta_\varepsilon) \,\mathrm{d}x \,\mathrm{d}s > \int_0^t \int_\Omega \xi(\nabla y, \partial_t \nabla y, \theta) \,\mathrm{d}x \,\mathrm{d}s$$

is possible under the available compactness results. In [2, Lemma 4.5] and [35, Propositions 5.1 and 6.6], equality was guaranteed by a chain rule for the mechanical energy (see [35, Proposition 3.6]) which also allowed to derive a mechanical energy balance. Due to the presence of the inertial term, it appears to be impossible to adapt this strategy to the current setting.

We overcome this difficulty by appealing to a weaker formulation of the mechanical and the heat-transfer equation. In (2.21), it suffices to perform an integration by parts in time to deal with the term $\partial^2_{tt} y_\varepsilon$. The passage to a weaker form of (2.22) is based on the observation that the delicate dissipation term cancels in the summation of (2.23) and (2.24). More precisely, this passage is achieved by testing (2.21) with $z = \partial_t y \varphi$ and adding the result to (2.22). This leads to the following notion of weak solution whose form will be explained in more detail by a formal computation in (2.29)–(2.33) below.

Definition 2.4 (Weak solutions to thermo-elastodynamic system for viscous solids)**.** Let $y_0 \in \mathcal{Y}_{\mathbf{id}}$, $y_0' \in H^1(\Omega;\mathbb{R}^d)$, $\theta_0 \in L^2_+(\Omega)$, $f \in W^{1,1}(I;L^2(\Omega;\mathbb{R}^d))$, and $\theta_\flat \in W^{1,1}(I;L^2_+(\partial\Omega))$. We say that a pair (y,θ) with

$$y \in L^\infty(I;\mathcal{Y}_{\mathbf{id}}) \cap H^1(I;H^1(\Omega;\mathbb{R}^d)), \quad \theta \in L^1(I;W^{1,1}_+(\Omega))$$

is a solution to the thermo-elastodynamic system with initial conditions (y_0, y_0', θ_0) if the following equations are satisfied for every $z \in C^\infty(I\times\overline{\Omega};\mathbb{R}^d)$ with $z=0$ on $I\times\partial\Omega$ and $z(T)=0$, and for every $\varphi \in C^\infty(I\times\overline{\Omega})$ with $\varphi(T)=0$:

$$\begin{aligned} 0 = &\int_I\int_\Omega \partial_F W(\nabla y,\theta) : \nabla z \,\mathrm{d}x\,\mathrm{d}t + \int_I\int_\Omega DH(\Delta y)\cdot\Delta z\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_I\int_\Omega \partial_{\dot F} R(\nabla y,\partial_t\nabla y,\theta) : \nabla z\,\mathrm{d}x\,\mathrm{d}t - \rho\int_I\int_\Omega \partial_t y\cdot\partial_t z\,\mathrm{d}x\,\mathrm{d}t - \int_I\int_\Omega f\cdot z\,\mathrm{d}x\,\mathrm{d}t - \rho\int_\Omega y_0'\cdot z(0)\,\mathrm{d}x, \end{aligned} \tag{2.26}$$

$$\begin{aligned} 0 = &\int_I\int_\Omega \mathcal{K}(\nabla y,\theta)\nabla\theta\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t - \kappa\int_I\int_{\partial\Omega}(\theta_\flat-\theta)\varphi\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t - \int_I\int_\Omega \varphi\, f\cdot\partial_t y\,\mathrm{d}x\,\mathrm{d}t \\ &- \int_I\int_\Omega \big(W^{\mathrm{el}}(\nabla y) + H(\Delta y) + w + \frac{\rho}{2}|\partial_t y|^2\big)\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \big(W^{\mathrm{el}}(\nabla y_0) + H(\Delta y_0) + w_0 + \frac{\rho}{2}|y_0'|^2\big)\varphi(0)\,\mathrm{d}x \\ &+ \int_I\int_\Omega \Big(\partial_F W(\nabla y,\theta) + \partial_{\dot F} R(\nabla y,\nabla\partial_t y,\theta)\Big) : (\partial_t y\otimes\nabla\varphi)\,\mathrm{d}x\,\mathrm{d}t \\ &- \int_I\int_\Omega DH(\Delta y)\cdot\partial_t y\Delta\varphi\,\mathrm{d}x\,\mathrm{d}t - 2\int_I\int_\Omega \nabla(DH(\Delta y)) : (\partial_t y\otimes\nabla\varphi)\,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{2.27}$$

where for shorthand we set $w := W^{\mathrm{in}}(\nabla y,\theta)$ and $w_0 := W^{\mathrm{in}}(\nabla y_0,\theta_0)$.

In particular, we observe that, due to the lack of regularity of $\partial^2_{tt}y$ and $\partial_t w$, the initial conditions of $\partial_t y$ and w are given implicitly in a weak form, relying on an integration by parts in time. An important aspect of the weak formulation (2.27) is that it directly guarantees the total energy balance. Indeed, for each $t\in I$ such that

$$\lim_{\delta\to 0} \fint_{t-\delta}^{t+\delta}\int_\Omega y\,\mathrm{d}x\,\mathrm{d}s = \int_\Omega y(t,x)\,\mathrm{d}x \in W^{2,p}(\Omega;\mathbb{R}^d), \quad \lim_{\delta\to 0} \fint_{t-\delta}^{t+\delta}\int_\Omega \partial_t y\,\mathrm{d}x\,\mathrm{d}s = \int_\Omega \partial_t y(t,x)\,\mathrm{d}x \in L^2(\Omega;\mathbb{R}^d),$$
$$\lim_{\delta\to 0} \fint_{t-\delta}^{t+\delta}\int_\Omega w\,\mathrm{d}x\,\mathrm{d}s = \int_\Omega w(t,x)\,\mathrm{d}x \in L^1(\Omega)$$

(and thus for a.e. $t\in I$), we can test (2.27) with φ given by $\varphi\equiv 1$ on $(0,t-\delta)$, $\varphi\equiv 0$ on $(t+\delta,T)$ and $\varphi'\equiv -\frac{1}{2\delta}$ on $(t-\delta,t+\delta)$. In the limit $\delta\to 0$, after rearranegment, this yields

$$\begin{aligned} &\int_\Omega \big(W^{\mathrm{el}}(\nabla y(t)) + H(\Delta y(t)) + w(t) + \frac{\rho}{2}|\partial_t y(t)|^2\big)\,\mathrm{d}x \\ &\quad = \int_\Omega \big(W^{\mathrm{el}}(\nabla y_0) + H(\Delta y_0) + w_0 + \frac{\rho}{2}|y_0'|^2\big)\,\mathrm{d}x + \kappa\int_0^t\int_{\partial\Omega}(\theta_\flat-\theta)\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}s + \int_0^t\int_\Omega f\cdot\partial_t y\,\mathrm{d}x\,\mathrm{d}s \end{aligned} \tag{2.28}$$

which is exactly the total energy balance, cf. also (2.25). Whereas a total energy balance still holds, the respective form of (2.23) and (2.24) may become *inequalities.* In some sense, this weaker form based on a replacement is inspired by fluid-mechanics for compressible heat conduction fluids, where the conservation of the total energy is guaranteed by transferring the heat equation into an inequality for the entropy [22, 21].

Theorem 2.5 (Existence and regularity of weak solutions)**.** *Let $p\in(2,+\infty)$ if $d=2$ or $p\in(3,6)$ for $d=3$. Assume that (W.1)–(W.3), (H.1)–(H.3), (C.1)–(C.5), (D.1)–(D.2), and* (2.13) *hold true. Let $y_0\in\mathcal{Y}_{\mathbf{id}}$, $y_0'\in H^1_0(\Omega;\mathbb{R}^d)$, $\theta_0\in L^2_+(\Omega)$, $f\in W^{1,1}(I;L^2(\Omega;\mathbb{R}^d))$, and $\theta_\flat\in W^{1,1}(I;L^2_+(\partial\Omega))$. Then, there exists a weak solution (y,θ) to the thermo-elastodynamic system with initial data (y_0,y_0',θ_0) in the sense of Definition 2.4. The weak solution satisfies $y\in L^2(I;H^3(\Omega;\mathbb{R}^d))$ and $(1+|\Delta y|)^{\frac{p-2}{2}}|\nabla\Delta y|^2\in L^2(I\times\Omega)$.*

Note that (2.28) holds and (2.18b) is satisfied in the sense of traces.

Formal derivation of the weak formulation. Let us close this section with a formal derivation of equation (2.27) which will be made precise below in Proposition 5.3 for the regularized system. Assuming sufficient regularity for y and θ, let us check that the formulations in (2.27) and (2.22) (for $\varepsilon=0$) coincide. First, an integration by parts in time shows that, for $\varepsilon = 0$, (2.22) is equivalent to

$$\begin{aligned} 0 = &\int_I \int_\Omega \mathcal{K}(\nabla y, \theta)\nabla\theta \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t - \int_I \int_\Omega \big(\xi(\nabla y, \partial_t \nabla y, \theta) + \partial_F W^{\mathrm{cpl}}(\nabla y, \theta) : \partial_t \nabla y\big)\varphi \,\mathrm{d}x\,\mathrm{d}t \\ &+ \kappa \int_I \int_{\partial\Omega} (\theta - \theta_\flat)\varphi \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t - \int_I \int_\Omega w\partial_t\varphi \,\mathrm{d}x\,\mathrm{d}t - \int_\Omega w_0\varphi(0)\,\mathrm{d}x \end{aligned} \tag{2.29}$$

for every $\varphi \in C^\infty(I \times \overline{\Omega})$ with $\varphi(T) = 0$. Now, we test (2.21) with $z := \partial_t y\varphi$ for $\varphi \in C^\infty(I \times \overline{\Omega})$ with $\varphi(T) = 0$. Using (2.8), (2.11), expanding $\nabla(\partial_t y\varphi)$, and rearranging the terms, we obtain

$$\begin{aligned} &-\int_I \int_\Omega \xi(\nabla y, \partial_t \nabla y, \theta)\varphi \,\mathrm{d}x\,\mathrm{d}t - \int_I \int_\Omega \partial_F W^{\mathrm{cpl}}(\nabla y, \theta) : \partial_t \nabla y\varphi \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_I \int_\Omega \partial_F W^{\mathrm{el}}(\nabla y, \theta) : \partial_t \nabla y\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_I \int_\Omega \partial_F W(\nabla y, \theta) : (\partial_t y \otimes \nabla\varphi)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_I \int_\Omega \partial_{\dot F} R(\nabla y, \partial_t \nabla y, \theta) : (\partial_t y \otimes \nabla\varphi)\,\mathrm{d}x\,\mathrm{d}t + \int_I \int_\Omega DH(\Delta y) \cdot \Delta(\partial_t y\varphi)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \int_I \int_\Omega f \cdot \partial_t y\varphi \,\mathrm{d}x\,\mathrm{d}t + \rho \int_I \int_\Omega \varphi\partial^2_{tt} y \cdot \partial_t y \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{2.30}$$

By the chain rule, the fundamental theorem of calculus, and $\varphi(T) = 0$ we get

$$\begin{aligned} \int_I \int_\Omega \varphi\partial^2_{tt} y \cdot \partial_t y \,\mathrm{d}x\,\mathrm{d}t &= \frac{1}{2}\int_I \int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}\big(\varphi|\partial_t y|^2\big)\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2}\int_I \int_\Omega \partial_t\varphi|\partial_t y|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &= -\frac{1}{2}\int_\Omega |y_0'|^2\varphi(0) - \frac{1}{2}\int_I \int_\Omega \partial_t\varphi|\partial_t y|^2 \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{2.31}$$

Moreover, by integration by parts in Ω and since $\partial_t y = 0$ in $I \times \partial\Omega$ (recall that $y(t) \in \mathcal{Y}_{\mathbf{id}}$), we have that

$$\begin{aligned} &\int_I \int_\Omega DH(\Delta y) \cdot \Delta(\partial_t y\varphi)\,\mathrm{d}x\,\mathrm{d}t = \int_I \int_\Omega DH(\Delta y) : \big(\varphi\partial_t\Delta y + 2\partial_t\nabla y\nabla\varphi + \partial_t y\Delta\varphi\big)\,\mathrm{d}x\,\mathrm{d}t \qquad (2.32) \\ &= \int_I \int_\Omega DH(\Delta y) : \partial_t\Delta y\varphi \,\mathrm{d}x\,\mathrm{d}t - \int_I \int_\Omega DH(\Delta y) : \partial_t y\Delta\varphi \,\mathrm{d}x\,\mathrm{d}t - 2\int_I \int_\Omega \nabla(DH(\Delta y)) : (\partial_t y \otimes \nabla\varphi)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Eventually, by the chain rule and by integration by parts with $\varphi(T) = 0$ we get

$$\begin{aligned} \int_I \int_\Omega \varphi\big(\partial_F W^{\mathrm{el}}(\nabla y) : \partial_t \nabla y + DH(\Delta y) \cdot \partial_t \Delta y\big)\,\mathrm{d}x\,\mathrm{d}t &= \int_I \int_\Omega \varphi\frac{\mathrm{d}}{\mathrm{d}t}\big(W^{\mathrm{el}}(\nabla y) + H(\Delta y)\big)\,\mathrm{d}x\,\mathrm{d}t \\ = -\int_\Omega \varphi(0)\big(W^{\mathrm{el}}(\nabla y_0) + H(\Delta y_0)\big)\,\mathrm{d}x &- \int_I \int_\Omega \partial_t\varphi\big(W^{\mathrm{el}}(\nabla y) + H(\Delta y)\big)\,\mathrm{d}x\,\mathrm{d}t\,. \end{aligned} \tag{2.33}$$

Combining (2.29)–(2.33) we infer (2.27).

Outline. The rest of the paper is structured as follows. In Section 3 we consider a time-delayed parabolic system for a time-delay $h > 0$ whose existence is established by a minimizing movement scheme with time discretization $\tau > 0$. In Section 4 we pass to the limit $h \to 0$ and prove existence of solutions to the regularized system, see Theorem 2.2. Eventually in Section 5 we pass to the limit $\varepsilon \to 0$ and show Theorem 2.5. Importantly, all relevant a priori estimates are established already on the level $\tau > 0$ in Section 3 and immediately transfer to the limits $\tau \to 0$, $h \to 0$, and $\varepsilon \to 0$.

3. Minimizing movements for time-delayed parabolic system

We fix a *regularization parameter* $\varepsilon > 0$ and a *time-delay* $h > 0$. For convenience, without further notice, we assume that $T/h \in \mathbb{N}$. In this section, we include the ε-dependent regularizing term in (2.17) and we suppose more regular initial conditions y_0, y_0', denoted by $y_{0,\varepsilon}$, $y_{0,\varepsilon}'$. As done in [5], by replacing the acceleration term $\rho\partial^2_{tt} y$ by a discrete difference $\rho\frac{\partial_t y - \partial_t y(\cdot - h)}{h}$, we turn the hyperbolic problem (2.17a) into a parabolic one. The main goal of this section is to prove the following existence result for the resulting problem, where for convenience we use the notation $I_h := [-h, T]$.

Theorem 3.1 (Weak solutions of the time-delayed regularized problem)**.** *Let T, h, $\varepsilon > 0$, $y_{0,\varepsilon} \in \mathcal{Y}^{\mathrm{reg}}_{\mathbf{id}}$, $y_{0,\varepsilon}' \in H^3(\Omega; \mathbb{R}^d) \cap H^1_0(\Omega; \mathbb{R}^d)$, and $\theta_0 \in L^2_+(\Omega)$. Then, there exist $y_h \in L^\infty(I; \mathcal{Y}^{\mathrm{reg}}_{\mathbf{id}}) \cap H^1(I_h; H^3(\Omega; \mathbb{R}^d))$ with $y_h(t) = y_{0,\varepsilon} + ty_{0,\varepsilon}'$ for all $t \in [-h, 0]$ and $\theta_h \in L^2(I; H^1_+(\Omega))$ such that $w_h := W^{\mathrm{in}}(\nabla y_h, \theta_h) \in L^2(I; H^1(\Omega)) \cap$*

$H^1(I;(H^1(\Omega))^*)$ *with* $w_h(0) = w_{0,\varepsilon} := W^{\rm in}(\nabla y_{0,\varepsilon}, \theta_0)$ *and the following holds true: For all* $z \in C^\infty(I \times \overline{\Omega}; \mathbb{R}^d)$ *satisfying* $z = 0$ *on* $I \times \partial\Omega$ *we have*

$$\begin{aligned}\int_I \int_\Omega \Big(\partial_F W(\nabla y_h, \theta_h) + \partial_{\dot F} R(\nabla y_h, \partial_t \nabla y_h, \theta_h)\Big) : \nabla z + DH(\Delta y_h) \cdot \Delta z + \varepsilon \partial_t \nabla \Delta y_h : \nabla \Delta z \,\mathrm{d}x\,\mathrm{d}t \\ = \int_I \int_\Omega f \cdot z \,\mathrm{d}x\,\mathrm{d}t - \frac{\rho}{h} \int_I \int_\Omega (\partial_t y_h(t) - \partial_t y_h(t-h)) \cdot z \,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{3.1a}$$

and for all $\varphi \in C^\infty(I \times \overline{\Omega})$ *it holds that*

$$\begin{aligned}\int_I \int_\Omega \mathcal{K}(\nabla y_h, \theta_h)\nabla\theta_h \cdot \nabla\varphi - \big(\xi(\nabla y_h, \partial_t \nabla y_h, \theta_h) + \partial_F W^{\rm cpl}(\nabla y_h, \theta_h) : \partial_t \nabla y_h\big)\,\varphi \,\mathrm{d}x\,\mathrm{d}t \\ - \varepsilon \int_0^T \int_\Omega |\partial_t \nabla \Delta y_h|^2 \varphi \,\mathrm{d}x\,\mathrm{d}t + \int_I \langle \partial_t w_h, \varphi\rangle \,\mathrm{d}t = \kappa \int_I \int_{\partial\Omega} (\theta_\flat - \theta_h)\varphi \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t,\end{aligned} \tag{3.1b}$$

where $\langle \cdot, \cdot \rangle$ *denotes the dual pairing between* $H^1(\Omega)$ *and* $(H^1(\Omega))^*$*.*

A similar notion of weak solutions has already been considered in [35, 2]. The main differences of the above equations (3.1a)–(3.1b) to, e.g., [2, (2.19)-(2.20)] is the presence of the additional h-dependent terms arising from the time-discretization of the acceleration as well as the regularizing terms depending on ε, which induce better regularity properties of the solutions. The solutions also depend on ε, which we however do not include in the notation for simplicity. The proof of existence follows along the lines of the reasoning from [2, Sections 3 and 4] and is based on a minimizing movement scheme. To keep the presentation concise, many proofs in this section will only be sketched, highlighting primarily the differences to the arguments in [2].

3.1. **Staggered minimizing movement scheme and its well-definedness.** We introduce a *discrete time-step* $\tau \in (0, h)$ and without further notice we assume that $h/\tau \in \mathbb{N}$. This also implies $T/\tau \in \mathbb{N}$. If not stated otherwise, all constants encountered in this section are independent of τ, h, and ε. Given any sequence $(a_k)_{k\in\mathbb{Z}}$, we introduce the notation for discrete differences as

$$\delta_\tau a_k := \frac{a_k - a_{k-1}}{\tau}, \qquad k \in \mathbb{Z}. \tag{3.2}$$

Theorem 3.1 will be shown via a *staggered* minimizing movements scheme. Let $y_{0,\varepsilon} \in \mathcal{Y}^{\rm reg}_{\mathbf{id}}$, $y'_{0,\varepsilon} \in H^3(\Omega;\mathbb{R}^d) \cap H^1_0(\Omega;\mathbb{R}^d)$, and $\theta_0 \in L^2_+(\Omega)$. We first define the initial conditions of the scheme by

$$y_\tau^{(k)} := y_{0,\varepsilon} + k\tau y'_{0,\varepsilon} \quad \text{for } k \in \{-h/\tau, \ldots, 0\} \quad \text{and} \quad \theta_\tau^{(0)} := \theta_0.$$

Note that the time-discrete deformation is also defined for negative times, which will allow us to prove that the solution y_h in (3.1a)–(3.1b) satisfies $y_h(t) = y_{0,\varepsilon} + t y'_{0,\varepsilon}$ for all $t \in [-h, 0]$.

Now, suppose that for $k \in \{1, \ldots, T/\tau\}$ we have already constructed $(y_\tau^{(0)}, \theta_\tau^{(0)}), \ldots, (y_\tau^{(k-1)}, \theta_\tau^{(k-1)})$. (The solutions also depend on h and ε, which we do not include in the notation for simplicity.) Let $f_\tau^{(k)} := ⨍_{(k-1)\tau}^{k\tau} f(t)\,\mathrm{d}t := \tau^{-1}\int_{(k-1)\tau}^{k\tau} f(t)\,\mathrm{d}t$, and for shorthand we denote by $(\cdot,\cdot)_2$ the scalar product in $L^2(\Omega;\mathbb{R}^d)$. Recalling also (2.6), (2.7), and (2.9), the next deformation $y_\tau^{(k)}$ is defined as a solution of the minimization problem

$$\begin{aligned}\min_{y \in \mathcal{Y}_{\mathbf{id}} \cap H^3(\Omega;\mathbb{R}^d)} \Bigg\{ \mathcal{M}(y) \; + \mathcal{W}^{\rm cpl}\big(y, \theta_\tau^{(k-1)}\big) + \frac{1}{\tau}\mathcal{R}\big(y_\tau^{(k-1)}, y - y_\tau^{(k-1)}, \theta_\tau^{(k-1)}\big) \\ + \frac{\varepsilon}{2\tau}\|\nabla\Delta y - \nabla\Delta y_\tau^{(k-1)}\|^2_{L^2(\Omega)} - (f_\tau^{(k)}, y)_2 + \frac{\rho\tau}{2h}\Big\| \frac{y - y_\tau^{(k-1)}}{\tau} - \delta_\tau y_\tau^{(k-h/\tau)} \Big\|^2_{L^2(\Omega)} \Bigg\}.\end{aligned} \tag{3.3}$$

Supposing that $y_\tau^{(k)}$ exists, we define $\theta_\tau^{(k)}$ as a solution to the minimization problem

$$\begin{aligned}\min_{\theta \in H^1_+(\Omega)} \Bigg\{ \int_\Omega \int_0^\theta \frac{1}{\tau}\Big(W^{\rm in}(\nabla y_\tau^{(k)}, s) - W^{\rm in}(\nabla y_\tau^{(k-1)}, \theta_\tau^{(k-1)})\Big)\,\mathrm{d}s\,\mathrm{d}x + \frac{1}{2}\int_\Omega \mathcal{K}(\nabla y_\tau^{(k-1)}, \theta_\tau^{(k-1)})\nabla\theta \cdot \nabla\theta \,\mathrm{d}x \\ - \int_\Omega \Big(\partial_F W^{\rm cpl}(\nabla y_\tau^{(k-1)}, \theta_\tau^{(k-1)}) : \delta_\tau \nabla y_\tau^{(k)} + \xi(\nabla y_\tau^{(k-1)}, \delta_\tau \nabla y_\tau^{(k)}, \theta_\tau^{(k-1)}) + \varepsilon|\delta_\tau \nabla\Delta y_\tau^{(k)}|^2 \wedge \tau^{-1}\Big)\,\theta \,\mathrm{d}x \\ + \frac{\kappa}{2}\int_{\partial\Omega} (\theta - \theta_{\flat,\tau}^{(k)})^2 \,\mathrm{d}\mathcal{H}^{d-1} \Bigg\},\end{aligned} \tag{3.4}$$

where $\theta_{\flat,\tau}^{(k)} := ⨍_{(k-1)\tau}^{k\tau} \theta_\flat(t)\,\mathrm{d}t$.

The minimization problem (3.3) differs from the one used in [5] due to the presence of the additional ε-regularizing term and a different discretization of the acceleration term. In [5, Definition 3.3, Theorem 3.5], the term

$$\frac{\rho\tau}{2h}\Big\|\frac{y-y_\tau^{(k-1)}}{\tau}-\fint_{(k-1)\tau}^{k\tau} v\,\mathrm{d}t\Big\|^2_{L^2(\Omega)}$$

for a generic $v\in L^2(0,h)$ is used. By replacing v with $\partial_t y(\cdot-h)$, one can then construct a τ-discretized solution in the small time interval $(0,h)$. Then, a successive repetition of the argument yields a time-discrete solution on $[0,T]$. Here, instead, we simply use the discretized solution $\delta_\tau y_\tau^{(k-h/\tau)}$ which directly allows us to construct time-discrete solutions in the entire time horizon $[0,T]$. The minimization problem (3.4) coincides with the thermal step used in [2], except for the regularizing term $\varepsilon|\delta_\tau\nabla\Delta y_\tau^{(k)}|^2\wedge\tau^{-1}$. In this context, the truncation by τ^{-1} is necessary to guarantee well-posedness of the problem, see Proposition 3.4 below.

We now show the well-definedness of the above minimization problems. In this regard, the following properties of the mechanical energy are useful.

