International Publications (USA)

Communications on Applied Nonlinear Analysis
Volume 14(2007), Number 2, 1–21

On the exact number of bifurcation branches in a square and in a cube[1]

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Communicated by Norman Dancer
(Received May 2006; Accepted August 2006)

Abstract

We study local bifurcation from an eigenvalue with multiplicity greater than one for a class of semilinear elliptic equations. In particular, we obtain the exact number of bifurcation branches of non trivial solutions at every eigenvalue of a square and at the second eigenvalue of a cube. We also compute the Morse index of the solutions in those branches.

Key words: local bifurcation, multiple branches, multiple eigenvalue, Morse index.

AMS Subject Classification: 35B32, 35J20, 35J60.

1 Introduction and main results

Let us consider the problem

$$\begin{cases} -\Delta u = |u|^{p-1}u + \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded open domain in $\mathbb{R}^N$, $N \geq 2$, $p > 1$ and $\lambda \in \mathbb{R}$.

[1]The research of the first author is supported by the M.I.U.R. National Project “Metodi variazionali ed equazioni differenziali nonlineari”. The research of the second author is supported by the M.I.U.R. National Project “Metodi variazionali e topologici nello studio di fenomeni non lineari”.

It has the trivial family of solutions $\{(\lambda, 0) \mid \lambda \in \mathbb{R}\}$. A point $(\lambda^*, 0)$ is called a *bifurcation point* for (1) if every neighborhood of $(\lambda^*, 0)$ contains nontrivial solutions of (1). It is easily seen that a necessary condition for $(\lambda^*, 0)$ to be a bifurcation point is that λ^* is an eigenvalue of the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

We denote by $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_j \leq \ldots$ the sequence of the eigenvalues of the problem (2).

Since problem (1) has a variational structure, the fact that λ^* is an eigenvalue of the problem (2) is not only necessary, but is also a sufficient condition for bifurcation to occur. More precisely in [5] and [13], it was proved that for any eigenvalue λ_j of (2) there exists $r_0 > 0$ such that for any $r \in (0, r_0)$ there are at least two distinct solutions $(\lambda_i(r), u_i(r))$, $i = 1, 2$ of (1) having $\|u_i(r)\| = r$ and in addition $(\lambda_i(r), u_i(r)) \to (\lambda_j, 0)$ as $r \to 0$.

As far as it concerns the structure of the bifurcation set at any eigenvalue λ_j, namely the set of nontrivial solutions (λ, u) of (1) in a neighborhood of λ_j, in [8] the authors provide an accurate description in the case of a simple eigenvalue, by showing that the bifurcation set is a C^1 curve crossing $(\lambda_j, 0)$. If the eigenvalue λ_j has higher multiplicity, in [16] (see also [2]) the author describes the possible behavior of the bifurcating set by showing that the following alternative occurs: either $(\lambda_j, 0)$ is not an isolated solution of (1) in $\{\lambda_j\} \times H_0^1(\Omega)$, or there is a one-sided neighborhood U of λ_j such that for all $\lambda \in U \setminus \{\lambda_j\}$ problem (1) has at least two distinct nontrivial solutions, or there is a neighborhood I of λ_j such that for all $\lambda \in I \setminus \{\lambda_j\}$ problem (1) has at least one nontrivial solution.

A natural question concerns the exact number of nontrivial solutions of (1) bifurcating from an eigenvalue λ_j which is not simple.

At this aim, we would like to quote the paper [9] of Dancer, where the author develops a method to study the small solutions of (1) in more detail. More precisely, he gives sufficient conditions in order to prove bifurcation from the right or from the left and he also describes the *growth* of the solutions in terms of the distance of $\lambda - \lambda_j$. He also obtains a complete count of the number of small solutions provided an abstract nondegeneracy condition is satisfied.

In this paper, we use the variational structure of the problem in order to count the number of solution branches bifurcating from a multiple eigenvalue in some special cases. Indeed, in what follows we focus on the following prototype problem

$$\begin{cases} -\Delta u = u^3 + \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

for some special bounded domain Ω. We have the following results.

Theorem 1.1 *Let Ω be a rectangle in $\mathbb{R}^2$. Let λ_j be an eigenvalue of (2) with multiplicity k. Then there exists $\delta > 0$ such that for any $\lambda \in (\lambda_j - \delta, \lambda_j)$ problem (3) has exactly $\frac{3^k - 1}{2}$ pairs of solutions $(u_\lambda, -u_\lambda)$ bifurcating from λ_j. In particular, if $k = 2$ problem (3) has two pairs of solutions with Morse index $j + 1$ and two pairs of solutions with Morse index j and if $k = 3$ problem (4) has three pairs of solutions with Morse index $j + 2$, six pairs of solutions with Morse index $j + 1$ and four pairs of solutions with Morse index j.*

Theorem 1.2 *Let Ω be a cube in $\mathbb{R}^3$. Let λ_2 be the second eigenvalue of* (2) *whose multiplicity is three. Then there exists $\delta > 0$ such that for any $\lambda \in (\lambda_2 - \delta, \lambda_2)$ problem* (3) *has exactly* 13 *pairs of solutions $(u_\lambda, -u_\lambda)$ bifurcating from λ_2. Moreover three pairs of solutions have Morse index* 3*, six pairs of solutions have Morse index* 2 *and four pairs of solutions have Morse index* 1.

The first theorem extends a result obtained in [10], where the authors study the bifurcation from the second eigenvalue λ_2 when Ω is a square in $\mathbb{R}^2$. In particular, they proved that the bifurcation set is constituted exactly by the union of four C^1 curves crossing $(\lambda_2, 0)$ from the left. We also remark that some exactness results in bifurcation theory for a different class of problems were obtained in [18].

It seems that most of the results obtained in this paper could follow from some old abstract result developed by Dancer in [9]. Nevertheless, we think that our approach, which strongly relies on the variational structure of the problem, allows to link the existence of small solutions to problem (1) with the existence of critical points to a very simple function (see (26)) defined on a finite dimensional space.

The proof of our results is based upon a well known Ljapunov-Schmidt reduction method (see, for example, [2], [3], [14], [16]).

The paper is organized as follows: in Section 2 we introduce some notation, in Section 3 we reduce problem (1) to a finite dimensional one, in Section 4 we examine the energy reduced to the eigenspace in a neighborhood of the bifurcation point and in Section 5 we deduce some information about Morse index of solutions. Finally, in Section 6 we prove Theorem 1.1 and Theorem 1.2.

In order to make the reading more fluent, in many calculations we have used the symbol c to denote different absolute constants which may vary from line to line.

Acknowledgement. The authors would like to thank Professor E.N. Dancer for many helpful comments and remarks.

2 Setting of the problem

First of all we rewrite problem (1) in a different way. We introduce a positive parameter ε. An easy computation shows that, if $u(x)$ solves problem (1), then for any $\varepsilon > 0$ the function $v(x) = \varepsilon^{-\frac{1}{p-1}} u(x)$ solves

$$\begin{cases} -\Delta v = \varepsilon |v|^{p-1} v + \lambda v & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

The parameter ε will be chosen in Lemma 4.2 as $\varepsilon = \lambda_j - \lambda > 0$.

Let $\mathrm{H}^1_0(\Omega)$ be the Hilbert space equipped with the usual inner product $\langle u, v \rangle = \int\limits_\Omega \nabla u \nabla v$, which induces the standard norm $\|u\| = \big(\int\limits_\Omega |\nabla u|^2 \big)^{1/2}$.

If $r \in [1, +\infty)$ and $u \in \mathrm{L}^r(\Omega)$, we will set $\|u\|_r = \big(\int\limits_\Omega |u|^r \big)^{1/r}$.

Definition 2.1 *Let us consider the embeddings* $i : \mathrm{H}^1_0(\Omega) \hookrightarrow \mathrm{L}^{\frac{2N}{N-2}}(\Omega)$ *if* $N \geq 3$ *and* $i : \mathrm{H}^1_0(\Omega) \hookrightarrow \bigcap_{q>1} \mathrm{L}^q(\Omega)$ *if* $N = 2$. *Let* $i^* : \mathrm{L}^{\frac{2N}{N+2}}(\Omega) \longrightarrow \mathrm{H}^1_0(\Omega)$ *if* $N \geq 3$ *and* $i^* : \bigcup_{q>1} \mathrm{L}^q(\Omega) \longrightarrow \mathrm{H}^1_0(\Omega)$ *if* $N = 2$ *be the adjoint operators defined by* $i^*(u) = v$ *if and only if* $\langle v, \varphi \rangle = \int_\Omega u(x)\varphi(x)dx$ *for any* $\varphi \in \mathrm{H}^1_0(\Omega)$.

