

OPTIMAL TRANSPORT ON NULL HYPERSURFACES AND THE NULL ENERGY CONDITION

FABIO CAVALLETTI, DAVIDE MANINI, AND ANDREA MONDINO

ABSTRACT. The goal of the present work is to study optimal transport on null hypersurfaces inside Lorentzian manifolds. The challenge here is that optimal transport along a null hypersurface is completely degenerate, as the cost takes only the two values 0 and $+\infty$. The tools developed in the manuscript enable to give an optimal transport characterization of the null energy condition (namely, non-negative Ricci curvature in the null directions) for Lorentzian manifolds in terms of convexity properties of the Boltzmann–Shannon entropy along null-geodesics of probability measures. We obtain as applications: a stability result under convergence of spacetimes, a comparison result for null-cones, and the Hawking area theorem (both in sharp form, for possibly weighted measures, and with apparently new rigidity statements).

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1. INTRODUCTION

Optimal transport revealed to be an extremely powerful tool in Riemannian signature. In particular, one can employ optimal transport in Riemannian manifolds in order to characterize Ricci curvature lower bounds in terms of convexity properties of the Boltzmann–Shannon entropy along W_2 -geodesics of probability measures, see McCann [30], Otto–Villani [37], Cordero Erasquin–McCann–Schmuckenschläger [11], von Renesse–Sturm [45]. Such a point of view has been extremely fruitful; for instance, it has been the starting point for developing a

E-mail addresses: `fabio.cavalletti@unimi.it`, `dmanini@campus.technion.ac.il`, `andrea.mondino@maths.ox.ac.uk`.

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theory of non-smooth metric measure spaces with Ricci curvature bounded below and dimension bounded above in a synthetic sense, the so-called $\text{CD}(K, N)$ spaces of Lott–Sturm–Villani [41, 42, 29].

The goal of the present work is to give an optimal transport characterization of Ricci curvature lower bounds in the null directions for Lorentzian manifolds in terms of convexity properties of the Boltzmann–Shannon entropy along null-geodesics of probability measures, in the spirit of the aforementioned [30, 37, 11, 45]. The challenge here (see later in the introduction for more details) is that optimal transport in the null directions is completely degenerate (the cost takes only the two values 0 and $+\infty$). Thus, a major task of the manuscript, is to analyse such a degenerate optimal transport problem and develop tools to characterize lower Ricci bounds (in particular, non-negative Ricci in the null directions, which corresponds to the renowned *null energy condition* in general relativity). In order to illustrate the power of the methods developed in the paper, we will obtain as applications: a stability result under convergence of space-times, a comparison result for null-cones and the Hawking area theorem (both in sharp form, for possibly weighted measures, and with apparently new rigidity statements). In a forthcoming work [6], we will use the optimal transport characterization of the null energy condition obtained in the present work as a starting point to develop a theory of non-smooth Lorentzian spaces satisfying such Ricci bounds in a synthetic setting, in the spirit of the $\text{CD}(K, N)$ spaces of Lott–Sturm–Villani [41, 42, 29].

Before discussing more in detail the main results of the paper, let us give some background and motivations.

The first part of the paper will be devoted to study volume distortion inside null hypersurfaces and optimal transport thereof. Recall that a hypersurface inside a Lorentzian manifold is said to be *null* if the restriction of the ambient Lorentzian metric has a 1-dimensional kernel. Null hypersurfaces are fundamental in general relativity. Indeed, they play an important role in understanding the propagation of light and gravitational waves, the causal structure of space-time, the nature of black holes and their horizons, and various theoretical constructs that underpin modern gravitational theory, such as holography and AdS/CFT correspondence.

The second part of the paper will connect optimal transport inside null hypersurfaces with the Null Energy Condition (NEC), which writes as non-negativity of the Ricci curvature in the null directions. It is essentially a mathematical formulation of the idea that “energy density” should be non-negative. The NEC is an important concept in general relativity, and it is expected to be satisfied by all forms of matter (see for instance [5, Ch. 4.6]), at least classically when quantum effects are not taken into account. For instance, it plays a pivotal role in:

- (1) *Singularity Theorems*: The NEC is a key assumption in the Penrose singularity theorem [38] predicting the formation of singularities under gravitational collapse, awarded the 2020 Nobel Prize in Physics.
- (2) *Black Hole Thermodynamics*: The NEC is used in proving important results in black hole thermodynamics, including the Hawking area theorem [19], which states that the area of the event horizon of a black hole cannot decrease over time, a striking analog to the second law of thermodynamics (see also the more recent revisits by Chruściel–Delay–Galloway–Howard [10] and Minguzzi [33], lowering the regularity assumptions).

Background on optimal transport and Ricci curvature in Lorentzian setting. Optimal transport in Lorentzian manifolds was studied, among others, by Eckstein–Miller [13], Suhr [43] and Kell–Suhr [21]. After developing further tools for *time-like* optimal transport of probability measures (i.e., pairs of probability measures admitting a coupling concentrated on the time-like relation) McCann [31] and Mondino–Suhr [35] gave an optimal transport characterization of *time-like Ricci curvature lower bounds* (resp. of the Einstein equations) in terms of convexity properties of the Boltzmann–Shannon entropy along time-like geodesics (with respect to a suitable Lorentz–Wasserstein structure) of probability measures. Building

on such a characterization in the smooth setting, Cavalletti–Mondino [8] developed a synthetic theory of non-smooth Lorentzian spaces satisfying time-like Ricci curvature lower bounds (see also the more recent developments [2, 4, 3]). The non-smooth setting is given by the class of Lorentzian (pre-)length spaces, introduced by Kunzinger–Sämman [26] (after the work by Kronheimer–Penrose [25] on causal spaces), as a Lorentzian counterpart of (length) metric spaces. For variants in the axiomatization of Lorentzian length spaces, see McCann [32] and Minguzzi–Suhr [34]. A related line of research has been to investigate convergence of (possibly non-smooth) Lorentzian spaces, see for instance [40, 1, 27, 34, 36].

Let us stress that the aforementioned works analyzed optimal transport in the *time-like* directions, i.e., the transport is performed along time-like geodesics. A special case of time-like Ricci bounds is the one of non-negative lower bound, also known as the *strong energy condition* (SEC). A weaker energy condition is the *null energy condition* (NEC), requiring the non-negativity of the Ricci curvature in null directions. Unlike the SEC, the NEC is expected to be satisfied by all forms of matter (see, for instance, Carroll [5, Ch. 4.6]), at least classically when quantum effects are not taken into account.

A first synthetic characterization of the NEC for smooth spacetimes was given by McCann [32], as a limiting case of timelike Ricci lower bounds. The advantage of such a characterization is that it makes sense for non-smooth Lorentzian length spaces; the drawback, as already remarked in [32], is that it is not stable under convergence of spacetimes. Moreover, it remains an open question to draw applications from such a characterization.

Another approach to characterize the NEC, by studying optimal transport in codimension 2 along null geodesics, was pursued by Ketterer [23] (in the spirit of the Riemannian higher codimension optimal transport of [24]). The rough idea is to study displacement convexity of the $(n-2)$ -Rényi entropy along Wasserstein geodesics of probability measures concentrated on space-like $(n-2)$ -dimensional submanifolds inside a null $(n-1)$ -dimensional hypersurface of an n -dimensional spacetime. The advantage of such an approach is the possibility to recover classical results, such as Penrose singularity theorem and Hawking’s area theorem, via optimal transport. The drawback being that the characterization uses quite heavily the smoothness of the spacetime and it is not clear how to extend it to the non-smooth setting of Lorentzian length spaces; moreover, the higher codimensional framework makes challenging the stability under convergence of such a characterization of the NEC.

The goal of the present paper (and the forthcoming companion [6]) is to propose a different approach to optimal transport inside null hypersurfaces and exploit it to give a synthetic characterization of the NEC which is fruitful in terms of applications, suitable to be extended to non-smooth Lorentzian spaces, and stable under convergence of spacetimes.

Challenges. The question at the heart of the optimal transport problem is, given two probability measures μ_0, μ_1 on a space X , find the “optimal way to transport μ_0 into μ_1 ”. A “transport” from μ_0 to μ_1 is a probability measure π on $X \times X$ (called *coupling*) whose marginals are μ_0 and μ_1 , and “optimal” is translated in mathematical terms by *minimizing* a suitable cost function. In Riemannian signature (see for instance [44]) one usually minimizes the integral of the squared (or another convex function of the) distance among all couplings from μ_0 to μ_1 . In Lorentzian signature, when studying optimal transport between causally related probability measures, one instead maximizes the integral of a suitable power (now less or equal than 1) of the time separation, among all couplings from μ_0 to μ_1 which are concentrated on the causal relation.

Such choices are motivated by the classical fact that geodesics minimize (resp. maximize) length in Riemannian (resp. Lorentzian) signature. Moreover, from the technical point of view, such a choice has the advantage that minimization (resp. maximization) problems are particularly well-behaved for convex (resp. concave) Lagrangians.

A first challenge in transporting two probability measures inside a null hypersurface is that the cost $c(\cdot, \cdot)$ is completely degenerate: $c(x, y)$ is either 0, if the pair (x, y) is causally related, or ∞ , if (x, y) is not causally related.

A second fundamental point is that the optimal transport characterization of Ricci lower bounds is formulated in terms of displacement convexity of suitable entropy functionals and, in order to compute the entropy of a probability measure, one needs a reference measure. From the physical point of view, this is because the entropy is a relative concept, of a (probability) measure with respect to a reference one. Here the challenge is that, if one restricts the ambient Lorentzian metric to a null hypersurface, the induced quadratic form has a one dimensional kernel, and thus the induced volume form is identically zero. Therefore, one has to define a natural non-trivial volume form on null hypersurfaces, which captures the geometric properties (in particular, the Ricci curvature) of the space-time.

Main results. In the next paragraphs, we briefly illustrate the main results of the paper. For the sake of the introduction, the discussion below will be in a simplified form; the reader is referred to the body of the work for the precise (and more general) statements.

Rigged measure and Ricci curvature, the $\text{NC}^1(N)$ condition. Recall that, given an n -dimensional Lorentzian manifold (M^n, g) , a hypersurface $H \subset M$ is said to be null if the restriction of g to H , denoted by g_H , has non-trivial kernel. In other terms, the tangent space to H at every point contains a null vector, i.e. a vector $v \neq 0$ such that $g(v, v) = 0$. As a consequence, the determinant (and thus the volume form) of g_H is identically zero.

For simplicity, throughout the introduction, we will assume that all the objects are smooth. Moreover, we assume that H is causal (i.e., there are no closed causal loops contained in H) and it admits a global space-like and acausal cross section S . We refer to the body of the paper for the more general statements.

In order to define a non-trivial volume form on H , we fix a future-directed null vector field L along H satisfying $\nabla_L L = 0$. Such fields L will be called *null-geodesic vector fields*. It is not hard to see that, locally, the class of null-geodesic vector fields is non-empty (Proposition 2.3).

We prove that, fixed a null hypersurface H and a null-geodesic vector field L , there is a canonical volume form Vol_L on H , depending on L (Corollary 3.11). Moreover, the dependence on L is controlled (Corollary 3.7): roughly, scaling L by a function φ will produce a scaling of the volume by $1/\varphi$.

The construction is compatible, but independent of, the rigging technique [17] and [33, Sect. 2.6] (see Sec. 2.2.1 for details). For this reason, Vol_L will be called *rigged measure*.

We point out that the rigged measure Vol_L is mutually absolutely continuous w.r.t. the volume measure induced on H by any auxiliary Riemannian metric. In the sequel, we will say that a measure $\mu \in \mathcal{M}^+(H)$ is absolutely continuous w.r.t. H , if it is absolutely continuous w.r.t. the volume measure on H induced by some (hence any) metric. Accordingly, the family of probability measures absolutely continuous w.r.t. H will be denoted by $\mathcal{P}_{ac}(H)$.

The distortion of the rigged measure Vol_L along null geodesics is strictly related to the Ricci curvature of the ambient space-time in null directions (Proposition 4.3). In Section 4.1, we also consider the case of a weighted rigged measure $\mathbf{m}_L = e^\Phi \text{Vol}_L$, for some continuous function $\Phi : M \rightarrow \mathbb{R}$.

In Section 5, in order to analyse the interplay between the rigged measure and Ricci curvature, we partition a null hypersurface into null geodesics. Such a partition will be phrased in a terminology borrowed from optimal transport, in order to be suitable for generalizations to the non-smooth setting (see the forthcoming [6]). The main result of the section, Theorem 5.11 expresses the (resp. weighted) rigged measure in terms of a disintegration compatible with the aforementioned partition in null geodesics, and makes manifest the connection with the (resp. weighted) Ricci curvature of the ambient space-time in the null directions: namely, the null

Ricci dictates the concavity properties of the densities of the measures on the null geodesics in the partition. Let us state the result in a simplified form, referring to Theorem 5.11 for the more general (and precise) statement.

Theorem 1.1. *Let (M^n, g) be a Lorentzian manifold, let H be a causal null hypersurface admitting a space-like and acausal cross-section S , and a null-geodesic vector field L .*

Then the following representation formula holds

$$\text{Vol}_L = \int_S (e^{a_z(t)} dt) \mathcal{H}^{n-2}(dz),$$

where:

- \mathcal{H}^{n-2} is the $(n-2)$ -Hausdorff measure on the space-like cross-section S with respect to the Riemannian metric induced by the restriction of g on tangent spaces of S ;
- for every $z \in S$, there is a unique (maximally extended) null geodesic γ_z contained in H , satisfying $\gamma_z(0) = z$, $\frac{d\gamma_z}{dt}(t) = L(\gamma_z(t))$ for all $t \in \text{Dom}(\gamma_z)$;
- for every $z \in S$, $e^{a_z(t)} dt$ is a weighted measure on $\text{Dom}(\gamma_z) \subset \mathbb{R}$;
- for every $z \in S$, the function $t \mapsto a_z(t)$ satisfies the concavity property

$$a_z''(t) + \frac{(a_z'(t))^2}{n-2} \leq -\text{Ric}_{\gamma_z(t)}^g(L, L).$$

Motivated by such a result, we introduce our first synthetic notion of null energy condition, denoted by NC^1 , which amounts to require that the densities of the measures on the null geodesics in the aforementioned partition are log-concave (in a quantified sense). Here, we give a simplified version, referring to Definition 5.12 for a more general one, admitting a C^0 -weight on the rigged measure and weaker assumptions on the null hypersurface.

Definition 1.2 ($\text{NC}^1(n)$ condition). Let (M^n, g) , H , S , L be as in Theorem 1.1. We say that the triplet (M, g, H) satisfies the null energy condition $\text{NC}^1(n)$ if, for all $z \in S$, the function $t \mapsto a_z(t)$ is concave (thus, in particular, locally-Lipschitz) and it satisfies

$$(1.1) \quad a_z'' + \frac{(a_z')^2}{N-2} \leq 0, \quad \text{in the sense of distributions.}$$

We say that the space-time (M^n, g) satisfies the null energy condition $\text{NC}^1(n)$ if, for any null hypersurface $H \subset M$ as above, the triplet (M, g, H) satisfies the null energy condition $\text{NC}^1(n)$.

Since a non-negative lower bound on the Ricci curvature is invariant under scaling, it is not hard to see that the $\text{NC}^1(n)$ condition is independent of L (see Remark 5.14). Clearly, Theorem 1.1 yields that the classical NEC implies $\text{NC}^1(n)$. The reverse implication is proved in Theorem 8.5. Let us stress that this reverse implication is proved by assuming the $\text{NC}^1(n)$ condition only on (local) future light-cones. This feature differs from [23] where the displacement convexity of the entropy is required for certain null hypersurfaces constructed ad-hoc, whereas here the $\text{NC}^1(n)$ condition is required only for light-cones, which are null hypersurfaces with a clean physical interpretation (namely the events spanned by the light radiating from a point). The advantage of the $\text{NC}^1(n)$ condition, with respect to the NEC, is that it is suitable to be generalised to the non-smooth setting (see the forthcoming paper [6]). Moreover, in Theorem 9.1 we prove that NC^1 is stable under C^1 -convergence of the Lorentzian metrics and C^0 -convergence of the weights on the rigged measures. More general stability results, allowing non-smooth limit spaces, will be established in the forthcoming paper [6].

Optimal transport inside a null hypersurface. One of the novel aspects of the present work is the one concerning optimal transport inside a null hypersurface. As mentioned above, the challenge here is that the cost is completely degenerate (taking only the values 0 and ∞). Nonetheless, the ruled geometry of null hypersurfaces allows to give a quite precise picture.

Let us start with a definition (see Definition 6.2). Let $e_t : C([0, 1]; H) \rightarrow H$ be the evaluation map, i.e., $e_t(\gamma) := \gamma(t)$; also, denote by $\tau : M \times M \rightarrow [0, \infty)$ the time separation function (i.e., the sup of the proper time of causal curves joining the two points in the argument of τ , with the convention that $\tau(x, y) = 0$ if $x \not\leq y$, where \leq denotes the causal relation), and by $\Pi_{\leq}(\mu_0, \mu_1)$ the set of causal couplings from μ_0 to μ_1 (i.e., the set of couplings from μ_0 to μ_1 which are concentrated on the causal relation). In the next definition, we give a slight variation of terminology introduced in [22].

Definition 1.3 (Transport inside a null hypersurface). Let (M, g) be a Lorentzian manifold, H a null hypersurface and $\mu_i \in \mathcal{P}(H)$, $i = 0, 1$, be two probability measures. We say that μ_0 is *null connected to μ_1 along H* if there exists a probability measure $\nu \in \mathcal{P}(C([0, 1]; H))$ such that

$$(1.2) \quad \pi := (e_0, e_1)_{\#}\nu \in \Pi_{\leq}(\mu_0, \mu_1), \quad \tau(x, y) = 0 \text{ for } \pi\text{-a.e. } (x, y), \quad \text{and} \quad \nu\text{-a.e. } \gamma \text{ is causal.}$$

We also denote

$$(1.3) \quad \text{OptCaus}^H(\mu_0, \mu_1) := \{\nu \in \mathcal{P}(C([0, 1]; H)) \text{ satisfying (1.2)}\}.$$

By the very definition, it holds that $\pi := (e_0, e_1)_{\#}\nu$ is an optimal coupling from μ_0 to μ_1 for any Lorentz–Wasserstein distance ℓ_p , for all $p \leq 1$, (see Eckstein–Miller [13], Suhr [43] for $p = 1$, McCann [31] and Cavalletti–Mondino [8] for $p < 1$). Moreover, from (1.2), it follows that ν -a.e. γ is a null pre-geodesic (i.e., it can be re-parameterized into a null geodesic).

When dealing with the dynamical approach to optimal transport in the Riemannian setting or in the setting of purely time-like transport, it is well-known that optimal dynamical transport plans are concentrated on geodesics. In the case of transport with degenerate cost, this is not the case. For instance, as observed above, one can reparameterize the geodesics where a null optimal dynamical transport plan is concentrated, still finding an admissible optimal dynamical transport plan. We therefore introduce the following definition (see Definition 6.3).

Definition 1.4 (Null-geodesic dynamical transport plan). Let (M, g) be a Lorentzian manifold and H a null hypersurface. Let $\mu_0, \mu_1 \in \mathcal{P}(H)$ be two probability measures null connected along H . We say that a dynamical transport plan $\nu \in \text{OptCaus}^H(\mu_0, \mu_1)$ is *null-geodesic* if it is concentrated on null geodesics. The set of null-geodesic dynamical transport plans from μ_0 to μ_1 is denoted by $\text{OptGeo}^H(\mu_0, \mu_1)$. In other terms,

$$\text{OptGeo}^H(\mu_0, \mu_1) := \{\nu \in \text{OptCaus}^H(\mu_0, \mu_1) : \nu\text{-a.e. } \gamma \text{ is a null geodesic}\}.$$

A particularly useful class of transports is the one of *monotone* couplings, defined below (see Definition 6.10). Let us denote by $J \subset X^2$ the subset of causally related pairs in $X \times X$.

Definition 1.5 (Monotonone couplings and plans). Let $\mu_0, \mu_1 \in \mathcal{P}(H)$ be two probability measures null connected along H . We say that an optimal coupling π between μ_0 and μ_1 is *monotone*, if:

- (1) $\pi = (e_0 \otimes e_1)_{\#}\nu$ for some $\nu \in \text{OptCaus}^H(\mu_0, \mu_1)$;
 - (2) there exists a Borel set $A \subset (H \times H) \cap J$ such that $\pi(A) = 1$ and
- $$(1.4) \quad \forall (x_1, y_1), (x_2, y_2) \in A, \quad x_1 \leq x_2, x_1 \neq x_2 \implies y_1 \leq y_2.$$

A null-geodesic dynamical transport plan $\nu \in \text{OptGeo}^H(\mu_0, \mu_1)$ is said to be *monotone* if the optimal coupling $\pi := (e_0 \otimes e_1)_{\#}\nu$ is monotone, in the above sense.

The main result of Section 6 is that for every pair $\mu_0, \mu_1 \in \mathcal{P}_{ac}(H)$ of probability measures absolutely continuous w.r.t. H , such that μ_0 is null connected to μ_1 along H , there exists a unique monotone, null-geodesic dynamical transport plan from μ_0 to μ_1 (see Theorem 6.12 for the precise statement).

NEC as displacement convexity of the Boltzmann–Shannon entropy. Building on top of the results about optimal transport inside a null hypersurface obtained in Section 6, in Section 7 we give a synthetic characterization of the NEC in terms of displacement convexity of the Boltzmann–Shannon entropy along monotone null-geodesic optimal transport plans. Below, we give a simplified version of the result, referring to Theorem 7.3 for the general (and more precise) statement. Recall that the Boltzmann–Shannon entropy $\text{Ent}(\mu|\mathbf{m})$ of a probability measure $\mu \in \mathcal{P}(M)$ with respect to the reference measure \mathbf{m} is defined by

$$\text{Ent}(\mu|\mathbf{m}) = \int \rho \log \rho \, d\mathbf{m},$$

if $\mu = \rho \mathbf{m}$ is absolutely continuous w.r.t. \mathbf{m} and $(\rho \log \rho)_+$ is \mathbf{m} -integrable; otherwise, we adopt the convention that $\text{Ent}(\mu|\mathbf{m}) = +\infty$. We will consider the dimensional variant

$$U_N(\mu|\mathbf{m}) := \exp\left(-\frac{1}{N} \text{Ent}(\mu|\mathbf{m})\right),$$

which was studied in [14] in connection to Ricci bounds. This kind of functional is well-known in information theory as the Shannon entropy power (see e.g., [12]). For instance, the entropy power on \mathbb{R}^N is the functional $U_{N/2}$.

Theorem 1.6. *Let (M, g) be a Lorentzian manifold of dimension n satisfying the NEC, and let $H \subset M$ be a null hypersurface endowed with a rigged measure Vol_L . Let $\mu_0, \mu_1 \in \mathcal{P}_{ac}(H)$ be two probability measures such that μ_0 is null connected to μ_1 along H and let $\nu \in \text{OptGeo}^H(\mu_0, \mu_1)$ be the unique monotone null-geodesic dynamical transport plan joining them. Denote by $\mu_t := (e_t)_\# \nu$, $t \in [0, 1]$, and let $u_{n-1}(t) := U_{n-1}(\mu_t | \text{Vol}_L)$.*

Then the function $[0, 1] \ni t \mapsto u_{n-1}(t)$ satisfies

$$(1.5) \quad u_{n-1}(t) \geq (1-t)u_{n-1}(0) + tu_{n-1}(1), \quad \forall t \in [0, 1].$$

Theorem 1.6 suggests the following two definitions, one concerning null hypersurfaces and one concerning the ambient space-time. We give here a simplified version, referring to Definition 7.6 and Definition 7.7 for the more general (and precise) notions, also allowing a C^0 -weight on the rigged measure.

Definition 1.7 (The $\text{NC}^e(n)$ condition). Let (M, g) be a Lorentzian manifold and $H \subset M$ be a null hypersurface endowed with a rigged measure Vol_L .

We say that the triplet (M, g, H) satisfies the $\text{NC}^e(n)$ condition if the following holds: For every pair of probability measures $\mu_0, \mu_1 \in \mathcal{P}_{ac}(H)$ such that μ_0 is null connected to μ_1 along H , there exists a null-geodesic dynamical transport plan $\nu \in \text{OptGeo}^H(\mu_0, \mu_1)$ satisfying (1.5).

We say that (M, g) satisfies the $\text{NC}^e(n)$ condition if every null hypersurface does so.

It is not hard to see that the definition of $\text{NC}^e(n)$ does not depend on the choice of L (Remark 7.8). Moreover NEC, $\text{NC}^1(n)$ and $\text{NC}^e(n)$ are all equivalent for smooth space-times: we already recalled that $\text{NC}^1(n)$ is equivalent to NEC, Theorem 7.3 proves the implication $\text{NC}^1(n) \Rightarrow \text{NC}^e(n)$ and, finally, the reverse implication $\text{NC}^e(n) \Rightarrow \text{NC}^1(n)$ is established in Theorem 8.4.

Applications. As applications of the theory developed in the present paper, we obtain weighted versions (in low regularity for the weight, assumed merely to be continuous) of the light-cone theorem and of the Hawking area theorem. Both in sharp form and with an apparently new rigidity statement for the former. The proofs build on the tools developed in the paper. Indeed, for a continuous weight, the classical NEC is not well-defined (as it involves a linear term in the second derivatives, and a quadratic term in the first derivatives of the weight); as a replacement, we assume its synthetic counterparts NC^1 or NC^e discussed above.

The weighted light-cone theorem. The light-cone theorem, first established by Choquet-Bruhat, Chruściel and Martín-García [9], states that the area of cross-sections of light-cones, in spacetimes satisfying the Einstein equations with vanishing cosmological constant and with the energy–momentum tensor obeying the dominant energy condition, is bounded above by the area of corresponding cross-sections of light-cones in Minkowski spacetime. The rigidity question is addressed in [9] under additional assumptions on the energy–momentum tensor.

In a subsequent work, Grant [16], revisited the light-cone theorem: the assumption was relaxed to asking the NEC, and the conclusion was sharpened to a monotonicity statement, reminiscent of the Bishop–Gromov volume comparison in Riemannian geometry. Moreover, [16] proves the sharpness of the result (equality is attained on model spaces), leaving open the question of the rigidity.

In Section 10.1, we extend the light-cone theorem in the sharp monotone form to Lorentzian manifolds with a continuous weight on the rigged measure satisfying $\text{NC}^e(N)$ (see Theorem 10.1), and we address the rigidity question by showing that equality is attained if and only if the light-cone is isometric to the light-cone in Minkowski spacetime (see Theorem 10.2).

The weighted Hawking area theorem. In Section 10.2, we provide an extension of the celebrated Hawking’s area theorem to Lorentzian manifolds with a continuous weight on the rigged measure, satisfying $\text{NC}^1(N)$. We also obtain an apparently new rigidity statement: equality is attained in the Hawking area theorem if and only if the metric on the horizon of the black hole is static (in the sense that it splits isometrically as a product). Let us briefly frame our result in the perspective of the existing literature.

The area theorem was proved by Hawking [19] in 1971; in its original formulation it states that, in the setting of smooth space-times satisfying the null energy condition, the area of a smooth black hole horizon can never decrease.

In classical general relativity, the result paved the way to black hole thermodynamics; indeed, the area of a black hole horizon is interpreted as a measure of its entropy, and the second law of thermodynamics states that entropy does not decrease in time.

The result was revisited by Chruściel–Delay–Galloway–Howard [10] and Minguzzi [33], lowering the regularity assumptions on the null hypersurface (the ambient spacetime is assumed to be smooth) and discussing rigidity: [10] proves that if equality is attained then the second fundamental form of the null hypersurface has to vanish; [33, Th. 12, Th. 13] show that equality holds in the area theorem if and only if the portion of the null hypersurface between the two cross-sections achieving equality can be written as a graph of a $W^{2,1}$ -function with vanishing hessian.

