EXISTENCE AND LOCAL ASYMPTOTICS FOR A SYSTEM OF CROSS-DIFFUSION EQUATIONS WITH NONLOCAL CAHN-HILLIARD TERMS

ELISA DAVOLI, GRETA MARINO, AND JAN-FREDERIK PIETSCHMANN

Abstract. We study a nonlocal Cahn-Hilliard model for a multicomponent mixture with cross-diffusion effects and degenerate mobility. The nonlocality is described by means of a symmetric singular kernel. We define a notion of weak solution adapted to possible degeneracies and prove, as our first main result, its global-in-time existence. The proof relies on an application of the formal gradient flow structure of the system (to overcome the lack of a-priori estimates), combined with an extension of the boundedness-by-entropy method, in turn involving a careful analysis of an auxiliary variational problem. This allows to obtain solutions to an approximate, time-discrete system. Letting the time step size go to zero, we recover the desired nonlocal weak solution where, due to their low regularity, the Cahn-Hilliard terms require a special treatment. Finally, we prove convergence of solutions for this class of nonlocal Cahn-Hilliard equations to their local counterparts.

1. Introduction

In this work we study a nonlocal Cahn-Hilliard model with degenerate mobility for a multicomponent mixture where cross-diffusion effects between the different species of the system are taken into account, and where the species do separate from each other. The motivation for considering such a model stems from multiphase systems modelling isothermal phase separation of miscible entities occupying an isolated region $\Omega \subset \mathbb{R}^d$ ($d = 1, 2, 3$), cf. [21] and the references therein. We consider a cross-diffusion system that models the interactions between $n + 1$ species, $n \in \mathbb{N} \setminus \{0\}$, in which all the species do separate from the others in a nonlocal way. More precisely, let Ω be the d-dimensional flat torus of $\mathbb{R}^d$, $d = 2, 3$ (which corresponds to imposing periodic boundary conditions), and let $T > 0$ be some final time. We assume that the $n + 1$ species in the mixture occupy the spatial domain Ω and for all $i = 0, \dots, n$, we denote by $u_i(t, x)$ the volume fraction of the i^{th} species at time $t \in [0, T]$ and point $x \in \Omega$ and set $\boldsymbol{u} := (u_0, u_1, \dots, u_n)$. Given a small parameter $\varepsilon > 0$ which accounts for the radius of the nonlocal interactions, we denote by $\boldsymbol{u}_\varepsilon = (u_{\varepsilon,0}, \dots, u_{\varepsilon,n})$ the nonlocal counterpart to $\boldsymbol{u}$.

We are then interested in the existence of weak solutions to the following system of cross-diffusion equations with nonlocal Cahn-Hilliard interactions

$$\partial_t \boldsymbol{u}_\varepsilon = \operatorname{div}\left(M(\boldsymbol{u}_\varepsilon)\nabla \boldsymbol{\mu}_\varepsilon\right), \tag{1.1}$$

such that

$$0 \le u_{\varepsilon,i}(t,x) \le 1 \text{ for every } i = 0, \dots, n \text{ and } \sum_{i=0}^{n} u_{\varepsilon,i}(t,x) = 1 \text{ for a.e. } (t,x) \in [0,T] \times \Omega.$$

Here, for all $\boldsymbol{u}_\varepsilon \in \mathbb{R}^{n+1}_+$, $M(\boldsymbol{u}_\varepsilon) \in \mathbb{R}^{(n+1)\times(n+1)}$ is a degenerate mobility matrix whose precise expression is given in Section 2, while $\boldsymbol{\mu}_\varepsilon$ is the chemical potential, defined as

$$\boldsymbol{\mu}_\varepsilon = D_{\boldsymbol{u}_\varepsilon} E_{NL}(\boldsymbol{u}_\varepsilon),$$

for E_{NL} given by

$$E_{\mathrm{NL}}(\boldsymbol{u}_\varepsilon) := \sum_{i=0}^{n} \int_\Omega u_i \ln u_{\varepsilon,i} - u_{\varepsilon,i} + 1 \, dx + \frac{1}{4} \sum_{i,j=0}^{n} \int_\Omega \int_\Omega c_{ij} K_\varepsilon(x,y)(u_{\varepsilon,i}(x) - u_{\varepsilon,i}(y))(u_{\varepsilon,j}(x) - u_{\varepsilon,j}(y)) \, dxdy, \tag{1.2}$$

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where K_ε are L^1 and symmetric convolution kernels and $C = (c_{ij})_{ij} \in \mathbb{R}^{(n+1)\times(n+1)}$ is a positive semidefinite matrix, see Section 2 for their definitions. Note that the logarithmic terms in this energy functional account for diffusion while the second integral is responsible for nonlocal phase separation. In the case when $K_\varepsilon \equiv 0$, system (1.1) boils down to a multi-species degenerate cross-diffusion system with size exclusion that was studied for example in [9, 3, 5, 12]. The local counterpart to (1.1), namely system

$$\partial_t \boldsymbol{u} = \operatorname{div}\left(M(\boldsymbol{u})\nabla\boldsymbol{\mu}\right), \tag{1.3}$$

is driven by the local energy functional E_{L} given by

$$E_{\mathrm{L}}(\boldsymbol{u}) := \sum_{i=0}^{n}\int_\Omega u_i \ln u_{\varepsilon,i} - u_{\varepsilon,i} + 1\; dx + \frac{1}{2}\sum_{i,j=0}^{n}\int_\Omega c_{ij}\nabla u_i\cdot\nabla u_j\; dx. \tag{1.4}$$

Indeed, as $\varepsilon \to 0$, it formally holds that

$$\frac{1}{4}\sum_{i,j=0}^{n}\int_\Omega\int_\Omega c_{ij}K_\varepsilon(x,y)(u_i(x)-u_i(y))(u_j(x)-u_j(y))\; dxdy \to \frac{1}{2}\sum_{i,j=0}^{n}\int_\Omega c_{ij}\nabla u_i\cdot\nabla u_j\; dx.$$

The main goals of our paper are to show existence of solution to the non-local system (1.1) as well as convergence of the non-local to the local equations. Before we proceed, let us put our work into perspective with respect to previous results.

Cross-diffusion systems with size exclusion. Systems of partial differential equations with cross-diffusion have gained a lot of interest in recent years [27, 13, 14, 28, 23] and appear in many applications, for instance the modeling of population dynamics of multiple species [10] or cell sorting or chemotaxis-like applications [31, 32]. One major difficulty in the analysis of such strongly coupled systems is the lack of a priori estimates, an issue that can be overcome by the *boundedness-by entropy method* if the system has the so called *formal gradient flow* structure, see [9, 23] for a more detailed presentation.

Cross-diffusion systems with Cahn-Hilliard contributions. These kind of systems has been recently studied in [19], see also [23] for a different choice of mobility, and describes the evolution of a multi-component mixture where cross-diffusion effects between the different species are taken into account, and (in this particular case) where only one species does separate from the others. This is motivated by multiphase systems where miscible entities may coexist in one single phase, see [26, 36] for examples. Within this phase, cross-diffusion between the different species is taken into account in order to correctly account for finite size effects that may occur at high concentrations.

Nonlocal Cahn-Hilliard equations. A nonlocal model for phase separation was originally proposed in [22]. Ever since the works [29, 17], the literature on nonlocal-to-local convergence of Cahn-Hilliard models has bloomed. We refer to [16, 1, 20] for an overview of the most recent developments.

Nonlocal cross-diffusion equations. Nonlocal interaction equations are continuum models for large systems of particles where every single particle can interact not only with its immediate neighbors but also with particles far away. These equations have a wide range of applications. In biology they are used to model the collective behavior of a large number of individuals [30], in physics they are used in models describing the evolution of vortex densities in superconductors [35], and they appear in simplified inelastic interaction models for granular media [4]. In their simplest form, nonlocal interaction equations can be written as

$$\frac{\partial\sigma}{\partial t} + \operatorname{div}(\sigma r) = 0, \quad r = -\nabla W * \sigma, \tag{1.5}$$

where $\sigma(t,x)$ is the probability or a mass density of particles at time t and at location $x \in \mathbb{R}^N$, $W\colon \mathbb{R}^N \to \mathbb{R}$ is the interaction potential and $r(t,x)$ is the velocity of the particles. It is by now well understood, see for example [2], that equation (1.5) is a gradient flow of the interaction energy

$$E(\sigma) = \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} W(x-y)\; d\sigma(x)d\sigma(y)$$

with respect to the 2-Wasserstein distance. Thus, stable steady states of (1.5) are expected to be local minimizers of the interaction energy. Much less is known however in the case of multi-species systems, which received most attention only recently. With different species, the modeling leads to nonlinear degenerate cross-diffusion systems for the densities of all species, again with some nonlocal terms. First rigorous studies of stationary problems show interesting phase separation phenomena, whose dynamics seems rather unexplored so far [11, 15]. In [6] the authors study a nonlocal cross-diffusion model for two species, which can be derived from a lattice-based microscopic model with size exclusion. In order to see the inherent phase separation in their model, they rewrite it as a system of nonlocal Cahn-Hilliard equations by means of a nonlocal Laplacian defined as a negative semidefinite operator. For the complementary setting in which the diffusion is driven by a local operator but further nonlocalities arising, e.g., in the mean-field limit of stochastic PDEs in the neurosciences, we refer to the two recent contributions [24, 25].

Degenerate cross-diffusion systems with nonlocal terms. Concerning nonlinear degenerate systems with non-local terms, very little is known on the dynamics, which seems rather unexplored so far [11, 15]. In [6] the authors study a nonlocal cross-diffusion model for two species, which can be derived from a lattice-based microscopic model with size exclusion. In order to see the inherent phase separation in their model, they rewrite it as a system of nonlocal Cahn-Hilliard equations by means of a nonlocal Laplacian defined as a negative semidefinite operator.

Contribution and structure of the paper. In this article we combine ideas mainly coming from [17] and [19] and prove the existence of global weak solutions to system (1.1) with energy (1.2) and supplemented with appropriate initial- and boundary conditions and then show that this solution, in the limit $\varepsilon \to 0$, converges to a solution to the local counterpart of (1.1).

The novelty of our work is threefold.

(a) This is, to the best of our knowledge, the first attempt to combine degenerate cross-diffusion system with non-local Cahn-Hilliard contributions.
(b) We are able to treat an energy that involves non-local Cahn-Hilliard terms acting on all species. Note that this requires an appropriate definition of weak solutions and a careful analysis when performing the limit of an approximate time-discrete non-local system, as the non-local terms yield less regularity in the a-priori estimates than their local counterparts.
(c) We apply the generalzed boundedness-by-entropy method exploited in [19] to this non-local context.

This manuscript is organized as follows. In Section 2, we provide the setting of the problem, introduce our notion of weak solutions and state the main existence theorems. The proof in the non-local case is based on the introduction of a regularized time discrete approximate problem, depending on a positive time step τ, which is presented in Section 3. We derive a priori estimates and prove the existence of time-discrete iterates via a Schauder fixed point argument. Then we recover some regularity properties and exploit them to pass to the limit as the time step $\tau \to 0$ and obtain a solution to the nonlocal system (1.1). Then, in Section 4 we show that the a-priori estimates previously obtained actually do not depend on the non-local parameter ε, which gives us enough regularity to pass to the limit as $\varepsilon \to 0$ and recover a solution to the local system (1.3).

2. Setting of the problem and main results

In this section we specify our setting, provide the notions of non-local and local weak solutions, and state the main results.

2.1. **Formulation of the model.** Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be the d-dimensional flat torus (this corresponds to imposing periodic boundary conditions) and let $u_i = u_i(t, x)$, $i = 0, 1, \ldots, n$, be the volume fractions of different atomic species that occupy the volume of Ω at the time $t > 0$. Furthermore, for every $\varepsilon > 0$ we consider the following family of convolution kernels,

$$K_\varepsilon(x,y) := \frac{\rho_\varepsilon(|x-y|)}{|x-y|^2}, \tag{2.1}$$

where $(\rho_\varepsilon)_\varepsilon \subset L^1_{\mathrm{loc}}(0,+\infty)$ is a suitable sequence of C^∞-maps satisfying the following properties (see [34, 33]): $\rho_\varepsilon : \mathbb{R} \to [0,+\infty)$ is defined as $\rho_\varepsilon(r) := \frac{1}{\varepsilon^d}\rho(r/\varepsilon)$, with $\rho \in C^\infty_c(0, \mathrm{diam}(\Omega))$ such that

$$\int_0^{+\infty} \rho(r) r^{d-3}\, dr < +\infty, \quad \int_0^{+\infty} \rho(r) r^{d-1}\, dr = \frac{2}{\int_{\mathbb{S}^{d-1}} |\sigma \cdot e_1|\, d\mathcal{H}^{d-1}(\sigma)}.$$

Note that under these assumptions $K_\varepsilon \in L^1(\Omega\times\Omega)$ and $K_\varepsilon(x,y) = K_\varepsilon(y,x)$ for almost every $x, y \in \Omega$.