It holds

$$\|i^*(u)\| \leq c\|u\|_{\frac{2N}{N+2}} \text{ for any } u \in \mathrm{L}^{\frac{2N}{N+2}}(\Omega), \text{ if } N \geq 3, \tag{5}$$

$$\|i^*(u)\| \leq c(q)\|u\|_q \text{ for any } u \in \mathrm{L}^q(\Omega),\ q > 1, \text{ if } N = 2. \tag{6}$$

Here the positive constants c and $c(q)$ depend only on Ω and N and Ω, N and q, respectively.

Let us recall the following regularity result proved in [1], which plays a crucial role when $p > \frac{N+2}{N-2}$ and $N \geq 3$.

Lemma 2.1 *Let* $N \geq 3$ *and* $s > \frac{2N}{N-2}$. *If* $u \in \mathrm{L}^{\frac{Ns}{N+2s}}(\Omega)$, *then* $i^*(u) \in \mathrm{L}^s(\Omega)$ *and* $\|i^*(u)\|_s \leq c\|u\|_{\frac{Ns}{N+2s}}$, *where the positive constant* c *depends only on* Ω, N *and* s.

Now, we introduce the space

$$\mathrm{X} = \mathrm{H}^1_0(\Omega) \text{ if either } N = 2 \text{ or } N \geq 3 \text{ and } 1 < p \leq \tfrac{N+2}{N-2} \tag{7}$$

or

$$\mathrm{X} = \mathrm{H}^1_0(\Omega) \cap \mathrm{L}^s(\Omega),\ s = \frac{N(p-1)}{2}, \text{ if } N \geq 3 \text{ and } p > \tfrac{N+2}{N-2}. \tag{8}$$

We remark that the choice of s is such that $\frac{pNs}{N+2s} = s$, a fact that will be used in the following.

X is a Banach space equipped with the norm $\|u\|_{\mathrm{X}} = \|u\|$ in the first case and $\|u\|_{\mathrm{X}} = \|u\| + \|u\|_s$ in the second case.

By means of the definition of the operator i^*, problem (4) turns out to be equivalent to

$$\begin{cases} u = i^*[\varepsilon f(u) + \lambda u] \\ u \in \mathrm{X}, \end{cases} \tag{9}$$

where $f(s) = |s|^{p-1}s$.

Now, let us fix an eigenvalue λ_j with multiplicity k, i.e. $\lambda_{j-1} < \lambda_j = \cdots = \lambda_{j+k-1} < \lambda_{j+k} \leq \ldots$. We denote by $e_1, \ldots, e_k$, k orthogonal eigenfunctions associated to the eigenvalue λ_j such that $\|e_i\|_2 = 1$ for $i = 1, \ldots, k$. We will look for solutions to (4), or to (9), having the form

$$u(x) = \sum_{i=1}^k a^i_\lambda e_i(x) + \phi_\lambda(x) = a_\lambda e + \phi_\lambda, \tag{10}$$

where $a^i_\lambda \in \mathbb{R}$, the function ϕ_λ is a lower order term and we have set $a := (a^1, \ldots, a^k)$, $e := (e_1, \ldots, e_k)$ and $ae := \sum_{i=1}^k a^i e_i$.

We consider the subspace of X given by $K_j = \text{span}\{e_i \mid i = 1, \ldots, k\}$ and its complementary space $K_j^\perp = \{\phi \in \mathrm{X} \mid \langle \phi, e_i \rangle = 0,\ i = 1, \ldots, k\}$.

Moreover let us introduce the operators $\Pi_j : \mathrm{X} \to K_j$ and $\Pi_j^\perp : \mathrm{X} \to K_j^\perp$ defined by $\Pi_j(u) = \sum_{i=1}^k \langle u, e_i \rangle e_i$ and $\Pi_j^\perp(u) = u - \Pi_j(u)$. We remark that there exists a positive constant c such that

$$\|\Pi_j(u)\|_{\mathrm{X}} \le c\|u\|_{\mathrm{X}}, \quad \|\Pi_j^\perp(u)\|_{\mathrm{X}} \le c\|u\|_{\mathrm{X}} \quad \forall\, u \in \mathrm{X}. \tag{11}$$

Our approach to solve problem (9) will be to find, for λ close enough to λ_j and ε small enough, real numbers $a^1, \ldots, a^k$ and a function $\phi \in K_j^\perp$ such that

$$\Pi_j^\perp \left\{ae + \phi - i^* \left[\varepsilon f\left(ae + \phi\right) + \lambda(ae + \phi)\right]\right\} = 0 \tag{12}$$

and

$$\Pi_j \left\{ae + \phi - i^* \left[\varepsilon f\left(ae + \phi\right) + \lambda(ae + \phi)\right]\right\} = 0. \tag{13}$$

3 Finite dimensional reduction

In this section we will solve equation (12). More precisely, we will prove that for any $a \in \mathbb{R}^k$, for λ close enough to λ_j and ε small enough, there exists a unique $\phi \in K_j^\perp$ such that (12) is fulfilled. Actually, this part of Theorem 4.4 was already proved in [9], so that we don't go into details. However, we sketch the proof in order to settle down some notations and because we get some uniform estimates we need for the proofs of Theorems 5.3 and 5.5.

Let us introduce the linear operator $L_\lambda : K_j^\perp \to K_j^\perp$ defined by

$$L_\lambda(\phi) = \phi - \Pi_j^\perp \left\{i^* \left[\lambda\phi\right]\right\}.$$

Lemma 3.1 *There exists $\delta > 0$ and a constant $c > 0$ such that for any $\lambda \in (\lambda_j - \delta, \lambda_j + \delta)$, the operator L_λ is invertible and it holds*

$$\|L_\lambda(\phi)\|_{\mathrm{X}} \ge c\|\phi\|_{\mathrm{X}} \qquad \forall\ \phi \in K_j^\perp. \tag{14}$$

Proof: First of all, we remark that L_λ is surjective.

Concerning the estimate, we prove our claim when $N \ge 3$ and $p > \frac{N+2}{N-2}$, and we argue in a similar way in the other cases.

Assume by contradiction that there are sequences $\delta_n \to 0$, $\lambda_n \to \lambda_j$ and $\phi_n \in K_j^\perp$ such that

$$\|L_{\lambda_n}(\phi_n)\|_{\mathrm{X}} < \frac{1}{n}\|\phi_n\|_{\mathrm{X}}.$$

Without loss of generality we can assume

$$\|\phi_n\|_{\mathrm{X}} = 1 \qquad \text{for any } n \in \mathbb{N}. \tag{15}$$

If $h_n := L_{\lambda_n}(\phi_n) \in \Pi_j^\perp$, then

$$\|h_n\|_{\mathrm{X}} \to 0 \tag{16}$$

and

$$\phi_n - i^*(\lambda_n\phi_n) = h_n - \Pi_j\{i^*[\lambda_n\phi_n]\} = h_n + w_n, \tag{17}$$

where $w_n \in K_j$.

First of all we point out that $w_n = 0$ for any $n \in \mathbb{N}$. Indeed, multiply equation (17) by e_i, $i = 1, \ldots, k$, so that $\langle w_n, e_i \rangle = -\lambda_n \int\limits_\Omega \phi_n e_i = 0$, so that $w_n \in K_j^\perp$, and then $w_n = 0$.

By (15), we can assume that, up to a subsequence, $\phi_n \to \phi$ weakly in X and strongly in $\mathrm{L}^q(\Omega)$ for any $q \in \left[1, \frac{2N}{N-2}\right)$. Multiplying (17) by a test function v, we get

$$\langle \phi_n, v \rangle - \lambda_n \int\limits_\Omega \phi_n v \, dx = \langle h_n, v \rangle,$$

and passing to the limit, by (16) we deduce that $\phi \in K_j$. Since $\phi \in K_j^\perp$, we conclude that $\phi = 0$.

On the other hand, multiplying (17) by ϕ_n, we get

$$\langle \phi_n, \phi_n \rangle - \lambda_n \int\limits_\Omega \phi_n^2 = \langle h_n, \phi_n \rangle,$$

which implies $\|\phi_n\| \to 0$. Moreover by (17), Lemma 2.1 and by interpolation (since $1 < \frac{Ns}{N+2s} < s$), we deduce that for some $\sigma \in (0,1)$

$$\|\phi_n\|_s \leq c \left(\|h_n\|_s + \|\phi_n\|_{\frac{Ns}{N+2s}} \right) \leq c \left(\|h_n\|_s + \|\phi_n\|^\sigma \right)$$

(recall that $\frac{Ns}{N+2s} = \frac{s}{p}$) and so $\|\phi_n\|_s \to 0$. Finally a contradiction arises, since $\|\phi_n\|_{\mathrm{X}} = 1$. $\square$

Now we can solve Equation (12).