In Theorem 10.3, the assumption required in the classical form of Hawking’s Theorem that the Ricci curvature is non-negative on all null vectors in the space-time, is replaced by the weaker null energy condition $\text{NC}^1(N)$ of Definition 1.2 (actually, the more general Definition 5.12). Moreover, we allow a C^0 weight on the rigged measure, and we obtain the rigidity in an apparently new formulation (see also Remark 10.4).

Let us mention that Ketterer [23] recently extended Hawking’s area theorem to smooth weighted space-times satisfying the NEC. The methods in [23] are also based on optimal transport; however, the approach in the present work is more tailored to extensions to non-smooth spacetimes. Indeed, building on the tools developed in the present paper, in the forthcoming work [6] we will push the results beyond the differentiable setting by extending the NC^1 condition to Lorentzian length spaces, and establishing Hawking’s area theorem in such a synthetic framework.

Plan of the paper. The paper is organized as follows. In Section 2 we recall a few facts about null hypersurfaces; in Section 3 we investigate the rigged volume measure, in particular its interplay with the Jacobi fields. Sections 4 and 5 investigate the distortion of the (weighted) rigged measure, culminating in the definition of $\text{NC}^1(N)$. In Sections 6 and 7 we study optimal

transport inside null hypersurfaces and the displacement convexity of the entropy. Section 8 proves the equivalence of $\text{NC}^1(N)$, $\text{NC}^e(N)$, and NEC; Section 9 presents a stability result for NC^1 . Finally, in Section 10 we present two applications: the light-cone theorem and Hawking area theorem.

2. CONVENTIONS AND PRELIMINARIES

In this section we recall a few facts regarding the geometry of Lorentzian manifolds.

2.1. General facts of Lorentzian geometry. A Lorentzian manifold is a couple (M, g) , where M is a smooth n -dimensional manifold (the case $n = 4$ corresponds to classical general relativity) and $g \in \mathfrak{X}(T^*M^{\otimes 2})$ is a Lorentzian product, i.e., a symmetric non-degenerate bilinear form, whose signature is $(- + \cdots +)$. Throughout the paper, g is assumed to be at least continuous. Note that, for C^0 Lorentzian metrics, the natural class of differentiability of the manifolds is C^1 . However, a C^1 manifold always possesses a C^∞ -sub-atlas, and one can choose some such sub-atlas whenever convenient. Thus, the smoothness assumption on M is not restrictive. A vector $X \in T_x M$ will be called

- time-like, if $g(X, X) < 0$;
- space-like, if $g(X, X) > 0$ or $X = 0$;
- light-like, if $g(X, X) = 0$ and $X \neq 0$;
- causal, if it is either time-like or light-like.

We say that a Lorentzian manifold (M, g) is *time-oriented*, if there exists a continuous time-like vector field. In this paper, all Lorentzian manifolds are assumed to be time-oriented. If $X \in T_x M$ is a causal vector, we say that X is *future-directed*, if $g(X, Y) < 0$, where Y is a vector giving the temporal orientation to the manifold.

One can always endow a Lorentzian manifold with an auxiliary smooth Riemannian metric; locally-Lipschitz regularity will always be understood w.r.t. some (hence any) auxiliary smooth Riemannian metric. We say that a locally-Lipschitz curve $\gamma : I \rightarrow M$ is *causal*, if $\dot{\gamma}$ is future-directed (at the differentiability points); we say that it is *chronological*, if it is causal and $g(\dot{\gamma}, \dot{\gamma}) < 0$. We say that a Lorentzian manifold is *causal* if there exists no periodic causal curve.

A causal curve is called *inextendible*, if its domain of definition I cannot be extended. If $x, y \in M$, we say $x \leq y$ (resp. $x \ll y$), if there exists a causal (resp. chronological) curve connecting x to y . The (Lorentzian) *length* of a causal curve $\gamma : I \rightarrow M$ is defined as

$$(2.1) \quad \text{Length}(\gamma) := \int_I \sqrt{-g(\dot{\gamma}_t, \dot{\gamma}_t)} dt;$$

the *time-separation* (or *Lorentzian distance*) between $x, y \in M$ is given by

$$(2.2) \quad \tau(x, y) := \begin{cases} \sup \text{Length}(\gamma) & \text{if } x \leq y, \\ 0 & \text{otherwise,} \end{cases}$$

where the supremum is taken among all causal curves γ connecting x to y .

A causal curve $\gamma : I \rightarrow M$ is called a *geodesic* if it is a critical point of the Length functional with respect to variations that do not move the end-points; in other words, if it satisfies the equation of the geodesics. If g is at least $C_{loc}^{1,1}$, a geodesic can be written in the form $\gamma_t = \exp(tv)$, for some $v \in TM$, where \exp is the (geodesic) exponential map. A *maximizing geodesic* $\gamma : [0, 1] \rightarrow M$ is a causal curve, such that

$$\tau(\gamma_t, \gamma_s) = (s - t) \tau(\gamma_0, \gamma_1), \quad \text{for all } t \leq s.$$

For $g \in C_{loc}^{1,1}$, any maximizing geodesic is also a geodesic in the above sense (as it is a maximum of the Length functional, with fixed extrema), while the converse implication holds only locally after reparametrizing by τ arc-length (see [28] for g of lower regularity, up to $C_{loc}^{0,1}$).

A submanifold $S \subset M$ is called *space-like* (resp. *null*) if the metric g restricted to the tangent space TS is positive definite (resp. degenerate). The normal bundle of a submanifold S is denoted by

$$\mathcal{N}S := \{v \in TM : g(v, w) = 0, \forall w \in TS\}.$$

If $S \subset M$ is of class C^{k+1} and g is of class C^h , the normal bundle will be of regularity $C^{\min\{h, k\}}$. Indeed, a local basis of $\mathcal{N}S$ can be built as follows: fix v_1, \dots, v_j a local basis for TS of class C^k and add v_{j+1}, \dots, v_n obtaining a local trivialization for $TM|_S$; then apply the Gram–Schmidt orthogonalization process and obtain a basis for $\mathcal{N}S$. Throughout the paper, sections of class C^k of a vector bundle F (of at least the same class) will be denoted by $\mathfrak{X}^k(F)$.

We now recall a classical fact for submanifolds of co-dimension 1 in Lorentzian signature.

Proposition 2.1 (Pag. 45 in [20]). *Let (M, g) be a Lorentzian manifold, with $g \in C^0$. Let $H \subset M$ be a C^1 hypersurface. Then the following are equivalent*

- (1) H is a null hypersurface, i.e., $g|_{TH^{\otimes 2}}$ is degenerate;
- (2) $\mathcal{N}H \subset TH$.

2.2. Null hypersurfaces. In this section we recall some basics on the geometry of null hypersurfaces. Let g be a C^1 Lorentzian metric on M , and let $H \subset M$ be a C^2 null hypersurface. Since g restricted to TH is degenerate, then H does not possess a Levi–Civita connection. However, we can restrict the Levi–Civita connection ∇ of g obtaining a bilinear operator, still denoted by ∇ , $\nabla : \mathfrak{X}^1(TH) \times \mathfrak{X}^1(TH) \rightarrow \mathfrak{X}^0(TM|_H)$.

More precisely, if $X, Y \in \mathfrak{X}^1(TH)$ are vector fields tangent to H , we define

$$\nabla_X Y := \nabla_X \tilde{Y} \in \mathfrak{X}^0(TM|_H),$$

where $\tilde{Y} \in \mathfrak{X}^1(TM)$ is any extension of Y . The definition is well-posed (see for instance [15]).

This operator inherits a few properties from the Levi–Civita connection. For instance, for all $X, Y, Z \in \mathfrak{X}^1(TH)$ and $f \in C^1(H)$, it holds:

$$\begin{aligned} \nabla_{fX} Y &= f \nabla_X Y, & \nabla_X (fY) &= X(f)Y + f \nabla_X Y, \\ X(g(Y, Z)) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \\ \nabla_X Y - \nabla_Y X &= [X, Y]. \end{aligned}$$

The following proposition states a fundamental property of vector fields orthogonal to a null hypersurface. The proof is included for readers' convenience.

Proposition 2.2. *Let g be a C^1 Lorentzian metric on the smooth manifold M . Let $H \subset M$ be a C^2 null hypersurface and let $L \in \mathfrak{X}^1(\mathcal{N}H)$ be a normal vector field. Then it holds that $\nabla_L L \in \mathfrak{X}^0(\mathcal{N}H)$, i.e., $\nabla_L L$ is normal to H .*

Proof. Let $X \in \mathfrak{X}^1(TH)$ and compute

$$\begin{aligned} g(X, \nabla_L L) &= L(g(X, L)) - g(\nabla_L X, L) = -g(\nabla_L X, L) = -g(\nabla_X L, L) - g([X, L], L) \\ &= -g(\nabla_X L, L) = -\frac{1}{2} X(g(L, L)) = 0, \end{aligned}$$

having used the properties of the operator ∇ , that $g(X, L) = 0$, and that $[X, L]$ is a vector field tangent to H (as $L \in TH$, by Proposition 2.1) hence normal to L . \square

Since g is non-degenerate, the null direction is one-dimensional and therefore Proposition 2.2 implies that $\nabla_L L = hL \in \mathfrak{X}^0(TH)$ for some real valued function h . Locally, one can reparametrize the vector field L and obtain $h = 0$, provided g is sufficiently regular.

Proposition 2.3. *Let g be a C^2 Lorentzian metric on the smooth manifold M , and let $H \subset M$ be a C^2 null hypersurface. Then locally there exists a normal vector field $L \in \mathfrak{X}^1(\mathcal{N}H)$, such that $\nabla_L L = 0$ and L is future-directed.*

Proof. The time-oriented-ness of (M, g) (recall this is a standing assumption) and the regularity of g and H imply the existence of a future-directed $\tilde{L} \in \mathfrak{X}^1(\mathcal{N}H)$ that never vanishes. Without loss of generality, we can assume that \tilde{L} has unit norm, w.r.t. an auxiliary smooth Riemannian metric. Let $S \subset H$ be a local space-like cross-section. Since we are looking for a local vector field, up to restricting H , we can assume that S is a global C^2 cross-section for H . Let $G : H \rightarrow \mathbb{R}$ be a C^2 function such that $G^{-1}(0) = S$ and $dG \neq 0$ along S . Consider the function

$$F : \text{Dom } F \subset H \times \mathbb{R} \rightarrow \mathbb{R}, \quad F(x, t) = G(\exp_x(t\tilde{L})).$$

Proposition 2.2 guarantees $\frac{d}{dt} \exp_x(t\tilde{L}) \in \mathcal{N}_{\exp_x(t\tilde{L})}H$, for t small enough, therefore the domain of definition of F contains a neighborhood of $S \times \{0\}$. Of course, F is of class C^1 and satisfies

$$(2.3) \quad F(x, 0) = 0 \quad \text{and} \quad \frac{\partial F}{\partial t}(x, 0) = dG(\tilde{L}) \neq 0, \quad \forall x \in S.$$

We are in position to apply the Implicit Function Theorem, yielding a neighborhood $U \subset H$ of S and a C^1 function $t : U \rightarrow \mathbb{R}$, such that $F(x, t(x)) = 0$.

Define the function $\lambda : U \rightarrow \mathbb{R}$, in such a way that

$$(2.4) \quad \left. \frac{d}{dt} \right|_{t=t(x)} \exp_x(t\tilde{L}) = \lambda(x) \tilde{L}_{\exp_x(t(x)\tilde{L})}.$$

The function λ is C^1 , because it coincides with the norm (in the auxiliary metric) of the vector $\left. \frac{d}{dt} \right|_{t=t(x)} \exp_x(t\tilde{L})$ (recall \tilde{L} is of unit norm). Moreover, $\lambda(x) = 1$ for all $x \in S$ and $\lambda > 0$ everywhere. Therefore, we can define $L := \lambda^{-1}\tilde{L} \in \mathfrak{X}^1(\mathcal{N}H)$. Its flow is geodesic, because it is the velocity field of the geodesic flow starting from points in S with initial velocity \tilde{L} . For this reason $\nabla_L L = 0$. \square

Remark 2.4. Proposition 2.3 proves the following: for any point $z \in H$ there exists an open (in H) neighborhood $Z \subset H$ of z and a section $L \in \mathfrak{X}^1(\mathcal{N}H)$ that is also future-directed, different from the zero vector in Z and $\nabla_L L = 0$ in Z . For the elements of this particular class of normal vector fields we will adopt the terminology of *null-geodesic vector fields*. They will be tacitly considered only on the open set where $\nabla_L L = 0$ holds true.

We now introduce the notion of transverse function.

Definition 2.5. Given a null hypersurface $H \subset M$, we say that a function $\varphi : H \rightarrow \mathbb{R}$ is *transverse* if for all $x \in H$ and all $v \in \mathcal{N}_x H \setminus \{0\}$, the composition $\varphi \circ \gamma$ is constant, where γ is the geodesic in M (and contained in H) with $\gamma_0 = x$ and $\dot{\gamma}_0 = v$.

The set of transverse functions is closed under point-wise sums and products. Note that if $\varphi : H \rightarrow \mathbb{R}$ is C^1 , being transverse is equivalent to $L(\varphi) = 0$, for some (hence any) normal vector field L .

Remark 2.6. If $L \in \mathfrak{X}^1(\mathcal{N}H)$ is a null-geodesic vector field and $\varphi \in C^1(H)$ is a transverse positive function, then φL is still a null-geodesic vector field.

Conversely, if $L_1, L_2 \in \mathfrak{X}^1(\mathcal{N}H)$ are two null-geodesic vector fields on the same open set, then there exists a C^1 function φ , such that $L_1 = \varphi L_2$.

Given a normal vector field $L \in \mathfrak{X}^0(\mathcal{N}H)$, it is natural to look for submanifolds whose tangent complement L in TH . These will be local space-like cross-sections of which we recall the definition.

Definition 2.7. A submanifold $S \subset H$ of dimension $n - 2$ is a *local cross-section* of H if the metric g is positive definite on TS . We will always assume that S is either closed without boundary, or an open $(n - 2)$ -manifold.

It is immediate to verify that at each point x of S , it holds that $T_x H = T_x S \oplus \text{span}\{L(x)\}$, where $L(x) \in \mathcal{N}_x H \setminus \{0\}$. We say that a cross-section S is *pre-compact* if \bar{S} is compact (the closure is intended in H) and there exists a cross-section \tilde{S} such that $\bar{S} \subset \tilde{S}$.

Given a cross-sections S and a normal vector field L (not necessarily null-geodesic), one can construct a new normal vector field \bar{L}^S along S in the following way. Define \bar{L}^S as the unique vector field in $\mathcal{N}S$ verifying the following conditions:

$$(2.5) \quad g(\bar{L}^S, X) = 0, \quad \forall X \in TS,$$

$$(2.6) \quad g(\bar{L}^S, L) = -1, \quad \text{and} \quad g(\bar{L}^S, \bar{L}^S) = 0.$$

Notice that the construction of \bar{L}^S depends both on L and S . If S is of class C^2 and L of class C^1 , then \bar{L}^S will be of class C^1 as well.

If L is a null-geodesic vector field (see Remark 2.4), using parallel transport along the flow of L , \bar{L}^S can be then defined in a suitable neighborhood $Z \subset H$ of any point of z of the submanifold S keeping (2.6) valid and preserving the C^1 regularity. Moreover, if φ is a transverse function, then $(\varphi L)^S = \frac{1}{\varphi} \bar{L}^S$.

Remark 2.8. Let S be a local cross-section for H such that the (null-)integral lines of L meet with S at most once. Then it is possible to construct a null-geodesic vector field, coinciding with L along S and whose integral lines have non-empty intersection with S , as follows. For each $x \in S$ consider $\gamma_{x,t} := \exp_x(tL)$. Then, setting $\tilde{L} := d\gamma_{x,t}/dt$, it is straightforward to check that \tilde{L} is null, see Proposition 2.2, and $\nabla_{\tilde{L}} \tilde{L} = 0$.

The three objects considered so far – L , S and \bar{L}^S – will be extensively used in the computations below. In the next section, we report on a slightly different method to build such objects.

2.2.1. The rigging technique. A popular approach used for investigating null hypersurfaces and to construct Riemannian metrics on them, is the *rigging technique*. The data are a null hypersurface and a transverse vector field to it called the *rigging vector*. The rigging vector field produces a tangent null vector field, called the *rigged vector*, and a Riemannian tensor, called the *rigged metric*. As a byproduct one also obtains a volume form.

The approach presented so far in Section 2.2 followed the opposite direction: from a normal vector field L and a local cross-section S , we obtained a vector field \bar{L}^S transverse to H .

Following [17] (see also [33, Sect. 2.6]), we now use the rigging technique by choosing as the rigging vector the transverse vector field $-\bar{L}^S$ that we assume to be defined on an open set $Z \subset H$; motivated by the previous discussion, we assume $-\bar{L}^S$ of class C^1 . As a final outcome, the rigged vector field will be L .

Let ω be the g -dual of $-\bar{L}^S$ in $TM|_Z$, i.e.,

$$\omega(v) = g(-\bar{L}^S, v), \quad \text{for all } v \in TM|_Z.$$

Let α be the restriction of ω to TH , i.e., $\alpha := i^* \omega$ where $i : Z \subset H \rightarrow M$ is the inclusion map. Notice that $\ker \alpha|_S \cap Z = TS$, see (2.5) and that α is of class C^1 .

Consider the rigged metric $\tilde{g}_L^S := g + \alpha \otimes \alpha \in \mathfrak{X}^1(T^*Z^{\otimes 2})$. Since \bar{L}^S is transverse to H , then \tilde{g}_L^S is a Riemannian metric for Z . A direct computation gives

$$(2.7) \quad \tilde{g}_L^S(v, w) = g(v, w), \quad \tilde{g}_L^S(v, L) = 0, \quad \tilde{g}_L^S(L, L) = 1, \quad \forall v, w \in \ker \alpha.$$

The rigged vector is obtained as the \tilde{g}_L^S -dual of the form α ; it is immediate to see that it is of class C^1 and it coincides with L :

$$(2.8) \quad \alpha(v) = g(-\bar{L}^S, v) = g(L, v) + g(-\bar{L}^S, L) g(-\bar{L}^S, v) = \tilde{g}_L^S(L, v), \quad \forall v \in TH,$$

having used the fact that L is orthogonal to all vectors in TH and (2.6).

Definition 2.9. We denote by $\tilde{\text{Vol}}_L^S \in \mathcal{M}^+(Z)$ the volume measure induced by the Riemannian rigged metric \tilde{g}_L^S defined over the open set $Z \subset H$.

We will prove that $\tilde{\text{Vol}}_L^S$ depends only on the null-geodesic vector L and not on the rigging vector $-\overline{L^S}$.

2.3. Local Parametrization. Let L be a local null-geodesic vector field on H . We write

$$(2.9) \quad \Psi_L(z, t) := \exp_z(tL), \quad \forall z \in H,$$

whenever this exponential makes sense and belongs to H . It is immediate to check that the map Ψ_L satisfies (on its domain):

$$\Psi_L(z, t + s) = \Psi_L(\Psi_L(z, t), s) \quad \text{and} \quad \Psi_{\varphi L}(z, t) = \Psi_L(z, \varphi(z)t), \quad \forall \varphi \text{ transverse.}$$

It is also clear that, for small t , $(z, t) \mapsto \Psi_L(z, t)$ is a local diffeomorphism on H .

We now construct the local parametrization that will be used in the computations of the next sections. Let $f : U \subset \mathbb{R}^{n-2} \rightarrow S$ be a local parametrization of S (recall that S is assumed to be C^2). We define

$$\zeta(x, t) := \Psi_L(f(x), t), \quad \forall x \in U,$$

giving a local parametrization for H . In particular, we denote by $V \subset U \times \mathbb{R}$ an open set where ζ is a diffeomorphism. The definition of ζ depends on the choice of L and S , however, to ease the notation, we omit these dependences.

With a slight abuse of notation, we extend the function Ψ_L to a larger domain by setting $\Psi_L(z, 0, s) := \exp_z(s\overline{L^S})$. We now transport the vector field L to $\Psi_L(z, 0, s)$ by parallel transport along the geodesic $s \mapsto \Psi_L(z, 0, s)$. Then we define

$$\Psi_L(z, t, s) := \exp_{\Psi_L(z, 0, s)}(tL).$$

This new definition of Ψ_L is compatible with the former one, as $\Psi_L(z, t, 0) = \Psi_L(z, t)$, and the newly defined map is a local diffeomorphism. Analogously, we extend ζ by defining

$$(2.10) \quad \zeta(x, t, s) := \Psi_L(f(x), t, s),$$

so that $\zeta(x, t, 0) = \zeta(x, t)$. Thus ζ is a local diffeomorphism on M . We let

$$(2.11) \quad \Omega_S := \zeta(V \times \{(0)\}) \subset Z \subset H.$$

It should be noticed that the definition of Ω_S does not depend on the choice of the parametrization f .

3. VOLUME MEASURES ON NULL HYPERSURFACES

The rigged volume measure $\tilde{\text{Vol}}_L^S$ is well-defined on the set Ω_S as in (2.11); by definition, in local coordinates, its density with respect to the Lebesgue measure can be written in terms of the square root of the determinant of the metric \tilde{g}_L^S . A convenient way to write this expression is based on Jacobi fields calculations. Throughout the section, we will assume the Lorentzian metric g and the null hypersurface H to be of class C^2 . Under such assumptions, Proposition 2.3 (see also Remark 2.4) guarantees the existence of a locally defined null-geodesic vector field L along H of class C^1 .

3.1. Jacobi fields in null hypersurfaces. Let $H \subset M$ be a null hypersurface and let $L \in \mathfrak{X}^1(\mathcal{N}H)$ be a null-geodesic vector field. Given $z \in H$, a Jacobi field starting from z is a C^2 -function $J : I \rightarrow TM|_H$, such that $J(t) \in T_{\Psi_L(z,t)}M$ and J solves the Jacobi equation

$$(3.1) \quad J''(t) + R(L, J(t))L = 0,$$

where R is the Riemann curvature tensor of the Lorentzian metric g . Since the ODE (3.1) is *linear* in J , and both R and L are continuous, the associated initial value problem has a unique local solution, thanks to the classical Cauchy-Lipschitz theorem. If $v \in T_zM$, we denote by $J_{v,L}(z, t)$ the Jacobi field solving (3.1), coupled with the initial conditions

$$(3.2) \quad J_{v,L}(z, 0) = v \quad \text{and} \quad J'_{v,L}(z, 0) = \nabla_v L.$$

Notice that if φ is a C^1 transverse function, then $J_{v,\varphi L}(z, t) = J_{v,L}(z, \varphi(z)t)$. To ease the notation, we will omit the dependence on z and L , whenever no confusion arise.

We recall a few facts about Jacobi fields. The first is that L itself is a Jacobi field, with $J_L(t) = L$. Moreover, Jacobi fields can be characterized as the solutions of the following first-order ordinary differential equation:

$$(3.3) \quad J'_v(t) = \nabla_{J_v(t)} L, \quad J_v(0) = v;$$

the proof of this fact can be obtained by deriving once and using the definition of the Riemann curvature tensor.

Recall the next useful result, often referred as Gauss' Lemma.

Lemma 3.1. *For all $v \in T_zM$, the map $t \mapsto g(J_v(t), L)$ is constant.*

Proof. If we derive twice $g(J_v(t), L)$, we obtain $-g(R(L, J_v(t))L, L)$, which is null by skew-symmetry of the Riemann curvature tensor. The first derivative computed at $t = 0$ is:

$$\left. \frac{d}{dt} \right|_{t=0} g(J_v(t), L) = g(J'_v(0), L) = g(\nabla_v L, L) = \frac{1}{2}v(g(L, L)) = 0. \quad \square$$

Corollary 3.2. *If $v \in TH$, then $J_v(t) \in TH$ for all t .*

Proof. Being tangent to the hypersurface is equivalent to be orthogonal to L , and the orthogonality to L is conserved by Lemma 3.1. \square

Given a basis $e_1(z) \dots e_n(z)$ for T_zM , using parallel transport along the curves $t \mapsto \Psi_L(z, t)$ we can extend $(e_i(z))_i$ to $\Psi_L(z, t)$. Define the i -th Jacobi vector field as $J_{i,L}(z, t) := J_{e_i,L}(z, t)$. We express the field $J_{i,L}$ in the basis $(e_j(z))_j$ as

$$(3.4) \quad J_{i,L}(z, t) = \sum_j J_{ij,L}(z, t)e_j(z).$$

The matrix $J_L(z, t) = (J_{ij,L}(z, t))_{i,j}$ satisfies the following linear ODE

$$(3.5) \quad J''_L(z, t) + J_L(z, t)R_L(t) = 0,$$

where R_L is a matrix representing the Riemann curvature tensor $R(L, e_i)L = \sum_j R_{ij,L}(t)e_j$. To ease the notation, we omit the dependence on z and L whenever no confusion arises.

The initial datum for the matrix J is $J(0) = I$ (here I is the identity matrix). We omit to compute the initial datum for the derivative J' , for it is not in our interest. Nonetheless, if

$$(3.6) \quad \eta(z) = (\eta_{ij})_{ij} = (g(e_i, e_j))_{ij},$$

i.e., η is the matrix representing the Lorentzian product in this basis, we can prove the following.

Proposition 3.3. *The matrix $J'(0)$ satisfies*

$$(J'(0)\eta)^T = J'(0)\eta.$$

Proof. First we notice that L is locally a gradient. Indeed, let u be the function defined by the equation

$$u(\zeta(x, t, s)) = -s,$$

where ζ is the local parametrization (2.10). Its differential is given by

$$du(e_i) = 0, \quad i = 1 \dots n-2, \quad du(e_{n-1}) = du(L) = 0, \quad du(e_n) = du(\overline{L^S}) = -1,$$

yielding that L is the only vector such that $g(L, e_i) = du(e_i)$, $i = 1 \dots n$. This implies the symmetry

$$g(\nabla_{e_i} L, e_j) = g(\nabla_{e_j} L, e_i).$$

Therefore

$$\sum_k J'_{ik}(0) \eta_{kj} = g(J'_i(0), e_j) = g(\nabla_{e_i} L, e_j) = g(\nabla_{e_j} L, e_i) = g(J'_j(0), e_i) = \sum_k J'_{jk}(0) \eta_{ki}. \quad \square$$

Lemma 3.4. *Let g be a Lorentzian metric of class C^2 and S be a local cross-section of class C^2 for the null hypersurface H .*

If $f : U \rightarrow S$ is a C^2 -diffeomorphism inducing the parametrization map ζ for H as in (2.10), such that $f(x) = z$ and $\zeta(x, 0, 0) = z$, then the vector field $Y_i(t) = d\zeta_{(x,t,0)} \left(\frac{\partial}{\partial x_i} \right)$ satisfies the Jacobi field equation (3.1) with initial conditions (3.3). In particular,

$$(3.7) \quad J_{d\zeta_{(x,0,0)} \left(\frac{\partial}{\partial x_i} \right)}(t) = d\zeta_{(x,t,0)} \left(\frac{\partial}{\partial x_i} \right), \quad i = 1 \dots n.$$

Proof. Since by Proposition 2.3 (see also Remark 2.4), the null geodesic vector field L is of class C^1 , then the parametrization map ζ for H given in (2.10) is of class C^1 as well. Denote $Y(t) := \frac{\partial}{\partial x_i} \zeta_{(x,t,0)}$ and notice this is a continuous vector field along H . Using that distributional derivative commute and the very definition of Riemann curvature tensor, we infer that

$$\begin{aligned} Y''(t) &= \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial x_i} \zeta_{(x,t,0)} \quad \text{in distributional sense} \\ &= \nabla_{\frac{\partial}{\partial x_i}} \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} \zeta_{(x,t,0)} - R \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_i} \right) \frac{\partial}{\partial t} \quad \text{in distributional sense.} \end{aligned}$$

Since, by construction, the curve $t \mapsto \zeta_{(x,t,0)}$ is a geodesic, then the first term in the right hand side vanishes. Therefore, Y is a distributional solution of the Jacobi fields ODE (3.1) with initial conditions (3.3). A standard bootstrap argument yields that $t \mapsto Y(t)$ is a C^2 classical solution of such a Cauchy problem, i.e. it is a Jacobi field. By the Cauchy-Lipschitz theorem, uniqueness for such an initial value problem holds, thus (3.7) holds. \square

Motivated by Lemma 3.4, we choose as basis for $T_z M$ the vectors $e_i(z) := d\zeta_{(x,0,0)} \left(\frac{\partial}{\partial x_i} \right)$, with $i = 1, \dots, n$; in particular $e_{n-1}(z) = L$ and $e_n(z) = \overline{L^S}$.