Set $\boldsymbol{u}_\varepsilon = \boldsymbol{u}_\varepsilon(t,x) := (u_{\varepsilon,0}, u_{\varepsilon,1}, \ldots, u_{\varepsilon,n})$. We are interested in the existence of weak solutions to the following system

$$\partial_t \boldsymbol{u}_\varepsilon = \mathrm{div}\left(M(\boldsymbol{u}_\varepsilon)\nabla\boldsymbol{\mu}_\varepsilon\right), \tag{2.2}$$

where the above expression is to be understood in the sense that $\partial_t u_{\varepsilon,i} = \mathrm{div}(M(\boldsymbol{u}_\varepsilon)\nabla\mu_\varepsilon)_i$ for $i = 0, \ldots, n$, and such that

$$0 \le u_{i,\varepsilon}(t,x) \le 1 \text{ and } \sum_{i=0}^n u_{i,\varepsilon}(t,x) = 1 \quad \text{for all } i = 0,\ldots,n \text{ and a.e. } (t,x) \in [0,T]\times\Omega. \tag{2.3}$$

Note that $M(\cdot) \in \mathbb{R}^{(n+1)\times(n+1)}$ in (2.2) is a degenerate mobility matrix with components

$$\begin{aligned} M_{ij}(\boldsymbol{u}_\varepsilon) &:= -L_{ij} u_{\varepsilon,i} u_{\varepsilon,j} && \forall\, i \neq j = 0,\ldots,n, \\ M_{ii}(\boldsymbol{u}_\varepsilon) &:= \sum_{0 \le j \neq i \le n} L_{ij} u_{\varepsilon,i} u_{\varepsilon,j} && \forall\, i = 0,\ldots,n, \end{aligned} \tag{2.4}$$

with $L_{ij} > 0$ such that $L_{ij} = L_{ji}$, for all $i \neq j = 0, \ldots, n$, while $\boldsymbol{\mu}_\varepsilon$ is the chemical potential, defined formally as

$$\boldsymbol{\mu}_\varepsilon := D_{\boldsymbol{u}_\varepsilon} E_{NL}(\boldsymbol{u}_\varepsilon),$$

where $D_{\boldsymbol{u}_\varepsilon} E_{NL}(\boldsymbol{u}_\varepsilon)$ denotes the variational derivative of the functional

$$E_{NL}(\boldsymbol{u}_\varepsilon) := \int_\Omega F(\boldsymbol{u}_\varepsilon)\, dx + \frac{1}{4}\sum_{i,j=0}^n \int_\Omega\int_\Omega c_{ij} K_\varepsilon(x,y)(u_{\varepsilon,i}(x) - u_{\varepsilon,i}(y))(u_{\varepsilon,j}(x) - u_{\varepsilon,j}(y))\, dxdy, \tag{2.5}$$

with

$$F(\boldsymbol{u}_\varepsilon) = \sum_{i=0}^n (u_{\varepsilon,i}\ln u_{\varepsilon,i} - u_{\varepsilon,i} + 1).$$

This in particular gives

$$\begin{aligned} \mu_{\varepsilon,i} = D_{u_{\varepsilon,i}} E(\boldsymbol{u}_\varepsilon) &= \ln u_{\varepsilon,i} + \sum_{j=0}^n c_{ij}\big((K_\varepsilon * 1)u_{\varepsilon,j} - K_\varepsilon * u_{\varepsilon,j}\big) \\ &=: \ln u_{\varepsilon,i} + \sum_{j=0}^n c_{ij} B_\varepsilon(u_{\varepsilon,j}), \end{aligned} \tag{2.6}$$

for all $i = 0, \ldots, n$, so that $\boldsymbol{\mu}_\varepsilon = (\mu_{\varepsilon,0}, \ldots, \mu_{\varepsilon,n})$, while the properties of the operator $B_\varepsilon(v) = (K_\varepsilon * 1)v - K_\varepsilon * v$ are as in [17, Section 2.2]. Note that we set

$$(K_\varepsilon * 1)(x) = \int_\Omega K_\varepsilon(x,y)\, dy \quad \text{as well as} \quad (K_\varepsilon * v)(x) = \int_\Omega K_\varepsilon(x,y) v(y)\, dy.$$

Under the previous assumptions on the interaction kernels, it was shown in [17, Proof of Theorem 2.2] that the following result holds true.

Lemma 2.1. *Let $(v_\varepsilon)_\varepsilon \subset H^1(0,T;(H^1)(\Omega)') \cap L^2(0,T;H^1(\Omega))$ be such that*

$$\|v_\varepsilon\|_{H^1(0,T;(H^1)(\Omega)')\cap L^2(0,T;H^1(\Omega))} + \|B_\varepsilon(v_\varepsilon)\|_{L^2(0,T;(H^1)(\Omega)')} \le C.$$

Then, up to subsequences,

$$\begin{aligned} &v_\varepsilon \to v \quad \text{weakly in } H^1(0,T;(H^1)(\Omega)') \cap L^2(0,T;H^1(\Omega)),\\ &B_\varepsilon(v_\varepsilon) \to \Delta v \quad \text{weakly in } L^2(0,T;(L^2)(\Omega)). \end{aligned}$$

Finally, the matrix $C = (c_{ij})_{i,j=0,\dots,n}$ has the following properties

(i) C is symmetric and positive semidefinite;
(ii) $c_{ii} > 0$ for all $i = 0,\dots,n$;
(iii) There holds

$$(n-1) \sup_{0\le j\ne i\le n} |c_{ij}| \ll \min_{0\le i\le n} c_{ii}. \tag{2.7}$$

Then, system (2.2) can be written in the scalar form

$$\begin{aligned} \partial_t u_{\varepsilon,i} &= \operatorname{div}\big((M(\boldsymbol{u}_\varepsilon)\nabla\boldsymbol{\mu}_\varepsilon)_i\big),\\ &= \operatorname{div}\Big(\sum_{0\le j\ne i\le n} L_{ij} u_{\varepsilon,i} u_{\varepsilon_j} \nabla(\mu_{\varepsilon,i} - \mu_{\varepsilon,j})\Big)\\ &= \operatorname{div}\Big(\sum_{0\le j\ne i\le n} L_{ij}\Big[u_{\varepsilon,j}\nabla u_{\varepsilon,i} - u_{\varepsilon,i}\nabla u_{\varepsilon,j} + u_{\varepsilon,i}u_{\varepsilon,j}\nabla\Big(\sum_{k=0}^n c_{ik}B_\varepsilon(u_{\varepsilon,k}) - \sum_{l=0}^n c_{jl}B_\varepsilon(u_{\varepsilon,j})\Big)\Big]\Big)\\ &= \operatorname{div}\Big(\sum_{0\le j\ne i\le n} L_{ij}\Big[u_{\varepsilon,j}\nabla u_{\varepsilon,i} - u_{\varepsilon,i}\nabla u_{\varepsilon,j} + u_{\varepsilon,i}u_{\varepsilon,j}\nabla(q_i(\boldsymbol{u}_\varepsilon) - q_j(\boldsymbol{u}_\varepsilon))\Big]\Big), \end{aligned}$$

for all $i = 0,\dots,n$, where

$$q_i(\boldsymbol{u}_\varepsilon) := \sum_{k=0}^n c_{ik}B_\varepsilon(u_{\varepsilon,k}) \quad \forall\, i = 0,\dots,n. \tag{2.8}$$

For all $i \ne j = 0,\dots,n$ we denote the part of the flux caused by the non-local contributions as

$$J_{\varepsilon,ij} := u_{\varepsilon,i}u_{\varepsilon,j}\nabla(q_i(\boldsymbol{u}_\varepsilon) - q_j(\boldsymbol{u}_\varepsilon)),$$

and set $J_\varepsilon := (J_{\varepsilon,ij})_{ij}$. Finally, let $\boldsymbol{u}_\varepsilon^0 = (u_{\varepsilon,0}^0,\dots,u_{\varepsilon,n}^0)$ be an initial condition such that

$$u_{\varepsilon,i}^0(x) \ge 0 \quad \text{for all } i = 0,\dots,n, \quad \sum_{i=0}^n u_{\varepsilon,i}^0(x) = 1, \quad \text{a.e. in } \Omega. \tag{2.9}$$

For every $\varepsilon > 0$, we then look for a solution $(\boldsymbol{u}_\varepsilon, J_\varepsilon)$ to the following problem

$$\begin{aligned} &\partial_t u_{\varepsilon,i} = \operatorname{div}\Big(\sum_{0\le j\ne i\le n} L_{ij}\Big[u_{\varepsilon,j}\nabla u_{\varepsilon,i} - u_{\varepsilon,i}\nabla u_{\varepsilon,j} + J_{\varepsilon,ij}\Big]\Big) && \text{in } \Omega\times(0,T),\\ &\sum_{i=0}^n u_{\varepsilon,i} = 1 && \text{in } \Omega\times(0,T),\\ &J_{\varepsilon,ij} = u_{\varepsilon,i}u_{\varepsilon,j}\nabla(q_i(\boldsymbol{u}_\varepsilon) - q_j(\boldsymbol{u}_\varepsilon)) && \text{in } \Omega\times(0,T),\\ &u_{\varepsilon,i}(\cdot\,,0) = u_{\varepsilon,i}^0 && \text{in } \Omega, \end{aligned} \tag{2.10}$$

for all $i = 0,\dots,n$. Similarly, for the local system, if $\boldsymbol{u}^0$ is an initial condition such that

$$u_i^0(x) \ge 0 \quad \text{for all } i = 0,\dots,n, \quad \sum_{i=0}^n u_i^0(x) = 1 \text{ a.e. in } \Omega,$$

by $\boldsymbol{u}$ we indicate a solution to the following local problem

$$\begin{aligned} &\partial_t u_i = \operatorname{div}\Big(\sum_{0\le j\ne i\le n} L_{ij}\Big[u_j\nabla u_i - u_i\nabla u_j + u_iu_j\nabla(\Delta u_i - \Delta u_j)\Big]\Big) && \text{in } \Omega\times(0,T),\\ &\sum_{i=0}^n u_i = 1 && \text{in } \Omega\times(0,T),\\ &u_i(\cdot\,,0) = u_i^0 && \text{in } \Omega. \end{aligned} \tag{2.11}$$

2.2. **Notions of weak solution and statement of main results.** In this subsection we state the main results of the paper. The idea is to pursue the following goals:

(1) To show that system (2.10) admits at least a weak solution $\boldsymbol{u}_\varepsilon$ in a suitable sense that will be specified below;
(2) To show that, since $(K_\varepsilon * 1)v - (K_\varepsilon * v) \to -\Delta v$ as $\varepsilon \to 0^+$, solutions to (2.10) converge to solutions to (2.11).

Definition 2.2 (Solution to the nonlocal system). *Let $\varepsilon > 0$ and $T > 0$ be fixed and let $\boldsymbol{u}_\varepsilon^0 \in H^1(\Omega)$ be an initial condition satisfying (2.9). We say that $(\boldsymbol{u}_\varepsilon, J_\varepsilon) = \big((u_{\varepsilon,i})_{i\in\{0,\dots,n\}}, (J_{\varepsilon,ij})_{ij\in\{0,\dots,n\}}\big)$ is a solution to the nonlocal system (2.10) if*

(1_{NL}) *$0 \le u_{\varepsilon,i} \le 1$ for all $i = 0,\dots,n$ and $\sum_{i=0}^n u_{\varepsilon,i} = 1$ a.e. in $\Omega\times(0,T)$;*
(2_{NL}) *$u_{\varepsilon,i} \in L^2(0,T;H^1(\Omega))$ and $\partial_t u_{\varepsilon,i} \in L^2(0,T;H^1(\Omega)')$ for all $i = 0,\dots,n$;*
(3_{NL}) *$u_{\varepsilon,i}(\cdot\,,0) = u^0_{\varepsilon,i}$ for all $i = 0,\dots,n$;*
(4_{NL}) *$J_{\varepsilon,ij} \in L^2((0,T)\times\Omega)^d$;*
(5_{NL}) *$J_{\varepsilon,ij} = u_{\varepsilon,i}u_{\varepsilon,j}\nabla(q_i(\boldsymbol{u}_\varepsilon) - q_j(\boldsymbol{u}_\varepsilon))$ in the following weak sense*

$$\int_0^T\int_\Omega J_{\varepsilon,ij}\cdot\eta\,dxdt = -\int_0^T\int_\Omega \operatorname{div}(u_{\varepsilon,i}u_{\varepsilon,j}\eta)(q_i(\boldsymbol{u}_\varepsilon) - q_j(\boldsymbol{u}_\varepsilon))\,dxdt$$

for every $\eta \in L^2(0,T;H^1(\Omega)^d)\cap L^\infty((0,T)\times\Omega;\mathbb{R}^d)$, with $\eta\cdot n = 0$ on $\partial\Omega$;
(6_{NL}) *for every $i = 0,\dots,n$ and every $\varphi_i \in L^2(0,T;H^1(\Omega))$ there holds*

$$\int_0^T \langle\partial_t u_{\varepsilon,i},\varphi_i\rangle_{H^1(\Omega)',H^1(\Omega)}\,dt = -\int_0^T\int_\Omega \sum_{0\le j\ne i\le n} L_{ij}\Big[u_{\varepsilon,j}\nabla u_{\varepsilon,i} - u_{\varepsilon,i}\nabla u_{\varepsilon,j} + J_{\varepsilon,ij}\Big]\cdot\nabla\varphi_i\,dxdt.$$

Our first result is to show global-in-time existence of such solutions.

Theorem 2.3. *Let $\boldsymbol{u}_\varepsilon^0 \in L^\infty(\Omega)^{n+1}$ be such that $u^0_{\varepsilon,i} \ge 0$ for all $i = 0,\dots,n$ and $\sum_{i=0}^n u^0_{\varepsilon,i} \le 1$. Then, there exists a solution $(\boldsymbol{u}_\varepsilon, J_\varepsilon)$ to (2.10) in the sense of Definition 2.2.*

For the proof of Theorem 2.3 we refer to Section 3 below.

Definition 2.4 (Solution to the local system). *Let $T > 0$ be fixed and let $\boldsymbol{u}^0 \in H^1(\Omega)$ be an initial condition. We say that $\boldsymbol{u} = (u_0,\dots,u_n)$ is a solution to the local system (2.11) if*

(1_L) *$0 \le u_i \le 1$ for all $i = 0,\dots,n$ and $\sum_{i=0}^n u_i = 1$ a.e. in $\Omega\times(0,T)$;*
(2_L) *$u_i \in L^2(0,T;H^2(\Omega))$, $\partial_t u_i \in L^2(0,T;H^2(\Omega)')$ as well as*

$$u_iu_j\nabla(\Delta u_i - \Delta u_j) \in L^2(0,T;L^2(\Omega))$$

for all $i,j = 0,\dots,n$;
(3_L) *$u_i(\cdot\,,0) = u_i^0$ for all $i = 0,\dots,n$;*

(4_L) *for every $i = 0, \dots, n$ and every $\varphi_i \in L^2(0, T; H^1(\Omega))$ there holds*

$$\begin{aligned} &\int_0^T \langle \partial_t u_i, \varphi_i \rangle_{H^2(\Omega)', H^2(\Omega)} \, dt \\ &= -\int_0^T \int_\Omega \sum_{0 \le j \ne i \le n} L_{ij} \Big[u_j \nabla u_i - u_i \nabla u_j + u_i u_j \nabla(\Delta u_i - \Delta u_j) \Big] \cdot \nabla \varphi_i \, dx dt. \end{aligned} \tag{2.12}$$

Our second main result concerns nonlocal-to-local convergence of solutions.

Theorem 2.5. *Let $\varepsilon > 0$, and let $\boldsymbol{u}_\varepsilon^0 \in H^1(\Omega)$ be such that $\boldsymbol{u}_\varepsilon^0$ satisfies (1_{NL}) in Definition 2.2 and $\boldsymbol{u}_\varepsilon^0 \to \boldsymbol{u}^0$ in $H^2(\Omega)'$, with $\boldsymbol{u}^0 \in H^1(\Omega)$. Let $\boldsymbol{u}_\varepsilon$ be a solution to the nonlocal system in the sense of Definition 2.2 and such that (4.1)–(4.5) are satisfied. Then, up to subsequences, there exists $\boldsymbol{u} \in L^2(0, T; H^1(\Omega)) \cap H^1(0, T; H^1(\Omega)')$, with $\boldsymbol{u}(0, x) = \boldsymbol{u}^0$ and such that*

$$u_{\varepsilon,i} \to u_i \quad \text{weakly in } L^2(0, T; H^1(\Omega)) \text{ and weakly in } H^1(0, T; H^1(\Omega)') \tag{2.13}$$

$$B_\varepsilon(u_{\varepsilon,i}) \to \Delta u_i \text{ weakly in } L^2(0, T; L^2(\Omega)) \tag{2.14}$$

and u_i is a solution to the local system in the sense of Definition 2.4.