Proposition 3.2 *For any compact set W in $\mathbb{R}^k$ there exist $\varepsilon_0 > 0$, $\delta > 0$ and $R > 0$ such that, for any $a \in W$, for any $\varepsilon \in (0, \varepsilon_0)$ and for any $\lambda \in (\lambda_j - \delta, \lambda_j + \delta)$, there exists a unique $\phi_\lambda(a) \in K_j^\perp$ such that*

$$\Pi_j^\perp \left\{ ae + \phi_\lambda(a) - i^* \left[\varepsilon f \left(ae + \phi_\lambda(a) \right) + \lambda (ae + \phi_\lambda(a)) \right] \right\} = 0. \tag{18}$$

Moreover

$$\|\phi_\lambda(a)\|_{\mathrm{X}} \leq R\varepsilon. \tag{19}$$

Finally, the map $a \mapsto \phi_\lambda(a)$ is an odd C^1–function from $\mathbb{R}^k$ to $K_j^\perp$.

Proof: We prove our claim when $N \geq 3$ and $p > \frac{N+2}{N-2}$. We argue in a similar way in the other cases.

Let us introduce the operator $T : K_j^\perp \longrightarrow K_j^\perp$ defined by

$$T(\phi) := \left(L_\lambda^{-1} \circ \Pi_j^\perp \circ i^* \right) \left[\varepsilon f(ae + \phi) \right].$$

We point out that ϕ solves equation (18) if and only if ϕ is a fixed point of T, i.e. $T(\phi) = \phi$.

Then, we will prove that there exist $\varepsilon_0 > 0$, $\delta > 0$ and $R > 0$ such that, for any $a \in W$, for any $\varepsilon \in (0, \varepsilon_0)$ and for any $\lambda \in (\lambda_j - \delta, \lambda_j + \delta)$

$$T : \{\phi \in K_j^\perp \mid \|\phi\|_{\mathrm{X}} \leq R\varepsilon\} \longrightarrow \{\phi \in K_j^\perp \mid \|\phi\|_{\mathrm{X}} \leq R\varepsilon\}$$

is a contraction mapping.

First of all, let us point out that by Lemma 3.1, (11), (5) and Lemma 2.1, we get that there exists $c = c(N, s, \Omega, W) > 0$ such that for any $\phi \in K_j^\perp$, $a \in W$

$$\begin{aligned} \|T(\phi)\|_{\mathrm{X}} &\leq c\varepsilon \left[\|f(ae+\phi)\|_{\frac{2N}{N+2}} + \|f(ae+\phi)\|_{\frac{Ns}{N+2s}} \right] \\ \text{(Hölder inequality)} \quad &\leq c\varepsilon \|f(ae+\phi)\|_{\frac{Ns}{N+2s}} \leq c\varepsilon \left(\|ae\|^p_{\frac{Nsp}{N+2s}} + \|\phi\|^p_{\frac{Nsp}{N+2s}} \right) \\ &\leq c\varepsilon (1 + \|\phi\|^p_{\mathrm{X}}). \end{aligned}$$

Finally, provided ε is small enough and R is suitable chosen, T maps $\{\phi \in K_j^\perp : \|\phi\|_{\mathrm{X}} \leq R\varepsilon\}$ into itself.

Now, let us show that T is a contraction, provided ε is even smaller. As before, by Lemma 3.1, (11), (5) and Lemma 2.1, we get that there exists $c > 0$ such that for any ϕ_1, $\phi_2 \in K_j^\perp$, $a \in K$

$$\begin{aligned} \|T(\phi_1) - T(\phi_2)\|_{\mathrm{X}} &\leq c\varepsilon \Big[\|f(ae+\phi_1) - f(ae+\phi_2)\|_{\frac{2N}{N+2}} \\ &\qquad + \|f(ae+\phi_1) - f(ae+\phi_2)\|_{\frac{Ns}{N+2s}} \Big] \\ &\leq c\varepsilon \|f(ae+\phi_1) - f(ae+\phi_2)\|_{\frac{Ns}{N+2s}} \\ &\leq c\varepsilon \left(\|\phi_1 - \phi_2\|_{\frac{Nsp}{N+2s}} + \|\phi_1\|^{p-1}_{\frac{Nsp}{N+2s}} \|\phi_1 - \phi_2\|_{\frac{Nsp}{N+2s}} + \|\phi_1 - \phi_2\|^p_{\frac{Nsp}{N+2s}} \right). \end{aligned}$$

Indeed, by the mean value theorem, it follows that there exists $\vartheta \in (0,1)$ such that $f(ae + \phi_1) - f(ae + \phi_2) = f'(ae + \phi_1 + \vartheta(\phi_2 - \phi_1))(\phi_1 - \phi_2)$.

Finally $\|T(\phi_1) - T(\phi_2)\|_{\mathrm{X}} \leq c\varepsilon \|\phi_1 - \phi_2\|_{\mathrm{X}}$ if $\|\phi_1\|_{\mathrm{X}}, \|\phi_2\|_{\mathrm{X}} \leq R\varepsilon$ and our claim immediately follows.

The oddness of the mapping $a \mapsto \phi_\lambda(a)$ i.e. $\phi_\lambda(a) = -\phi_\lambda(-a)$, is a straightforward consequence of the uniqueness of solutions of problem (18).

The regularity of the mapping can be proved using standard arguments. □

4 The reduced problem

In this section we will solve equation (13). More precisely, we will prove that if λ is close enough to λ_j, there exists $a_\lambda \in \mathbb{R}^k$ such that equation (13) is fulfilled.

Let $I_\lambda : \mathrm{H}^1_0(\Omega) \longrightarrow \mathbb{R}$ be defined by

$$I_\lambda(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \frac{\lambda}{2} \int_\Omega u^2 \, dx - \frac{\varepsilon}{p+1} \int_\Omega |u|^{p+1} \, dx. \tag{20}$$

It is well known that critical points of I_λ are solutions of problem (4). Let us consider the reduced functional $J_\lambda : \mathbb{R}^k \longrightarrow \mathbb{R}$ defined by

$$J_\lambda(a) := I_\lambda\left(ae + \phi_\lambda(a)\right), \tag{21}$$

where $\phi_\lambda(a)$ is the unique solution of (18).

Lemma 4.1 *A function $u_\lambda := ae + \phi_\lambda(a)$ is a solution to (4) if and only if a is a critical point of J_λ.*

Proof: We point out that

$$\frac{\partial J_\lambda}{\partial a_i}(a) = J'_\lambda\left(ae + \phi_\lambda(a)\right)\left(e_i + \frac{\partial \phi_\lambda}{\partial a_i}(a)\right) = J'_\lambda\left(ae + \phi_\lambda(a)\right)(e_i),$$

since $\phi_\lambda(a)$ solves equation (18) and $\frac{\partial \phi_\lambda}{\partial a_i}(a) \in K_j^\perp$. Then the claim easily follows. $\square$

From now on we set

$$\varepsilon := \lambda_j - \lambda > 0. \tag{22}$$

Lemma 4.2 *It holds*

$$J_\lambda(a) = (\lambda_j - \lambda)\left[J_{\lambda_j}(a) + \Phi_\lambda(a)\right], \tag{23}$$

where J_{λ_j} is defined in (26) and $\Phi_\lambda : \mathbb{R}^k \longrightarrow \mathbb{R}$ is an even C^1-function such that Φ_λ goes to zero C^1-uniformly on compact sets of $\mathbb{R}^k$ as $\lambda \to \lambda_j$.