The orthogonality properties of L and $\overline{L^S}$ imply that, under this choice of coordinates, the matrix η as in (3.6) takes the form

$$(3.8) \quad \eta := \begin{bmatrix} \eta_{11} & \dots & \eta_{1,n-2} & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \eta_{n-2,1} & \dots & \eta_{n-2,n-2} & 0 & 0 \\ 0 & \dots & 0 & 0 & -1 \\ 0 & \dots & 0 & -1 & 0 \end{bmatrix}.$$

Moreover with this choice of basis, we deduce some additional property for the matrix J .

Lemma 3.5. *The matrix $J(t)$ satisfies the following identities*

$$(3.9) \quad J(t)^T \mathbf{f}_{n-1} = \mathbf{f}_{n-1}, \quad J(t) \mathbf{f}_n = \mathbf{f}_n,$$

with the convention that $(\mathbf{f}_i)_i$ is the canonical basis of \mathbb{R}^n . In other words, the matrix $J(t)$ is of the form

$$(3.10) \quad J(t) := \begin{bmatrix} J_{11} & \cdots & J_{1,n-2} & J_{1,n-1} & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ J_{n-2,1} & \cdots & J_{n-2,n-2} & J_{n-2,n-1} & 0 \\ 0 & \cdots & 0 & 1 & 0 \\ J_{n,1} & \cdots & J_{n,n-2} & J_{n,n-1} & 1 \end{bmatrix}.$$

Proof. Since $J_L(t) = L$, then $J_{n-1}(t) = L$ or in coordinates $J_{n-1,j}(t) = \delta_{n-1,j}$, which proves the first identity in (3.9). Then, using Lemma 3.1 and (3.8), we compute

$$\begin{aligned} g(J_i(t), L) &= g(J_i(0), L) = g(J_i(0), e_{n-1}) = g(e_i, e_{n-1}) = \eta_{in-1} \\ g(J_i(t), L) &= \sum_j g(J_{ij}(t) e_j, e_{n-1}) = \sum_j J_{ij}(t) \eta_{jn-1} = -J_{in}(t), \end{aligned}$$

thus $J_{in}(t) = \delta_{in}$, which proves the second identity of (3.9). \square

As a consequence of (3.9) we have that

$$(3.11) \quad J(t)^{-1} \mathbf{f}_n = \mathbf{f}_n, \quad (J(t)^{-1})^T \mathbf{f}_{n-1} = \mathbf{f}_{n-1}, \quad J'(t) \mathbf{f}_n = 0, \quad \text{and} \quad J'(t)^T \mathbf{f}_{n-1} = 0.$$

3.2. Local expression of the rigged Volume. We now write in a convenient way the rigged volume measure induced by the rigged metric over H . As it involves the $(n-2)$ -dimensional Hausdorff measure on space-like cross-sections, we point out that on a space-like cross-section, the restriction of the Lorentzian metric and of the rigged metric associated to it coincide; thus also the Hausdorff measures coincide.

Proposition 3.6. *Let H be a null hypersurface and S be a space-like local cross-section. Assume that g, H , and S are of class C^2 . Consider L a C^1 null-geodesic vector field over H and Ω_S the open coordinate patch of H defined in (2.11). Denote by Ψ_L the flow map of L as in (2.9), by $\tilde{\text{Vol}}_L^S$ the rigged volume form as in Definition 2.9, and by $J = J_L$ the Jacobi field matrix as in (3.4). Then*

$$\tilde{\text{Vol}}_{L \lrcorner \Omega_S}^S = (\Psi_L)_\# (\det(J(z, t)) \mathcal{H}^{n-2} \llcorner_S \otimes \mathcal{L}^1) \llcorner_{\Omega_S},$$

where \mathcal{H}^{n-2} denotes the Hausdorff measure of dimension $n-2$ (over S) and \mathcal{L}^1 the one-dimensional Lebesgue measure.

In particular, for all $\varphi \in C_c^0(\Omega_S)$

$$(3.12) \quad \int_{\Omega_S} \varphi(z) \tilde{\text{Vol}}_L^S(dz) = \int_S \int_{\mathbb{R}} \varphi(\Psi_L(z, t)) \det(J(z, t)) dt \mathcal{H}^{n-2}(dz).$$

Proof. The integral w.r.t. the rigged volume can be computed in coordinates with the formula

$$(3.13) \quad \int_{\Omega_S} \varphi(z) \tilde{\text{Vol}}_L^S(dz) = \int_U \int_{\mathbb{R}} \varphi(\zeta(x, t)) D(x, t) dt dx,$$

where, using (3.7),

$$(3.14) \quad \begin{aligned} D(x, t) &:= \left(\det \left(\tilde{g}_L^S \left(\frac{\partial \zeta}{\partial x_i}(x, t), \frac{\partial \zeta}{\partial x_j}(x, t) \right)_{i,j=1, \dots, n-1} \right) \right)^{\frac{1}{2}} \\ &= \det \left(\left(\tilde{g}_L^S (J_{e_i(z)}(t), J_{e_j(z)}(t))_{i,j=1, \dots, n-1} \right) \right)^{\frac{1}{2}}. \end{aligned}$$

By Equation (2.7) and (3.8), we deduce that the matrix $\tilde{\eta}$ representing the rigged metric in the basis $(e_i)_{i=1,\dots,n-1}$ is of the form

$$(3.15) \quad \tilde{\eta} := \begin{bmatrix} \eta_{11} & \cdots & \eta_{1,n-2} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \eta_{n-2,1} & \cdots & \eta_{n-2,n-2} & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

In particular, from (3.8), it follows that $\det \tilde{\eta} = -\det \eta$. Denote by \tilde{J} the upper-left $(n-1) \times (n-1)$ minor of the matrix J . Taking into account (3.10), if $f(x) = z$, we deduce that

$$(3.16) \quad \begin{aligned} D(x, t) &= (\det(\tilde{J}^T(z, t)\tilde{\eta}(\Psi_L(z, t))\tilde{J}(z, t)))^{\frac{1}{2}} = \det \tilde{J}^T(z, t)(\det \tilde{\eta}(\Psi_L(z, t)))^{\frac{1}{2}} \\ &= \det J^T(z, t)(|\det \eta(\Psi_L(z, t))|)^{\frac{1}{2}} = \det J(z, t)(|\det \eta(\Psi_L(z, t))|)^{\frac{1}{2}}. \end{aligned}$$

Plugging the identity (3.16) into (3.13), yields

$$\begin{aligned} \int_{\Omega_S} \varphi(z) \tilde{\text{Vol}}_L^S(dz) &= \int_U \int_{\mathbb{R}} \varphi(\zeta(x, t)) D(x, t) dt dx \\ &= \int_U \int_{\mathbb{R}} \varphi(\Psi_L(f(x), t)) \det J(f(x), t) (|\det \eta(f(x))|)^{\frac{1}{2}} dt dx. \end{aligned}$$

The thesis follows immediately by observing that

$$\int_S \psi(z) \mathcal{H}^{n-2}(dz) = \int_U \psi(f(x)) (\det \tilde{\eta}(f(x)))^{\frac{1}{2}} dt dx, \quad \forall \psi \in C_c^0(S),$$

which is the integration formula in the parametrization given by f for the volume on the submanifold S . \square

We now point out that if φ is a transverse function (recall Definition 2.5), then it is not hard to check that

$$(3.17) \quad \det J_{\varphi L}(z, t) = \det J_L(z, \varphi(z)t).$$

Combining then the representation formula (3.12) with (3.17) it is immediate to establish the following:

Corollary 3.7. *Let L be a local null-geodesic vector field of class C^1 , S a local space-like cross-section for H and φ a C^1 transverse function. Assume also that g, S and H are of class C^2 . Then it holds that $\tilde{\text{Vol}}_{\varphi L}^S = \frac{1}{\varphi} \tilde{\text{Vol}}_L^S$.*

3.3. Independence on local cross-sections. By its very definition (see Definition 2.9), the rigged volume measure $\tilde{\text{Vol}}_L^S$ on H depends both on L and on the chosen cross-section S . We will now prove that it actually does not depend on S . Subsequently, we will show that its dependency on L is controlled. As a consequence, $\tilde{\text{Vol}}_L^S$ will be globally well-defined on H .

We start by establishing a representation formula for the Hausdorff measure of cross-sections that are graphs over other cross-sections.

Proposition 3.8. *Let H be a null hypersurface and let S be a local space-like cross-section. Assume also that g, S and H are of class C^2 . Let $\bar{S} \subset \Omega_S$ be another local space-like cross-section of class C^2 . Assume that there exists a C^2 function $t_L : \text{Dom}(t_L) \subset S \rightarrow \mathbb{R}$, such that $\Psi_L(z, t_L(z)) \in \bar{S}$ and that $z \mapsto \Psi_L(z, t_L(z))$ is a surjection onto \bar{S} . Then, for every $\varphi \in C_c^0(\bar{S})$, the following identity holds :*

$$(3.18) \quad \int_{\bar{S}} \varphi(z) \mathcal{H}^{n-2}(dz) = \int_{\text{Dom}(t_L)} \varphi(\Psi_L(z, t_L(z))) \det J(z, t_L(z)) \mathcal{H}^{n-2}(dz).$$

Proof. Consider the local parametrization of H induced by $f : U \subset \mathbb{R}^{n-2} \rightarrow S$ (i.e., $\zeta(x, t) = \Psi_L(f(x), t)$). Denote by $s(x) := t_L \circ f$ and notice that $\bar{f}(x) := \zeta(x, s(x))$ induces a parametrization for \bar{S} . Then

$$(3.19) \quad \int_{\bar{S}} \varphi(z) \mathcal{H}^{n-2}(dz) = \int_{\text{Dom } \bar{f}} \varphi(\bar{f}(x)) D(x) dx,$$

where $D(x)$ is the determinant of the matrix

$$(3.20) \quad \begin{aligned} A_{ij} &:= g \left(\frac{\partial \bar{f}}{\partial x_i}(x), \frac{\partial \bar{f}}{\partial x_j}(x) \right) \\ &= g \left(\frac{\partial \zeta}{\partial x_i}(x, s(x)) + \frac{\partial \zeta}{\partial t}(x, s(x)) \frac{\partial s}{\partial x_i}(x), \frac{\partial \zeta}{\partial x_j}(x, s(x)) + \frac{\partial \zeta}{\partial t}(x, s(x)) \frac{\partial s}{\partial x_j}(x) \right) \\ &= g \left(J_{\frac{\partial f}{\partial x_i}}(x)(s(x)) + \frac{\partial s}{\partial x_i}(x)L, J_{\frac{\partial f}{\partial x_j}}(x)(s(x)) + \frac{\partial s}{\partial x_j}(x)L \right) \\ &= g \left(J_{\frac{\partial f}{\partial x_i}}(x)(s(x)), J_{\frac{\partial f}{\partial x_j}}(x)(s(x)) \right), \quad \forall i, j = 1, \dots, n-2. \end{aligned}$$

In the last identity, we have tacitly used the same notation for the metric g restricted to $T\bar{S}$ and to TH ; notice indeed that the vectors $J_{\partial f / \partial x_i}(x)(s(x))$ are no longer tangent to \bar{S} . Expanding the identity and setting $z = f(x)$, we obtain:

$$\begin{aligned} A_{ij} &= g \left(\sum_{h \leq n-1} J_{i,h}(z, s(x)) e_h, \sum_{k \leq n-1} J_{j,k}(z, s(x)) e_k \right) \\ &= \sum_{h, k \leq n-1} J_{i,h}(z, s(x)) \eta_{h,k}(\bar{f}(x)) J_{j,k}(z, s(x)). \end{aligned}$$

Hence A is the upper left $(n-2) \times (n-2)$ minor of $J(z, s(x)) \eta(\bar{f}(x)) J^T(z, s(x))$. As L is a null vector for g , one can directly check that $A = \bar{J}(s(x)) \bar{\eta}(\bar{f}(x)) \bar{J}^T(z, s(x))$, where \bar{J} and $\bar{\eta}$ denotes the the upper left $(n-2) \times (n-2)$ minors of J and η , respectively. Therefore $D(x) = \det J(f(x), s(x)) (|\det \eta(\bar{f}(x))|)^{\frac{1}{2}}$. Plugging this identity in (3.19), we conclude that

$$\begin{aligned} \int_{\bar{S}} \varphi(z) \mathcal{H}^{n-2}(dz) &= \int_{\text{Dom } \bar{f}} \varphi(\bar{f}(x)) D(x) dx \\ &= \int_{\text{Dom } \bar{f}} \varphi(\Psi_L(f(x), t_L(f(x)))) \det J(f(x), t_L(f(x))) (|\det \eta(\bar{f}(x))|)^{\frac{1}{2}} dx \\ &= \int_{\text{Dom } t_L} \varphi(\Psi_L(z, t_L(z))) \det J(z, t_L(z)) \mathcal{H}^{n-2}(dz), \end{aligned}$$

having used that $\det \eta(\bar{f}(x)) = \det \eta(f(x)) = \det \bar{\eta}(f(x))$, which follows by parallel transport. \square

Next, we show that the rigged volume induced by two different cross-sections coincides on the intersection of their domain of definition (recall Section 2.3). To prove this claim, we will tacitly use the standard fact that the determinant of the matrix representing the Jacobi field $\det(J(z, t))$ used to represent the rigged volume measure does not depend on the choice of the basis $(e_i)_i$. We will also use the following semi-group property of the determinant of the matrix given by the linearity of the Jacobi equation

$$(3.21) \quad \det(J(\Psi_L(z, s), t)) \det(J(z, s)) = \det(J(z, t + s)),$$

for all $s \in \text{Dom}(J(z, \cdot))$ and $t \in \text{Dom}(J(\Psi_L(z, s), \cdot))$.

Proposition 3.9 (Independence of S). *Let H a null hypersurface, let L be a null geodesic vector field and let $S_1, S_2 \subset H$ be two space-like cross-sections. Assume g, H, S_1, S_2 are of class C^2 , and L is of class C^1 . Then it holds that*

$$\tilde{\text{Vol}}_L^{S_1} \llcorner_{\Omega_{S_1} \cap \Omega_{S_2}} = \tilde{\text{Vol}}_L^{S_2} \llcorner_{\Omega_{S_1} \cap \Omega_{S_2}}$$

Proof. Assume first that $S_2 \subset \Omega_{S_1}$, and that S_2 can be parameterized by a certain function t_L , i.e., $\Psi_L(z, t_L(z)) \in S_2$, as in the hypothesis of Proposition 3.8. Fix $\varphi \in C_c^0(\Omega_{S_1} \cap \Omega_{S_2})$. A direct computation gives

$$\begin{aligned} & \int_{\Omega_{S_1} \cap \Omega_{S_2}} \varphi \, d\tilde{\text{Vol}}_L^{S_2} \\ & \stackrel{(3.12)}{=} \int_{S_2} \int_{\mathbb{R}} \varphi(\Psi_L(z, t)) \det(J(z, t)) \, dt \, \mathcal{H}^{n-2}(dz) \\ & \stackrel{(3.18)}{=} \int_{S_1} \int_{\mathbb{R}} \varphi(\Psi_L(\Psi_L(z, t_L(z)), t)) \det(J(\Psi_L(z, t_L(z)), t)) \, dt \det(J(z, t_L(z))) \, \mathcal{H}^{n-2}(dz) \\ & = \int_{S_1} \int_{\mathbb{R}} \varphi(\Psi_L(z, t_L(z) + t)) \det(J(z, t + t_L(z))) \, dt \, \mathcal{H}^{n-2}(dz) \\ & = \int_{S_1} \int_{\mathbb{R}} \varphi(\Psi_L(z, t)) \det(J(z, t)) \, dt \, \mathcal{H}^{n-2}(dz) \\ & \stackrel{(3.12)}{=} \int_{\Omega_{S_1} \cap \Omega_{S_2}} \varphi \, d\tilde{\text{Vol}}_L^{S_1}. \end{aligned}$$

We now consider the general case.

By the following Lemma 3.10, it is sufficient to show that, for every $z \in \Omega_{S_1} \cap \Omega_{S_2}$ there exists a neighborhood U of z such that the two measures coincide therein. Fix $z \in \Omega_{S_1} \cap \Omega_{S_2}$ and let S_3 be a space-like local cross-section containing z . Up to taking a smaller cross-section, we can assume that S_3 is parameterized by $t_L^1 : \text{Dom}(t_L^1) \subset S_1 \rightarrow \mathbb{R}$, w.r.t. the cross-section S_1 . Analogously, we can also assume that there exists $t_L^2 : \text{Dom}(t_L^2) \subset S_2 \rightarrow \mathbb{R}$ a parametrization of S_3 w.r.t. S_2 . The cross-section S_3 produces the rigged volume $\tilde{\text{Vol}}_L^{S_3} \in \mathcal{M}^+(\Omega_{S_3})$. We choose as a neighborhood of z the open set $U = \Omega_{S_1} \cap \Omega_{S_2} \cap \Omega_{S_3}$. The fact that the two measures coincide on U follows from the previous part. \square

The following well-known gluing lemma can be proved by a standard argument via continuous partition of unity (see for instance [39, Th. 2.13]).

Lemma 3.10. *Let X be a locally-compact Hausdorff space and let $(U_i)_{i \in I}$ be an open cover. Let $\mu_i \in \mathcal{M}^+(U_i)$ be a family of locally finite positive measures. Assume that*

$$\mu_i \llcorner_{U_i \cap U_j} = \mu_j \llcorner_{U_i \cap U_j}, \quad \text{for all } i, j \in I.$$

Then there exists a positive, locally-finite measure $\mu \in \mathcal{M}^+(X)$, such that

$$\mu \llcorner_{U_i} = \mu_i, \quad \text{for all } i \in I.$$

As highlighted by Corollary 3.11 below, Proposition 3.9 implies in particular that the rigged volume measure depends only on the null-geodesic vector field, and not on the cross-section S .

Corollary 3.11. *Let g be a Lorentzian metric of class C^2 on M . Given a null hypersurface H of class C^2 and a global null-geodesic vector field L of class C^1 , there exists a unique measure Vol_L , such that for any space-like local cross-section S of class C^2 , it holds that $\tilde{\text{Vol}}_L^S = \text{Vol}_L \llcorner_{\Omega_S}$.*

We point out that the measure Vol_L is mutually absolutely continuous w.r.t. the volume measure induced on H by any auxiliary Riemannian metric. In the sequel, we will say that a measure $\mu \in \mathcal{M}^+(H)$ is absolutely continuous w.r.t. H , if it is absolutely continuous w.r.t.

the volume measure on H induced by some (hence any) metric. Accordingly, the family of probability measures absolutely continuous w.r.t. H will be denoted by $\mathcal{P}_{ac}(H)$.

4. VOLUME DISTORTION OF THE (WEIGHTED) RIGGED MEASURE

Throughout the section, we will assume the following regularity properties: g is a C^2 Lorentzian metric on M , S is a C^2 space-like cross-section for the C^2 null hypersurface H , and L is a C^1 null-geodesic vector field.

As we will have to deal with weights for the volume measures, it will be convenient to adopt the following notation.

Definition 4.1. Let (M, g) be a Lorentzian manifold and let $H \subset M$ be a null hypersurface. Let L be a null-geodesic vector field. We define the function $W_L : \text{Dom}(W_L) \subset H \times \mathbb{R} \rightarrow \mathbb{R}$ as

$$(4.1) \quad W_L(z, t) := \log(\det J(z, t)).$$

The domain of definition of W_L is the maximal set where the definition makes sense, i.e., where $\Psi_L(z, t) \in H$ and where the determinant is not null; in principle, this set could be larger than Ω_S , the set where the rigged volume is defined. We next show that

$$\text{Dom}(W_L) = \{(z, t) : \det J(z, t) > 0\} = \text{Dom}(\Psi_L).$$

Lemma 4.2. *If we define I_z to be the connected component of $(\Psi_L(z, \cdot))^{-1}(H)$ containing 0, then $\det(J(z, t)) > 0$ in the interval I_z .*

Proof. Assume on the contrary that $\det J(z, t) = 0$ for some t . Without loss of generality, we can assume $t > 0$. Let $\bar{t} := \inf\{t > 0 : \det J(z, t) = 0\} > 0$ and let $\bar{z} = \Psi_L(z, \bar{t})$. By continuity, there exists $\varepsilon > 0$, such that $\det J(\bar{z}, t) > 0$ for all $t \in (-2\varepsilon, 2\varepsilon)$. An iterated use of the semi-group property of the determinant gives a contradiction

$$\begin{aligned} \det J(z, \bar{t}) &= \det J(z, \bar{t} - \varepsilon) \det J(\Psi_L(z, \bar{t} - \varepsilon), \varepsilon) = \det J(z, \bar{t} - \varepsilon) \det J(\Psi_L(\bar{z}, -\varepsilon), \varepsilon) \\ &= \det J(z, \bar{t} - \varepsilon) (\det J(\bar{z}, -\varepsilon))^{-1}. \end{aligned} \quad \square$$

Capitalising on Section 3.1, we show that W_L satisfies a Riccati-type inequality.

Proposition 4.3. *The function $W_L(z, \cdot)$ is twice differentiable and it satisfies the following differential inequality*

$$(4.2) \quad W_L''(z, t) + \frac{(W_L'(z, t))^2}{n-2} \leq -\text{Ric}_{\Psi_L(z, t)}(L, L).$$

Proof. First of all, since $t \mapsto J(z, t)$ is of class C^2 , then also the map $t \mapsto W_L(z, t)$ is so.

Differentiate W_L once and obtain

$$W_L'(z, t) := \text{tr}(J(t)^{-1} J'(t)).$$

Define the matrix

$$U(t) := J(t)^{-1} J'(t).$$

Differentiating U , we obtain

$$(4.3) \quad U'(t) = -J(t)^{-1} J'(t) J(t)^{-1} J'(t) + J(t)^{-1} J''(t) = -U(t)^2 - R(t).$$

We now claim that $\text{tr}(U^2) \geq \frac{(\text{tr } U)^2}{n-2}$. If this is the case, one can differentiate W_L twice, obtaining

$$\begin{aligned} W_L''(t) &= \text{tr}(U'(t)) = -[\text{tr}(U(t)^2) + \text{tr}(R(t))] \leq -\frac{(\text{tr } U(t))^2}{n-2} - \text{Ric}_{\Psi_L(z, t)}(L, L) \\ &= -\frac{W_L'(t)^2}{n-2} - \text{Ric}_{\Psi_L(z, t)}(L, L). \end{aligned}$$

At this point we prove the claim. First notice that Equation (3.11) yields

$$(4.4) \quad U(t)\mathbf{f}_n = 0 \quad \text{and} \quad U(t)^T\mathbf{f}_{n-1} = 0,$$

and the same holds true for the matrix $U(t)^2$. We deduce that the matrix U is of the form

$$(4.5) \quad U(t) := \begin{bmatrix} U_{11} & \cdots & U_{1,n-2} & U_{1,n-1} & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ U_{n-2,1} & \cdots & U_{n-2,n-2} & U_{n-2,n-1} & 0 \\ 0 & \cdots & 0 & 0 & 0 \\ U_{n,1} & \cdots & U_{n,n-2} & U_{n,n-1} & 0 \end{bmatrix},$$

and that $\text{tr } U(t) = \text{tr } \bar{U}(t)$, where $\bar{U}(t)$ is the upper-left $(n-2) \times (n-2)$ -minor of the matrix $U(t)$. Fix $i, j = 1 \dots n-2$ and compute

$$(U(t)^2)_{i,j} = \sum_{k=1}^{n-2} U_{ik}(t)U_{kj}(t) + U_{i,n-1}(t)U_{n-1,j}(t) + U_{in}(t)U_{nj}(t) = \sum_{k=1}^{n-2} \bar{U}_{ik}(t)\bar{U}_{kj}(t) = (\bar{U}(t)^2)_{i,j},$$

that is, the upper-left $(n-2) \times (n-2)$ -minor of the matrix $U(t)^2$ is precisely $\bar{U}(t)^2$. The identity (4.3) implies that \bar{U} solves the following ODE (here \bar{R} is defined accordingly)

$$(4.6) \quad \bar{U}'(t) = -\bar{U}(t)^2 - \bar{R}(t).$$

Since the ODE (4.6) is solved also by \bar{U}^T and the initial datum $\bar{U}(0)$ is a symmetric matrix (recall that $U(0) = J'(0)$ and that $(J'(0)\eta)^T = J'(0)\eta$), then $\bar{U}(t)$ is symmetric for all t , by the uniqueness of the solution for the initial value problem for (4.6). In particular its eigenvalues are real. Therefore, one can apply Jensen inequality finding that $\text{tr}(\bar{U}(t)^2) \geq \frac{(\text{tr } \bar{U}(t))^2}{n-2}$. \square

4.1. Weighted rigged measure. As Ricci curvature is often used to describe the interplay between the volume measure Vol_g and the Riemmanina metric, in the case an additional weight is considered, the generalized N -Ricci tensor (also known as Baryk-Émery tensor) is the natural object to consider.

In the case of a null hypersurface H endowed with a null-geodesic vector field L , given a weight $\Phi : H \rightarrow \mathbb{R}$ of class C^2 , we consider the generalized N -Ricci tensor

$$(4.7) \quad \text{Ric}^{g,\Phi,N} = \begin{cases} \text{Ric} - \text{Hess}_g \Phi - \frac{1}{N-n} \nabla_g \Phi \otimes \nabla_g \Phi & \text{if } N > n, \\ \text{Ric} & \text{if } N = n \text{ and } d\Phi = 0 \\ -\infty & \text{otherwise;} \end{cases}$$

where N plays the role of upper bound on the dimension. We will consider the weighted rigged measure $\mathbf{m}_L := e^\Phi \text{Vol}_L$ on the set Ω_S . In particular, if S is a local cross-section and the flow map Ψ_L is a diffeomorphism on Ω_S , then

$$(4.8) \quad \int_{\Omega_S} \varphi(z) \mathbf{m}_L(dz) = \int_S \int_{\mathbb{R}} \varphi(\Psi_L(z, t)) e^{\Phi(\Psi_L(z, t)) + W_L(z, t)} dt \mathcal{H}^{n-2}(dz), \quad \forall \varphi \in C_c^0(\Omega_S).$$

We now complement Proposition 4.3 in the weighted case.

Proposition 4.4. *Let L be a C^1 null-geodesic vector field and S be a C^2 space-like cross section. Then, for every $z \in S$, the function $a(t) := \Phi(\Psi_L(z, t)) + W_L(z, t)$ is twice differentiable and it satisfies the following differential inequality*

$$(4.9) \quad a''(t) + \frac{(a'(t))^2}{N-2} \leq -\text{Ric}_{\Psi_L(z, t)}^{g,\Phi,N}(L, L).$$

Proof. By differentiating $a(\cdot)$, we obtain

$$(4.10) \quad a'(t) = W'_L(t) + g(L, \nabla \Phi)_{\Psi_L(z, t)} \quad \text{and} \quad a''(t) = W''_L(t) + \text{Hess}(\Phi)_{\Psi_L(z, t)}[L, L].$$

A direct computation gives

$$\begin{aligned}
a''(t) &= W_L''(t) + \text{Hess}(\Phi)_{\Psi_L(z,t)}[L, L] \leq -\frac{(W_L'(t))^2}{n-2} - \text{Ric}_{\Psi_L(z,t)}(L, L) + \text{Hess}(\Phi)_{\Psi_L(z,t)}[L, L] \\
&= -\frac{(W_L'(t))^2}{n-2} - \text{Ric}_{\Psi_L(z,t)}^{g,\Phi,N}(L, L) - \frac{(g(\nabla\Phi, L)_{\Psi_L(z,t)})^2}{N-n} \\
&\leq -\frac{(W_L'(t) + g(\nabla\Phi, L)_{\Psi_L(z,t)})^2}{N-2} - \text{Ric}_{\Psi_L(z,t)}^{g,\Phi,N}(L, L) = -\frac{(a'(t))^2}{N-2} - \text{Ric}_{\Psi_L(z,t)}^{g,\Phi,N}(L, L),
\end{aligned}$$

having taken into account Proposition 4.3 and Lemma 4.5. \square

Lemma 4.5. *For all $\alpha, \beta > 0$, and $x, y \in \mathbb{R}$, it holds that*

$$(4.11) \quad \frac{x^2}{\alpha} + \frac{y^2}{\beta} \geq \frac{(x+y)^2}{\alpha+\beta}.$$

Proof. The claimed bound (4.11) is equivalent to the trivial inequality $(\beta x - \alpha y)^2 \geq 0$. \square

5. RIGGED MEASURE AND RICCI CURVATURE: THE $\text{NC}^1(N)$ CONDITION

In order to analyse the interplay between the rigged measure and Ricci curvature, it is convenient to partition a null hypersurface into null geodesics also known as the generators of the null hypersurface. Such a partition will be phrased in a terminology borrowed from optimal transport.