Lemma 3.2 (Coercivity of $\mathcal{M}$)**.** *Given $M>0$ there exists a constant $C_M>0$ such that for all $y\in\mathcal{Y}_{\mathbf{id}}$ with $\mathcal{M}(y)\le M$ it holds that*

$$\|y\|_{W^{2,p}(\Omega)}\le C_M,\qquad \|y\|_{C^{1,1-d/p}(\Omega)}\le C_M,\qquad \|(\nabla y)^{-1}\|_{C^{1-d/p}(\Omega)}\le C_M,\qquad \det(\nabla y)\ge\frac{1}{C_M}\ \text{in } \Omega. \tag{3.5}$$

Proof. By the definition of $\mathcal{H}$, the first inequality in (H.3) and (W.3) we see that

$$-C_0|\Omega|+\int_\Omega|\Delta y|^p\,\mathrm{d}x\le\mathcal{M}(y)\le M,$$

and hence

$$\|\Delta y\|^p_{L^p(\Omega)}\le M+C_0|\Omega|=:\tilde M.$$

As $y-\mathbf{id}\in W^{2,p}(\Omega;\mathbb{R}^d)\cap W^{1,p}_0(\Omega;\mathbb{R}^d)$ by the definition of $\mathcal{Y}_{\mathbf{id}}$, we then derive from [30, Lemma 9.17] and the regularity of $\partial\Omega$ that

$$\|y-\mathbf{id}\|_{W^{2,p}(\Omega)}\le C\|\Delta y\|_{L^p(\Omega)}\le C\tilde M^{1/p}$$

for a constant C independent of M. Our choice of the elastic potential W^{el} satisfies all assumptions imposed in [35]. Consequently, [35, Theorem 3.1] applies which directly leads to (3.5). ☐

Proposition 3.3 (Existence of the mechanical step)**.** *For any $M>0$ there exists $\tau_0\in(0,1]$ such that for all $\tau\in(0,\tau_0)$ and $k\in\{1,\dots,T/\tau\}$ the following holds: Let $y_\tau^{(k-1)}\in\mathcal{Y}_{\mathbf{id}}\cap H^3(\Omega;\mathbb{R}^d)$ satisfy $\mathcal{M}(y_\tau^{(k-1)})\le M$ and let $\theta_\tau^{(k-1)}\in H^1_+(\Omega)$. Then, the minimization problem (3.3) attains a solution $y_\tau^{(k)}\in\mathcal{Y}_{\mathbf{id}}\cap H^3(\Omega;\mathbb{R}^d)$ solving the corresponding Euler-Lagrange equation, i.e., for all $z\in H^3(\Omega;\mathbb{R}^d)\cap H^1_0(\Omega;\mathbb{R}^d)$ it holds that*

$$\begin{aligned}\int_\Omega\Big(\partial_F W(\nabla y_\tau^{(k)},\theta_\tau^{(k-1)})+\partial_{\dot F}R(\nabla y_\tau^{(k-1)},\delta_\tau\nabla y_\tau^{(k)},\theta_\tau^{(k-1)})\Big):\nabla z+DH(\Delta y_\tau^{(k)})\cdot\Delta z+\varepsilon\delta_\tau\nabla\Delta y_\tau^{(k)}:\nabla\Delta z\,\mathrm{d}x\\ =\int_\Omega f_\tau^{(k)}\cdot z\,\mathrm{d}x-\frac{\rho}{h}\int_\Omega(\delta_\tau y_\tau^{(k)}-\delta_\tau y_\tau^{(k-h/\tau)})\cdot z\,\mathrm{d}x.\end{aligned}\tag{3.6}$$

Moreover, there exists a constant $C_M>0$, possibly depending on M, such that

$$\mathcal{M}(y_\tau^{(k)})+\frac{\tau}{C_M}\|\delta_\tau\nabla y_\tau^{(k)}\|^2_{L^2(\Omega)}+\frac{\varepsilon\tau}{2}\|\delta_\tau\nabla\Delta y_\tau^{(k)}\|^2_{L^2(\Omega)}\le C_M(1+\|f\|^2_{L^2(I\times\Omega)})+\frac{\rho\tau}{h}\|\delta_\tau y_\tau^{(k-h/\tau)}\|^2_{L^2(\Omega)}. \tag{3.7}$$

Proof. The proof is similar to the one in [2, Proposition 3.5]. We start by showing compactness. To this end, let $(y_n)_n\subset\mathcal{Y}_{\mathbf{id}}\cap H^3(\Omega;\mathbb{R}^d)$ be a minimizing sequence for the minimization problem in (3.3). Using $y_\tau^{(k-1)}$ as a competitor in (3.3), we may suppose that each y_n satisfies

$$\begin{aligned}&\mathcal{M}(y_n)+\mathcal{W}^{\mathrm{cpl}}(y_n,\theta_\tau^{(k-1)})+\frac{1}{\tau}\mathcal{R}(y_\tau^{(k-1)},y_n-y_\tau^{(k-1)},\theta_\tau^{(k-1)})+\frac{\varepsilon}{2\tau}\|\nabla\Delta y_n-\nabla\Delta y_\tau^{(k-1)}\|^2_{L^2(\Omega)}\\&\quad-(f_\tau^{(k)},y_n)_2+\frac{\rho\tau}{2h}\Big\|\frac{y_n-y_\tau^{(k-1)}}{\tau}-\delta_\tau y_\tau^{(k-h/\tau)}\Big\|^2_{L^2(\Omega)}\\&\qquad\le\mathcal{M}(y_\tau^{(k-1)})+\mathcal{W}^{\mathrm{cpl}}(y_\tau^{(k-1)},\theta_\tau^{(k-1)})-(f_\tau^{(k)},y_\tau^{(k-1)})_2+\frac{\rho\tau}{2h}\|\delta_\tau y_\tau^{(k-h/\tau)}\|^2_{L^2(\Omega)}.\end{aligned}$$

As the mechanical energy $\mathcal{M}$ satisfies the same coercivity properties as the one in [2], see Lemma 3.2, we can apply the generalized Korn's inequality in the form [35, Corollary 3.4] (see also [36] for its original formulation). Consequently, reasoning similarly to the proof of [2, Proposition 3.5] for the terms involving $\mathcal{M}$, $\mathcal{W}^{\mathrm{cpl}}$, $\mathcal{R}$, and

$f_\tau^{(k)}$, see [2, Equation (3.9)], we can find $C_M > 0$ and $\tau_0 \in (0,1)$, possibly depending on M, such that for $\tau \in (0, \tau_0)$

$$
\begin{aligned}
(1 - C_M\tau)\mathcal{M}(y_n) &+ \frac{1}{C_M \tau}\|\nabla y_n - \nabla y_\tau^{(k-1)}\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2\tau}\|\nabla\Delta y_n - \nabla\Delta y_\tau^{(k-1)}\|^2_{L^2(\Omega)} \\
&+ \frac{\rho\tau}{2h}\left\|\frac{y_n - y_\tau^{(k-1)}}{\tau} - \delta_\tau y_\tau^{(k-h/\tau)}\right\|^2_{L^2(\Omega)} \\
&\le (1 + C_M\tau)\mathcal{M}(y_\tau^{(k-1)}) + C_M\tau\big(1 + \|f_\tau^{(k)}\|^2_{L^2(\Omega)}\big) + \frac{\rho\tau}{2h}\|\delta_\tau y_\tau^{(k-h/\tau)}\|^2_{L^2(\Omega)}.
\end{aligned}
$$

By the definition of $f_\tau^{(k)}$ and Jensen's inequality we see that

$$
\tau\|f_\tau^{(k)}\|^2_{L^2(\Omega)} = \tau\int_\Omega \Big|\fint_{(k-1)\tau}^{k\tau} f\,\mathrm{d}t\Big|^2\,\mathrm{d}x \le \tau\int_\Omega \fint_{(k-1)\tau}^{k\tau} |f|^2\,\mathrm{d}t\,\mathrm{d}x \le \|f\|^2_{L^2(I\times\Omega)}.
$$

Then, choosing τ_0 small enough such that $C_M\tau_0 \le \frac{1}{2}$ and using $\mathcal{M}(y_\tau^{(k-1)}) \le M$, we see that, after possibly increasing C_M, it follows

$$
\begin{aligned}
\mathcal{M}(y_n) &+ \frac{1}{C_M\tau}\|\nabla y_n - \nabla y_\tau^{(k-1)}\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2\tau}\|\nabla\Delta y_n - \nabla\Delta y_\tau^{(k-1)}\|^2_{L^2(\Omega)} + \frac{\rho\tau}{2h}\left\|\frac{y_n - y_\tau^{(k-1)}}{\tau} - \delta_\tau y_\tau^{(k-h/\tau)}\right\|^2_{L^2(\Omega)} \\
&\le C_M(1 + \|f\|^2_{L^2(I\times\Omega)}) + \frac{\rho\tau}{h}\|\delta_\tau y_\tau^{(k-h/\tau)}\|^2_{L^2(\Omega)}.
\end{aligned}
\tag{3.8}
$$

By Young's inequality with power 2 and constant $\lambda \in (0,1)$ we derive that

$$
\begin{aligned}
\|\nabla\Delta y_n - \nabla\Delta y_\tau^{(k)}\|^2_{L^2(\Omega)} &= \|\nabla\Delta y_n\|^2_{L^2(\Omega)} - 2\int_\Omega \nabla\Delta y_n : \nabla\Delta y_\tau^{(k-1)}\,\mathrm{d}x + \|\nabla\Delta y_\tau^{(k-1)}\|^2_{L^2(\Omega)} \\
&\ge (1-\lambda)\|\nabla\Delta y_n\|^2_{L^2(\Omega)} - (-1 + 1/\lambda)\|\nabla\Delta y_\tau^{(k-1)}\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Choosing $\lambda = 1/2$ above and combining with (3.8) this leads to

$$
\mathcal{M}(y_n) + \frac{\varepsilon}{4\tau}\|\Delta\nabla y_n\|^2_{L^2(\Omega)} \le +C_M(1 + \|f\|^2_{L^2(I\times\Omega)}) + \frac{\varepsilon}{2\tau}\|\nabla\Delta y_\tau^{(k-1)}\|^2_{L^2(\Omega)} + \frac{\rho\tau}{h}\|\delta_\tau y_\tau^{(k-h/\tau)}\|^2_{L^2(\Omega)}.
$$

This implies $\sup_{n\in\mathbb{N}}\|\nabla\Delta y_n\|_{L^2(\Omega)} < \infty$ and, in view of (H.3), in particular shows $\sup_{n\in\mathbb{N}}\|\Delta y_n\|_{H^1(\Omega)} < \infty$. As Ω was assumed to have a C^5-boundary and $(y_n)_n \subset \mathcal{Y}_{\mathbf{id}}$, elliptic regularity for the operator Δ implies

$$
\|y_n - \mathbf{id}\|_{H^3(\Omega)} \le C\|\Delta y_n\|_{H^1(\Omega)},
$$

and thus $\sup_{n\in\mathbb{N}}\|y_n\|_{H^3(\Omega)} < \infty$. Consequently, as $p < 2^*$, we can select a subsequence (without relabeling) such that $y_n \to y$ strongly in $W^{2,p}(\Omega;\mathbb{R}^d)$ as well as $y_n \rightharpoonup y$ weakly in $H^3(\Omega;\mathbb{R}^d)$.

Existence of minimizers then follows by standard lower semicontinuity arguments. Also, recalling (3.2), the derivation of the Euler-Lagrange equation (3.6) is standard, see the proof of [2, Proposition 3.5] for some details. Eventually, estimate (3.7) directly follows from (3.8), after passing to the limit $n \to \infty$ and using standard lower semicontinuity arguments. □

Proposition 3.4 (Existence of the thermal step). *For any $M > 0$ there exists $\tau_0 \in (0,1]$ such that for all $\tau \in (0,\tau_0)$ and $k \in \{1,\dots,T/\tau\}$ the following holds: Let $y_\tau^{(k-1)}, y_\tau^{(k)} \in \mathcal{Y}_{\mathbf{id}}$ be such that $\mathcal{M}(y_\tau^{(k-1)}) \le M$ and let $\theta_\tau^{(k-1)} \in H^1_+(\Omega)$. Then, the minimization problem (3.4) attains a solution $\theta_\tau^{(k)} \in H^1_+(\Omega)$ solving the corresponding Euler-Lagrange equation, i.e., for all $\varphi \in H^1(\Omega)$ it holds that*

$$
\begin{aligned}
0 = \int_\Omega \delta_\tau w_\tau^{(k)}\varphi\,\mathrm{d}x &+ \int_\Omega \mathcal{K}(\nabla y_\tau^{(k-1)}, \theta_\tau^{(k-1)})\nabla\theta_\tau^{(k)}\cdot\nabla\varphi\,\mathrm{d}x + \kappa\int_{\partial\Omega}(\theta_\tau^{(k)} - \theta_{\flat,\tau}^{(k)})\varphi\,\mathrm{d}\mathcal{H}^{d-1} \\
&- \int_\Omega \big(\partial_F W^{\mathrm{cpl}}(\nabla y_\tau^{(k-1)}, \theta_\tau^{(k-1)}) : \delta_\tau\nabla y_\tau^{(k)} + \xi(\nabla y_\tau^{(k-1)}, \delta_\tau\nabla y_\tau^{(k)}, \theta_\tau^{(k-1)}) + \varepsilon|\delta_\tau\nabla\Delta y_\tau^{(k)}|^2 \wedge \tau^{-1}\big)\varphi\,\mathrm{d}x\,,
\end{aligned}
\tag{3.9}
$$

where we shortly write $w_\tau^{(k-1)} := W^{\mathrm{in}}(\nabla y_\tau^{(k-1)}, \theta_\tau^{(k-1)})$, $w_\tau^{(k)} := W^{\mathrm{in}}(\nabla y_\tau^{(k)}, \theta_\tau^{(k)})$, and $\delta_\tau w_\tau^{(k)} := \frac{w_\tau^{(k)} - w_\tau^{(k-1)}}{\tau}$.

Proof. The thermal step differs from the one used in [2] only by the regularization of the dissipation term $\xi(\nabla y_\tau^{(k-1)}, \delta_\tau\nabla y_\tau^{(k)}, \theta_\tau^{(k-1)})$ by $\varepsilon|\delta_\tau\nabla\Delta y_\tau^{(k)}|^2 \wedge \tau^{-1}$. Therefore, the statement immediately follows from [2, Proposition 3.8] since an inspection of its proof shows that for the existence of $\theta_\tau^{(k)}$ it is enough that the dissipation term lies in $L^\infty(\Omega)$ (see Steps 1 and 2) and for nonnegativity of $\theta_\tau^{(k)}$ it is enough that the dissipation is larger than $c|(\delta_\tau\nabla y_\tau^{(k)})^T\nabla y_\tau^{(k-1)} + (\nabla y_\tau^{(k-1)})^T\delta_\tau\nabla y_\tau^{(k)}|^2$, see [2, Remark 3.9] and (D.1). □

3.2. **Existence of time-discrete solutions.** In the previous subsection, we focused on one step in the staggered scheme. Our next goal is to prove the existence of time-discrete solutions and to derive first a priori bounds independent of τ, h, and ε. We set

$$C_f := \|f\|_{W^{1,1}(I;L^2(\Omega))}, \tag{3.10}$$

and note that by the fundamental theorem of calculus it holds that

$$\|f(t)\|_{L^2(\Omega)} \leq C_T C_f \quad \text{for all } t \in I, \tag{3.11}$$

where here and in the following C_T denotes a constant possibly depending on T.

Given the sequences $y_\tau^{(0)}, \ldots, y_\tau^{(k)}$ and $\theta_\tau^{(0)}, \ldots, \theta_\tau^{(k)}$ for some $k \in \{1, \ldots, T/\tau\}$, as described in Subsection 3.1, for $l \in \{0, \ldots, k\}$ we define

$$\mathcal{F}^{(l)} := \mathcal{E}(y_\tau^{(l)}, \theta_\tau^{(l)}) - (f(l\tau), y_\tau^{(l)})_2,$$

where the total internal energy $\mathcal{E}$ is defined in (2.16). Then, using (W.3) we find

$$|(f(l\tau), y_\tau^{(l)})_2| \leq \min\{\mathcal{F}^{(l)}, \mathcal{E}(y_\tau^{(l)}, \theta_\tau^{(l)})\} + C_T C_f^2 + C(1 + C_0), \tag{3.12}$$

see also [2, Lemma 3.10]. Finally, for $l \in \{0, \ldots, k\}$ we also define

$$\mathcal{G}^{(l)} := \mathcal{F}^{(l)} + \frac{\rho}{2}\frac{\tau}{h} \sum_{m=l-h/\tau+1}^{l} \|\delta_\tau y_\tau^{(m)}\|_{L^2(\Omega)}^2, \tag{3.13}$$

which corresponds to adding also a suitably averaged kinetic energy. The main result of this subsection reads as follows.

Proposition 3.5 (Existence of time-discrete solutions)**.** *Let C_f be as in* (3.10) *and $\mathcal{G}^{(0)}$ be as in* (3.13)*. For any $T > 0$, there exists a constant $\overline{C}_T > 0$, corresponding constants*

$$M' := 2e^{\overline{C}_T C_f}\Big(\mathcal{G}^{(0)} + \overline{C}_T(1 + C_f^3) + \kappa \int_I \int_{\partial\Omega} \theta_\flat \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t\Big), \qquad M := 2M' + \overline{C}_T C_f^2,$$

as well as a constant $C_M > 0$ and scalar $\tau_0 \in (0, h]$ only depending on M above such that the following holds true: For each $\tau \in (0, \tau_0)$ such that $h/\tau \in \mathbb{N}$ the sequences $y_\tau^{(0)}, \ldots, y_\tau^{(T/\tau)}$ and $\theta_\tau^{(0)}, \ldots, \theta_\tau^{(T/\tau)}$ constructed in Subsection 3.1 exist, and for all $k \in \{0, \ldots, T/\tau\}$ it holds that

$$\mathcal{E}(y_\tau^{(k)}, \theta_\tau^{(k)}) + \frac{\rho}{2}\frac{\tau}{h} \sum_{l=k-h/\tau+1}^{k} \|\delta_\tau y_\tau^{(l)}\|_{L^2(\Omega)}^2 \leq M, \tag{3.14}$$

$$\sum_{l=1}^{k} \tau\Big(\|\delta_\tau \nabla y_\tau^{(l)}\|_{L^2(\Omega)}^2 + \varepsilon\|\delta_\tau \nabla\Delta y_\tau^{(l)}\|_{L^2(\Omega)}^2\Big) \leq C_M(M(1+T) + \overline{C}_T C_f^2). \tag{3.15}$$

The first estimate corresponds to a bound on the total energy and the second one is a bound on the (regularized) strain rate. The proof relies on the following two lemmas.

Lemma 3.6 (Inductive bound on the total energy)**.** *For any $M, T > 0$ there exist constants $C_M > 0$ and $C_T > 0$ only depending on M and T, respectively, such that the following holds true: Suppose $\tau \in (0, 1)$ is chosen such that for $k \in \{1, \ldots, T/\tau\}$ the sequences $y_\tau^{(0)}, \ldots, y_\tau^{(k)}$ and $\theta_\tau^{(0)}, \ldots, \theta_\tau^{(k)}$ constructed in Subsection 3.1 exist. Moreover, assume that $\mathcal{G}^{(l)} \leq M$ for all $l \in \{0, \ldots, k-1\}$ with $\mathcal{G}^{(l)}$ as in* (3.13)*. Then, it holds that*

$$\mathcal{G}^{(k)} \leq \mathcal{G}^{(0)} + C_M \tau V_k + C_T(1 + C_f^3) + \kappa \int_0^{k\tau} \int_{\partial\Omega} \theta_\flat \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t + C \sum_{l=0}^{k} \mathcal{G}^{(l)} \int_{(l-1)\tau}^{(l+1)\tau} \|\partial_t f(t)\|_{L^2(\Omega)}\,\mathrm{d}t,$$

where $C > 0$ is a universal constant, and

$$V_k := \sum_{m=1}^{k} \tau \int_\Omega |\delta_\tau \nabla y_\tau^{(m)}|^2 \,\mathrm{d}x. \tag{3.16}$$

Lemma 3.7 (Inductive bound on the strain rates)**.** *Given $M, T > 0$, there exist a constant $C_M > 0$ and $\tau_0 \in (0, 1]$ only depending on M, and a constant $C_T > 0$ only depending on T such that for $\tau \in (0, \tau_0)$ the following holds: Suppose that the sequences $y_\tau^{(0)}, \ldots, y_\tau^{(k)}$ and $\theta_\tau^{(0)}, \ldots, \theta_\tau^{(k)}$ for some $k \in \{1, \ldots, T/\tau\}$*

constructed in Subsection 3.1 exist. Moreover, suppose that $\mathcal{G}^{(l)} \le M$ *for all* $l \in \{0, \dots, k-1\}$ *with* $\mathcal{G}^{(l)}$ *as defined in* (3.13)*. Then,*

$$\begin{aligned}&\sum_{l=1}^{k} \tau\Big(\|\delta_\tau \nabla y_\tau^{(l)}\|^2_{L^2(\Omega)} + \varepsilon\|\delta_\tau \nabla\Delta y_\tau^{(l)}\|^2_{L^2(\Omega)}\Big)\\&\quad\le C_M\Big(\mathcal{M}(y_{0,\varepsilon}) + \frac{\rho}{2}\|y'_{0,\varepsilon}\|^2_{L^2(\Omega)} + C_T C_f^2\Big) + C_M\tau\sum_{l=0}^{k-1}\big(1+\mathcal{M}(y_\tau^{(l)})\big).\end{aligned} \tag{3.17}$$

Proof of Proposition 3.5. Once Lemmas 3.6–3.7 are proved, the proof of Proposition 3.5 follows by an inductive argument using the discrete Gronwall's inequality and Propositions 3.3–3.4. We refer to [2, Theorem 3.13] for all details (cf. also [2, Lemma 3.11, Lemma 3.12]). ☐

We now proceed with the proof of the lemmas. As an auxiliary result, we show Λ-convexity of $W^{\rm el}$ and $W^{\rm cpl}$. The result is closely related to the estimate in [14, Subsection 2.3], which improved upon [35, Proposition 3.2], where *local* Λ-convexity has been shown.

Lemma 3.8 (Λ-convexity of $W^{\rm el}$ and $W^{\rm cpl}$)**.** *For any* $M > 0$ *there exists a constant* $C_M > 0$ *such that for all* $y_1, y_2 \in \mathcal{Y}_{\mathbf{id}}$ *with* $\mathcal{M}(y_1), \mathcal{M}(y_2) \le M$ *and* $\theta \in L^1(\Omega)$*, we have*

$$\int_\Omega W^{\rm el}(y_2)\,{\rm d}x \ge \int_\Omega W^{\rm el}(y_1)\,{\rm d}x + \int_\Omega \partial_F W^{\rm el}(\nabla y_1) : \big(\nabla y_2 - \nabla y_1\big)\,{\rm d}x - C_M\|\nabla y_2 - \nabla y_1\|^2_{L^2(\Omega)}.$$

$$\int_\Omega W^{\rm cpl}(y_2,\theta)\,{\rm d}x \ge \int_\Omega W^{\rm cpl}(y_1,\theta)\,{\rm d}x + \int_\Omega \partial_F W^{\rm cpl}(\nabla y_1,\theta) : \big(\nabla y_2 - \nabla y_1\big)\,{\rm d}x - C_M\|\nabla y_2 - \nabla y_1\|^2_{L^2(\Omega)}.$$

Proof. We first prove the statement for $W^{\rm el}$. By Lemma 3.2 there exists a constant C^*_M depending on M such that for all $x \in \Omega$ and $l \in \{1,2\}$ it holds that

$$|\nabla y_l(x)| \le C^*_M, \quad \det(\nabla y_l(x)) \ge \frac{1}{C^*_M}. \tag{3.18}$$

By the continuity of the determinant we can find $\delta_M > 0$ such that for all $\lambda \in (0,1)$ and F_1, $F_2 \in GL^+(d)$ with $|F_1|, |F_2| \le C^*_M$, $\det(F_1)$, $\det(F_2) \ge \frac{1}{C^*_M}$ and $|F_1 - F_2| \le \delta_M$ we have that $\det(\lambda F_1 + (1-\lambda)F_2) \ge \frac{1}{2C^*_M}$. The set

$$K := \Big\{F \in GL^+(d) \colon |F| \le C^*_M,\ \det(F) \ge \frac{1}{2C^*_M}\Big\}$$

is a compact subset of $GL^+(d)$. Hence, by the C^2-regularity of $W^{\rm el}$ it follows that

$$C_M := \sup_{F\in K} |\partial^2_{FF} W^{\rm el}(F)| < \infty.$$

With δ_M as above, let us define the set $G := \{x \in \Omega\colon |\nabla y_2(x) - \nabla y_1(x)| \le \delta_M\}$. Moreover, we define $y_t = (1-t)y_1 + t y_2$ for $t \in [0,1]$. By the previous reasoning, for every $x \in G$ it holds that $\nabla y_t(x) \in K$ and

$$|\partial_F W^{\rm el}(\nabla y_t(x)) - \partial_F W^{\rm el}(\nabla y_1(x))| \le C_M|\nabla y_t(x) - \nabla y_1(x)| \le C_M|\nabla y_2(x) - \nabla y_1(x)|. \tag{3.19}$$

On the one hand, by the Gateaux differentiability of $\mathcal{W}^{\rm el}$ (see [35, Proposition 3.2]) and the chain rule we have that

$$\int_G W^{\rm el}(\nabla y_2)\,{\rm d}x - \int_G W^{\rm el}(\nabla y_1)\,{\rm d}x = \int_0^1\int_G \partial_F W^{\rm el}(\nabla y_t) : \big(\nabla y_2 - \nabla y_1\big)\,{\rm d}x\,{\rm d}t.$$

On the other hand, using (3.19) we can estimate

$$\left|\int_0^1\int_G \partial_F W^{\rm el}(\nabla y_t) : \big(\nabla y_2 - \nabla y_1\big)\,{\rm d}x\,{\rm d}t - \int_0^1\int_G \partial_F W^{\rm el}(\nabla y_1) : \big(\nabla y_2 - \nabla y_1\big)\,{\rm d}x\,{\rm d}t\right| \le C_M\|\nabla y_2 - \nabla y_1\|^2_{L^2(\Omega)}.$$

By the definition of G we also see that, by possibly increasing C_M, it holds

$$\begin{aligned}&\Big|\int_{\Omega\setminus G} W^{\rm el}(\nabla y_2)\,{\rm d}x - \int_{\Omega\setminus G} W^{\rm el}(\nabla y_1)\,{\rm d}x - \int_{\Omega\setminus G} \partial_F W^{\rm el}(\nabla y_1) : \big(\nabla y_2 - \nabla y_1\big)\,{\rm d}x\Big|\\&\quad\le \int_{\Omega\setminus G} \frac{|W^{\rm el}(\nabla y_1)| + |W^{\rm el}(\nabla y_2)| + \delta_M|\partial_F W^{\rm el}(\nabla y_1)|}{\delta_M^2}|\nabla y_2 - \nabla y_1|^2\,{\rm d}x \le C_M\|\nabla y_2 - \nabla y_1\|^2_{L^2(\Omega)},\end{aligned} \tag{3.20}$$

where we used that $W^{\mathrm{el}}(\nabla y_1)$, $W^{\mathrm{el}}(\nabla y_2)$, and $|\partial_F W^{\mathrm{el}}(\nabla y_1)|$ are uniformly bounded by (3.18). The combination of the last three estimates gives the statement for W^{el}.

The argument for W^{cpl} is similar. However, the bounds in (3.19) and (3.20) do not follow immediately from the compactness of K due to the presence of the temperature $\theta \in L^1(\Omega)$. To obtain the analogous bounds, we use (C.4), the first inequality in (C.5), and (3.18). $\square$

As a second auxiliary result, we establish a bound on the mechanical energy.