Proof: Set $\phi := \phi_\lambda(a)$. By (22) we get

$$\begin{aligned} J_\lambda(a) \quad &= \frac{1}{2}\int_\Omega |\nabla(ae+\phi)|^2\,dx - \frac{\varepsilon}{p+1}\int_\Omega |ae+\phi|^{p+1}\,dx - \frac{\lambda}{2}\int_\Omega (ae+\phi)^2\,dx \\ &= \frac{1}{2}(\lambda_j - \lambda)\left(a_1^2 + \cdots + a_k^2\right) - \frac{\varepsilon}{p+1}\int_\Omega |a_1 e_1 + \cdots + a_k e_k|^{p+1}\,dx \\ &+ \frac{1}{2}\int_\Omega |\nabla\phi|^2\,dx - \frac{\lambda}{2}\int_\Omega \phi^2\,dx - \frac{\varepsilon}{p+1}\int_\Omega \left[|ae+\phi|^{p+1} - |ae|^{p+1}\right]dx \\ &= (\lambda_j - \lambda)\left[J_{\lambda_j}(a) + \Phi_\lambda(a)\right], \end{aligned}$$

where J_{λ_j} is defined in (26) and

$$\begin{aligned} \Phi_\lambda(a) := \frac{1}{\lambda_j - \lambda}&\left[\frac{1}{2}\int_\Omega |\nabla\phi|^2\,dx - \frac{\lambda}{2}\int_\Omega \phi^2\,dx\right] \\ &- \frac{1}{p+1}\int_\Omega \left[|ae+\phi|^{p+1} - |ae|^{p+1}\right]dx. \end{aligned}$$

Here we used the fact that $\phi_{\lambda_j}(a) = 0$ for any a, as it is clear from (19).

Of course Φ_λ is even and of class C^1. It remains to prove that it goes to zero C^1-uniformly on every compact subset W of $\mathbb{R}^k$ as $\lambda \to \lambda_j$, that is, as $\varepsilon \to 0$.

Let us fix a compact set W in $\mathbb{R}^k$. It is easy to check that

$$|\Phi_\lambda(a)| \leq \frac{c}{\varepsilon}\|\phi\|^2 + c\|\phi\|_{\rm X} \leq c\varepsilon, \text{ for any } a \in W.$$

Indeed, by the mean value theorem we deduce that there exists $\vartheta \in (0,1)$ such that

$$\frac{1}{p+1}\int\limits_\Omega \left[|ae+\phi|^{p+1} - |ae|^{p+1}\right] dx = \int\limits_\Omega f\,(ae+\vartheta\phi)\,\phi\,dx.$$

Therefore Φ_λ goes to zero uniformly on W as $\lambda \to \lambda_j$, since $\|\boldsymbol{\phi}\|_{\mathbf{X}} \leq \boldsymbol{R\varepsilon}$ by (19).

Now, let us prove that also $\nabla\Phi_\lambda$ goes to zero as $\lambda \to \lambda_j$ uniformly on W. Indeed, fix $i = 1, \ldots, k$ and evaluate

$$\begin{aligned}\frac{\partial\Phi_\lambda(a)}{\partial a_i} =& \frac{1}{\lambda_j-\lambda}\left[\int\limits_\Omega \nabla\phi\cdot\frac{\partial\nabla\phi}{\partial a_i}\,dx - \lambda\int\limits_\Omega \phi\frac{\partial\phi}{\partial a_i}\,dx\right] \\ & - \int\limits_\Omega\left[|ae+\phi|^{p-1}(ae+\phi)\Big(e_i+\frac{\partial\phi}{\partial a_i}\Big) - |ae|^{p-1}aee_i\right] dx.\end{aligned} \tag{24}$$

By (18), for every $z \in K_j^\perp$ we have

$$\int\limits_\Omega \nabla\phi\cdot\nabla z\,dx - \lambda\int\limits_\Omega \phi z\,dx - \varepsilon\int\limits_\Omega |ae+\phi|^{p-1}(ae+\phi)z\,dx = 0.$$

Then, taking $z = \frac{\partial\phi}{\partial a_i} \in K_j^\perp$, by (24) we deduce

$$\frac{\partial\Phi_\lambda(a)}{\partial a_i} = \int\limits_\Omega [f(ae+\phi) - f(ae)]\,e_i\,dx$$

and so

$$\left|\frac{\partial\Phi_\lambda(a)}{\partial a_i}(a)\right| \leq c\|\phi\|_{\rm X} \leq c\varepsilon, \text{ for any } a \in W.$$

Indeed, again by the mean value theorem we deduce that there exists $\vartheta \in (0,1)$ such that

$$\int\limits_\Omega [f(ae+\phi) - f(ae)]\,e_i\,dx = \int\limits_\Omega f'\,(ae+\vartheta\phi)\,\phi e_i\,dx.$$

Therefore, also $\nabla\Phi_\lambda$ goes to zero uniformly on W as $\lambda \to \lambda_j$. □

Proposition 4.3 *There exists $\delta > 0$ such that for any $\lambda \in (\lambda_j - \delta, \lambda_j)$ the function J_λ has at least k pairs $(a_\lambda, -a_\lambda)$ of distinct critical points. Moreover $a_\lambda \to a$ as λ goes to λ_j and a is a critical point of J_{λ_j} (see* (26)*).*

Proof: First of all, we note that $J_\lambda(0)=0$ and also that J_λ is an even function. Moreover (see (26)), it is clear that there exist $R>r>0$ such that

$$\inf_{|a|=r} J_{\lambda_j}(a) > J_{\lambda_j}(0) = 0 > \sup_{|a|=R} J_{\lambda_j}(a).$$

Therefore, by Lemma 4.2 we deduce that, if λ is close enough to λ_j, it holds

$$\inf_{|a|=r} J_\lambda(a) > J_\lambda(0) = 0 > \sup_{|a|=R} J_\lambda(a).$$

Then J_{λ_j} has at least k pairs of distinct critical points $(a_\lambda, -a_\lambda)$ in $B(0,R)$. We can assume that $a_\lambda \to a \in \overline{B(0,R)}$ as $\lambda \to \lambda_j$. By (23) we get $\nabla J_{\lambda_j}(a_\lambda) = \frac{1}{\varepsilon}\nabla J_\lambda(a_\lambda) - \nabla\Phi_\lambda(a_\lambda) = -\nabla\Phi_\lambda(a_\lambda)$, and since Φ_λ goes to zero C^1- uniformly on $\overline{B(0,R)}$ as $\lambda\to\lambda_j$, we get $\nabla J_{\lambda_j}(a)=0$. That proves our claim. □

Finally, we state the main result of this section, which covers [9, Theorem 4] and other previous results in the asymptotic description of bifurcation solutions, but which also strongly relates such solutions with critical points of a function defined on $\mathbb{R}^k$.

Theorem 4.4 *There exists $\delta=\delta(\lambda_j)>0$ such that for any $\lambda\in(\lambda_j-\delta,\lambda_j)$ problem* (1) *has at least k pairs of solutions $(u_\lambda,-u_\lambda)$ bifurcating from λ_j. Moreover, associated to each u_λ there exist real numbers $a^1_\lambda,\dots,a^k_\lambda$ and a function $\phi_\lambda\in \mathrm{X}$ (see* (7) *and* (8)*), with $(\phi_\lambda,e_i)=0$ for $i=1,\dots,k$ such that*

$$u_\lambda = (\lambda_j-\lambda)^{\frac{1}{p-1}}\left[\sum_{i=1}^k a^i_\lambda e_i + \phi_\lambda\right], \quad \lim_{\lambda\to\lambda_j}\|\phi_\lambda\|_{\mathrm{X}} = 0 \ \text{and} \ \lim_{\lambda\to\lambda_j} a^i_\lambda = a^i, \tag{25}$$

where $a:=(a^1,\dots,a^k)$ is a critical point of the function $J_{\lambda_j}:\mathbb{R}^k\longrightarrow\mathbb{R}$, defined by

$$J_{\lambda_j}(a) = \frac{1}{2}\left(a_1^2+\cdots+a_k^2\right) - \frac{1}{p+1}\int_\Omega |a_1e_1+\cdots+a_ke_k|^{p+1}\,dx. \tag{26}$$

Proof: The claim follows by Lemma 4.1 and Proposition 4.3. □

We point out that the existence of at least $2k$ nontrivial solutions bifurcating from the eigenvalue λ_j was already known (see [5], [13], [16] and [6], [7], [11] and the references therein for different multiplicity results in the critical case), as well as the asymptotic behaviour of the solutions as λ goes to λ_j (see [9]).

5 Some uniqueness results

In Theorem 4.4 we find out a relation between solutions to problem (1) bifurcating from the eigenvalue λ_j and critical points of the function J_{λ_j}: the solution u_λ which satisfies (25) is "generated" by the critical point a. This suggests that the solution u_λ "generated" by a can inherit some properties of a. At this aim, first of all we prove that any non degenerate critical point a of J_{λ_j} generates a unique solution u_λ bifurcating from the eigenvalue λ_j which satisfies (25).