Definition 5.1. Fix $H \subset M$ a null hypersurface in M , of class C^1 . Define the relation

(5.1)

$$\Gamma := \{(x, y) \in H \times H : y = \exp_x(v), v \in \mathcal{N}_x H \text{ future-directed}, \forall s \in [0, 1] \exp_x(sv) \in H\}.$$

Proposition 5.2. *Let (M, g) be a Lorentzian manifold, H a null hypersurface and Γ as above. Assume $g \in C_{loc}^{1,1}$ and $H \in C^1$. Then the set Γ is a preorder on H , i.e., it is reflexive and transitive.*

Proof. The fact that Γ is reflexive is trivial. We next show the transitivity. Let $x_1, x_2, x_3 \in H$, such that $(x_1, x_2), (x_2, x_3) \in \Gamma$. Let $v_i \in \mathcal{N}_{x_i} H$, such that $x_{i+1} = \exp_{x_i}(v_i)$, $i = 1, 2$. Let $w \in \mathcal{N}_{x_2} H$ be the parallel transport of v_1 along the geodesic $t \mapsto \exp_{x_1}(tv_1)$. Since $\dim \mathcal{N}H = 1$, then $v_2 = aw$, for some $a > 0$. Therefore $x_3 = \exp_{x_1}((1+a)v_1)$ and $\exp_{x_1}(t(1+a)v_1) \in H$, for all $t \in [0, 1]$. \square

We say that a null hypersurface $H \subset M$ is *causal* if it does not contain periodic causal curves.

Proposition 5.3. *Let (M, g) be a Lorentzian manifold, let H be a causal null hypersurface and Γ as above. Assume $g \in C_{loc}^{1,1}$ and $H \in C^1$. Then:*

- (1) *The relation Γ as in (5.1) is anti-symmetric, hence a partial order on H .*
- (2) *Let $(x, y) \in \Gamma$, and $v_1, v_2 \in \mathcal{N}_x H$ be such that*

$$y = \exp_x(v_1) \quad \text{and} \quad y = \exp_x(v_2).$$

Then $v_1 = v_2$. In particular for any $(x, y) \in \Gamma$ the choice of v in (5.1) is unique, and one can define the map $V : \Gamma \rightarrow \mathcal{N}H$ where $V(x, y)$ is the unique element in $\mathcal{N}_x H$, such that $\exp_x(V(x, y)) = y$. In other words, V is the left inverse of the map $(P_H \otimes \exp) : \mathcal{N}H \rightarrow H \times H$, $v \mapsto (P_H(v), \exp(v))$, where $P_H : \mathcal{N}H \rightarrow H$ is the projection on the base point. Finally, the map V is Borel.

Proof. First we prove that Γ is anti-symmetric. Assume on the contrary that $(x, y), (y, x) \in \Gamma$, but $x \neq y$. Then $y = \exp_x(v_1)$ and $x = \exp_y(v_2)$. Let $w \in \mathcal{N}_y H$ be the parallel transport of v_1

along $t \mapsto \exp_x(tv_1)$; of course $v_2 = aw$, for some $a > 0$. We deduce that the causal geodesic $t \mapsto \exp_x(tv_1)$ is periodic of period $1 + a$ and is contained in H , contradicting that H is causal.

Concerning the second part, if $v_1 \neq v_2$, then it is clear that $v_1 = bv_2$, for some $b \neq 1$. Up to swapping v_1 and v_2 , we can assume that $b > 1$. The curve $t \mapsto \exp(tv_2)$ intersects y for $t = 1$ and $t = b$; therefore, the restriction $(1, b) \ni t \mapsto \exp(tv_2)$ defines a causal periodic curve in H , contradicting that H is causal.

Finally, we prove that V is Borel. Let $G := \text{Graph}(P_H \otimes \exp)$ be the graph of the map $v \in \mathcal{N}H \mapsto (P_H(v), \exp(v))$; G is closed, for it is a graph of a continuous function. Moreover, since it is a closed subset of a manifold, it is also σ -compact. By swapping the entries of G and intersecting with the set $\Gamma \times \mathcal{N}H$, we obtain the graph of V that therefore is a σ -compact set. Since a map with a σ -compact graph is F_σ -measurable, then V is F_σ -measurable, thus Borel measurable. \square

Definition 5.4 (Transport Relation). Define $\mathcal{R} = \Gamma \cup \Gamma^{-1}$ (here Γ^{-1} is given by swapping the two entries of all the couples in Γ). The relation \mathcal{R} can also be characterized as

$$(5.2) \quad \mathcal{R} = \{(x, y) \in H \times H : \exists v \in \mathcal{N}_x H : y = \exp_x(v), \text{ and } \forall s \in [0, 1] : \exp_x(sv) \in H\}.$$

Proposition 5.5. *Let (M, g) be a Lorentzian manifold and H a causal null hypersurface. Assume $g \in C_{loc}^{1,1}$ and $H \in C^1$. Then the set \mathcal{R} is an equivalence relation on H .*

Proof. Thanks to Proposition 5.2, the only non-trivial part is the transitivity. Let $x_1, x_2, x_3 \in H$ be such that

$$(x_1, x_2), (x_2, x_3) \in \mathcal{R}.$$

Let $v_i \in \mathcal{N}_{x_i} H$ be such that $x_{i+1} = \exp_{x_i}(v_i)$, $i = 1, 2$. Let $w \in \mathcal{N}_{x_2} H$ be the parallel transport of v_1 along $t \mapsto \exp_{x_1}(tv_1)$. Since $\dim \mathcal{N}H = 1$, then $v_2 = aw$, for some $a \neq 0$. Therefore $x_3 = \exp_{x_1}((1+a)v_1)$ and $\exp_{x_1}(t(1+a)v_1) \in H$, for $t \in [0, 1]$. \square

Accordingly, if $\alpha \in H$, we will denote by H_α the equivalence class of \mathcal{R} containing α . Each equivalence class is the image of a maximal (in H) light-like geodesic. The sets H_α are usually called the *generators* of H . The set parametrizing the family of equivalence classes will be denoted by Q , i.e. $Q = H/\mathcal{R}$, and the partition will be written as $\{H_\alpha\}_{\alpha \in Q}$.

Remark 5.6. Notice that the C^1 -regularity of H and $C_{loc}^{1,1}$ regularity of g are key to have that \mathcal{R} is an equivalence relation on all H . For instance, in the light cone in Minkowski, \mathcal{R} would fail transitivity, as two different causal geodesics can intersect at the tip of the cone, in case it is the initial point of both. The setting of lower regularity will be investigated in a forth-coming work [6].

Proposition 5.7. *Let (M, g) be a Lorentzian manifold and $H \subset M$ be a causal null hypersurface. Assume $g \in C_{loc}^{1,1}$ and $H \in C^1$. Then, for all $(x, y) \in \mathcal{R}$ it holds that*

$$(5.3) \quad (x, y) \in \Gamma \iff x \leq y.$$

Proof. The “ \implies implication” is trivial: given $(x, y) \in \Gamma$, let $v \in \mathcal{N}_x H$ be as in (5.1); then the causal curve $s \mapsto \exp_x(sv)$ connects x to y .

Regarding the “ \impliedby implication”, assume that $x \leq y$, but $(x, y) \notin \Gamma$. Then $x \neq y$ and, since $(x, y) \in \mathcal{R}$, $(y, x) \in \Gamma$. Therefore, by the “ \implies implication”, $y \leq x$ hence, by the causality of H , $x = y$, which is a contradiction. \square

We next introduce the following notation: if H is a causal null hypersurface and S a local space-like cross-section, then

$$H_S := \bigcup_{\alpha \in S} H_\alpha,$$

i.e. H_S is the set of all points y such that $(x, y) \in \mathcal{R}$, for some $x \in S$. We next show that H_S is an open subset of H .

Proposition 5.8. *Let (M, g) be a Lorentzian manifold, let H be a casual null hypersurface and let S be a local space-like cross-section for H . Assume $g \in C_{loc}^{1,1}$ and $H, S \in C^1$. Then H_S is an open subset of H .*

Proof. It is sufficient to prove that for all $z \in S$, there exists a local cross-section $S' \subset S$ still containing z , such that $H_{S'}$ is open in H . Fix $z \in S$ and find S' , a coordinated open set (inside S), whose coordinates are given by $f : U' \subset \mathbb{R}^{n-2} \rightarrow S'$. Take $L \in \mathfrak{X}(\mathcal{N}H|_{S'})$ of class C^1 . We can construct the standard coordinates system by taking $\zeta(x, t) := \Psi_L(f(x), t) = \exp_{f(x)}(tL)$, as described in Section 2.3. Let $U \subset \mathbb{R}^{n-1}$ be the maximal definition domain of ζ ; using that H is causal, one can prove that $U \subset \mathbb{R}^{n-1}$ is open and $\zeta(U) = H_{S'}$.

If ζ is a local diffeomorphism, then its image is open and we conclude. The fact that ζ is a local diffeomorphism is a consequence of Lemma 4.2. \square

Motivated by Proposition 5.8, we give the following:

Definition 5.9. Let (M, g) be a Lorentzian manifold, let H be a causal null hypersurface and let $(S_k)_k$ be a sequence of local space-like acausal cross sections for H . We say that S_k is a *dense sequence of local space-like acausal cross sections for H* if $H = \bigcup_k H_{S_k}$.

Notice that if H admits a global space-like and acausal cross section S , then S is dense in the sense of definition Definition 5.9.

Remark 5.10 (Global definition of \mathbf{m}_L and L). If S is acausal inside H , i.e. any causal curve in H intersects S at most once, as pointed out in Remark 2.8, one can produce a global null-geodesic vector field L on H_S , by defining it on S and then extending it by parallel transport. The null hypersurface H_S can be globally parameterized using the map Ψ_L taking $\alpha \in S$ and $t \in \mathbb{R}$ as parameters, implying that S is a global cross-section for H_S . Below, we use results from the previous sections; it is then convenient to assume g, H, S of class C^2 . Lemma 4.2 guarantees moreover that W_L as in (4.1) is defined on the whole domain of definition of Ψ_L . Hence we can choose H_S to play the role of Ω_S . We recall that the open set Ω_S , introduced in Section 2.3, is a neighborhood of S in H , where the parametrization given by Ψ_L is a diffeomorphism.

Therefore, the representation formula for the rigged volume holds true for the whole set H_S :

$$(5.4) \quad \int_{H_S} \varphi(z) \text{Vol}_L(dz) = \int_S \int_{\mathbb{R}} \varphi(\Psi_L(z, t)) e^{W_L(z, t)} dt \mathcal{H}^{n-2}(dz), \quad \forall \varphi \in C_c^0(H_S).$$

Combining this observation with Definition 5.9 we deduce that: if H is causal and $(S_k)_k$ is a dense sequence of local space-like and acausal cross-sections, we obtained the following objects:

$$(5.5) \quad H = \bigcup_k H_{S_k}, \quad H_{S_k} \subset H \text{ open}, \quad L_k \in \mathfrak{X}^1(\mathcal{N}H_{S_k}) \text{ null-geodesic vector field.}$$

such that $\nabla_{L_k} L_k = 0$ and $L_k \neq 0$ over H_{S_k} . Hence the measures $\text{Vol}_{L_k}, \mathbf{m}_{L_k} \in \mathcal{M}_+(H_{S_k})$ (not depending on the choice of S_k by Corollary 3.11) are well defined, they can be represented by (4.8) and verify the conclusion of Proposition 4.4.

By considering the (saturated) set $B_k := H_{S_k} \setminus \bigcup_{h=1}^{k-1} H_{S_h}$ one can then define

$$(5.6) \quad L := \sum_k \mathbf{1}_{B_k} L_k, \quad \text{Vol}_L := \sum_k \text{Vol}_{L_k \lrcorner B_k}.$$

The previous set of assumptions on H yields that L is a global null-geodesic vector field of Borel regularity with a well-defined associated flow map Ψ_L ; finally $\text{Vol}_L \in \mathcal{M}_+(H)$. (retaining (4.8) and the conclusion of Proposition 4.4).

Accordingly, one defines the weighted measure \mathbf{m}_L . We summarize this properties in the following

Theorem 5.11. *Let (M, g) be a Lorentzian manifold, let H be a causal null-hypersurface and $(S_k)_k$ be a dense sequence of local space-like and acausal cross-sections for H . Assume g, H, S_k to be of class C^2 . Let also L be a null-geodesic vector field of class C^1 .*

Then the following representation formula

$$\text{Vol}_L = \sum_k \int_{B_k \cap S_k} (\Psi_L(z, \cdot))_{\#} (e^{W_L(z,t)} dt) \mathcal{H}^{n-2}(dz),$$

is well-posed and defines a non-negative Radon measure over H (see Section 2.3 for the definition of Ψ_L and (4.1) for the definition of W_L). Moreover if $\Phi : H \rightarrow \mathbb{R}$ is a C^2 -function, defining for each $z \in S_k$ the map $a_z(t) := \Phi(\Psi_L(z, t)) + W_L(z, t)$, it holds that

$$a_z''(t) + \frac{(a_z'(t))^2}{N-2} \leq -\text{Ric}_{\Psi_L(z,t)}^{g,\Phi,N}(L, L).$$

Theorem 5.11 suggests the following two definitions, one for null hypersurfaces and one for manifolds.

Definition 5.12 ($\text{NC}^1(N)$ for a null hypersurface). Let (M, g) be a Lorentzian manifold, let $H \subset M$ be a causal null hypersurface, $(S_k)_k$ be a dense sequence of local space-like and acausal cross-sections for H and $\Phi : H \rightarrow \mathbb{R}$ be a C^0 function. Assume g, H, S_k to be of class C^2 .

We say that the quadruple (M, g, H, Φ) satisfies the null energy condition $\text{NC}^1(N)$ if and only if for all $z \in S_k$ the function $a_z(t) := \Phi(\Psi_L(z, t)) + W_L(z, t)$ is concave (thus, in particular, locally-Lipschitz) and it satisfies

$$(5.7) \quad a_z'' + \frac{(a_z')^2}{N-2} \leq 0, \quad \text{in the sense of distributions.}$$

Definition 5.13 ($\text{NC}^1(N)$ for a space-time). Let (M, g) be a Lorentzian manifold with $g \in C^2$ and let $\Phi : M \rightarrow \mathbb{R}$ be a C^0 function.

We say that the triple (M, g, Φ) satisfies the null energy condition $\text{NC}^1(N)$, if and only if, for any causal null hypersurface $H \subset M$ of class C^2 admitting a dense sequence of local space-like and acausal cross-sections of class C^2 , the quadruple (M, g, H, Φ) satisfies the null energy condition $\text{NC}^1(N)$.

Remark 5.14 (Independence on L of Definition 5.12). Definition 5.12 is well-posed. To verify this statement it is first necessary to show its independence on the choice of L . So given $M, H \subset M$ and $(S_k)_k$ be a dense sequence of local space-like and acausal cross-sections for H .

It is clear that is sufficient to argue for each single S_k and H_{S_k} . By Remark 2.6, any two null-geodesic vector fields L_1 and L_2 (firstly defined over S_k and the extended by parallel transport to the whole H_{S_k}) differ by a transverse function φ . Then the results in Section 2.3 give that

$$\Psi_{L_2}(z, t) = \Psi_{\varphi L_1}(z, t) = \Psi_{L_1}(z, \varphi(z)t).$$

Analogously, from (4.1) and (3.17):

$$W_{L_2}(z, t) = \log(\det(J_{\varphi L_1}(z, t))) = \log(\det(J_{L_1}(z, \varphi(z)t))) = W_{L_1}(z, \varphi(z)t).$$

Therefore, if $a_{L_i}(t) := \Phi(\Psi_{L_i}(z, t)) + W_{L_i}(z, t)$ for $i = 1, 2$, we conclude that

$$a_{L_2}'' + \frac{(a_{L_2}')^2}{N-2} = \varphi^2(z) \left(a_{L_1}'' + \frac{(a_{L_1}')^2}{N-2} \right),$$

proving the claim. To conclude, we recall that the independence of the construction on a particular choice of local cross-sections was already established in Proposition 3.9.

Remark 5.15. The global null-geodesic vector field L of Theorem 5.11, can be used as a ‘‘gauge’’ for the map V . This means that we can uniquely determine the function $t_L : \Gamma \rightarrow [0, \infty)$ such that

$$(5.8) \quad V(x, y) = t_L(x, y) L.$$

In other words, $\Psi_L(x, t_L(x, y)) = y$. If L is Borel regular, then t_L is Borel as well and, moreover, if φ is a C^1 transverse function, $t_{\varphi L} = \frac{1}{\varphi} t_L$. Finally, we extend t_L to Γ^{-1} , by setting $t_L(y, x) = -t_L(x, y)$.

6. OPTIMAL TRANSPORT INSIDE NULL HYPERSURFACES

In Section 5, we gave a synthetic characterization of the null energy condition (see Theorem 5.11 and Definitions 5.12–5.13), based on concavity properties of 1-dimensional densities along rays of the transport set. In the next Section 7, we will give another synthetic characterization of the null energy condition in terms of displacement convexity of the entropy. To this aim, in the present section, we establish some results about optimal transport inside null hypersurfaces.

For some basics on optimal transport in the Lorentzian setting, see, e.g., Eckstein–Miller [13], Suhr [43], McCann [31], and Cavalletti–Mondino [8].

Let us start with an example, which will serve as a motivation for the next definitions.

Example 6.1. Consider the two-dimensional Minkowsky space-time $M = \mathbb{R}^2$, whose coordinates x_1 and x_2 are the space and time coordinates respectively. We take as null hypersurface H the disjoint union of the lines $\{x_2 = x_1\}$ and $\{x_2 = x_1 + 2\}$. Let $\mu_0 = \delta_{(0,0)}$ and $\mu_1 = \delta_{(-1,1)}$ be the starting and ending measures, respectively. It is trivial to see that $\ell_p(\mu_0, \mu_1) = 0$; in particular, the measure μ_0 can be transported along causal curves to μ_1 . Nonetheless, μ_0 cannot be transported to μ_1 along curves inside H .

The example above suggests the following definition.

Definition 6.2. Let (M, g) be a Lorentzian manifold, with g of class C^0 . Let H be a C^1 null hypersurface and let $\mu_i \in \mathcal{P}(H)$, $i = 0, 1$ be two probability measures. We say that μ_0 is *null connected to μ_1 along H* if there exists a probability measure $\nu \in \mathcal{P}(C([0, 1]; H))$ such that

$$(6.1) \quad \pi := (e_0, e_1)_\# \nu \in \Pi_{\leq}(\mu_0, \mu_1), \quad \tau(x, y) = 0 \text{ for } \pi\text{-a.e. } (x, y), \quad \text{and } \nu\text{-a.e. } \gamma \text{ is causal.}$$

We also denote

$$(6.2) \quad \text{OptCaus}^H(\mu_0, \mu_1) := \{\nu \in \mathcal{P}(C([0, 1]; H)) \text{ satisfying (6.1)}\}.$$

By the very definition, it holds that $\pi := (e_0, e_1)_\# \nu$ is an optimal coupling from μ_0 to μ_1 for any Lorentz–Wasserstein distance ℓ_p , for all $p \leq 1$.

From (6.1), it follows that ν -a.e. γ is a null pre-geodesic (i.e. it can be re-parameterized into a null geodesic), provided that g is sufficiently regular, say $g \in C_{loc}^{1,1}$.

When dealing with the dynamical approach to optimal transport in the Riemannian setting or in the setting of purely time-like transport, it is well-known that optimal dynamical transport plans are concentrated on geodesics. In the case of transport with null cost this is not the case. For instance, as observed above, one can reparameterize the geodesics where a null optimal dynamical transport plan is concentrated, still finding an admissible optimal dynamical transport plan. We therefore introduce the following definition.

Definition 6.3. Let (M, g) be a Lorentzian manifold, with g of class $C_{loc}^{1,1}$. Let H be a C^1 null hypersurface. Let $\mu_0, \mu_1 \in \mathcal{P}(H)$ be two probability measures null connected along H . We say that a dynamical transport plan $\bar{\nu} \in \text{OptCaus}^H(\mu_0, \mu_1)$ is *null-geodesic* if it is concentrated on the set

$$D = \{\gamma \in C([0, 1]; H), \text{ such that } \exists V \in \mathcal{N}_{\gamma_0} H \text{ light-like and future-directed,} \\ \text{such that } \gamma_t = \exp(tV), t \in [0, 1]\}.$$

The set of null-geodesic dynamical transport plans from μ_0 to μ_1 is denoted as $\text{OptGeo}^H(\mu_0, \mu_1)$. In other terms,

$$(6.3) \quad \text{OptGeo}^H(\mu_0, \mu_1) := \{\nu \in \text{OptCaus}^H(\mu_0, \mu_1) : \nu\text{-a.e. } \gamma \text{ is a null geodesic}\}.$$

The next two lemmas recall some elementary properties of optimal transport on the real line, that will be useful in the rest of the work. We include the proofs for the reader's convenience.

Lemma 6.4. *Let $\mathcal{B} = \{(x, y) \in \mathbb{R}^2 : y \geq x\}$. Let $\mu_0, \mu_1 \in \mathcal{P}(\mathbb{R})$ be two probability measures. Assume that there exists a coupling $\pi \in \Pi(\mu_0, \mu_1)$, such that $\pi(\mathcal{B}) = 1$. Let ν be the monotone rearrangement plan of the two measures. Then $\nu(\mathcal{B}) = 1$.*

Moreover, if ν is induced by a map T , then $T(t) \geq t$, for all $t \in \text{supp } \mu_0$.

Proof. Consider first the case when μ_0 and μ_1 are of the form

$$(6.4) \quad \mu_0 = \frac{1}{n} \sum_{i=1}^n \delta_{x_i}, \quad \mu_1 = \frac{1}{n} \sum_{i=1}^n \delta_{y_i},$$

with $x_i \leq x_{i+1}$ and $y_i \leq y_{i+1}$. In this case

$$(6.5) \quad \pi = \sum_{i,j} A_{ij} \delta_{(x_i, y_j)},$$

where $(A_{ij})_{i,j}$ is a bistochastic matrix. We have then that $A = \sum_{\sigma \in \mathcal{S}_n} b_\sigma \sigma$, where $b_\sigma \geq 0$, $\sum_{\sigma \in \mathcal{S}_n} b_\sigma = 1$; the sums are taken over the set of rank- n permutations \mathcal{S}_n and a permutation is identified by its matricial representation. Let $\tau \in \mathcal{S}_n$ be such that $b_\tau > 0$. We claim that $y_{\tau(k)} \geq x_k$, for all $k = 1, \dots, n$; if on the contrary $y_{\tau(k)} < x_k$ for some k , then

$$\pi(\mathbb{R}^2 \setminus \mathcal{B}) = \sum_{i,j} A_{ij} \delta_{(x_i, y_j)}(\mathbb{R}^2 \setminus \mathcal{B}) = \sum_{\sigma \in \mathcal{S}_n} b_\sigma \sum_i \delta_{(x_i, y_{\sigma(i)})}(\mathbb{R}^2 \setminus \mathcal{B}) \geq b_\tau \delta_{(x_k, y_{\tau(k)})}(\mathbb{R}^2 \setminus \mathcal{B}) = b_\tau > 0,$$

a contradiction. We can therefore apply Lemma 6.5 below to τ , deducing that $y_k \geq x_k$, for all k . Let $\nu = \sum_{i=1}^n \frac{1}{n} \delta_{(x_i, y_i)}$; since ν is clearly the monotone rearrangement measure, we conclude.

We consider the general case by an approximation argument. Fix π as in the statement, and let $\pi_n \in \mathcal{P}(\mathcal{B})$ be a sequence such that $\pi_n \rightharpoonup \pi$ and $\pi = \frac{1}{k_n} \sum_{i=1}^{k_n} \delta_{(x_{n,i}, y_{n,i})}$; the fact that such approximation exists is standard and can be found in any textbook on Optimal Transport. Let $\mu_{n,i} := (P_i)_\# \pi_n$, $i = 1, 2$ be the marginal probability measures for π_n . We are in position to apply the previous part deducing that ν_n , the monotone rearrangement plan between $\mu_{n,0}$ and $\mu_{n,1}$, is concentrated on \mathcal{B} . We can take the limit, noting that (details can be found in any textbook on Optimal Transport): 1) $\mu_{n,i} \rightharpoonup \mu_i$; 2) $\nu_n \rightharpoonup \nu$, for some ν ; 3) ν is monotone, because monotonicity pass to the weak limit; 4) $\nu(\mathcal{B}) \geq \limsup_n \nu_n(\mathcal{B}) = 1$. \square

Lemma 6.5. *Let $n \in \mathbb{N}$. Let*

$$x_1 \leq \dots \leq x_i \leq x_{i+1} \leq \dots \leq x_n \quad \text{and} \quad y_1 \leq \dots \leq y_i \leq y_{i+1} \leq \dots \leq y_n$$

be two non-decreasing sequences in \mathbb{R} . Assume that there exists a permutation $\sigma \in \mathcal{S}_n$ such that $y_{\sigma(k)} \geq x_k$, for all $k = 1, \dots, n$. Then $y_k \geq x_k$, for all $k = 1, \dots, n$.

Proof. Assume on the contrary that $y_k < x_k$, for some k . For all $l > k$, it holds that

$$(6.6) \quad y_k < x_k \leq x_l \leq y_{\sigma(l)},$$

therefore $\sigma(l) > k$, for all $l > k$. This means that σ splits in two permutations, one of them permuting the elements $\{k+1, \dots, n\}$. On the other hand $y_k < x_k \leq y_{\sigma(k)}$, therefore $\sigma(k) > k$, which is a contradiction. \square

Lemma 6.6. *Let (M, g) be a Lorentzian manifold, with g of class $C_{loc}^{1,1}$. Let H be a C^1 null hypersurface and let $\nu \in \mathcal{P}(C([0, 1]; H))$ be a measure concentrated on the set of causal curves. Let C be the set*

$$(6.7) \quad C = \{\gamma \in C([0, 1]; H) : (\gamma_s, \gamma_t) \in \Gamma, \forall 0 \leq s \leq t \leq 1\},$$

where Γ was defined in (5.1). Then $\nu(C) = 1$.

Next we prove a useful decomposition of an optimal transport plan along a null hypersurface in a countable family of optimal transport plans, each of them along a null hypersurface having a global cross-section.

Proposition 6.7. *Let (M, g) be a Lorentzian manifold, with g of class $C_{loc}^{1,1}$. Let H be a C^1 a null hypersurface, such that there exists a dense sequence of local space-like and acausal C^1 cross-sections $(S_k)_k$, in the sense of Definition 5.9. Let $\mu_0, \mu_1 \in \mathcal{P}(H)$ be two probability measures that are null connected along H .*

Then, up to taking a not relabeled subsequence in $(S_k)_k$, there exists a (possibly finite) real sequence $(w_k)_k \subset (0, \infty)$ and two sequences of probability measures $(\mu_i^k)_k$, $i = 0, 1$, such that the following hold. For all $k \in \mathbb{N}$, the measure μ_0^k is null connected to μ_1^k along H_{S_k} . For $i \in [0, 1]$, the measures μ_i^k and μ_i^h are orthogonal and $\mu_i = \sum_k w_k \mu_i^k$.

