H-COMPACTNESS FOR NONLOCAL LINEAR OPERATORS IN FRACTIONAL DIVERGENCE FORM

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Abstract. We study the H-convergence problem for a class of nonlocal linear operators in fractional divergence form, where the oscillations of the matrices are prescribed outside the reference domain. Our compactness argument bypasses the failure of the classical localisation techniques, that mismatch with the nonlocal nature of the operators involved. If symmetry is also assumed, we show that the H-compactness can be equivalently obtained through the Γ-convergence of the associated energies.

1. Introduction

At the beginning of the '70s the French school of J.-L. Lions set the stage for the mathematical theory of the *homogenisation of composite materials*, made by the superposition of sheets of different materials, or considering homogeneous materials with holes filled by another material. For a given source term f, this problem is mathematically modelled by a symmetric matrix-valued function $A(x)$, identifying the different materials at each point x, and expressed through partial differential equations in divergence form

$$-\operatorname{div}(A(x)\nabla u(x)) = f(x). \tag{1.1}$$

An equivalent statement of the problem (1.1) considers the momentum $p(x) \coloneqq A(x)\nabla u(x)$ and is formulated as

$$-\operatorname{div}(p(x)) = f(x).$$

Such equations appear in many applications in physics and engineering: in the *equation of the electrostatics*, u is the electric potential, p is the electric displacement, and A is the dielectric constant, while, in the *equation of magnetostatics*, u is the magnetic potential, p is the magnetic induction, and A is the magnetic permeability. Other examples are the *equation of the time-independent heat transfer*, where u is the temperature, p is the heat flux, and A represents the thermal conductivity and, in the vectorial case, the *equation of the linear elasticity for composite materials*, where u is the displacement field, p (usually denoted by σ) is the Cauchy stress tensor, and A represents the elasticity tensor.

From a numerical point of view, when the coefficients of the matrix A oscillate too fast, it can be very difficult to deal with the above equations, for example when the heterogeneous material has a periodic structure. A possible way to overcome this problem is to approximate the solutions by the (unique) solution of a limit problem, much easier to deal with numerically, in which the matrix A is independent on x, the so-called *homogenised* problem.

The first relevant contributions to this theory are by Sánchez-Palencia [28, 29, 30], through applications of the *asymptotic expansions* method, consisting in approximating solutions of (1.1) with a series depending on the layers of the material, and then estimating the resulting error.

A more general approach, introduced by Spagnolo in the pioneering works [36, 37, 38], is the so-called G-convergence of linear partial differential equations of the form

$$-\operatorname{div}(A_h(x)\nabla u(x)) = f(x), \quad h \in \mathbb{N}. \tag{1.2}$$

The G-convergence theory studies the asymptotic behaviour of the solutions of problems (1.2), assuming very mild hypotheses on the coefficients of any matrix A_h, and it is based on the distributional nature of the problems (1.2) and some standard tools of functional analysis.

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Nevertheless, at its early stage it was not clear what properties the limit with respect to the G-convergence would have satisfied. A first characterisation of the limit operator (the so-called G-limit) was only later obtained by De Giorgi and Spagnolo [17], who show that the G-limit still acts as a divergence form operator depending on a limit homogenised matrix A_∞ (sometimes called *effective* matrix and denoted by A^{eff}). We underline that, whenever $A_h(x) = A(hx)$, we recover the former theory of the homogenisation of composite materials but, differently from it, the limit matrix of the G-convergence, $A_\infty = A_\infty(x)$, may in general still be x-dependent.

By its very nature, the G-convergence is sensible according to the class of operators considered and, without further requirements, it appears incomplete when dealing with *non-symmetric* matrices, needed in applications involving e.g. porous media. Tartar observes in [41] that the divergence operator does not in fact recognise perturbations from skew-symmetric matrices leading, in this case, to the non-uniqueness of A_∞.

This lack of uniqueness is bypassed by Tartar [40] and Murat [25, 26] at the end of the '70s, and this theory in nowadays called H-convergence, where H stands for *homogenisation*. The main difference with the former G-convergence naturally appears in the non-symmetric case: the uniqueness of the homogenised matrix A_∞ is now guaranteed, whenever the sequence of the momenta $p_h = A_h \nabla u_h$ converges (in the weak topology) to the momentum of the limit solution $p_\infty = A_\infty \nabla u_\infty$.

In the last decade there has been an increasing interest in extending these results to a *nonlocal* framework, motivated by the huge number of applications in fluid mechanics, image denoising, nonlinear elasticity, nonlocal minimal surfaces, anomalous diffusion, and stable Lévy processes. First attempts to deal with nonlocal H-convergence appear in [18], where the authors study scalar perturbations of the fractional p-Laplace operator in the linear and superlinear cases, and in [8], regarding the H-convergence of fractional powers of elliptic operators in divergence form. We also mention [5, 19, 27] for the homogenisation of more general nonlocal energies and [9, 10] for other applications involving nonlocal operators.

All the aforementioned contributions in nonlocal H-convergence consider, however, only the case of scalar weights, in contrast to the classical results by Tartar [41]. On the other hand, the case of matrix weights is of great interest in applications, as it allows to study anisotropic heterogeneous materials. We observe that it is not immediately clear how to get a momentum operator in problems involving Gagliardo seminorms, since the latters are defined via integration of scalar energies.

We hence wonder whether it might be possible to formulate a nonlocal H-convergence-type problem à la Tartar. To this aim, we exploit a suitable notion of fractional-order divergence div^s and fractional-order gradient ∇^s (see Definition 2.2), in a way that the fractional divergence acts on the nonlocal momentum $p(x) = A(x)\nabla^s u(x)$. The construction of this class of fractional-order operators is explained in details in Section 2, and relies on the pioneering contributions by Shieh and Spector [33, 34], Šilhavý [42], and the subsequent works [7, 11, 12, 13].

This leads to the study of the H-convergence of the following sequence of elliptic problems

$$(P_h^f) \qquad \begin{cases} -\operatorname{div}^s(A_h \nabla^s u_h) = f & \text{in } \Omega, \\ u_h = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

$h \in \mathbb{N}$, on the same spirit of Tartar and Murat.

When dealing with the asymptotics of problems (P_h^f), there are two main issues to be taken into account. The first one is to overcome the localisation techniques needed in the H-convergence (see e.g. [41] for a deep discussion on this topic), and this is done by exploiting the distributional nature of (P_h^f) and using a suitable Leibniz rule (see Proposition 2.7).

The second question concerns the behaviour of the matrices $A_h(x)$ outside the reference domain Ω. Indeed, differently from the local scenario, since the fractional gradient ∇^s must be defined on the whole space $\mathbb{R}^n$, then the fractional divergence div^s acts on vector fields globally defined over $\mathbb{R}^n$ (see Definition 2.2). Hence, the matrix-valued functions $A_h(x)$ must be defined on the whole space $\mathbb{R}^n$.

On the other hand, $-\operatorname{div}^s(A_h\nabla^s u_h)$ is bounded only in Ω, and so we cannot hope to obtain compactness of $(A_h)_h$ outside the reference domain. In order to tackle this lack of compactness, the values of any matrix $A_h(x)$ are prescribed outside Ω by a fixed matrix $A_0(x)$ which satisfies standard growth conditions.

In details, we study the limiting behaviour, in the sense of the H-convergence, of the class of matrix-valued functions

$$\mathcal{M}(\lambda,\Lambda,\Omega,A_0) := \{A \in \mathcal{M}(\lambda,\Lambda,\mathbb{R}^n) : A = A_0 \text{ a.e. in } \mathbb{R}^n\setminus\Omega\}$$

for positive constants $\lambda \leq \Lambda$ and $A_0 \in \mathcal{M}(\lambda,\Lambda,\mathbb{R}^n)$, where $\mathcal{M}(\lambda,\Lambda,\mathbb{R}^n)$ denotes the class of matrix-valued measurable functions $A\colon \mathbb{R}^n \to \mathbb{R}^{n\times n}$ satisfying the growth conditions

$$A(x)\xi\cdot\xi \geq \lambda|\xi|^2 \qquad \text{for all } \xi\in\mathbb{R}^n \text{ and for a.e. } x\in\mathbb{R}^n,$$
$$A(x)\xi\cdot\xi \geq \Lambda^{-1}|A(x)\xi|^2 \qquad \text{for all } \xi\in\mathbb{R}^n \text{ and for a.e. } x\in\mathbb{R}^n.$$

Our first main result Theorem 3.1 is the H-compactness of the class $\mathcal{M}(\lambda,\Lambda,\Omega,A_0)$, with respect to a suitable notion of nonlocal H-convergence, which is Definition 2.13. More precisely, we prove that any sequence of matrix-valued functions $(A_h)_h$ in $\mathcal{M}(\lambda,\Lambda,\Omega,A_0)$ admits, up to a subsequence, a limit matrix A_∞ such that $(A_h)_h$ H-converges to A_∞. Moreover, A_∞ still belongs to $\mathcal{M}(\lambda,\Lambda,\Omega,A_0)$, leading to the compactness of the class.

We point out that our H-limit A_∞ coincides within the reference domain with its local counterpart, showing a consistency between the local and nonlocal H-convergence. As a by-product, in Theorem 3.2 we show that also the subclass of symmetric matrices, here denoted by $\mathcal{M}^{\mathrm{sym}}(\lambda,\Lambda,\Omega,A_0)$, is H-compact, i.e. if $(A_h)_h$ is in $\mathcal{M}^{\mathrm{sym}}(\lambda,\Lambda,\Omega,A_0)$, then A_∞ belongs to $\mathcal{M}^{\mathrm{sym}}(\lambda,\Lambda,\Omega,A_0)$.