Proposition 5.1 *Suppose that a is a non degenerate nontrivial critical point of the function J_{λ_j} defined in (26). Then there exists $\delta = \delta(\lambda_j) > 0$ such that for any $\lambda \in (\lambda_j - \delta, \lambda_j)$ problem (4) with $\epsilon = \lambda_j - \lambda$ has a unique solution u_λ such that $u_\lambda = a_\lambda e + \phi_\lambda$, where $a_\lambda \to a$ in $\mathbb{R}^k$, $\langle \phi_\lambda, e_i \rangle = 0$ for any $i = 1, \dots, k$ and $\|\phi_\lambda\|_{\mathrm{X}} \to 0$ as $\lambda \to \lambda_j$.*

Proof: As far as it concerns the existence result, we remark that, since a is a non degenerate critical point of J_{λ_j}, by Lemma 4.2 we deduce that there exists $\delta > 0$ such that for any $\lambda \in (\lambda_j - \delta, \lambda_j)$ the function J_λ has a critical point a_λ such that a_λ goes to a as λ goes to λ_j. Then by Lemma 4.1 we deduce that the function $u_\lambda = a_\lambda e + \phi_\lambda(a_\lambda)$ is a solution to problem (4), with $\langle \phi_\lambda, e_i \rangle = 0$ for any $i = 1, \dots, k$ and $\|\phi_\lambda\|_{\mathrm{X}} \to 0$ as $\lambda \to \lambda_j$.

Let us show the uniqueness result. Let u_λ and v_λ be two solutions of (4) such that $u_\lambda = a_\lambda e + \phi_\lambda$ and $v_\lambda = b_\lambda e + \psi_\lambda$, where $\phi_\lambda, \psi_\lambda \in K_j^\perp$, a_λ, b_λ go to a and $\|\phi_\lambda\|_{\mathrm{X}}, \|\psi_\lambda\|_{\mathrm{X}}$ go to zero as λ goes to λ_j.

Assume by contradiction that $u_\lambda \neq v_\lambda$ and consider the function

$$z_\lambda := \frac{u_\lambda - v_\lambda}{\|u_\lambda - v_\lambda\|}.$$

It is clear that z_λ satisfies the problem

$$\begin{cases} -\Delta z_\lambda = \lambda z_\lambda + (\lambda_j - \lambda) \dfrac{f(u_\lambda) - f(v_\lambda)}{\|u_\lambda - v_\lambda\|} & \text{in } \Omega \\ z_\lambda = 0 & \text{on } \partial\Omega. \end{cases} \tag{27}$$

We point out that by the Mean Value Theorem there exists $\vartheta \in (0, 1)$ such that

$$\frac{f(u_\lambda) - f(v_\lambda)}{\|u_\lambda - v_\lambda\|} = f'\left(u_\lambda + \vartheta(u_\lambda - v_\lambda)\right) z_\lambda. \tag{28}$$

We also remark that $f'\left(u_\lambda + \vartheta(u_\lambda - v_\lambda)\right)$ converges to $f'(ae)$ strongly in $\mathrm{L}^{\mathrm{N}/2}(\Omega)$ as λ goes to λ_j.

Up to a subsequence, we can assume that $z_\lambda \to z$ weakly in $\mathrm{H}_0^1(\Omega)$ and strongly in $\mathrm{L}^q(\Omega)$ for any $1 < q < \frac{2N}{N-2}$. Moreover, by (27) we deduce that there exists $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{R}^k$ such that $z = \alpha e = \sum_{i=1}^k \alpha_i e_i$.

Now, multiplying (27) by e_i, $i = 1, \dots, k$, and using (28), we deduce

$$\int\limits_\Omega z_\lambda e_i \, dx = \int\limits_\Omega f'\left(u_\lambda + \vartheta(u_\lambda - v_\lambda)\right) z_\lambda e_i \, dx$$

and passing to the limit, as λ goes to λ_j, we get $\alpha_i = \int\limits_\Omega f'(ae) z e_i \, dx$ for any $i = 1, \dots, k$. Therefore α is a solution of the linear system $\mathcal{H}J_{\lambda_j}(a)\alpha = 0$, where $\mathcal{H}J_{\lambda_j}(a)$ denotes the Hessian matrix of $\tilde{J}$ at a. Since a is a non degenerate critical point of J_{λ_j}, we deduce that $\alpha = 0$, namely $z = 0$.

On the other hand, multiplying (27) by z_λ, and using (28), we deduce

$$\int\limits_\Omega |\nabla z_\lambda|^2 \, dx = \lambda \int\limits_\Omega z_\lambda^2 \, dx + (\lambda_j - \lambda) \int\limits_\Omega f'\left(u_\lambda + \vartheta(u_\lambda - v_\lambda)\right) z_\lambda^2 \, dx$$

and passing to the limit, as λ goes to λ_j, we get $\|z_\lambda\| \to 0$. Finally, a contradiction arises since $\|z_\lambda\| = 1$. □

Secondly, we compute the Morse index of the solution u_λ generated by a critical point a of J_{λ_j} (see (26)) in terms of the Morse index of a.

We recall that the Morse index of a solution u of problem (1) is the number of negative eigenvalues μ of the linear problem

$$v - i^*\left[\lambda v + f'(u)v\right] = \mu v, \; v \in \mathrm{H}^1_0(\Omega)$$

or equivalently

$$\begin{cases} -(1-\mu)\Delta v = \lambda v + f'(u)v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

We point out that the function u, which solves problem (1), and the function $v = \varepsilon^{-\frac{1}{p-1}}u$, which solves problem (4), have the same Morse index.

Proposition 5.2 *Let $u_\lambda = \sum_{i=1}^{k} a^i_\lambda e_i + \phi_\lambda$ be a solution to (4) such that $\lim_{\lambda\to\lambda_j} \|\phi_\lambda\|_{\mathrm{X}} = 0$, $\lim_{\lambda\to\lambda_j} a^i_\lambda = a^i$ and $(a^1, \dots, a^k)$ is a non trivial critical point of J_{λ_j} (see (26)). If the Morse index of a is m, then the Morse index of u_λ is at least $m + j - 1$. Moreover if a is also non degenerate, then the solution u_λ is non degenerate and its Morse index is exactly $m + j - 1$.*

Proof: We denote by $\mu^1_\lambda < \mu^2_\lambda \le \dots \le \mu^i_\lambda \le \dots$ the sequence of the eigenvalues, counted with their multiplicities, of the linear problem

$$\begin{cases} -(1-\mu)\Delta v = \lambda v + (\lambda_j - \lambda) f'(u_\lambda) v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{29}$$

We also denote by $v^i_\lambda \in \mathrm{H}^1_0(\Omega)$, with $\|v^i_\lambda\|_2 = 1$, the eigenfunction associated to the eigenvalue μ^i_λ.

It is clear that, as λ goes to λ_j, eigenvalues and eigenfunctions of (29) converge to eigenvalues and eigenfunctions of the linear problem

$$\begin{cases} -(1-\mu)\Delta v = \lambda_j v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

whose set of eigenvalues is

$$\left\{ 1 - \frac{\lambda_j}{\lambda_1}, \dots, 1 - \frac{\lambda_j}{\lambda_{j-1}}, \underbrace{0, \dots, 0}_{k}, 1 - \frac{\lambda_j}{\lambda_{j+k}}, \dots \right\}.$$

Therefore, if λ is close enough to λ_j, we can claim that $\mu^1_\lambda, \dots, \mu^{j-1}_\lambda$ are negative and they are close to $1 - \frac{\lambda_j}{\lambda_1}, \dots, 1 - \frac{\lambda_j}{\lambda_{j-1}}$, respectively, and that μ^{j+k}_λ is positive and close to $1 - \frac{\lambda_j}{\lambda_{j+k}}$.

Therefore, it remains to understand what happens to the k eigenvalues $\mu_\lambda^j, \dots, \mu_\lambda^{j+k-1}$, which go to zero as λ goes to λ_j.