Moreover, if $\nu \in \text{OptCaus}^H(\mu_0, \mu_1)$ (i.e., ν realizes Definition 6.2), then we can choose a measure $\nu_k \in \text{OptCaus}^{H_{S_k}}(\mu_0^k, \mu_1^k)$ so that $\nu = \sum_k w_k \nu_k$. Furthermore, having defined $\mu_t^k := (e_t)_\# \bar{\nu}^k$, $t \in [0, 1]$, it holds that the measures μ_t^k and μ_t^h are orthogonal.

Proof. Let $B_k := H_{S_k} \setminus \bigcup_{h=1}^{k-1} H_{S_h}$, and let C_k the set of causal curves contained in B_k . Since $H = \bigcup_k B_k$ and causal curves in H can only move inside the generators of H , then $(C_k)_k$ covers the set of causal curves in H ; in particular, $\nu(\bigcup_k C_k) = 1$. Moreover, as the sets $(B_k)_k$ are pairwise disjoint, also the sets $(C_k)_k$ are pairwise disjoint and therefore the sequence $w_k := \nu(C_k)$ sums to 1.

We extract (without relabeling) a subsequence, by dropping the elements such that $w_k = 0$. Let $\nu^k := \nu_{\setminus C_k} / w_k$, and let $\mu_t^k := (e_t)_\#(\bar{\nu}^k)$, $t \in [0, 1]$. The fact that μ_0^k is null connected to μ_1^k along H_{S_k} is trivial, since $\nu^k \in \text{OptGeo}_{\ell_p}^{H_{S_k}}(\mu_0^k, \mu_1^k)$. The orthogonality of μ_t^k and μ_t^h follows from the fact that ν_k gives full measure to C_k , thus μ_t^k gives full measure to B_k and the sets B_k are disjoint. \square

Proposition 6.7 guarantees that, in presence of a dense sequence of local space-like and acausal cross-sections, a light-like optimal transport plan on a null hypersurface can be decomposed into a family of optimal transport plans, each on a null hypersurface possessing a global, space-like, acausal cross-section. Conversely, it is trivial to see that, given a sequence of null-cost optimal transport plans on the same null hypersurface, we can sum them obtaining a null-cost optimal transport transport plan.

6.1. Existence and uniqueness of monotone plans along H . We will now single out a class of well-behaved optimal dynamical plans. We start with two technical facts.

Lemma 6.8. *Let (M, g) be a Lorentzian manifold, with g of class $C_{loc}^{1,1}$. Let H be a C^1 null hypersurface and let $\mu_0, \mu_1 \in \mathcal{P}(H)$ be two probability measures, such that μ_0 is null connected to μ_1 along H . Let $\nu \in \text{OptCaus}^H(\mu_0, \mu_1)$, and let $\mu_t := (e_t)_\# \nu$, for all $t \in [0, 1]$.*

If $\varphi : H \rightarrow \mathbb{R}$ is a transverse function, then it holds that

$$(6.8) \quad \int_H \varphi d\mu_t = \int_H \varphi d\mu_s, \quad \text{for all } t, s \in [0, 1].$$

Assume also that φ is non-negative and the integrals in (6.8) are equal to 1. Define $\tilde{\mu}_t := \varphi \mu_t$. Then $\tilde{\mu}_0$ is null connected to $\tilde{\mu}_1$ along H , and $\tilde{\nu} := (\varphi \circ e_0) \nu \in \text{OptCaus}^H(\tilde{\mu}_0, \tilde{\mu}_1)$ is a dynamical transport plan, such that $\tilde{\mu}_t = (e_t)_\# \tilde{\nu}$ for all $t \in [0, 1]$.

Proof. Let C be the set of causal curves in H . Since φ is transverse, then $\varphi \circ \gamma$ is constant for all $\gamma \in C$. Clearly $\nu(C) = 1$, thus $\varphi \circ e_t = \varphi \circ e_s$, ν -a.e.. By definition of μ_t and μ_s , we have that

$$(6.9) \quad \int_H \varphi d\mu_t - \int_H \varphi d\mu_s = \int_{C([0,1];H)} (\varphi \circ e_t - \varphi \circ e_s) d\nu = 0.$$

For the second part, a direct computation gives

$$(6.10) \quad \tilde{\mu}_t = \varphi\mu_t = \varphi(e_t)_\#(\nu) = (e_t)_\#(\varphi \circ e_t\nu) = (e_t)_\#(\varphi \circ e_0\nu) = (e_t)_\#(\tilde{\nu}). \quad \square$$

Before stating the next lemma, recall that a subset $B \subset \mathbb{R} \times \mathbb{R}$ is said to be *monotone*, provided the following holds: if $(s_1, t_1) \in B$ and $(s_2, t_2) \in B$ with $s_1 \leq s_2$, then $t_1 \leq t_2$.

Lemma 6.9. *Let (M, g) be a Lorentzian manifold, with g of class $C_{loc}^{1,1}$. Let H be a C^1 null hypersurface. Let $A \subset \Gamma$. The following are equivalent.*

- (1) For all $(x_1, y_1), (x_2, y_2) \in A$, if $(x_1, x_2) \in \Gamma$ and $x_1 \neq x_2$ then $(y_1, y_2) \in \Gamma$.
- (2) for all $\alpha \in Q$, there exists a monotone set $B_\alpha \subset \mathbb{R} \times \mathbb{R}$, such that

$$(6.11) \quad A \cap (H_\alpha \times H_\alpha) \subset \Psi_L(\alpha, \cdot) \otimes \Psi_L(\alpha, \cdot)(B_\alpha)$$

Proof. (1) \implies (2). Take as $B_\alpha := (\Psi_L(\alpha, \cdot) \otimes \Psi_L(\alpha, \cdot))^{-1}(A)$. The fact that B_α is monotone is trivial.

(2) \implies (1). Fix $(x_1, y_1), (x_2, y_2) \in A$. If $(x_1, x_2) \in \Gamma$, then all these four points belong to the same class H_α . Then $x_i = \Psi_L(\alpha, s_i)$ and $y_i = \Psi_L(\alpha, t_i)$, for some $(s_i, t_i) \in B_\alpha$, $i = 1, 2$. The assumption $(x_1, x_2) \in \Gamma$ gives $s_1 \leq s_2$ and the monotonicity of B_α gives $t_1 \leq t_2$, concluding the proof. \square

We now introduce the concept of monotone transport plan.

Definition 6.10. Let (M, g) be a Lorentzian manifold, with g of class $C_{loc}^{1,1}$. Let H be a C^1 null hypersurface. Let $\mu_0, \mu_1 \in \mathcal{P}(H)$ be two probability measures null connected along H . We say that an optimal coupling π between μ_0 and μ_1 is *monotone*, if:

- (1) $\pi = (e_0 \otimes e_1)_\#\nu$ for some $\nu \in \text{OptCaus}^H(\mu_0, \mu_1)$;
- (2) there exists a Borel set $A \subset \Gamma$ such that $\pi(A) = 1$ and

$$(6.12) \quad \forall (x_1, y_1), (x_2, y_2) \in A, \quad (x_1, x_2) \in \Gamma, x_1 \neq x_2 \implies (y_1, y_2) \in \Gamma.$$

The next proposition guarantees the uniqueness of monotone optimal couplings. Their existence will be proved in Theorem 6.12.

Proposition 6.11. *Let (M, g) be a Lorentzian manifold, with g of class $C_{loc}^{1,1}$. Let H be a C^1 causal null hypersurface such that there exists a countable dense family of local space-like and acausal cross-sections of H of class C^1 . Let $\mu_0, \mu_1 \in \mathcal{P}(H)$ be two probability measures, such that $\ell_p(\mu_0, \mu_1) = 0$, for some (and thus every) $p \in (0, 1]$.*

If a monotone optimal coupling between μ_0 and μ_1 exists, then it is unique.

Proof. Let π^1 and π^2 be two monotone optimal transport plans, induced by ν^1 and ν^2 , respectively, i.e., $\pi^k = (e_0 \otimes e_1)_\#\nu^k$, $k = 1, 2$.

We assume first that there exists S , a global space-like and acausal cross-section of H . Let L be a global null-geodesic vector field over H . Let $A^1, A^2 \subset \Gamma$, be two subsets given by the monotonicity of π^1 and π^2 respectively. Let $K := \bigcup_{\alpha \in S} H_\alpha \times H_\alpha$. Lemma 6.6 implies that $(e_0, e_1)(\gamma) \in K$ for ν^k -a.e. γ , $k = 1, 2$, therefore $\pi^1(K) = \pi^2(K) = 1$. Let $\mathfrak{Q} : H \rightarrow S$ be the map such that $\mathfrak{Q}(\Psi_L(\alpha, t)) = \alpha$, for all $(\alpha, t) \in \text{Dom } \Psi_L$. With a slight abuse of notation, we also denote by \mathfrak{Q} the map $\mathfrak{Q} : K \rightarrow S$ given by $\mathfrak{Q} \circ P_1$. It is easy to see that

$$(6.13) \quad \mathfrak{Q}_\#\pi^1 = \mathfrak{Q}_\#\pi^2 = \mathfrak{Q}_\#\mu^0 = \mathfrak{Q}_\#\mu^1 =: \mathfrak{q}.$$

We can therefore apply the disintegration theorem, finding that

$$(6.14) \quad \omega = \int_S \omega_\alpha \mathfrak{q}(d\alpha),$$

where ω can be either π^1, π^2, μ_0 , or μ_1 , and the measure ω_α is concentrated on $H_\alpha \times H_\alpha$ or H_α , accordingly. Notice that $\pi_\alpha^k(A^k) = 1$ for \mathfrak{q} -a.e. α . Let $\tilde{\pi}_\alpha^k := ((\Psi_L(\alpha, \cdot))^{-1} \otimes (\Psi_L(\alpha, \cdot))^{-1})_\#\pi^k$, and $\tilde{\mu}_{i,\alpha} := (\Psi_L(\alpha, \cdot))_\#\mu_{i,\alpha}$. It is then clear that $\tilde{\pi}_\alpha^k$ is a transport plan between the measures

$\tilde{\mu}_{0,\alpha}$ and $\tilde{\mu}_{1,\alpha}$. Lemma 6.9 implies that $\tilde{\pi}_\alpha^k$ is concentrated on some monotone set B_α^k , therefore both $\tilde{\pi}_\alpha^1$ and $\tilde{\pi}_\alpha^2$ are monotone rearrangements of $\tilde{\mu}_{0,\alpha}$ into $\mu_{1,\alpha}$, thus these two transport plans coincide. By integrating, we conclude that $\pi^1 = \pi^2$.

We now drop the assumption that a global acausal cross-section exists. In this case we use Proposition 6.7, obtaining the sequences $(w_j)_j$, μ_i^j , ν_1^j , and ν_2^j . In particular $\nu = \sum_j w_j \nu_k^j$, $k = 1, 2$. If we define $\pi_j^k := (e_0 \otimes e_1)_\# \nu_j^k$, $k = 1, 2$, then both π_j^1 and π_j^2 are monotone optimal transport plans. Applying the previous part, we get that $\pi_j^1 = \pi_j^2$, which implies $\pi^1 = \pi^2$. \square

We are now in position to prove the existence of a null-geodesic dynamical transport plan and a corresponding monotone optimal coupling between two probability measures null connected along H .

Theorem 6.12. *Let (M, g) be a Lorentzian manifold, with g of class $C_{loc}^{1,1}$. Let H be a C^1 causal, null hypersurface such that there exists a countable dense family of local space-like and acausal cross-sections of class C^1 . Let $\mu_0, \mu_1 \in \mathcal{P}_{ac}(H)$ be two probability measures, absolutely continuous w.r.t. H , such that μ_0 is null connected to μ_1 along H .*

Then there exists a unique null-geodesic dynamical transport plan $\nu \in \text{OptGeo}^H(\mu_0, \mu_1)$, such that the optimal coupling $\pi := (e_0 \otimes e_1)_\# \nu$ is monotone. Such a ν will be called monotone null-geodesic dynamical transport plan.

Moreover, there exists a unique future-directed vector field V , such that $\dot{\gamma}_0 = V(\gamma_0)$, for ν -a.e. γ . If L is a global null-geodesic vector field as defined in (5.6), then $V(\gamma_0) = t_L(\gamma_0, \gamma_1)L$ for ν -a.e. γ , where t_L is defined by (5.8). Finally, the measure $(e_t)_\# \nu$ is absolutely continuous w.r.t. H , for all $t \in [0, 1]$.

Proof. We consider the case when there exists a global acausal cross-section S . Since it follows the line presented in the previous proofs (see Theorem 5.11), we omit the proof of the general case.

Let L be a global null-geodesic vector field. Let ρ_i be the density of μ_i w.r.t. the measure Vol_L , $i = 0, 1$. First we claim that

$$(6.15) \quad \int_{\mathbb{R}} \rho_0(\Psi_L(z, t)) e^{W_L(z, t)} dt = \int_{\mathbb{R}} \rho_1(\Psi_L(z, t)) e^{W_L(z, t)} dt, \quad \text{for a.e. } z \in S.$$

Indeed, fix φ a transverse function and compute

$$\begin{aligned} & \int_S \varphi(z) \int_{\mathbb{R}} \rho_0(\Psi_L(z, t)) e^{W_L(z, t)} dt \mathcal{H}^{d-2}(dz) \\ &= \int_S \int_{\mathbb{R}} \varphi(\Psi_L(z, t)) \rho_0(\Psi_L(z, t)) e^{W_L(z, t)} dt \mathcal{H}^{d-2}(dz) \\ &= \int_H \varphi(z) \rho_0(z) \text{Vol}_L(dz) = \int_H \varphi(z) \rho_1(z) \text{Vol}_L(dz) \\ &= \int_S \int_{\mathbb{R}} \varphi(\Psi_L(z, t)) \rho_1(\Psi_L(z, t)) e^{W_L(z, t)} dt \mathcal{H}^{d-2}(dz) \\ &= \int_S \varphi(z) \int_{\mathbb{R}} \rho_0(\Psi_L(z, t)) e^{W_L(z, t)} dt \mathcal{H}^{d-2}(dz). \end{aligned}$$

By arbitrariness of φ , we deduce the validity of (6.15). For $z \in S$, let $\tilde{\mu}_i^z \in \mathcal{M}^+(\mathbb{R})$ be the measure given by

$$(6.16) \quad \tilde{\mu}_i^z := \rho_i(\Psi_L(z, \cdot)) e^{W_L(z, \cdot)} \mathcal{L}^1(dt), \quad i = 0, 1.$$

Let $T(z, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ be the map transporting $\tilde{\mu}_0^z$ to $\tilde{\mu}_1^z$ given by the monotone rearrangement, and let $R(z, t) = T(z, t) - t$. Define V in the following way: if $z = \Psi_L(z', t)$, for some $z' \in S$,

then $V(z) = R(z', t)L$. In other words, $V(z) = R(\mathfrak{Q}(z), t_L^S(z))L$, where $\mathfrak{Q} : H \rightarrow S$ and $t_L^S : H \rightarrow \mathbb{R}$ are the maps uniquely defined by the relations

$$(6.17) \quad \mathfrak{Q}(z) \in S, \quad (\mathfrak{Q}(z), z) \in \mathcal{R}, \quad \Psi_L(\mathfrak{Q}(z), t_L^S(z)) = z.$$

Let $\text{Exp}(V) : H \rightarrow C([0, 1]; H)$ be the function mapping z to the curve $s \mapsto \exp_z(sV(z))$. Define

$$\nu := (\text{Exp}(V))_{\#}\mu_0.$$

We now check that ν and V satisfy the desired properties. First we notice that

$$\begin{aligned} \text{Exp}(V)(\Psi_L(z, t))(s) &= \exp_{\Psi_L(z, t)}(sV) = \exp_{\Psi_L(z, t)}(sR(z, t)L) \\ &= \Psi_L(z, t + sR(z, t)) = \Psi_L(z, T(z, t, s)), \end{aligned}$$

where $T(z, t, s) = t + sR(z, t) = (1 - s)t + sT(z, t)$, or in other words

$$(e_s \circ \text{Exp}(V))(\Psi_L(z, t)) = \Psi_L(z, T(z, t, s)).$$

It is clear that

$$(6.18) \quad \mu_i = \int_S \Psi_L(z, \cdot)_{\#} \tilde{\mu}_i^z \mathcal{H}^{n-2}(dz), \quad i = 0, 1.$$

We can therefore compute

$$\begin{aligned} (6.19) \quad (e_s)_{\#}\nu &= (e_s \circ \text{Exp}(V))_{\#}\mu_0 = \int_S (e_s \circ \text{Exp}(V))_{\#} \Psi_L(z, \cdot)_{\#} \tilde{\mu}_0^z \mathcal{H}^{n-2}(dz) \\ &= \int_S (\Psi_L(z, T(z, \cdot, s)))_{\#} \tilde{\mu}_0^z \mathcal{H}^{n-2}(dz) = \int_S \Psi_L(z, \cdot)_{\#} T(z, \cdot, s)_{\#} \tilde{\mu}_0^z \mathcal{H}^{n-2}(dz). \end{aligned}$$

The formula above has two consequences. If we specialize to the case $s = 0, 1$, we see that $(e_s)_{\#}\nu = \mu_s$, and therefore $\nu \in \text{OptGeo}^H(\mu_0, \mu_1)$ (the fact that ν is optimal is trivial). Moreover, if $s \in [0, 1]$, the formula above implies that $(e_s)_{\#}\nu$ is absolutely continuous. Following similar computations, one can see that the plan $\pi = (e_0 \otimes e_1)_{\#}\nu$ can be represented as

$$\pi = (e_s \circ \text{Exp}(V))_{\#}\mu_0 \int_S \Psi_L(z, \cdot)_{\#} (\text{Id} \otimes T(z, \cdot))_{\#} \tilde{\mu}_0^z \mathcal{H}^{n-2}(dz).$$

Therefore, Lemma 6.9 implies that π is monotone. The fact that $V(\gamma_0) = t_L(\gamma_0, \gamma_1)L = \dot{\gamma}_0$ for ν -a.e. γ follows almost by definition. The fact that μ_t is absolutely continuous is a consequence of (6.19) and the fact that monotone interpolation in \mathbb{R} preserves the absolute continuity w.r.t. the Lebesgue measure.

Finally, we check that V is future directed. Clearly, it is sufficient to check that it is future-directed μ_0 -a.e., otherwise, one can change V on a negligible set, preserving the already-proven properties. Consider a dynamical transport plan $\hat{\nu} \in \text{OptCaus}^H(\hat{\mu}_0, \hat{\mu}_1)$. Define $\hat{\pi} := (e_0 \otimes e_1)_{\#}\hat{\nu}$. By disintegrating the measure π with respect to the projection on the first factor $P_1 : H \times H \rightarrow H$, following the argument in the proof of Proposition 6.11, one can produce a family of coupling $\hat{\pi}_z$, between their marginals $\tilde{\mu}_0^z$ and $\tilde{\mu}_1^z$. Since π gives full measure to Γ , it follows that for μ_0 -a.e. z , π_z gives full measure to the set $\{(x, y) : y \geq x\}$. Therefore, Lemma 6.4 implies that the monotone rearrangement of $\tilde{\mu}_0^z$ into $\tilde{\mu}_1^z$, given by the function $T(z, \cdot)$, satisfies $T(z, t) \geq t$. In other words $R(z, t) \geq 0$, and since V is obtained by multiplying R with L , we conclude that V is future directed. \square

7. NULL ENERGY CONDITION AS DISPLACEMENT CONVEXITY

Building on top of the results about optimal transport along a null hypersurface obtained in Section 6, in the present section we will provide a synthetic characterization of the null energy condition based on displacement convexity of the Boltzmann–Shannon entropy along monotone null-geodesic optimal transport plans (cf. Theorem 6.12). Recall that the Boltzmann–Shannon

entropy $\text{Ent}(\mu|\mathbf{m})$ of a probability measure $\mu \in \mathcal{P}(M)$ with respect to the reference measure \mathbf{m} is defined by

$$(7.1) \quad \text{Ent}(\mu|\mathbf{m}) = \int \rho \log \rho \, d\mathbf{m},$$

if $\mu = \rho \mathbf{m}$ is absolutely continuous w.r.t. \mathbf{m} and $(\rho \log \rho)_+$ is \mathbf{m} -integrable; otherwise, we adopt the convention that $\text{Ent}(\mu|\mathbf{m}) = +\infty$.

Firstly we notice that the convexity properties of the entropy along monotone null-geodesic optimal transport plans do not depend on the choice of null-geodesic vector field L , used to define the volume measure Vol_L on H (cf. Theorem 5.11).

Proposition 7.1. *Let (M, g) be a Lorentzian manifold with $g \in C^2$, and let H be a null C^2 -hypersurface. Let L_1, L_2 be two null-geodesic vector fields of class C^1 and let $\text{Vol}_{L_1}, \text{Vol}_{L_2}$ be the associated volume measures on H , as in Theorem 5.11. Let $\Phi : H \rightarrow \mathbb{R}$ be a C^0 function and let $\mathbf{m}_{L_1} := e^\Phi \text{Vol}_{L_1}$, $\mathbf{m}_{L_2} := e^\Phi \text{Vol}_{L_2}$ be the corresponding weighted measures on H . Let $\mu_0, \mu_1 \in \mathcal{P}(H)$ be two probability measures null connected along H and let $\mu_t := (e_t)_\# \nu$, $t \in [0, 1]$, for some $\text{OptCaus}^H(\mu_0, \mu_1)$. Let $e_k(t) := \text{Ent}(\mu_t|\mathbf{m}_{L_k})$, $k = 1, 2$, and assume that both functions are well-defined on $[0, 1]$.*

Then the function $e_1 - e_2$ is constant on $[0, 1]$. In particular, e_1 and e_2 enjoy the same convexity properties.

Proof. Since L_1 and L_2 are null-geodesic vector fields, we can assume without any loss of generality that there exists a positive transverse function φ , such that $L_2 = \varphi L_1$ on the union of the supports of the measures μ_t , $t \in [0, 1]$. Let $\rho_{k,t}$ be the density of μ_t w.r.t. the measure \mathbf{m}_{L_k} , $k = 1, 2$, $t \in [0, 1]$. Since $\rho_{2,t} = \rho_{1,t}/\varphi$, a direct computation gives

$$\begin{aligned} e_2(t) &= \text{Ent}(\mu_t|\mathbf{m}_{L_2}) = \int_H \log(\rho_{2,t}) \, d\mu_t = \int_H \log(\rho_{1,t}) \, d\mu_t - \int_H \log \varphi \, d\mu_t \\ &= \text{Ent}(\mu_t|\mathbf{m}_{L_1}) - \int_H \log \varphi \, d\mu_t \stackrel{(6.8)}{=} \text{Ent}(\mu_t|\mathbf{m}_{L_1}) - \int_H \log \varphi \, d\mu_0 = e_1(t) - \int_H \log \varphi \, d\mu_0. \quad \square \end{aligned}$$

We now recall a classical one-dimensional result on the displacement convexity of the entropy functional. We include a quick proof for readers' convenience.

Lemma 7.2. *Let $\mu_0 \in \mathcal{P}_{ac}(\mathbb{R})$ be an absolutely continuous probability measure, and let $T : \mathbb{R} \rightarrow \mathbb{R}$ be a non-decreasing function. Let $\mu_t := ((1-t)\text{Id} + tT)_\#(\mu_0)$, for all $t \in (0, 1]$. Then μ_t is absolutely continuous for all $t \in (0, 1)$.*

Moreover, if $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ is a locally-Lipschitz function satisfying

$$(7.2) \quad \Phi'' + \frac{(\Phi')^2}{N-1} \leq -K \quad \text{in the sense of distributions,}$$

then, having defined $e(t) := \text{Ent}(\mu_t|e^\Phi \mathcal{L}^1)$, it holds that $e(\cdot)$ is locally-Lipschitz and it satisfies

$$(7.3) \quad e'' - \frac{(e')^2}{N} \geq K \int_{\mathbb{R}} |x - T(x)|^2 \mu_0(dx), \quad \text{in the sense of distributions.}$$

Conversely, if for any (μ_t) as above the entropic inequality (7.3) holds, then Φ is locally-Lipschitz and satisfies (7.2).

Proof. The first claim follows from the fact that $(\mu_t)_{t \in [0,1]}$ is a W_2 -geodesic along $\mathcal{P}_2(\mathbb{R})$ (the space of probability measures with finite second moment over \mathbb{R}) as soon as $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{R})$; since the claim is about a local property, it is not restrictive to assume $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{R})$. Hence the first claim follows, see [44].

The equivalence of (7.2) and (7.3) can be checked by a direct computation (since it involves optimal transport on the Real line with a weight). For sake of brevity, we refer to existing literature. The inequality (7.2) is equivalent to asking that $(\mathbb{R}, |\cdot|, e^\Phi \mathcal{L}^1)$ satisfies the $\text{CD}(K, N)$

condition. From [14, Th. 3.12], this is equivalent to $(\mathbb{R}, |\cdot|, e^\Phi \mathcal{L}^1)$ satisfying the entropic $\text{CD}^\epsilon(K, N)$ (see [14, Definition 3.1]) that by [14, Lemma 2.8, Lemma 2.2] is equivalent to (7.3). \square

We are now in position to prove the convexity of the entropy along a Wasserstein geodesic induced by a monotone null-geodesic dynamical transport plan, provided a null Ricci lower bound holds true.

Theorem 7.3. *Let (M, g) be a Lorentzian manifold of dimension n , with $g \in C^2$, and let $H \subset M$ be a causal null C^2 -hypersurface that admits a dense sequence $(S_i)_i$ of local space-like and acausal cross-sections of class C^2 . Consider $\Phi : H \rightarrow \mathbb{R}$ a C^0 weight function and assume that H satisfies the $\text{NC}^1(N)$ condition (see Definition 5.12).*

Let $\mu_0, \mu_1 \in \mathcal{P}_{ac}(H)$ be two probability measures such that μ_0 is null connected to μ_1 along H and let $\nu \in \text{OptGeo}^H(\mu_0, \mu_1)$ be the unique monotone null-geodesic dynamical transport plan (see Theorem 6.12).

Let L be a null-geodesic vector field of class C^1 and let Vol_L be the associated volume measure on H , as in Theorem 5.11. Set $\mathbf{m}_L := e^\Phi \text{Vol}_L$. Denote by $\mu_t := (e_t)_\# \nu$, $t \in [0, 1]$, and let $e(t) := \text{Ent}(\mu_t | \mathbf{m}_L)$.

Then the function $[0, 1] \ni t \mapsto e(t)$ is convex (thus, in particular, locally-Lipschitz on $(0, 1)$) and it satisfies

$$(7.4) \quad e'' - \frac{(e')^2}{N-1} \geq 0, \quad \text{in the sense of distributions on } (0, 1).$$

The l.h.s. in Equation (7.4) does not depend on the choice of L : Proposition 7.1 guarantees that the convexity properties of the entropy along Wasserstein geodesics do not depend on the null-geodesic vector field L .