The proof of Theorem 2.5 is the focus of Section 4.

3. Existence of nonlocal solutions

This section is devoted to the proof of Theorem 2.3 whose starting point is a, still non-linear, regularized time-discrete scheme. Existence of iterates is guaranteed by a fixed-point argument involving the chemical potentials as unknowns. Finally, using the fact that the formal gradient flow structure of the system is preserved in the discrete scheme, we obtain a-priori estimates on the iterates which are sufficient to pass to the limit as $\tau \to 0^+$ and recover the desired weak solution.

Let us emphasize the fact that a key role is played by Lemma 3.11, in which we treat the convergence of the least regular terms of our system. To obtain the desired convergence, we will need to use a careful truncation argument along with the strong convergence in $L^2(0, T; L^2(\Omega))$ as well as the continuity properties of the operator B_ε.

3.1. **Time-discrete approximation.** We start by considering the closed, convex and non-empty subset of $L^2(\Omega)^{n+1}$ given by

$$\mathcal{A} = \left\{ \boldsymbol{u} \in L^2(\Omega)^{n+1} : u_i \ge 0, \ i = 0, \dots, n, \text{ and } \sum_{i=0}^n u_i = 1 \text{ a.e. in } \Omega \right\}$$

endowed with the L^2-topology. Note that in particular if $\boldsymbol{u} \in \mathcal{A}$, then for every $i = 0, \dots, n$ it holds $0 \le u_i \le 1$ almost everywhere.

We first prove an existence result for iterates of a regularized, time-discrete approximation of (2.12).

Proposition 3.1. *Let $\tau > 0$ be a discrete time step, let $p \in \mathbb{N}$ and let $\boldsymbol{u}_\varepsilon^p \in \mathcal{A}$. Then, there exists a solution $(\boldsymbol{u}_\varepsilon^{p+1}, \boldsymbol{\mu}_\varepsilon^{p+1}) \in \mathcal{A} \times H^2(\Omega)^{n+1}$ to the following coupled system: for every $i = 0, \dots, n$ and $\varphi_i \in H^2(\Omega)$,*

$$\begin{aligned} \int_\Omega \frac{u_{\varepsilon,i}^{p+1} - u_{\varepsilon,i}^p}{\tau} \varphi_i \, dx = &-\int_\Omega \Big(\sum_{0 \le j \ne i \le n} L_{ij} u_{\varepsilon,i}^{p+1} u_{\varepsilon_j}^{p+1} \nabla(\mu_{\varepsilon,i}^{p+1} - \mu_{\varepsilon,j}^{p+1}) \Big) \cdot \nabla \varphi_i \, dx \\ &- \tau \langle \mu_{\varepsilon,i}^{p+1}, \varphi_i \rangle_{H^2(\Omega)}, \end{aligned} \tag{3.1}$$

and for every $\phi_i \in L^\infty(\Omega)$,

$$\sum_{i=0}^n \int_\Omega \Big(\ln u_{\varepsilon,i} + \sum_{k=0}^n c_{ik} B_\varepsilon(u_{\varepsilon,k}) \Big) \phi_i \, dx = \sum_{i=0}^n \int_\Omega \mu_{\varepsilon,i} \phi_i \, dx. \tag{3.2}$$

Additionally, there exists $\delta_p > 0$ *such that*

$$u^p_{\varepsilon,i} \geq \delta_p \quad \textit{for all } i = 0, \ldots, n, \ \textit{a.e. in } \Omega \times (0,T).$$

The idea of the proof is to first linearize (3.1), with respect to the unknowns $\mu_{\varepsilon,i}$, by replacing $\boldsymbol{u}^{p+1}_\varepsilon$ by a given $\tilde{\boldsymbol{u}} \in \mathcal{A}$. This gives rise to an operator

$$\mathcal{S}_1 \colon \mathcal{A} \subset L^2(\Omega)^{n+1} \to H^2(\Omega)^{n+1}, \quad \mathcal{S}_1(\tilde{\boldsymbol{u}}) = \boldsymbol{\mu},$$

which we show to be well-defined. Next, using variational methods, we establish that for every given $\boldsymbol{\mu}$ there exists a solution $\boldsymbol{u}$ to (3.2) defining an operator

$$\mathcal{S}_2 \colon H^2(\Omega)^{n+1} \to \mathcal{A}, \quad \mathcal{S}_2(\boldsymbol{\mu}) = \boldsymbol{u}.$$

A fixed point of the composed operator $\mathcal{S}_2 \circ \mathcal{S}_1$ is then a weak solution to (3.1) and its existence will be a consequence of Schauder's fixed point theorem.

We will structure the proof in several lemmas below, starting with the linearized version of (3.1).

Lemma 3.2. *Let* $\widetilde{\boldsymbol{u}} \in \mathcal{A}$. *Then, there exists a unique solution* $\boldsymbol{\mu} \in H^2(\Omega)^{n+1}$ *to*

$$\int_\Omega \frac{\widetilde{u}_i - u^p_{\varepsilon,i}}{\tau} \varphi_i \, dx = -\int_\Omega \Big(\sum_{0 \leq j \neq i \leq n} L_{ij} \widetilde{u}_i \widetilde{u}_j \nabla(\mu_i - \mu_j) \Big) \cdot \nabla \varphi_i \, dx - \tau \langle \mu_i, \varphi_i \rangle_{H^2(\Omega)} \tag{3.3}$$

for every $i = 0, \ldots, n$ *and every* $\varphi_i \in H^2(\Omega)$. *Moreover, there exists* $C_1 = C_1(n, \tau, \Omega) > 0$ *such that*

$$\|\boldsymbol{\mu}\|_{(H^2(\Omega))^{n+1}} \leq C_1. \tag{3.4}$$

Proof. Thanks to (2.4), system (3.3) can be written in matrix form as follows

$$-\frac{1}{\tau} \int_\Omega (\widetilde{\boldsymbol{u}} - \boldsymbol{u}^p_\varepsilon) \cdot \varphi \, dx = \int_\Omega \nabla \varphi \cdot M(\widetilde{\boldsymbol{u}}) \nabla \boldsymbol{\mu} \, dx + \tau \langle \varphi, \boldsymbol{\mu} \rangle_{H^2(\Omega)^{n+1}}, \tag{3.5}$$

for every $\varphi \in H^2(\Omega)^{n+1}$. Moreover,

$$0 \leq M(\widetilde{\boldsymbol{u}}) \leq (n+1)\, L\, I_{n+1} \quad \text{a.e. in } \Omega \tag{3.6}$$

in the sense of symmetric matrices, where $L = \max_{0 \leq j \neq i \leq n} L_{ij}$ and I_{n+1} is the identity matrix of $\mathbb{R}^{(n+1)\times(n+1)}$. Then, existence and uniqueness to (3.5) is a consequence of the Lax-Milgram's theorem, [8]. Choosing $\varphi = \boldsymbol{\mu}$ in (3.5) and using the semidefiniteness of M as well as Hölder inequality gives

$$\tau \|\boldsymbol{\mu}\|^2_{H^2(\Omega)^{n+1}} \leq \frac{1}{\tau} \sum_{i=0}^n \|\widetilde{u}_i - u^p_{\varepsilon,i}\|_{L^2(\Omega)} \|\mu_i\|_{L^2(\Omega)} \leq \frac{C}{\tau} \Big(\sum_{i=0}^n \|\widetilde{u}_i - u^p_{\varepsilon,i}\|_{L^2(\Omega)} \Big) \|\boldsymbol{\mu}\|_{H^2(\Omega)^{n+1}}.$$

Since $\widetilde{\boldsymbol{u}}, \boldsymbol{u}^p \in \mathcal{A}$ it follows that

$$\|\boldsymbol{\mu}\|_{(H^2(\Omega))^{n+1}} \leq \frac{C}{\tau^2} 2(n+1)|\Omega|^{1/2},$$

and therefore (3.4) is satisfied. □

This result shows that the operator $\mathcal{S}_1 \colon \mathcal{A} \subset L^\infty(\Omega)^{n+1} \to H^2(\Omega)^{n+1}$ which associates to any $\widetilde{\boldsymbol{u}} \in \mathcal{A}$ the unique solution $\boldsymbol{\mu}$ to (3.3) is well-defined. The following result guarantees its continuity and as it follows almost verbatim as in [19, Lemma 3.5], we omit its proof here. Note that the set $\mathcal{A}$ here is endowed with the L^2-topology whereas in [19, Lemma 3.5] it is considered as a subset of $L^\infty(\Omega)^{n+1}$. Nevertheless, arguing as in the proof of [19, Lemma 3.5] yields Lemma 3.3 below.

Lemma 3.3. *The map* $\mathcal{S}_1 \colon \mathcal{A} \to H^2(\Omega)^{n+1}$ *is continuous.*

As next step we now aim to recover $\boldsymbol{u} \in \mathcal{A}$ from a given $\boldsymbol{\mu} \in H^2(\Omega)^{n+1}$ by means of (3.2). We will tackle this problem by using a variational approach. To be precise, we will identify $\boldsymbol{u} \in \mathcal{A}$ as the unique solution to the minimization problem

$$\min_{\boldsymbol{w} \in \mathcal{A}} F_{\boldsymbol{\mu}}(\boldsymbol{w}), \tag{3.7}$$

where $F_{\boldsymbol{\mu}}\colon \mathcal{A} \to \mathbb{R}$ is defined by

$$F_{\boldsymbol{\mu}}(\boldsymbol{w}) = \sum_{i=0}^{n} \int_\Omega w_i \ln w_i + \sum_{k=0}^{n} c_{ik} B_\varepsilon(w_k) w_i - \mu_i w_i \, dx.$$

We first show that this problem indeed admits a unique minimum.

Lemma 3.4. *Let $\boldsymbol{\mu} \in H^2(\Omega)^{n+1}$. Then, there exists a unique solution $\boldsymbol{u}$ to (3.7).*

Proof. Existence. We use the direct methods of the Calculus of Variations. First of all, given $\boldsymbol{u} \in \mathcal{A}$ we show that $F_{\boldsymbol{\mu}}$ is bounded from below. Indeed, we use the definition of $\mathcal{A}$, the fact that $x \ln x \geq 1/e$ in $[0,1]$, a Young's inequality in the nonlocal terms and $\boldsymbol{\mu} \in H^2(\Omega)^{n+1} \subset L^\infty(\Omega)^{n+1}$ to infer

$$F_{\boldsymbol{\mu}}(\boldsymbol{w}) \geq \frac{n}{e}|\Omega| - 2n \max_{0 \leq i,k \leq n} |c_{ik}| \|K_\varepsilon\|_{L^1(\Omega^2)} - \sum_{i=0}^{n} \|\mu_i\|_{L^\infty(\Omega)} |\Omega| > -\infty.$$

Observe further that the function $\boldsymbol{w}^*$ defined as

$$w_i^* = \frac{e^{-|x|^2}}{1 + \sum_{i=0}^{n} e^{-|x|^2}} \quad \text{for every } i = 0, \ldots, n.$$

is an element of $\mathcal{A}$ and satisfies $F_{\boldsymbol{\mu}}(\boldsymbol{w}^*) < +\infty$. Thus, it holds that

$$-\infty < \inf_{\mathcal{A}} F_{\boldsymbol{\mu}} \leq F_{\boldsymbol{\mu}}(\boldsymbol{w}^*) < +\infty$$

which implies the existence of a minimizing sequence $(\boldsymbol{u}^m)_{m\in\mathbb{N}} \subset \mathcal{A}$ such that

$$\lim_{m\to\infty} F_{\boldsymbol{\mu}}(\boldsymbol{u}^m) = \inf_{\mathcal{A}} F_{\boldsymbol{\mu}}.$$

Moreover, the L^∞-bounds arising from the fact that $\boldsymbol{u}^m$ belongs to $\mathcal{A}$ imply the existence of $\boldsymbol{u} \in \mathcal{A}$ such that $u_i^m \rightharpoonup u_i$ weakly in $L^2(\Omega)$ for any $i = 0, \ldots, n$. By convexity it follows that

$$\int_\Omega u_i \ln u_i \, dx \leq \liminf_{m\to\infty} \int_\Omega u_i^m \ln u_i^m \, dx \quad \text{for all } i = 0, \ldots, n.$$

Concerning the nonlocal terms we first observe that

$$\begin{aligned}
\sum_{i,k=0}^{n} \int_\Omega c_{ik} B_\varepsilon(u_k) u_i \, dx &= \sum_{i,k=0}^{n} \int_\Omega c_{ik}((K_\varepsilon * 1)u_k - K_\varepsilon * u_k) u_i \, dx \\
&= \sum_{i,k=0}^{n} \int_\Omega \int_\Omega c_{ik} K_\varepsilon(x,y)(u_k(x) - u_k(y))(u_i(x) - u_i(y)) \, dxdy \\
&= \int_\Omega \int_\Omega K_\varepsilon(x,y)(\boldsymbol{u}(x) - \boldsymbol{u}(y))^t C(\boldsymbol{u}(x) - \boldsymbol{u}(y)) \, dxdy,
\end{aligned}$$

and in turn the positive semidefiniteness of the matrix C implies the convexity of the above term. This allows us to conclude that

$$\begin{aligned}
&\int_\Omega \int_\Omega K_\varepsilon(x,y)(\boldsymbol{u}(x) - \boldsymbol{u}(y))^t C(\boldsymbol{u}(x) - \boldsymbol{u}(y)) \, dxdy \\
&\quad \leq \liminf_{m\to\infty} \int_\Omega \int_\Omega K_\varepsilon(x,y)(\boldsymbol{u}^m(x) - \boldsymbol{u}^m(y))^t C(\boldsymbol{u}^m(x) - \boldsymbol{u}^m(y)) \, dxdy.
\end{aligned}$$

Finally, for the last term we have

$$\int_\Omega \mu_i u_i^m \, dx \to \int_\Omega \mu_i u_i \, dx \quad \text{for all } i = 0, \ldots, n.$$

Summarizing gives

$$F_{\boldsymbol{\mu}}(\boldsymbol{u}) \leq \liminf_{m\to\infty} F_{\boldsymbol{\mu}}(\boldsymbol{u}^m) = \inf_{\mathcal{A}} F_{\boldsymbol{\mu}},$$

from which we infer that $\boldsymbol{u}$ is a minimizer of $F_{\boldsymbol{\mu}}$ on $\mathcal{A}$.