In the second part of this paper, we focus exclusively on the symmetric case and prove the Γ-compactness of the class of nonlocal energies associated with $\mathcal{M}^{\mathrm{sym}}(\lambda,\Lambda,\Omega,A_0)$, which is Theorem 3.3. Similarly to before, this is achieved by exploiting the classical local Γ-compactness result for quadratic functionals and using the properties of the Riesz potential to pass to the nonlocal scenario. A very non-exhaustive list of recent references about the Γ-convergence of fractional quadratic energies is given by [1, 6, 31, 35]. In particular, in [6], the Γ-compactness is obtained using the Beurling-Deny criteria for Dirichlet forms, which are unfortunately not available in our setting since it is not known whether quadratic forms of the fractional gradient are Dirichlet forms.

In the last part of the paper, we present an alternative variational characterization of the H-compactness of $\mathcal{M}^{\mathrm{sym}}(\lambda,\Lambda,\Omega,A_0)$ (Theorem 3.2) via Γ-convergence. First, in Proposition 3.4, we prove that the convergence of momenta, required in the H-convergence, can be obtained through Γ-convergence. Finally, we use the aforementioned Γ-compactness result (Theorem 3.3) to recover the H-compactness of $\mathcal{M}^{\mathrm{sym}}(\lambda,\Lambda,\Omega,A_0)$.

The paper is structured as follows. In Section 2, all the preliminaries are provided and discussed in details. In Section 3, we first state and prove our main H-compactness results for nonlocal linear operators both in the general case and in the case of symmetric matrices, which are Theorem 3.1 and Theorem 3.2, respectively. Later, we focus on the symmetric case and prove the Γ-compactness Theorem 3.3. Finally, we prove that the H-compactness Theorem 3.2 can be equivalently obtained as a consequence of the previous Γ-compactness Theorem 3.3 and the (variational) convergence of the momenta Proposition 3.4. Finally, a list of some open problems and new research directions drawn from this work is presented in Section 4.

2. Preliminaries

2.1. **Notation.** We assume that $s\in(0,1)$, $n\geq 2$, and that Ω is a bounded open subset of $\mathbb{R}^n$. The space of $n\times n$ real matrices is denoted by $\mathbb{R}^{n\times n}$, and the subspace of symmetric matrices is denoted by $\mathbb{R}^{n\times n}_{\mathrm{sym}}$. We adopt standard notation for Lebesgue spaces on measurable subsets $E\subseteq\mathbb{R}^n$ and Sobolev spaces on open subsets $O\subseteq\mathbb{R}^n$. The norm of a generic Banach space X is denoted by $\|\cdot\|_X$. We denote by X' the dual space of X, and by $\langle\cdot,\cdot\rangle_{X'\times X}$ the duality product between X' and X.

2.2. **The functional setting.** For any $\alpha \in (0,n)$, the α-Riesz potential of a measurable function $f\colon \mathbb{R}^n \to \mathbb{R}$ is defined as

$$I_\alpha f(x) \coloneqq \frac{1}{\gamma_\alpha} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, \mathrm{d}y, \quad \text{with } \gamma_\alpha \coloneqq 2^\alpha \pi^{\frac{n}{2}} \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}.$$

Notice that I_α is a Fourier multiplier that acts in the Fourier space as

$$\mathcal{F}\left(I_\alpha f\right)(\xi) = |\xi|^{-\alpha} \mathcal{F}\left(f\right)(\xi),$$

being $\mathcal{F}$ the Fourier transform. Moreover, as shown in [39, Theorem 1, pag. 119], we have the following result.

Proposition 2.1. *Let $\alpha \in (0,n)$ and $p \in \left(1, \frac{n}{\alpha}\right)$. For all $f \in L^p(\mathbb{R}^n)$, the α-Riesz potential $I_\alpha f$ is well-defined and there exists a positive constant C, depending only on α, n, and p, such that*

$$\|I_\alpha f\|_{L^{p^*_\alpha}(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)},$$

where $p^*_\alpha \coloneqq \frac{np}{n-\alpha p}$.

Let us recall the notions of fractional divergence and fractional gradient that will be used to introduce the H-convergence problem in our framework.

Definition 2.2. *Let $\psi \in C^\infty_c(\mathbb{R}^n)$ and $\phi \in C^\infty_c(\mathbb{R}^n;\mathbb{R}^n)$ be fixed. We define the s-fractional gradient $\nabla^s\psi$ and the s-fractional divergence $\operatorname{div}^s \phi$, respectively, as*

$$\nabla^s\psi(x) \coloneqq \frac{n-1+s}{\gamma_{1-s}} \int_{\mathbb{R}^n} \frac{(\psi(x)-\psi(y))(x-y)}{|x-y|^{n+s+1}}\, \mathrm{d}y \quad \textit{for all } x \in \mathbb{R}^n,$$
$$\operatorname{div}^s \phi(x) \coloneqq \frac{n-1+s}{\gamma_{1-s}} \int_{\mathbb{R}^n} \frac{(\phi(x)-\phi(y))\cdot(x-y)}{|x-y|^{n+s+1}}\, \mathrm{d}y \quad \textit{for all } x \in \mathbb{R}^n.$$

The next result links the notions of fractional gradient and fractional divergence with the classical ones. A proof can be found in [33, Theorem 1.2] and [11, Proposition 2.2].

Proposition 2.3. *Let $\psi \in C^\infty_c(\mathbb{R}^n)$ and $\phi \in C^\infty_c(\mathbb{R}^n;\mathbb{R}^n)$. It holds that*

$$\nabla^s\psi = \nabla(I_{1-s}\psi) = I_{1-s}(\nabla\psi) \quad \textit{in } \mathbb{R}^n,$$
$$\operatorname{div}^s \phi = I_{1-s}(\operatorname{div}\phi) = \operatorname{div}(I_{1-s}\phi) \quad \textit{in } \mathbb{R}^n.$$

We now introduce the functional framework for our problems in fractional divergence form.

Definition 2.4. *Let $O \subseteq \mathbb{R}^n$ be an open set. We denote by $H^s_0(O)$ the Hilbert space defined as the closure of $C^\infty_c(O)$ with respect to the following norm*

$$\|\cdot\|_{H^s(\mathbb{R}^n)} \coloneqq \left(\|\cdot\|^2_{L^2(\mathbb{R}^n)} + \|\nabla^s\cdot\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^n)}\right)^{1/2}$$

and, by $H^{-s}(O)$, the dual space of $H^s_0(O)$. When $O=\mathbb{R}^n$, we have that $H^s_0(\mathbb{R}^n) = H^s(\mathbb{R}^n)$.

As a consequence of Proposition 2.3, one can immediately show, through Fourier transform, that for all $\psi \in C^\infty_c(\mathbb{R}^n)$ and $s,\sigma \in (0,1)$ we have

$$-\operatorname{div}^s(\nabla^\sigma\psi) = (-\Delta)^{\frac{s+\sigma}{2}}\psi. \tag{2.1}$$

Moreover, by Fubini's Theorem, it holds that

$$\int_{\mathbb{R}^n} \nabla^s\psi(x)\cdot\phi(x)\,\mathrm{d}x = -\int_{\mathbb{R}^n} \psi(x)\operatorname{div}^s\phi(x)\,\mathrm{d}x \tag{2.2}$$

for all $\phi \in C^\infty_c(\mathbb{R}^n;\mathbb{R}^n)$ and $\psi \in C^\infty_c(\mathbb{R}^n)$.

If we extend the operators ∇^s and div^s respectively to $H^s(\mathbb{R}^n)$ and $H^s(\mathbb{R}^n;\mathbb{R}^n)$, then (2.2) holds for all $\phi \in H^s(\mathbb{R}^n;\mathbb{R}^n)$ and $\psi \in H^s(\mathbb{R}^n)$. In particular, we can define the operator $\operatorname{div}^s\colon L^2(\mathbb{R}^n;\mathbb{R}^n) \to H^{-s}(\Omega)$ as

$$\langle \operatorname{div}^s u, v\rangle_{H^{-s}(\Omega)\times H^s_0(\Omega)} \coloneqq -\int_{\mathbb{R}^n} u(x)\cdot\nabla^s v(x)\,\mathrm{d}x$$

for all $u \in L^2(\mathbb{R}^n;\mathbb{R}^n)$ and $v \in H^s_0(\Omega)$.

The following two Propositions provide a useful connection between the Sobolev spaces $H^1(\mathbb{R}^n)$ and $H^s(\mathbb{R}^n)$. For the proof of the next Proposition we refer the interested reader to [7, Lemma A.4].

Proposition 2.5. *Let $w \in H^1(\mathbb{R}^n)$ and set $u := (-\Delta)^{\frac{1-s}{2}} w$. It holds that*

$$u \in H^s(\mathbb{R}^n) \quad \textit{and} \quad \nabla^s u = \nabla w \ \textit{a.e. in } \mathbb{R}^n.$$

Moreover, there exists a positive constant C, depending only on n and s, such that the following estimate holds true

$$\|u\|_{L^2(\mathbb{R}^n)} \le C(n,s)\|w\|^s_{L^2(\mathbb{R}^n)}\|\nabla w\|^{1-s}_{L^2(\mathbb{R}^n;\mathbb{R}^n)}. \tag{2.3}$$

Proposition 2.6. *Let $u \in H^s(\mathbb{R}^n)$ and set $w := I_{1-s}u$. It holds that*

$$w \in \left\{ v \in L^{2^*_{1-s}}(\mathbb{R}^n) \, : \, \nabla v \in L^2(\mathbb{R}^n) \right\} \quad \textit{and} \quad \nabla w = \nabla^s u \ \textit{a.e. in } \mathbb{R}^n,$$

where $2^*_{1-s} = \frac{2n}{n-2+2s}$.

In particular, $w \in H^1_{\mathrm{loc}}(\mathbb{R}^n)$ and, for every open set $O \Subset \mathbb{R}^n$, there exists a positive constant C, depending only on n, s and O, such that the following estimate holds true

$$\|w\|_{H^1(O)} \le C(n,s)\|u\|_{H^s(\mathbb{R}^n)}. \tag{2.4}$$

Proof. If $u \in C^\infty_c(\mathbb{R}^n)$, the result is a consequence of Proposition 2.1 (with $p = 2$ and $\alpha = 1-s$) and of Proposition 2.3.