We claim that

$$\begin{cases} \lim\limits_{\lambda\to\lambda_j} \frac{\mu_\lambda^{j+l-1}}{\lambda-\lambda_j}\lambda_j = \Lambda^l, \quad \text{where } \Lambda^1 \le \dots \le \Lambda^k,\ l = 1,\dots,k, \text{ and} \\ \Lambda^l \text{ are the eigenvalues of the Hessian matrix } \mathcal{H}J_{\lambda_j}(a). \end{cases} \tag{30}$$

For any $l = 1,\dots,k$ we denote by v_λ^l an eigenfunction associated to μ_λ^{j+l-1}, with $\|v_\lambda^l\|_2 = 1$, i.e.

$$\begin{cases} -(1-\mu_\lambda^{j+l-1})\Delta v_\lambda^l = \lambda v_\lambda^l + (\lambda_j - \lambda) f'(u_\lambda) v_\lambda^l & \text{in } \Omega, \\ v_\lambda^l = 0 & \text{on } \partial\Omega. \end{cases} \tag{31}$$

Then we can write

$$\begin{cases} v_\lambda^l = \sum\limits_{i=1}^k b_\lambda^{l,i} e_i + \psi_\lambda^l,\ b_\lambda^{l,i} \in \mathbb{R}, \\ \langle \psi_\lambda^l, e_i\rangle = 0,\ i = 1,\dots,k,\ \langle v_\lambda^l, v_\lambda^s\rangle = 0 \text{ if } l \ne s, \\ \sum\limits_{i=1}^k \left(b_\lambda^{l,i}\right)^2 + \|\psi_\lambda^l\|_2^2 = 1. \end{cases} \tag{32}$$

Now, up to a subsequence, we can assume that for any $l = 1,\dots,k$ and $i = 1,\dots,k$, $\psi_\lambda^l \to \psi^l$ and $b_\lambda^{l,i} \to b^{l,i}$ as λ goes to λ_j. Then $v_\lambda^l \to v^l := \sum\limits_{i=1}^k b^{l,i} e_i + \psi^l$ as λ goes to λ_j. We point out that the convergence in $\mathrm{H}_0^1(\Omega)$ is strong, since v_λ^l solves equation (31) and μ_λ^{j+l-1} does not go to 1 as λ goes to λ_j.

First of all we claim that $\psi^l = 0$ for any $l = 1,\dots,k$. In fact by (31) we deduce that for any $l = 1,\dots,k$ and for all $v \in \mathrm{H}_0^1(\Omega)$ it holds

$$(1-\mu_\lambda^{j+l-1}) \int_\Omega \nabla v_\lambda^l \nabla v\, dx = \lambda \int_\Omega v_\lambda^l v\, dx + (\lambda_j - \lambda) \int_\Omega f'(u_\lambda) v_\lambda^l v\, dx,$$

and passing to the limit as λ goes to λ_j

$$\int_\Omega \nabla v^l \nabla v\, dx = \lambda_j \int_\Omega v_\lambda^l v\, dx \quad \forall\ v \in \mathrm{H}_0^1(\Omega),$$

that is v^l is an eigenfunction associated to the eigenvalue λ_j, so that $\psi^l = 0$. Therefore by (32) we deduce that

$$\sum_{i=1}^k b^{l,i} b^{s,i} = 0 \text{ if } l \ne s \text{ and } \sum_{i=1}^k \left(b^{l,i}\right)^2 = 1 \quad \forall\, l = 1,\dots,k. \tag{33}$$

Now, if we multiply (31) by e_i, we get for any $i = 1,\dots,k$ and for any $l = 1,\dots,k$,

$$(1-\mu_\lambda^{j+l-1})\lambda_j b_\lambda^{l,i} = \lambda b_\lambda^{l,i} + (\lambda_j - \lambda) \int_\Omega f'(u_\lambda) v_\lambda^l e_i\, dx,$$

that is

$$b_\lambda^{l,i}\left(1-\frac{\mu_\lambda^{j+l-1}}{\lambda_j-\lambda}\lambda_j\right)=\int\limits_\Omega f'(u_\lambda)\left(\sum_{i=1}^k b_\lambda^{l,i}e_i+\psi_\lambda^l\right)e_i\,dx. \tag{34}$$

Now, since as λ goes to λ_j

$$\int\limits_\Omega f'\left(\sum_{i=1}^k a_\lambda^i e_i+\phi_\lambda\right)\left(\sum_{i=1}^k b_\lambda^{l,i}e_i+\psi_\lambda^l\right)e_i\,dx$$
$$\to\int\limits_\Omega f'\left(\sum_{i=1}^k a^i e_i\right)\left(\sum_{i=1}^k b^{l,i}e_i\right)e_i\,dx,$$

by (34) and by (33) we deduce that for any $l=1,\dots,k$, when λ goes to λ_j, $\frac{\mu_\lambda^{j+l-1}}{\lambda_j-\lambda}\lambda_j$ converges to an eigenvalue Λ^l of the matrix $\mathcal{H}J(a)$ and also that $b_\lambda^l e$ converges to the associated eigenfunction $b^l e$, since

$$b^{l,i}\left(1-\Lambda^l\right)=\int\limits_\Omega f'\left(\sum_{i=1}^k a^i e_i\right)\left(\sum_{i=1}^k b^{l,i}e_i\right)e_i\,dx \quad \forall\, i=1,\dots,k,$$

i.e. $HJ(a)(b^l e)=\Lambda^l(b^l e)$.

Since a has Morse index m, there are m eigenvalues Λ^l which are negative, so that at least m eigenvalues μ_λ^l are negative as well, provided λ is close to λ_j.

Finally, if a is non degenerate, all the Λ^l's are different from 0, so that, if λ is near λ_j, there are exactly m negative eigenvalues μ_λ^l, as claimed. □

Now, we can prove the first main result of this section.

Theorem 5.3 *Assume a is a non degenerate critical point of the function J_{λ_j} with Morse index m. Then there exists $\delta=\delta(\lambda_j)>0$ such that for any $\lambda\in(\lambda_j-\delta,\lambda_j)$ problem* (1) *has a unique solution u_λ bifurcating from λ_j which satisfies* (25). *Moreover u_λ is non degenerate and its Morse index is $m+j-1$.*

Proof: The claim follows by Proposition 5.1 and Proposition 5.2. □

The previous result suggests that the number of nontrivial solutions to (1) bifurcating from the eigenvalue λ_j coincides with the number of nontrivial critical points of J_{λ_j}. In fact, we can describe the asymptotic behaviour of the solution u_λ of problem (1) bifurcating from the eigenvalue λ_j as λ goes to λ_j.

Proposition 5.4 *Let $u_\lambda\in \mathrm{X}$ be a solution to problem* (1) *such that $\|u_\lambda\|_{\mathrm{X}}$ goes to zero as λ goes to λ_j. Then for any λ sufficiently close to λ_j there exist $a_\lambda^1,\dots,a_\lambda^k\in\mathbb{R}$ and $\phi_\lambda\in K_j^\perp$ such that*

$$u_\lambda=(\lambda_j-\lambda)^{\frac{1}{p-1}}\left(\sum_{i=1}^k a_\lambda^i e_i+\phi_\lambda\right),$$

where $\phi_\lambda\to 0$ in X and $a_\lambda^i\to a^i$, $i=1,\dots,k$, as λ goes to λ_j. Moreover $a=(a^1,\dots,a^k)$ is a critical point of J_{λ_j}.

Proof: We prove our claim when $N \geq 3$ and $p > \frac{N+2}{N-2}$. We argue in a similar way in the other cases.

The function $v_\lambda = \frac{u_\lambda}{\|u_\lambda\|_{\mathrm{X}}}$ solves the problem

$$\begin{cases} -\Delta v = \lambda v + \|u_\lambda\|_{\mathrm{X}}^{p-1} |v|^{p-1} v & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{35}$$

Up to a subsequence, we can assume that, as λ goes to λ_j, $v_\lambda \to v$ weakly in X and strongly in $\mathrm{L}^q(\Omega)$ for any $q \in \left[1, \frac{2N}{N-2}\right)$. By (35) we deduce that v solves $-\Delta v = \lambda_j v$ in Ω, $v = 0$ on $\partial\Omega$ and, so, $v \in K_j$.

We claim that $v \neq 0$. Indeed, if $v = 0$ by (35) we get

$$\int_\Omega |\nabla v_\lambda|^2 \, dx = \lambda \int_\Omega v_\lambda^2 \, dx + \|u_\lambda\|_{\mathrm{X}}^{p-1} \int_\Omega |v_\lambda|^{p+1} \, dx$$

and passing to the limit as λ goes to λ_j we deduce that $\|v_\lambda\|$ goes to zero, since $\|v_\lambda\|_{p+1}$ is bounded. Moreover by (35) and Lemma 2.1 we get

$$\|v_\lambda\|_s \leq c \left(\lambda \|v_\lambda\|_{\frac{Ns}{N+2s}} + \|u_\lambda\|_{\mathrm{X}}^{p-1} \|v_\lambda\|^p_{\frac{pNs}{N+2s}} \right),$$

and by interpolation (since $1 < \frac{Ns}{N+2s} < s$) $\|v_\lambda\|_s \to 0$ as λ goes to λ_j.