Lemma 3.9 (Mechanical energy bound in the time-discrete setting)**.** *Given $M > 0$, there exists a constant $C_M > 0$ such that the following holds: Suppose that for $\tau \in (0, \tau_0)$ and $k \in \{1, \ldots, T/\tau\}$ the sequences $y_\tau^{(0)}, \ldots, y_\tau^{(k)}$ and $\theta_\tau^{(0)}, \ldots, \theta_\tau^{(k)}$ constructed in Subsection 3.1 exist, and that $\mathcal{G}^{(l)} \leq M$ for all $l \in \{0, \ldots, k-1\}$. Then, it holds that*

$$\mathcal{M}(y_\tau^{(k)}) + \frac{\rho\tau}{2h} \sum_{l=k-h/\tau+1}^{k} \|\delta_\tau y_\tau^{(l)}\|^2_{L^2(\Omega)} + \sum_{l=1}^{k} \tau \Big(2\mathcal{R}_\varepsilon(y_\tau^{(l-1)}, \delta_\tau y_\tau^{(l)}, \theta_\tau^{(l-1)}) + \frac{\rho}{2h} \|\delta_\tau y_\tau^{(l)} - \delta_\tau y_\tau^{(l-h/\tau)}\|^2_{L^2(\Omega)} \Big)$$
$$\leq \mathcal{M}(y_{0,\varepsilon}) + \frac{\rho}{2}\|y'_{0,\varepsilon}\|^2_{L^2(\Omega)} - \tau \sum_{l=1}^{k} \int_\Omega \partial_F W^{\mathrm{cpl}}(\nabla y_\tau^{(l-1)}, \theta_\tau^{(l-1)}) : \delta_\tau \nabla y_\tau^{(l)} \,\mathrm{d}x + \tau \sum_{l=1}^{k} (f_\tau^{(l)}, \delta_\tau y_\tau^{(l)})_2 + C_M \tau V_k,$$

where $\mathcal{R}_\varepsilon$ is given in (2.12) *and V_k in* (3.16)*.*

Proof. Using Proposition 3.3 for l in place of k, (2.11), and testing (3.6) with $z = \delta_\tau y_\tau^{(l)}$ it follows that

$$\begin{aligned} 0 = \int_\Omega & \partial_F W(\nabla y_\tau^{(l)}, \theta_\tau^{(l-1)}) : \delta_\tau \nabla y_\tau^{(l)} + DH(\Delta y_\tau^{(l)}) \cdot \delta_\tau \Delta y_\tau^{(l)} + \varepsilon \delta_\tau \nabla \Delta y_\tau^{(l)} : \delta_\tau \nabla \Delta y_\tau^{(l)} \,\mathrm{d}x \\ &+ \int_\Omega \xi(\nabla y_\tau^{(l-1)}, \delta_\tau \nabla y_\tau^{(l)}, \theta_\tau^{(l-1)}) \,\mathrm{d}x - \int_\Omega f_\tau^{(l)} \cdot \delta_\tau y_\tau^{(l)} \,\mathrm{d}x + \frac{\rho}{h} \int_\Omega \big(\delta_\tau y_\tau^{(l)} - \delta_\tau y_\tau^{(l-h/\tau)}\big) \cdot \delta_\tau y_\tau^{(l)} \,\mathrm{d}x. \end{aligned} \tag{3.21}$$

By the convexity of H (see (H.1)) it follows for $l \in \{1, \ldots, k\}$ that

$$\mathcal{H}(\Delta y_\tau^{(l-1)}) \geq \mathcal{H}(\Delta y_\tau^{(l)}) + \int_\Omega DH(\Delta y_\tau^{(l)}) \cdot (\Delta y_\tau^{(l-1)} - \Delta y_\tau^{(l)}) \,\mathrm{d}x = \mathcal{H}(\Delta y_\tau^{(l)}) - \tau \int_\Omega DH(\Delta y_\tau^{(l)}) \cdot \delta_\tau \Delta y_\tau^{(l)} \,\mathrm{d}x, \tag{3.22}$$

where we recall the notation in (2.3). We now perform a similar argument for the elastic energy. Using (3.10), (3.12)–(3.13), and $\mathcal{G}^{(l)} \leq M$ for $l \in \{0, \ldots, k-1\}$, we get

$$\mathcal{M}(y_\tau^{(l-1)}) \leq \mathcal{G}^{(l-1)} + (f((l-1)\tau), y_\tau^{(l-1)})_2 \leq 2\mathcal{G}^{(l-1)} + C + C_T C_f^2 \leq 2M + C + C_T C_f^2 \tag{3.23}$$

for all $l \in \{1, \ldots, k\}$. Thus, we can apply Proposition 3.3 for all $l \in \{1, \ldots, k\}$, where now C_M may also depend on T and f. Then, using again $\mathcal{G}^{(l)} \leq M$ for $l \in \{0, \ldots, k-1\}$, by (3.7) we find for all $l \in \{1, \ldots, k\}$ that

$$\mathcal{M}(y_\tau^{(l)}) \leq C_M\big(1 + \|f\|^2_{L^2(I\times\Omega)}\big) + 2\mathcal{G}^{(l-1)} \leq 2M + C_M(1 + C_T C_f^2). \tag{3.24}$$

Consequently, by (3.2), (3.23)–(3.24), and Lemma 3.8 applied for $y_1 = y_\tau^{(l)}$ and $y_2 = y_\tau^{(l-1)}$ we find

$$\tau \int_\Omega \partial_F W^{\mathrm{el}}(\nabla y_\tau^{(l)}) : \delta_\tau \nabla y_\tau^{(l)} \,\mathrm{d}x \geq \mathcal{W}^{\mathrm{el}}(y_\tau^{(l)}) - \mathcal{W}^{\mathrm{el}}(y_\tau^{(l-1)}) - C_M \tau^2 \int_\Omega |\delta_\tau \nabla y_\tau^{(l)}|^2 \,\mathrm{d}x \tag{3.25}$$

for a possibly larger C_M, where we recall the definition in (2.2). Multiplying (3.21) by τ, using (3.22) and (3.25), and summing over $l \in \{1, \ldots, k\}$, we conclude

$$\begin{aligned} \mathcal{M}(y_\tau^{(k)}) &+ \frac{\tau\rho}{h} \sum_{l=1}^{k} \int_\Omega \big(\delta_\tau y_\tau^{(l)} - \delta_\tau y_\tau^{(l-h/\tau)}\big) \cdot \delta_\tau y_\tau^{(l)} \,\mathrm{d}x \\ &+ \sum_{l=1}^{k} \tau \Big(\int_\Omega \xi(\nabla y_\tau^{(l-1)}, \delta_\tau \nabla y_\tau^{(l)}, \theta_\tau^{(l-1)}) \,\mathrm{d}x + \varepsilon \|\delta_\tau \nabla \Delta y_\tau^{(l)}\|^2_{L^2(\Omega)} \Big) \\ &\leq \mathcal{M}(y_{0,\varepsilon}) - \tau \sum_{l=1}^{k} \int_\Omega \partial_F W^{\mathrm{cpl}}(\nabla y_\tau^{(l)}, \theta_\tau^{(l-1)}) : \delta_\tau \nabla y_\tau^{(l)} \,\mathrm{d}x + \tau \sum_{l=1}^{k} (f_\tau^{(l)}, \delta_\tau y_\tau^{(l)})_2 + C_M \tau V_k, \end{aligned} \tag{3.26}$$

where we employed the definitions in (2.6) and (3.16). Using the identity

$$\Pi_l := \int_\Omega \big(\delta_\tau y_\tau^{(l)} - \delta_\tau y_\tau^{(l-h/\tau)}\big) \cdot \delta_\tau y_\tau^{(l)} \,\mathrm{d}x = \frac{1}{2}\big(\|\delta_\tau y_\tau^{(l)}\|^2_{L^2(\Omega)} - \|\delta_\tau y_\tau^{(l-h/\tau)}\|^2_{L^2(\Omega)} + \|\delta_\tau y_\tau^{(l)} - \delta_\tau y_\tau^{(l-h/\tau)}\|^2_{L^2(\Omega)}\big),$$

summing over $l \in \{1, \dots, k\}$, and recalling the definition $y_\tau^{(m)} = y_{0,\varepsilon} + m\tau y'_{0,\varepsilon}$ for $m \in \{-h/\tau, \dots, 0\}$ we derive that

$$\begin{aligned}\sum_{l=1}^k 2\tau \Pi_l &= \sum_{l=1}^k \tau \|\delta_\tau y_\tau^{(l)} - \delta_\tau y_\tau^{(l-h/\tau)}\|^2_{L^2(\Omega)} + \sum_{l=k-h/\tau+1}^k \tau \|\delta_\tau y_\tau^{(l)}\|^2_{L^2(\Omega)} - \sum_{l=-h/\tau+1}^0 \tau \|\delta_\tau y_\tau^{(l)}\|^2_{L^2(\Omega)} \\ &= \sum_{l=1}^k \tau \|\delta_\tau y_\tau^{(l)} - \delta_\tau y_\tau^{(l-h/\tau)}\|^2_{L^2(\Omega)} + \sum_{l=k-h/\tau+1}^k \tau \|\delta_\tau y_\tau^{(l)}\|^2_{L^2(\Omega)} - h\|y'_{0,\varepsilon}\|^2_{L^2(\Omega)}. \end{aligned} \tag{3.27}$$

We apply Lemma 3.8 first for $y_1 = y_\tau^{(l-1)}$ and $y_2 = y_\tau^{(l)}$, then for $y_1 = y_\tau^{(l)}$ and $y_2 = y_\tau^{(l-1)}$, and sum the equations to get

$$0 \geq \int_\Omega \Big(\partial_F W^{\rm cpl}(\nabla y_\tau^{(l)}, \theta_\tau^{(l-1)}) - \partial_F W^{\rm cpl}(\nabla y_\tau^{(l-1)}, \theta_\tau^{(l-1)})\Big) : \big(\nabla y_\tau^{(l-1)} - \nabla y_\tau^{(l)}\big)\,\mathrm{d}x - 2C_M\tau^2 \int_\Omega |\delta_\tau \nabla y_\tau^{(l)}|^2\,\mathrm{d}x.$$

Rearranging and summing over $l \in \{1, \dots, k\}$ yields

$$\tau \sum_{l=1}^k \int_\Omega \Big(\partial_F W^{\rm cpl}(\nabla y_\tau^{(l)}, \theta_\tau^{(l-1)}) - \partial_F W^{\rm cpl}(\nabla y_\tau^{(l-1)}, \theta_\tau^{(l-1)})\Big) : \delta_\tau \nabla y_\tau^{(l)}\,\mathrm{d}x \geq -2C_M\tau V_k. \tag{3.28}$$

Eventually, recalling (2.11)–(2.12), the combination of (3.26), (3.27), and (3.28) concludes the proof. □

We now proceed with the proofs of Lemma 3.6 and Lemma 3.7.

Proof of Lemma 3.6. Recalling the argument in (3.23), we observe that Proposition 3.4 is applicable by passing to a larger value of M. We test (3.9) (for l in place of k) with $\varphi = 1$ to obtain

$$\begin{aligned} 0 = \int_\Omega \Big(&\delta_\tau w_\tau^{(l)} - \partial_F W^{\rm cpl}(\nabla y_\tau^{(l-1)}, \theta_\tau^{(l-1)}) : \delta_\tau \nabla y_\tau^{(l)} - \xi(\nabla y_\tau^{(l-1)}, \delta_\tau \nabla y_\tau^{(l)}, \theta_\tau^{(l-1)})\,\mathrm{d}x - \varepsilon|\delta_\tau \nabla \Delta y_k^{(\tau)}|^2 \wedge \tau^{-1}\Big)\,\mathrm{d}x \\ &+ \kappa \int_{\partial\Omega} (\theta_\tau^{(l)} - \theta_{\flat,\tau}^{(l)})\,\mathrm{d}\mathcal{H}^{d-1}. \end{aligned}$$

Multiplying this equation by τ, summing over $l \in \{1, \dots, k\}$, and adding to the estimate in Lemma 3.9, by (2.11)–(2.12) we discover that

$$\begin{aligned} &\mathcal{M}(y_\tau^{(k)}) + \frac{\rho\tau}{2h} \sum_{l=k-h/\tau+1}^k \|\delta_\tau y_\tau^{(l)}\|^2_{L^2(\Omega)} + \int_\Omega w_\tau^{(k)}\,\mathrm{d}x \\ &\leq \mathcal{M}(y_{0,\varepsilon}) + \int_\Omega w_{0,\varepsilon}\,\mathrm{d}x + \frac{\rho}{2}\|y'_{0,\varepsilon}\|^2_{L^2(\Omega)} + \tau \sum_{l=1}^k (f_\tau^{(l)}, \delta_\tau y_\tau^{(l)})_2 + \tau \sum_{l=1}^k \kappa \int_{\partial\Omega} (\theta_{\flat,\tau}^{(l)} - \theta_\tau^{(l)})\,\mathrm{d}\mathcal{H}^{d-1} + C_M\tau V_k, \end{aligned}$$

where $w_{0,\varepsilon} = W^{\rm in}(\nabla y_{0,\varepsilon}, \theta_0)$. Recalling the definition of $\mathcal{E}$ in (2.16), we conclude that

$$\begin{aligned} &\mathcal{E}(y_\tau^{(k)}, \theta_\tau^{(k)}) + \frac{\rho\tau}{2h} \sum_{l=k-h/\tau+1}^k \|\delta_\tau y_\tau^{(l)}\|^2_{L^2(\Omega)} \\ &\leq \mathcal{E}(y_{0,\varepsilon}, \theta_0) + \frac{\rho}{2}\|y'_{0,\varepsilon}\|^2_{L^2(\Omega)} + C_M\tau V_k + \tau \sum_{l=1}^k (f_\tau^{(l)}, \delta_\tau y_\tau^{(l)})_2 + \tau \sum_{l=1}^k \kappa \int_{\partial\Omega} (\theta_{\flat,\tau}^{(l)} - \theta_\tau^{(l)})\,\mathrm{d}\mathcal{H}^{d-1}. \end{aligned} \tag{3.29}$$

Now, we estimate the last two terms on the right-hand side of (3.29). By the nonnegativity of $\theta_\tau^{(l)}$ and the definition of $\theta_{\flat,\tau}^{(l)}$ we can bound

$$\tau \sum_{l=1}^k \kappa \int_{\partial\Omega} (\theta_{\flat,\tau}^{(l)} - \theta_\tau^{(l)})\,\mathrm{d}\mathcal{H}^{d-1} \leq \tau \sum_{l=1}^k \kappa \int_{\partial\Omega} \theta_{\flat,\tau}^{(l)}\,\mathrm{d}\mathcal{H}^{d-1} = \kappa \int_0^{k\tau} \int_{\partial\Omega} \theta_\flat\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t. \tag{3.30}$$

We define the piecewise affine function $\hat{y}_\tau(t) = \frac{t-(l-1)\tau}{\tau} y_\tau^{(l)} + \frac{l\tau - t}{\tau} y_\tau^{(l-1)}$ for $t \in [(l-1)\tau, l\tau]$ and $l \in \{1, \dots, k\}$, and note that $\delta_\tau y_\tau^{(l)} = \partial_t \hat{y}_\tau(t)$ for $t \in ((l-1)\tau, l\tau)$. Consequently, integration by parts yields

$$\begin{aligned} \sum_{l=1}^k \tau (f_\tau^{(l)}, \delta_\tau y_\tau^{(l)})_2 &= \int_0^{k\tau} (f(t), \partial_t \hat{y}_\tau(t))_2 \,\mathrm{d}t = (f(k\tau), \hat{y}_\tau(k\tau))_2 - (f(0), y_{0,\varepsilon})_2 - \int_0^{k\tau} (\partial_t f(t), \hat{y}_\tau(t))_2 \,\mathrm{d}t \\ &\le (f(k\tau), \hat{y}_\tau(k\tau))_2 - (f(0), y_{0,\varepsilon})_2 + \int_0^{k\tau} \|\partial_t f(t)\|_{L^2(\Omega)} \|\hat{y}_\tau(t)\|_{L^2(\Omega)} \,\mathrm{d}t. \end{aligned} \tag{3.31}$$

By Poincaré's inequality and (W.3) we have for every $t \in ((l-1)\tau, l\tau)$ that

$$\|\hat{y}_\tau(t)\|^2_{L^2(\Omega)} \le C(\|\nabla y_\tau^{(l-1)}\|^2_{L^2(\Omega)} + \|\nabla y_\tau^{(l)}\|^2_{L^2(\Omega)}) \le C\big(1 + \mathcal{W}^{\mathrm{el}}(y_\tau^{(l-1)}) + \mathcal{W}^{\mathrm{el}}(y_\tau^{(l)})\big).$$

Therefore, by (3.11), (3.12), the definition of $\mathcal{G}^{(l)}$ in (3.13), and $\sqrt{s} \le 1 + s$ for all $s \ge 0$ we get

$$\begin{aligned} \int_0^{k\tau} \|\partial_t f(t)\|_{L^2(\Omega)} \|\hat{y}_\tau(t)\|_{L^2(\Omega)} \,\mathrm{d}t &\le C \sum_{l=1}^k \Big(1 + \mathcal{W}^{\mathrm{el}}(y_\tau^{(l-1)}) + \mathcal{W}^{\mathrm{el}}(y_\tau^{(l)})\Big) \int_{(l-1)\tau}^{l\tau} \|\partial_t f(t)\|_{L^2(\Omega)} \,\mathrm{d}t \\ &\le C \sum_{l=1}^k \Big((\mathcal{G}^{(l-1)} + \mathcal{G}^{(l)}) \int_{(l-1)\tau}^{l\tau} \|\partial_t f(t)\|_{L^2(\Omega)} \,\mathrm{d}t \Big) + C_T(C_f + C_f^3). \end{aligned}$$

Then, using an index shift and $C_f \le \frac{2}{3} + \frac{1}{3} C_f^3$ we get

$$\int_0^{k\tau} \|\partial_t f(t)\|_{L^2(\Omega)} \|\hat{y}_\tau(t)\|_{L^2(\Omega)} \,\mathrm{d}t \le C \sum_{l=0}^k \Big(\mathcal{G}^{(l)} \int_{(l-1)\tau}^{l\tau} \big(\|\partial_t f(t)\|_{L^2(\Omega)} + \|\partial_t f(t+\tau)\|_{L^2(\Omega)}\big) \,\mathrm{d}t\Big) + C_T(1 + C_f^3)$$

for a possibly larger $C_T > 0$. Plugging this into (3.31) and using (3.30) to estimate the terms on the right-hand side of (3.29), we conclude the proof by the definition of $\mathcal{G}^{(l)}$ in (3.13). □

Proof of Lemma 3.7. Applying Lemma 3.9 and using the nonnegativity of $\mathcal{M}$ we get

$$\begin{aligned} &\sum_{l=1}^k \tau \Big(\int_\Omega \xi(\nabla y_\tau^{(l-1)}, \delta_\tau \nabla y_\tau^{(l)}, \theta_\tau^{(l-1)}) \,\mathrm{d}x + \varepsilon \|\delta_\tau \nabla \Delta y_\tau^{(l)}\|^2_{L^2(\Omega)} \Big) \\ &\le \mathcal{M}(y_{0,\varepsilon}) + \frac{\rho}{2} \|y'_{0,\varepsilon}\|^2_{L^2(\Omega)} - \tau \sum_{l=1}^k \int_\Omega \partial_F W^{\mathrm{cpl}}(\nabla y_\tau^{(l-1)}, \theta_\tau^{(l-1)}) : \delta_\tau \nabla y_\tau^{(l)} \,\mathrm{d}x + \tau \sum_{l=1}^k (f_\tau^{(l)}, \delta_\tau y_\tau^{(l)})_2 + C_M \tau V_k. \end{aligned} \tag{3.32}$$

As $\mathcal{M}(y_\tau^{(l-1)}) \le 2M + C + C_T C_f^2$ for all $l \in \{1, \dots, k\}$, see the argument in (3.23), employing also Lemma 3.2 we can apply the generalized Korn's inequality in the form [35, Corollary 3.4], leading to

$$\int_\Omega \xi(\nabla y_\tau^{(l-1)}, \delta_\tau \nabla y_\tau^{(l)}, \theta_\tau^{(l-1)}) \,\mathrm{d}x \ge \frac{1}{C_M} \|\delta_\tau \nabla y_\tau^{(l)}\|^2_{L^2(\Omega)}, \tag{3.33}$$

for a constant C_M depending on M, T, and f. By Hölder's inequality, Poincaré's inequality, and Young's inequality with constant $\lambda \in (0,1)$ we derive that

$$|(f_\tau^{(l)}, \delta_\tau y_\tau^{(l)})_2| \le C \|f_\tau^{(l)}\|_{L^2(\Omega)} \|\delta_\tau \nabla y_\tau^{(l)}\|_{L^2(\Omega)} \le \frac{C}{\lambda} \|f_\tau^{(l)}\|^2_{L^2(\Omega)} + \lambda \|\delta_\tau \nabla y_\tau^{(l)}\|^2_{L^2(\Omega)}. \tag{3.34}$$

Choosing $\lambda < \frac{1}{2C_M}$ above, summing over $l \in \{1, \dots, k\}$ in (3.33)–(3.34), and plugging into (3.32), we find

$$\begin{aligned} &\frac{1}{2C_M} V_k + \sum_{l=1}^k \tau\varepsilon \|\delta_\tau \nabla \Delta y_\tau^{(l)}\|^2_{L^2(\Omega)} = \frac{1}{2C_M} \sum_{l=1}^k \tau \|\delta_\tau \nabla y_\tau^{(l)}\|^2_{L^2(\Omega)} + \sum_{l=1}^k \tau\varepsilon \|\delta_\tau \nabla \Delta y_\tau^{(l)}\|^2_{L^2(\Omega)} \\ &\le \mathcal{M}(y_{0,\varepsilon}) + \frac{\rho}{2} \|y'_{0,\varepsilon}\|^2_{L^2(\Omega)} - \tau \sum_{l=1}^k \int_\Omega \partial_F W^{\mathrm{cpl}}(\nabla y_\tau^{(l-1)}, \theta_\tau^{(l-1)}) : \delta_\tau \nabla y_\tau^{(l)} \,\mathrm{d}x + C_M \tau V_k + C_M C_T C_f^2, \end{aligned} \tag{3.35}$$

where we also used (3.11). Since $|\partial_F W^{\mathrm{cpl}}(F,\theta)| \le 2C_0(1+|F|) \le 2C_0(2+|F|^2)$ for all $F \in GL^+(d)$ and $\theta > 0$, see [2, Lemma 3.4], applying Young's inequality, (W.3), and (2.6) we get for some $C_M^* > 0$ that

$$\tau \sum_{l=1}^{k} \int_\Omega \partial_F W^{\mathrm{cpl}}(\nabla y_\tau^{(l-1)}, \theta_\tau^{(l-1)}) : \delta_\tau \nabla y_\tau^{(l)} \,\mathrm{d}x \le \frac{1}{8C_M} \sum_{l=1}^{k} \tau \|\delta_\tau \nabla y_\tau^{(l)}\|^2_{L^2(\Omega)} + C_M^* \tau \sum_{l=1}^{k} \int_\Omega (1 + |\nabla y_\tau^{(l-1)}|^2) \,\mathrm{d}x$$
$$\le \frac{1}{8C_M} V_k + C_M^* \tau \sum_{l=1}^{k} \left(1 + \mathcal{M}(y_\tau^{(l-1)})\right).$$

Plugging this into (3.35), and possibly decreasing τ_0 such that $C_M^2 \tau_0 \le \frac{1}{8}$ holds true, we get

$$\frac{1}{4C_M} V_k + \sum_{l=1}^{k} \tau\varepsilon \|\delta_\tau \nabla \Delta y_\tau^{(l)}\|^2_{L^2(\Omega)} \le \mathcal{M}(y_{0,\varepsilon}) + \frac{\rho}{2}\|y'_{0,\varepsilon}\|^2_{L^2(\Omega)} + C_M \tau \sum_{l=1}^{k} \left(1 + \mathcal{M}(y_\tau^{(l-1)})\right) + C_M C_T C_f^2$$

for $C_M > 0$ sufficiently large. Multiplying both sides with $4C_M$ then leads to the desired estimate (3.17). ☐

3.3. **A priori bounds, compactness, and regularity.** Given $y_\tau^{(0)}, \dots, y_\tau^{(T/\tau)}$ and $\theta_\tau^{(0)}, \dots, \theta_\tau^{(T/\tau)}$ from Proposition 3.5, we define the following interpolations: for $k \in \{-h/\tau, \dots, T/\tau\}$, let $\overline{y}_\tau(k\tau) = \underline{y}_\tau(k\tau) = \hat{y}_\tau(k\tau) := y_\tau^{(k)}$ and for $t \in ((k-1)\tau, k\tau)$ let

$$\overline{y}_\tau(t) := y_\tau^{(k)}, \qquad \underline{y}_\tau(t) := y_\tau^{(k-1)}, \qquad \hat{y}_\tau(t) := \frac{k\tau - t}{\tau} y_\tau^{(k-1)} + \frac{t-(k-1)\tau}{\tau} y_\tau^{(k)}. \tag{3.36}$$

A similar notation is employed for $\overline{\theta}_\tau$, $\underline{\theta}_\tau$, and $\hat{\theta}_\tau$, for $\overline{w}_\tau$, $\underline{w}_\tau$, and $\hat{w}_\tau$, and for $\overline{f}_\tau$. The next proposition lists several a priori bounds for the sequences of interpolations.