Uniqueness. Let us now assume that $\boldsymbol{u}$ and $\overline{\boldsymbol{u}}$ are two solutions to (3.7), that is, they solve (3.2) for every $\phi \in L^\infty(\Omega)^{n+1}$. Taking the difference of the corresponding equations and choosing $\phi = \boldsymbol{u} - \overline{\boldsymbol{u}}$ gives

$$\sum_{i=0}^{n} \int_\Omega (\ln u_i - \ln \overline{u}_i)(u_i - \overline{u}_i) + \sum_{k=0}^{n} c_{ik}(B_\varepsilon(u_k) - B_\varepsilon(\overline{u}_k))(u_i - \overline{u}_i)\, dx = 0. \tag{3.8}$$

Since C is positive semidefinite we deduce

$$\begin{aligned}
&\sum_{i,k=0}^{n} \int_\Omega c_{ik}(B_\varepsilon(u_k) - B_\varepsilon(\overline{u}_k))(u_i - \overline{u}_i)\, dx \\
&= \sum_{i,k=0}^{n} \int_\Omega c_{ik}\big[\big((K_\varepsilon * 1)u_k - K_\varepsilon * u_k\big) - \big((K_\varepsilon * 1)\overline{u}_k - K_\varepsilon * \overline{u}_k\big)\big](u_i - \overline{u}_i)\, dx \\
&= \sum_{i,k=0}^{n} \int_\Omega \int_\Omega K_\varepsilon(x,y) c_{ik}\big[(u_k - \overline{u}_k)(x) - (u_k - \overline{u}_k)(y)\big]\big[(u_i - \overline{u}_i)(x) - (u_i - \overline{u}_i)(y)\big] dx \\
&= \int_\Omega \int_\Omega K_\varepsilon(x,y)\big((\boldsymbol{u} - \overline{\boldsymbol{u}})(x) - (\boldsymbol{u} - \overline{\boldsymbol{u}})(y)\big)^t C\big((\boldsymbol{u} - \overline{\boldsymbol{u}})(x) - (\boldsymbol{u} - \overline{\boldsymbol{u}})(y)\big) dx \geq 0,
\end{aligned} \tag{3.9}$$

which, together with the strict monotonicity of the logarithm implies that from (3.8) we have

$$u_i = \overline{u}_i \quad \text{for all } i = 0, \dots, n,$$

which, in turn, yields the assertion. □

In order to identify (3.2) as the Euler-Lagrange equation to (3.7), and to obtain that minimizers of $F_{\boldsymbol{\mu}}$ are weak solutions to (3.2), we need to be able to construct perturbations in such a way that we do not leave the set $\mathcal{A}$. This will be possible due to pointwise bounds of the minimizers which only depend on $\boldsymbol{\mu}$ through its H^2-norm as the following result shows.

Lemma 3.5. *For every $\boldsymbol{\mu} \in H^2(\Omega)^{n+1}$ there exists $\delta_{\boldsymbol{\mu}} > 0$ such that for any minimizer $\boldsymbol{u}$ to $F_{\boldsymbol{\mu}}$ in $\mathcal{A}$ there holds*

$$u_i \geq \delta_{\boldsymbol{\mu}} \quad \textit{for all } i = 0, \dots, n \textit{ and almost everywhere in } \Omega.$$

Additionally, for all $N > 0$, there exists $\delta > 0$ which only depends on N, $\max_{0 \leq i,j \leq n} |c_{ij}|$, $\|K_\varepsilon\|_{L^1(\Omega^2)}$, such that for all $\boldsymbol{\mu} \in H^2(\Omega)^{n+1}$ with $\|\boldsymbol{\mu}\|_{H^2(\Omega)^{n+1}} \leq N$ and for any minimizer $\boldsymbol{u}$ to $F_{\boldsymbol{\mu}}$ there holds

$$u_i \geq \delta \quad \textit{for all } i = 0, \dots, n \textit{ and almost everywhere in } \Omega.$$

Proof. Let $\boldsymbol{\mu} \in H^2(\Omega)^{n+1} \subset L^\infty(\Omega)^{n+1}$ and let $\boldsymbol{u}$ be a minimizer of $F_{\boldsymbol{\mu}}$ on $\mathcal{A}$. We aim to show that there exists $\delta > 0$ such that $\delta \leq u_i$ almost everywhere in Ω for all $i = 0, \dots, n$. The precise dependence of δ on the data will be specified later in the proof.
To fix the ideas, we will show the result for u_0, being understood that the same reasoning applies to the other components. We reason by contradiction and assume that the Lebesgue measure of the set $\mathcal{M}_\delta := \{x \in \Omega \,:\, u_0(x) < \delta\}$ is positive. Now, let us define

$$u_0^\delta := \max(u_0, \delta) \quad \text{as well as} \quad u_i^\delta := u_i - (u_0^\delta - u_0)\frac{u_i}{1 - u_0}, \quad i = 1, \dots, n, \tag{3.10}$$

and set $\boldsymbol{u}^\delta := (u_1^\delta, \dots, u_n^\delta)$. In (3.10), since $1 - u_0 = \sum_{j=1}^{n} u_j \geq u_i \geq 0$, the function $\frac{u_i}{1-u_0}$ is well-defined almost everywhere using the convention that $\frac{u_i}{1-u_0} = 0$ as soon as $u_i = 0$. By definition, we have $1 \geq u_0^\delta \geq 0$ and $u_0^\delta + \sum_{i=1}^{n} u_i^\delta = 1$. Furthermore, $u_i^\delta(x) = 0$ for all $x \in \Omega$ such that $u_i(x) = 0$. For all $x \in \Omega$ such that $u_i(x) > 0$, it follows that $1 - u_0(x) \geq u_i(x) > 0$ and

$$u_i^\delta(x) = u_i(x)\left(1 - \frac{u_0^\delta(x) - u_0(x)}{1 - u_0(x)}\right) \geq 0, \qquad \text{since} \quad \frac{u_0^\delta(x) - u_0(x)}{1 - u_0(x)} \leq \frac{1 - u_0(x)}{1 - u_0(x)} = 1.$$

As a consequence, $\boldsymbol{u}^\delta \in \mathcal{A}$ and $u_0^\delta = 1 - \sum_{i=1}^n u_i^\delta$. We now prove that for δ sufficiently small, $F_{\boldsymbol{\mu}}(\boldsymbol{u}^\delta) < F_{\boldsymbol{\mu}}(\boldsymbol{u})$. Using the fact that $u_i^\delta = u_i$ on $\mathcal{M}_\delta^c = \{x \in \Omega \,:\, u_0(x) \geq \delta\}$ yields

$$\begin{aligned} &F_{\boldsymbol{\mu}}(\boldsymbol{u}^\delta) - F_{\boldsymbol{\mu}}(\boldsymbol{u}) \\ &\leq \sum_{i=0}^n \int_{\mathcal{M}_\delta} \big(u_i^\delta \ln u_i^\delta - u_i \ln u_i\big) + \sum_{k=0}^n c_{ik}\big(B_\varepsilon(u_k^\delta)u_i^\delta - B_\varepsilon(u_k)u_i\big) - \mu_i(u_i^\delta - u_i)\, dx. \end{aligned} \tag{3.11}$$

For the nonlocal terms, we find

$$\begin{aligned} &\sum_{i,k=0}^n \int_{\mathcal{M}_\delta} c_{ik}\big(B_\varepsilon(u_k^\delta)u_i^\delta - B_\varepsilon(u_k)u_i\big)\, dx \\ &= \int_{\mathcal{M}_\delta} c_{00}\big(B_\varepsilon(u_0^\delta)u_0^\delta - B_\varepsilon(u_0)u_0\big)\, dx + \sum_{k=1}^n \int_{\mathcal{M}_\delta} c_{0k}\big(B_\varepsilon(u_k^\delta)u_0^\delta - B_\varepsilon(u_k)u_0\big)\, dx \\ &\quad + \sum_{i=1}^n \int_{\mathcal{M}_\delta} c_{i0}\big(B_\varepsilon(u_0^\delta)u_i^\delta - B_\varepsilon(u_0)u_i\big)\, dx + \sum_{i,k=1}^n \int_{\mathcal{M}_\delta} c_{ik}\big(B_\varepsilon(u_k^\delta)u_i^\delta - B_\varepsilon(u_k)u_i\big)\, dx. \end{aligned}$$

We estimate only the first term on the right-hand side of the equality above, because the other terms can be bounded in a similar fashion, with the additional information (3.10). Using the definition in (2.6) gives

$$\begin{aligned} &\int_{\mathcal{M}_\delta} c_{00}\big(B_\varepsilon(u_0^\delta)u_0^\delta - B_\varepsilon(u_0)u_0\big)\, dx \\ &= \int_{\mathcal{M}_\delta} c_{00}\Big[\big((K_\varepsilon * 1)u_0^\delta - K_\varepsilon * u_0^\delta\big)u_0^\delta - \big((K_\varepsilon * 1)u_0 - K_\varepsilon * u_0\big)u_0\Big]\, dx \\ &= \int_{\mathcal{M}_\delta} c_{00}\Big[(K_\varepsilon * 1)\big((u_0^\delta)^2 - u_0^2\big) - \big((K_\varepsilon * u_0^\delta)u_0^\delta - (K_\varepsilon * u_0)u_0\big)\Big]\, dx \\ &\leq c_{00}\Big(2\|K_\varepsilon * 1\|_{L^\infty(\Omega)} + \|K_\varepsilon * u_0^\delta\|_{L^\infty(\Omega)} + \|K_\varepsilon * u_0\|_{L^\infty(\Omega)}\Big) \int_{\mathcal{M}_\delta} (u_0^\delta - u_0)\, dx \\ &\leq 4c_{00}\|K_\varepsilon\|_{L^1(\Omega^2)} \int_{\mathcal{M}_\delta} (u_0^\delta - u_0)\, dx, \end{aligned}$$

where in the last step we used Young's inequality for convolution and the fact that both $u_0^\delta, u_0 \leq 1$. Repeating a similar argument for the other terms gives eventually

$$\sum_{i,k=0}^n \int_{\mathcal{M}_\delta} c_{ik}\big(B_\varepsilon(u_k^\delta)u_i^\delta - B_\varepsilon(u_k)u_i\big)\, dx \leq C \int_{\mathcal{M}_\delta} (u_0^\delta - u_0)\, dx, \tag{3.12}$$

where $C > 0$ is a constant depending on n, $\|K_\varepsilon\|_{L^1(\Omega^2)}$ and $\max_{i,k=0,\ldots,n} |c_{ik}|$.

The last terms in (3.11) are easily estimated by means of (3.10) while for the logarithmic terms we follow the proof of [19, Lemma 3.7] so that (3.11) reduces to

$$F_{\boldsymbol{\mu}}(\boldsymbol{u}^\delta) - F_{\boldsymbol{\mu}}(\boldsymbol{u}) \leq (\ln\delta + C) \int_{\mathcal{M}_\delta} (u_0^\delta - u_0)\, dx,$$

where $C > 0$ is a constant depending on the data but not on δ. Moreover, note that $\int_{\mathcal{M}_\delta} (u_0^\delta - u_0)\, dx > 0$, because the function $u_0^\delta - u_0 > 0$ on $\mathcal{M}_\delta$, which has positive measure. Therefore, if δ is chosen small enough to guarantee that $\ln\delta + C < 0$, it follows that

$$F_{\boldsymbol{\mu}}(\boldsymbol{u}^\delta) - F_{\boldsymbol{\mu}}(\boldsymbol{u}) < 0,$$

which contradicts the fact that $\boldsymbol{u}$ is a minimizer to $F_{\boldsymbol{\mu}}$.

Since the same argument applies to all the remaining components, we can conclude that for every minimizer $\boldsymbol{u} \in \mathcal{A}$ to $F_{\boldsymbol{\mu}}$ there exists $\delta_{\boldsymbol{\mu}} > 0$ such that

$$u_i \geq \delta_{\boldsymbol{\mu}} \quad \text{for all } i = 0, \dots, n.$$

Arguing as in the proof of [19, Lemma 3.7], end of Step 1, we see that δ can be chosen to be dependent only on the data, as soon as $\boldsymbol{\mu}$ is assumed to satisfy $\|\boldsymbol{\mu}\|_{H^2(\Omega)^{n+1}} \leq N$, thanks to the compact embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$. This concludes the proof of the lemma. □

Lemma 3.6. *Let $\boldsymbol{u} \in \mathcal{A}$ be the unique minimizer of $F_{\boldsymbol{\mu}}$. Then, $\boldsymbol{u}$ is a weak solution to (3.2). In particular,*

$$\ln u_i + \sum_{k=0}^{n} c_{ik} B_\varepsilon(u_k) - \mu_i = 0 \quad \text{a.e. in } \Omega \quad \text{for all } i = 0, \dots, n. \tag{3.13}$$

Proof. Since $\boldsymbol{u}$ minimizes $F_{\boldsymbol{\mu}}$ it follows that $\langle F'_{\boldsymbol{\mu}}(u), \boldsymbol{\phi} \rangle = 0$ for any $\boldsymbol{\phi} \in L^\infty(\Omega)^{n+1}$, which implies that (3.2) is satisfied. The arbitrariness of $\boldsymbol{\phi}$ then gives that (3.13) is satisfied as well. □

Let now $\mathcal{S}_2 \colon H^2(\Omega)^{n+1} \to \mathcal{A}$ be the map that associates each $\boldsymbol{\mu} \in H^2(\Omega)^{n+1}$ to the unique minimizer of $F_{\boldsymbol{\mu}}$ in $\mathcal{A}$, that is, the unique weak solution $\boldsymbol{u} \in \mathcal{A}$ to (3.2). We have the following regularity result.

Lemma 3.7. *The operator $\mathcal{S}_2 \colon H^2(\Omega)^{n+1} \to \mathcal{A}$ is continuous.*

Proof. Let $(\boldsymbol{\mu}^m)_{m\in\mathbb{N}} \subset H^2(\Omega)^{n+1}$ be such that $\boldsymbol{\mu}^m \to \boldsymbol{\mu}$ strongly in $H^2(\Omega)^{n+1}$. Set $\boldsymbol{u}^m = \mathcal{S}_2(\boldsymbol{\mu}^m)$ as well as $\boldsymbol{u} = \mathcal{S}_2(\boldsymbol{\mu})$. We want to show that $\mathcal{S}_2(\boldsymbol{\mu}^m) \to \mathcal{S}_2(\boldsymbol{\mu})$ strongly in $L^2(\Omega)^{n+1}$ as $m \to \infty$. We argue as in the existence step of the proof of Lemma 3.4 and consider test functions $\phi_i = u_i^m - u_i$, $i = 0, \dots, n$.

Exploiting the positive semidefiniteness of C as in equation (3.9), and from the definition of $\mathcal{A}$ and the consequent boundedness of its elements, we infer

$$\int_\Omega (\ln u_i^m - \ln u_i)(u_i^m - u_i)\, dx \leq 2 \int_\Omega |\mu_i^m - \mu_i|\, dx.$$

We proceed by splitting the left-hand side of the above inequality into four regions.