Finally a contradiction arises since $\|v_\lambda\|_{\mathrm{X}} = 1$.

Now it is easy to check that there exist $b_\lambda^1, \ldots, b_\lambda^k \in \mathbb{R}$ and $\psi_\lambda \in K_j^\perp$ such that $v_\lambda = \sum_{i=1}^k b_\lambda^i e_i + \psi_\lambda$ and, as λ goes to λ_j, $\|\psi_\lambda\|_{\mathrm{X}} \to 0$ (using Lemma 2.1) and $b_\lambda^i \to b^i$, $i = 1, \ldots, k$ (see also [15] for analogous properties in presence of more general nonlinearities).

We want to prove that there exists Λ such that

$$\frac{\|u_\lambda\|_{\mathrm{X}}^{p-1}}{\lambda_j - \lambda} \to \Lambda > 0 \text{ as } \lambda \text{ goes to } \lambda_j. \tag{36}$$

Multiplying (35) by e_i, we deduce that

$$b_\lambda^i (\lambda_j - \lambda) = \|u_\lambda\|_{\mathrm{X}}^{p-1} \int_\Omega |v_\lambda|^{p-1} v_\lambda e_i \, dx, \quad i = 1, \ldots, k. \tag{37}$$

We recall that, as λ goes to λ_j, $b_\lambda^i \to b^i$ and $b^i \neq 0$ for some i, since $v \neq 0$. We also point out that, as λ goes to λ_j, $\int_\Omega |v_\lambda|^{p-1} v_\lambda e_i \to \int_\Omega |v|^{p-1} v e_i$ (since $v_\lambda \to v$ strongly in X), and $\int_\Omega |v|^{p-1} v e_i \neq 0$ for some i (since $\int_\Omega |v|^{p+1} \neq 0$). Then by (37) we deduce that, as λ goes to λ_j, $\frac{\|u_\lambda\|_{\mathrm{X}}^{p-1}}{\lambda_j - \lambda}$ goes to $\Lambda \in \mathbb{R}$, $\Lambda \neq 0$ and also that

$$b^i = \Lambda \int_\Omega |v|^{p-1} v e_i \, dx = \Lambda \int_\Omega \left| \sum_{h=1}^k b^h e_h \right|^{p-1} \left(\sum_{h=1}^k b^h e_h \right) e_i \, dx. \tag{38}$$

By (38) we easily deduce that $\Lambda > 0$ and (36) is proved.

Finally, we write $u_\lambda = (\lambda_j - \lambda)^{\frac{1}{p-1}} \left(\sum\limits_{i=1}^{k} a^i_\lambda e_i + \phi_\lambda \right)$, where $a^i_\lambda = \|u_\lambda\|_X (\lambda_j - \lambda)^{-\frac{1}{p-1}} b^i_\lambda$ and $\phi_\lambda = \|u_\lambda\|_X (\lambda_j - \lambda)^{-\frac{1}{p-1}} \psi_\lambda$. Moreover, as λ goes to λ_j, a_λ goes to $a = \Lambda^{\frac{1}{p-1}} b$ and by (38) we deduce that a is a critical point of J_{λ_j}. That proves our claim. $\square$

Finally, we can prove the second main result of this section.

Theorem 5.5 *Assume the function J_{λ_j} defined in* (26) *has exactly $2h$ non trivial critical points which are non degenerate. Then there exists $\delta = \delta(\lambda_j) > 0$ such that for any $\lambda \in (\lambda_j - \delta, \lambda_j)$ problem* (1) *has exactly h pairs of solutions $(u_\lambda, -u_\lambda)$ bifurcating from λ_j and all of them are non degenerate.*

Proof: The claim follows by Proposition 5.4 and Theorem 5.3. $\square$

6 Applications

Firstly, we study solutions bifurcating from any multiple eigenvalues when Ω is a rectangle in $\mathbb{R}^2$.

Proof of Theorem 1.1*:* Without loss of generality, we can assume that $\Omega = (0, L) \times (0, M)$. We know that $\lambda_j = \pi^2 \left(\frac{n_i^2}{L^2} + \frac{m_i^2}{M^2} \right)$, $i = 1, \dots, k$, where $n_i, m_i \in \mathbb{N}$ with $n_i \neq n_l$ and $m_i \neq m_l$ if $i \neq l$. The eigenspace associated to λ_j is spanned by the functions

$$e_i(x, y) = \sqrt{\frac{4}{LM}} \sin \frac{n_i \pi}{L} x \sin \frac{m_i \pi}{M} y, \; i = 1, \dots, k.$$

It is easy to check that

$$\alpha := \int\limits_\Omega e_i^4 \, dxdy = \frac{9}{4} \frac{\pi^2}{LM} \quad \text{and} \quad \beta := \int\limits_\Omega e_i^2 e_l^2 \, dxdy = \frac{\pi^2}{LM} \text{ if } i \neq l.$$

Moreover, we have that

$$\int\limits_\Omega e_i^2 e_h e_l \, dxdy = 0 \text{ if } h \neq l.$$

Indeed,

$$
\begin{aligned}
&\int_\Omega e_i^2 e_h e_l \, dxdy \\
&= \frac{16}{(LM)^2} \int_0^L \sin^2 \frac{n_i \pi}{L} x \sin \frac{n_h \pi}{L} x \sin \frac{n_l \pi}{L} x \, dx \int_0^M \sin^2 \frac{m_i \pi}{L} y \sin \frac{m_h \pi}{L} y \sin \frac{m_l \pi}{L} y \, dy \\
&= \frac{4}{(LM)^2} \int_0^L \left(1 - \cos \frac{2n_i \pi}{L} x\right) \left\{ \cos \frac{(n_h - n_l)\pi x}{L} - \cos \frac{(n_h + n_l)\pi x}{L} \right\} dx \\
&\quad \cdot \int_0^M \left(1 - \cos \frac{2m_i \pi}{M} y\right) \left\{ \cos \frac{(m_h - m_l)\pi y}{M} - \cos \frac{(m_h + m_l)\pi y}{M} \right\} dy \\
&= \frac{4}{(LM)^2} \int_0^L \cos \frac{2n_i \pi}{L} x \left\{ \cos \frac{(n_h - n_l)\pi x}{L} - \cos \frac{(n_h + n_l)\pi x}{L} \right\} dx \\
&\quad \cdot \int_0^M \cos \frac{2m_i \pi}{M} y \left\{ \cos \frac{(m_h - m_l)\pi y}{M} - \cos \frac{(m_h + m_l)\pi y}{M} \right\} dy,
\end{aligned}
$$

because $n_h \neq n_l$ and $m_h \neq m_l$. Finally, the claim follow, since we can prove that $2n_i \neq n_h - n_l$ and $2n_i \neq n_h + n_l$ or $2m_i \neq m_h - m_l$ and $2m_i \neq m_h + m_l$. For example, assume, by contradiction, that $2n_i = n_h + n_l$ and $2m_i = m_h + m_l$. The other cases can be treated in a similar way. We have

$$
\begin{aligned}
4\lambda_i &= 4\pi^2 \left(\frac{n_i^2}{L^2} + \frac{m_i^2}{M^2} \right) \\
&= \pi^2 \left(\frac{n_h^2 + n_l^2 + 2n_h n_l}{L^2} + \frac{m_h^2 + m_l^2 + 2m_h m_l}{M^2} \right) \\
&= 2\lambda_i + \pi^2 \left(\frac{2n_h n_l}{L^2} + \frac{2m_h m_l}{M^2} \right) \\
&< 2\lambda_i + \pi^2 \left(\frac{n_h^2 + n_l^2}{L^2} + \frac{m_h^2 + m_l^2}{M^2} \right) \\
&< 4\lambda_i
\end{aligned}
$$

and a contradiction arises.