Proof. Following Proposition 6.7, we first assume the existence a global space-like and acausal cross-section S for H . For each $t \in [0, 1]$, let ρ_t be the Radon–Nikodym derivative of μ_t w.r.t. \mathbf{m}_L . The entropy is given by

$$(7.5) \quad \begin{aligned} e(t) &= \text{Ent}(\mu_t | \mathbf{m}_L) = \int_H \rho_t \log \rho_t \, d\mathbf{m}_L \\ &= \int_S \int_{\mathbb{R}} \rho_t(\Psi_L(z, s)) \log(\rho_t(\Psi_L(z, s))) e^{W_L(z, s) + \Phi(\Psi_L(z, s))} \, ds \, \mathcal{H}^{n-2}(dz). \end{aligned}$$

Let $\tilde{\mathbf{m}}_L^z := e^{W_L(z, \cdot) + \Phi(\Psi_L(z, \cdot))} \mathcal{L}^1(ds)$ and $\tilde{\mu}_{t,z} := \rho_t(\Psi_L(z, \cdot)) \tilde{\mathbf{m}}_L^z$. If we define

$$e_z(t) := \text{Ent}(\tilde{\mu}_{t,z} | \tilde{\mathbf{m}}_L^z) = \int_{\mathbb{R}} \rho_t(\Psi_L(z, s)) \log(\rho_t(\Psi_L(z, s))) e^{W_L(z, s) + \Phi(\Psi_L(z, s))} \, ds,$$

then (7.5) can be rewritten as

$$e(t) = \int_S e_z(t) \, \mathcal{H}^{n-2}(dz).$$

Let V be the vector field given by Theorem 6.12. Let $h : H \rightarrow [0, \infty)$ be the function such that $V = hL$ and let $R(z, s) = h(\Psi_L(z, s))$, for $z \in S$. We claim that the function $[0, 1] \ni t \rightarrow e_z(t)$ is convex (thus, in particular, locally-Lipschitz on $(0, 1)$) and it satisfies

$$(7.6) \quad e_z'' - \frac{(e_z')^2}{\tilde{\mu}_{0,z}(\mathbb{R})(N-1)} \geq 0 \quad \text{in the sense of distributions on } (0, 1).$$

Let $T(z, s) = s + R(z, s)$. The map $T(z, \cdot)$ is monotone by construction and one can see that $\tilde{\mu}_{t,z} = ((1-t)\text{Id} + tT(z, \cdot))_\#(\tilde{\mu}_{0,z})$ (this computation was indeed done in the proof of Theorem 6.12). Therefore the Ricci lower bound, Lemma 7.2 and Proposition 4.3 imply the

convexity of $e_z(\cdot)$ and the validity of (7.6). In case of a global cross-section, the thesis follows by combining (7.6) with Lemma 7.5 below.

In the general case, Proposition 6.7 produces a sequence of mutually orthogonal probability measures $(\mu_i^k)_k$, $i = 0, 1$, such that $\mu_i = \sum_{k=1}^{\infty} w_k \mu_i^k$, for some sequence w_k . Each measure μ_0^k is null connected to μ_1^k along H by the dynamical transport plan $\nu_k \in \text{OptGeo}^H(\mu_0^k, \mu_1^k)$ and $\sum_k w_k \nu_k = \nu$. Let $\mu_t^k := (e_t)_\# \nu$ and notice that for all $t \in [0, 1]$ the measure μ_t^k and μ_t^h are mutually orthogonal. Define $\pi^k = (e_0 \otimes e_1)_\# \nu^k$ and notice that $\pi = \sum_k w_k \pi^k$. If we define $e_k(t) := \text{Ent}(\mu_t^k | \mathbf{m}_L)$, then the first part of the proof implies that

$$(7.7) \quad e_k'' - \frac{(e_k')^2}{N-1} \geq 0.$$

Let $\tilde{e}_k(t) := \text{Ent}(w_k \mu_t^k | \mathbf{m}_L) = w_k e_k(t) + w_k \log w_k$, and notice that

$$(7.8) \quad \tilde{e}_k'' - \frac{(e_k')^2}{w_k(N-1)} \geq 0.$$

Since the measures μ_t^k are mutually orthogonal, then $e(t) = \sum_{k=1}^{\infty} \tilde{e}_k(t)$, and we conclude the proof by invoking Lemma 7.5 below. \square

The following are two technical lemmas used in the proof of Theorem 7.3.

Lemma 7.4. *Let $e : (0, 1) \rightarrow \mathbb{R}$ be a continuous function; let $a \geq 0$ and $b \in \mathbb{R}$. Let $e_\varepsilon := e * \rho_\varepsilon$, where ρ_ε is a mollifier with support in $[-\varepsilon, \varepsilon]$. Then*

$$(7.9) \quad e \text{ is locally-Lipschitz} \quad \text{and} \quad e'' \geq a(e')^2 + b, \quad \text{in the sense of distributions}$$

if and only if

$$(7.10) \quad e_\varepsilon''(t) \geq a(e_\varepsilon'(t))^2 + b, \quad \text{for all } t \in (\varepsilon, 1 - \varepsilon), \text{ for all } \varepsilon > 0.$$

Proof. “If” part. Clearly, $e_\varepsilon \rightarrow e$ pointwise. Since $a \geq 0$, then each e_ε is b -convex. Therefore, on every compact sub-interval of $(0, 1)$, the functions e_ε are equi-Lipschitz, for every $\varepsilon > 0$. As a consequence e is locally-Lipschitz, the convergence is locally uniform, and $e'_\varepsilon = e' * \rho_\varepsilon$. Regarding the derivative, u'_ε is in L^∞_{loc} , thus in L^2_{loc} and $e'_\varepsilon \rightarrow e'$ in L^2_{loc} . Therefore $\int_0^1 (e'_\varepsilon(t))^2 \psi \rightarrow \int_0^1 (e'(t))^2 \psi$, $\forall \psi \in C_c^\infty((0, 1))$, i.e., $(e'_\varepsilon)^2 \rightarrow (e')^2$ in the sense of distributions. Thus we can take the limit in (7.10) and obtain (7.9).

“Only if” part. Fix $\psi \in C_c^\infty((0, 1))$ and $\varepsilon > 0$. A direct computation gives

$$\begin{aligned} \int e_\varepsilon''(t) \psi(t) dt &\geq \int (a[(e')^2]_\varepsilon + b) \psi(t) dt = \int a \int (e'(t))^2 \rho_\varepsilon(s-t) ds \psi(t) dt + b \int \psi(t) dt \\ &\geq \int a \left(\int e'(t) \rho_\varepsilon(s-t) ds \right)^2 \psi(t) dt + b \int \psi(t) dt \\ &= \int (a (e'_\varepsilon(t))^2 + b) \psi(t) dt, \end{aligned}$$

having used the convention that $[(e')^2]_\varepsilon$ is the mollification of $(e')^2$, the fact that e' is an L^1_{loc} function, and Jensen inequality. \square

Lemma 7.5. *Let $(Q, \mathcal{Q}, \mathbf{q})$ be a measure space and let $e : Q \times (0, 1) \rightarrow \mathbb{R}$, be a function satisfying the following assumptions:*

- For \mathcal{L}^1 -a.e. $t \in \mathbb{R}$, the map $Q \ni \alpha \mapsto e_\alpha(t)$ is \mathbf{q} -integrable;
- For \mathbf{q} -a.e. $\alpha \in Q$, the map $t \mapsto e_\alpha(t)$ is locally-Lipschitz on $(0, 1)$ and it satisfies

$$(7.11) \quad e''_\alpha \geq a \frac{(e'_\alpha)^2}{c(\alpha)} + b(\alpha), \quad \text{in the sense of distributions,}$$

where $a \geq 0$, $b \in L^1(\mathbf{q})$, $b \geq 0$ \mathbf{q} -a.e., $\int_Q c(\alpha) \mathbf{q}(d\alpha) = 1$ with $c > 0$ \mathbf{q} -a.e..

Denote by $e(t) := \int_Q e_\alpha(t) \mathbf{q}(d\alpha)$ and $B := \int_Q b(\alpha) \mathbf{q}(d\alpha)$. Then the map $t \mapsto e(t)$ is locally-Lipschitz on $(0, 1)$ and it satisfies

$$(7.12) \quad e'' \geq a(e')^2 + B, \quad \text{in the sense of distributions.}$$

Proof. Since a, b, c are non-negative, then each e_α is convex on $[0, 1]$, hence e is convex on $[0, 1]$, thus locally-Lipschitz on $(0, 1)$. Let ρ_ε be a mollifier and define

$$e_{\alpha,\varepsilon}(t) := e_\alpha * \rho_\varepsilon(t) = \int_{\mathbb{R}} e_\alpha(t-s) \rho_\varepsilon(s) ds, \quad e_\varepsilon(t) := e * \rho_\varepsilon(t) = \int_{\mathbb{R}} \int_Q e_\alpha(t-s) \rho_\varepsilon(s) \mathbf{q}(d\alpha) ds.$$

We claim that $e_\varepsilon(t) = \int_Q e_{\alpha,\varepsilon}(t) \mathbf{q}(d\alpha)$. Indeed, it is sufficient to check that e is integrable in $Q \times [t-\varepsilon, t+\varepsilon]$, and then apply Fubini theorem. This fact is a consequence of the following two bounds given by the convexity in the second variable:

$$(7.13) \quad e_\alpha(s) \geq e_\alpha(t-\varepsilon) + (s-t+\varepsilon) \frac{e_\alpha(t-\varepsilon) - e_\alpha(t-2\varepsilon)}{2\varepsilon}, \quad \forall s \geq t-\varepsilon,$$

$$(7.14) \quad e_\alpha(s) \leq e_\alpha(t-\varepsilon) \frac{t+\varepsilon-s}{2\varepsilon} + e_\alpha(t+\varepsilon) \frac{s-t+\varepsilon}{2\varepsilon}, \quad \forall s \in [t-\varepsilon, t+\varepsilon].$$

Lemma 7.4 gives that $e_{\alpha,\varepsilon}$ solves the equation

$$(7.15) \quad e''_{\alpha,\varepsilon}(t) \geq a \frac{(e'_{\alpha,\varepsilon}(t))^2}{c(\alpha)} + b(\alpha).$$

We integrate over Q , obtaining

$$\begin{aligned} e''_\varepsilon(t) &= \int_Q e''_{\alpha,\varepsilon}(t) \mathbf{q}(d\alpha) \geq a \int_Q \frac{(e'_{\alpha,\varepsilon}(t))^2}{c(\alpha)} \mathbf{q}(d\alpha) + \int_Q b(\alpha) \mathbf{q}(d\alpha) \\ &= a \int_Q \left(\frac{e'_{\alpha,\varepsilon}(t)}{c(\alpha)} \right)^2 c(\alpha) \mathbf{q}(d\alpha) + B \\ &\geq a \left(\int_Q \frac{e'_{\alpha,\varepsilon}(t)}{c(\alpha)} c(\alpha) \mathbf{q}(d\alpha) \right)^2 + B = a(e'_\varepsilon(t))^2 + B, \end{aligned}$$

where we have used the Jensen inequality. We conclude by taking the limit as $\varepsilon \rightarrow 0$ using Lemma 7.4. \square

Theorem 7.3 suggests the following two definitions, one concerning null hypersurfaces and one concerning the ambient manifold. In order to state them in a more concise form, let us consider the following dimensional variant of the Boltzmann–Shannon entropy

$$(7.16) \quad U_N(\mu|\mathbf{m}) := \exp\left(-\frac{1}{N} \text{Ent}(\mu|\mathbf{m})\right),$$

which was studied in [14] in connection to Ricci bounds. This kind of functional is well-known in information theory as the Shannon entropy power, see e.g. [12]. For instance, the entropy power on \mathbb{R}^N is the functional $U_{N/2}$.

Definition 7.6. Let (M, g) be a Lorentzian manifold with $g \in C^2$ and $H \subset M$ be a causal null C^2 -hypersurface admitting a dense sequence of local space-like and acausal cross-sections of class C^2 . Let Vol_L be as in Theorem 5.11. Let $\Phi : H \rightarrow \mathbb{R}$ be a C^0 function and define $\mathbf{m}_L := e^\Phi \text{Vol}_L$.

We say that the quadruple (M, g, H, Φ) satisfies the null energy condition $\text{NC}^e(N)$ if and only if the following holds:

For every $\mu_0, \mu_1 \in \mathcal{P}_{ac}(H)$ probability measures such that μ_0 is null connected to μ_1 along H , there exists a null-geodesic dynamical transport plan $\nu \in \text{OptGeo}^H(\mu_0, \mu_1)$, such that the

function $u_{N-1}(t) := U_{N-1}((e_t)_\# \nu | \mathbf{m}_L)$ satisfies

$$(7.17) \quad u_{N-1}(t) \geq (1-t)u_{N-1}(0) + t u_{N-1}(1), \quad \forall t \in [0, 1].$$

Definition 7.7. Let (M, g) be a Lorentzian manifold with $g \in C^2$ and $\Phi : M \rightarrow \mathbb{R}$ a C^0 function. We say that the triple (M, g, Φ) satisfies the null energy condition $\text{NC}^e(N)$ if and only if for any null causal C^2 -hypersurface $H \subset M$ admitting a dense sequence of local space-like and acausal cross-sections of class C^2 , the quadruple (M, g, H, Φ) satisfies the null energy condition $\text{NC}^e(N)$.

Remark 7.8. Definition 7.6 is well-posed, i.e. it does not depend on the particular choice of the global Borel null vector field L . Indeed, Proposition 7.1 implies that on each saturated set H_{S_k} the convexity properties of the entropy along μ_t do not depend on L . To obtain the full claim, one can argue as in the proof of Theorem 7.3.

Remark 7.9. With the just-introduced definitions, Theorem 7.3 can be stated as $\text{NC}^1(N)$ implies $\text{NC}^e(N)$ for null hypersurfaces, and therefore for spaces. In next section we will prove the converse implication.

Remark 7.10. Note that Definition 7.6 is slightly different from the thesis of Theorem 7.3, in three aspects. The first one is merely cosmetic: Theorem 7.3 is phrased as a convexity property of the Boltzmann–Shannon entropy Ent , see (7.1), while Definition 7.6 writes as a concavity condition on the Shannon entropy power U_N , see (7.16). However one can easily pass from one formulation to the other via a change of variables. The two slightly more substantial differences are the following. On the one hand, Theorem 7.3 deals with monotone plans, whereas Definition 7.6 does not require plans to be monotone. In this regard, Definition 7.6 is less restrictive on the allowed transports. On the other hand, Theorem 7.3 guarantees a full convexity of the entropy along monotone transports (which always exist, thanks to Theorem 6.12), whereas the condition requested in Definition 7.6 is a concavity-type inequality only at the end-points.

The choice for Definition 7.6 is motivated by the subsequent Theorem 8.4 showing that checking convexity along a causal plan is enough to obtain the NC^1 condition

8. EQUIVALENCE OF THE VARIOUS FORMULATIONS

8.1. Equivalence of the synthetic energy conditions $\text{NC}^e(N)$ and $\text{NC}^1(N)$. In the previous section (see Theorem 7.3 and Remark 7.9), we proved that $\text{NC}^1(N)$ implies $\text{NC}^e(N)$. In the present section, we prove the converse implication $\text{NC}^e(N) \implies \text{NC}^1(N)$, under the assumptions $\Phi \in C^0$ and $g \in C^2$. We start with a few technical lemmas.

Lemma 8.1. *Let $(Q, \mathcal{Q}, \mathbf{q})$ be a measure space, $t \in (0, 1)$, and $e : Q \times \{0, t, 1\} \rightarrow \mathbb{R}$ a measurable function. Fix $G : \overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}$ a continuous, strictly decreasing function. Let $F = \{w \in L^\infty(Q, \mathbf{q}) : w \geq 0, \int_Q w d\mathbf{q} = 1\}$. If $w \in F$, let $e_w(t) = \int_Q w(\alpha) e(\alpha, t) \mathbf{q}(d\alpha)$. Define*

$$(8.1) \quad u_w = G(e_w) \quad \text{and} \quad u_\alpha = G(e_\alpha).$$

Assume that for all $w \in F$ satisfies

$$(8.2) \quad u_w(t) \geq (1-t)u_w(0) + t u_w(1).$$

Then, for \mathbf{q} -a.e. $\alpha \in Q$, it holds that

$$(8.3) \quad u_\alpha(t) \geq (1-t)u_\alpha(0) + t u_\alpha(1).$$

Proof. Fix $h, k \in G(\overline{\mathbb{R}})$ and let

$$A_{h,k} = \{\alpha \in Q : u_\alpha(0) \leq h, u_\alpha(1) \leq k\} = \{\alpha \in Q : e_\alpha(0) \geq G^{-1}(h), e_\alpha(1) \geq G^{-1}(k)\}.$$

Fix $B \subset A_{h,k}$ and take $w = \frac{1}{\mathfrak{q}(B)} \mathbf{1}_B$. A direct computation gives

$$\begin{aligned} u_w(t) &\leq (1-t)u_w(0) + tu_w(0) \\ &= (1-t)G\left(\frac{1}{\mathfrak{q}(B)} \int_B e_\alpha(0) \mathfrak{q}(d\alpha)\right) + tG\left(\frac{1}{\mathfrak{q}(B)} \int_B e_\alpha(1) \mathfrak{q}(d\alpha)\right) \\ &\leq (1-t)h + tk, \end{aligned}$$

thus

$$\frac{1}{\mathfrak{q}(B)} \int_B e_\alpha(t) \geq G^{-1}((1-t)h + tk).$$

By arbitrariness of B , we deduce

$$e_\alpha(t) \geq G^{-1}((1-t)h + tk), \quad \text{for } \mathfrak{q}\text{-a.e. } \alpha \in A_{h,k}.$$

Hence the set

$$\tilde{A}_{h,k} := \{\alpha \in Q : \alpha \in A_{h,k} \implies u_\alpha(z) \leq (1-t)h + tk\}$$

has full measure in Q . Define the full-measure set $\tilde{A} := \bigcap_{h,k \in G(\mathbb{R}) \cap \mathbb{Q}} \tilde{A}_{h,k}$. Assume by contradiction that for some $\varepsilon > 0$, for some $\alpha \in \tilde{A}$ it holds that

$$(8.4) \quad u_\alpha(t) \geq (1-t)u_\alpha(0) + tu_\alpha(1) + 2\varepsilon.$$

In this case, choose $h \in [u_\alpha(0), u_\alpha(0) + \varepsilon] \cap \mathbb{Q} \subset G(\mathbb{R})$ and $k \in [u_\alpha(1), u_\alpha(1) + \varepsilon] \cap \mathbb{Q} \subset G(\mathbb{R})$. By construction, $\alpha \in \tilde{A}_{h,k}$, therefore

$$u_\alpha(t) \leq (1-t)h + tk \leq (1-t)(u_\alpha(0) + \varepsilon) + t(u_\alpha(1) + \varepsilon) = (1-t)u_\alpha(0) + tu_\alpha(1) + \varepsilon,$$

which contradicts (8.4). \square

The next lemma is a sort of ‘‘Brunn–Minkowski inequality implies log-concavity’’ that suits our setting.

Lemma 8.2. *Fix $N > 1$. Let $I \subset \mathbb{R}$ be an interval and $\Phi : I \rightarrow \mathbb{R}$ be continuous function and let $\mathbf{m} := e^{-\Phi} \mathcal{L}^1$. Given $x_0 < x_1$ in $I \cap \mathbb{Q}$, define $x_t := (1-t)x_0 + tx_1$. Similarly, for $\varepsilon_0, \varepsilon_1 > 0$ in \mathbb{Q} , define $\varepsilon_t = (1-t)\varepsilon_0 + t\varepsilon_1$. Assume that for all x_i and ε_i as above, $i = 0, 1$, it holds that*

$$(8.5) \quad \mathbf{m}([x_t, x_t + \varepsilon_t])^{\frac{1}{N}} \geq (1-t)\mathbf{m}([x_t, x_t + \varepsilon_t])^{\frac{1}{N}} + t\mathbf{m}([x_t, x_t + \varepsilon_t])^{\frac{1}{N}}, \quad \forall t \in [0, 1] \cap \mathbb{Q}.$$

Then Φ is locally-Lipschitz and satisfies

$$(8.6) \quad \Phi'' + \frac{(\Phi')^2}{N-1} \leq 0, \quad \text{in the sense of distributions.}$$

Proof. Fix $x_0 < x_1$ as in the hypothesis. Let $\lambda = (\lambda_0, \lambda_1)$, such that $\lambda_0, \lambda_1 > 0$ to be fixed later. Let $(\lambda^n)_n$ be a sequence in \mathbb{Q}^2 converging to λ and let $\lambda_t^n = (1-t)\lambda_0^n + t\lambda_1^n$. Let $(\delta_n)_n$ be an infinitesimal sequence. We define

$$(8.7) \quad u_n(t) := \delta_n^{-\frac{1}{N}} \mathbf{m}([x_t, x_t + \delta_n \lambda_t^n])^{\frac{1}{N}}.$$

We can easily compute the limit

$$u(t) := \lim_{n \rightarrow \infty} u_n(t) = \left(\lim_{n \rightarrow \infty} \delta_n^{-1} \int_{x_t}^{x_t + \delta_n \lambda_t^n} \Phi(y) dy \right)^{\frac{1}{N}} = (\lambda_t e^{\Phi(x_t)})^{\frac{1}{N}}.$$

Since the concavity passes to pointwise limit, we have that

$$e^{\Phi(x_t)/N} \geq \lambda_t^{-\frac{1}{N}} \left((1-t)(\lambda_0 e^{\Phi(x_0)})^{\frac{1}{N}} + t(\lambda_1 e^{\Phi(x_1)})^{\frac{1}{N}} \right).$$

The choice $\lambda_i = e^{\Phi(x_i)/(N-1)}$, for $i = 0, 1$, yields

$$\begin{aligned} e^{\Phi(x_t)/N} &\geq \lambda_t^{-\frac{1}{N}} \left((1-t)(e^{\Phi(x_0)})^{\frac{1}{N-1}+1} + t(e^{\Phi(x_1)})^{\frac{1}{N-1}+1} \right) \\ &= \lambda_t^{-\frac{1}{N}} \left((1-t)e^{\Phi(x_0)/(N-1)} + te^{\Phi(x_1)/(N-1)} \right) \\ &= \lambda_t^{-\frac{1}{N}} \left((1-t)\lambda_0 + t\lambda_1 \right) = \lambda_t^{\frac{N-1}{N}} = \left((1-t)e^{\Phi(x_0)/(N-1)} + te^{\Phi(x_1)/(N-1)} \right)^{\frac{N-1}{N}}. \end{aligned}$$

By arbitrariness of x_0 and x_1 , we deduce

$$e^{\Phi((1-t)x_0+tx_1)/(N-1)} \geq (1-t)e^{\Phi(x_0)/(N-1)} + te^{\Phi(x_1)/(N-1)}, \quad \forall x_0, x_1 \in I \cap \mathbb{Q}, \forall t \in [0, 1] \cap \mathbb{Q}.$$

The continuity of Φ guarantees that the constraint of x_0, x_1, t to be rational can be dropped, or in other words $e^{\Phi/(N-1)}$ is concave, which is the thesis. \square

Lemma 8.3. *Let $(Q, \mathfrak{Q}, \mathfrak{q})$ be a measure space, $I \subset \mathbb{R}$ an interval, and $\Phi : Q \times I \rightarrow \mathbb{R}$ a function, measurable in the first variable and continuous in the second variable. Let $\mathfrak{m} := e^\Phi \mathfrak{q} \otimes \mathcal{L}^1 \llcorner_I \in \mathcal{M}^+(Q \times I)$ and fix $N > 0$. Define $F_t : Q \times I \times I \rightarrow Q \times I$ as $F_t(\alpha, x, y) = (\alpha, (1-t)x + ty)$. For a given measure $\pi \in \mathcal{P}(Q \times I \times I)$, define*

$$e_\pi(t) := \text{Ent}((F_t)_\# \pi | \mathfrak{m}) \in [-\infty, \infty], \quad u_\pi(t) := \exp\left(-\frac{e_\pi(t)}{N}\right).$$

Assume that, for every $w \in L^\infty(Q)$, with $\int_Q w d\mathfrak{q} = 1$ and $w \geq 0$, and for every $\hat{\mu}_0, \hat{\mu}_1 \in \mathcal{P}(I)$, such that $\text{supp } \mu_0 \subset (-\infty, a)$ and $\text{supp } \mu_1 \subset (a, +\infty)$, for some $a \in I$, there exists $\pi \in \mathcal{P}(Q \times I \times I)$, such that

$$(8.8) \quad (P_1, P_2)_\# \pi = w\mathfrak{q} \otimes \hat{\mu}_0 \quad \text{and} \quad (P_1, P_3)_\# \pi = w\mathfrak{q} \otimes \hat{\mu}_1$$

and that the function u_π is satisfies

$$(8.9) \quad u_\pi(t) \geq (1-t)u_\pi(0) + tu_\pi(1), \quad \forall t \in [0, 1].$$

Then, for \mathfrak{q} -a.e. $\alpha \in Q$, it holds that the function $t \mapsto \Phi(\alpha, t)$ is concave and it satisfies

$$(8.10) \quad \Phi(\alpha, \cdot)'' + \frac{(\Phi(\alpha, \cdot)')^2}{N-1} \leq 0, \quad \text{in the sense of distributions.}$$

Proof. Fix $x = (x_0, x_1) \in (Q \cap I)^2$, with $x_0 < x_1$ and $\varepsilon = (\varepsilon_0, \varepsilon_1) \in \mathbb{Q}^2$ with $\varepsilon_0, \varepsilon_1 > 0$ small enough; define $x_t = (1-t)x_0 + tx_1$ and $\varepsilon_t = (1-t)\varepsilon_0 + t\varepsilon_1$. Fix also $w \in L^\infty(Q)$, as above. We define

$$\hat{\mu}_i^{x,\varepsilon} = \frac{1}{\varepsilon_i} \mathcal{L}^1 \llcorner_{[x_i, x_i + \varepsilon_i]}, \quad \mu_i^{x,\varepsilon,w} := w\mathfrak{q} \otimes \hat{\mu}_i^{x,\varepsilon}, \quad i = 0, 1.$$

The measures $\hat{\mu}_i^{x,\varepsilon,w}$, $i = 0, 1$, satisfy the assumption, thus there exists π_w , as above. In particular, since $\mu_i^{x,\varepsilon,w}$, $i = 0, 1$, are absolutely continuous, then inequality (8.9) implies that also $\mu_t^{x,\varepsilon,w} := (F_t)_\# \pi_w$, $t \in [0, 1]$, is absolutely continuous w.r.t. \mathfrak{m} . We can therefore write

$$(8.11) \quad \mu_t^{x,\varepsilon,w} = w(\alpha) \rho_{t,\alpha}^{x,\varepsilon,w} \mathfrak{q} \otimes \mathcal{L}^1,$$

for some function $\rho_{t,\alpha}^{x,\varepsilon,w}$ which satisfies the following immediate properties

$$(8.12) \quad \int_{x_t}^{x_t + \varepsilon_t} \rho_{t,\alpha}^{x,\varepsilon,w}(y) dy = 1 \quad \text{and} \quad \text{supp } \rho_{t,\alpha}^{x,\varepsilon,w} \subset [x_t, x_t + \varepsilon_t].$$

We now compute the entropy

$$\begin{aligned} e^{x,\varepsilon,w}(t) &:= \text{Ent}(\mu_t^{x,\varepsilon,w} | \mathbf{m}) = \int_Q \int_{x_t}^{x_t+\varepsilon_t} w(\alpha) \rho_{t,\alpha}^{x,\varepsilon,w}(y) \log(w(\alpha) \rho_{t,\alpha}^{x,\varepsilon,w}(y) e^{-\Phi(\alpha,y)}) \, dy \, d\mathbf{q} \\ &= \int_Q w(\alpha) \log w(\alpha) \, \mathbf{q}(d\alpha) + \int_Q \int_{x_t}^{x_t+\varepsilon_t} w(\alpha) \rho_{t,\alpha}^{x,\varepsilon,w}(y) \log(\rho_{t,\alpha}^{x,\varepsilon,w}(y) e^{-\Phi(\alpha,y)}) \, dy \, d\mathbf{q} \\ &= \text{Ent}(w\mathbf{q} | \mathbf{q}) + \int_Q w(\alpha) \int_{x_t}^{x_t+\varepsilon_t} \rho_{t,\alpha}^{x,\varepsilon,w}(y) \log(\rho_{t,\alpha}^{x,\varepsilon,w}(y) e^{-\Phi(\alpha,y)}) \, dy \, d\mathbf{q}. \end{aligned}$$

Next, we define $\mathbf{m}_\alpha := e^{\Phi(\alpha,\cdot)} \mathcal{L}^1_{\lfloor I}$ and compute (using Jensen's inequality for the function $\rho \log \rho$, w.r.t. the measure \mathbf{m}_α)

$$\begin{aligned} e_\alpha^{x,\varepsilon}(t) &:= -\log(\mathbf{m}_\alpha([x_t, x_t + \varepsilon_t])) \\ &= \int_{x_t}^{x_t+\varepsilon_t} \rho_{t,\alpha}^{x,\varepsilon,w}(y) e^{-\Phi(\alpha,y)} \, \mathbf{m}_\alpha(dy) \log \left(\frac{\int_{x_t}^{x_t+\varepsilon_t} \rho_{t,\alpha}^{x,\varepsilon,w}(y) e^{-\Phi(\alpha,y)} \, \mathbf{m}_\alpha(dy)}{\mathbf{m}_\alpha([x_t, x_t + \varepsilon_t])} \right) \\ &\leq \int_{x_t}^{x_t+\varepsilon_t} \rho_{t,\alpha}^{x,\varepsilon,w} e^{-\Phi(\alpha,y)} \log(\rho_{t,\alpha}^{x,\varepsilon,w} e^{-\Phi(\alpha,y)}) \, \mathbf{m}_\alpha(dy) \\ &= \int_{x_t}^{x_t+\varepsilon_t} \rho_{t,\alpha}^{x,\varepsilon,w} \log(\rho_{t,\alpha}^{x,\varepsilon,w} e^{-\Phi(\alpha,y)}) \, dy, \end{aligned}$$

with equality for $t = 0, 1$. Therefore, it holds that

$$(8.13) \quad e^{x,\varepsilon,w}(t) \geq \text{Ent}(w\mathbf{q} | \mathbf{q}) + \int_Q w(\alpha) e_\alpha^{x,\varepsilon}(t) \, \mathbf{q}(d\alpha),$$

with equality at the endpoints. Let

$$u^{x,\varepsilon,w}(t) := \exp\left(-\frac{e^{x,\varepsilon,w}(t)}{N}\right), \quad u^{x,\varepsilon,\alpha}(t) := \exp\left(-\frac{e_\alpha^{x,\varepsilon}(t)}{N}\right) = \mathbf{m}_\alpha([x_t, x_t + \varepsilon_t]).$$

Fix $t \in (0, 1)$. By assumption, $u^{x,\varepsilon,w}$ verifies (8.9), thus we can apply Lemma 8.1 (with $G(b) = \exp(-b/(N-1))$) and deduce that for \mathbf{q} -a.e. α it holds that

$$u^{x,\varepsilon,\alpha}(t) \geq (1-t)u^{x,\varepsilon,\alpha}(0) + tu^{x,\varepsilon,\alpha}(1).$$

By taking a countable intersection, we deduce that the inequality above holds true for all $t \in [0, 1] \cap \mathbb{Q}$. We can thus conclude by applying Lemma 8.2. \square

We are now in position to prove that the displacement convexity of the entropy implies the $\text{NC}^1(N)$ condition.