Proposition 3.10 (A priori bounds and compactness). *Let T, h, $\varepsilon > 0$ and τ_0 be as in Proposition 3.5. Then, there exists a constant C only depending on T, $y_{0,\varepsilon}$, $y'_{0,\varepsilon}$, θ_0, f, and $\theta_\flat$ such that for all $\tau \in (0,\tau_0)$ with $T/h \in \mathbb{N}$ and $h/\tau \in \mathbb{N}$ the following bounds hold true:*

$$\|\overline{y}_\tau\|_{L^\infty(I_h;W^{2,p}(\Omega))} + \|\det(\nabla \overline{y}_\tau)^{-1}\|_{L^\infty(I_h\times\Omega)} \le C, \tag{3.37a}$$
$$\|\hat{y}_\tau\|_{H^1(I_h;H^1(\Omega))} + \sup_{t\in[0,T]} \fint_{t-h}^{t} \|\partial_t \hat{y}_\tau(s)\|^2_{L^2(\Omega)} \,\mathrm{d}s + \sqrt{\varepsilon}\|\partial_t \hat{y}_\tau\|_{L^2(I_h;H^3(\Omega))} \le C, \tag{3.37b}$$
$$\|\overline{\theta}_\tau\|_{L^\infty(I;L^1(\Omega))} + \|\overline{w}_\tau\|_{L^\infty(I;L^1(\Omega))} \le C. \tag{3.37c}$$

Moreover, for each $q \in [1, \frac{d+2}{d})$ and $r \in [1, \frac{d+2}{d+1})$ we can find constants $C_q > 0$ and $C_r > 0$ such that

$$\|\overline{\theta}_\tau\|_{L^q(I\times\Omega)} + \|\overline{w}_\tau\|_{L^q(I\times\Omega)} \le C_q, \tag{3.37d}$$
$$\|\nabla\overline{\theta}_\tau\|_{L^r(I\times\Omega)} + \|\nabla\overline{w}_\tau\|_{L^r(I\times\Omega)} \le C_r. \tag{3.37e}$$

Moreover, there exist $y_h \in C(I_h; \mathcal{Y}_{\mathbf{id}}) \cap H^1(I_h; H^3(\Omega;\mathbb{R}^d))$ and $\theta_h \in L^1(I;W^{1,1}(\Omega))$ with $y_h(t) = y_{0,\varepsilon} + t y'_{0,\varepsilon}$ for all $t \in [-h,0]$ and $\theta \ge 0$ a.e. such that, up to taking subsequences (not relabeled), as $\tau \to 0$ it holds that

$$\hat{y}_\tau \rightharpoonup y_h \quad \textit{weakly in } H^1(I_h;H^3(\Omega;\mathbb{R}^d)), \tag{3.38a}$$
$$\overline{y}_\tau \to y_h \quad \textit{strongly in } L^\infty(I_h;W^{1,\infty}(\Omega;\mathbb{R}^d)) \quad \textit{and strongly in } L^\infty(I_h;W^{2,p}(\Omega;\mathbb{R}^d)), \tag{3.38b}$$
$$\overline{\theta}_\tau \rightharpoonup \theta_h \quad \textit{and} \quad \overline{w}_\tau \rightharpoonup w_h \quad \textit{weakly in } L^r(I;W^{1,r}(\Omega)) \textit{ for any } r \in [1, \tfrac{d+2}{d+1}), \tag{3.38c}$$
$$\overline{\theta}_\tau \to \theta_h \quad \textit{and} \quad \overline{w}_\tau \to w_h \quad \textit{strongly in } L^s(I\times\Omega) \textit{ for any } s \in [1, \tfrac{d+2}{d}), \tag{3.38d}$$

where $w_h := W^{\mathrm{in}}(\nabla y_h, \theta_h)$. Note that the convergence in (3.38b) *also hold true for $\underline{y}_\tau$ or $\hat{y}_\tau$ instead of $\overline{y}_\tau$. Moreover, the convergences in* (3.38c) *and* (3.38d) *remain true after replacing $\overline{\theta}_\tau$ ($\overline{w}_\tau$) with $\underline{\theta}_\tau$ ($\underline{w}_\tau$) or $\hat{\theta}_\tau$ ($\hat{w}_\tau$), respectively.*

Proof. Except for the estimate on $\|\partial_t \hat{y}_\tau\|_{L^2(I_h;H^3(\Omega))}$, the bounds (3.37a)–(3.37c) are a direct consequence of the a priori bounds (3.14)–(3.15) from Proposition 3.5, Lemma 3.2, (2.15), and our definition of the different interpolations in time, where we particularly use that $\hat{y}_\tau(t) = y_{0,\varepsilon} + t y'_{0,\varepsilon}$ for $t \in (-h,0)$. The remaining estimate in (3.37b) is based on the bound

$$\|\partial_t \nabla \hat{y}_\tau\|_{L^2(I_h\times\Omega)} + \sqrt{\varepsilon}\|\partial_t \nabla\Delta \hat{y}_\tau\|_{L^2(I_h\times\Omega)} \le C \tag{3.39}$$

provided by (3.15). Elliptic regularity for the operator Δ and the fact that $\partial_t \hat{y}_\tau = 0$ on $I_h \times \partial\Omega$ imply

$$\|\partial_t \hat{y}_\tau\|_{L^2(I_h;H^3(\Omega))} \le C \|\partial_t \Delta \hat{y}_\tau\|_{L^2(I_h;H^1(\Omega))}. \tag{3.40}$$

Eventually, we use the interpolation inequality $\|\Delta v\|_{L^2(\Omega)} \le C\|\nabla \Delta v\|_{L^2(\Omega)} + C\|\nabla v\|_{L^2(\Omega)}$ for all $v \in H^3(\Omega;\mathbb{R}^d)$ which can be shown by the standard contradiction-compactness argument. This along with (3.39)–(3.40) indeed yields the remaining bound in (3.37b).

The proof of (3.37d)–(3.37e) relies on a proof of a weighted L^2-bound on the temperature gradient, namely

$$\int_0^T \int_\Omega \frac{\eta}{(1+\overline{w}_\tau)^{1+\eta}} |\nabla \overline{w}_\tau|^2 \,\mathrm{d}x\,\mathrm{d}t \le C \tag{3.41}$$

for any $\eta \in (0,1)$, where C is a constant only depending on T. We refer to e.g. [2, Theorem 3.20] for further details. It thus remains to show (3.41). As noticed in the proof of Proposition 3.4, our thermal step coincides with the one from [2] up to adding the term $\varepsilon|\delta_\tau \nabla \Delta y_\tau^{(k)}|^2 \wedge \tau^{-1}$ to the dissipation $\xi(\nabla y_\tau^{(k-1)}, \delta_\tau \nabla y_\tau^{(k)}, \theta_\tau^{(k-1)})$. The proof of the weighted L^2-bound (3.41) in [2, Lemma 3.19] relies only on a uniform L^1-bound on the dissipation. In view of (3.37b), a uniform L^1-bound (in τ, h, and ε) is still available in the current setting and the arguments in [2, Lemma 3.19] also apply here.

The a priori bounds along with a diagonal sequence argument show (3.38a)–(3.38d). More precisely, the convergence (3.38a) is a straightforward consequence of the a priori bound (3.37b) and Banach's selection principle. The convergences in (3.38b) follow from the embeddings $H^3(\Omega;\mathbb{R}^d) \subset\subset W^{2,p}(\Omega;\mathbb{R}^d) \subset\subset W^{1,\infty}(\Omega;\mathbb{R}^d)$ (recall that $p \in (3,6)$ for $d = 3$), and (3.37b) along with the Aubin-Lions' lemma. Next, (3.38c) follows from (3.37d)–(3.37e). Eventually, the convergence in (3.38d) and the identification $w_h = W^{\mathrm{in}}(\nabla y_h, \theta_h)$ can be shown using the bounds (3.37d)–(3.37e), the Aubin-Lions' lemma, and interpolation with the bound in (3.37c). For further details we refer, e.g., to [2, Lemma 4.2]. □

We proceed with further regularity results for $\hat{y}_\tau$ which hinge on a special case of elliptic regularity, see Lemma A.1 in the appendix. This will allow us to prove better regularity for the temperature variable, see Corollary 3.13 below, and it will also be instrumental for the passage $\varepsilon \to 0$ in Section 5.

Proposition 3.11 (Higher regularity of the deformation). *Let T, h, $\varepsilon > 0$ and $\tau \in (0,\tau_0)$ be as in Proposition 3.5. Then, $\hat{y}_\tau \in L^2(I;H^4(\Omega;\mathbb{R}^d))$, $\partial_t \hat{y}_\tau \in L^2(I;H^5(\Omega;\mathbb{R}^d))$, and for each $t \in I$ the time derivative $\partial_t \hat{y}_\tau$ satisfies the boundary conditions*

$$\partial_\nu \Delta \partial_t \hat{y}_\tau(t) = \Delta \partial_t \hat{y}_\tau(t) = \Delta^2 \partial_t \hat{y}_\tau(t) = 0 \qquad \mathcal{H}^{d-1}\text{-a.e. in } \partial\Omega. \tag{3.42}$$

Note that $\hat{y}_\tau$ only has H^4-regularity in space due to the regularity of the initial condition $y_{0,\varepsilon}$, see (2.20). The next lemma provides some useful bounds on $\Delta \hat{y}_\tau$ and on $\Delta^2 \hat{y}_\tau$, which will be particularly crucial for passing to the limit $\varepsilon \to 0$ in Section 5. Define

$$\varrho = \frac{2p}{4p-(p-2)d} \tag{3.43}$$

and note that $\varrho \in (\frac{1}{2}, 1)$ since $p > 2$ for $d = 2$ and $p \in (3,6)$ for $d = 3$.

Lemma 3.12 (Bounds from regularity). *Let T, h, $\varepsilon > 0$. Then, for τ_0 sufficiently small depending on ε and h such that Proposition 3.5 is applicable and for $\tau \in (0,\tau_0)$, there exists a constant $C > 0$ only depending on T, $y_{0,\varepsilon}$, $y'_{0,\varepsilon}$, θ_0, f, and $\theta_\flat$ such that*

$$\|\Delta \overline{y}_\tau\|_{L^2(I;H^1(\Omega))} + \||\Delta \overline{y}_\tau|^{\frac{p-2}{2}} \nabla(\Delta \overline{y}_\tau)\|_{L^2(I\times\Omega)} \le C\big(1 + \sqrt{\varepsilon}\|y_{0,\varepsilon}\|_{H^4(\Omega)} + \|y'_{0,\varepsilon}\|_{H^1(\Omega)}\big), \tag{3.44}$$

$$\|\,\Delta^2 \hat{y}_\tau\|_{L^2(I\times\Omega)} \le C\varepsilon^{-1/2}\big(1 + \sqrt{\varepsilon}\|y_{0,\varepsilon}\|_{H^4(\Omega)} + \|y'_{0,\varepsilon}\|_{H^1(\Omega)}\big), \tag{3.45}$$

$$\|\nabla DH(\Delta \overline{y}_\tau)\|_{L^2(I;L^{p'}(\Omega))} \le C\big(1 + \sqrt{\varepsilon}\|y_{0,\varepsilon}\|_{H^4(\Omega)} + \|y'_{0,\varepsilon}\|_{H^1(\Omega)}\big), \tag{3.46}$$

$$\|\,\Delta \partial_t \hat{y}_\tau\|_{L^2(I;H^2(\Omega))} \le C\varepsilon^{-\frac{1+\varrho}{2}}\big(1 + \sqrt[4]{\varepsilon}\|y_{0,\varepsilon}\|_{H^4(\Omega)} + \|y'_{0,\varepsilon}\|_{H^1(\Omega)}\big). \tag{3.47}$$

We postpone the proofs of the two results to the end of the subsection and first present the following consequence.

Corollary 3.13 (Further a priori bounds). *Let T, h, $\varepsilon > 0$ and τ_0 be as in Proposition 3.5. Then, there exists a constant $C_\varepsilon > 0$ only depending on T, ε, $y_{0,\varepsilon}$, $y'_{0,\varepsilon}$, θ_0, f, and $\theta_\flat$ such that for all $\tau \in (0,\tau_0)$ with $T/h \in \mathbb{N}$ and $h/\tau \in \mathbb{N}$ the following bounds hold true:*

$$\|\overline{\theta}_\tau\|_{L^2(I;H^1(\Omega))} + \|\overline{w}_\tau\|_{L^2(I;H^1(\Omega))} \le C_\varepsilon, \tag{3.48}$$

$$\|\hat{w}_\tau\|_{H^1(I;(H^1(\Omega))^*)} \le C_\varepsilon, \tag{3.49}$$

$$\|\hat{y}_\tau\|_{L^2(I;H^4(\Omega))} \le C_\varepsilon. \tag{3.50}$$

In particular, θ_h and w_h from Proposition 3.10 satisfy $\theta_h, w_h \in L^2(I; H^1(\Omega))$ and $w_h \in H^1(I; (H^1(\Omega))^*)$ with $w_h(0) = W^{\rm in}(\nabla y_{0,\varepsilon}, \theta_0)$. Moreover, y_h lies in $L^\infty(I; \mathcal{Y}^{\rm reg}_{\mathbf{id}})$.

Proof. First, by elliptic regularity along with the boundary conditions $\hat{y}_\tau = \mathbf{id}$ and $\Delta\hat{y}_\tau = 0$ on $I_h \times \partial\Omega$ (see (2.1), (2.20), and (3.42)), and the fact that Ω has C^5-boundary, we get

$$\|\hat{y}_\tau(t)\|_{H^4(\Omega)} \le C\|\Delta\hat{y}_\tau(t)\|_{H^2(\Omega)}, \qquad \|\Delta\hat{y}_\tau(t)\|_{H^2(\Omega)} \le C\|\Delta^2\hat{y}_\tau(t)\|_{L^2(\Omega)}$$

for a.e. $t \in I_h$. This along with (3.45) shows (3.50). Moreover, $y_h \in L^\infty(I; \mathcal{Y}^{\rm reg}_{\mathbf{id}})$ (see (2.20)) follows from (3.37b), (3.42), (3.47), and the fact that $y_{0,\varepsilon} \in \mathcal{Y}^{\rm reg}_{\mathbf{id}}$.

Due to (3.47), we find that $\varepsilon|\partial_t\nabla\Delta\hat{y}_\tau|^2$ is bounded in $L^2(I_h; L^2(\Omega))$ for a bound depending on ε, but independent of τ and h. Therefore, the term $\partial_F W^{\rm cpl}(\nabla\underline{y}_\tau, \underline{\theta}_\tau) : \partial_t\nabla\hat{y}_\tau + \xi(\nabla\underline{y}_\tau, \partial_t\nabla\hat{y}_\tau, \underline{\theta}_\tau) + \varepsilon|\partial_t\nabla\Delta\hat{y}_\tau|^2 \wedge \tau^{-1}$ appearing in the second line of (3.9) is bounded in $L^2(I_h; L^2(\Omega))$ for a bound depending on ε, but independent of τ and h. This regularity allows us to apply the a priori estimates in [35, Proposition 4.2]. This yields (3.48)–(3.49), and then the regularity of the limits θ_h and w_h is a direct consequence of weak compactness. By [4, Lemma 4.5(iii)] we get $\theta_h, w_h \in C(I; L^2(\Omega))$ which along with (C.1) and $y_h \in C(I_h; \mathcal{Y}_{\mathbf{id}})$ also shows $w_h(0) = W^{\rm in}(\nabla y_{0,\varepsilon}, \theta_0)$. This concludes the proof. □

We now come to the proofs of Proposition 3.11 and Lemma 3.12. As they are purely of technical nature, the reader might want to skip these proofs on first reading of the paper.

Proof of Proposition 3.11. We recall the notation in (3.36) and for convenience we drop the index τ in the entire proof. As a preliminary step, we first show $\hat{y} \in H^1(I; H^4(\Omega; \mathbb{R}^d))$, and afterwards the statement.

Step 1 ($\hat{y} \in H^1(I; H^4(\Omega; \mathbb{R}^d))$): We show that, if $\overline{y} \in L^2(I; W^{2,q}(\Omega; \mathbb{R}^d))$ for some $q \in [p, 2(p-1)]$, then

$$\hat{y} \in H^1(I; W^{4,\frac{q}{p-1}}(\Omega; \mathbb{R}^d)), \qquad \overline{y} \in L^2(I; W^{2,q(1+\eta)}(\Omega; \mathbb{R}^d)), \tag{3.51}$$

where $\eta := \infty$ for $d = 2$ and $\eta := \frac{5}{p-1} - 1 > 0$ for $d = 3$, as well as

$$\partial_\nu\Delta\partial_t\hat{y}(t) = 0 \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega. \tag{3.52}$$

Once this has been shown, this argument can first be applied for $q = p$ by (3.37a), and then by a bootstrapping argument, after a finite number of repetitions depending on η, we get $\hat{y} \in H^1(I; H^4(\Omega; \mathbb{R}^d))$.

Let us show (3.51)–(3.52). Suppose that $\overline{y} \in L^2(I; W^{2,q}(\Omega; \mathbb{R}^d))$ for some $q \in [p, 2(p-1)]$. For $t \in I$, we define $g^{\rm 3rd}(t) \in X_q^*$ in the dual space of $X_q := W^{2,(\frac{q}{p-1})'}(\Omega; \mathbb{R}^d) \cap H^1_0(\Omega; \mathbb{R}^d)$ by

$$\begin{aligned} \langle g^{\rm 3rd}(t), z\rangle := &\int_\Omega DH(\Delta\overline{y}(t)) \cdot \Delta z + \Big(\partial_F W(\nabla\overline{y}(t), \underline{\theta}(t)) + \partial_{\dot{F}} R(\nabla\underline{y}(t), \partial_t\nabla\hat{y}(t), \underline{\theta}(t))\Big) : \nabla z \,\mathrm{d}x \\ &- \int_\Omega \overline{f}(t) \cdot z \,\mathrm{d}x + \frac{\rho}{h}\int_\Omega (\partial_t\hat{y}(t) - \partial_t\hat{y}(t-h)) \cdot z \,\mathrm{d}x \end{aligned} \tag{3.53}$$

for all $z \in X_q$. By (H.3) and the assumption $\overline{y} \in L^2(I; W^{2,q}(\Omega; \mathbb{R}^d))$ we get $DH(\Delta\overline{y}) \in L^2(I; L^{\frac{q}{p-1}}(\Omega; \mathbb{R}^d))$ and thus $g^{\rm 3rd} \in L^2(I; X_q^*)$ by the bounds from Proposition 3.10. Fixing $z \in X_q \cap H^3(\Omega; \mathbb{R}^d)$ and using z as a test function in (3.6), we derive that

$$-\varepsilon\int_\Omega \nabla\Delta\partial_t\hat{y}(t) : \nabla\Delta z \,\mathrm{d}x = \langle g^{\rm 3rd}(t), z\rangle, \tag{3.54}$$

i.e., $g^{\rm 3rd}$ represents the regularization of third order. Due to (3.54), for every $t \in I$ we can apply Lemma A.1(a) for $u := \partial_t\hat{y}(t)$. With (A.2)–(A.3) this shows $\partial_t\hat{y} \in L^2(I; W^{4,\frac{q}{p-1}}(\Omega; \mathbb{R}^d))$ as well as (3.52). As $y_{0,\varepsilon} \in \mathcal{Y}^{\rm reg}_{\mathbf{id}}$, the above statements together with (2.20) directly lead to $\hat{y} \in H^1(I; W^{4,\frac{q}{p-1}}(\Omega; \mathbb{R}^d))$. By Sobolev embedding we get that $\overline{y} \in L^2(I; W^{2,r}(\Omega; \mathbb{R}^d))$, where $r = (\frac{q}{p-1})^{**}$. Since $q \ge p$ and thus $\frac{q}{p-1} \ge \frac{6}{5}$, we get $r = \infty$ for $d = 2$ and for $d = 3$ we have $r = (3\frac{q}{p-1})(3 - 2\frac{q}{p-1})^{-1} \ge \frac{5q}{p-1} = q(1+\eta)$. This concludes the proof of (3.51).

Step 2 (Proof of the statement): From now on we can suppose that $\hat{y} \in H^1(I; H^4(\Omega; \mathbb{R}^d))$. Due to this improved regularity of $\hat{y}$, we have for each $t \in I$ and any $z \in C_c^\infty(\Omega; \mathbb{R}^d)$ that

$$\int_\Omega DH(\Delta\overline{y}(t)) \cdot \Delta z \,\mathrm{d}x = -\int_\Omega \nabla(DH(\Delta\overline{y}(t))) : \nabla z \,\mathrm{d}x. \tag{3.55}$$

Recalling (2.4)–(2.5), an elementary computation for a general $v \in H^4(\Omega; \mathbb{R}^d)$ yields that pointwise a.e. in Ω it holds that

$$\nabla(DH(\Delta v)) = \begin{cases} 2\nabla\Delta v & \text{if } p|\Delta v|^{p-2} \le 2, \\ p(p-2)|\Delta v|^{p-4}\Delta v \otimes \big((\nabla\Delta v)^T\Delta v\big) + p|\Delta v|^{p-2}\nabla\Delta v & \text{else.} \end{cases} \tag{3.56}$$

Taking the Frobenius norm on both sides of (3.56) we see a.e. in Ω that

$$|\nabla(DH(\Delta v))| \leq \begin{cases} 2|\nabla\Delta v| & \text{if } p|\Delta v|^{p-2} \leq 2, \\ p(p-1)|\Delta v|^{p-2}|\nabla\Delta v| & \text{else.} \end{cases} \tag{3.57}$$

By Sobolev embedding we have that $H^2(\Omega;\mathbb{R}^d) \subset L^\infty(\Omega;\mathbb{R}^d)$ for $d = 2,3$. Hence, $\Delta\overline{y} \in L^\infty(I \times \Omega;\mathbb{R}^d)$. In particular, this shows $\nabla(DH(\Delta\overline{y})) \in L^2(I \times \Omega;\mathbb{R}^{d\times d})$ and therefore, recalling the definition in (3.53) and (3.55), we derive $g^{3\text{rd}} \in L^2(I;H^{-1}(\Omega;\mathbb{R}^d))$ by arbitrariness of z, where we have used that $C_c^\infty(\Omega;\mathbb{R}^d)$ is dense in $H_0^1(\Omega;\mathbb{R}^d)$. (Now, $\langle\cdot,\cdot\rangle$ stands for the dual pairing between $H_0^1(\Omega;\mathbb{R}^d)$ and $H^{-1}(\Omega;\mathbb{R}^d)$.) By Lemma A.1(b), (3.52), and the bounds from Proposition 3.10 we derive that $\partial_t\hat{y} \in L^2(I;H^5(\Omega;\mathbb{R}^d))$ and that $\partial_t\hat{y}$ satisfies

$$\|\partial_t\hat{y}\|_{L^2(I;H^5(\Omega))} \leq C_{\varepsilon,h} \tag{3.58}$$

for a constant $C_{\varepsilon,h}$ depending on ε and h. Moreover, we have the boundary condition

$$\Delta^2\partial_t\hat{y}(t) = 0 \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega \text{ for } t \in I. \tag{3.59}$$

To conclude the proof, it remains to show $\Delta\partial_t\hat{y}(t) = 0$ $\mathcal{H}^{d-1}$-a.e. on $\partial\Omega$ for $t \in I$. Defining for $z \in H := H^3(\Omega;\mathbb{R}^d) \cap H_0^1(\Omega;\mathbb{R}^d)$

$$\begin{aligned}\langle g^{2\text{nd}}(t), z\rangle := -\varepsilon\int_\Omega \nabla\Delta\partial_t\hat{y}(t) : \nabla\Delta z\,\mathrm{d}x - \int_\Omega \Big(\partial_F W(\nabla\overline{y}(t),\underline{\theta}(t)) + \partial_{\dot F} R(\nabla\underline{y}(t),\partial_t\nabla\hat{y}(t),\underline{\theta}(t))\Big) : \nabla z\,\mathrm{d}x \\ + \int_\Omega \overline{f}(t)\cdot z\,\mathrm{d}x - \frac{\rho}{h}\int_\Omega (\partial_t\hat{y}(t) - \partial_t\hat{y}(t-h))\cdot z\,\mathrm{d}x,\end{aligned}$$

we can rewrite (3.53)–(3.54) as

$$\int_\Omega DH(\Delta\overline{y}(t))\cdot\Delta z\,\mathrm{d}x = \langle g^{2\text{nd}}(t), z\rangle \tag{3.60}$$

for $t \in I$, i.e., $g^{2\text{nd}}$ represents the term with second derivative. Using the improved regularity of $\partial_t\hat{y}$ and the boundary conditions (3.52) and (3.59), the first integral in the definition of $g^{2\text{nd}}(t)$ can be written as

$$\begin{aligned}\int_\Omega \nabla\Delta\partial_t\hat{y}(t) : \nabla\Delta z\,\mathrm{d}x &= -\int_\Omega \Delta^2\partial_t\hat{y}(t)\cdot\Delta z\,\mathrm{d}x + \int_{\partial\Omega} \partial_\nu\Delta\partial_t\hat{y}(t)\cdot\Delta z\,\mathrm{d}\mathcal{H}^{d-1} \\ &= \int_\Omega \nabla\Delta^2\partial_t\hat{y}(t) : \nabla z\,\mathrm{d}x - \int_{\partial\Omega} \Delta^2\partial_t\hat{y}(t)\cdot\partial_\nu z\,\mathrm{d}\mathcal{H}^{d-1} = \int_\Omega \nabla\Delta^2\partial_t\hat{y}(t) : \nabla z\,\mathrm{d}x\end{aligned}$$

for each $z \in H$. As $\partial_t\hat{y} \in L^2(I;H^5(\Omega;\mathbb{R}^d))$, this shows $g^{2\text{nd}}(t) \in H^{-1}(\Omega;\mathbb{R}^d)$ for each $t \in I$. Now, it is standard to find $v(t) \in H_0^1(\Omega;\mathbb{R}^d)$ such that $\langle g^{2\text{nd}}(t), z\rangle = \int_\Omega v(t)\cdot\Delta z\,\mathrm{d}x$ for all $z \in H^2(\Omega;\mathbb{R}^d) \cap H_0^1(\Omega;\mathbb{R}^d)$, see (A.16)–(A.18) below for details. Given an arbitary $\varphi \in L^2(\Omega;\mathbb{R}^d)$ and choosing $z \in H^2(\Omega;\mathbb{R}^d) \cap H_0^1(\Omega;\mathbb{R}^d)$ with $\Delta z = \varphi$, this along with (3.60) shows

$$\int_\Omega \big(DH(\Delta\overline{y}(t)) - v(t)\big)\cdot\varphi\,\mathrm{d}x = 0.$$

This yields $DH(\Delta\overline{y}(t)) = v(t)$ a.e. in Ω and thus it holds that $DH(\Delta\overline{y}(t)) = 0$ $\mathcal{H}^{d-1}$-a.e. on $\partial\Omega$ for $t \in I$. Since

$$DH(v) = \max\{2, p|v|^{p-2}\}v = 0 \;\Leftrightarrow\; v = 0 \in \mathbb{R}^d$$

and $y_{0,\varepsilon} \in \mathcal{Y}_{\mathbf{id}}^{\text{reg}}$, this concludes the proof of (3.42). □

Proof of Lemma 3.12. For notational convenience, we define the weighted Bochner space

$$\|\cdot\|_{L^2_T(I;L^q(\Omega))} := \|\sqrt{T-t}\cdot\|_{L^2(I;L^q(\Omega))},$$

and employ a similar notation for Sobolev spaces. We first note that it is not restrictive to establish the bounds (3.44)–(3.47) only for the weighted space. Indeed, by extending $\theta_\flat$ and f suitably on the time interval $[T, T+\eta]$ for some $\eta > 0$ small, we can establish time-discrete solutions on the time interval $[-h, T+\eta]$, see Proposition 3.5. Then, a control on $\|\cdot\|_{L^2_{T+\eta}([0,T+\eta];L^q(\Omega))}$ will directly imply a control on $\|\cdot\|_{L^2(I;L^q(\Omega))}$ for a constant additionally depending on η. To simplify notation, we use the interval $I = [0,T]$ instead of $[0, T+\eta]$ in the sequel. As in the proof of Proposition 3.11, we omit the index τ.