Let now $(u_h)_h \subset C^\infty_c(\mathbb{R}^n)$ be such that $u_h \to u$ strongly in $H^s(\mathbb{R}^n)$. Then,

$$w_h := I_{1-s}u_h \to w := I_{1-s}u \quad \text{strongly in } L^{2^*_{1-s}}(\mathbb{R}^n) \text{ as } h \to \infty$$

and

$$\nabla w_h = \nabla^s u_h \to \nabla^s u \quad \text{strongly in } L^2(\mathbb{R}^n;\mathbb{R}^n) \text{ as } h \to \infty,$$

which gives the existence of $\nabla w = \nabla^s u \in L^2(\mathbb{R}^n;\mathbb{R}^n)$. Thus, the estimate (2.4) follows again in virtue of Proposition 2.1. □

We conclude this subsection recalling some extensions of classical results to the framework of fractional calculus, that will be used throughout the paper. They are a Leibniz-type rule for the fractional gradient [12, Eq. (1.5), (1.6)] and [20, Eq. (2.11)], a Poincaré-type inequality [33, Theorem 3.3], and a Rellich-type Theorem [34, Theorem 2.2].

Proposition 2.7 (Leibniz rule)**.** *Let $\varphi \in C^1_c(\Omega)$ and $u \in H^s(\mathbb{R}^n)$. Then, $\varphi u \in H^s_0(\Omega)$ and*

$$\nabla^s(\varphi u) = \varphi\nabla^s u + u\nabla^s\varphi + \nabla^s_{\mathrm{NL}}(\varphi, u)$$

where, for every $x \in \mathbb{R}^n$, the remainder term $\nabla^s_{\mathrm{NL}}(\varphi,u)(x)$ is

$$\nabla^s_{\mathrm{NL}}(\varphi,u)(x) := \frac{n-1+s}{\gamma_{1-s}}\int_{\mathbb{R}^n} \frac{(\varphi(x)-\varphi(y))(u(x)-u(y))(x-y)}{|x-y|^{n+s+1}}\,\mathrm{d}y.$$

Moreover, there exists a positive constant C, depending only on n and s, such that

$$\|\nabla^s_{\mathrm{NL}}(\varphi,u)\|_{L^2(\mathbb{R}^n;\mathbb{R}^n)} \le C\|u\|_{L^2(\mathbb{R}^n)}\|\varphi\|^{1-s}_{L^\infty(\Omega)}\|\nabla\varphi\|^s_{L^\infty(\Omega;\mathbb{R}^n)}.$$

Proposition 2.8 (Poincaré inequality)**.** *For every set $O \Subset \mathbb{R}^n$ there exists a positive constant C, depending only on n, s and O, such that*

$$\|u\|_{L^2(O)} \le C\|\nabla^s u\|_{L^2(\mathbb{R}^n;\mathbb{R}^n)} \quad \textit{for all } u \in H^s(\mathbb{R}^n).$$

Proposition 2.9 (Rellich Theorem)**.** *For every set $O \Subset \mathbb{R}^n$ the space $H^s_0(O)$ is compactly embedded into $L^2(O)$.*

2.3. **The H-convergence problem.** We now want to introduce our notion of H-convergence associated with the nonlocal operators introduced above. As in the local counterpart (see e.g. [41]), we begin by defining the class of matrices in which we are interested.

Definition 2.10. *Given $0 < \lambda \leq \Lambda < \infty$ and a measurable subset E of $\mathbb{R}^n$, we define $\mathcal{M}(\lambda, \Lambda, E)$ as the collection of all matrix-valued measurable functions $A\colon E \to \mathbb{R}^{n\times n}$ satisfying*

$$A(x)\xi \cdot \xi \geq \lambda|\xi|^2 \qquad \text{for all } \xi \in \mathbb{R}^n \text{ and for a.e. } x \in E, \tag{2.5}$$

$$A(x)\xi \cdot \xi \geq \Lambda^{-1}|A(x)\xi|^2 \qquad \text{for all } \xi \in \mathbb{R}^n \text{ and for a.e. } x \in E. \tag{2.6}$$

We also set

$$\mathcal{M}^{\mathrm{sym}}(\lambda, \Lambda, E) \coloneqq \{A \in \mathcal{M}(\lambda, \Lambda, E) : A = A^T \text{ a.e. in } E\}.$$

The estimate (2.6) above is needed to obtain the compactness of the class $\mathcal{M}(\lambda, \Lambda, \Omega)$ with respect to the H-convergence topology, even in the local setting. More precisely, it is well-known that the H-limit of sequences of non-symmetric matrices satisfying the standard growth condition

$$\lambda|\xi|^2 \leq A(x)\xi \cdot \xi \leq \Lambda|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n \text{ and for a.e. } x \in \Omega, \tag{2.7}$$

instead of (2.5) and (2.6), may belong to a wider class $\mathcal{M}(\lambda, \Lambda', \Omega)$ with $\Lambda' \geq \Lambda$ (see e.g. the observations of Tartar in [41, Chapter 6, Pag. 81]).

We also point out that, in the symmetric case $A = A^T$, conditions (2.5)–(2.6) are actually equivalent to (2.7), as shown in the following result.

Lemma 2.11. *Given $0 < \lambda \leq \Lambda < \infty$, let $B \in \mathbb{R}^{n\times n}$ satisfy*

$$B\xi \cdot \xi \geq \lambda|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n. \tag{2.8}$$

Then, B is invertible and condition

$$B\xi \cdot \xi \geq \Lambda^{-1}|B\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n \tag{2.9}$$

is equivalent to

$$B^{-1}\xi \cdot \xi \geq \Lambda^{-1}|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n. \tag{2.10}$$

In particular, B satisfies

$$B\xi \cdot \xi \leq \Lambda|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n. \tag{2.11}$$

Moreover, if B is symmetric ($B = B^T$), then conditions (2.9), (2.10) and (2.11) are all equivalent.

Proof. Let $B \in \mathbb{R}^{n\times n}$ satisfy (2.8). By Lax-Milgram Theorem, for all $\eta \in \mathbb{R}^n$ there exists a unique vector $\xi \in \mathbb{R}^n$ such that $B\xi = \eta$, i.e. B is invertible. Hence, (2.9) implies

$$\Lambda^{-1}|\xi|^2 = \Lambda^{-1}|BB^{-1}\xi|^2 \leq BB^{-1}\xi \cdot B^{-1}\xi = B^{-1}\xi \cdot \xi \quad \text{for all } \xi \in \mathbb{R}^n.$$

Conversely, by (2.10), we get

$$\Lambda^{-1}|B\xi|^2 \leq B^{-1}B\xi \cdot B\xi = B\xi \cdot \xi \quad \text{for all } \xi \in \mathbb{R}^n.$$

Assume now that B satisfies condition (2.9). By Cauchy-Schwartz inequality, we get

$$\Lambda^{-1}|B\xi|^2 \leq B\xi \cdot \xi \leq |B\xi||\xi| \quad \text{for all } \xi \in \mathbb{R}^n,$$

leading to

$$|B\xi| \leq \Lambda|\xi|. \tag{2.12}$$

Therefore, by applying again Cauchy-Schwartz inequality, we get

$$B\xi \cdot \xi \leq |B\xi||\xi| \leq \Lambda|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n$$

and so condition (2.9) implies (2.11).

Finally, assume that B is symmetric. It is enough to show that (2.11) implies (2.9). We start by observing that, by the symmetry of B, the bilinear form $(\xi, \eta) \mapsto (B\xi, \eta)$ is a scalar product in $\mathbb{R}^n$, in virtue of (2.11). Hence, by Cauchy-Schwartz inequality and (2.11), it holds that

$$|B\xi \cdot \eta|^2 \leq (B\xi \cdot \xi)(B\eta \cdot \eta) \leq (B\xi \cdot \xi)\Lambda|\eta|^2 \quad \text{for all } \xi, \eta \in \mathbb{R}^n.$$

In particular, for $\eta = B\xi$, we get

$$|B\xi|^4 = |B\xi \cdot B\xi|^2 \leq (B\xi \cdot \xi)\Lambda|B\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n,$$

which implies (2.9). □

From now on, we fix $0 < \lambda \leq \Lambda < \infty$. Given a sequence of matrices $(A_h)_h \subset \mathcal{M}(\lambda, \Lambda, \mathbb{R}^n)$, for all $f \in H^{-s}(\Omega)$ and $h \in \mathbb{N}$ we consider the following elliptic problems

$$(P_h^f) \qquad \begin{cases} -\operatorname{div}^s(A_h \nabla^s u_h) = f & \text{in } \Omega, \\ u_h = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where div^s and ∇^s are the fractional differential operators introduced in Definition 2.2. The following result ensures that the problem (P_h^f) is well-defined.

Lemma 2.12. *Let $A \in \mathcal{M}(\lambda, \Lambda, \mathbb{R}^n)$. For every $f \in H^{-s}(\Omega)$ there exists a unique (weak) solution $u \in H_0^s(\Omega)$ of the elliptic problem*

$$\begin{cases} -\operatorname{div}^s(A \nabla^s u) = f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

i.e. satisfying

$$\int_{\mathbb{R}^n} A(x)\nabla^s u(x) \cdot \nabla^s v(x) \, dx = \langle f, v \rangle_{H^{-s}(\Omega) \times H_0^s(\Omega)} \quad \text{for all } v \in H_0^s(\Omega).$$

Moreover, the solution u satisfies the following estimate

$$\|\nabla^s u\|_{L^2(\mathbb{R}^n;\mathbb{R}^n)} \leq \lambda^{-1} \|f\|_{H^{-s}(\Omega)}.$$

Proof. The proof is a direct application of Lax-Milgram Theorem and Proposition 2.8. Indeed, the bilinear form $a : H_0^s(\Omega) \times H_0^s(\Omega) \to \mathbb{R}$, defined as

$$a(u, v) := \int_{\mathbb{R}^n} A(x)\nabla^s u(x) \cdot \nabla^s v(x) \, dx \quad \text{for any } u, v \in H_0^s(\Omega),$$

is continuous and coercive, being $A \in \mathcal{M}(\lambda, \Lambda, \mathbb{R}^n)$. □

The notion of nonlocal H-convergence of $(A_h)_h \subset \mathcal{M}(\lambda, \Lambda, \mathbb{R}^n)$ we propose consists in finding a limit matrix $A_\infty \in \mathcal{M}(\lambda', \Lambda', \mathbb{R}^n)$, with $0 < \lambda' \leq \lambda \leq \Lambda \leq \Lambda' < \infty$, such that the sequence of problems $(P_h^f)_h$ is related to the limit problem

$$(P_\infty^f) \qquad \begin{cases} -\operatorname{div}^s(A_\infty \nabla^s u_\infty) = f & \text{in } \Omega, \\ u_\infty = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

in the sense of the next definition. In what follows, we denote by $u_h = u_h(f) \in H_0^s(\Omega)$, $h \in \mathbb{N}$, and by $u_\infty = u_\infty(f) \in H_0^s(\Omega)$ the unique weak solutions of (P_h^f) and (P_∞^f), respectively.