Theorem 8.4 ($\text{NC}^e(N) \Rightarrow \text{NC}^1(N)$). *Let (M, g) be a Lorentzian manifold, let H be a causal null-hypersurface and $(S_k)_k$ be a dense sequence of local space-like and acausal cross-sections for H . Assume g, H, S_k to be of class C^2 . Consider $\Phi : H \rightarrow \mathbb{R}$ a C^0 weight function and define $\mathbf{m}_L := e^\Phi \text{Vol}_L$.*

If (M, g, H, Φ) satisfies the null energy condition $\text{NC}^e(N)$, then, for every $z \in S$, the function $a_z(t) := W_L(z, t) + \Phi(\Psi_L(z, t))$ is concave on $[0, 1]$ (thus, in particular, locally-Lipschitz on $(0, 1)$) and it satisfies

$$(8.14) \quad a_z'' + \frac{(a_z')^2}{N-2} \leq 0, \quad \text{in the sense of distributions on } (0, 1).$$

Proof. We will consider only the case for t in a neighborhood of 0: if t is far away from 0, we can take an auxiliary cross section and apply Lemma 4.2. We can also restrict H so that S is a global space-like and acausal cross-section (Theorem 5.11). Up to multiplying L by a

transverse function (no change in the derivatives of the densities of the measures) and further restricting H , we can also assume that $\Psi_L(S \times (-1, 1)) = H$.

Our aim is now to apply Lemma 8.3. Take $I = (-1, 1)$, $Q = S$, $\mathbf{q} = \mathcal{H}^{d-2}$, and $\hat{\mathbf{m}} = a_z(t)\mathbf{q}(dz) \otimes dt \in \mathcal{M}^+(Q \times I)$. Fix $\hat{\mu}_i \in \mathcal{P}(I)$, $i = 0, 1$, absolutely continuous, such that $\text{supp } \hat{\mu}_0 \subset (-\infty, b)$ and $\hat{\mu}_1 \subset (b, +\infty)$, for some $b \in I$. Fix $w \in L^\infty(Q)$ non-negative, with $\int_Q w = 1$. Define $\mu_i := (\Psi_L)_\#(w\mathbf{q} \otimes \hat{\mu}_i)$, $i = 0, 1$. Let $\nu \in \text{OptGeo}^H(\mu_0, \mu_1)$ by null-geodesics dynamical transport plan given by the definition of $\text{NC}^e(N)$; in other words, ν satisfies

$$(8.15) \quad u(t) \geq (1-t)u(0) + tu(1), \quad \text{where } u(t) := \exp\left(-\frac{\text{Ent}((e_t)_\#\nu|\mathbf{m})}{N-1}\right).$$

We now construct π . Let $P_S : H \rightarrow S$, given by $P_S(x) = \alpha$, if $(x, \alpha) \in \mathcal{R}$; let $G_L : H \rightarrow \mathbb{R}$, given by $\Psi_L(P_S(x), G_L(x)) = x$ (in other words, (P_S, G_L) is the inverse of Ψ_L). Let Γ a map that given a causal curve γ it returns $\Gamma(\gamma) = (P_S(\gamma_0), G_L(\gamma_0), G_L(\gamma_1))$. We can thus define $\pi := \Gamma_\#\gamma$.

We need to check that π satisfies the hypothesis of Lemma 8.3. Let F_t as in the hypothesis of Lemma 8.3. Since $F_t \circ \Gamma = (P_S, G_L) \circ e_t$, it holds that

$$\begin{aligned} \text{Ent}((F_t)_\#\pi|\hat{\mathbf{m}}) &= \text{Ent}((F_t \circ \Gamma)_\#\nu|\hat{\mathbf{m}}) = \text{Ent}(((P_S, G_L) \circ e_t)_\#\nu|\hat{\mathbf{m}}) \\ &= \text{Ent}(((P_S, G_L) \circ e_t)_\#\nu|(P_S, G_L)_\#\mathbf{m}_L) = \text{Ent}((e_t)_\#\nu|\mathbf{m}_L), \end{aligned}$$

having used the fact that $\hat{\mathbf{m}} = (P_S, G_L)_\#\mathbf{m}_L$. We can thus deduce from (8.15) the assumption (8.9) of Lemma 8.3.

Applying this Lemma, we deduce that for \mathbf{q} -a.e. z , the function $I \ni t \mapsto W_L(z, t) + \Phi(\Psi_L(z, t))$ is concave and it satisfies (8.14). Finally, the ‘‘for \mathbf{q} -a.e. $z \in S$ ’’ improves to ‘‘for all $z \in S$ ’’ thanks to the continuity of both W_L and Φ . \square

8.2. From displacement convexity to classical lower Ricci bounds. Let us start by reviewing the implications proved so far. Under the regularity assumptions $\Phi \in C^0$ and $g \in C^2$, we showed that the conditions $\text{NC}^e(N)$ and $\text{NC}^1(N)$ are equivalent, for a fixed weighted null hypersurface (see Theorem 7.3 and Theorem 8.4); as a consequence, $\text{NC}^e(N)$ and $\text{NC}^1(N)$ are two equivalent conditions for spaces.

Both conditions, under the regularity assumptions $\Phi \in C^2$ and $g \in C^2$ are implied by $\text{Ric}^{g, \Phi, N}(v) \geq 0$, for every light-like vector v .

We now close this circle of implications, by showing that if a Lorentzian manifold enjoys the convexity of the entropy in the way specified above, then $\text{Ric}(v, v) \geq 0$ in the null directions.

To obtain the classical null-energy condition, we will merely use the convexity of the entropy inside future light-cones.

We fix few notations that will be used in the sequel. Given a Lorentzian manifold M and a point $p \in M$, we consider the set

$$(8.16) \quad \hat{H} := \{v \in T_p M : v \text{ is future-directed and } g(v, v) = 0\},$$

i.e. the future light-cone in the tangent space without the tip. Let $U \subset T_p M$ be a neighborhood of the origin such that $\exp_p|_U$ is a diffeomorphism on its image. We define $H := \exp_p(\hat{H} \cap U)$. It is clear that H is a null hypersurface. Any null hypersurface constructed in this way will be called *local future light-cone*.

Theorem 8.5 ($\text{NC}^1(n)$ on light-cones \Rightarrow NEC). *Let (M, g) be a Lorentzian manifold of dimension n , with g of class C^2 . Assume that every local future light-cone H satisfies the null energy condition $\text{NC}^1(n)$.*

Then $\text{Ric}_g(v, v) \geq 0$ for any $v \in TM$, such that $g(v, v) = 0$.

Proof. By the the definition of W_L , the hypothesis coincides with the convexity of the function

$$t \mapsto \exp(W_L(z, t)/(n-2)) = (\det(J(t)))^{1/(n-2)}.$$

Recall also that from (3.10) we have $\det J(t) = \det \bar{J}(t)$, where \bar{J} is the upper-left $(n-2) \times (n-2)$ -minor of J .

Fix $p \in M$ and assume by contradiction that there exists $v \in T_p M$ such that $g(v, v) = 0$ and $\text{Ric}(v, v) < 0$. Let $(e_i)_i$ be a basis of $T_p M$, such that:

$$(8.17) \quad g(e_i, e_j) = \delta_{ij}, \quad i = 1, \dots, n-2, \quad j = 1, \dots, n$$

$$(8.18) \quad e_{n-1} = v, \quad g(e_{n-1}, e_n) = -1, \quad g(e_n, e_n) = 0.$$

We endow $T_p M$ with a (Riemannian) scalar product \hat{g} , such that $(e_i)_i$ is an orthonormal basis for \hat{g} ; in particular $\hat{g}(v, v) = 1$. Up to rescaling v (and the corresponding $(e_i)_i$), we can assume that

$$\hat{S} := \{w \in \hat{H} : \hat{g}(w, w) = 1\} \cap B_\varepsilon^{\hat{g}}(v) \subset U \quad \text{and} \quad S := \exp_p(\hat{S}) \subset H,$$

for some $\varepsilon > 0$ small enough. Clearly, S is a global space-like cross-section for H . We define the map $\hat{L} : \hat{H} \rightarrow T_p M$ as $\hat{L}(w) = w / \sqrt{\hat{g}(w, w)}$. Using the standard identification of $T_p M$ with $T_w(T_p M)$, we can define $L(\exp_p(w)) := d(\exp_p)_w[\hat{L}]$. It is clear that L is a null-geodesic vector field on H . Let $\gamma_t = \exp_p(tv)$, and let $z = \gamma_1 = \exp_p(v) \in S$.

With a slight abuse of notation, we still denote with $(e_i)_i$ the basis of $T_z S$ given by $e_i = d(\exp_p)_v[e_i]$, $i = 1, \dots, n$; in other words $(e_i)_i$ is obtained via parallel transport along the curve γ . Notice that $e_{n-1} = L$ and $e_n = \bar{L}^S$. We now consider the Jacobi fields $J_i = J_{e_i}$ (recall the notation of Section 3.1).

It is a standard fact of Lorentzian (and also Riemannian) geometry (see, e.g., [15, Sec. 3.C.3]) that J_{e_i} can be characterized as

$$(8.19) \quad J_{e_i}(t-1) = d(\exp_p)_{tv}[te_i], \quad i = 1, \dots, n-2.$$

Moreover, for $t = 0$, it holds that

$$(8.20) \quad J_{e_i}(-1) = 0 \quad \text{and} \quad J'_{e_i}(-1) = e_i, \quad i = 1, \dots, n-2.$$

Using the coordinate system introduced in Section 3.1 (we recall that R represents the Riemann curvature tensor, J represents the Jacobi flow, and J_i is the i -th row of J), we can write

$$J''_i = -J_i(t)R(t), \quad J_i(-1) = 0, \quad J'_i(-1) = \mathbf{f}_i^T, \quad i = 1, \dots, n-2,$$

and thus $J''_i(-1) = 0$. The C^2 -regularity of the metric guarantees that R is continuous, and thus J_i is C^2 . We now compute the third derivative $J'''_i(-1)$:

$$\begin{aligned} J'''_i(-1) &= \lim_{t \rightarrow -1^+} \frac{J''_i(t) - J''_i(-1)}{t+1} = \lim_{t \rightarrow -1^+} \frac{-J_i(t)R(t)}{t+1} \\ &= - \lim_{t \rightarrow -1^+} \frac{J_i(t)}{t+1} R(-1) = -J'(-1)R(-1) = -\mathbf{f}_i^T R(-1), \quad i = 1, \dots, n-2. \end{aligned}$$

Denoting by \bar{R} and \bar{J} the upper-left $(n-2) \times (n-2)$ -minor of R and J respectively, the equation above becomes $\bar{J}'''(-1) = -\bar{R}(-1)$. Therefore, the following Taylor expansion holds:

$$\bar{J}(h-1) = hI - \frac{h^3}{6}\bar{R}(-1) + o(h^3), \quad h \rightarrow 0^+.$$

The determinant of \bar{J} has the expansion

$$\det \bar{J}(h-1) = h^{n-2} \det \left(I - \frac{h^2}{6}\bar{R}(-1) + o(h^2) \right) = h^{n-2} \left(1 - \frac{h^2}{12} \text{tr}(\bar{R}(-1)) + o(h^2) \right),$$

hence

$$(\det \bar{J}(h-1))^{\frac{1}{n-2}} = h \left(1 - \frac{h^2}{12} \text{tr}(\bar{R}(-1)) + o(h^2) \right)^{\frac{1}{n-2}} = h - \frac{h^3}{12(n-2)} \text{tr}(\bar{R}(-1)) + o(h^3).$$

The trace of $\bar{R}(-1)$ is given by

$$\operatorname{tr} \bar{R}(-1) = \operatorname{tr} R(-1) - R_{n-1,n-1}(-1) - R_{n,n}(-1) = \operatorname{Ric}_p(v, v) - R_{n-1,n-1}(-1) - R_{n,n}(-1).$$

Using the orthogonality properties of the basis $(e_i)_i$, we can compute

$$\begin{aligned} R_{n-1,n-1}(-1) &= -g(R(e_{n-1}, v)v, e_n) = -g(R(v, v)v, e_n) = 0, \\ R_{n,n}(-1) &= -g(R(e_n, v)v, e_{n-1}) = -g(R(e_n, v)v, v) = 0. \end{aligned}$$

Thus $\operatorname{tr} \bar{R}(-1) = \operatorname{Ric}_p(v, v)$, yielding

$$(\det J(h-1))^{\frac{1}{n-2}} = (\det \bar{J}(h-1))^{\frac{1}{n-2}} = h - \frac{h^3}{12(n-2)} \operatorname{Ric}_p(v, v) + o(h^3).$$

Since we assumed by contradiction that $\operatorname{Ric}_p(v, v) < 0$, then the function $t \mapsto (\det J(t))^{\frac{1}{n-2}}$ is not concave in a neighborhood of -1 , contradicting that H satisfies the $\operatorname{NC}^1(n)$ condition. \square

9. STABILITY OF THE NULL-ENERGY CONDITION

In this section, we prove that the null-energy condition $\operatorname{NC}^1(N)$ is stable under C_{loc}^1 -convergence of the Lorentzian metrics and C_{loc}^0 -convergence of the weights on the volume measures. Here, by C_{loc}^1 -convergence we mean that the sequence of Lorentzian metrics, when expressed in local coordinates, converges locally uniformly, together with first derivatives.

Theorem 9.1. *Let M be a smooth manifold, $(g_j)_j$ a sequence of Lorentzian metrics and $\Phi_j : M \rightarrow \mathbb{R}$ a sequence of C^0 -functions converging to Φ , locally uniformly. Assume that $g_j \rightarrow g$ in C_{loc}^1 and that $\Phi_j \rightarrow \Phi$ locally uniformly.*

If, for every j , the triple $(M, g_j, e^{\Phi_j} \operatorname{Vol}_g)$ satisfies the null energy condition $\operatorname{NC}^1(N)$, then also $(M, g, e^{\Phi} \operatorname{Vol}_g)$ satisfies the $\operatorname{NC}^1(N)$ condition as well.

Proof. Fix, once and for all, an auxiliary Riemannian metric h . Fix a null hypersurface H for g , a null-geodesic vector field $L \in \mathfrak{X}(\mathcal{N}^g H)$ w.r.t. g , a space-like (for g) cross-section $S \subset H$, and $z \in S$. Since the $\operatorname{NC}^1(N)$ property is local, without loss of generality, we can restrict S so that it is pre-compact and assume also that S is a global cross-section. We can also assume that $\Psi_L(z, t)$ is well-defined and belongs to H , for all $z \in S$ and all $t \in (-2, 2)$.

As S is compact and space-like for g , it is clear that S is space-like for g_j , for j large enough. Let $L \in \mathfrak{X}(\mathcal{N}^g H)$ be a null-geodesic vector field (w.r.t. g). For $p \in S$, define

$$(9.1) \quad L_j(p) := \arg \min \{ h(v - L, v - L) : v \in T_p M, v \perp_{g_j} T_p S, g_j(v, v) = 0 \}.$$

Notice that $\{v \in T_p M : v \perp_{g_j} T_p S, g_j(v, v) = 0\}$ defines a pair of null lines in $T_p M$, and v is the projection (w.r.t. h) of L on such a set; since $g_j(p) \rightarrow g(p)$ and the above construction for g gives back L , it is clear that, for j large, the minimization problem (9.1) admits a unique solution and that this solution defines a null vector field L_j on S w.r.t. g_j . Using that $g_j \rightarrow g$ in C_{loc}^1 it also follows that $L_j \rightarrow L$ in C_{loc}^1 . Define

$$(9.2) \quad H_j := \{ \exp_p(tL_j) : p \in S, t \in \mathbb{R} \},$$

whenever the expression makes sense. It is clear that H_j is a null hypersurface for g_j .

Let us now discuss the convergence. Since $g_j \rightarrow g$ in C_{loc}^1 , the Christoffel symbols converge locally uniformly. Therefore, the geodesics converge in C_{loc}^0 (to see this, one can pass in coordinates and apply some standard stability theorem for ODEs, see [18, Lemma 3.1, p. 24]) and the exponential map converges in C_{loc}^0 , as well. It follows that $\Psi_{L_j}(p_j, t_j) \rightarrow \Psi_L(p, t)$, whenever $p_j \rightarrow p$ in S and $t_j \rightarrow t$ in \mathbb{R} , provided all these expressions are well-defined.

We now claim that there exists \bar{j} such that for all $j > \bar{j}$ and all $t \in (-1, 1)$, $\Psi_{L_j}(p, t)$ is well-defined. Suppose not, i.e., there exist sequences $j_k, p_k \in S, t_k \in (-1, 1)$, such that $\Psi_{L_{j_k}}(p_k, t_k)$ is not well-defined. By compactness, up to taking a not-relabeled subsequence, it holds that $p_k \rightarrow p \in S$ and $t_k \rightarrow t \in [-1, 1]$. By the assumptions we made at the beginning, we have

that $\Psi_L(p, t) = \exp_p^g(tL)$ is well-defined. Let $U \subset TM$ be a neighborhood of $\{tL : t \in [-1, 1]\}$, compact in the domain of definition of the exponential map \exp^g . Since $g_j \rightarrow g$ in C_{loc}^1 , the exponential maps convergence locally uniformly, and in particular there exists $\hat{j} > 0$ such that $U \subset \text{Dom}(\exp^{g_j})$, for all $j > \hat{j}$. Moreover, $t_k L_{j_k}(p_k) \rightarrow tL(p) \in U$, therefore $t_k L_{j_k}(p_k) \in U$ for k large enough, i.e., $\Psi_{L_{j_k}}(p_k, t_k)$ is well-defined, which is a contradiction.

As an immediate consequence of the above argument, we obtain that the functions $t \mapsto \Phi_j(\Psi_{L_j}(p, t))$ converge locally-uniformly to $t \mapsto \Phi(\Psi_L(p, t))$, for every $p \in S$.

Fix now a vector $p \in S$ and $v \in T_p M$. For each j , we can consider the Jacobi field given by the solution of the Cauchy problem

$$(9.3) \quad J''_{v,j}(t) = -R^{g_j}(L_j, J_{v,j}(t))L_j, \quad J_{v,j}(0) = v, \quad J'_{v,j}(0) = \nabla_v^{g_j} L_j.$$

It is clear that $J_{v,j}$ is a solution of the following first-order ODE (compare with (3.3))

$$(9.4) \quad J'_{v,j}(t) = \nabla_{J_{v,j}(t)}^{g_j}(L_j).$$

Since the Christoffel symbols of g_j converge in C_{loc}^0 , it follows that $J_{v,j}$ converges in C_{loc}^0 to a certain vector field J_v satisfying $J_v(0) = v$ and

$$(9.5) \quad J'_v(t) = \nabla_{J_v(t)}^g(L)$$

and therefore

$$(9.6) \quad J''_v(t) = -R^g(L_j, J_v(t))L, \quad J_v(0) = v, \quad J'_v(0) = \nabla_v L.$$

If we now fix a basis $(e_i)_i$ for $T_p M$, we deduce that $g_j(J_{e_i,j}(t), J_{e_k,j}(t)) \rightarrow g(J_{e_i}(t), J_{e_k}(t))$, locally uniformly. Since $W_L^{g_j}(p, t)$ is the determinant of the matrix $(g_j(J_{e_i,j}(t), J_{e_k,j}(t)))_{i,k}$, we deduce that $W_L^{g_j}(p, t)$ converge locally-uniformly to $W_L^g(p, t)$.

Finally, recalling that $a_j(t) = W_{L_j}^{g_j}(p, t) + \Phi_j(g_{L_j}(p, t))$, we can pass to the limit in the $\text{NC}^1(N)$ condition (5.12), concluding the proof. \square

10. APPLICATIONS

10.1. The weighted light-cone theorem. The goal of this section is to extend the light-cone theorem in the sharp monotone form (see [9, 16]) to weighted Lorentzian manifolds (see Theorem 10.1), and to address the rigidity question (see Theorem 10.2). The proofs build on the tools developed in the paper.

Let (M, g) be a Lorentzian manifold, fix $p \in M$ and, given a unit-length, future-directed, time-like vector $v \in T_p M$, define the co-dimension two subset $S_1^+(0) \subset T_p M$

$$S_1^+(0) := \{\ell \in T_p M : g(\ell, \ell) = 0, g(v, \ell) = -1\}.$$

Given $\ell \in S_1^+(0)$, one denotes by $\gamma_\ell : [0, \beta_\ell) \rightarrow M$ the unique future-directed affinely parameterized maximising geodesic (future maximally defined) with $\gamma_\ell(0) = p$ and $\gamma'_\ell(0) = \ell$. Finally,

$$S_s := \{\gamma_\ell(s) : \ell \in S_1^+(0) \text{ and } \beta_\ell > s\}.$$

The set S_s is a smooth co-dimension 2 submanifold of M inheriting a Riemannian metric σ_s , and it is contained inside the future light-cone at p , see Section 8.2, that we denote by $H(p)$.

Theorem 10.1 (Weighted light-cone theorem). *Let (M^n, g) be a Lorentzian manifold with g of class C^2 and fix $p \in M$. Consider $\Phi : H(p) \rightarrow \mathbb{R}$ a C^0 weight function and assume $(M, g, H(p), \Phi)$ satisfies the null energy condition $\text{NC}^e(N)$. Then the map*

$$(10.1) \quad s \mapsto \frac{1}{\omega_{N-2} s^{N-1}} \int_{S_s} e^{\Phi(z)} \text{Vol}_{\sigma_s}(dz)$$

is non-increasing and converges to $e^{\Phi(p)}$ as $s \rightarrow 0^+$. In particular,

$$(10.2) \quad \int_{S_s} e^{\Phi(z)} \text{Vol}_{\sigma_s}(dz) \leq e^{\Phi(p)} \omega_{N-2} s^{N-2}, \quad \text{for all } s > 0.$$

Proof. Choosing as null-geodesic vector field L the extension by parallel transport of ℓ along all $H(p)$ and following Proposition 3.8 (see in particular (3.18)), we have for each $0 < s < s'$

$$\text{Vol}_{\sigma_{s'}} = \Psi_L(\cdot, s' - s)_\#(\det J_L(\cdot, s' - s) \text{Vol}_{\sigma_s}),$$

giving

$$\int_{S_{s'}} e^{\Phi(z)} \text{Vol}_{\sigma_{s'}}(dz) = \int_{S_s} e^{\Phi(\Psi_L(z, s' - s))} \det J_L(z, s' - s) \text{Vol}_{\sigma_s}(dz).$$

Posing as before for each $z \in S_s$, $a_z(t) := \Phi(\Psi_L(z, t)) + W_L(z, t)$, the $\text{NC}^1(N)$ condition states (adopting the terminology of [7, Lemma A.9]) that the map

$$t \mapsto h(t) := e^{\Phi(\Psi_L(z, t))} \det J_L(z, t)$$

is a $\text{CD}(0, N-1)$ -density, i.e., the function $h^{1/(N-2)}$ is concave where it is positive. We use the convention that $h(t) = 0$ for all $t \geq \beta_{\ell(z)}$, where $\beta_{\ell(z)}$ is the supremum of the maximal interval of definition of the maximal null geodesic $\gamma_{\ell(z)}$, with $\gamma_{\ell(z)}(s) = z$. Then, for each $0 < t_1 \leq t_2$, we have $h(t_1) \geq (t_1/t_2)^{N-2} h(t_2)$. It follows that, for $0 < s < s_1 < s_2$ and for any measurable set $A \subset S_{s_2}$, it holds

$$\begin{aligned} \int_A e^{\Phi(z)} \text{Vol}_{\sigma_{s_2}}(dz) &= \int_{\Psi_L(\cdot, s_2 - s)^{-1}(A)} e^{\Phi(\Psi_L(z, s_2 - s))} \det J_L(z, s_2 - s) \text{Vol}_{\sigma_s}(dz) \\ &\leq \left(\frac{s_2 - s}{s_1 - s} \right)^{N-2} \int_{\Psi_L(\cdot, s_2 - s)^{-1}(A)} e^{\Phi(\Psi_L(z, s_1 - s))} \det J_L(z, s_1 - s) \text{Vol}_{\sigma_s}(dz) \\ &= \left(\frac{s_2 - s}{s_1 - s} \right)^{N-2} \int_{\Psi_L(\cdot, s_2 - s_1)^{-1}(A)} e^{\Phi(z)} \text{Vol}_{\sigma_{s_1}}(dz). \end{aligned}$$

Sending $s \rightarrow 0^+$ yields the following monotonicity property:

$$(10.3) \quad e^{\Phi(\cdot)} \text{Vol}_{\sigma_{s_2}} \leq \left(\frac{s_2}{s_1} \right)^{N-2} \Psi_L(\cdot, s_2 - s_1)_\#(e^{\Phi(\cdot)} \text{Vol}_{\sigma_{s_1}}), \quad \text{for all } 0 < s_1 < s_2.$$

Evaluating (10.3) over S_{s_2} gives (10.1). The convergence to $e^{\Phi(p)}$ follows by the continuity of Φ , concluding the proof. \square

Notice that $\omega_{N-2} s^{N-2}$ equal the area of the canonical slices of the null-cone in flat Minkowski. Thus, Theorem 10.1 can be read as a comparison result between the area of cross-sections in a null-cone satisfying $\text{NC}^e(N)$ and the canonical cross-sections in a null-cone in flat Minkowski.