Fix $\delta > 0$, and assume first that $x \in \Omega$ is such that $u_i^m(x) - u_i(x) \geq \delta u_i(x)$. Then,

$$\ln u_i^m(x) - \ln u_i(x) = \ln\left(1 + \frac{u_i^m(x) - u_i(x)}{u_i(x)}\right) \geq \ln(1+\delta),$$

so that

$$\ln(1+\delta) \int_{\{x\in\Omega \colon u_i^m(x) - u_i(x) \geq \delta u_i(x)\}} u_i^m(x) - u_i(x)\, dx \leq 2 \int_\Omega |\mu_i^m - \mu_i|\, dx. \tag{3.14}$$

Analogously, assuming that $u_i^m(x) - u_i(x) \leq -\delta u_i(x)$ gives

$$\ln(1+\delta) \int_{\{x\in\Omega \colon u_i(x) - u_i^m(x) \geq \delta u_i(x)\}} u_i(x) - u_i^m(x)\, dx \leq 2 \int_\Omega |\mu_i^m - \mu_i|\, dx. \tag{3.15}$$

Finally, for the remainder, using that $\boldsymbol{u} \in \mathcal{A}$, we have

$$\int_{\{x\in\Omega \colon |u_i^m(x) - u_i(x)| \leq \delta u_i(x)\}} |u_i(x) - u_i^m(x)|\, dx \leq \delta|\Omega|. \tag{3.16}$$

By combining (3.14)–(3.16), using again the definition of $\mathcal{A}$ and by the convergence of the chemical potentials we deduce

$$\limsup_{m\to\infty} \int_\Omega |u_i^m - u_i|\, dx \leq \delta|\Omega| + \limsup_{m\to\infty} \frac{4}{\ln(1+\delta)} \int_\Omega |\mu_i^m - \mu_i|\, dx = \delta|\Omega|.$$

In view of the arbitrariness of δ we conclude that $u_i^m \to u_i$ strongly in $L^1(\Omega)$ for every $i \in \{0, \dots, n\}$. Since $\boldsymbol{u}^m, \boldsymbol{u} \in \mathcal{A}$, by the Dominated Convergence Theorem we also find $u_i^m \to u_i$ strongly in $L^r(\Omega)$ for every $i \in \{0, \dots, n\}$ and for every $r \in [1, \infty)$. □

We now have all the necessary ingredients at hand to prove the existence of iterates.

Proof of Proposition 3.1. By the results of Lemmas 3.2, 3.3, 3.4 and 3.7, together with the compact embedding of $H^2(\Omega)$ into $L^2(\Omega)$ for $d \leq 3$, it follows that the operator $\mathcal{S} := \mathcal{S}_2 \circ \mathcal{S}_1 : \mathcal{A} \to \mathcal{A}$ is both compact and continuous and thus, by Schauder's fixed point theorem, it admits a fixed point $\boldsymbol{u}_\varepsilon^{p+1}$ which is the solution to (3.1). □

3.2. **A-priori estimates.** In view of Proposition 3.1, we deduce that for any initial datum $\boldsymbol{u}_\varepsilon^0 \in \mathcal{A}$ and any $\tau \in (0,1)$ there exists a sequence $(\boldsymbol{u}_\varepsilon^p, \mu_\varepsilon^p)_p \subset \mathcal{A} \times H^2(\Omega)^{n+1}$, defined recursively as solutions to (3.1) and (3.2) for every $p \in \mathbb{N}$.

We then define several piecewise-constant-in-time functions as follows: for all $p \in \mathbb{N} \setminus \{0\}$, for all $i = 0, \dots, n$ and all $t \in (t_{p-1}, t_p]$, we set

$$
\begin{aligned}
&\boldsymbol{u}_\varepsilon^{(\tau)}(t) = \boldsymbol{u}_\varepsilon^p, \quad u_{\varepsilon,i}^{(\tau)}(t) = u_{\varepsilon,i}^p, \\
&\quad \mu_{\varepsilon,i}^{(\tau)} = \mu_{\varepsilon,i}^p = \ln u_{\varepsilon,i}^p + \sum_{j=0}^n c_{ij}\big((K_\varepsilon * 1)u_{\varepsilon,j}^p - K_\varepsilon * u_{\varepsilon,j}^p\big) = \ln u_{\varepsilon,i}^p + \sum_{j=0}^n c_{ij} B_\varepsilon(u_{\varepsilon,j}^p).
\end{aligned}
\tag{3.17}
$$

At time $t = 0$ we define $\boldsymbol{u}^{(\tau)}(0) = \boldsymbol{u}^0$. Let $P^{(\tau)} \in \mathbb{N} \setminus \{0\}$ be the lowest integer such that $t_{P^{(\tau)}} \geq T$. We additionally introduce the time-shifted solution $\sigma_\tau \boldsymbol{u}_\varepsilon^{(\tau)}$ as

$$
\sigma_\tau \boldsymbol{u}_\varepsilon^{(\tau)}(t) = \boldsymbol{u}_\varepsilon^{p-1} \text{ for all } t \in (t_{p-1}, t_p],\ p \in \mathbb{N} \setminus \{0\},
$$

whose components are given by $(\sigma_\tau u_{\varepsilon,0}^{(\tau)}, \dots, \sigma_\tau u_{\varepsilon,n}^{(\tau)})$. For all $\boldsymbol{u}_\varepsilon = (u_{\varepsilon,0}, u_{\varepsilon,1}, \dots, u_{\varepsilon,n}) \in \mathcal{A}$, for all $\tau > 0$ and $t > 0$ we define the (convex) entropy functional

$$
E_{NL}^{(\tau)}(t) := \sum_{i=0}^n \int_\Omega u_i^{(\tau)}(t) \ln u_i^{(\tau)}(t)\, dx + \frac{1}{4} \sum_{i,j=0}^n \int_\Omega \int_\Omega c_{ij} K_\varepsilon(x,y)(u_{\varepsilon,i}^{(\tau)}(t,x) - u_{\varepsilon,i}^{(\tau)}(t,y))(u_{\varepsilon,j}^{(\tau)}(t,x) - u_{\varepsilon,j}^{(\tau)}(t,y))\, dxdy,
$$

so that, for all $p \in \mathbb{N}$,

$$
E_{NL}^{(\tau)}(t_{p+1}) = \sum_{i=0}^n \int_\Omega u_i^{p+1} \ln u_i^{p+1}\, dx + \frac{1}{4} \sum_{i,j=0}^n \int_\Omega \int_\Omega c_{ij} K_\varepsilon(x,y)(u_{\varepsilon,i}^{p+1}(x) - u_{\varepsilon,i}^{p+1}(y))(u_{\varepsilon,j}^{p+1}(x) - u_{\varepsilon,j}^{p+1}(y))\, dxdy.
$$

The main contribution of this subsection is to establish some a-priori bounds on the interpolants $(u_{\varepsilon,i}^{(\tau)}, \mu_{\varepsilon,i}^{(\tau)})$, with constants independent of τ and ε.

Remark 1. *Note that the energy functional E_{NL} in (2.5) is bounded from below, since the matrix C is positive semidefinite and the function $x \mapsto x \ln x - x + 1$ is bounded from below in $[0,1]$.*

We begin this subsection by stating the monotonicity of E_{NL}.

Lemma 3.8. *Let $(\boldsymbol{u}_\varepsilon^p)_{p\in\mathbb{N}}$ be a sequence of solutions to (3.1). Then, the sequence $(E_{NL}^{(\tau)}(t_p))_{p\in\mathbb{N}}$ is decreasing. Moreover, there exists a constant $C > 0$ such that*

$$
\frac{1}{4} \int_\Omega \int_\Omega K_\varepsilon(x,y)(u_{\varepsilon,i}^{(\tau)}(x) - u_{\varepsilon,i}^{(\tau)}(y))(u_{\varepsilon,j}^{(\tau)}(x) - u_{\varepsilon,j}^{(\tau)}(y))\, dxdy \leq C \quad \text{for all } \tau > 0,\ t > 0,\ i,j = 0, \dots, n.
$$

Proof. We reason as in [19, Lemma 4.3]. We test each equation of (3.1) with $\varphi_i = \mu_{\varepsilon,i}^{p+1}$ and then sum for $i = 0, \dots, n$ to have

$$
\begin{aligned}
\sum_{i=0}^n \int_\Omega \frac{u_{\varepsilon,i}^{p+1} - u_{\varepsilon,i}^p}{\tau} \mu_{\varepsilon,i}^{p+1}\, dx = &- \sum_{i=0}^n \int_\Omega \sum_{0\leq j\neq i\leq n} L_{ij} \Big[u_{\varepsilon,j}^{p+1} \nabla u_{\varepsilon,i}^{p+1} - u_{\varepsilon,i}^{p+1} \nabla u_{\varepsilon,j}^{p+1} + J_{ij}\Big] \cdot \nabla \mu_{\varepsilon,i}^{p+1}\, dx \\
&- \tau \sum_{i=0}^n \|\mu_{\varepsilon,i}^{p+1}\|_{H^2(\Omega)}^2.
\end{aligned}
\tag{3.18}
$$

For the right-hand side, we exploit the fact the mobility M is positive semidefinite, see [19, Remark 4.1], to have

$$\begin{aligned}
&-\sum_{i=0}^{n}\int_{\Omega}\sum_{0\le j\ne i\le n}L_{ij}\Big[u_{\varepsilon,j}^{p+1}\nabla u_{\varepsilon,i}^{p+1}-u_{\varepsilon,i}^{p+1}\nabla u_{\varepsilon,j}^{p+1}+J_{ij}\Big]\cdot\nabla\mu_{\varepsilon,i}^{p+1}\,dx-\tau\sum_{i=0}^{n}\|\mu_{\varepsilon,i}^{p+1}\|_{H^2(\Omega)}^2\\
&\le-\sum_{i=0}^{n}\int_{\Omega}\Big(\sum_{0\le j\ne i\le n}L_{ij}u_{\varepsilon,i}^{p+1}u_{\varepsilon,j}^{p+1}\nabla(\mu_{\varepsilon,i}^{p+1}-\mu_{\varepsilon,j}^{p+1})\Big)\cdot\nabla\mu_{\varepsilon,i}^{p+1}\,dx\\
&\le-(\nabla\boldsymbol{\mu}_{\varepsilon}^{p+1})^{T}M(\boldsymbol{u}_{\varepsilon}^{p+1})\nabla\boldsymbol{\mu}_{\varepsilon}^{p+1}\le 0.
\end{aligned}\tag{3.19}$$

On the left-hand side we use (2.6) together with the convexity of the energy to get

$$\sum_{i=0}^{n}\int_{\Omega}\frac{u_{\varepsilon,i}^{p+1}-u_{\varepsilon,i}^{p}}{\tau}\mu_{\varepsilon,i}^{p+1}\,dx=\frac{1}{\tau}\sum_{i=0}^{n}\int_{\Omega}(u_{\varepsilon,i}^{p+1}-u_{\varepsilon,i}^{p})D_{u_{\varepsilon,i}^{p+1}}E_{NL}(\boldsymbol{u}^{p+1})\,dx\ge\frac{1}{\tau}\big(E_{NL}(\boldsymbol{u}_{\varepsilon}^{p+1})-E_{NL}(\boldsymbol{u}_{\varepsilon}^{p})\big).\tag{3.20}$$

From (3.18)-(3.20) we infer

$$\frac{1}{\tau}\big(E_{NL}(\boldsymbol{u}_{\varepsilon}^{p+1})-E_{NL}(\boldsymbol{u}_{\varepsilon}^{p})\big)\le 0,$$

that is, the sequence $(E_{NL}(\boldsymbol{u}_{\varepsilon}^{p}))_{p\in\mathbb{N}}$ is non-increasing. In particular, there exists $C>0$ such that $E_{NL}(\boldsymbol{u}_{\varepsilon}^{p+1})\le E_{NL}(\boldsymbol{u}_{\varepsilon}^{0})\le C$ which, taking (2.5) into account, entails

$$\frac{1}{4}\int_{\Omega}\int_{\Omega}K_{\varepsilon}(x,y)(u_{\varepsilon,i}^{p+1}(x)-u_{\varepsilon,i}^{p+1}(y))^2dxdy\le C.$$

This completes the proof. □

We now exploit this property to establish some a-priori estimates which play a central role in order to pass to the limit as $\tau\to 0^+$.

Proposition 3.9. *Let $(\boldsymbol{u}_{\varepsilon}^{p})_{p\in\mathbb{N}}$ be a sequence of solutions to (3.1). Then, there exists a constant $C>0$ such that*

$$\sum_{i=0}^{n}\int_{0}^{T}\int_{\Omega}\frac{|\nabla u_{\varepsilon,i}^{(\tau)}|^2}{u_{\varepsilon,i}^{(\tau)}}\,dxdt\le C,\tag{3.21}$$

$$\sum_{i=0}^{n}\int_{0}^{T}\int_{\Omega}\int_{\Omega}c_{ii}K_{\varepsilon}(x,y)|\nabla u_{\varepsilon,i}^{(\tau)}(x)-\nabla u_{\varepsilon,i}^{(\tau)}(y)|^2\,dxdydt\le C,\tag{3.22}$$

$$\sum_{i=0}^{n}\sum_{0\le j\ne i\le n}\int_{0}^{T}\int_{\Omega}|u_{\varepsilon,i}^{(\tau)}u_{\varepsilon,j}^{(\tau)}\nabla(q_i(\boldsymbol{u}_{\varepsilon}^{(\tau)})-q_j(\boldsymbol{u}_{\varepsilon}^{(\tau)}))|^2\,dxdt\le C,\tag{3.23}$$

$$\tau\sum_{i=0}^{n}\int_{0}^{T}\|\mu_{\varepsilon,i}^{(\tau)}\|_{H^2(\Omega)}^2\,dt\le C.\tag{3.24}$$

Proof. We use an argument already exploited in [19]. We first test each equation in (3.1) with $\varphi_i=\mu_{\varepsilon,i}^{p+1}$ and sum for $i=0,\dots,n$. This gives

$$\begin{aligned}\sum_{i=0}^{n}\int_{\Omega}\frac{u_{\varepsilon,i}^{p+1}-u_{\varepsilon,i}^{p}}{\tau}\mu_{\varepsilon,i}^{p+1}\,dx=&-\sum_{i=0}^{n}\int_{\Omega}\sum_{0\le j\ne i\le n}L_{ij}u_{\varepsilon,i}^{p+1}u_{\varepsilon,j}^{p+1}\nabla(\mu_{\varepsilon,i}^{p+1}-\mu_{\varepsilon,j}^{p+1})\cdot\nabla\mu_{\varepsilon,i}^{p+1}\,dx\\
&-\tau\sum_{i=0}^{n}\|\mu_{\varepsilon,i}^{p+1}\|_{H^2(\Omega)}^2.\end{aligned}$$