Definition 2.13 (Nonlocal H-convergence)**.** *Let $0 < \lambda' \leq \lambda \leq \Lambda \leq \Lambda' < \infty$ and consider $(A_h)_h \subset \mathcal{M}(\lambda, \Lambda, \mathbb{R}^n)$ and $A_\infty \in \mathcal{M}(\lambda', \Lambda', \mathbb{R}^n)$. We say that*

$$(A_h)_h \ H\text{-converges to } A_\infty \text{ in } H_0^s(\Omega)$$

if for all $f \in H^{-s}(\Omega)$ the following convergences simultaneously hold as $h \to \infty$:

(2.13) ***convergence of solutions:*** $u_h \to u_\infty$ *weakly in* $H_0^s(\Omega)$,

(2.14) ***convergence of momenta:*** $A_h \nabla^s u_h \to A_\infty \nabla^s u_\infty$ *weakly in* $L^2(\mathbb{R}^n; \mathbb{R}^n)$.

Definition 2.13 is the natural counterpart in the nonlocal setting of the local H-convergence, see e.g. [41, Definition 6.4]. For the readers' convenience, we recall such a notion in what follows.

Let $(B_h)_h \subset \mathcal{M}(\lambda, \Lambda, \Omega)$ be fixed. For all $h \in \mathbb{N}$ and $g \in H^{-1}(\Omega)$, we consider the following sequence of elliptic problems

$$(Q_h^g) \qquad \begin{cases} -\operatorname{div}(B_h \nabla w_h) = g & \text{in } \Omega, \\ w_h = 0 & \text{on } \partial\Omega. \end{cases}$$

Given $B_\infty \in \mathcal{M}(\lambda', \Lambda', \Omega)$, for some $0 < \lambda' \leq \lambda \leq \Lambda \leq \Lambda' < \infty$, we also consider the problem

$$(Q_\infty^g) \qquad \begin{cases} -\operatorname{div}(B_\infty \nabla w_\infty) = g & \text{in } \Omega, \\ w_\infty = 0 & \text{on } \partial\Omega, \end{cases}$$

and denote by $w_h = w_h(g) \in H_0^1(\Omega)$, $h \in \mathbb{N}$, and by $w_\infty = w_\infty(g) \in H_0^1(\Omega)$ the unique weak solutions of (Q_h^g) and (Q_∞^g), respectively.

Definition 2.14 (Local H-convergence). *Let $0 < \lambda' \leq \lambda \leq \Lambda \leq \Lambda' < \infty$, and consider $(B_h)_h \subset \mathcal{M}(\lambda, \Lambda, \Omega)$ and $B_\infty \in \mathcal{M}(\lambda', \Lambda', \Omega)$. We say that*

$$(B_h)_h \ H\text{-converges to } B_\infty \text{ in } H_0^1(\Omega),$$

if for all $g \in H^{-1}(\Omega)$ the following convergences simultaneously hold as $h \to \infty$:

$$\textbf{convergence of solutions:} \quad w_h \to w_\infty \quad \text{weakly in } H_0^1(\Omega), \tag{2.15}$$

$$\textbf{convergence of momenta:} \quad B_h \nabla w_h \to B_\infty \nabla w_\infty \quad \text{weakly in } L^2(\Omega; \mathbb{R}^n). \tag{2.16}$$

We recall that the class $\mathcal{M}(\lambda, \Lambda, \Omega)$ is compact with respect to the local H-convergence in $H_0^1(\Omega)$. A proof of this classical result can be found e.g. in [41, Theorem 6.5].

Proposition 2.15 (Local H-compactness). *For any $(B_h)_h \subset \mathcal{M}(\lambda, \Lambda, \Omega)$, there exist a not relabeled subsequence and a matrix-valued function $B_\infty \in \mathcal{M}(\lambda, \Lambda, \Omega)$ such that*

$$(B_h)_h \ H\text{-converges to } B_\infty \text{ in } H_0^1(\Omega).$$

We also recall that the local H-convergence is stable with respect to the transpose operation. More precisely, we have the following result, due to Tartar, whose proof can be found in [41, Lemma 10.2].

Proposition 2.16 (H-convergence of the transpose). *Let $B_h, B_\infty \in \mathcal{M}(\lambda, \Lambda, \Omega)$, $h \in \mathbb{N}$, and assume that*

$$(B_h)_h \ H\text{-converges to } B_\infty \text{ in } H_0^1(\Omega).$$

Then,

$$(B_h^T)_h \ H\text{-converges to } B_\infty^T \text{ in } H_0^1(\Omega).$$

The first goal of this paper is to show that suitable subclasses of $\mathcal{M}(\lambda, \Lambda, \mathbb{R}^n)$ are compact with respect to the *nonlocal* H-convergence. The classes of matrices we are interested in are defined as follows.

Definition 2.17. *Fix $0 < \lambda \leq \Lambda < \infty$ and $A_0 \in \mathcal{M}(\lambda, \Lambda, \mathbb{R}^n)$. We define*

$$\mathcal{M}(\lambda, \Lambda, \Omega, A_0) := \{A \in \mathcal{M}(\lambda, \Lambda, \mathbb{R}^n) : A = A_0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}.$$

For $A_0 \in \mathcal{M}^{\mathrm{sym}}(\lambda, \Lambda, \mathbb{R}^n)$, we also set

$$\mathcal{M}^{\mathrm{sym}}(\lambda, \Lambda, \Omega, A_0) := \{A \in \mathcal{M}(\lambda, \Lambda, \Omega, A_0) : A = A^T \text{ a.e. in } \mathbb{R}^n\}.$$

In the next section we show that any sequence $(A_h)_h$ in $\mathcal{M}(\lambda, \Lambda, \Omega, A_0)$ (respectively in $\mathcal{M}^{\mathrm{sym}}(\lambda, \Lambda, \Omega, A_0)$), for a fixed matrix $A_0 \in \mathcal{M}(\lambda, \Lambda, \mathbb{R}^n)$ (respectively in $\mathcal{M}^{\mathrm{sym}}(\lambda, \Lambda, \mathbb{R}^n)$), admits a not relabeled subsequence and a limit matrix A_∞ in $\mathcal{M}(\lambda, \Lambda, \Omega, A_0)$ (respectively in $\mathcal{M}^{\mathrm{sym}}(\lambda, \Lambda, \Omega, A_0)$) such that

$$(A_h)_h \ H\text{-converges to } A_\infty \text{ in } H_0^s(\Omega)$$

in the sense of Definition 2.13. In particular, the H-limit A_∞ satisfies (2.5)–(2.6) with the same constants λ and Λ of the sequence $(A_h)_h$, leading to the compactness of both classes.

2.4. **The Γ-convergence problem.** Let us fix $A_0 \in \mathcal{M}^{\mathrm{sym}}(\lambda, \Lambda, \mathbb{R}^n)$ and a sequence $(A_h)_h$ in $\mathcal{M}^{\mathrm{sym}}(\lambda, \Lambda, \Omega, A_0)$. For all $h \in \mathbb{N}$, we consider the nonlocal energies $F_h \colon L^2(\mathbb{R}^n) \to [0, \infty]$ associated with A_h, represented by

$$F_h(u) := \begin{cases} \frac{1}{2} \int_{\mathbb{R}^n} A_h(x) \nabla^s u(x) \cdot \nabla^s u(x) \, \mathrm{d}x & \text{if } u \in H^s_0(\Omega), \\ \infty & \text{if } u \in L^2(\mathbb{R}^n) \setminus H^s_0(\Omega). \end{cases} \tag{2.17}$$

The second goal of this paper is to show the Γ-compactness in the strong topology of $L^2(\mathbb{R}^n)$ for the class of nonlocal energies (2.17). In particular, we prove the existence of a limit symmetric matrix-valued function $A_\infty \in \mathcal{M}^{\mathrm{sym}}(\lambda, \Lambda, \Omega, A_0)$ such that, up to a not relabeled subsequence,

$$(F_h)_h \ \Gamma\text{-converges to } F_\infty \text{ strongly in } L^2(\mathbb{R}^n),$$

where the limit nonlocal energy $F_\infty \colon L^2(\mathbb{R}^n) \to [0, \infty]$ is represented by

$$F_\infty(u) := \begin{cases} \frac{1}{2} \int_{\mathbb{R}^n} A_\infty(x) \nabla^s u(x) \cdot \nabla^s u(x) \, \mathrm{d}x & \text{if } u \in H^s_0(\Omega), \\ \infty & \text{if } u \in L^2(\mathbb{R}^n) \setminus H^s_0(\Omega). \end{cases} \tag{2.18}$$

As a consequence, in Subsection 3.3 we provide a second (equivalent) proof of the H-compactness for the class $\mathcal{M}^{\mathrm{sym}}(\lambda, \Lambda, \Omega, A_0)$, which relies exclusively on variational techniques.

We start recalling the notion of Γ-convergence for functionals. For a complete treatment of the topic, we refer the interested reader to the monographs [4, 15].