Therefore the function $J_{\lambda_j} : \mathbb{R}^k \to \mathbb{R}$ introduced in (26) with $p = 3$ reduces to

$$
J_{\lambda_j}(a) = \frac{1}{2} \sum_{i=1}^k a_i^2 - \frac{1}{4} \alpha \sum_{i=1}^k a_i^4 - \frac{3}{4} \beta \sum_{\substack{i,l=1 \\ i \neq l}}^k a_i^2 a_l^2 .
$$

We can compute

$$
\frac{\partial J_{\lambda_j}}{\partial a_i}(a) = a_i - \alpha a_i^3 - 3\beta a_i \sum_{\substack{l=1 \\ l \neq i}}^k a_l^2, \; i = 1, \ldots, k \tag{39}
$$

and also the Hessian matrix $\mathcal{H}J_{\lambda_j}(a)$

$$\begin{pmatrix} 1-3\alpha a_1^2-3\beta\sum\limits_{\substack{l=1\\ l\neq 1}}^{k} a_l^2 & -6\beta a_1 a_2 & \dots & -6\beta a_1 a_k \\ -6\beta a_1 a_2 & 1-3\alpha a_2^2-3\beta\sum\limits_{\substack{l=1\\ l\neq 2}}^{k} a_l^2 & \dots & -6\beta a_2 a_k \\ \vdots & \vdots & \ddots & \vdots \\ -6\beta a_1 a_k & -6\beta a_2 a_k & \dots & 1-3\alpha a_k^2-3\beta\sum\limits_{\substack{l=1\\ l\neq k}}^{k} a_l^2 \end{pmatrix}. \tag{40}$$

Let us consider the case $k=2$. It is clear that (39) reduces to

$$\frac{\partial J_{\lambda_j}}{\partial a_1}(a_1,a_2) = a_1-\alpha a_1^3-3\beta a_1 a_2^2$$
$$\frac{\partial J_{\lambda_j}}{\partial a_2}(a_1,a_2) = a_2-\alpha a_2^3-3\beta a_2 a_1^2$$

and also that (40) reduces to

$$\mathcal{H}J_{\lambda_j}(a_1,a_2) = \begin{pmatrix} 1-3\alpha a_1^2-3\beta a_2^2 & -6\beta a_1 a_2 \\ -6\beta a_1 a_2 & 1-3\alpha a_2^2-3\beta a_1^2 \end{pmatrix}.$$

Set $\gamma_1 := \alpha^{-1/2}$ and $\gamma_2 := (\alpha+3\beta)^{-1/2}$. Therefore J_{λ_j} has exactly the following (non trivial) critical points: $(0,\pm\gamma_1)$, $(\pm\gamma_1,0)$, which have Morse index 2, and $(\gamma_2,\pm\gamma_2)$ and $(-\gamma_2,\pm\gamma_2)$, which have Morse index 1. Therefore the claim follows by Theorem 5.3.

Let us consider the case $k=3$. It is easy to check that (39) reduces to

$$\frac{\partial J_{\lambda_j}}{\partial a_1}(a_1,a_2,a_3) = a_1-\alpha a_1^3-3\beta a_1(a_2^2+a_3^2)$$
$$\frac{\partial J_{\lambda_j}}{\partial a_2}(a_1,a_2,a_3) = a_2-\alpha a_2^3-3\beta a_2(a_1^2+a_3^2)$$
$$\frac{\partial J_{\lambda_j}}{\partial a_3}(a_1,a_2,a_3) = a_3-\alpha a_3^3-3\beta a_3(a_1^2+a_2^2)$$

and also that the Hessian matrix $\mathcal{H}J_{\lambda_j}(a_1,a_2,a_3)$ (40) reduces to

$$\begin{pmatrix} 1-3\alpha a_1^2-3\beta(a_2^2+a_3^2) & -6\beta a_1 a_2 & -6\beta a_1 a_3 \\ -6\beta a_1 a_2 & 1-3\alpha a_2^2-3\beta(a_1^2+a_3^2) & -6\beta a_2 a_3 \\ -6\beta a_1 a_3 & -6\beta a_2 a_3 & 1-3\alpha a_3^2-3\beta(a_1^2+a_2^2) \end{pmatrix}.$$

Set $\gamma_1 := \alpha^{-1/2}$, $\gamma_2 := (\alpha+3\beta)^{-1/2}$ and $\gamma_3 := (\alpha+6\beta)^{-1/2}$. Therefore J_{λ_j} has exactly the following (non trivial) pairs of critical points:

$$
\begin{aligned}
&A_1 := (0,0,\gamma_1),\ A_2 := (0,\gamma_1,0),\ A_3 := (\gamma_1,0,0)\\
&A_4 := (0,\gamma_2,\gamma_2),\ A_5 := (0,\gamma_2,-\gamma_2),\ A_6 := (\gamma_2,0,\gamma_2),\\
&A_7 := (\gamma_2,0,-\gamma_2),\ A_8 := (\gamma_2,\gamma_2,0),\ A_9 := (\gamma_2,-\gamma_2,0),\\
&A_{10} := (\gamma_3,\gamma_3,\gamma_3),\ A_{11} := (\gamma_3,\gamma_3,-\gamma_3),\\
&A_{12} := (\gamma_3,-\gamma_3,\gamma_3),\ A_{13} := (-\gamma_3,\gamma_3,\gamma_3)
\end{aligned}
$$

and $-A_i$ for $i = 1,\dots,13$. A simple computation shows that A_i and $-A_i$ have Morse index 3 if $i = 1,2,3$, that they have Morse index 2 if $i = 4,\dots,9$ and that they have Morse index 1 if $i = 10,\dots,13$. Therefore the claim follows by Theorem 5.3.

In the general case, the function J_{λ_j} has 3^k-1 non degenerate critical points of the form $(0,\dots,0,\underbrace{\pm\gamma_i,\dots,\pm\gamma_i}_{i},0,\dots,0)$, where $\gamma_i := [\alpha+3(i-1)\beta]^{-1/2}$ for any $i = 1,\dots,k$. Let us compute the Hessian matrix $\mathcal{H}J_{\lambda_j}(a)$ (40) at the point $a = (0,\dots,0,\underbrace{\pm\gamma_i,\dots,\pm\gamma_i}_{i},0,\dots,0)$. We have

$$
\frac{1}{\alpha+3(i-1)\beta}\begin{pmatrix} A_i & 0\\ 0 & B_i\end{pmatrix},
$$

where the $i\times i$ matrix A_i is given by

$$
A_i := \begin{pmatrix} -2\alpha & -6\beta & \dots & -6\beta\\ -6\beta & -2\alpha & \dots & -6\beta\\ \vdots & \vdots & \ddots & \vdots\\ -6\beta & -6\beta & \dots & -2\alpha \end{pmatrix} = (LM)^i \begin{pmatrix} -\frac{9}{32} & -\frac{3}{8} & \dots & -\frac{3}{8}\\ -\frac{3}{8} & -\frac{9}{32} & \dots & -\frac{3}{8}\\ \vdots & \vdots & \ddots & \vdots\\ -\frac{3}{8} & -\frac{3}{8} & \dots & -\frac{9}{32} \end{pmatrix}
$$

and the $(k-i)\times(k-i)$ matrix B_i is given by

$$
B_i := \begin{pmatrix} \alpha-3\beta & 0 & \dots & 0\\ 0 & \alpha-3\beta & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \alpha-3\beta \end{pmatrix} = \begin{pmatrix} -\frac{3}{64} & 0 & \dots & 0\\ 0 & -\frac{3}{64} & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & -\frac{3}{64} \end{pmatrix}.
$$

□

Secondly, we study solutions bifurcating from the second eigenvalue when Ω is a cube in $\mathbb{R}^3$.

Proof of Theorem 1.2*:* Without loss of generality, we can assume that $\Omega = (0,\pi)\times(0,\pi)\times(0,\pi)$. The eigenspace associated to λ_2 is spanned by the functions

$$
\begin{aligned}
e_1(x,y,z) &= \sqrt{\frac{8}{\pi^3}}\sin x\sin y\sin 2z,\\
e_2(x,y,z) &= \sqrt{\frac{8}{\pi^3}}\sin x\sin 2y\sin z,\\
e_3(x,y,z) &= \sqrt{\frac{8}{\pi^3}}\sin 2x\sin y\sin z.
\end{aligned}
$$

Taking in account that

$$\int_\Omega e_1^3 e_2 dxdydz = \int_\Omega e_1^2 e_2 e_3 dxdydz = 0$$

and also that

$$\alpha := \int_\Omega e_1^4 dxdydz = \frac{27}{8\pi^3} \text{ and } \beta := \int_\Omega e_1^2 e_2^2 dxdydz = \frac{3}{2\pi^3},$$

the function J_{λ_j} with $p = 3$ reduces to

$$J_{\lambda_j}(a_1, a_2, a_3) = \frac{1}{2}(a_1^2 + a_2^2 + a_3^2) - \frac{1}{4}\alpha(a_1^4 + a_2^4 + a_3^4) - \frac{3}{2}\beta(a_1^2 a_2^2 + a_1^2 a_3^2 + a_2^2 a_3^2).$$

Therefore the claim follows by Theorem 5.3, arguing exactly as in the previous example in the case $k = 3$. □

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