The left-hand side can be handled as in Lemma the proof of 3.8, cf. (3.20). Concerning the right-hand side, we first set $\ell := \min_{\substack{0\le i,j\le n\\ i\neq j}} L_{ij}$. We have

$$
\begin{aligned}
&-\sum_{i=0}^{n}\int_\Omega \sum_{0\le j\neq i\le n} L_{ij}u_{\varepsilon,i}^{p+1}u_{\varepsilon,j}^{p+1}\nabla(\mu_{\varepsilon,i}^{p+1}-\mu_{\varepsilon,j}^{p+1})\cdot\nabla\mu_{\varepsilon,i}^{p+1}\,dx-\tau\sum_{i=0}^{n}\|\mu_{\varepsilon,i}^{p+1}\|_{H^2(\Omega)}^2 \qquad (3.25)\\
&=-\sum_{i=0}^{n}\int_\Omega \sum_{0\le j\neq i\le n} (L_{ij}-\ell)u_{\varepsilon,i}^{p+1}u_{\varepsilon,j}^{p+1}\nabla(\mu_{\varepsilon,i}^{p+1}-\mu_{\varepsilon,j}^{p+1})\cdot\nabla\mu_{\varepsilon,i}^{p+1}\,dx\\
&\quad-\ell\sum_{i=0}^{n}\int_\Omega \sum_{0\le j\neq i\le n} u_{\varepsilon,i}^{p+1}u_{\varepsilon,j}^{p+1}\nabla(\mu_{\varepsilon,i}^{p+1}-\mu_{\varepsilon,j}^{p+1})\cdot\nabla\mu_{\varepsilon,i}^{p+1}\,dx-\tau\sum_{i=0}^{n}\|\mu_{\varepsilon,i}^{p+1}\|_{H^2(\Omega)}^2.
\end{aligned}
$$

The first term reads as

$$
\begin{aligned}
&-\sum_{i=0}^{n}\int_\Omega \sum_{0\le j\neq i\le n} (L_{ij}-\ell)u_{\varepsilon,i}^{p+1}u_{\varepsilon,j}^{p+1}\nabla(\mu_{\varepsilon,i}^{p+1}-\mu_{\varepsilon,j}^{p+1})\cdot\nabla\mu_{\varepsilon,i}^{p+1}\,dx \qquad (3.26)\\
&=-\int_\Omega (\nabla\boldsymbol{\mu}_\varepsilon^{p+1})^T\widetilde{M}(\boldsymbol{u}_\varepsilon^{p+1})\nabla\boldsymbol{\mu}_\varepsilon^{p+1}\le 0,
\end{aligned}
$$

where $\widetilde{M}(\boldsymbol{u}_\varepsilon^{p+1})$ is given as in (2.4), but with the coefficients L_{ij} replaced by $L_{ij}-\ell$. Using (2.6), the second term reads as follows

$$
\begin{aligned}
&-\ell\sum_{i=0}^{n}\int_\Omega \sum_{0\le j\neq i\le n} u_{\varepsilon,i}^{p+1}u_{\varepsilon,j}^{p+1}\nabla(\mu_{\varepsilon,i}^{p+1}-\mu_{\varepsilon,j}^{p+1})\cdot\nabla\mu_{\varepsilon,i}^{p+1}\,dx \qquad (3.27)\\
&=-\ell\sum_{i=0}^{n}\int_\Omega \sum_{0\le j\neq i\le n} u_{\varepsilon,i}^{p+1}u_{\varepsilon,j}^{p+1}\nabla(\ln u_{\varepsilon,i}^{p+1}-\ln u_{\varepsilon,j}^{p+1})\cdot\nabla\ln u_{\varepsilon,i}^{p+1}\,dx\\
&\quad-\ell\sum_{i=0}^{n}\int_\Omega \sum_{0\le j\neq i\le n} u_{\varepsilon,i}^{p+1}u_{\varepsilon,j}^{p+1}\nabla(\ln u_{\varepsilon,i}^{p+1}-\ln u_{\varepsilon,j}^{p+1})\cdot\nabla q_i(\boldsymbol{u}^{p+1})\,dx\\
&\quad-\ell\sum_{i=0}^{n}\int_\Omega \sum_{0\le j\neq i\le n} u_{\varepsilon,i}^{p+1}u_{\varepsilon,j}^{p+1}\nabla\ln u_{\varepsilon,i}^{p+1}\cdot\nabla(q_i(\boldsymbol{u}^{p+1})-q_j(\boldsymbol{u}^{p+1}))\,dx\\
&\quad-\ell\sum_{i=0}^{n}\int_\Omega \sum_{0\le j\neq i\le n} u_{\varepsilon,i}^{p+1}u_{\varepsilon,j}^{p+1}\nabla(q_i(\boldsymbol{u}^{p+1})-q_j(\boldsymbol{u}^{p+1}))\cdot\nabla q_i(\boldsymbol{u}^{p+1})\,dx.
\end{aligned}
$$

We are going to estimate the several terms in the above equation separately. First of all, using the constraint (2.3) gives

$$
-\ell\sum_{i=0}^{n}\int_\Omega \sum_{0\le j\neq i\le n} u_{\varepsilon,i}^{p+1}u_{\varepsilon,j}^{p+1}\nabla(\ln u_{\varepsilon,i}^{p+1}-\ln u_{\varepsilon,j}^{p+1})\cdot\nabla\ln u_{\varepsilon,i}^{p+1}\,dx=-\ell\sum_{i=0}^{n}\int_\Omega \frac{|\nabla u_{\varepsilon,i}^{p+1}|^2}{u_{\varepsilon,i}^{p+1}}\,dx.
$$

For the second term we reason as in [17, inequality after (3.14)] to have

$$
\begin{aligned}
&-\ell\sum_{i=0}^{n}\int_\Omega\sum_{0\le j\ne i\le n}L_{ij}u_{\varepsilon,i}^{p+1}u_{\varepsilon,j}^{p+1}\nabla(\ln u_{\varepsilon,i}^{p+1}-\ln u_{\varepsilon,j}^{p+1})\cdot\nabla q_i(\boldsymbol{u}_\varepsilon^{p+1})\,dx\\
&=-\ell\sum_{i=0}^{n}\int_\Omega\nabla u_{\varepsilon,i}^{p+1}\cdot\nabla\Big(\sum_{k=0}^{n}c_{ik}B_\varepsilon(u_{\varepsilon,k}^{p+1})\Big)=-\sum_{i,k=0}^{n}\int_\Omega c_{ik}\nabla u_{\varepsilon,i}^{p+1}\cdot B_\varepsilon(\nabla u_{\varepsilon,k}^{p+1})\,dx\\
&=-\frac{\ell}{2}\sum_{i=0}^{n}\int_\Omega\int_\Omega c_{ii}K_\varepsilon(x,y)|\nabla u_{\varepsilon,i}^{p+1}(x)-\nabla u_{\varepsilon,i}^{p+1}(y)|^2\,dxdy\\
&\qquad-\frac{\ell}{2}\sum_{i=0}^{n}\sum_{0\le k\ne i\le n}\int_\Omega\int_\Omega c_{ik}K_\varepsilon(x,y)\nabla(u_{\varepsilon,i}^{p+1}(x)-u_{\varepsilon,i}^{p+1}(y))\cdot\nabla(u_{\varepsilon,k}^{p+1}(x)-u_{\varepsilon,k}^{p+1}(y))\,dxdy\\
&\le-\frac{\ell}{2}\sum_{i=0}^{n}\int_\Omega\int_\Omega c_{ii}K_\varepsilon(x,y)|\nabla u_{\varepsilon,i}^{p+1}(x)-\nabla u_{\varepsilon,i}^{p+1}(y)|^2\,dxdy\\
&\qquad+\frac{\ell}{4}\sum_{i=0}^{n}\sum_{0\le k\ne i\le n}\int_\Omega\int_\Omega |c_{ik}|K_\varepsilon(x,y)|\nabla u_{\varepsilon,i}^{p+1}(x)-\nabla u_{\varepsilon,i}^{p+1}(y)|^2\,dxdy\\
&\qquad+\frac{\ell}{4}\sum_{i=0}^{n}\sum_{0\le k\ne i\le n}\int_\Omega\int_\Omega |c_{ik}|K_\varepsilon(x,y)|\nabla u_{\varepsilon,k}^{p+1}(x)-\nabla u_{\varepsilon,k}^{p+1}(y)|^2\,dxdy\\
&\le-\widetilde{C}\sum_{i=0}^{n}\int_\Omega\int_\Omega c_{ii}K_\varepsilon(x,y)|\nabla u_{\varepsilon,i}^{p+1}(x)-\nabla u_{\varepsilon,i}^{p+1}(y)|^2\,dxdy,
\end{aligned}
$$

with $\widetilde{C}>0$ thanks to (2.7). For the third term, we first multiply and divide by $(u_{\varepsilon,i}^{p+1})^{1/2}$ and then apply a Young's inequality to have

$$
\begin{aligned}
&-\ell\sum_{i=0}^{n}\int_\Omega\sum_{0\le j\ne i\le n}u_{\varepsilon,i}^{p+1}u_{\varepsilon,j}^{p+1}\nabla\ln u_{\varepsilon,i}^{p+1}\cdot\nabla(q_i(\boldsymbol{u}_\varepsilon^{p+1})-q_j(\boldsymbol{u}_\varepsilon^{p+1}))\,dx\\
&=-\ell\sum_{i=0}^{n}\int_\Omega\sum_{0\le j\ne i\le n}u_{\varepsilon,j}^{p+1}\nabla u_{\varepsilon,i}^{p+1}\cdot\nabla(q_i(\boldsymbol{u}_\varepsilon^{p+1})-q_j(\boldsymbol{u}_\varepsilon^{p+1}))\,dx\\
&\le\frac{\ell}{2}\sum_{i=0}^{n}\sum_{0\le j\ne i\le n}\int_\Omega u_{\varepsilon,i}^{p+1}u_{\varepsilon,j}^{p+1}|\nabla(q_i(\boldsymbol{u}_\varepsilon^{p+1})-q_j(\boldsymbol{u}_\varepsilon^{p+1}))|^2\,dx+\frac{\ell}{2}\sum_{i=0}^{n}\sum_{0\le j\ne i\le n}\int_\Omega u_{\varepsilon,j}^{p+1}\frac{|\nabla u_{\varepsilon,i}^{p+1}|^2}{u_{\varepsilon,i}^{p+1}}\,dx\\
&\le\frac{\ell}{2}\sum_{i=0}^{n}\sum_{0\le j\ne i\le n}\int_\Omega u_{\varepsilon,i}^{p+1}u_{\varepsilon,j}^{p+1}|\nabla(q_i(\boldsymbol{u}_\varepsilon^{p+1})-q_j(\boldsymbol{u}_\varepsilon^{p+1}))|^2\,dx+\frac{\ell}{2}\sum_{i=0}^{n}\int_\Omega\frac{|\nabla u_{\varepsilon,i}^{p+1}|^2}{u_{\varepsilon,i}^{p+1}}\,dx,
\end{aligned}
$$

while exploiting symmetries in the last term gives

$$
\begin{aligned}
&-\ell\sum_{i=0}^{n}\int_\Omega\sum_{0\le j\ne i\le n}u_{\varepsilon,i}^{p+1}u_{\varepsilon,j}^{p+1}\nabla(q_i(\boldsymbol{u}_\varepsilon^{p+1})-q_j(\boldsymbol{u}_\varepsilon^{p+1}))\cdot\nabla q_i(\boldsymbol{u}_\varepsilon^{p+1})\,dx\\
&=-\ell\sum_{i=0}^{n}\sum_{0\le j\ne i\le n}\int_\Omega u_{\varepsilon,i}^{p+1}u_{\varepsilon,j}^{p+1}|\nabla\big(q_i(\boldsymbol{u}_\varepsilon^{p+1})-q_j(\boldsymbol{u}_\varepsilon^{p+1})\big)|^2\,dx.
\end{aligned}
$$

Thus, recalling again (3.20), we obtain

$$\begin{aligned}&\frac{1}{\tau}\big(E(\boldsymbol{u}_\varepsilon^{p+1})-E(\boldsymbol{u}_\varepsilon^{p})\big)\\&\le -\frac{\ell}{2}\sum_{i=0}^n\int_\Omega \frac{|\nabla u_{\varepsilon,i}^{p+1}|^2}{u_{\varepsilon,i}^{p+1}}\,dx-\ell\sum_{i=0}^n\int_\Omega\int_\Omega c_{ii}K_\varepsilon(x,y)|\nabla u_{\varepsilon,i}^{p+1}(x,s)-\nabla u_{\varepsilon,i}^{p+1}(y,s)|^2\,dxdy\\&\quad-\frac{\ell}{2}\sum_{i=0}^n\sum_{0\le j\ne i\le n}\int_\Omega u_{\varepsilon,i}^{p+1}u_{\varepsilon,j}^{p+1}|\nabla\big(q_i(\boldsymbol{u}_\varepsilon^{p+1})-q_j(\boldsymbol{u}_\varepsilon^{p+1})\big)|^2\,dx-\tau\sum_{i=0}^n\|\mu_{\varepsilon,i}^{p+1}\|_{H^2(\Omega)}^2.\end{aligned}$$

We then multiply this expression by τ, sum for $0\le p\le P^{(\tau)}-1$ and use Remark 1 to obtain

$$\begin{aligned}&\frac{\ell}{2}\sum_{i=0}^n\int_0^T\int_\Omega\frac{|\nabla u_{\varepsilon,i}^{(\tau)}|^2}{u_{\varepsilon,i}^{(\tau)}}\,dxdt+\ell\sum_{i=0}^n\int_0^T\int_\Omega\int_\Omega c_{ii}K_\varepsilon(x,y)|\nabla u_{\varepsilon,i}^{(\tau)}(x)-\nabla u_{\varepsilon,i}^{(\tau)}(y)|^2\,dxdydt\\&\quad+\frac{\ell}{2}\sum_{i=0}^n\sum_{0\le j\ne i\le n}\int_0^T\int_\Omega u_{\varepsilon,i}^{(\tau)}u_{\varepsilon,j}^{(\tau)}|\nabla\big(q_i(\boldsymbol{u}_\varepsilon^{(\tau)})-q_j(\boldsymbol{u}_\varepsilon^{(\tau)})\big)|^2\,dxdt+\tau\sum_{i=0}^n\int_0^T\|\mu_{\varepsilon,i}^{(\tau)}\|_{H^2(\Omega)}^2\,dt\\&\le C(T+1)+E(\boldsymbol{u}_\varepsilon^0).\end{aligned}$$

Since all the quantities on the left-hand side are nonnegative, the bounds (3.21), (3.22) and (3.24) are satisfied, while the bound (3.23) follows by noticing that, since $u_{\varepsilon,i}^{(\tau)},u_{\varepsilon,j}^{(\tau)}\in(0,1)$, then

$$\int_0^T\int_\Omega|u_{\varepsilon,i}^{(\tau)}u_{\varepsilon,j}^{(\tau)}\nabla\big(q_i(\boldsymbol{u}_\varepsilon^{(\tau)})-q_j(\boldsymbol{u}_\varepsilon^{(\tau)})\big)|^2\,dxdt\le\int_0^T\int_\Omega u_{\varepsilon,i}^{(\tau)}u_{\varepsilon,j}^{(\tau)}|\nabla\big(q_i(\boldsymbol{u}_\varepsilon^{(\tau)})-q_j(\boldsymbol{u}_\varepsilon^{(\tau)})\big)|^2\,dxdt,$$

for all $i=0,\dots,n$ and all $j\ne i=0,\dots,n$. This ends the proof. □

Lemma 3.10. *There exists $C>0$, independent of $\tau>0$ and $\varepsilon>0$, such that*

$$\int_0^T\Big\|\frac{u_{\varepsilon,i}^{(\tau)}-\sigma_\tau u_{\varepsilon,i}^{(\tau)}}{\tau}\Big\|_{H^2(\Omega)'}^2\,dt\le C\quad\textit{for all } i=0,\dots,n.$$

Proof. It works as in [19, Lemma 4.7]. □

3.3. **The limit as $\tau\to0^+$.** In this subsection we will recover a weak solution to (2.10) in the sense of Definition 2.2 as the weak limit of some extracted subsequence $(\boldsymbol{u}_\varepsilon^{(\tau)})_{\tau>0}$ as $\tau\to0^+$.