Definition 2.18. *Let X be a Banach space and let $E_h, E_\infty \colon X \to [0, \infty]$, $h \in \mathbb{N}$. We say that*

$$(E_h)_h \ \Gamma\text{-converges to } E_\infty \text{ strongly in } X$$

if the following two conditions simultaneously hold:

- **Γ-liminf inequality:** *for all $x \in X$ and for any sequence $(x_h)_h \subset X$ strongly converging to x in X one has*
$$E_\infty(x) \le \liminf_{h \to \infty} E_h(x_h);$$
- **Γ-limsup inequality:** *for all $x \in X$ there exists a recovery sequence $(y_h)_h \subset X$ strongly converging to x in X and such that*
$$E_\infty(x) \ge \limsup_{h \to \infty} E_h(y_h).$$

In analogy with Subsection 2.3, we remind that the class of local energies $G_h \colon L^2(\Omega) \to [0, \infty]$, $h \in \mathbb{N}$, associated with a sequence $(B_h)_h$ in $\mathcal{M}^{\mathrm{sym}}(\lambda, \Lambda, \Omega)$ and represented by

$$G_h(v) := \begin{cases} \frac{1}{2} \int_{\Omega} B_h(x) \nabla v(x) \cdot \nabla v(x) \, \mathrm{d}x & \text{if } v \in H^1(\Omega), \\ \infty & \text{if } v \in L^2(\Omega) \setminus H^1(\Omega), \end{cases} \tag{2.19}$$

is compact with respect to the Γ-convergence in the strong topology of $L^2(\Omega)$. A proof of the next result can be found in [32] and [15, Theorem 22.2].

Proposition 2.19 (Γ-compactness of local functionals)**.** *Let $(B_h)_h \subset \mathcal{M}^{\mathrm{sym}}(\lambda, \Lambda, \Omega)$ and, for any $h \in \mathbb{N}$, let $G_h \colon L^2(\Omega) \to [0, \infty]$ be the local energy associated with B_h as in (2.19). Then, there exists $G_\infty \colon L^2(\Omega) \to [0, \infty]$ such that, up to a not relabeled subsequence,*

$$(G_h)_h \ \Gamma\text{-converges to } G_\infty \text{ strongly in } L^2(\Omega).$$

Moreover, there exists a matrix-valued function $B_\infty \in \mathcal{M}^{\mathrm{sym}}(\lambda, \Lambda, \Omega)$ such that the Γ-limit G_∞ has the following integral representation

$$G_\infty(v) := \begin{cases} \frac{1}{2} \int_{\Omega} B_\infty(x) \nabla v(x) \cdot \nabla v(x) \, \mathrm{d}x & \text{if } v \in H^1(\Omega), \\ \infty & \text{if } v \in L^2(\Omega) \setminus H^1(\Omega). \end{cases} \tag{2.20}$$

Remark 2.20. In order to derive Proposition 2.19 from [32] and [15, Theorem 22.2], we recall that, according to Lemma 2.11, any symmetric matrix-valued function B belongs to the space $\mathcal{M}^{\mathrm{sym}}(\lambda, \Lambda, \Omega)$ if and only if the following growth condition is satisfied

$$\lambda|\xi|^2 \leq B(x)\xi \cdot \xi \leq \Lambda|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n \text{ and for a.e. } x \in \Omega. \tag{2.21}$$

3. Main results

3.1. H-compactness of nonlocal operators.

This first subsection is devoted to the H-compactness result for nonlocal operators. The proof of the following Theorem 3.1 relies on Proposition 2.15 and Proposition 2.16.

Theorem 3.1. *Fix $A_0 \in \mathcal{M}(\lambda, \Lambda, \mathbb{R}^n)$. For any $(A_h)_h \subset \mathcal{M}(\lambda, \Lambda, \Omega, A_0)$, there exist a not relabeled subsequence and a matrix-valued function $A_\infty \in \mathcal{M}(\lambda, \Lambda, \Omega, A_0)$ such that*

$$(A_h)_h \ H\text{-converges to } A_\infty \text{ in } H_0^s(\Omega).$$

Proof. For all $h \in \mathbb{N}$, we define

$$B_h := A_h|_\Omega \in \mathcal{M}(\lambda, \Lambda, \Omega). \tag{3.1}$$

By Proposition 2.15, there exists a limit matrix

$$B_\infty \in \mathcal{M}(\lambda, \Lambda, \Omega) \tag{3.2}$$

such that, up to subsequences,

$$(B_h)_h \ H\text{-converges to } B_\infty \text{ in } H_0^1(\Omega).$$

For a.e. $x \in \mathbb{R}^n$ we define

$$A_\infty(x) := \begin{cases} B_\infty(x) & \text{if } x \in \Omega, \\ A_0(x) & \text{if } x \in \mathbb{R}^n \setminus \Omega. \end{cases} \tag{3.3}$$

By construction, $A_\infty \in \mathcal{M}(\lambda, \Lambda, \Omega, A_0)$. To conclude the proof, we show that A_∞ is the H-limit of $(A_h)_h$ in $H_0^s(\Omega)$, in the sense of Definition 2.13.

Fix $f \in H^{-s}(\Omega)$ and consider the sequence $(u_h)_h \subset H_0^s(\Omega)$ of unique solutions of $(P_h^f)_h$. Since $(u_h)_h$ and $(A_h \nabla^s u_h)_h$ are respectively bounded in $H_0^s(\Omega)$ and $L^2(\mathbb{R}^n; \mathbb{R}^n)$, then there exist $u^* \in H_0^s(\Omega)$ and $m \in L^2(\mathbb{R}^n; \mathbb{R}^n)$ such that, up to a not relabeled subsequence,

$$u_h \to u^* \quad \text{weakly in } H_0^s(\Omega) \quad \text{and} \quad A_h \nabla^s u_h \to m \quad \text{weakly in } L^2(\mathbb{R}^n; \mathbb{R}^n) \text{ as } h \to \infty. \tag{3.4}$$

Our goal is to show that

$$m = A_\infty \nabla^s u^* \quad \text{a.e. in } \mathbb{R}^n. \tag{3.5}$$

Indeed, once obtained (3.5), by passing to the limit (as $h \to \infty$) in the weak formulation of (P_h^f), we obtain that u^* is a solution of (P_∞^f). Thus, by uniqueness, $u^* = u_\infty$ and, by (3.4),

$$(A_h)_h \ H\text{-converges to } A_\infty \text{ in } H_0^s(\Omega),$$

as desired.

First, notice that

$$m = A_0 \nabla^s u^* = A_\infty \nabla^s u^* \quad \text{a.e. in } \mathbb{R}^n \setminus \Omega. \tag{3.6}$$

In fact, by (3.4), for all $\Phi \in L^2(\mathbb{R}^n \setminus \Omega; \mathbb{R}^n)$ it holds that

$$\begin{aligned} \int_{\mathbb{R}^n \setminus \Omega} m(x) \cdot \Phi(x) \, \mathrm{d}x &= \lim_{h \to \infty} \int_{\mathbb{R}^n \setminus \Omega} A_h(x) \nabla^s u_h(x) \cdot \Phi(x) \, \mathrm{d}x \\ &= \lim_{h \to \infty} \int_{\mathbb{R}^n \setminus \Omega} A_0(x) \nabla^s u_h(x) \cdot \Phi(x) \, \mathrm{d}x = \int_{\mathbb{R}^n \setminus \Omega} A_0(x) \nabla^s u^*(x) \cdot \Phi(x) \, \mathrm{d}x, \end{aligned}$$

which implies (3.6).

It remains to show that

$$m = A_\infty \nabla^s u^* \quad \text{a.e. in } \Omega. \tag{3.7}$$

Fix $g \in H^{-1}(\Omega)$ and consider $(w_h)_h \subset H^1_0(\Omega)$ and $w_\infty \in H^1_0(\Omega)$, respectively (unique) solutions of the transpose problems

$$\begin{cases} -\operatorname{div}(B_h^T \nabla w_h) = g & \text{in } \Omega, \\ w_h = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} -\operatorname{div}(B_\infty^T \nabla w_\infty) = g & \text{in } \Omega, \\ w_\infty = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

By Proposition 2.16,

$$w_h \to w_\infty \text{ weakly in } H^1_0(\Omega) \text{ and } B_h^T \nabla w_h \to B_\infty^T \nabla w_\infty \text{ weakly in } L^2(\Omega;\mathbb{R}^n) \text{ as } h \to \infty. \tag{3.9}$$

Let $\varphi \in C_c^\infty(\Omega)$ and define

$$M_h := \int_{\mathbb{R}^n} \varphi(x) A_h(x) \nabla^s u_h(x) \cdot \nabla w_h(x)\, dx.$$

We claim that for all $\varphi \in C_c^\infty(\Omega)$

$$\lim_{h\to\infty} M_h = \int_\Omega \varphi(x) m(x) \cdot \nabla w_\infty(x)\, dx = \int_\Omega \varphi(x) A_\infty(x) \nabla^s u^*(x) \cdot \nabla w_\infty(x)\, dx. \tag{3.10}$$

Once the claim is proved, we obtain (3.7). Indeed, since the operator $-\operatorname{div}(B_\infty^T \nabla\,\cdot\,)$ defines a bijection between the spaces $H^1_0(\Omega)$ and $H^{-1}(\Omega)$ and, since $g \in H^{-1}(\Omega)$ can be arbitrarily taken in (3.8), then for all $\varphi \in C_c^\infty(\Omega)$ and $w \in H^1_0(\Omega)$ the following identity holds

$$\int_\Omega \varphi(x) m(x) \cdot \nabla w(x)\, dx = \int_\Omega \varphi(x) A_\infty(x) \nabla^s u^*(x) \cdot \nabla w(x)\, dx.$$

Hence,

$$m(x) \cdot \nabla w(x) = A_\infty(x) \nabla^s u^*(x) \cdot \nabla w(x) \quad \text{for a.e. } x \in \Omega \text{ and for all } w \in H^1_0(\Omega) \tag{3.11}$$

and the collections of points of Ω where (3.11) fails can be chosen independent of w by a density argument. Therefore, by fixing $\Omega' \Subset \Omega$ and $\phi \in C_c^1(\Omega)$, such that $\phi = 1$ on Ω', and by defining

$$w(x) := \phi(x) \xi \cdot x \quad \text{for a.e. } x \in \Omega \text{ and for all } \xi \in \mathbb{R}^n,$$

by (3.11), we get that

$$m(x) \cdot \xi = A_\infty(x) \nabla^s u^*(x) \cdot \xi \quad \text{for a.e. } x \in \Omega' \text{ and for all } \xi \in \mathbb{R}^n,$$

which implies the validity of (3.7) in Ω'. Moreover, since this is true for every $\Omega' \Subset \Omega$, we get (3.7) in all Ω, which completes the proof.