Step 1 (Proof of (3.44) *and* (3.45)*):* As $\partial_t\hat{y} \in L^2(I;H^4(\Omega;\mathbb{R}^d))$ with $\partial_\nu\Delta\partial_t\hat{y}(t) = 0$ $\mathcal{H}^{d-1}$-a.e. on $\partial\Omega$ for $t \in I$ by Proposition 3.11, by an integration by part we can rewrite the ε-dependent term in (3.6) as

$$\int_\Omega \varepsilon\nabla\Delta\partial_t\hat{y} : \nabla\Delta z\,\mathrm{d}x = -\int_\Omega \varepsilon\Delta^2\partial_t\hat{y} : \Delta z\,\mathrm{d}x + \int_{\partial\Omega} \varepsilon\partial_\nu\Delta\partial_t\hat{y} : \Delta z\,\mathrm{d}\mathcal{H}^{d-1} = -\int_\Omega \varepsilon\Delta^2\partial_t\hat{y} : \Delta z\,\mathrm{d}x \tag{3.61}$$

for each $t \in ((k-1)\tau, k\tau)$ and $z \in H^3(\Omega;\mathbb{R}^d) \cap H^1_0(\Omega;\mathbb{R}^d)$. Thus, by approximation we can test (3.6) with functions in $H^2(\Omega;\mathbb{R}^d) \cap H^1_0(\Omega;\mathbb{R}^d)$. As $\Delta\hat{y} \in L^2(I;H^2(\Omega;\mathbb{R}^d))$ with $\Delta\hat{y}(t) = 0$ $\mathcal{H}^{d-1}$-a.e. on $\partial\Omega$ for $t \in I$ by Proposition 3.11 and the fact that $y_{0,\varepsilon} \in \mathcal{Y}^{\rm reg}_{\mathbf{id}}$ (see (2.20)), we discover that $z(t,x) := (T-t)\Delta\hat{y}(t,x)$ for $t \in ((k-1)\tau, k\tau)$ is a valid test function in (3.6). After summation and rearranging terms, and employing (3.61) this yields

$$\begin{aligned}
&\varepsilon \int_I \int_\Omega \Delta^2 \partial_t \hat{y} : \Delta^2 \hat{y}(T-t)\,\mathrm{d}x\,\mathrm{d}t - \int_I\int_\Omega DH(\Delta\overline{y}) \cdot \Delta^2\hat{y}(T-t)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_I\int_\Omega \big(\partial_F W(\nabla\overline{y},\underline{\theta}) + \partial_{\dot F} R(\nabla \underline{y}, \partial_t\nabla\hat{y},\underline{\theta})\big) : \nabla\Delta\hat{y}(T-t)\,\mathrm{d}x\,\mathrm{d}t \qquad (3.62)\\
&\qquad - \int_I\int_\Omega \overline{f}\cdot\Delta\hat{y}(t)(T-t)\,\mathrm{d}x\,\mathrm{d}t + \frac{\rho}{h}\int_I\int_\Omega \big(\partial_t\hat{y}(t) - \partial_t\hat{y}(t-h)\big)\cdot\Delta\hat{y}(t)(T-t)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

(The advantage of multiplying with $(T-t)$ will become apparent in (3.68) below.) Using (3.42) and $y_{0,\varepsilon} \in \mathcal{Y}^{\rm reg}_{\mathbf{id}}$ we integrate by parts in the second term on the left-hand side above, which leads to

$$\begin{aligned}
&-\int_I\int_\Omega DH(\Delta\overline{y})\cdot\Delta^2\hat{y}(T-t)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_I\int_\Omega \nabla(DH(\Delta\overline{y})) : \nabla\Delta\hat{y}(T-t)\,\mathrm{d}x\,\mathrm{d}t - \int_I\int_{\partial\Omega} DH(\Delta\overline{y})\cdot\partial_\nu\Delta\hat{y}(T-t)\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\
&\quad \geq \int_I\int_\Omega \nabla(DH(\Delta\overline{y})) : \nabla\Delta\overline{y}(T-t)\,\mathrm{d}x\,\mathrm{d}t - C\tau\|\nabla(DH(\Delta\overline{y}))\|_{L^2(I\times\Omega)}\|\nabla\Delta\partial_t\hat{y}\|_{L^2(I\times\Omega)},
\end{aligned}$$

where in the last step we exploited the definition in (3.36). In view of (3.56), we get that

$$\begin{aligned}
&\int_I\int_\Omega \nabla(DH(\Delta\overline{y})) : \nabla\Delta\overline{y}(T-t)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_I\int_\Omega \max\{2, p|\Delta\overline{y}|^{p-2}\}|\nabla\Delta\overline{y}|^2(T-t)\,\mathrm{d}x\,\mathrm{d}t + \int_I\int_\Omega \mathbb{1}_{\{p|\Delta\overline{y}|^{p-2}\geq 2\}} p(p-2)|(\nabla\Delta\overline{y})^T\Delta\overline{y}|^2|\Delta\overline{y}|^{p-4}(T-t)\,\mathrm{d}x\,\mathrm{d}t \\
&\geq \int_I\int_\Omega \max\{2, p|\Delta\overline{y}|^{p-2}\}|\nabla\Delta\overline{y}|^2(T-t)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

By $y_{0,\varepsilon} \in \mathcal{Y}^{\rm reg}_{\mathbf{id}}$ and (3.58) we find $\|\Delta\overline{y}\|_{L^\infty(I;L^\infty(\Omega))} \leq C_{\varepsilon,h}$ and $\|\overline{y}\|_{L^2(I;H^3(\Omega))} \leq C_{\varepsilon,h}$. Then, again using (3.56)–(3.57) and (3.37b) we eventually find

$$-\int_I\int_\Omega DH(\Delta\overline{y})\cdot\Delta^2\hat{y}(T-t)\,\mathrm{d}x\,\mathrm{d}t \geq \int_I\int_\Omega \max\{2,p|\Delta\overline{y}|^{p-2}\}|\nabla\Delta\overline{y}|^2(T-t)\,\mathrm{d}x\,\mathrm{d}t - \tau C_{\varepsilon,h}. \qquad (3.63)$$

By the chain rule we write

$$\begin{aligned}
\int_I\int_\Omega \Delta^2\partial_t\hat{y}\cdot\Delta^2\hat{y}(T-t)\,\mathrm{d}x\,\mathrm{d}t &= \int_I \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\int_\Omega |\Delta^2\hat{y}|^2(T-t)\,\mathrm{d}x\Big)\,\mathrm{d}t + \frac{1}{2}\int_I\int_\Omega |\Delta^2\hat{y}|^2\,\mathrm{d}x\,\mathrm{d}t \\
&= \frac{1}{2}\int_I\int_\Omega |\Delta^2\hat{y}|^2\,\mathrm{d}x\,\mathrm{d}t - \frac{T}{2}\|\Delta^2 y_{0,\varepsilon}\|^2_{L^2(\Omega)}. \qquad (3.64)
\end{aligned}$$

By the definition of $\mathcal{Y}_{\mathbf{id}}$ and $\hat{y} \in L^\infty(I_h;\mathcal{Y}_{\mathbf{id}})$, it follows that $\partial_t\hat{y}(t) = 0$ $\mathcal{H}^{d-1}$-a.e. in $\partial\Omega$ for $t \in I$. Hence, integrating by parts leads to

$$\int_I\int_\Omega (\partial_t\hat{y}(t) - \partial_t\hat{y}(t-h))\cdot\Delta\hat{y}(t)(T-t)\,\mathrm{d}x\,\mathrm{d}t = -\int_I\int_\Omega (\partial_t\nabla\hat{y}(t) - \partial_t\nabla\hat{y}(t-h)) : \nabla\hat{y}(t)(T-t)\,\mathrm{d}x\,\mathrm{d}t.$$

Now, using the substitution $t \mapsto t-h$ we derive

$$\begin{aligned}
\frac{\rho}{h}\int_I\int_\Omega (\partial_t\hat{y}(t) - \partial_t\hat{y}(t-h))\cdot\Delta\hat{y}(t)(T-t)\,\mathrm{d}x\,\mathrm{d}t &= \Pi + \rho\int_\Omega \nabla y'_{0,\varepsilon} : \fint_0^h \nabla\hat{y}(t)(T-t)\,\mathrm{d}t\,\mathrm{d}x \qquad (3.65)\\
&\quad - \rho\fint_{T-h}^T \int_\Omega \partial_t\nabla\hat{y}(t) : \nabla\hat{y}(t)(T-t)\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

where

$$\Pi := \rho\int_0^{T-h}(T-t)\int_\Omega \partial_t\nabla\hat{y}(t) : \frac{\nabla\hat{y}(t+h) - \nabla\hat{y}(t)}{h}\,\mathrm{d}x\,\mathrm{d}t - \rho\int_0^{T-h}\int_\Omega \partial_t\nabla\hat{y}(t) \colon \nabla\hat{y}(t+h)\,\mathrm{d}x\,\mathrm{d}t.$$

Using that $\frac{\nabla\hat{y}(t+h) - \nabla\hat{y}(t)}{h} = \fint_t^{t+h}\partial_t\nabla\hat{y}(s)\,\mathrm{d}s$ and applying Hölder's inequality, we can check that

$$|\Pi| \leq C\|\partial_t\nabla\hat{y}\|^2_{L^2(I\times\Omega)} + C\|\partial_t\nabla\hat{y}\|_{L^2(I\times\Omega)}\|\nabla\hat{y}\|_{L^2(I\times\Omega)} \leq C\|\nabla\hat{y}\|^2_{H^1(I;L^2(\Omega))}.$$

Combining (3.62)–(3.65) then leads to

$$\begin{aligned}
&\int_I\int_\Omega \max\{2,p|\Delta\overline{y}|^{p-2}\}|\nabla\Delta\overline{y}|^2(T-t)\,\mathrm{d}x\,\mathrm{d}t+\frac{\varepsilon}{2}\int_I\int_\Omega|\Delta^2\hat{y}|^2\,\mathrm{d}x\,\mathrm{d}t-\frac{T\varepsilon}{2}\|\Delta^2 y_{0,\varepsilon}\|^2_{L^2(\Omega)}\\
&\quad\le \int_I\int_\Omega\big(\partial_F W(\nabla\overline{y},\underline{\theta})+\partial_{\dot F}R(\nabla\underline{y},\partial_t\nabla\hat{y},\underline{\theta})\big):\nabla\Delta\hat{y}(T-t)\,\mathrm{d}x\,\mathrm{d}t-\int_I\int_\Omega\overline{f}\cdot\Delta\hat{y}(T-t)\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+\tau C_{\varepsilon,h}+C\|\nabla\hat{y}\|^2_{H^1(I;L^2(\Omega))}+\rho\int_\Omega\nabla y'_{0,\varepsilon}:\fint_0^h\nabla\hat{y}(t)(T-t)\,\mathrm{d}t\,\mathrm{d}x\\
&\qquad-\rho\fint_{T-h}^T\int_\Omega\partial_t\nabla\hat{y}(t):\nabla\hat{y}(t)(T-t)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}\tag{3.66}$$

In view of the bounds (3.37a)–(3.37e), we have that

$$\begin{aligned}
&\sup_{h>0,\,\varepsilon\in(0,1)}\|\partial_F W(\nabla\overline{y},\underline{\theta})+\partial_{\dot F}R(\nabla\underline{y},\partial_t\nabla\hat{y},\underline{\theta})\|_{L^2(I\times\Omega)}<+\infty\,,\\
&\sup_{h>0,\,\varepsilon\in(0,1)}\|\overline{y}\|_{L^\infty(I;W^{2,p}(\Omega))}+\|\hat{y}\|_{H^1(I;H^1(\Omega))}<+\infty\,,
\end{aligned}\tag{3.67}$$

where we use (W.1), (C.4), and (D.1). Moreover, we choose τ_0 small enough such that $\tau_0 C_{\varepsilon,h}\le 1$. Hence, by Hölder's inequality, the weighted Young's inequality, and by the boundary condition $\Delta\hat{y}(t)=0$ for a.e. $t\in[0,T]$ and $\mathcal{H}^{d-1}$-a.e. in $\partial\Omega$ (see (2.20) and Proposition 3.11), we infer from (3.66) that

$$\sqrt{\varepsilon}\|\Delta^2\hat{y}\|_{L^2(I\times\Omega)}+\|\Delta\overline{y}\|_{L^2_T(I;H^1(\Omega))}+\|\,|\Delta\overline{y}|^{\frac{p-2}{2}}\nabla(\Delta\overline{y})\|_{L^2_T(I;L^2(\Omega))}\le C\big(1+\sqrt{\varepsilon}\|y_{0,\varepsilon}\|_{H^4(\Omega)}+\|y'_{0,\varepsilon}\|_{H^1(\Omega)}\big)\tag{3.68}$$

for some constant $C>0$ independent of $\varepsilon\in(0,1)$ and $h>0$, where the notation $L^2_T(I;H^1(\Omega))$ has been introduced at the beginning of the proof. Here, the factor $(T-t)$ guarantees that the last term on the right-hand side of (3.67) can be controlled uniformly in h. This shows (3.44) and (3.45).

Step 2 (Proof of (3.46)*):* We proceed with (3.46). In view of (3.57), it holds that

$$|\,\nabla(DH(\Delta\overline{y}))|\le C\big(1+|\Delta\overline{y}|\big)^{\frac{p-2}{2}}\left|\big(1+|\Delta\overline{y}|\big)^{\frac{p-2}{2}}\nabla\Delta\overline{y}\right|$$

a.e. in Ω. By using Hölder's inequality for $\frac{p'}{2}+\frac{p-2}{2(p-1)}=1$ we estimate

$$\begin{aligned}
\|\nabla DH(\Delta\overline{y})\|^2_{L^2_T(I;L^{p'}(\Omega))}&\le C\int_I\left(\int_\Omega\big(1+|\Delta\overline{y}|\big)^{\frac{(p-2)p}{2(p-1)}}\big(\sqrt{T-t}\,\big(1+|\Delta\overline{y}|\big)^{\frac{p-2}{2}}|\nabla\Delta\overline{y}|\big)^{p'}\,\mathrm{d}x\right)^{\frac{2}{p'}}\,\mathrm{d}t\\
&\le C\|1+|\Delta\overline{y}|\|^{p-2}_{L^\infty(I;L^p(\Omega))}\|(1+|\Delta\overline{y}|)^{\frac{p-2}{2}}\nabla\Delta\overline{y}\|^2_{L^2_T(I;L^2(\Omega))}.
\end{aligned}\tag{3.69}$$

Note that $|\Delta\overline{y}|$ is uniformly bounded in $L^\infty(I;L^p(\Omega;\mathbb{R}^d))$ by (3.37a). Hence, (3.46) follows from (3.68).

Step 3 (Proof of (3.47)*):* Finally, we prove (3.47). In view of Corollary 3.11, we may test the mechanical equation (3.6) with $z=(T-t)\partial_t\Delta\hat{y}(t)$ for $t\in((k-1)\tau,k\tau)$. After summation and rearranging terms this yields

$$\begin{aligned}
&-\varepsilon\int_I\int_\Omega\partial_t\nabla\Delta\hat{y}:\partial_t\nabla\Delta^2\hat{y}(T-t)\,\mathrm{d}x\,\mathrm{d}t-\int_I\int_\Omega DH(\Delta\overline{y})\cdot\partial_t\Delta^2\hat{y}(T-t)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\int_I\int_\Omega\big(\partial_F W(\nabla\overline{y},\underline{\theta})+\partial_{\dot F}R(\nabla\underline{y},\partial_t\nabla\hat{y},\underline{\theta})\big):\partial_t\nabla\Delta\hat{y}(T-t)\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad-\int_I\int_\Omega\overline{f}\cdot\partial_t\Delta\hat{y}(T-t)\,\mathrm{d}x\,\mathrm{d}t+\frac{\rho}{h}\int_I\int_\Omega(\partial_t\hat{y}(t)-\partial_t\hat{y}(t-h))\cdot\partial_t\Delta\hat{y}(t)(T-t)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Integrating by parts the left-hand side and the last term on the right-hand side above, by the boundary conditions $\partial_\nu\Delta\partial_t\hat{y}=0$ and $\partial_t\hat{y}=0$ on $\partial\Omega$ for $t\in I$, we get that

$$\begin{aligned}
&\varepsilon\int_I\int_\Omega|\partial_t\Delta^2\hat{y}|^2(T-t)\,\mathrm{d}x\,\mathrm{d}t+\int_I\int_\Omega\nabla(DH(\Delta\overline{y})):\partial_t\nabla\Delta\hat{y}(T-t)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\int_I\int_\Omega\big(\partial_F W(\nabla\overline{y},\underline{\theta})+\partial_{\dot F}R(\nabla\underline{y},\partial_t\nabla\hat{y},\underline{\theta})\big):\partial_t\nabla\Delta\hat{y}(T-t)\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad-\int_I\int_\Omega\overline{f}\cdot\partial_t\Delta\hat{y}(T-t)\,\mathrm{d}x\,\mathrm{d}t-\frac{\rho}{h}\int_0^T\int_\Omega(\partial_t\nabla\hat{y}(t)-\partial_t\nabla\hat{y}(t-h)):\partial_t\nabla\hat{y}(t)(T-t)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}\tag{3.70}$$

We denote the last term on the right-hand side of (3.70) (without negative sign) by Π_T. An expansion yields

$$\begin{aligned}\Pi_T &= \frac{\rho}{2h}\int_I \big(\|\partial_t\nabla\hat{y}(t)\|^2_{L^2(\Omega)} - \|\partial_t\nabla\hat{y}(t-h)\|^2_{L^2(\Omega)} + \|\partial_t\nabla\hat{y}(t)-\partial_t\nabla\hat{y}(t-h)\|^2_{L^2(\Omega)}\big)(T-t)\,\mathrm{d}t\\ &= \frac{\rho}{2}\int_0^{T-h}\int_\Omega |\partial_t\nabla\hat{y}(t)|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{\rho}{2}\fint_{T-h}^{T}\int_\Omega |\partial_t\nabla\hat{y}(t)|^2(T-t)\,\mathrm{d}x\,\mathrm{d}t - \frac{\rho}{2}\|\nabla y'_{0,\varepsilon}\|^2_{L^2(\Omega)}\fint_0^h (T-t)\,\mathrm{d}t\\ &\quad + \frac{\rho}{2h}\int_I\int_\Omega |\partial_t\nabla\hat{y}(t)-\partial_t\nabla\hat{y}(t-h)|^2(T-t)\,\mathrm{d}x\,\mathrm{d}t\,. \end{aligned} \tag{3.71}$$

Combining (3.70) and (3.71) and applying Hölder inequality, we deduce that

$$\begin{aligned}\varepsilon\|\partial_t\Delta^2\hat{y}\|^2_{L^2_T(I;L^2(\Omega))} &\le \|\nabla(DH(\Delta\overline{y}))\|_{L^2_T(I;L^{p'}(\Omega))}\|\partial_t\nabla\Delta\hat{y}\|_{L^2_T(I;L^p(\Omega))}\\ &\quad + \sqrt{T}\|(\partial_F W(\nabla\overline{y},\underline{\theta}) + \partial_{\dot F}R(\nabla\underline{y},\partial_t\nabla\hat{y},\underline{\theta}))\|_{L^2(I\times\Omega)}\|\partial_t\nabla\Delta\hat{y}\|_{L^2_T(I;L^2(\Omega))}\\ &\quad + \sqrt{T}\|f\|_{L^2(I\times\Omega)}\|\partial_t\Delta\hat{y}\|_{L^2_T(I;L^2(\Omega))} + \frac{\rho T}{2}\|y'_{0,\varepsilon}\|^2_{H^1(\Omega)}\,.\end{aligned}$$

Then, by (3.67)–(3.69), Hölder's inequality, and Poincaré's inequality along with $\Delta\partial_t\hat{y}=0$ on $I\times\partial\Omega$ we derive

$$\varepsilon\|\partial_t\Delta^2\hat{y}\|^2_{L^2_T(I;L^2(\Omega))} \le C(1+\sqrt{\varepsilon}\|y_{0,\varepsilon}\|_{H^4(\Omega)} + \|y'_{0,\varepsilon}\|_{H^1(\Omega)})\|\partial_t\nabla\Delta\hat{y}\|_{L^2_T(I;L^p(\Omega))} + C\|y'_{0,\varepsilon}\|^2_{H^1(\Omega)}. \tag{3.72}$$

By the Gagliardo-Nirenberg interpolation inequality with $\theta = \frac{p-2}{2p}d$ ($\theta\in(0,1)$ as $p\in(3,6)$ for $d=3$), see e.g. [39, Theorem 1.24]) for $r=p$, $\beta=0$, $k=1$, and $p=q=2$, we get

$$\|\partial_t\nabla\Delta\hat{y}\|_{L^2_T(I;L^p(\Omega))} \le \|\partial_t\nabla\Delta\hat{y}\|_{L^2_T(I;L^2(\Omega))} + \|\partial_t\nabla\Delta\hat{y}\|^{1-\theta}_{L^2_T(I;L^2(\Omega))}\|\partial_t\nabla^2\Delta\hat{y}\|^{\theta}_{L^2_T(I;L^2(\Omega))}.$$

By elliptic regularity for the operator Δ and the fact that $\Delta\partial_t\hat{y}=0$ on $I\times\partial\Omega$ we get $\|\partial_t\nabla^2\Delta\hat{y}\|_{L^2_T(I;L^2(\Omega))}\le C\|\partial_t\Delta^2\hat{y}\|_{L^2_T(I;L^2(\Omega))}$. Then, using the weighted Young's inequality for $\lambda>0$ and exponent $\frac{2}{\theta}$ we get

$$\|\partial_t\nabla\Delta\hat{y}\|_{L^2_T(I;L^p(\Omega))} \le \|\partial_t\nabla\Delta\hat{y}\|_{L^2_T(I;L^2(\Omega))} + C(\varepsilon\lambda)^{-\frac{\theta}{2-\theta}}\|\partial_t\nabla\Delta\hat{y}\|^{\frac{2(1-\theta)}{2-\theta}}_{L^2_T(I;L^2(\Omega))} + C\varepsilon\lambda\|\partial_t\Delta^2\hat{y}\|^2_{L^2_T(I;L^2(\Omega))}.$$

The bound $\sqrt{\varepsilon}\|\partial_t\hat{y}\|_{L^2(I;H^3(\Omega))}\le C$ given by (3.37b) and the fact that $\frac{1}{2-\theta}=\varrho\in(\frac12,1)$ (see (3.43)) yield

$$\|\partial_t\nabla\Delta\hat{y}\|_{L^2_T(I;L^p(\Omega))} \le C\varepsilon^{-1/2} + C_\lambda\varepsilon^{-\frac{\theta}{2-\theta}}\varepsilon^{-\frac{1-\theta}{2-\theta}} + C\varepsilon\lambda\|\partial_t\Delta^2\hat{y}\|^2_{L^2_T(I;L^2(\Omega))} \le C_\lambda\varepsilon^{-\varrho} + C\varepsilon\lambda\|\partial_t\Delta^2\hat{y}\|^2_{L^2_T(I;L^2(\Omega))}.$$

Multiplying inequality (3.72) by ε^ϱ and choosing λ sufficiently small we get

$$\varepsilon^{1+\varrho}\|\partial_t\Delta^2\hat{y}\|^2_{L^2_T(I;L^2(\Omega))} \le C(1+\sqrt{\varepsilon}\|y_{0,\varepsilon}\|_{H^4(\Omega)} + \|y'_{0,\varepsilon}\|_{H^1(\Omega)}) + C\varepsilon^\varrho\|y'_{0,\varepsilon}\|^2_{H^1(\Omega)}$$

for some constant $C>0$ independent of $\varepsilon\in(0,1)$ and of $h>0$. This concludes the proof of (3.47) by using (3.42) and an elliptic regularity estimate. ☐

3.4. **Proof of Theorem 3.1.** In this subsection, we separately discuss the limiting passage $\tau\to 0$ for the mechanical and the heat-transfer equation, leading to (3.1a) and (3.1b), respectively.

Lemma 3.14 (Convergence of the mechanical equation)**.** *Let (y_h,θ_h) be as in Proposition 3.10. Then, for any test function $z\in C^\infty(I\times\overline{\Omega};\mathbb{R}^d)$ satisfying $z=0$ on $I\times\partial\Omega$ we have that* (3.1a) *holds.*

Proof. By (3.38a) we have that $\partial_t\hat{y}_\tau \rightharpoonup y_h$ weakly in $L^2(I_h;H^3(\Omega;\mathbb{R}^d))$ and by (3.38b) it holds that $\hat{y}_\tau(t),\overline{y}_\tau(t)$, $\underline{y}_\tau(t)\to y_h$ strongly in $W^{2,p}(\Omega;\mathbb{R}^d)$ for a.e. $t\in I$. Moreover, $(\hat{y}_\tau)_\tau$, $(\overline{y}_\tau)_\tau$, and $(\underline{y}_\tau)_\tau$ are bounded in $L^\infty(I;W^{2,p}(\Omega;\mathbb{R}^d))$ independently of τ and h. Hence, we deduce (3.1a) by testing (3.6) with z, summing over k, and passing to the limit as $\tau\to0$ using the generalized dominated convergence theorem. Here, we crucially use that $\partial_{\dot F}R$ is linear in the second entry, cf. (D.1). ☐

Before we concern ourselves with the limiting passage in the heat-transfer equation, we establish a mechanical energy balance for (y_h,θ_h).