10.1.1. Rigidity. We next investigate the rigidity in the weighted light-cone theorem. We will show that, if equality is attained in (10.2) for some $s_0 > 0$, then one has several rigidity properties: in the unweighted case, we obtain a metric rigidity result, stating that $H(p)$ is isometric to the future null-cone in Minkowski; in the weighted case, we obtain that the weighted Ricci tensor vanishes in the $\frac{\partial}{\partial s}$ directions and, under the additional assumption that $H(p)$ is isometric to the future null-cone in Minkowski, we prove a measure rigidity result, stating that $e^{\Phi(z, s)} = s^{N-n}$.

Theorem 10.2 (Rigidity in the weighted light-cone theorem). *Let (M^n, g) be a Lorentzian manifold with g of class C^2 and fix $p \in M$. Consider $\Phi : H(p) \rightarrow \mathbb{R}$ a C^0 weight function and assume $(M, g, H(p), \Phi)$ satisfies the null energy condition $\text{NC}^e(N)$.*

Assume that there exists $s_0 > 0$ such that

$$(10.4) \quad \int_{S_{s_0}} e^{\Phi(z)} \text{Vol}_{\sigma_{s_0}}(dz) = e^{\Phi(p)} \omega_{N-2} s_0^{N-2}.$$

Then:

- Unweighted case, i.e., $\Phi \equiv 0$ and $N = n$. The exponential map based at p induces an isometry between $(H(p) \cap J^-(S_{s_0}), g|_{H(p)})$ and the future null-cone in n -dimensional Minkowski space-time, i.e.

$$(10.5) \quad g|_{H(p)}(s, z) = \omega_{N-2}^{1/(N-2)} s^2 g_{S^{n-2}}, \quad \text{for all } s \in (0, s_0)$$

and the null vector $\frac{\partial}{\partial s}$ lies in the kernel of both sides.

- Weighted case. The identity (10.4) is valid for all $s \in (0, s_0]$ and $\text{Ric}_{(z,s)}^{g,\Phi,N}(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}) = 0$ for all $s \in (0, s_0)$.

Under the additional assumption that the identity (10.5) holds, then $e^{\Phi(z,s)} = s^{N-n}$, for all $s \in (0, s_0)$.

Proof. From Theorem 10.1 it follows that the identity (10.4) is valid for all $s \in (0, s_0)$:

$$(10.6) \quad \int_{S_s} e^{\Phi(z)} \text{Vol}_{\sigma_s}(dz) = e^{\Phi(p)} \omega_{N-2} s^{N-2}.$$

Step 1 Proof that $r_0 \geq s_0$, where r_0 is the null injectivity radius at p .

Assume by contradiction that $r_0 < s_0$. In this case, (10.6) gives that

$$e^{\Phi} \text{Vol}_{\sigma_s} = \left(\frac{s}{r_0}\right)^{N-2} \Psi_L(\cdot, s - r_0)_{\sharp} (e^{\Phi} \text{Vol}_{\sigma_{r_0}}), \quad \text{for all } 0 < s < r_0.$$

From Proposition 3.8, we infer that, for any measurable subset $B \subset S_{r_0}$:

$$\begin{aligned} \int_B e^{\Phi(\Psi_L(z, s-r_0))} \det J_L(z, s - r_0) \text{Vol}_{\sigma_{r_0}}(dz) &= \int_{\Psi_L(B, s-r_0)} e^{\Phi(z)} \text{Vol}_{\sigma_s}(dz) \\ &= \left(\frac{s}{r_0}\right)^{N-2} \int_B e^{\Phi(z)} \text{Vol}_{\sigma_{r_0}}(dz), \end{aligned}$$

implying that, for all $z \in S_{r_0}$ and $s \in (0, r_0]$:

$$(10.7) \quad e^{\Phi(\Psi_L(z, s-r_0))} \det J_L(z, s - r_0) = \left(\frac{s}{r_0}\right)^{N-2} e^{\Phi(z)}.$$

Moreover, since the $\text{CD}(0, N-1)$ condition yields that the function

$$s \mapsto (e^{\Phi(\Psi_L(z, s))} \det J_L(z, s))^{1/(N-2)}$$

is concave on its domain of definition (see above, in the proof of Theorem 10.1), we infer that

$$(10.8) \quad e^{\Phi(\Psi_L(z, s-r_0))} \det J_L(z, s - r_0) \leq \left(\frac{s}{r_0}\right)^{N-2} e^{\Phi(z)}, \quad \forall s > 0, \forall z \in S_{r_0},$$

with the convention that the l.h.s. is null, if it is not defined (i.e., when $\Psi_L(z, s - r_0)$ is not defined). We can thus compute (the first inequality is due to the fact that we are integrating on a set containing the one in (3.18))

$$\begin{aligned} e^{\Phi(p)} \omega_{N-2} s_0^{N-2} &\stackrel{(10.4)}{=} \int_{S_{s_0}} e^{\Phi(z)} \text{Vol}_{\sigma_{s_0}}(dz) \stackrel{(3.18)}{\leq} \int_{S_{r_0}} e^{\Phi(\Psi_L(z, s_0-r_0))} \det J_L(z, s_0 - r_0) \text{Vol}_{\sigma_{r_0}}(dz) \\ &\stackrel{(10.8)}{\leq} \left(\frac{s_0}{r_0}\right)^{N-2} \int_{S_{r_0}} e^{\Phi(z)} \text{Vol}_{\sigma_{r_0}}(dz) = \left(\frac{s_0}{r_0}\right)^{N-2} e^{\Phi(p)} \omega_{N-2} r_0^{N-2}. \end{aligned}$$

It follows that inequality (10.8) is saturated for $s \in (0, s_0]$:

$$(10.9) \quad e^{\Phi(\Psi_L(z, s-r_0))} \det J_L(z, s-r_0) = \left(\frac{s}{r_0}\right)^{N-2} e^{\Phi(z)}, \quad \forall s \in (0, s_0], \forall z \in S_{r_0}.$$

In particular, $\Psi_L(z, s_0 - r_0)$ is well-defined, proving that the null injectivity radius is at least $s_0 > r_0$, a contradiction.

Step 2 Conclusion.

From Proposition 4.4, we know that, for every $z \in S_{s_0}$, the function $a(t) = \Phi(\Psi_L(z, t)) + \log \det J_L(z, t)$ is twice differentiable and it satisfies the following differential inequality

$$a''(t) + \frac{(a'(t))^2}{N-2} \leq 0, \quad \text{in the sense of distributions.}$$

On the other hand, (10.7) implies that $a''(t) + (a'(t))^2/(N-2) = 0$. Expanding the calculations and denoting by $W_L(t) = \log \det J_L(z, t)$, we get:

$$(10.10) \quad \begin{aligned} -\frac{(a'(t))^2}{N-2} = a''(t) &= W_L''(t) + \text{Hess}(\Phi)_{\Psi_L(z,t)}[L, L] \\ &\leq -\frac{(W_L'(t))^2}{n-2} - \text{Ric}_{\Psi_L(z,t)}(L, L) + \text{Hess}(\Phi)_{\Psi_L(z,t)}[L, L] \end{aligned}$$

$$(10.11) \quad \begin{aligned} &= -\frac{(W_L'(t))^2}{n-2} - \text{Ric}_{\Psi_L(z,t)}^{g,\Phi,N}(L, L) - \frac{(g(\nabla\Phi, L)_{\Psi_L(z,t)})^2}{N-n} \\ &\leq -\frac{(W_L'(t) + g(\nabla\Phi, L)_{\Psi_L(z,t)})^2}{N-2} = -\frac{(a'(t))^2}{N-2}. \end{aligned}$$

We will exploit the two inequalities (10.10)–(10.11) turning into identities. Firstly, from Proposition 4.3 and equality in (10.10), we deduce

$$W_L''(z, t) + \frac{(W_L'(z, t))^2}{n-2} = -\text{Ric}_{\Psi_L(z,t)}(L, L).$$

Inspecting the proof of Proposition 4.3, since $W_L'(z, t) := \text{tr}(J(t)^{-1}J'(t))$, for the matrix $U(t) := J(t)^{-1}J'(t)$ we obtain $\text{tr}(U^2) = \text{tr}(U)^2/n-2$. Equality in Cauchy–Schwartz inequality implies that U has to be a multiple of the identity matrix, giving that

$$(10.12) \quad J_L'(z, t) = \alpha(z, t)J_L(z, t).$$

Since $J_L(z, 0) = \text{Id}$, we infer that

$$(10.13) \quad J_L(z, t) = \lambda(z, t) \text{Id},$$

for some positive function $\lambda(z, t)$. From the equality in (10.11), we deduce that

$$(10.14) \quad \alpha(z, t) = \frac{W_L'(z, t)}{n-2} = \frac{g(\nabla\Phi, L)_{\Psi_L(z,t)}}{N-n}.$$

Finally,

$$(10.15) \quad \text{Ric}_{\Psi_L(z,t)}^{g,\Phi,N}(L, L) = 0, \quad \text{for all } z \in S_{s_0} \text{ and } t \in (-s_0, 0].$$

Recall that $J_L(z, t)$ is the matrix of Jacobi fields obtained as follows. Fix $p \in M$ and denote by $\text{Exp}_p^g : \mathcal{U} \rightarrow M$ the exponential map based at p , which is a diffeomorphism in a neighbourhood $\mathcal{U} \subset T_pM$ of the origin $0 \in T_pM$. Up to scaling, we can assume that \mathcal{U} contains the coordinate ball of radius 2. The coordinates on T_pM are chosen so that $g_{ij}(p)$ is the Minkowski Lorentzian metric.

Let $C_{\mathbb{M}} \subset J^+(0)$ be the future null-cone in Minkowski, identified with the future null-cone in T_pM . We endowed $C_{\mathbb{M}} \setminus \{0\}$ with coordinates (z, t) with $z \in S^{n-2}$ and $t \in (0, \infty)$. For each $z_0 \in S^{n-2}$ we fixed an orthonormal basis $\{e_1(z_0), \dots, e_{n-2}(z_0)\}$ of $(T_{z_0}S^{n-2}, g_{S^{n-2}})$. Each $z_0 \in S^{n-2}$ identifies a null vector $L = L(z_0)$. Notice that $\Psi_L(z_0, t-1) = \text{Exp}_p^g(tz_0)$.

The matrix of Jacobi fields $J = J_L(z_0, t)$ was constructed in such a way that the i -th column J_i satisfies $J_i(z_0, 0) = e_i(z, 1)$. and $J'_i(z_0, 0) = \nabla_{e_i} L$. Such a choice of Jacobi fields yields

$$(10.16) \quad J_i(z_0, t) = \frac{\partial}{\partial z^i} \Big|_{(z_0, t)} \text{Exp}_p^g, \quad \text{for all } i = 1, \dots, n-2,$$

where (z^1, \dots, z^{n-2}) are local coordinates on S^{n-2} such that $\frac{\partial}{\partial z^i} \Big|_{z_0} = e_i(z_0)$.

The combination of Equations (10.13) and (10.16) implies that

$$(10.17) \quad g \left(\frac{\partial}{\partial z^i} \Big|_{(z_0, t)} \text{Exp}_p^g, \frac{\partial}{\partial z^j} \Big|_{(z_0, t)} \text{Exp}_p^g \right) = g(J_i(z_0, t), J_j(z_0, t)) = \lambda(z_0, t)^2 \delta_{ij}.$$

Since $z_0 \in S^{n-2}$ was arbitrary it follows that, if we endow

$$(10.18) \quad H(p) \setminus \{p\} = \text{Exp}_p^g((C_{\mathbb{M}} \setminus \{0\}) \cap \mathcal{U})$$

with the coordinates (z, t) of $C_{\mathbb{M}} \setminus \{0\}$, the metric takes the form

$$(10.19) \quad g|_{H(p)} = \lambda(z, t)^2 g_{S^{n-2}},$$

where $L = L(z)$ is in the kernel of both sides. Notice that, by the smoothness of Exp_p^g at 0, it holds that

$$(10.20) \quad \lim_{t \rightarrow 0^+} \omega_{N-2}^{1/(2(N-2))} t \frac{1}{\lambda(z, t)} = 1.$$

We next split the discussion in two cases: the unweighted and the weighted ones.

The unweighted case: This corresponds to the case when $\Phi \equiv 0$ and $N = n$. From (10.7) and (10.13), we infer that λ is independent of the z variable, i.e., $\lambda = \lambda(t)$ is S^{n-2} -spherically symmetric. The identity (10.7) implies that $t \mapsto \lambda(t)$ is linear. Recalling (10.20), we conclude that $\lambda(t) = \omega_{N-2}^{\frac{1}{2(N-2)}} t$ and thus $g|_{H(p)} = g_{C_{\mathbb{M}}}$, i.e. Exp_p^g induces an isometry between (suitable neighbourhoods of the tips of) the null-cone in Minkowski $C_{\mathbb{M}}$ and $H(p)$.

The weighted case: We already proved (see (10.19)) that the metric on $H(p)$ is a warped product of the form $g|_{H(p)} = \lambda(z, t)^2 g_{S^{n-2}}$, with the warping function λ satisfying the asymptotics (10.20). The warping function λ and the weight function Φ are related via the null-Ricci flat condition (10.15) and the identity (10.14). Without further assumptions this is the best one can say, indeed Φ and λ need not be S^{n-2} -spherically symmetric and just compensate each other in the various identities.

If we assume in addition that $(H(p), g|_{H(p)})$ is isometric to the null-cone in Minkowski $C_{\mathbb{M}} \cap \mathcal{U}$ via the exponential map, then $\lambda(t) = \omega_{N-2}^{1/(2(N-2))} t$. From (10.13), it follows that J_L is independent of $z \in S_{s_0}$ and thus the identity (10.7) yields

$$(10.21) \quad e^{\Phi(\Psi_L(z, s-s_0)) - \Phi(z)} = \frac{1}{\det J_L(s-s_0)} \left(\frac{s}{s_0} \right)^{N-2} \quad \text{is independent of } z \in S_{s_0}, \text{ for all } s.$$

Using in (10.21) that

$$\lim_{s \rightarrow 0^+} e^{\Phi(\Psi_L(z, s-s_0))} = e^{\Phi(p)} \quad \text{is independent of } z \in S_{s_0},$$

we get that $\Phi|_{S_{s_0}}$ is constant. Using again (10.21), we deduce that Φ is S^{n-2} -spherically symmetric, i.e., $\Phi = \Phi(t)$. Plugging the expression of λ into (10.12) and (10.13) gives that $\alpha(t) = \frac{1}{t}$; inserting the expression of α into (10.14) finally implies that $e^{\Phi(t)} = t^{N-n}$, as claimed. \square

10.2. The weighted Hawking's Area Theorem. As a second application of the theory developed in the paper, in this section we provide an extension of the celebrated Hawking's Area Theorem to a setting of low regularity.

Before stating the result, recall that a null hypersurface $H \subset M$ is said to be *future geodesically complete* if every future-directed null-geodesic $\gamma : [a, b] \rightarrow H \subset M$ can be extended to the half-line $[a, \infty)$ as a future-directed null-geodesic.

Theorem 10.3. *Let (M, g) be a Lorentzian manifold of dimension n , $H \subset M$ be a null hypersurface, and $S_1, S_2 \subset H$ be two space-like cross-sections for H . Assume g, H, S_1, S_2 to be of class C^2 . Assume that $S_1 \subset J^-(S_2)$ and that S_2 is a global cross-section for H . Let $\Phi : H \rightarrow \mathbb{R}$ be a C^0 function and let $\mathbf{m} = e^\Phi \text{Vol}_L$ be the corresponding weighted measure on H . Assume that H is future geodesically complete and that (M, g, H, Φ) satisfies $\text{NC}^1(N)$.*

Then it holds that

$$(10.22) \quad \int_{S_1} e^\Phi d\mathcal{H}^{n-2} \leq \int_{S_2} e^\Phi d\mathcal{H}^{n-2}.$$

Rigidity. *Assume there exists S_1, S_2 with $S_1 \cap S_2 = \emptyset$ satisfying equality in (10.22). Then there exists a null-geodesic vector field along H such that the flow map $z \mapsto \Psi_L(z, t)$ is measure preserving in following sense: for every $t > 0$, the section $(S_1, e^\Phi \mathcal{H}^{n-2})$ and the section $(\Psi_L(S_1, t), e^\Phi \mathcal{H}^{n-2})$ are isomorphic as measure spaces, i.e.,*

$$\Psi_L(\cdot, t)_\#(e^\Phi \mathcal{H}^{n-2} \llcorner S_1) = e^\Phi \mathcal{H}^{n-2} \llcorner \Psi_L(S_1, t).$$

Moreover, $\text{Ric}_{\Psi_L(z, t)}^{g, \Phi, N}(L, L) = 0$, for all $t > 0$ and $z \in S_1$.

In the unweighted case, i.e., when $\Phi \equiv 0$, $N = n$, the above measure rigidity shall be promoted to the following metric rigidity. The map $\Psi_L : S_1 \times (0, \infty) \rightarrow (J^+(S_1) \cap H) \setminus S_1$ defines an isometry between the (degenerate) product metric $g_{S_1} \oplus (0 dt \otimes dt)$ and g_H . In particular, $(J^+(S_1) \cap H) \setminus S_1, g_H$ splits metrically, and L is in the kernel of the degenerate non-negative definite $(0, 2)$ -tensor field g_H .

Remark 10.4 (Physical interpretation). The area of a black-hole horizon is classically interpreted as a measure of its entropy. The Hawking Area Theorem then is classically interpreted as a second law of thermodynamics: namely, the entropy does not decrease in time, provided the null energy condition holds. The rigidity part in Theorem 10.3 might be interpreted as follows: if the entropy of the event horizons does not increase in time, then the metric on the corresponding null hypersurface is static, in the sense that it is isometric to a (degenerate) product $g_S \oplus (0 dt \otimes dt)$, where g_S is the Riemannian metric on one of the $(n - 2)$ -dimensional event horizons, whose entropy do not change.

Proof. Let L be a global null-geodesic vector field. Since S_2 is a global cross-section, then $\Omega_{S_2} = H_{S_2} = H$. As $S_1 \subset J^-(S_2)$, there exists a map

$$t_L : \text{Dom}(t_L) \subset S_2 \rightarrow (-\infty, 0] \quad \text{such that} \quad \Psi_L(z, t_L(z)) \in S_1.$$

This means that S_1 is a ‘‘graph surface’’, w.r.t. S_2 ; then we can apply Proposition 3.8, obtaining

$$(10.23) \quad \int_{S_1} \varphi(z) \mathcal{H}^{n-2}(dz) = \int_{\text{Dom}(t_L)} \varphi(\Psi_L(z, t_L(z))) e^{W_L(z, t_L(z))} \mathcal{H}^{n-2}(dz), \quad \forall \varphi \in C_c^0(S_1).$$

Since we can approximate e^Φ monotonically by functions with compact support, the monotone convergence theorem yields

$$\begin{aligned} \int_{S_1} e^{\Phi(z)} \mathcal{H}^{n-2}(dz) &= \int_{\text{Dom}(t_L)} e^{\Phi(\Psi_L(z, t_L(z))) + W_L(z, t_L(z))} \mathcal{H}^{n-2}(dz) \\ &= \int_{\text{Dom}(t_L)} e^{a_z(t_L(z))} \mathcal{H}^{n-2}(dz), 1 \end{aligned}$$

where $a_z(t) := \Phi(\Psi_L(z, t)) + W_L(z, t)$. The $\text{NC}^1(N)$ condition states that $a_z'' + (a_z')^2/(N-2) \leq 0$ or, equivalently, that $t \mapsto e^{a_z(t)/(N-2)}$ is concave. Since H is future geodesically complete, $a_z(t)$ is well-defined for all $t > 0$. We deduce that $t \mapsto e^{a_z(t)/(N-2)}$ is non-decreasing, thus $t \mapsto e^{a_z(t)}$ is non-decreasing. Since t_L is non-positive, we can directly compute (recall that $W_L(z, 0) = 0$)

$$\begin{aligned} \int_{S_1} e^{\Phi(z)} \mathcal{H}^{n-2}(dz) &= \int_{\text{Dom}(t_L)} e^{a_z(t_L(z))} \mathcal{H}^{n-2}(dz) \leq \int_{\text{Dom}(t_L)} e^{a_z(0)} \mathcal{H}^{n-2}(dz) \\ &\leq \int_{S_2} e^{a_z(0)} \mathcal{H}^{n-2}(dz) = \int_{S_2} e^{\Phi(z) + W_L(z, 0)} \mathcal{H}^{n-2}(dz) \\ &= \int_{S_2} e^{\Phi(z)} \mathcal{H}^{n-2}(dz). \end{aligned}$$

This completes the proof of (10.22).

Proof of the rigidity. Assume there exists S_1, S_2 with $S_1 \cap S_2 = \emptyset$. From the above arguments it follows that, for every $z \in S_1$, there exists a non-empty open interval where the function $t \mapsto e^{a_z(t)/(N-2)}$ is constant. Since such a function is also concave and non-negative on an half-line, it must be constant all the way. It follows that the flow map $\Psi_L(\cdot, t)$ is measure preserving, in the sense that $\Psi_L(\cdot, t)_\#(e^\Phi \mathcal{H}^{n-2} \llcorner S_1) = e^\Phi \mathcal{H}^{n-2} \llcorner \Psi_L(S_1, t)$, and that $\text{Ric}_{\Psi_L(z, t)}^{g, \Phi, N}(L, L) = 0$, for all $t > 0$ and $z \in S_1$.

In the *unweighted* case, we claim that $\Psi_L : S_1 \times (0, \infty) \rightarrow (J^+(S_1) \cap H) \setminus S_1$ defines an isometry between the (degenerate) product metric $g_{S_1} \oplus (0 dt \otimes dt)$ and g_H . To this aim, using that $\Phi \equiv 0$, we have that

$$(10.24) \quad a_z(t) = \log \det(J_L(z, t)) = W_L(z, t) \text{ is constant in the variable } t.$$

Inspecting the proof of Proposition 4.3, since $0 = W_L'(z, t) := \text{tr}(J_L(t)^{-1} J_L'(t))$, for the matrix $U(t) := J_L(t)^{-1} J_L'(t)$ we obtain $\text{tr}(U^2) = \text{tr}(U)^2/n-2$. Equality in Cauchy-Schwartz inequality implies that U has to be a multiple of the identity, giving that $J_L'(z, t) = \alpha(z, t) J_L(z, t)$. Since $J_L(z, 0) = \text{Id}$, we infer that $J_L(z, t) = \lambda(z, t) \text{Id}$, for some positive function $\lambda(z, t)$. Recalling (10.24), we obtain that $\lambda(z, t)$ does not depend on t . Since $J_L(z, 0) = \text{Id}$, we conclude that

$$(10.25) \quad J_L(z, t) = \text{Id}, \quad \text{for all } t > 0.$$

Recall that the matrix of Jacobi fields $J_L(z, t)$ was obtained as follows. For each $z_0 \in S_1$, we fixed an orthonormal basis $\{e_1(z_0), \dots, e_{n-2}(z_0)\}$ of $(T_{z_0} S_1, g_{S_1})$. The matrix of Jacobi fields $J = J_L(z_0, t)$ was constructed such that the i -th column J_i satisfies $J_i(z_0, 0) = e_i(z_0)$ and $J_i'(z_0, 0) = \nabla_{e_i} L$. Recall also that, by construction, L is a null-geodesic vector field of g along H , i.e., its flow lines $t \mapsto \Psi_L(z, t)$ are null-geodesics of g along H , for each $z \in S_1$. Such a choice of Jacobi fields yields

$$(10.26) \quad J_i(z_0, t) = \left. \frac{\partial}{\partial z^i} \right|_{(z_0, t)} \Psi_L(z_0, t), \quad \text{for all } i = 1, \dots, n-2,$$

where (z^1, \dots, z^{n-2}) are local coordinates on S_1 , such that $\left. \frac{\partial}{\partial z^i} \right|_{z_0} = e_i(z_0)$.

The combination of (10.25) and (10.26) implies that

$$(10.27) \quad g \left(\left. \frac{\partial}{\partial z^i} \right|_{(z_0, t)} \Psi_L, \left. \frac{\partial}{\partial z^j} \right|_{(z_0, t)} \Psi_L \right) = g(J_i(z, t), J_j(z, t)) = \delta_{ij}.$$

Since (10.27) holds for all $z_0 \in S_1$, it follows that $\Psi_L(\cdot, t)$ is an isometry from S_1 to $\Psi_L(S_1, t)$, for all $t > 0$. Recalling that L is a null-geodesic vector field and it remains orthogonal to $\Psi_L(S_1, t)$, for all $t > 0$, we conclude that $\Psi_L : S_1 \times (0, \infty) \rightarrow (J^+(S_1) \cap H) \setminus S_1$ defines an isometry between the (degenerate) product metric $g_{S_1} \oplus (0 dt \otimes dt)$ and g_H . \square

Example 10.5 ($M = \Sigma^2 \times \mathbb{M}^2$). Let us discuss an easy example attaining equality in (10.22) and thus exhibiting the isometric splitting illustrated in the rigidity part of Theorem 10.3. Let $M = \Sigma \times \mathbb{R}^2$, where Σ is a 2-dimensional closed surface and \mathbb{R}^2 is endowed with coordinates (x, t) . Let g_Σ be a Riemannian metric on Σ and $g_{\mathbb{M}_2}$ be the Minkowski metric on \mathbb{R}^2 in diagonal form $(1, -1)$. Endow M with the Lorentzian metric $g_M := g_\Sigma \oplus g_{\mathbb{M}_2}$. Let $H := \Sigma \times \{x = t\} \subset M$. It is easily seen that H is a null hypersurface, that the slices $S_1 := \Sigma \times \{t = t_1\}$ and $S_2 := \Sigma \times \{t = t_2\}$ have the same \mathcal{H}^2 measure, and that the metric induced on H is a (degenerate) product in the sense of the rigidity part in Theorem 10.3, where the null-geodesic vector field is $L = \frac{\partial}{\partial x} + \frac{\partial}{\partial t}$.

Example 10.6 (Schwarzschild event horizon). A more interesting example of null hypersurface attaining equality in (10.22) and thus exhibiting the isometric splitting illustrated in the rigidity part of Theorem 10.3 is the event horizon of Schwarzschild space-time.

Since the standard coordinates of Schwarzschild space-time are singular on the event horizon, in order to illustrate the example we use the Lemaître coordinates $(\tau, \rho, \theta, \varphi)$ in which the Schwarzschild metric takes the form

$$ds^2 = -d\tau^2 + \frac{r_S}{r} d\rho^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2), \quad \text{where } r = \left[\frac{3}{2}(\rho - \tau) \right]^{2/3} r_S^{1/3}.$$

Here r_S is the Schwarzschild radius (i.e., $r_S = 2GM/c^2$ where M is the mass of the central body). The metric induced on the event horizon $H = \{r = r_S\} = \{\frac{3}{2}(\rho - \tau) = r_S\}$ writes as $g_H = r_S^2(d\theta^2 + \sin^2 \theta d\varphi^2)$. Note that g_H is a (degenerate) product in the sense of the rigidity part in Theorem 10.3, where the null-geodesic vector field is $L = \frac{\partial}{\partial \tau} + \frac{\partial}{\partial \rho}$.

Example 10.7 (Kerr–Newmann event horizon). An even more interesting example of null hypersurface attaining equality in (10.22) is the event horizon of a Kerr–Newmann black hole. The Kerr–Newmann metric models a black hole of mass M , angular momentum $J = aM$, and charge Q . It is the most general asymptotically flat and stationary solution of the Einstein–Maxwell equations and, as such, satisfies the null energy condition. One can compute¹ that the area of the event horizon is independent of the natural time coordinate, and depends only on M, a and Q ; in particular, equality holds in (10.22). Therefore, the rigidity part in Theorem 10.3 implies that the corresponding null hypersurface is metrically a (degenerate) product.

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¹See for instance <https://physics.stackexchange.com/questions/482962/area-of-kerr-newman-event-horizon>

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