After interpolation, (3.1) reads as

$$\begin{aligned}&\int_0^T\int_\Omega\frac{u_{\varepsilon,i}^{(\tau)}-\sigma_\tau u_{\varepsilon,i}^{(\tau)}}{\tau}\varphi_i\,dxdt\\&=-\int_0^T\int_\Omega\sum_{0\le j\ne i\le n}L_{ij}\Big[u_{\varepsilon,j}^{(\tau)}\nabla u_{\varepsilon,i}^{(\tau)}-u_{\varepsilon,i}^{(\tau)}\nabla u_{\varepsilon,j}^{(\tau)}+u_{\varepsilon,i}^{(\tau)}u_{\varepsilon,j}^{(\tau)}\nabla\Big(\sum_{k=0}^n c_{ik}B_\varepsilon(u_{\varepsilon,k}^{(\tau)})-\sum_{l=0}^n c_{jl}B_\varepsilon(u_{\varepsilon,l}^{(\tau)})\Big)\Big]\cdot\nabla\varphi_i\,dxdt\\&\quad-\tau\int_0^T\langle\mu_{\varepsilon,i}^{(\tau)},\varphi_i\rangle_{H^2(\Omega)}\,dt,\end{aligned}\tag{3.28}$$

for every $\varphi_i\in L^2(0,T;H^2(\Omega))$. The estimates collected in the previous section imply the existence of a function $\boldsymbol{u}_\varepsilon\in L^2(0,T;H^1(\Omega))^{n+1}$ such that, up to the extraction of a subsequence,

$$u_{\varepsilon,i}^{(\tau)}\rightharpoonup u_{\varepsilon,i}\quad\text{weakly in } L^2(0,T;H^1(\Omega))\quad\text{as well as weak-}*\text{ in } L^\infty(0,T;L^\infty(\Omega)),\tag{3.29}$$

$$\frac{u_{\varepsilon,i}^{(\tau)}-\sigma_\tau u_{\varepsilon,i}^{(\tau)}}{\tau}\rightharpoonup\partial_t u_{\varepsilon,i}\quad\text{weakly in } L^2(0,T;H^2(\Omega)').\tag{3.30}$$

Taking the compact embedding $H^1(\Omega)\hookrightarrow L^2(\Omega)$ into account, [18, Theorem 1] implies that

$$u_{\varepsilon,i}^{(\tau)}\to u_{\varepsilon,i}\qquad L^2(0,T;L^2(\Omega)),\tag{3.31}$$

while the uniform bound of $(u^\tau_{\varepsilon,i})_{\tau>0}$ in $L^\infty(0,T;L^\infty(\Omega))$ implies that $u^{(\tau)}_{\varepsilon,i} \to u_{\varepsilon,i}$ strongly in $L^r(0,T;L^r(\Omega))$, for any $r\in[1,\infty)$ and for any $i=0,\dots,n$.

From [17, Lemma 2] we know that

$$\int_0^T \|B_\varepsilon(u^{(\tau)}_{\varepsilon,i})\|_{H^1(\Omega)'}\,dt \le C\int_0^T \|\nabla u^{(\tau)}_{\varepsilon,i}\|^2_{L^2(\Omega)}\,dt \le C, \tag{3.32}$$

which implies the existence of $\psi\in L^2(0,T;H^1(\Omega)')$ such that $B_\varepsilon(u^{(\tau)}_{\varepsilon,i}) \rightharpoonup \psi_i$ weakly in $L^2(0,T;H^1(\Omega)')$. Let now $\varphi_i\in L^2(0,T;C^\infty_c(\Omega))$. We have

$$\langle B_\varepsilon(u^{(\tau)}_{\varepsilon,i}),\varphi_i\rangle_{L^2(\Omega),L^2(\Omega)} = \frac12\int_0^T\int_\Omega\int_\Omega K_\varepsilon(x,y)(u^{(\tau)}_{\varepsilon,i}(x)-u^{(\tau)}_{\varepsilon,i}(y))(\varphi_i(x)-\varphi_i(y))\,dxdydt.$$

Since $(u^\tau_{\varepsilon,i})_{\tau>0}$ is uniform bounded in $L^\infty(0,T;L^\infty(\Omega))$ while $K_\varepsilon(x,y)(\varphi_i(x)-\varphi_i(y))\in L^1(\Omega^2)$ thanks to [7, Theorem 1], by (3.31) it follows by dominated convergence that

$$\langle B_\varepsilon(u^\tau_{\varepsilon,i}),\varphi_i\rangle_{L^2(\Omega),L^2(\Omega)} \to \langle B_\varepsilon(u_{\varepsilon,i}),\varphi_i\rangle_{L^2(\Omega),L^2(\Omega)}.$$

In particular, we infer that $B_\varepsilon(u^\tau_{\varepsilon,i})\to B_\varepsilon(u_{\varepsilon,i})$ weakly in $L^2(0,T;L^2(\Omega))$.

Next we prove that

$$\sqrt{\tau}\mu^{(\tau)}_{\varepsilon,i}\to 0 \quad \text{weakly in } L^2(0,T;H^2(\Omega)). \tag{3.33}$$

From (3.24) we have that $\big(\sqrt{\tau}\mu^{(\tau)}_{\varepsilon,i}\big)_{\tau>0}$ is bounded in $L^2(0,T;H^2(\Omega))$. Additionally, from (2.6) and Lemma 3.5 we infer that $\mu^{(\tau)}_{\varepsilon,i}$ is bounded in $L^2(0,T;H^1(\Omega)')$. As a result, in the limit as $\tau\to0$ this gives (3.33).

These convergences enable us to pass to the limit in (3.28) as $\tau\to0$. Indeed, for all $i=0,\dots,n$ and as $\tau\to0$, up to the extraction of subsequences we have

$$\begin{aligned}
&\int_0^T\int_\Omega \frac{u^{(\tau)}_{\varepsilon,i}-\sigma_\tau u^{(\tau)}_{\varepsilon,i}}{\tau}\varphi_i\,dxdt \to \int_0^T\langle\partial_t u_{\varepsilon,i},\varphi_i\rangle_{H^2(\Omega)',H^2(\Omega)}\,dt,\\
&\int_0^T\int_\Omega\sum_{0\le j\ne i\le n} L_{ij}\Big[u^{(\tau)}_{\varepsilon,j}\nabla u^{(\tau)}_{\varepsilon,i}-u^{(\tau)}_{\varepsilon,i}\nabla u^{(\tau)}_{\varepsilon,j}\Big]\cdot\nabla\varphi_i\,dxdt\\
&\qquad\to\int_0^T\int_\Omega\sum_{0\le j\ne i\le n} L_{ij}\Big[u_{\varepsilon,j}\nabla u_{\varepsilon,i}-u_{\varepsilon,i}\nabla u_{\varepsilon,j}\Big]\cdot\nabla\varphi_i\,dxdt,\\
&\tau\int_0^T\langle\mu^{(\tau)}_{\varepsilon,i},\varphi_i\rangle_{H^2(\Omega)}\,dt\to0.
\end{aligned}\tag{3.34}$$

Passing to the limit in the term

$$\begin{aligned}
&\int_0^T\int_\Omega\sum_{0\le j\ne i\le n} L_{ij}u^{(\tau)}_{\varepsilon,i}u^{(\tau)}_{\varepsilon,j}\nabla\Big(\sum_{k=0}^n c_{ik}B_\varepsilon(u^{(\tau)}_{\varepsilon,k})-\sum_{l=0}^n c_{jl}B_\varepsilon(u^{(\tau)}_{\varepsilon,l})\Big)\cdot\nabla\varphi_i\,dxdt\\
&\quad=\int_0^T\int_\Omega\sum_{0\le j\ne i\le n} L_{ij}u^{(\tau)}_{\varepsilon,i}u^{(\tau)}_{\varepsilon,j}\nabla\Big(q_i(\boldsymbol{u}^{(\tau)}_\varepsilon)-q_j(\boldsymbol{u}^{(\tau)}_\varepsilon)\Big)\cdot\nabla\varphi_i\,dxdt
\end{aligned}$$

is more delicate and will be treated in details in the following lemma.

Lemma 3.11. *For all $i\ne j=0,\dots,n$ there exists $J_{\varepsilon,ij}\in L^2(0,T;L^2(\Omega)^d)$ such that $J_{\varepsilon,ij}=u_{\varepsilon,i}u_{\varepsilon,j}\nabla(q_i(\boldsymbol{u}_\varepsilon)-q_j(\boldsymbol{u}_\varepsilon))$ holds in the weak sense, that is,*

$$\int_0^T\int_\Omega J_{\varepsilon,ij}\cdot\eta\,dxdt=-\int_0^T\int_\Omega(q_i(\boldsymbol{u}_\varepsilon)-q_j(\boldsymbol{u}_\varepsilon))\,\mathrm{div}(u_{\varepsilon,i}u_{\varepsilon,j}\eta)\,dxdt,$$

for every $\eta\in L^2(0,T;H^1(\Omega)^d)\cap L^\infty((0,T)\times\Omega;\mathbb{R}^d)$, with $\eta\cdot n=0$ on $\partial\Omega$ and such that, up to the extraction of a subsequence,

$$u^{(\tau)}_{\varepsilon,i}u^{(\tau)}_{\varepsilon,j}\nabla(q_i(\boldsymbol{u}^{(\tau)}_\varepsilon)-q_j(\boldsymbol{u}^{(\tau)}_\varepsilon))\to J_{\varepsilon,ij}\quad \textit{weakly in } L^2(0,T;L^2(\Omega)^d).$$

Proof. Fix $i, j \in \{0, \dots, n\}$ such that $j \neq i$ and set, for simplicity,

$$J^{(\tau)}_{\varepsilon,ij} := u^{(\tau)}_{\varepsilon,i} u^{(\tau)}_{\varepsilon,j} \nabla(q_i(\boldsymbol{u}^{(\tau)}_\varepsilon) - q_j(\boldsymbol{u}^{(\tau)}_\varepsilon)). \tag{3.35}$$

Estimate (3.23) implies the existence of $J_{\varepsilon,ij} \in L^2(0,T;L^2(\Omega)^d)$ such that

$$J^{(\tau)}_{\varepsilon,ij} \rightharpoonup J_{\varepsilon,ij} \quad \text{weakly in } L^2(0,T;L^2(\Omega)^d). \tag{3.36}$$

For every $\kappa > 0$ we split

$$J^{(\tau)}_{\varepsilon,ij} = \chi_{\{u^{(\tau)}_{\varepsilon,i} u^{(\tau)}_{\varepsilon,j} > \kappa\}} J^{(\tau)}_{\varepsilon,ij} + \chi_{\{u^{(\tau)}_{\varepsilon,i} u^{(\tau)}_{\varepsilon,j} \le \kappa\}} J^{(\tau)}_{\varepsilon,ij}. \tag{3.37}$$

Let us consider the first term on the right-hand side of (3.37). Thanks to (3.36) and the strong convergence $\chi_{\{u^{(\tau)}_{\varepsilon,i} u^{(\tau)}_{\varepsilon,j} > \kappa\}} \to \chi_{\{u_{\varepsilon,i} u_{\varepsilon,j} \ge \kappa\}}$ in $L^2(0,T;L^2(\Omega))$ we have further that

$$\chi_{\{u^{(\tau)}_{\varepsilon,i} u^{(\tau)}_{\varepsilon,j} > \kappa\}} J^{(\tau)}_{\varepsilon,ij} \rightharpoonup \chi_{\{u_{\varepsilon,i} u_{\varepsilon,j} \ge \kappa\}} J_{\varepsilon,ij} \quad \text{weakly in } L^2(0,T;L^2(\Omega)^d). \tag{3.38}$$

We now claim that

$$\chi_{\{u_{\varepsilon,i} u_{\varepsilon,j} \ge \kappa\}} J_{\varepsilon,ij} = \chi_{\{u_{\varepsilon,i} u_{\varepsilon,j} \ge \kappa\}} u_{\varepsilon,i} u_{\varepsilon,j} \nabla(q_i(\boldsymbol{u}_\varepsilon) - q_j(\boldsymbol{u}_\varepsilon)) \quad \text{a.e. and for all } \kappa > 0. \tag{3.39}$$

Indeed, from (3.35) we find

$$\chi_{\{u^{(\tau)}_{\varepsilon,i} u^{(\tau)}_{\varepsilon,j} > \kappa\}} \nabla(q_i(\boldsymbol{u}^{(\tau)}_\varepsilon) - q_j(\boldsymbol{u}^{(\tau)}_\varepsilon)) = \chi_{\{u^{(\tau)}_{\varepsilon,i} u^{(\tau)}_{\varepsilon,j} > \kappa\}} \frac{J^{(\tau)}_{\varepsilon,ij}}{u^{(\tau)}_{\varepsilon,i} u^{(\tau)}_{\varepsilon,j}}. \tag{3.40}$$

Thanks to (3.38), (3.35) and the fact that $u^{(\tau)}_{\varepsilon,i} u^{(\tau)}_{\varepsilon,j} \to u_{\varepsilon,i} u_{\varepsilon,j}$ strongly in $L^2(0,T;L^2(\Omega))$ we have

$$\chi_{\{u^{(\tau)}_{\varepsilon,i} u^{(\tau)}_{\varepsilon,j} > \kappa\}} \frac{J^{(\tau)}_{\varepsilon,ij}}{u^{(\tau)}_{\varepsilon,i} u^{(\tau)}_{\varepsilon,j}} \rightharpoonup \chi_{\{u_{\varepsilon,i} u_{\varepsilon,j} \ge \kappa\}} \frac{J_{\varepsilon,ij}}{u_{\varepsilon,i} u_{\varepsilon,j}}.$$

On the other hand, taking the continuity of the operator B_ε as well as (2.8) into account gives

$$\nabla(q_i(\boldsymbol{u}^{(\tau)}_\varepsilon) - q_j(\boldsymbol{u}^{(\tau)}_\varepsilon)) \rightharpoonup \nabla(q_i(\boldsymbol{u}_\varepsilon) - q_j(\boldsymbol{u}_\varepsilon)) \quad \text{weakly in } L^2(0,T;H^{-1}(\Omega)).$$

Therefore, taking the limit as $\tau \to 0^+$ in (3.40) implies the claim (3.39).