Lemma 3.15 (Mechanical energy balance)**.** *Let (y_h,θ_h) be as in Proposition 3.10 with $y_h(0,\cdot)=y_{0,\varepsilon}$ satisfying* (3.1a)*. Then, for any $t\in I$ we have the mechanical energy balance*

$$\begin{aligned}&\mathcal{M}(y_h(t)) + \frac{\rho}{2}\fint_{t-h}^t \|\partial_t y_h(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s + \int_0^t 2\mathcal{R}_\varepsilon(y_h,\partial_t y_h,\theta_h)\,\mathrm{d}s + \frac{\rho}{2h}\int_0^t \|\partial_t y_h(s)-\partial_t y_h(s-h)\|^2_{L^2(\Omega)}\,\mathrm{d}s\\ &\quad = \mathcal{M}(y_{0,\varepsilon}) + \frac{\rho}{2}\|y'_{0,\varepsilon}\|^2_{L^2(\Omega)} - \int_0^t\int_\Omega \partial_F W^{\mathrm{cpl}}(\nabla y_h,\theta_h):\partial_t\nabla y_h\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega f\cdot\partial_t y_h\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{3.73}$$

Proof. By the regularity $y_h \in H^1(I_h; H^3(\Omega;\mathbb{R}^d))$, using the chain rule for Λ-convex functionals (see [35, Proposition 3.6] and Lemma 3.8), we first observe that the mechanical energy defined in (2.6) satisfies that $t \mapsto \mathcal{M}(y_h(t))$ lies in $W^{1,1}(I)$ and that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}(y_h) = \int_\Omega DH(\Delta y_h) : \partial_t \Delta y_h \,\mathrm{d}x + \int_\Omega \partial_F W^{\mathrm{el}}(\nabla y_h) \cdot \partial_t \nabla y_h \,\mathrm{d}x \quad \text{for a.e. } t \in I. \tag{3.74}$$

We test the equation (3.1a) with $z := \partial_t y_h \mathbb{1}_{[0,t]}$, and obtain by (2.11)

$$\begin{aligned}
&\int_0^t \int_\Omega \big(\partial_F W^{\mathrm{el}}(\nabla y_h) + \partial_F W^{\mathrm{cpl}}(\nabla y_h, \theta_h)\big) : \partial_t \nabla y_h \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_\Omega DH(\Delta y_h) \cdot \partial_t \Delta y_h \,\mathrm{d}x\,\mathrm{d}s \\
&- \int_0^t \int_\Omega f \cdot \partial_t y_h \,\mathrm{d}x\,\mathrm{d}s + \frac{\rho}{h}\int_0^t \int_\Omega \big(\partial_t y_h(s) - \partial_t y_h(s-h)\big) \cdot \partial_t y_h(s) \,\mathrm{d}x\,\mathrm{d}s \\
&\qquad = -\int_0^t \int_\Omega 2R(\nabla y_h, \partial_t \nabla y_h, \theta_h) \,\mathrm{d}x\,\mathrm{d}s - \varepsilon \int_0^t \int_\Omega |\partial_t \nabla \Delta y_h|^2 \,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

Applying the chain rule (3.74) we find

$$\begin{aligned}
&\mathcal{M}(y_h(t)) - \mathcal{M}(y_{0,\varepsilon}) + \int_0^t \int_\Omega \partial_F W^{\mathrm{cpl}}(\nabla y_h, \theta_h) : \partial_t \nabla y_h \,\mathrm{d}x\,\mathrm{d}s - \int_0^t \int_\Omega f \cdot \partial_t y_h \,\mathrm{d}x\,\mathrm{d}s \qquad (3.75)\\
&\quad = -\int_0^t \int_\Omega \Big(2R(\nabla y_h, \partial_t \nabla y_h, \theta_h) + \varepsilon |\partial_t \nabla \Delta y_h|^2\Big) \,\mathrm{d}x\,\mathrm{d}s - \frac{\rho}{h}\int_0^t \int_\Omega \big(\partial_t y_h(s) - \partial_t y_h(s-h)\big) \cdot \partial_t y_h(s) \,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

Denoting the last term on the right-hand side by Π (without negative sign), and expanding it as in (3.71), we derive

$$\begin{aligned}
\Pi &= \frac{\rho}{2h}\int_0^t \big(\|\partial_t y_h(s)\|^2_{L^2(\Omega)} - \|\partial_t y_h(s-h)\|^2_{L^2(\Omega)} + \|\partial_t y_h(s) - \partial_t y_h(s-h)\|^2_{L^2(\Omega)}\big) \,\mathrm{d}s \\
&= \frac{\rho}{2}\fint_{t-h}^t \|\partial_t y_h(s)\|^2_{L^2(\Omega)} \,\mathrm{d}s - \frac{\rho}{2}\fint_{-h}^0 \|\partial_t y_h(s)\|^2_{L^2(\Omega)} \,\mathrm{d}s + \frac{\rho}{2h}\int_0^t \|\partial_t y_h(s) - \partial_t y_h(s-h)\|^2_{L^2(\Omega)} \,\mathrm{d}s.
\end{aligned}$$

Plugging this into (3.75) and using (2.12) as well as $\partial_t y_h(s) = y'_{0,\varepsilon}$ for $s \in (-h, 0)$, the proof of the mechanical energy balance is concluded. □

Lemma 3.16 (Convergence of the heat-transfer equation)**.** *Let* (y_h, θ_h) *be as in Proposition 3.10. Then, for any test function* $\varphi \in C^\infty(I \times \overline{\Omega})$ *equation* (3.1b) *is satisfied.*

Proof. The essential point is to show strong convergence of the strain rates, namely

$$\nabla \partial_t \hat{y}_\tau \to \nabla \partial_t y_h \text{ and } \nabla \Delta \partial_t \hat{y}_\tau \to \nabla \Delta \partial_t y_h \quad \text{strongly in } L^2(I; L^2(\Omega;\mathbb{R}^{d\times d})). \tag{3.76}$$

Once this is achieved, using the convergences in (3.38a)–(3.38d), we can almost verbatim follow the proof of [2, Proposition 4.6], recalling that the scheme for the heat-transfer equation differs from the one in [2] only by the regularizing term $\varepsilon|\partial_t \nabla \Delta \hat{y}_\tau|^2 \wedge \tau^{-1}$, see (3.9) and [2, Equation (3.11)]. The only difference is that for the term

$$\int_\Omega \partial_t \hat{w}_\tau \varphi \,\mathrm{d}x$$

in (3.9) we do not perform an integration by parts in time, but directly pass to the limit using that $\partial_t \hat{w}_\tau \rightharpoonup \partial_t w_h$ in $L^2(I; (H^1(\Omega))^*)$ by Corollary 3.13. This gives (3.1b).

The argument for showing (3.76) is along the lines of [2, Lemma 4.5] or [35, Proposition 5.1], and relies on passing to the limit in the mechanical energy balance. We briefly sketch the argument. In view of the notation in (3.36), we can write the discrete mechanical energy estimate in Lemma 3.9 as

$$\begin{aligned}
&\mathcal{M}(\overline{y}_\tau(T)) + \frac{\rho}{2}\fint_{T-h}^T \|\partial_t \hat{y}_\tau(t)\|^2_{L^2(\Omega)} \,\mathrm{d}t + \int_0^T 2\mathcal{R}_\varepsilon(\underline{y}_\tau, \partial_t \hat{y}_\tau, \underline{\theta}_\tau) \,\mathrm{d}t + \frac{\rho}{2h}\int_0^T \|\partial_t \hat{y}_\tau(t) - \partial_t \hat{y}_\tau(t-h)\|^2_{L^2(\Omega)} \,\mathrm{d}t \\
&\quad \le \mathcal{M}(y_{0,\varepsilon}) + \frac{\rho}{2}\|y'_{0,\varepsilon}\|^2_{L^2(\Omega)} - \int_0^T \int_\Omega \partial_F W^{\mathrm{cpl}}(\underline{y}_\tau, \underline{\theta}_\tau) : \partial_t \nabla \hat{y}_\tau \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \big(\overline{f}_\tau(t), \partial_t \hat{y}_\tau(t)\big)_2 \,\mathrm{d}t + C_M \tau V_{T/\tau}.
\end{aligned}$$

Employing (3.38a)–(3.38d), standard lower semicontinuity arguments (see [2, Equation (4.15)] and also [23, Theorem 7.5] for a general result) imply,

$$\liminf_{\tau\to 0} \mathcal{M}(\overline{y}_\tau(T)) \geq \mathcal{M}(y_h(T)), \tag{3.77a}$$

$$\liminf_{\tau\to 0} \int_0^T \int_\Omega \xi(\nabla \underline{y}_\tau, \partial_t \nabla \hat{y}_\tau, \underline{\theta}_\tau)\,\mathrm{d}x\,\mathrm{d}t \geq \int_0^T \int_\Omega \xi(\nabla y_h, \partial_t \nabla y_h, \theta_h)\,\mathrm{d}x\,\mathrm{d}t, \tag{3.77b}$$

$$\liminf_{\tau\to 0} \int_0^T \int_\Omega \varepsilon|\partial_t \nabla \Delta \hat{y}_\tau|^2\,\mathrm{d}x\,\mathrm{d}t \geq \int_0^T \int_\Omega \varepsilon|\partial_t \nabla \Delta y_h|^2\,\mathrm{d}x\,\mathrm{d}t, \tag{3.77c}$$

$$\liminf_{\tau\to 0} \frac{\rho}{2} \fint_{T-h}^T \|\partial_t \hat{y}_\tau(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t \geq \frac{\rho}{2} \fint_{T-h}^T \|\partial_t y_h(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t, \tag{3.77d}$$

$$\liminf_{\tau\to 0} \frac{\rho}{2h} \int_0^T \|\partial_t \hat{y}_\tau(t) - \partial_t \hat{y}_\tau(t-h)\|^2_{L^2(\Omega)}\,\mathrm{d}t \geq \frac{\rho}{2h} \int_0^T \|\partial_t y_h(t) - \partial_t y_h(t-h)\|^2_{L^2(\Omega)}\,\mathrm{d}t. \tag{3.77e}$$

Since $\int_0^T (\overline{f}_\tau(t), \partial_t \hat{y}_\tau(t))_2\,\mathrm{d}t \to \int_0^T \int_\Omega f \cdot \partial_t y_h\,\mathrm{d}x\,\mathrm{d}t$, as well as $\int_0^T \int_\Omega \partial_F W^{\mathrm{cpl}}(\underline{y}_\tau, \underline{\theta}_\tau) : \partial_t \nabla \hat{y}_\tau\,\mathrm{d}x\,\mathrm{d}t$ converges to $\int_0^T \int_\Omega \partial_F W^{\mathrm{cpl}}(\nabla y_h, \theta_h) : \partial_t \nabla y_h\,\mathrm{d}x\,\mathrm{d}t$ by (3.38a)–(3.38d), and $\tau V_{T/\tau} \to 0$ as $\tau \to 0$ by (3.37b), using the mechanical energy balance (3.73), we find that all estimates (3.77a)–(3.77e) are actually equalities, see [2, Equation (4.15)] for details on this argument. In particular, the convergence in (3.77b) implies the first part of (3.76), by repeating the arguments in [2, Equation (4.16)ff.]. Eventually, the convergence in (3.77c) provides the second part of (3.76). ☐

Proof of Theorem 3.1. The statement follows by collecting the regularity for y_h, θ_h, and w_h given in Proposition 3.10 and Corollary 3.13, and the identification of the limiting equations in Lemmas 3.14 and 3.16. ☐

4. Vanishing time-delay

We recall that for each $T, h, \varepsilon > 0$, Theorem 3.1 guarantees the existence of (y_h, θ_h) such that $y_h \in L^\infty(I; \mathcal{Y}^{\mathrm{reg}}_{\mathbf{id}}) \cap H^1(I_h; H^3(\Omega; \mathbb{R}^d))$, $\theta_h \in L^2(I; H^1_+(\Omega))$, and such that equations (3.1a)–(3.1b) hold. We also recall that $y_h(t) = y_{0,\varepsilon} + t y'_{0,\varepsilon}$ for $t \in [-h, 0]$. The goal of this section is to pass to the limit $h \to 0$ in (3.1a)–(3.1b). We start by a compactness result.

Lemma 4.1 (Compactness). *There exist $y_\varepsilon \in L^\infty(I; \mathcal{Y}^{\mathrm{reg}}_{\mathbf{id}}) \cap H^1(I; H^3(\Omega; \mathbb{R}^d))$ with $y_\varepsilon(0) = y_{0,\varepsilon}$, and $\theta_\varepsilon \in L^2(I; H^1_+(\Omega))$, as well as $w_\varepsilon := W^{\mathrm{in}}(\nabla y_\varepsilon, \theta_\varepsilon) \in L^2(I; H^1(\Omega)) \cap H^1(I; (H^1(\Omega))^*)$ with $w_\varepsilon(0) = w_{0,\varepsilon}$ such that, up to selecting a subsequence (not relabeled), it holds that*

$$y_h \rightharpoonup y_\varepsilon \quad \text{weakly in } H^1(I; H^3(\Omega; \mathbb{R}^d)), \tag{4.1a}$$

$$y_h \to y_\varepsilon \quad \text{strongly in } C(I; W^{1,\infty}(\Omega; \mathbb{R}^d)) \quad \text{and strongly in } C(I; W^{2,p}(\Omega; \mathbb{R}^d)), \tag{4.1b}$$

$$\theta_h \to \theta_\varepsilon \quad \text{and} \quad w_h \to w_\varepsilon \text{ strongly in } L^s(I \times \Omega) \text{ for any } s \in [1, \tfrac{d+2}{d}), \tag{4.1c}$$

$$\theta_h \rightharpoonup \theta_\varepsilon \quad \text{and} \quad w_h \rightharpoonup w_\varepsilon \text{ weakly in } L^2(I; H^1(\Omega)), \tag{4.1d}$$

$$w_h \rightharpoonup w_\varepsilon \text{ weakly in } H^1(I; (H^1(\Omega))^*). \tag{4.1e}$$

Proof. As the a priori bounds derived in Proposition 3.10 and Corollary 3.13 are independent of h, we see by the lower semicontinuity of norms that the same bounds hold true for (y_h, θ_h) in place of (y_τ, θ_τ). Then, (4.1a)–(4.1c) can be obtained exactly as in the proof of Proposition 3.10, and (4.1d) –(4.1e) follow from the bounds in Corollary 3.13 by weak compactness and the Aubin-Lions' lemma, see also [35, Proposition 5.1]. ☐

We collect a priori bounds for the limits $(y_\varepsilon, \theta_\varepsilon)$ which directly follow from Proposition 3.10, Lemma 3.12, Lemma 4.1, and lower semicontinuity: for each $q \in [1, \frac{d+2}{d})$ and $r \in [1, \frac{d+2}{d+1})$ we can find constants C, C_q, and C_r independent of ε such that

$$\|y_\varepsilon\|_{L^\infty(I; W^{2,p}(\Omega))} + \|\det(\nabla y_\varepsilon)^{-1}\|_{L^\infty(I\times\Omega)} \leq C, \tag{4.2a}$$

$$\|y_\varepsilon\|_{H^1(I\times\Omega)} + \sqrt{\varepsilon}\|\partial_t y_\varepsilon\|_{L^2(I; H^3(\Omega))} \leq C, \tag{4.2b}$$

$$\|\theta_\varepsilon\|_{L^\infty(I; L^1(\Omega))} + \|w_\varepsilon\|_{L^\infty(I; L^1(\Omega))} \leq C \tag{4.2c}$$

$$\|\theta_\varepsilon\|_{L^q(I\times\Omega)} + \|w_\varepsilon\|_{L^q(I\times\Omega)} \leq C_q, \tag{4.2d}$$

$$\|\nabla\theta_\varepsilon\|_{L^r(I\times\Omega)} + \|\nabla w_\varepsilon\|_{L^r(I\times\Omega)} \leq C_r. \tag{4.2e}$$

Furthermore, there exist $\varrho \in (\frac{1}{2}, 1)$ and $C > 0$ independently of ε with

$$\|\Delta y_\varepsilon\|_{L^2(I;H^1(\Omega))} + \||\Delta y_\varepsilon|^{\frac{p-2}{2}} \nabla(\Delta y_\varepsilon)\|_{L^2(I\times\Omega)} \le C\big(1 + \sqrt{\varepsilon}\|y_{0,\varepsilon}\|_{H^4(\Omega)} + \|y'_{0,\varepsilon}\|_{H^1(\Omega)}\big), \tag{4.3a}$$

$$\|\nabla DH(\Delta y_\varepsilon)\|_{L^2(I;L^{p'}(\Omega))} \le C\big(1 + \sqrt{\varepsilon}\|y_{0,\varepsilon}\|_{H^4(\Omega)} + \|y'_{0,\varepsilon}\|_{H^1(\Omega)}\big), \tag{4.3b}$$

$$\|\Delta \partial_t y_\varepsilon\|_{L^2(I;H^2(\Omega))} \le C\varepsilon^{-\frac{1+\varrho}{2}}\big(1 + \sqrt[4]{\varepsilon}\|y_{0,\varepsilon}\|_{H^4(\Omega)} + \|y'_{0,\varepsilon}\|_{H^1(\Omega)}\big). \tag{4.3c}$$

Moreover, we have

$$\Delta y_\varepsilon(t) = \partial_\nu \Delta y_\varepsilon(t) = 0 \qquad \text{on } \partial\Omega \text{ for a.e. } t \in I, \tag{4.4a}$$

$$\Delta \partial_t y_\varepsilon(t) = \partial_\nu \Delta \partial_t y_\varepsilon(t) = 0 \qquad \text{on } \partial\Omega \text{ for a.e. } t \in I. \tag{4.4b}$$

In particular, (4.4b) follows from the boundary conditions deduced in Proposition 3.11, the estimates obtained in Lemma 3.12 on $(\partial_t y_h)_h$, and a compactness argument, while (4.4a) is a consequence of (4.4b) and the fact that $y_{0,\varepsilon} \in \mathcal{Y}^{\rm reg}_{\mathbf{id}}$, see (2.20).

The rest of the section is devoted to the proof of Theorems 2.2 and 2.3. In particular, we show that $(y_\varepsilon, \theta_\varepsilon)$ is a solution to the regularized system (2.21)–(2.22). As the second bound in (3.37b) transfers uniformly to $\partial_t y_h$, there exists $y'_{T,\varepsilon} \in L^2(\Omega;\mathbb{R}^d)$ such that, as $h \to 0$,

$$\fint_{T-h}^{T} \partial_t y_h(s)\,\mathrm{d}s \rightharpoonup y'_{T,\varepsilon} \quad \text{weakly in } L^2(\Omega;\mathbb{R}^d). \tag{4.5}$$

Proposition 4.2 (Auxiliary mechanical equation)**.** *Let $(y_\varepsilon, \theta_\varepsilon)$ be as in Lemma 4.1. Then, $(y_\varepsilon, \theta_\varepsilon)$ satisfies*

$$\begin{aligned} 0 = &\int_I\int_\Omega \partial_F W(\nabla y_\varepsilon, \theta_\varepsilon) : \nabla z \,\mathrm{d}x\,\mathrm{d}t + \int_I\int_\Omega DH(\Delta y_\varepsilon)\cdot \Delta z\,\mathrm{d}x\,\mathrm{d}t + \varepsilon\int_I\int_\Omega \partial_t \nabla\Delta y_\varepsilon : \nabla\Delta z\,\mathrm{d}x\,\mathrm{d}t \\ &- \int_I\int_\Omega f\cdot z\,\mathrm{d}x\,\mathrm{d}t + \int_I\int_\Omega \partial_{\dot F} R(\nabla y_\varepsilon, \partial_t\nabla y_\varepsilon, \theta_\varepsilon) : \nabla z\,\mathrm{d}x\,\mathrm{d}t - \rho\int_I\int_\Omega \partial_t y_\varepsilon\cdot\partial_t z\,\mathrm{d}x\,\mathrm{d}t \\ &+ \rho\int_\Omega \big(y'_{T,\varepsilon}\cdot z(T) - y'_{0,\varepsilon}\cdot z(0)\big)\,\mathrm{d}x, \end{aligned} \tag{4.6}$$

for every $z \in C^\infty(I\times\overline{\Omega};\mathbb{R}^d)$ with $z = 0$ on $I\times\partial\Omega$.

Proof. Notice that by a change of variables we can rewrite (3.1a) for every $z \in C^\infty(I\times\overline{\Omega};\mathbb{R}^d)$ with $z = 0$ on $I\times\partial\Omega$ as

$$\begin{aligned} &\int_I\int_\Omega DH(\Delta y_h)\cdot\Delta z + \Big(\partial_F W(\nabla y_h, \theta_h) + \partial_{\dot F} R(\nabla y_h, \partial_t\nabla y_h, \theta_h)\Big) : \nabla z\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \varepsilon\int_I\int_\Omega \partial_t\nabla\Delta y_h\cdot\nabla\Delta z\,\mathrm{d}x\,\mathrm{d}t - \rho\int_0^{T-h}\int_\Omega \partial_t y_h(t)\cdot\frac{z(t+h)-z(t)}{h}\,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \rho\int_\Omega y'_{0,\varepsilon}\cdot\fint_0^h z(t)\,\mathrm{d}t\,\mathrm{d}x + \rho\fint_{T-h}^T\int_\Omega \partial_t y_h(t)\cdot z(t)\,\mathrm{d}x\,\mathrm{d}t = \int_I\int_\Omega f\cdot z\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

By the smoothness of z and (4.5) it directly follows that

$$\int_\Omega y'_{0,\varepsilon}\cdot\fint_0^h z(t)\,\mathrm{d}t\,\mathrm{d}x \to \int_\Omega y'_{0,\varepsilon}\cdot z(0)\,\mathrm{d}x \qquad \fint_{T-h}^T\int_\Omega \partial_t y_h(t)\cdot z(t)\,\mathrm{d}x\,\mathrm{d}t \to \int_\Omega y'_{T,\varepsilon}\cdot z(T)\,\mathrm{d}x \quad \text{as } h\to 0.$$

Moreover, the convergence in (4.1a) and the smoothness of z show by weak-strong convergence

$$\int_0^{T-h}\int_\Omega \partial_t y_h(t)\cdot\frac{z(t+h)-z(t)}{h}\,\mathrm{d}x\,\mathrm{d}t \to \int_0^T\int_\Omega \partial_t y_\varepsilon\cdot\partial_t z\,\mathrm{d}x\,\mathrm{d}t \qquad \text{as } h\to 0.$$

By (4.1b), assumption (H.3), and the generalized dominated convergence theorem, we derive that

$$\int_I\int_\Omega DH(\Delta y_h)\cdot\Delta z\,\mathrm{d}x\,\mathrm{d}t \to \int_I\int_\Omega DH(\Delta y_\varepsilon)\cdot\Delta z\,\mathrm{d}x\,\mathrm{d}t \qquad \text{as } h\to 0.$$

Finally, by (4.1a)–(4.1c) the convergence of all remaining terms follows, leading to the desired equation (4.6). □

Proposition 4.3 (Mechanical equation)**.** *Let $(y_\varepsilon, \theta_\varepsilon)$ be as in Lemma 4.1. Then, $(y_\varepsilon, \theta_\varepsilon)$ satisfies* (2.21)*. Moreover, it holds that $\partial^2_{tt} y_\varepsilon \in L^2(I;(H^3(\Omega;\mathbb{R}^d)\cap H^1_0(\Omega;\mathbb{R}^d))^*)$ with $\|\partial^2_{tt} y_\varepsilon\|_{L^2(I;(H^3(\Omega)\cap H^1_0(\Omega))^*)} \le C$ for a constant $C > 0$ independent of ε. In particular, $\partial_t y_\varepsilon \in C(I;L^2(\Omega;\mathbb{R}^d))$ with $\partial_t y_\varepsilon(0) = y'_{0,\varepsilon}$ and $\partial_t y_\varepsilon(T) = y'_{T,\varepsilon}$.*

Proof. In view of (4.2a)–(4.2e), all time integrals in (4.6) except for $\rho \int_I \int_\Omega \partial_t y_\varepsilon \cdot \partial_t z \, \mathrm{d}x \, \mathrm{d}t$ lie in $L^2(I; (H^3(\Omega) \cap H^1_0(\Omega))^*)$, where for the nonlinear term $\int_I \int_\Omega DH(\Delta y) \cdot \Delta z \, \mathrm{d}x \, \mathrm{d}t$ we particularly use (H.3), (4.2a), and the fact that $\Delta z \in L^2(I; L^p(\Omega))$ for each $z \in L^2(I; H^3(\Omega) \cap H^1_0(\Omega))$ ($p \in (3, 6)$ for $d = 3$). More precisely, the corresponding operator norms are uniformly bounded independently of ε. Then, by definition of weak derivatives we get that $\partial^2_{tt} y_\varepsilon \in L^2(I; (H^3(\Omega; \mathbb{R}^d) \cap H^1_0(\Omega; \mathbb{R}^d))^*)$ exists and is bounded independently of ε. Moreover, an integration by parts in time shows (2.21) for $z \in C^\infty(I \times \overline{\Omega}; \mathbb{R}^d)$ with $z = 0$ on $I \times \partial\Omega$ and $z(0) = z(T) = 0$. Then, by a density argument we observe that the assumption $z(0) = z(T) = 0$ can be dropped. Eventually, $\partial_t y_\varepsilon \in C(I; L^2(\Omega; \mathbb{R}^d))$ follows from [39, Lemma 7.3], and $\partial_t y_\varepsilon(0) = y'_{0,\varepsilon}$ as well as $\partial_t y_\varepsilon(T) = y'_{T,\varepsilon}$ follow by integration by parts in time of (2.21) for general $z \in C^\infty(I \times \overline{\Omega}; \mathbb{R}^d)$ and a comparison with (4.6). □

We proceed with a mechanical energy balance which is the analog to the one in (3.73). We note that the balance can be formulated for each $t \in I$ since $y_\varepsilon \in C(I; W^{2,p}(\Omega; \mathbb{R}^d))$ and $\partial_t y_\varepsilon \in C(I; L^2(\Omega; \mathbb{R}^d))$.