We then conclude by showing the validity of the claim (3.10). Its proof is divided into two steps.

Step 1. We first show that

$$\lim_{h\to\infty} M_h = \int_\Omega \varphi(x) A_\infty(x) \nabla^s u^*(x) \cdot \nabla w_\infty(x)\, dx. \tag{3.12}$$

In virtue of Proposition 2.3, we have

$$\begin{aligned} M_h &= \int_\Omega \varphi(x) \nabla^s u_h(x) \cdot B_h^T(x) \nabla w_h(x)\, dx \\ &= \int_\Omega \varphi(x) \nabla(I_{1-s} u_h)(x) \cdot B_h^T(x) \nabla w_h(x)\, dx \\ &= \int_\Omega \nabla(\varphi I_{1-s} u_h)(x) \cdot B_h^T(x) \nabla w_h(x)\, dx - \int_\Omega I_{1-s} u_h(x) \nabla\varphi(x) \cdot B_h^T(x) \nabla w_h(x)\, dx. \end{aligned} \tag{3.13}$$

By (3.4) and Proposition 2.9,

$$u_h \to u^* \quad \text{strongly in } L^2(\mathbb{R}^n) \text{ as } h \to \infty, \tag{3.14}$$

and so, by Proposition 2.1, we get

$$I_{1-s} u_h \to I_{1-s} u^* \quad \text{strongly in } L^2(\Omega) \text{ as } h \to \infty.$$

This last convergence, coupled with (3.9), implies that

$$\lim_{h\to\infty}\int_\Omega I_{1-s}u_h(x)\nabla\varphi(x)\cdot B_h^T(x)\nabla w_h(x)\,\mathrm{d}x=\int_\Omega I_{1-s}u^*(x)\nabla\varphi(x)\cdot B_\infty^T(x)\nabla w_\infty(x)\,\mathrm{d}x. \tag{3.15}$$

We observe that $\varphi I_{1-s}u_h\in H_0^1(\Omega)$. Hence, by (3.4), (3.14), and Proposition 2.6,

$$\varphi I_{1-s}u_h\to\varphi I_{1-s}u^*\quad\text{weakly in } H_0^1(\Omega)\text{ as } h\to\infty.$$

Thus, since w_h solves problem (3.8), we get

$$\begin{aligned}\lim_{h\to\infty}\int_\Omega\nabla(\varphi I_{1-s}u_h)(x)\cdot B_h^T(x)\nabla w_h(x)\,\mathrm{d}x&=\lim_{h\to\infty}\langle g,\varphi I_{1-s}u_h\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}\\&=\langle g,\varphi I_{1-s}u^*\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}\\&=\int_\Omega\nabla(\varphi I_{1-s}u^*)(x)\cdot B_\infty^T(x)\nabla w_\infty(x)\,\mathrm{d}x.\end{aligned}\tag{3.16}$$

By combining (3.13), (3.15), and (3.16), we then obtain (3.12), being

$$\begin{aligned}\lim_{h\to\infty}M_h&=\int_\Omega\nabla(\varphi I_{1-s}u^*)(x)\cdot B_\infty^T(x)\nabla w_\infty(x)\,\mathrm{d}x-\int_\Omega I_{1-s}u^*(x)\nabla\varphi(x)\cdot B_\infty^T(x)\nabla w_\infty(x)\,\mathrm{d}x\\&=\int_\Omega\varphi(x)\nabla(I_{1-s}u^*)(x)\cdot B_\infty^T(x)\nabla w_\infty(x)\,\mathrm{d}x=\int_\Omega\varphi(x)A_\infty(x)\nabla^s u^*(x)\cdot\nabla w_\infty(x)\,\mathrm{d}x.\end{aligned}$$

Step 2. We conclude by showing that

$$\lim_{h\to\infty}M_h=\int_\Omega\varphi(x)m(x)\cdot\nabla w_\infty(x)\,\mathrm{d}x. \tag{3.17}$$

By Proposition 2.5 and Proposition 2.7, we have

$$\begin{aligned}M_h&=\int_\Omega\varphi(x)A_h(x)\nabla^s u_h(x)\cdot\nabla w_h(x)\,\mathrm{d}x\\&=\int_{\mathbb{R}^n}\varphi(x)A_h(x)\nabla^s u_h(x)\cdot\nabla^s((-\Delta)^{\frac{1-s}{2}}w_h)(x)\,\mathrm{d}x\\&=\int_{\mathbb{R}^n}A_h(x)\nabla^s u_h(x)\cdot\nabla^s(\varphi(-\Delta)^{\frac{1-s}{2}}w_h)(x)\,\mathrm{d}x\\&\quad-\int_{\mathbb{R}^n}A_h(x)\nabla^s u_h(x)\cdot\nabla^s\varphi(x)(-\Delta)^{\frac{1-s}{2}}w_h(x)\,\mathrm{d}x\\&\quad-\int_{\mathbb{R}^n}A_h(x)\nabla^s u_h(x)\cdot\nabla^s_{\mathrm{NL}}(\varphi,(-\Delta)^{\frac{1-s}{2}}w_h)(x)\,\mathrm{d}x.\end{aligned}\tag{3.18}$$

For what concerns the second integral in (3.18), in view of (3.9) and Proposition 2.9, we have

$$w_h\to w_\infty\quad\text{strongly in } L^2(\mathbb{R}^n)\text{ as } h\to\infty.$$

Moreover, the sequence $(w_h)_h\subset H_0^1(\Omega)$ is uniformly bounded. Hence, by Proposition 2.5,

$$(-\Delta)^{\frac{1-s}{2}}w_h\to(-\Delta)^{\frac{1-s}{2}}w_\infty\quad\text{strongly in } L^2(\mathbb{R}^n) \tag{3.19}$$

and, since $\nabla^s\varphi\in L^\infty(\mathbb{R}^n)$, by (3.4) we get

$$\lim_{h\to\infty}\int_{\mathbb{R}^n}A_h(x)\nabla^s u_h(x)\cdot\nabla^s\varphi(x)(-\Delta)^{\frac{1-s}{2}}w_h(x)\,\mathrm{d}x=\int_{\mathbb{R}^n}m(x)\cdot\nabla^s\varphi(x)(-\Delta)^{\frac{1-s}{2}}w_\infty(x)\,\mathrm{d}x. \tag{3.20}$$

Regarding the third integral in (3.18), by Proposition 2.7 and (3.19), since ∇^s_{NL} is a bilinear operator, we deduce that

$$\nabla^s_{\mathrm{NL}}(\varphi,(-\Delta)^{\frac{1-s}{2}}w_h)\to\nabla^s_{\mathrm{NL}}(\varphi,(-\Delta)^{\frac{1-s}{2}}w_\infty)\quad\text{strongly in } L^2(\mathbb{R}^n;\mathbb{R}^n)\text{ as } h\to\infty.$$

Therefore, in virtue of (3.4), we have

(3.21)
$$\lim_{h\to\infty}\int_{\mathbb{R}^n} A_h(x)\nabla^s u_h(x)\cdot\nabla^s_{\mathrm{NL}}(\varphi,(-\Delta)^{\frac{1-s}{2}}w_h)(x)\,\mathrm{d}x=\int_{\mathbb{R}^n} m(x)\cdot\nabla^s_{\mathrm{NL}}(\varphi,(-\Delta)^{\frac{1-s}{2}}w_\infty)(x)\,\mathrm{d}x.$$

Finally, $(-\Delta)^{\frac{1-s}{2}}w_h\in H^s(\mathbb{R}^n)$ by Proposition 2.5, which implies that $\varphi(-\Delta)^{\frac{1-s}{2}}w_h\in H^s_0(\Omega)$. Then,

$$\varphi(-\Delta)^{\frac{1-s}{2}}w_h\to\varphi(-\Delta)^{\frac{1-s}{2}}w_\infty\quad\text{weakly in } H^s_0(\Omega)\text{ as } h\to\infty.$$

Therefore, regarding the first integrals in (3.18), since u_h is a solution of (P^f_h) then, by (3.4),

$$\begin{aligned}&\lim_{h\to\infty}\int_{\mathbb{R}^n} A_h(x)\nabla^s u_h(x)\cdot\nabla^s(\varphi(-\Delta)^{\frac{1-s}{2}}w_h)(x)\,\mathrm{d}x\\&=\lim_{h\to\infty}\langle f,\varphi(-\Delta)^{\frac{1-s}{2}}w_h\rangle_{H^{-s}(\Omega)\times H^s_0(\Omega)}\\&=\langle f,\varphi(-\Delta)^{\frac{1-s}{2}}w_\infty\rangle_{H^{-s}(\Omega)\times H^s_0(\Omega)}\\&=\lim_{h\to\infty}\int_{\mathbb{R}^n} A_h(x)\nabla^s u_h(x)\cdot\nabla^s(\varphi(-\Delta)^{\frac{1-s}{2}}w_\infty)(x)\,\mathrm{d}x\\&=\int_{\mathbb{R}^n} m(x)\cdot\nabla^s(\varphi(-\Delta)^{\frac{1-s}{2}}w_\infty)(x)\,\mathrm{d}x.\end{aligned}\tag{3.22}$$