We now consider the case then the product $u_{\varepsilon,i} u_{\varepsilon,j}$ vanishes which is captured by the second term on the right-hand side of (3.37). Thanks to (3.23), there holds

$$\int_0^T \|\chi_{\{u^{(\tau)}_{\varepsilon,i} u^{(\tau)}_{\varepsilon,j} \le \kappa\}} J^{(\tau)}_{\varepsilon,ij}\|_{L^2(\Omega)} \, dt \le \int_0^T \|J^{(\tau)}_{\varepsilon,ij}\|_{L^2(\Omega)} \, dt \le C, \tag{3.41}$$

therefore there exists $g^\kappa \in L^2(0,T;L^2(\Omega)^d)$ such that

$$\chi_{\{u^{(\tau)}_{\varepsilon,i} u^{(\tau)}_{\varepsilon,j} \le \kappa\}} J^{(\tau)}_{\varepsilon,ij} \rightharpoonup g^\kappa \quad \text{weakly in } L^2(0,T;L^2(\Omega)^d).$$

By (3.41) we infer that $\|g^\kappa\|_{L^2(0,T;L^2(\Omega)^d)} \le C$, which gives the existence of $g \in L^2(0,T;L^2(\Omega)^d)$ such that

$$g^\kappa \to g \quad \text{weakly in } L^2(0,T;L^2(\Omega)^d).$$

We claim that $g \equiv 0$, and in particular

$$\int_0^T \int_\Omega g \cdot \nabla\phi \, dxdt = 0 \quad \text{for all } \phi \in L^2(0,T;H^1(\Omega)). \tag{3.42}$$

To this end, fix $\phi \in L^2(0,T;H^1(\Omega))$ and let $(\phi_n)_{n\in\mathbb{N}} \subset L^\infty(0,T;C^\infty(\Omega))$ be such that

$$\|\phi_n - \phi\|_{L^2(0,T;H^1(\Omega))} < \frac{1}{n} \quad \text{for all } n \in \mathbb{N}.$$

We have

$$
\begin{aligned}
&\Big|\int_0^T g\cdot\nabla\phi\,dxdt\Big| \le \lim_{\kappa\to 0}\Big|\int_0^T\int_\Omega g^\kappa\cdot\nabla\phi\,dxdt\Big| \\
&\le \lim_{\kappa\to 0}\lim_{\tau\to 0}\Big|\int_0^T\int_\Omega \chi_{\{u^{(\tau)}_{\varepsilon,i}u^{(\tau)}_{\varepsilon,j}\le\kappa\}} J^{(\tau)}_{\varepsilon,ij}\cdot\nabla\phi\,dxdt\Big| \le \lim_{\kappa\to 0}\lim_{\tau\to 0}\int_0^T\int_{\{u^{(\tau)}_{\varepsilon,i}u^{(\tau)}_{\varepsilon,j}\le\kappa\}}|J^{(\tau)}_{\varepsilon,ij}\cdot\nabla\phi|\,dxdt \\
&\le \lim_{\kappa\to 0}\lim_{\tau\to 0}\Big(\int_0^T\int_{\{u^{(\tau)}_{\varepsilon,i}u^{(\tau)}_{\varepsilon,j}\le\kappa\}}|J^{(\tau)}_{\varepsilon,ij}\cdot\nabla(\phi-\phi_n)|\,dxdt + \int_0^T\int_{\{u^{(\tau)}_{\varepsilon,i}u^{(\tau)}_{\varepsilon,j}\le\kappa\}}|J^{(\tau)}_{\varepsilon,ij}\cdot\nabla\phi_n|\,dxdt\Big) \\
&\le \lim_{\kappa\to 0}\lim_{\tau\to 0}\Big(\int_0^T\int_{\{u^{(\tau)}_{\varepsilon,i}u^{(\tau)}_{\varepsilon,j}\le\kappa\}} u^{(\tau)}_{\varepsilon,i}u^{(\tau)}_{\varepsilon,j}|\nabla(q_i(\boldsymbol{u}^{(\tau)}_\varepsilon)-q_j(\boldsymbol{u}^{(\tau)}_\varepsilon))|^2\,dxdt\Big)^{1/2}\|\nabla(\phi-\phi_n)\|_{L^2(0,T;L^2(\Omega))} \\
&\qquad + \lim_{\kappa\to 0}\lim_{\tau\to 0}\|\nabla\phi_n\|_{L^\infty(0,T;L^\infty(\Omega))}\int_0^T\int_{\{u^{(\tau)}_{\varepsilon,i}u^{(\tau)}_{\varepsilon,j}\le\kappa\}}\sqrt{u^{(\tau)}_{\varepsilon,i}u^{(\tau)}_{\varepsilon,j}}\Big(\sqrt{u^{(\tau)}_{\varepsilon,i}u^{(\tau)}_{\varepsilon,j}}|\nabla(q_i(\boldsymbol{u}^{(\tau)}_\varepsilon)-q_j(\boldsymbol{u}^{(\tau)}_\varepsilon))|\Big)\,dxdt \\
&\le \frac{C}{n} + \lim_{\kappa\to 0}\lim_{\tau\to 0}\|\nabla\phi_n\|_{L^\infty(0,T;L^\infty(\Omega))}\Big(\int_0^T\int_{\{u^{(\tau)}_{\varepsilon,i}u^{(\tau)}_{\varepsilon,j}\le\kappa\}} u^{(\tau)}_{\varepsilon,i}u^{(\tau)}_{\varepsilon,j}\,dxdt\Big)^{1/2} \\
&\qquad\quad \times\|u^{(\tau)}_{\varepsilon,i}u^{(\tau)}_{\varepsilon,j}\nabla(q_i(\boldsymbol{u}^{(\tau)}_\varepsilon)-q_j(\boldsymbol{u}^{(\tau)}_\varepsilon))\|_{L^2(0,T;L^2(\Omega))} \\
&\le \frac{C}{n} + \lim_{\kappa\to 0}\sqrt{\kappa}C\|\nabla\phi_n\|_{L^\infty(0,T;L^\infty(\Omega))} \\
&\le \frac{C}{n}.
\end{aligned}
$$

Taking the limit as $n\to\infty$ in the previous inequality yields (3.42).

We now consider again (3.37). Passing to the limit as $\tau\to 0$ and $\kappa\to 0$ and taking into account what discussed so far gives from (3.39) that

$$J_{\varepsilon,ij} = u_{\varepsilon,i}u_{\varepsilon,j}\nabla(q_i(\boldsymbol{u}_\varepsilon)-q_j(\boldsymbol{u}_\varepsilon)) \quad \text{a.e. in } \Omega\times(0,T).$$

The proof is thus complete. □

Proof of Theorem 2.3. In order to obtain the desired result, it remains to pass to the limit as $\tau\to 0$ in (3.28) using the convergences in (3.34) and the result of Lemma 3.11.

It follows that there exists $\boldsymbol{u}_\varepsilon\in L^2(0,T;H^1(\Omega))^{n+1}$ with $\partial_t\boldsymbol{u}_\varepsilon\in L^2(0,T;H^2(\Omega)')^{n+1}$ such that

$$
\begin{aligned}
&\int_0^T\langle\partial_t u_{\varepsilon,i},\varphi_i\rangle_{H^2(\Omega)',H^2(\Omega)}\,dt \\
&= -\int_0^T\int_\Omega\sum_{0\le j\ne i\le n}L_{ij}\Big[u_{\varepsilon,j}\nabla u_{\varepsilon,i}-u_{\varepsilon,i}\nabla u_{\varepsilon,j}+u_{\varepsilon,i}u_{\varepsilon,j}\nabla\Big(\sum_{k=0}^n c_{ik}B_\varepsilon(u_{\varepsilon,k})-\sum_{l=0}^n c_{jl}B_\varepsilon(u_{\varepsilon,l})\Big)\Big]\cdot\nabla\varphi_i\,dxdt,
\end{aligned}
$$

for all $i=0,\dots,n$ and for all $\varphi_i\in L^2(0,T;H^2(\Omega))$. The previous weak formulation implies that $\partial_t u_{\varepsilon,i}\in L^2(0,T;H^1(\Omega)')$ and therefore by density we can extend the whole formulation to any $\varphi_i\in L^2(0,T;H^1(\Omega))$ as follows

$$
\begin{aligned}
&\int_0^T\langle\partial_t u_{\varepsilon,i},\varphi_i\rangle_{H^1(\Omega)',H^1(\Omega)}\,dt \\
&= -\int_0^T\int_\Omega\sum_{0\le j\ne i\le n}L_{ij}\Big[u_{\varepsilon,j}\nabla u_{\varepsilon,i}-u_{\varepsilon,i}\nabla u_{\varepsilon,j}+u_{\varepsilon,i}u_{\varepsilon,j}\nabla\Big(\sum_{k=0}^n c_{ik}B_\varepsilon(u_{\varepsilon,k})-\sum_{l=0}^n c_{jl}B_\varepsilon(u_{\varepsilon,l})\Big)\Big]\cdot\nabla\varphi_i\,dxdt.
\end{aligned}
$$

Finally, we have that necessarily it holds $u_{\varepsilon,i}(0,x)=u^0_{\varepsilon,i}(x)$, reasoning as in [19]. Therefore, it turns out that $(\boldsymbol{u}_\varepsilon,J_\varepsilon)$ is a weak solution to (2.10) in the sense of Definition 2.2. The proof is thus complete. □

4. From nonlocal to local

In this section we perform the limit as $\varepsilon \to 0$. We start by collecting some estimates coming from the results obtained in the previous section.

Lemma 4.1. *There exists a constant $C > 0$, independent of ε, such that for any $i = 0, \dots, n$ it holds*

$$\int_0^T \|\partial_t u_{\varepsilon,i}\|_{H^2(\Omega)'}^2 \, dt \le C, \tag{4.1}$$

$$\int_0^T \int_\Omega |\nabla u_{\varepsilon,i}|^2 \, dxdt \le C, \tag{4.2}$$

$$\|B_\varepsilon(u_{\varepsilon,i})\|_{L^2(0,T;H^1(\Omega)')} \le C, \tag{4.3}$$

$$\int_0^T \int_\Omega \int_\Omega c_{ii} K_\varepsilon(x,y) |\nabla u_{\varepsilon,i}(x) - \nabla u_{\varepsilon,i}(y)|^2 \, dxdydt \le C, \tag{4.4}$$

$$\sum_{0 \le j \ne i \le n} \int_0^T \int_\Omega |J_{\varepsilon,ij}|^2 \, dxdt \le C. \tag{4.5}$$

Proof. Property (4.1) follows from Lemma 3.10 and (3.30). Estimates (3.21) and (3.29) imply (4.2). Equation (3.32) gives (4.3). Equations (3.22) and (3.29) yield (4.4). From Lemma 3.11 and equation (3.23) we infer (4.5). □

We are finally in a position to prove Theorem 2.5.

Proof of Theorem 2.5. In view of (4.1) and (4.2) we infer (2.13), while (2.14) follows from (4.3) and Lemma 2.1.

It remains to show that such $\boldsymbol{u}$ actually satisfies the requirements in Definition 2.4.

We first observe that, thanks to (2.13), there holds

$$u_{\varepsilon,i} \to u_i \quad \text{strongly in } L^2(0,T;L^2(\Omega)), \tag{4.6}$$

while (4.4) and [17, Lemma 4] imply that

$$u_{\varepsilon,i} \to u_i \quad \text{strongly in } L^2(0,T;H^1(\Omega)). \tag{4.7}$$

This implies that (1_L) and (2_L) are satisfied. Condition (3_L) follows from (2.13) and the Aubin-Lions' lemma. In order to verify condition (4_L) we note the following: for every $\varphi_i \in H^1(\Omega)$, thanks to (2.13) there holds

$$\int_0^T \langle \partial_t u_{\varepsilon,i}, \varphi_i \rangle_{H^1(\Omega)', H^1(\Omega)} \, dt \to \int_0^T \langle \partial_t u_i, \varphi_i \rangle_{H^1(\Omega)', H^1(\Omega)} \, dt.$$

Moreover, using again (2.13) gives

$$\int_0^T \int_\Omega \sum_{0 \le j \ne i \le n} L_{ij} (u_{\varepsilon,j} \nabla u_{\varepsilon,i} - u_{\varepsilon,i} \nabla u_{\varepsilon,j}) \cdot \nabla \varphi_i \, dxdt \to \int_0^T \int_\Omega \sum_{0 \le j \ne i \le n} L_{ij} (u_j \nabla u_i - u_i \nabla u_j) \cdot \nabla \varphi_i \, dxdt.$$

Concerning the last term in (2.12), thanks to Lemma 3.11 we have

$$\int_0^T \int_\Omega \sum_{0 \le j \ne i \le n} L_{ij} J_{\varepsilon,ij} \cdot \nabla \varphi_i = - \int_0^T \int_\Omega \operatorname{div}(u_{\varepsilon,i} u_{\varepsilon,j} \nabla \varphi_i)(q_i(u_\varepsilon) - q_j(u_\varepsilon)).$$

Combining (4.6), (4.7) and (2.14) and from the uniform boundedness of $u_{\varepsilon,i}$ in $L^\infty((0,T) \times \Omega)$, it follows that

$$\int_0^T \int_\Omega \sum_{0 \le j \ne i \le n} L_{ij} J_{\varepsilon,ij} \cdot \nabla \varphi_i \to \int_0^T \int_\Omega \sum_{0 \le j \ne i \le n} L_{ij} J_{ij} \cdot \nabla \varphi_i,$$

with

$$J_{ij} = u_i u_j \nabla(\Delta u_i - \Delta u_j),$$

and $J_{ij} \in L^2(0,T;L^2(\Omega))$, due to (4.5). This completes the proof of (4_L) and of the theorem. □

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(E. Davoli) Tu Wien, Institute for Analysis and Scientific Computing, Wiedner Hauptstrasse 8-10, 1040 Wien, Austria

Email address: elisa.davoli@tuwien.ac.at

(G. Marino) Universität Augsburg, Institut für Mathematik, Universitätsstrasse 12a, 86159 Augsburg, Germany

Email address: greta.marino@uni-a.de

(J.-F. Pietschmann) Universität Augsburg, Institut für Mathematik, Universitätsstrasse 12a, 86159 Augsburg, Germany and Centre for Advanced Analytics and Predictive Sciences (CAAPS), University of Augsburg, Universitätsstr. 12a, 86159 Augsburg, Germany.

Email address: jan-f.pietschmann@uni-a.de