Lemma 4.4 (Mechanical energy balance)**.** *Let $(y_\varepsilon, \theta_\varepsilon)$ be as in Lemma 4.1 satisfying* (2.21)*. Then, for any $t \in I$ we have the mechanical energy balance* (2.23)*.*

Proof. Since $\partial^2_{tt} y_\varepsilon$ lies in $L^2(I; (H^3(\Omega; \mathbb{R}^d) \cap H^1_0(\Omega; \mathbb{R}^d))^*)$, see Proposition 4.3, by an approximation argument we can test (2.21) with $z = \partial_t y_\varepsilon \mathbb{1}_{[0,t]}$ and obtain

$$\int_0^t \int_\Omega \Big(\partial_F W^{\mathrm{el}}(\nabla y_\varepsilon) + \partial_F W^{\mathrm{cpl}}(\nabla y_\varepsilon, \theta_\varepsilon) \Big) : \partial_t \nabla y_\varepsilon \, \mathrm{d}x \, \mathrm{d}t + \int_0^t \int_\Omega DH(\Delta y_\varepsilon) : \partial_t \Delta y_\varepsilon \, \mathrm{d}x \, \mathrm{d}s$$
$$= - \int_0^t 2\mathcal{R}_\varepsilon(y_\varepsilon, \partial_t y_\varepsilon, \theta_\varepsilon) \, \mathrm{d}s - \rho \int_0^t \langle \partial^2_{tt} y_\varepsilon, \partial_t y_\varepsilon \rangle \, \mathrm{d}s + \int_0^t \int_\Omega f \cdot \partial_t y_\varepsilon \, \mathrm{d}x \, \mathrm{d}s. \tag{4.7}$$

Using the chain rule we find

$$\rho \int_0^t \langle \partial^2_{tt} y_\varepsilon, \partial_t y_\varepsilon \rangle \, \mathrm{d}s = \frac{\rho}{2} \int_0^t \int_\Omega \frac{\mathrm{d}}{\mathrm{d}t} |\partial_t y_\varepsilon|^2 \, \mathrm{d}x \, \mathrm{d}s = \frac{\rho}{2} \|\partial_t y_h(t)\|^2_{L^2(\Omega)} - \frac{\rho}{2} \|y'_{0,\varepsilon}\|^2_{L^2(\Omega)}.$$

Combining this with the chain rule in (3.74) (for y_ε in place of y_h) and plugging into (4.7), the proof is concluded, using again (2.11)–(2.12). □

Proposition 4.5 (Heat-transfer equation)**.** *Let $(y_\varepsilon, \theta_\varepsilon)$ be as in Lemma 4.1. Then, $(y_\varepsilon, \theta_\varepsilon)$ satisfies* (2.22)*.*

Proof. As in the proof of Lemma 3.16, the essential point is to show strong convergence of the strain rates, namely

$$\nabla \partial_t y_h \to \nabla \partial_t y_\varepsilon \text{ and } \nabla \Delta \partial_t y_h \to \nabla \Delta \partial_t y_\varepsilon \quad \text{strongly in } L^2(I; L^2(\Omega; \mathbb{R}^{d \times d})), \tag{4.8}$$

as then one can pass to the limit in each term by using the convergences in (4.1a)–(4.1e). Rearranging the terms in (3.73), dropping one nonnegative term, and passing to the liminf as $h \to 0$, by the convergences in Lemma 4.1 we get

$$\liminf_{h \to 0} \Big(\mathcal{M}(y_h(T)) + \frac{\rho}{2} \fint_{T-h}^T \|\partial_t y_h(s)\|^2_{L^2(\Omega)} \, \mathrm{d}s + \int_I 2\mathcal{R}_\varepsilon(y_h, \partial_t y_h, \theta_h) \, \mathrm{d}t \Big)$$
$$\leq \mathcal{M}(y_{0,\varepsilon}) + \frac{\rho}{2} \|y'_{0,\varepsilon}\|^2_{L^2(\Omega)} + \lim_{h \to 0} \int_I \int_\Omega f \cdot \partial_t y_h \, \mathrm{d}x \, \mathrm{d}t - \lim_{h \to 0} \int_I \int_\Omega \partial_F W^{\mathrm{cpl}}(\nabla y_h, \theta_h) : \partial_t \nabla y_h \, \mathrm{d}x \, \mathrm{d}t$$
$$= \mathcal{M}(y_{0,\varepsilon}) + \frac{\rho}{2} \|y'_{0,\varepsilon}\|^2_{L^2(\Omega)} + \int_I \int_\Omega f \cdot \partial_t y_\varepsilon \, \mathrm{d}x \, \mathrm{d}t - \int_I \int_\Omega \partial_F W^{\mathrm{cpl}}(\nabla y_\varepsilon, \theta_\varepsilon) : \partial_t \nabla y_\varepsilon \, \mathrm{d}x \, \mathrm{d}t. \tag{4.9}$$

By the convergences in Lemma 4.1 and standard lower semicontinuity arguments we get

$$\liminf_{h \to 0} \mathcal{M}(y_h(T)) \geq \mathcal{M}(y_\varepsilon(T)),$$
$$\liminf_{h \to 0} \int_I 2\mathcal{R}_\varepsilon(y_h, \partial_t y_h, \theta_h) \, \mathrm{d}t \geq \int_I 2\mathcal{R}_\varepsilon(y_\varepsilon, \partial_t y_\varepsilon, \theta_\varepsilon) \, \mathrm{d}t,$$
$$\liminf_{h \to 0} \frac{\rho}{2} \fint_{T-h}^T \|\partial_t y_h(s)\|^2_{L^2(\Omega)} \, \mathrm{d}s \geq \frac{\rho}{2} \|\partial_t y_\varepsilon(T)\|^2_{L^2(\Omega)}. \tag{4.10}$$

For the first two estimates we also refer to [2, Equation (4.15)] and the last one follows from (4.5) and Proposition 4.3. Combining (4.9)–(4.10) with the mechanical energy balance in Lemma 4.4, we conclude that all inequalities in (4.10) are actually equalities. Then, (4.8) follows exactly as in the final argument of the proof of Lemma 3.16. □

Proofs of Theorems 2.2 and 2.3. The weak formulation, the regularity properties of $(y_\varepsilon, \theta_\varepsilon, w_\varepsilon)$, and the initial conditions for $y_\varepsilon, \partial_t y_\varepsilon, w_\varepsilon$ follow from Lemma 4.1, Proposition 4.3, and Proposition 4.5. The mechanical energy balance is given in Lemma 4.4.

Concerning (2.24), we observe that by density we can test (2.22) with functions $\varphi \in W^{1,1}(I)$ which are independent of the space variable x and satisfy $\varphi(T) = 0$. For $t \in (0,T)$ fixed, we define the test function φ with $\varphi \equiv 1$ on $(0, t-\delta)$, $\varphi \equiv 0$ on $(t+\delta, T)$, and $\varphi' \equiv -\frac{1}{2\delta}$ on $(t-\delta, t+\delta)$. Then, by an integration by parts in time for the term $\int_I \langle \partial_t w_\varepsilon, \varphi\rangle \,\mathrm{d}t$, in the limit $\delta \to 0$, (2.22) yields

$$0 = \int_0^t -\Big(\xi(\nabla y_\varepsilon, \partial_t \nabla y_\varepsilon, \theta_\varepsilon) + \partial_F W^{\mathrm{cpl}}(\nabla y_\varepsilon, \theta_\varepsilon) : \partial_t \nabla y_\varepsilon + \varepsilon |\partial_t \nabla \Delta y_\varepsilon|^2\Big) \,\mathrm{d}x\,\mathrm{d}t$$
$$- \kappa \int_0^t \int_{\partial\Omega} (\theta_\flat - \theta_\varepsilon) \,\mathrm{d}\mathcal{H}^{d-1} \,\mathrm{d}t + \lim_{\delta\to 0} \frac{1}{2\delta} \int_{t-\delta}^{t+\delta} \int_\Omega w_\varepsilon \,\mathrm{d}x\,\mathrm{d}t - \int_\Omega w_{0,\varepsilon} \,\mathrm{d}x.$$

As $w_\varepsilon \in C(I; L^2(\Omega))$ by Lemma 4.1 and [39, Lemma 7.3], we find $\lim_{\delta\to 0} \frac{1}{2\delta} \int_{t-\delta}^{t+\delta} \int_\Omega w_\varepsilon \,\mathrm{d}x\,\mathrm{d}s = \int_\Omega w_\varepsilon(t,x)\,\mathrm{d}x$, which concludes the proof of (2.24). Eventually, (2.25) follows by summation of (2.23) and (2.24). ☐

5. Vanishing regularization: Proof of Theorem 2.5

This section is devoted to the analysis of the limit of the regularized thermo-elastodynamic system (2.21)–(2.22) as $\varepsilon \to 0$. This will show existence of solutions to the system (2.26)–(2.27), i.e., Theorem 2.5. First, given initial data $y_0 \in \mathcal{Y}_{\mathbf{id}}$ and $y_0' \in H_0^1(\Omega;\mathbb{R}^d)$, we consider suitable regularizations $y_{0,\varepsilon} \in \mathcal{Y}_{\mathbf{id}}^{\mathrm{reg}}$ (see (2.20)) and $y_{0,\varepsilon}' \in H^3(\Omega;\mathbb{R}^d)$ such that

$$y_{0,\varepsilon} \to y_0 \text{ in } W^{2,p}(\Omega;\mathbb{R}^d) \quad \text{and} \quad y_{0,\varepsilon}' \to y_0' \text{ in } H^1(\Omega;\mathbb{R}^d), \tag{5.1}$$
$$\limsup_{\varepsilon\to 0} \sqrt[4]{\varepsilon}\|y_{0,\varepsilon}\|_{H^4(\Omega)} < +\infty\,. \tag{5.2}$$

This can be achieved by considering regularizations $(\varphi_\varepsilon)_\varepsilon \subset C_c^\infty(\Omega;\mathbb{R}^d)$ with $\varphi_\varepsilon \to \Delta y_0 \in L^p(\Omega;\mathbb{R}^d)$, and choosing $y_{0,\varepsilon} \in C^\infty(\Omega;\mathbb{R}^d) \cap \mathcal{Y}_{\mathbf{id}}$ as the solution to $\Delta y_{0,\varepsilon} = \varphi_\varepsilon$. Then, $\partial_\nu \Delta y_{0,\varepsilon} = \Delta y_{0,\varepsilon} = 0$ on $\partial\Omega$ holds by construction and (5.1)–(5.2) can be achieved by the elliptic regularity estimate $\|y_{0,\varepsilon} - y_0\|_{W^{2,p}(\Omega)} \le C\|\Delta(y_{0,\varepsilon} - y_0)\|_{L^p(\Omega)}$, see [30, Lemma 9.17].

For every $\varepsilon > 0$, in Theorem 2.2 we have shown the existence of a solution $(y_\varepsilon, \theta_\varepsilon)$ to the regularized thermo-elastodynamic system (2.21)–(2.22). In the following lemma, we summarize the compactness properties of such sequences.

Lemma 5.1. *Let $(y_\varepsilon, \theta_\varepsilon)$ be a sequence of solutions to (2.21)–(2.22) with initial data $(y_{0,\varepsilon}, y_{0,\varepsilon}', \theta_0)$ given by Theorem 2.2. Then, there exists $(y,\theta) \in (L^\infty(I;\mathcal{Y}_{\mathbf{id}}) \cap H^1(I;H^1(\Omega;\mathbb{R}^d))) \times L^1(I;W_+^{1,1}(\Omega))$ such that, up to a subsequence, it holds for any $q \in (1, 2^*)$ that*

$$y_\varepsilon \overset{*}{\rightharpoonup} y \text{ weakly* in } L^\infty(I;W^{2,p}(\Omega;\mathbb{R}^d)) \text{ and weakly in } H^1(I;H^1(\Omega;\mathbb{R}^d)), \tag{5.3a}$$
$$y_\varepsilon \to y \text{ in } L^\infty(I;W^{1,\infty}(\Omega;\mathbb{R}^d)) \text{ and in } L^2(I;W^{2,p}(\Omega;\mathbb{R}^d)), \tag{5.3b}$$
$$\partial_t y_\varepsilon \to \partial_t y \text{ in } L^2(I;L^q(\Omega;\mathbb{R}^d)), \tag{5.3c}$$
$$\theta_\varepsilon \rightharpoonup \theta \quad \text{and} \quad w_\varepsilon \rightharpoonup w \text{ weakly in } L^r(I;W^{1,r}(\Omega)) \text{ for any } r \in [1, \tfrac{d+2}{d+1}), \tag{5.3d}$$
$$\theta_\varepsilon \to \theta \quad \text{and} \quad w_\varepsilon \to w \text{ in } L^s(I\times\Omega) \text{ for any } s \in [1, \tfrac{d+2}{d}). \tag{5.3e}$$

where $w := W^{\mathrm{in}}(\nabla y, \theta)$.

Proof. The convergences in (5.3a) and (5.3d)–(5.3e) as well as the first convergence in (5.3b) are obtained arguing as in the proof of Proposition 3.10, relying on the estimates (4.2a)–(4.2e). As for (5.3c), we apply the Aubin-Lions' lemma as follows: by (4.2b) we have that $\partial_t y_\varepsilon$ is bounded in $L^2(I;H^1(\Omega;\mathbb{R}^d))$ and Proposition 4.3 yields that $\partial_{tt}^2 y_\varepsilon$ is bounded in $L^2(I;(H^3(\Omega;\mathbb{R}^d) \cap H_0^1(\Omega;\mathbb{R}^d))^*)$. Hence, (5.3c) holds by the compact embedding $H^1(\Omega;\mathbb{R}^d) \subset\subset L^q(\Omega;\mathbb{R}^d)$ for $q < 2^*$. The identification $w = W^{\mathrm{in}}(\nabla y, \theta)$ follows as in the proof of Proposition 3.10, cf. also [2, Lemma 4.2]. Note that by (4.3a) and elliptic regularity we follow that y_ε is bounded in $L^2(I;H^3(\Omega;\mathbb{R}^d))$. Consequently, yet another application of the Aubin-Lions' lemma using also the boundedness of $\partial_t y_\varepsilon$ in $L^2(I\times\Omega;\mathbb{R}^d)$ shows the second convergence in (5.3b). ☐

5.1. **The mechanical equation.** We recall that the mechanical equation (2.21) is equivalent to the formulation in (4.6).

Proposition 5.2. *Let (y,θ) be as in Lemma 5.1. Then, (y,θ) satisfies* (2.26).

Proof. We test (4.6) with $z \in C^\infty(I \times \overline{\Omega}; \mathbb{R}^d)$ with $z = 0$ in $I \times \partial\Omega$ and $z(T) = 0$. Thanks to the convergences in (5.3a)–(5.3e) and to assumptions ((W.1)), ((H.1)) and ((H.3)), ((C.1)), ((D.1)), and (5.1), we have that

$$\int_I \int_\Omega \partial_F W(\nabla y_\varepsilon, \theta_\varepsilon) : \nabla z \,\mathrm{d}x\,\mathrm{d}t \to \int_I \int_\Omega \partial_F W(\nabla y, \theta) : \nabla z \,\mathrm{d}x\,\mathrm{d}t\,, \tag{5.4a}$$

$$\int_I \int_\Omega DH(\Delta y_\varepsilon) \cdot \Delta z \,\mathrm{d}x\,\mathrm{d}t \to \int_I \int_\Omega DH(\Delta y) \cdot \Delta z \,\mathrm{d}x\,\mathrm{d}t\,, \tag{5.4b}$$

$$\int_I \int_\Omega \partial_{\dot F} R(\nabla y_\varepsilon, \partial_t \nabla y_\varepsilon, \theta_\varepsilon) : \nabla z \,\mathrm{d}x\,\mathrm{d}t \to \int_I \int_\Omega \partial_{\dot F} R(\nabla y, \partial_t \nabla y, \theta) : \nabla z \,\mathrm{d}x\,\mathrm{d}t\,, \tag{5.4c}$$

$$\int_I \int_\Omega \partial_t y_\varepsilon \cdot \partial_t z \,\mathrm{d}x\,\mathrm{d}t \to \int_I \int_\Omega \partial_t y \cdot \partial_t z \,\mathrm{d}x\,\mathrm{d}t\,, \tag{5.4d}$$

$$\int_\Omega y_{0,\varepsilon} \cdot z(0) \,\mathrm{d}x \to \int_\Omega y_0 \cdot z(0) \,\mathrm{d}x\,. \tag{5.4e}$$

In particular, in (5.4b) we have used (5.3b) and in (5.4c) we have exploited the linear structure of $\partial_{\dot F} R$ with respect to $\partial_t y_\varepsilon$. Finally, estimate (4.2b) implies that

$$\varepsilon \int_I \int_\Omega \partial_t \nabla \Delta y_\varepsilon : \nabla \Delta z \,\mathrm{d}x\,\mathrm{d}t \to 0\,.$$

Hence, the pair (y,θ) satisfies (2.26). □

5.2. **The heat-transfer equation.** We are left to consider the limit as $\varepsilon \to 0$ in the heat-transfer equation (2.22). To this purpose, we now derive a weaker form of the regularized heat equation which is suitable for the limit procedure. This relies on integration by parts and on the chain rule for the mechanical energy. For notational convenience, for $\psi \in C^\infty(\overline{\Omega})$ we define

$$\mathcal{E}(y,\theta;\psi) = \int_\Omega \Big(W^{\mathrm{el}}(\nabla y) + H(\Delta y) + W^{\mathrm{in}}(y,\theta) \Big) \psi \,\mathrm{d}x. \tag{5.5}$$

Notice that for the limiting passage we will also crucially use the bounds in (4.3).

Proposition 5.3. *For every $\varepsilon > 0$, every $\varphi \in C^\infty(I \times \overline{\Omega})$ of the form $\varphi = \psi\eta$ for $\psi \in C^\infty(\overline{\Omega})$ and $\eta \in C^\infty(I)$ with $\eta(T) = 0$ it holds*

$$\begin{aligned}
0 = &\int_I \int_\Omega \eta\, \mathcal{K}(\nabla y_\varepsilon, \theta_\varepsilon) \nabla \theta_\varepsilon \cdot \nabla \psi - \kappa \int_I \int_{\partial\Omega} \eta\psi\, (\theta_\flat - \theta_\varepsilon) \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t - \int_I \int_\Omega \psi\eta\, f \cdot \partial_t y_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \\
&- \int_I \partial_t \eta \Big(\mathcal{E}(y_\varepsilon, \theta_\varepsilon; \psi) + \int_\Omega \frac{\rho}{2} |\partial_t y_\varepsilon|^2 \psi \,\mathrm{d}x \Big) \mathrm{d}t - \eta(0) \Big(\mathcal{E}(y_{0,\varepsilon}, \theta_0; \psi) + \int_\Omega \frac{\rho}{2} |y'_{0,\varepsilon}|^2 \psi \,\mathrm{d}x \Big) \\
&+ \int_I \int_\Omega \eta \Big(\partial_F W(\nabla y_\varepsilon, \theta_\varepsilon) + \partial_{\dot F} R(\nabla y_\varepsilon, \partial_t \nabla y_\varepsilon, \theta_\varepsilon) \Big) : (\partial_t y_\varepsilon \otimes \nabla \psi) \,\mathrm{d}x\,\mathrm{d}t \\
&- \int_I \int_\Omega \eta\, DH(\Delta y_\varepsilon) \cdot \partial_t y_\varepsilon \Delta \psi \,\mathrm{d}x\,\mathrm{d}t - 2 \int_I \int_\Omega \eta\, \nabla (DH(\Delta y_\varepsilon)) : (\partial_t y_\varepsilon \otimes \nabla \psi) \,\mathrm{d}x\,\mathrm{d}t \\
&- \varepsilon \int_I \int_\Omega \eta\, \partial_t \Delta^2 y_\varepsilon \cdot \big(2 \partial_t \nabla y_\varepsilon \nabla \psi + \mathrm{div}(\partial_t y_\varepsilon \otimes \nabla \psi) \big) \,\mathrm{d}x\,\mathrm{d}t \\
&+ \varepsilon \int_I \int_\Omega \eta\, \partial_t \nabla \Delta y_\varepsilon : \partial_t \nabla y_\varepsilon \Delta \psi \,\mathrm{d}x\,\mathrm{d}t - 2\varepsilon \int_I \int_\Omega \eta\, \partial_t \nabla \Delta y_\varepsilon : \partial_t \nabla y_\varepsilon \nabla^2 \psi \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{5.6}$$

Note that except for the ε-dependent terms this formulation coincides with the one in (2.27).

Proof. For $\varepsilon > 0$, we define $z := \eta\psi\partial_t y_\varepsilon$ and, by (4.2b), we note that $z \in L^2(I; H^3(\Omega; \mathbb{R}^d))$ and $z = 0$ in $I \times \partial\Omega$. Recall that $\partial^2_{tt} y_\varepsilon \in L^2(I; (H^3(\Omega; \mathbb{R}^d) \cap H^1_0(\Omega; \mathbb{R}^d))^*)$ by Proposition 4.3. Testing the heat-transfer equation (2.22) and performing an integration by parts in time we may write

$$\Pi := \int_I\int_\Omega \eta\psi\,\xi(\nabla y_\varepsilon,\partial_t\nabla y_\varepsilon,\theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t + \varepsilon\int_I\int_\Omega \eta\psi\,|\partial_t\nabla\Delta y_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t + \int_I\int_\Omega \eta\psi\,\partial_F W^{\mathrm{cpl}}(\nabla y_\varepsilon,\theta_\varepsilon) : \partial_t\nabla y_\varepsilon\;\mathrm{d}x\,\mathrm{d}t$$
$$= \int_I\int_\Omega \Big(\eta\,\mathcal{K}(\nabla y_\varepsilon,\theta_\varepsilon)\nabla\theta_\varepsilon\cdot\nabla\psi - \psi\,w_\varepsilon\partial_t\eta\Big)\;\mathrm{d}x\,\mathrm{d}t - \kappa\int_I\int_{\partial\Omega}\eta\psi\,(\theta_\flat-\theta_\varepsilon)\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t - \int_\Omega w_{0,\varepsilon}\psi\eta(0)\,\mathrm{d}x. \quad (5.7)$$

Our goal is to rewrite the terms on the left hand side, i.e., Π. To this end, we test the regularized mechanical equation (2.21) with $z = \eta\psi\partial_t y_\varepsilon$: this yields

$$\int_I\int_\Omega \eta\Big(\partial_F W^{\mathrm{el}}(\nabla y_\varepsilon) + \partial_F W^{\mathrm{cpl}}(\nabla y_\varepsilon,\theta_\varepsilon)\Big) : \nabla(\psi\partial_t y_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t + \int_I\int_\Omega \eta\partial_{\dot F}R(\nabla y_\varepsilon,\partial_t\nabla y_\varepsilon,\theta_\varepsilon) : \nabla(\psi\partial_t y_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\,\varepsilon\int_I\int_\Omega \eta\,\partial_t\nabla\Delta y_\varepsilon : \nabla\Delta(\psi\partial_t y_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t + \int_I\int_\Omega \eta DH(\Delta y_\varepsilon)\cdot\Delta(\psi\partial_t y_\varepsilon)\;\mathrm{d}x\,\mathrm{d}t$$
$$= \int_I\int_\Omega \psi\eta\, f\cdot\partial_t y_\varepsilon\,\mathrm{d}x\,\mathrm{d}t - \rho\int_I \big\langle\partial^2_{tt}y_\varepsilon, \partial_t y_\varepsilon\,\eta\psi\big\rangle\,\mathrm{d}t. \quad (5.8)$$

We expand the terms on the left-hand side of (5.8) by expanding $\nabla(\psi\partial_t y_\varepsilon)$, $\nabla\Delta(\psi\partial_t y_\varepsilon)$, and $\Delta(\psi\partial_t y_\varepsilon)$. This yields

$$\int_I\int_\Omega \eta\Big(\partial_F W^{\mathrm{el}}(\nabla y_\varepsilon) + \partial_F W^{\mathrm{cpl}}(\nabla y_\varepsilon,\theta_\varepsilon)\Big) : \nabla(\psi\partial_t y_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t = \int_I\int_\Omega \eta\psi\,\partial_F W^{\mathrm{cpl}}(\nabla y_\varepsilon,\theta_\varepsilon) : \partial_t\nabla y_\varepsilon\,\mathrm{d}x\,\mathrm{d}t + J_0 + J_1,$$
$$\int_I\int_\Omega \eta\partial_{\dot F}R(\nabla y_\varepsilon,\partial_t\nabla y_\varepsilon,\theta_\varepsilon) : \nabla(\psi\partial_t y_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t = \int_I\int_\Omega \eta\psi\,\partial_{\dot F}R(\nabla y_\varepsilon,\partial_t\nabla y_\varepsilon,\theta_\varepsilon) : \partial_t\nabla y_\varepsilon\,\mathrm{d}x\,\mathrm{d}t + J_2,$$
$$\varepsilon\int_I\int_\Omega \eta\,\partial_t\nabla\Delta y_\varepsilon : \nabla\Delta(\psi\partial_t y_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t = \varepsilon\int_I\int_\Omega \eta\psi\,|\partial_t\nabla\Delta y_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t + J_3 + J_4,$$
$$\int_I\int_\Omega \eta DH(\Delta y_\varepsilon)\cdot\Delta(\psi\partial_t y_\varepsilon)\;\mathrm{d}x\,\mathrm{d}t = \int_I\int_\Omega \eta\psi\,DH(\Delta y_\varepsilon)\cdot\partial_t\Delta y_\varepsilon\;\mathrm{d}x\,\mathrm{d}t + J_5, \quad (5.9)$$

where for brevity we have written

$$J_0 := \int_I\int_\Omega \eta\psi\,\partial_F W^{\mathrm{el}}(\nabla y_\varepsilon) : \partial_t\nabla y_\varepsilon\,\mathrm{d}x\,\mathrm{d}t,$$
$$J_1 := \int_I\int_\Omega \eta\Big(\partial_F W^{\mathrm{el}}(\nabla y_\varepsilon) + \partial_F W^{\mathrm{cpl}}(\nabla y_\varepsilon,\theta_\varepsilon)\Big) : (\partial_t y_\varepsilon\otimes\nabla\psi)\;\mathrm{d}x\;\mathrm{d}t,$$
$$J_2 := \int_I\int_\Omega \eta\,\partial_{\dot F}R(\nabla y_\varepsilon,\partial_t\nabla y_\varepsilon,\theta_\varepsilon) : (\partial_t y_\varepsilon\otimes\nabla\psi)\,\mathrm{d}x\,\mathrm{d}t,$$
$$J_3 := \varepsilon\int_I\int_\Omega \eta\,\partial_t\nabla\Delta y_\varepsilon : (\partial_t\Delta y_\varepsilon\otimes\nabla\psi)\,\mathrm{d}x\,\mathrm{d}t,$$
$$J_4 := \varepsilon\int_I\int_\Omega \eta\,\partial_t\nabla\Delta y_\varepsilon : \nabla(\partial_t\nabla y_\varepsilon\nabla\psi)\,\mathrm{d}x\,\mathrm{d}t + \varepsilon\int_I\int_\Omega \eta\,\partial_t\nabla\Delta y_\varepsilon : \nabla(\mathrm{div}(\partial_t y_\varepsilon\otimes\nabla\psi))\,\mathrm{d}x\,\mathrm{d}t,$$
$$J_5 := \int_I\int_\Omega \eta\,DH(\Delta y_\varepsilon)\cdot\partial_t y_\varepsilon\Delta\psi\;\mathrm{d}x\,\mathrm{d}t + 2\int_I\int_\Omega \eta\,DH(\Delta y_\varepsilon)\cdot\partial_t\nabla y_\varepsilon\nabla\psi\;\mathrm{d}x\,\mathrm{d}t. \quad (5.10)$$

Recalling (2.11), we see that the first three terms on the right-hand sides of (5.9) correspond to the terms in (5.7) whose sum is denoted by Π. This along with (5.8) yields

$$\Pi + \sum_{i=0}^4 J_i = \int_I\int_\Omega \psi\eta\, f\cdot\partial_t y_\varepsilon\,\mathrm{d}x\,\mathrm{d}t - \int_I\int_\Omega \eta DH(\Delta y_\varepsilon)\cdot\Delta(\psi\partial_t y_\varepsilon)\;\mathrm{d}x\,\mathrm{d}t - \rho\int_I \big\langle\partial^2_{tt}y_\varepsilon, \partial_t y_\varepsilon\,\eta\psi\big\rangle\,\mathrm{d}t.$$

Using the last equation in (5.9) and the definition of J_0, we get

$$\Pi + \sum_{i=1}^5 J_i = \int_I\int_\Omega \psi\eta\,\Big(f\cdot\partial_t y_\varepsilon - \partial_F W^{\mathrm{el}}(\nabla y_\varepsilon) : \partial_t\nabla y_\varepsilon - DH(\Delta y_\varepsilon)\cdot\partial_t\Delta y_\varepsilon\Big)\;\mathrm{d}x\,\mathrm{d}t - \rho\int_I \big\langle\partial^2_{tt}y_\varepsilon, \partial_t y_\varepsilon\,\eta\psi\big\rangle\,\mathrm{d}t\,. \quad (5.11)$$