By combining (3.18), (3.20), (3.21), and (3.22), we finally get (3.17), being

$$\begin{aligned}\lim_{h\to\infty}M_h&=\int_{\mathbb{R}^n} m(x)\cdot\nabla^s(\varphi(-\Delta)^{\frac{1-s}{2}}w_\infty)(x)\,\mathrm{d}x\\&\quad-\int_{\mathbb{R}^n} m(x)\cdot\nabla^s\varphi(x)(-\Delta)^{\frac{1-s}{2}}w_\infty(x)\,\mathrm{d}x-\int_{\mathbb{R}^n} m(x)\cdot\nabla^s_{\mathrm{NL}}(\varphi,(-\Delta)^{\frac{1-s}{2}}w_\infty)(x)\,\mathrm{d}x\\&=\int_{\mathbb{R}^n}\varphi(x)m(x)\cdot\nabla^s((-\Delta)^{\frac{1-s}{2}}w_\infty)(x)\,\mathrm{d}x=\int_\Omega\varphi(x)m(x)\cdot\nabla w_\infty(x)\,\mathrm{d}x.\end{aligned}$$

Hence, the claim (3.10) holds true and the proof of the Theorem is accomplished. □

As a consequence of Proposition 2.16, the H-compactness Theorem 3.1 also applies to the subclass of symmetric matrices $\mathcal{M}^{\mathrm{sym}}(\lambda,\Lambda,\Omega,A_0)$, for a fixed $A_0\in\mathcal{M}^{\mathrm{sym}}(\lambda,\Lambda,\mathbb{R}^n)$. More precisely, the H-limit A_∞ of sequences $(A_h)_h$ in $\mathcal{M}^{\mathrm{sym}}(\lambda,\Lambda,\Omega,A_0)$ still lies in $\mathcal{M}^{\mathrm{sym}}(\lambda,\Lambda,\Omega,A_0)$.

Theorem 3.2. *Assume that $A_0\in\mathcal{M}^{\mathrm{sym}}(\lambda,\Lambda,\mathbb{R}^n)$. For any $(A_h)_h\subset\mathcal{M}^{\mathrm{sym}}(\lambda,\Lambda,\Omega,A_0)$, there exist a not relabeled subsequence and a matrix-valued function $A_\infty\in\mathcal{M}^{\mathrm{sym}}(\lambda,\Lambda,\Omega,A_0)$ such that*

$$(A_h)_h\ H\text{-converges to } A_\infty \text{ in } H^s_0(\Omega).$$

Proof. The proof of the Theorem in the symmetric case follows verbatim the construction already presented in the proof of Theorem 3.1 for the general case, combined with the following observations.

The limit matrix B_∞ in the local H-convergence of the sequence $(B_h)_h$, introduced in (3.1), whose existence is shown in (3.2), is now symmetric in view of Proposition 2.16 and by the uniqueness of the local H-limit. This implies that the matrix A_∞, defined in (3.3), belongs to the class $\mathcal{M}^{\mathrm{sym}}(\lambda,\Lambda,\Omega,A_0)$, and it is the H-limit of $(A_h)_h$, in virtue of Theorem 3.1. □

3.2. **Γ-compactness of nonlocal energies.** The goal of this subsection is to show the following Γ-compactness Theorem, whose proof is inspired by some recent ideas presented in [14, 20].

Theorem 3.3. *Assume that $A_0\in\mathcal{M}^{\mathrm{sym}}(\lambda,\Lambda,\mathbb{R}^n)$. Let $(A_h)_h\subset\mathcal{M}^{\mathrm{sym}}(\lambda,\Lambda,\Omega,A_0)$ and let $(F_h)_h$ be the nonlocal energies introduced in (2.17). Then, there exist a not relabeled subsequence of $(A_h)_h$ and $A_\infty\in\mathcal{M}^{\mathrm{sym}}(\lambda,\Lambda,\Omega,A_0)$ such that*

$$(F_h)_h\ \Gamma\text{-converges to } F_\infty \text{ strongly in } L^2(\mathbb{R}^n),$$

where $F_\infty\colon L^2(\mathbb{R}^n)\to[0,\infty]$ is the nonlocal energy associated with A_∞, as in (2.18).

Proof. For all $h \in \mathbb{N}$, we define the matrix-valued functions

$$B_h := A_h|_\Omega \in \mathcal{M}^{\mathrm{sym}}(\lambda, \Lambda, \Omega)$$

and the functionals $G_h : L^2(\Omega) \to [0,\infty]$, which are the local energies associated with B_h, as in (2.19). By Proposition 2.19, there exists a limit matrix $B_\infty \in \mathcal{M}^{\mathrm{sym}}(\lambda, \Lambda, \Omega)$ such that, up to a not relabeled subsequence,

$$(G_h)_h \ \Gamma\text{-converges to } G_\infty \text{ strongly in } L^2(\Omega), \tag{3.23}$$

where $G_\infty : L^2(\Omega) \to [0,\infty]$ is the limit local energy associated with B_∞, as in (2.20).

We define $A_\infty \in \mathcal{M}^{\mathrm{sym}}(\lambda, \Lambda, \Omega, A_0)$ as

$$A_\infty(x) := \begin{cases} B_\infty(x) & \text{if } x \in \Omega, \\ A_0(x) & \text{if } x \in \mathbb{R}^n \setminus \Omega, \end{cases}$$

and denote by $F_\infty : L^2(\Omega) \to [0,\infty]$ the limit nonlocal energy associated with A_∞, as in (2.18).

To conclude the proof, we show that

$$(F_h)_h \ \Gamma\text{-converges to } F_\infty \text{ strongly in } L^2(\mathbb{R}^n),$$

in accordance with Definition 2.18.

Γ-liminf inequality. Let $u_h, u \in L^2(\mathbb{R}^n)$, $h \in \mathbb{N}$, be such that $(u_h)_h$ strongly converges to u in $L^2(\mathbb{R}^n)$ as $h \to \infty$. We show that

$$F_\infty(u) \le \liminf_{h\to\infty} F_h(u_h).$$

Without loss of generality, we assume that

$$\liminf_{h\to\infty} F_h(u_h) < \infty,$$

the conclusion being otherwise trivial, and that the limit is actually achieved up to a not relabeled subsequence, i.e.

$$\liminf_{h\to\infty} F_h(u_h) = \lim_{h\to\infty} F_h(u_h).$$

According to its own definition, $(F_h)_h$ is finite only on $H^s_0(\Omega)$, thus forcing the sequence $(u_h)_h$ to lie therein. Since $(A_h)_h \subset \mathcal{M}^{\mathrm{sym}}(\lambda, \Lambda, \Omega, A_0)$, there exists a positive constant C such that

$$\sup_{h\in\mathbb{N}} \|\nabla^s u_h\|_{L^2(\mathbb{R}^n;\mathbb{R}^n)} \le C,$$

which yields that $(u_h)_h$ is uniformly bounded in $H^s_0(\Omega)$. Then, the limit u also lies on $H^s_0(\Omega)$ and

$$u_h \to u \quad \text{weakly in } H^s_0(\Omega) \text{ as } h \to \infty. \tag{3.24}$$

For any $h \in \mathbb{N}$, we define

$$v_h := I_{1-s} u_h \quad \text{and} \quad v := I_{1-s} u.$$

By Proposition 2.6, $v_h, v \in H^1(\Omega)$ for any $h \in \mathbb{N}$ and, by (3.24) and the continuity of the linear operator $I_{1-s} : H^s(\mathbb{R}^n) \to H^1(\Omega)$,

$$v_h \to v \quad \text{strongly in } L^2(\Omega) \text{ and weakly in } H^1(\Omega) \text{ as } h \to \infty.$$

By (3.23),

$$G_\infty(v) \le \liminf_{h\to\infty} G_h(v_h), \tag{3.25}$$

in virtue of the Γ-liminf inequality. We also note that, by Proposition 2.6,

$$\nabla v_h = \nabla^s u_h \quad \text{and} \quad \nabla v = \nabla^s u \quad \text{a.e. in } \mathbb{R}^n.$$

Thus, we can rephrase (3.25) as

$$\frac{1}{2}\int_\Omega B_\infty(x)\nabla^s u(x) \cdot \nabla^s u(x)\,\mathrm{d}x \le \liminf_{h\to\infty} \frac{1}{2}\int_\Omega A_h(x)\nabla^s u_h(x) \cdot \nabla^s u_h(x)\,\mathrm{d}x.$$

On the other hand, since $A_0 \in \mathcal{M}(\lambda, \Lambda, \mathbb{R}^n)$, by (3.24) we get

$$\frac{1}{2}\int_{\mathbb{R}^n\setminus\Omega} A_0(x)\nabla^s u(x)\cdot\nabla^s u(x)\,\mathrm{d}x \le \liminf_{h\to\infty}\frac{1}{2}\int_{\mathbb{R}^n\setminus\Omega} A_0(x)\nabla^s u_h(x)\cdot\nabla^s u_h(x)\,\mathrm{d}x.$$

Hence,

$$\begin{aligned}F_\infty(u) &= \frac{1}{2}\int_\Omega B_\infty(x)\nabla^s u(x)\cdot\nabla^s u(x)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^n\setminus\Omega} A_0(x)\nabla^s u(x)\cdot\nabla^s u(x)\,\mathrm{d}x\\ &\le \liminf_{h\to\infty}\frac{1}{2}\int_\Omega A_h(x)\nabla^s u_h(x)\cdot\nabla^s u_h(x)\,\mathrm{d}x + \liminf_{h\to\infty}\frac{1}{2}\int_{\mathbb{R}^n\setminus\Omega} A_0(x)\nabla^s u_h(x)\cdot\nabla^s u_h(x)\,\mathrm{d}x\\ &\le \liminf_{h\to\infty} F_h(u_h).\end{aligned}$$

Γ-limsup inequality. We fix $u \in L^2(\mathbb{R}^n)$ and show the existence of a recovery sequence $(u_h)_h \subset L^2(\mathbb{R}^n)$ such that $(u_h)_h$ strongly converges to u in $L^2(\mathbb{R}^n)$, as $h \to \infty$, and

$$F_\infty(u) \ge \limsup_{h\to\infty} F_h(u_h). \tag{3.26}$$

The proof of the Γ-limsup inequality is rather technical and for the readers' convenience, we indicate the main steps below.