Using the chain rule and $\eta(T) = 0$, we rewrite the last term on the right-hand side as

$$\rho\int_I \big\langle\partial^2_{tt}y_\varepsilon, \partial_t y_\varepsilon\psi\eta\big\rangle\,\mathrm{d}t = \frac{\rho}{2}\int_I\int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}(|\partial_t y_\varepsilon|^2\eta)\psi\,\mathrm{d}x\,\mathrm{d}t - \frac{\rho}{2}\int_I\int_\Omega \psi|\partial_t y_\varepsilon|^2\partial_t\eta\,\mathrm{d}x\,\mathrm{d}t \quad (5.12)$$
$$= -\frac{\rho}{2}\int_\Omega \psi\eta(0)\,|y'_{0,\varepsilon}|^2\;\mathrm{d}x - \frac{\rho}{2}\int_I\int_\Omega \psi|\partial_t y_\varepsilon|^2\partial_t\eta\,\mathrm{d}x\,\mathrm{d}t.$$

By the chain rule for the mechanical energy, see (3.74) (for y_ε in place of y_h), and integration by parts we have

$$\int_I\int_\Omega \eta\psi\,\Big(\partial_F W^{\rm el}(\nabla y_\varepsilon):\partial_t\nabla y_\varepsilon + DH(\Delta y_\varepsilon)\cdot\partial_t\Delta y_\varepsilon\Big)\,\mathrm{d}x\,\mathrm{d}t = \int_I \eta\,\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \psi\big(W^{\rm el}(\nabla y_\varepsilon)+H(\Delta y_\varepsilon)\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$= -\int_\Omega \eta(0)\,\psi\big(W^{\rm el}(\nabla y_{0,\varepsilon})+H(\Delta y_{0,\varepsilon})\big)\,\mathrm{d}x - \int_I\int_\Omega \partial_t\eta\,\psi\big(W^{\rm el}(\nabla y_\varepsilon)+H(\Delta y_\varepsilon)\big)\,\mathrm{d}x\,\mathrm{d}t. \tag{5.13}$$

Next, we manipulate J_3, J_4, J_5. Recall that $\Delta y_\varepsilon = 0$ and $\partial_t y_\varepsilon = 0$ on $\partial\Omega$ for a.e. $t\in I$ (see (4.4)) and $DH(0)=0$ by (2.4)–(2.5). We perform an integration by parts in J_5 to get

$$J_5 = \int_I\int_\Omega \eta\,DH(\Delta y_\varepsilon)\cdot\partial_t y_\varepsilon\Delta\psi\;\mathrm{d}x\,\mathrm{d}t - 2\int_I\int_\Omega \eta\,\nabla(DH(\Delta y_\varepsilon)):(\partial_t y_\varepsilon\otimes\nabla\psi)\;\mathrm{d}x\,\mathrm{d}t$$
$$- 2\int_I\int_\Omega \eta\,DH(\Delta y_\varepsilon)\cdot\partial_t y_\varepsilon\Delta\psi\;\mathrm{d}x\,\mathrm{d}t$$
$$= -\int_I\int_\Omega \eta\,DH(\Delta y_\varepsilon)\cdot\partial_t y_\varepsilon\Delta\psi\;\mathrm{d}x\,\mathrm{d}t - 2\int_I\int_\Omega \eta\,\nabla(DH(\Delta y_\varepsilon)):(\partial_t y_\varepsilon\otimes\nabla\psi)\;\mathrm{d}x\,\mathrm{d}t. \tag{5.14}$$

By integrating by parts and recalling the boundary conditions $\Delta\partial_t y_\varepsilon = \partial_\nu\Delta\partial_t y_\varepsilon = 0$ on $\partial\Omega$ for a.e. $t\in I$ (see (4.4)), we rewrite J_4 as

$$J_4 = -\varepsilon\int_I\int_\Omega \eta\,\partial_t\Delta^2 y_\varepsilon\cdot\big(\partial_t\nabla y_\varepsilon\nabla\psi + \mathrm{div}(\partial_t y_\varepsilon\otimes\nabla\psi)\big)\,\mathrm{d}x\,\mathrm{d}t, \tag{5.15}$$

and, by elementary but tedious computations, J_3 can be written as

$$\frac{1}{\varepsilon}J_3 = \int_I\int_\Omega \eta\,\partial_t\nabla\Delta y_\varepsilon:(\partial_t\Delta y_\varepsilon\otimes\nabla\psi)\,\mathrm{d}x\,\mathrm{d}t = -\int_I\int_\Omega \eta\,\partial_t\Delta y_\varepsilon\cdot\big((\partial_t\nabla\Delta y_\varepsilon\nabla\psi)+\partial_t\Delta y_\varepsilon\Delta\psi\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_I\int_\Omega \eta\,\partial_t\nabla\Delta y_\varepsilon:\big(\partial_t\nabla^2 y_\varepsilon\nabla\psi + \partial_t\nabla y_\varepsilon\Delta\psi\big)\,\mathrm{d}x\,\mathrm{d}t + \int_I\int_\Omega \eta\,\partial_t\Delta y_\varepsilon\cdot\big(\partial_t\nabla^2 y_\varepsilon\nabla^2\psi + \partial_t\nabla y_\varepsilon\nabla\Delta\psi\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$= -\int_I\int_\Omega \eta\,\partial_t\Delta^2 y_\varepsilon\cdot(\partial_t\nabla y_\varepsilon\nabla\psi)\,\mathrm{d}x\,\mathrm{d}t + \int_I\int_\Omega \eta\,\partial_t\nabla\Delta y_\varepsilon:\partial_t\nabla y_\varepsilon\Delta\psi\,\mathrm{d}x\,\mathrm{d}t$$
$$- 2\int_I\int_\Omega \eta\,\partial_t\nabla\Delta y_\varepsilon:\partial_t\nabla y_\varepsilon\nabla^2\psi\,\mathrm{d}x\,\mathrm{d}t\,. \tag{5.16}$$

Recalling the definition in (5.5) and combining (5.7) with (5.10)–(5.16) we obtain the statement. More precisely, (5.12) and (5.13) contribute to the second line in (5.6), the third line corresponds to J_1+J_2 (see (5.10)) and the last three lines correspond to $J_3+J_4+J_5$ (see (5.14)–(5.16)). ☐

We are now in a position to pass to the limit as $\varepsilon\to 0$ in the modified heat-transfer equation (5.6). This will conclude the proof of Theorem 2.5.

Proposition 5.4. *Let (y,θ) be given by Lemma 5.1. Then, (y,θ) satisfies* (2.27).

Proof. In order to show (2.27), by a density argument it suffices to consider test functions of the form $\varphi=\psi\eta$ for $\psi\in C^\infty(\overline{\Omega})$ and $\eta\in C^\infty(I)$ with $\eta(T)=0$. We test (5.6) with φ and pass to the limit term by term.

Thanks to the convergences stated in Lemma 5.1, the estimates (4.2)–(4.3), and the assumptions (W.1)–(W.3), (H.1)–(H.3), (C.1)–(C.5), (D.1)–(D.2), and (5.1), we have that each of the terms in the first three lines of (5.6) converges to the respective term in the first three lines of (2.27) (with $\eta\psi$ in place of φ). Here, we againg exploit that $\partial_{\dot F}R$ is linear in the second entry, cf. (D.1). Therefore, it suffices to show

$$\lim_{\varepsilon\to 0}\Big(\int_I\int_\Omega \eta\,DH(\Delta y_\varepsilon)\cdot\partial_t y_\varepsilon\Delta\psi\;\mathrm{d}x\,\mathrm{d}t + 2\int_I\int_\Omega \eta\,\nabla(DH(\Delta y_\varepsilon)):(\partial_t y_\varepsilon\otimes\nabla\psi)\;\mathrm{d}x\,\mathrm{d}t\Big)$$
$$= \int_I\int_\Omega \eta\,DH(\Delta y)\cdot\partial_t y\Delta\psi\;\mathrm{d}x\,\mathrm{d}t + 2\int_I\int_\Omega \eta\,\nabla(DH(\Delta y)):(\partial_t y\otimes\nabla\psi)\;\mathrm{d}x\,\mathrm{d}t, \tag{5.17}$$

$$\lim_{\varepsilon\to 0}\varepsilon\int_I\int_\Omega \eta\,\partial_t\Delta^2 y_\varepsilon\cdot\big(2\partial_t\nabla y_\varepsilon\nabla\psi + \mathrm{div}(\partial_t y_\varepsilon\otimes\nabla\psi)\big)\,\mathrm{d}x\,\mathrm{d}t = 0, \tag{5.18}$$

$$\lim_{\varepsilon\to 0}\Big(\varepsilon\int_I\int_\Omega \eta\,\partial_t\nabla\Delta y_\varepsilon:\partial_t\nabla y_\varepsilon\Delta\psi\,\mathrm{d}x\,\mathrm{d}t - 2\varepsilon\int_I\int_\Omega \eta\,\partial_t\nabla\Delta y_\varepsilon:\partial_t\nabla y_\varepsilon\nabla^2\psi\,\mathrm{d}x\,\mathrm{d}t\Big) = 0. \tag{5.19}$$

We start with (5.17). Recalling that $\partial_t y_\varepsilon$ converges strongly in $L^2(I; L^p(\Omega;\mathbb{R}^d))$ by (5.3c) (recall $p < 6$ for $d = 3$), the key point is to show the strong (resp. weak) convergence of $DH(\Delta y_\varepsilon)$ (resp. $\nabla(DH(\Delta y_\varepsilon))$) in $L^2(I; L^{p'}(\Omega))$. Since $y_\varepsilon \to y$ in $L^2(I; W^{2,p}(\Omega;\mathbb{R}^d))$ (cf. (5.3b)), we infer by (H.3) that

$$\eta\, DH(\Delta y_\varepsilon) \;\to\; \eta DH(\Delta y) \qquad \text{in } L^2(I; L^{p'}(\Omega;\mathbb{R}^d)). \tag{5.20}$$

In view of (4.3b), by weak compactness we also have that

$$\eta\, \nabla(DH(\Delta y_\varepsilon)) \;\rightharpoonup\; \eta\nabla(DH(\Delta y)) \qquad \text{in } L^2(I; L^{p'}(\Omega;\mathbb{R}^{d\times d})). \tag{5.21}$$

Now, (5.20)–(5.21) along with (5.3c) show (5.17).

Eventually, by Hölder's inequality, we get that

$$\begin{aligned}
&\left|\int_I\int_\Omega \eta\, \partial_t\Delta^2 y_\varepsilon \cdot \big(2\partial_t\nabla y_\varepsilon\nabla\psi + \mathrm{div}(\partial_t y_\varepsilon \otimes \nabla\psi)\big)\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\qquad \le C\|\partial_t\Delta^2 y_\varepsilon\|_{L^2(I\times\Omega)}\big(\|\eta\nabla\psi\|_{L^\infty(\Omega)} + \|\eta\nabla^2\psi\|_{L^\infty(\Omega)}\big)\big(\|\partial_t y_\varepsilon\|_{L^2(I\times\Omega)} + \|\,\partial_t\nabla y_\varepsilon\|_{L^2(I\times\Omega)}\big)\,, \\
&\left|\int_I\int_\Omega \eta\, \partial_t\nabla\Delta y_\varepsilon : \big(2\partial_t\nabla y_\varepsilon\nabla^2\psi - \partial_t\nabla y_\varepsilon\Delta\psi\big)\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\qquad \le C\|\partial_t\nabla\Delta y_\varepsilon\|_{L^2(I\times\Omega)}\|\eta\nabla^2\psi\|_{L^\infty(\Omega)}\|\partial_t\nabla y_\varepsilon\|_{L^2(I\times\Omega)}\,.
\end{aligned}$$

Thus, we infer from (4.2b) and (4.3c) (recall $\varrho < 1$) that (5.18)–(5.19) hold. This concludes the proof. ☐

Proof of Theorem 2.5. The weak formulation follows from Propositions 5.2 and 5.4. We deduce the regularity $y \in L^2(I; H^3(\Omega;\mathbb{R}^d))$ and $(1+|\Delta y|)^{\frac{p-2}{2}}|\nabla\Delta y|^2 \in L^2(I\times\Omega)$ by the bound (4.3a) and lower semicontinuity of norms as $\varepsilon \to 0$, again applying an elliptic regularity estimate. ☐

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Appendix A. A special case of elliptic regularity

We formulate and prove the lemma used in Subsection 3.3.

Lemma A.1 (A special case of elliptic regularity). *Consider the Banach space $X := W^{2,q}(\Omega;\mathbb{R}^d) \cap H^1_0(\Omega;\mathbb{R}^d)$ for some $q > 1$. Moreover, let $u \in H^3(\Omega;\mathbb{R}^d) \cap H^1_0(\Omega;\mathbb{R}^d)$ and $g \in X^*$ be such that for all $z \in C^\infty(\overline{\Omega};\mathbb{R}^d)$ with $z = 0$ on $\partial\Omega$ it holds that*

$$\int_\Omega \nabla\Delta u : \nabla\Delta z = \langle g, z\rangle, \tag{A.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the dual pairing between X and X^*. Then, the following holds true:

(a) We have that $u \in W^{4,q'}(\Omega;\mathbb{R}^d)$ with

$$\|u\|_{W^{4,q'}(\Omega)} \le C\|g\|_{X^*} + C|\mu| \tag{A.2}$$

for a constant $C > 0$ only depending on Ω, where $\mu := \fint_\Omega \Delta u\,\mathrm{d}x$. Moreover, the following boundary condition holds true:

$$\partial_\nu\Delta u = 0 \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega. \tag{A.3}$$

(b) *If we additionally have* $g \in H^{-1}(\Omega;\mathbb{R}^d)$, *then* $u \in H^5(\Omega;\mathbb{R}^d)$ *with*

$$\|u\|_{H^5(\Omega)} \le C\|g\|_{H^{-1}(\Omega)} + C|\mu|, \tag{A.4}$$

satisfying the boundary condition

$$\Delta^2 u = 0 \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega. \tag{A.5}$$

Proof. Step 1 ($W^{4,q'}$-regularity): Using (A.1) and integrating by parts we see that for all $z \in C^\infty(\overline{\Omega};\mathbb{R}^d)$ with $z = \partial_\nu \Delta z = 0$ on $\partial\Omega$ it holds that

$$\langle g, z\rangle = \int_\Omega \nabla\Delta u : \nabla\Delta z \,\mathrm{d}x = -\int_\Omega \Delta u\cdot\Delta^2 z\,\mathrm{d}x + \int_{\partial\Omega}\Delta u\cdot\partial_\nu\Delta z\,\mathrm{d}\mathcal{H}^{d-1} = -\int_\Omega \Delta u\cdot\Delta^2 z\,\mathrm{d}x. \tag{A.6}$$

By representation of the dual space we find $G, G_j, G_{ij} \in L^{q'}(\Omega;\mathbb{R}^d)$ such that

$$\langle g, z\rangle = \int_\Omega G\cdot z\,\mathrm{d}x + \sum_{j=1}^d G_j\cdot\partial_j z\,\mathrm{d}x + \sum_{i,j=1}^d G_{ij}\cdot\partial^2_{ij} z\,\mathrm{d}x \tag{A.7}$$

with $\|G\|_{L^{q'}(\Omega)}+\sum_{j=1}^d \|G_j\|_{L^{q'}(\Omega)}+\sum_{i,j=1}^d \|G_{ij}\|_{L^{q'}(\Omega)} \le C\|g\|_{X^*}$. We approximate G, G_j, and G_{ij} by sequences $G^k, G^k_j, G^k_{ij} \in C_c^\infty(\Omega;\mathbb{R}^d)$ converging to the respective functions in $L^{q'}(\Omega;\mathbb{R}^d)$. We set

$$g_k \coloneqq G_k - \sum_{j=1}^d \partial_j G^k_j + \sum_{i,j=1}^d \partial^2_{ij} G^k_{ij},$$

and let $v_k \in H^1_0(\Omega;\mathbb{R}^d)$ be the weak solution to

$$\begin{cases} -\Delta v_k = g_k & \text{in } \Omega,\\ \quad v_k = 0 & \text{on } \partial\Omega.\end{cases}$$

In particular, by integration by parts, for all $z \in C^\infty(\overline{\Omega};\mathbb{R}^d)$ with $z = 0$ on $\partial\Omega$ it holds that

$$\int_\Omega v_k\cdot\Delta z\,\mathrm{d}x = -\int_\Omega g_k\cdot z\,\mathrm{d}x = -\int_\Omega\Big(G_k\cdot z + \sum_{j=1}^d G^k_j\cdot\partial_j z + \sum_{i,j=1}^d G^k_{ij}\cdot\partial^2_{ij} z\Big)\,\mathrm{d}x. \tag{A.8}$$

Moreover, let $w_k \in H^1_0(\Omega;\mathbb{R}^d)$ be the weak solution to

$$\begin{cases} \Delta w_k = |v_k|^{\frac{2-q}{q-1}} v_k & \text{in } \Omega,\\ \quad w_k = 0 & \text{on } \partial\Omega.\end{cases}$$

As Ω has C^5-boundary, by elliptic regularity we see that $w_k \in W^{2,q}(\Omega;\mathbb{R}^d)$ with

$$\|w_k\|_{W^{2,q}(\Omega)} \le C\||v_k|^{\frac{2-q}{q-1}} v_k\|_{L^q(\Omega)} = C\|v_k\|^{1/(q-1)}_{L^{q'}(\Omega)},$$

where the constant C only depends on Ω. With (A.8), this shows

$$\begin{aligned}\|v_k\|^{q'}_{L^{q'}(\Omega)} &= \int_\Omega v_k\cdot\Delta w_k\,\mathrm{d}x = -\int_\Omega g_k\cdot w_k\,\mathrm{d}x\\ &\le C\Big(\|G_k\|_{L^{q'}(\Omega)} + \sum_{j=1}^d\|G^k_j\|_{L^{q'}(\Omega)} + \sum_{i,j=1}^d \|G^k_{ij}\|_{L^{q'}(\Omega)}\Big)\|w_k\|_{W^{2,q}(\Omega)}\\ &\le C\|g\|_{X^*}\|v_k\|^{1/(q-1)}_{L^{q'}(\Omega)}.\end{aligned} \tag{A.9}$$

Consequently, there exists $v \in L^{q'}(\Omega;\mathbb{R}^d)$ such that, up to selecting a subsequence, $v_k \rightharpoonup v$ weakly in $L^{q'}(\Omega;\mathbb{R}^d)$. Passing to the limit $k\to\infty$ in (A.8) and (A.9), and recalling (A.7), we discover that v satisfies

$$\int_\Omega v\cdot\Delta z\,\mathrm{d}x = -\langle g, z\rangle, \tag{A.10}$$

$$\|v\|_{L^{q'}(\Omega)} \le C\|g\|_{X^*}, \tag{A.11}$$

for all $z \in C^\infty(\overline{\Omega};\mathbb{R}^d)$ with $z = 0$ on $\partial\Omega$. Now, let $w \in H^1(\Omega;\mathbb{R}^d)$ with $⨍_\Omega w\,\mathrm{d}x = 0$ be the weak solution to

$$\begin{cases} \Delta w = v - m & \text{in } \Omega,\\ \partial_\nu w = 0 & \text{on } \partial\Omega,\end{cases} \tag{A.12}$$

with $m := \fint_\Omega v \,\mathrm{d}x$, i.e., $\int_\Omega w \,\mathrm{d}x = 0$ and $-\int_\Omega \nabla w : \nabla z \,\mathrm{d}x = \int_\Omega (v - m) \cdot z \,\mathrm{d}x$ for all $z \in H^1(\Omega;\mathbb{R}^d)$. As Ω has C^5-boundary and $v \in L^{q'}(\Omega;\mathbb{R}^d)$, by elliptic regularity (see for instance [34, Chapter 2, Section 5]) and (A.11) we derive that $w \in W^{2,q'}(\Omega;\mathbb{R}^d)$ (i.e., (A.12) holds in a pointwise sense) and

$$\|w\|_{W^{2,q'}(\Omega)} \le C\|v - m\|_{L^{q'}(\Omega)} \le C\|g\|_{X^*}. \tag{A.13}$$

Consequently, for z with $z = \partial_\nu \Delta z = 0$ on $\partial\Omega$ we derive, due to $\partial_\nu w = 0$ for $\mathcal{H}^{d-1}$-a.e. point in $\partial\Omega$, (A.10), and (A.12) that

$$\begin{aligned}\int_\Omega w \cdot \Delta^2 z \,\mathrm{d}x &= -\int_\Omega \nabla w : \nabla \Delta z \,\mathrm{d}x + \int_{\partial\Omega} w \cdot \partial_\nu \Delta z \,\mathrm{d}\mathcal{H}^{d-1} = \int_\Omega \Delta w \cdot \Delta z \,\mathrm{d}x - \int_{\partial\Omega} \partial_\nu w \cdot \Delta z \,\mathrm{d}\mathcal{H}^{d-1} \\ &= \int_\Omega v \cdot \Delta z \,\mathrm{d}x - \int_\Omega m \cdot \Delta z \,\mathrm{d}x = -\langle g, z\rangle - \int_\Omega m \cdot \Delta z \,\mathrm{d}x\,.\end{aligned}$$

Using (A.6) we get

$$\int_\Omega (\Delta u - w) \cdot \Delta^2 z \,\mathrm{d}x = 0 \tag{A.14}$$

for all $z \in C^\infty(\Omega;\mathbb{R}^d)$ with $z = \partial_\nu \Delta z = 0$ at $\mathcal{H}^{d-1}$-a.e. point in $\partial\Omega$ and $\fint_\Omega \Delta z \,\mathrm{d}x = 0$. We now show that $\Delta u - w$ constant a.e. in Ω. To this end, let $\varphi \in C^\infty(\Omega;\mathbb{R}^d)$ with $\fint_\Omega \varphi \,\mathrm{d}x = 0$ be arbitrary. As Ω has C^5-boundary, by elliptic regularity we can find $\tilde z \in H^2(\Omega;\mathbb{R}^d)$ such that $\fint_\Omega \tilde z \,\mathrm{d}x = 0$ and

$$\begin{cases} -\Delta \tilde z = \varphi & \text{in } \Omega, \\ \partial_\nu \tilde z = 0 & \text{on } \partial\Omega, \end{cases}$$

and, subsequently, we can find $z \in H^4(\Omega;\mathbb{R}^d)$ satisfying

$$\begin{cases} -\Delta z = \tilde z & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases}$$

Consequently, with (A.14) this leads to $\int_\Omega (\Delta u - w) \cdot \varphi \,\mathrm{d}x = 0$, and by the arbitrariness of φ to $\Delta u - w$ constant a.e. in Ω. As $\fint_\Omega w \,\mathrm{d}x = 0$, we get $\Delta u - w = \mu = \fint_\Omega \Delta u \,\mathrm{d}x$. We let $\nu \in H^1_0(\Omega;\mathbb{R}^d)$ such that $\Delta\nu = \mu$. As by assumption $u \in H^1_0(\Omega;\mathbb{R}^d)$ and Ω has C^5-boundary, and since $w \in W^{2,q'}(\Omega;\mathbb{R}^d)$, by elliptic regularity we see that $u \in W^{4,q'}(\Omega;\mathbb{R}^d)$ and

$$\|u - \nu\|_{W^{4,q'}(\Omega)} \le C\|\Delta u - \mu\|_{W^{2,q'}(\Omega)} = C\|w\|_{W^{2,q'}(\Omega)}.$$

This along with (A.13) and the elliptic regularity estimate $\|\nu\|_{W^{4,q'}(\Omega)} \le C\|\Delta\nu\|_{W^{2,q'}(\Omega)} \le C|\mu|$ shows (A.2). Finally, (A.3) directly follows from $\Delta u - w$ constant and (A.12). This concludes the proof of (a).

Step 2 (H^5-regularity): From now on, we assume that $g \in H^{-1}(\Omega;\mathbb{R}^d)$. Since then also $g \in H^*$, Step 1 yields $u \in W^{4,q'}(\Omega;\mathbb{R}^d)$. Thus, we can integrate by parts in (A.1) and use (A.3) to derive

$$-\langle g, z\rangle = -\int_\Omega \nabla\Delta u : \nabla\Delta z \,\mathrm{d}x = \int_\Omega \Delta^2 u : \Delta z \,\mathrm{d}x - \int_{\partial\Omega} \partial_\nu \Delta u \cdot \Delta z \,\mathrm{d}\mathcal{H}^{d-1} = \int_\Omega \Delta^2 u : \Delta z \,\mathrm{d}x \tag{A.15}$$

for all $z \in C^\infty(\overline{\Omega};\mathbb{R}^d)$ with $z = 0$ on $\partial\Omega$, where $\langle\cdot,\cdot\rangle$ now denotes the dual pairing between $H^1_0(\Omega;\mathbb{R}^d)$ and $H^{-1}(\Omega;\mathbb{R}^d)$. Let $v \in H^1_0(\Omega;\mathbb{R}^d)$ be the weak solution to

$$\begin{cases} -\Delta v = g & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{A.16}$$

In particular, we have with Poincaré's inequality that

$$\|v\|^2_{H^1(\Omega)} \le C\|\nabla v\|^2_{L^2(\Omega)} = \langle g, v\rangle \le \|g\|_{H^{-1}(\Omega)}\|v\|_{H^1(\Omega)}. \tag{A.17}$$

Furthermore, for all $z \in C^\infty(\overline{\Omega};\mathbb{R}^d)$ with $z = 0$ on $\partial\Omega$ it holds that

$$-\langle g, z\rangle = -\int_\Omega \nabla v : \nabla z \,\mathrm{d}x = \int_\Omega v \cdot \Delta z - \int_{\partial\Omega} v \cdot \partial_\nu z \,\mathrm{d}\mathcal{H}^{d-1} = \int_\Omega v \cdot \Delta z. \tag{A.18}$$

Subtracting this from (A.15), we arrive at

$$\int_\Omega (\Delta^2 u - v) \cdot \Delta z \,\mathrm{d}x = 0$$

for all $z \in C^\infty(\overline{\Omega};\mathbb{R}^d)$ with $z = 0$ on $\partial\Omega$. By an argument similar to the one from Step 1 this leads to $\Delta^2 u = v$ a.e. in Ω. This also shows $\int_\Omega v \,\mathrm{d}x = \int_{\partial\Omega} \partial_\nu \Delta u \,\mathrm{d}\mathcal{H}^{d-1} = 0$ by (A.3). Then, in view of (A.3) and (A.17), we derive by elliptic regularity for Neumann problems (see for instance [34, Chapter 2, Section 5]) that $\Delta u \in H^3(\Omega;\mathbb{R}^d)$ such that

$$\|\Delta u - \mu\|_{H^3(\Omega)} \leq C\|v\|_{H^1(\Omega)} \leq C\|g\|_{H^{-1}(\Omega)},$$

where as before $\mu = \fint_\Omega \Delta u \, \mathrm{d}x$. Hence, as $u = 0$ for $\mathcal{H}^{d-1}$-a.e. point on $\partial\Omega$ and Ω has C^5-boundary, yet another application of elliptic regularity leads to $u \in H^5(\Omega;\mathbb{R}^d)$ and the bound (A.4). Then, (A.5) follows from $\Delta^2 u = v$ a.e. in Ω and (A.16). ☐

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