- First, we exploit the Riesz potential to move to the local setting and we obtain the existence of a recovery sequence $(v_h)_h$ for the Γ-convergence of the local energies $(G_h)_h$ to G_∞.
- Then, through a cut-off argument, we adapt the sequence $(v_h)_h$ to the boundary data of our problem and we come back to the nonlocal setting, obtaining the existence of a sequence $(u_h^\varepsilon)_h$ satisfying the Γ-limsup inequality up to a reminder term, which depends on a parameter $\varepsilon > 0$.
- In the last part of the proof, we let the reminder term to zero by a diagonal argument, which provides the existence of a recovery sequence $(u_h)_h$ for our problem.

Without loss of generality, we consider only the case of $u \in H_0^s(\Omega)$, the conclusion being otherwise trivial.

We define

$$v := I_{1-s}u.$$

Then, by Proposition 2.6,

$$v \in H^1(\Omega) \quad \text{and} \quad \nabla v = \nabla^s u \quad \text{a.e. in } \mathbb{R}^n. \tag{3.27}$$

Moreover, by (3.23), there exists a recovery sequence $(v_h)_h \subset H^1(\Omega)$ for v, i.e. such that

$$v_h \to v \text{ strongly in } L^2(\Omega) \text{ as } h\to\infty \quad \text{and} \quad \lim_{h\to\infty} G_h(v_h) = G_\infty(v) < \infty \tag{3.28}$$

(we recall, in fact, that the Γ-liminf and the Γ-limsup inequalities imply that the limit is achieved at least for the recovery sequence). In particular, by the definition of G_h (see (2.19)), $(v_h)_h$ is bounded in $H^1(\Omega)$, which gives that

$$v_h \to v \quad \text{weakly in } H^1(\Omega) \text{ as } h\to\infty.$$

Let $\varepsilon > 0$ be fixed and let $K^\varepsilon \Subset \Omega$ be a compact set such that

$$\int_{\Omega\setminus K^\varepsilon} |\nabla v(x)|^2\,\mathrm{d}x < \varepsilon. \tag{3.29}$$

We fix an open set U^ε such that $K^\varepsilon \Subset U^\varepsilon \Subset \Omega$, consider a cut-off function $\varphi^\varepsilon \in C_c^\infty(U^\varepsilon)$ satisfying $0 \le \varphi^\varepsilon \le 1$ on U^ε and $\varphi^\varepsilon \equiv 1$ on K^ε and, for all $h \in \mathbb{N}$, we define

$$v_h^\varepsilon := \varphi^\varepsilon v_h + (1-\varphi^\varepsilon)v. \tag{3.30}$$

By construction,

$$v_h^\varepsilon \to v \quad \text{strongly in } L^2(\Omega) \text{ and weakly in } H^1(\Omega) \text{ as } h\to\infty. \tag{3.31}$$

Moreover, by (2.12), (2.21), (3.29), and the convexity of the map $\xi \mapsto A_h(x)\xi \cdot \xi$, it holds that

$$
\begin{aligned}
G_h(v_h^\varepsilon) &= \frac{1}{2}\int_\Omega A_h(x)[\varphi^\varepsilon(x)\nabla v_h(x) + (1-\varphi^\varepsilon(x))\nabla v(x)] \cdot [\varphi^\varepsilon(x)\nabla v_h(x) + (1-\varphi^\varepsilon(x))\nabla v(x)]\,\mathrm{d}x \\
&\quad + \int_\Omega A_h(x)[\nabla\varphi^\varepsilon(x)(v_h(x) - v(x))] \cdot [\varphi^\varepsilon(x)\nabla v_h(x) + (1-\varphi^\varepsilon(x))\nabla v(x)]\,\mathrm{d}x \\
&\quad + \frac{1}{2}\int_\Omega A_h(x)[\nabla\varphi^\varepsilon(x)(v_h(x) - v(x))] \cdot [\nabla\varphi^\varepsilon(x)(v_h(x) - v(x))]\,\mathrm{d}x \\
&\le \frac{1}{2}\int_\Omega \varphi^\varepsilon(x) A_h(x)\nabla v_h(x)\cdot\nabla v_h(x)\,\mathrm{d}x + \frac{1}{2}\int_\Omega (1-\varphi^\varepsilon(x))A_h(x)\nabla v(x)\cdot\nabla v(x)\,\mathrm{d}x \\
&\quad + \Lambda\|\nabla\varphi^\varepsilon\|_{L^\infty(\Omega;\mathbb{R}^n)}\|v_h - v\|_{L^2(\Omega)}(\|\nabla v_h\|_{L^2(\Omega;\mathbb{R}^n)} + \|\nabla v\|_{L^2(\Omega;\mathbb{R}^n)}) \\
&\quad + \frac{\Lambda}{2}\|\nabla\varphi^\varepsilon\|^2_{L^\infty(\Omega;\mathbb{R}^n)}\|v_h - v\|^2_{L^2(\Omega)} \\
&\le G_h(v_h) + \frac{\Lambda}{2}\varepsilon + \Lambda\|\nabla\varphi^\varepsilon\|_{L^\infty(\Omega;\mathbb{R}^n)}\|v_h - v\|_{L^2(\Omega)}(\|\nabla v_h\|_{L^2(\Omega;\mathbb{R}^n)} + \|\nabla v\|_{L^2(\Omega;\mathbb{R}^n)}) \\
&\quad + \frac{\Lambda}{2}\|\nabla\varphi^\varepsilon\|^2_{L^\infty(\Omega;\mathbb{R}^n)}\|v_h - v\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Hence, by (3.28) and the boundedness of $(v_h)_h$ in $H^1(\Omega)$, we conclude that

$$
\limsup_{h\to\infty} G_h(v_h^\varepsilon) \le \lim_{h\to\infty} G_h(v_h) + \frac{\Lambda}{2}\varepsilon = G_\infty(v) + \frac{\Lambda}{2}\varepsilon. \tag{3.32}
$$

We trivially extend $v_h^\varepsilon - v \in H^1_0(\Omega)$ to a function in $H^1(\mathbb{R}^n)$ and, for all $h \in \mathbb{N}$, we define

$$
w_h^\varepsilon := (-\Delta)^{\frac{1-s}{2}}(v_h^\varepsilon - v).
$$

By Proposition 2.5, we have that

$$
w_h^\varepsilon \in H^s(\mathbb{R}^n) \quad \text{and} \quad \nabla^s w_h^\varepsilon = \nabla(v_h^\varepsilon - v) \quad \text{a.e. in } \mathbb{R}^n \tag{3.33}
$$

and, by (2.3), (3.31), and (3.33), there exist two positive constants C and C_ε such that

$$
\|w_h^\varepsilon\|^2_{H^s(\mathbb{R}^n)} = \|w_h^\varepsilon\|^2_{L^2(\mathbb{R}^n)} + \|\nabla^s w_h^\varepsilon\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^n)} \le C\|v_h^\varepsilon - v\|^2_{H^1_0(\Omega)} \le C_\varepsilon \quad \text{for all } h \in \mathbb{N}.
$$

Therefore, by (2.1), (2.2) and (3.31), for all $\psi \in C_c^\infty(\mathbb{R}^n)$ we get that

$$
\int_{\mathbb{R}^n} w_h^\varepsilon(x)\psi(x)\,\mathrm{d}x = \int_{\mathbb{R}^n}(v_h^\varepsilon(x) - v(x))(-\Delta)^{\frac{1-s}{2}}\psi(x)\,\mathrm{d}x \to 0 \quad \text{as } h\to\infty,
$$

which yields that

$$
w_h^\varepsilon \to 0 \quad \text{weakly in } H^s(\mathbb{R}^n) \text{ as } h\to\infty. \tag{3.34}
$$

In particular, by (2.3) and (3.31),

$$
w_h^\varepsilon \to 0 \quad \text{strongly in } L^2(\mathbb{R}^n) \text{ as } h\to\infty. \tag{3.35}
$$

Let $\chi^\varepsilon \in C_c^\infty(\Omega)$ satisfy $0 \le \chi^\varepsilon \le 1$ on Ω and $\chi^\varepsilon = 1$ on $\overline{U^\varepsilon}$. We define

$$
u_h^\varepsilon := u + \chi^\varepsilon w_h^\varepsilon \in H^s_0(\Omega).
$$

By (3.34) and (3.35),

$$
u_h^\varepsilon \to u \quad \text{strongly in } L^2(\mathbb{R}^n) \text{ and weakly in } H^s_0(\Omega) \text{ as } h\to\infty. \tag{3.36}
$$

For all $h \in \mathbb{N}$, we also set

$$
R_h^\varepsilon := \nabla^s(\chi^\varepsilon w_h^\varepsilon) - \chi^\varepsilon\nabla^s w_h^\varepsilon.
$$

By Proposition 2.7, there exists a positive constant C such that

$$
\|R_h^\varepsilon\|_{L^2(\mathbb{R}^n;\mathbb{R}^n)} \le C\,\|\chi^\varepsilon\|_{W^{1,\infty}(\mathbb{R}^n)}\,\|w_h^\varepsilon\|_{L^2(\mathbb{R}^n)} \quad \text{for all } h \in \mathbb{N}.
$$

Then, by (3.35),

$$
R_h^\varepsilon \to 0 \quad \text{strongly in } L^2(\mathbb{R}^n;\mathbb{R}^n) \text{ as } h\to\infty \tag{3.37}
$$

and, by (3.33),

$$\nabla^s u_h^\varepsilon = \nabla^s u + \chi^\varepsilon \nabla^s w_h^\varepsilon + R_h^\varepsilon = \nabla^s u + \chi^\varepsilon \nabla (v_h^\varepsilon - v) + R_h^\varepsilon \quad \text{a.e. in } \mathbb{R}^n. \tag{3.38}$$

We consider the following decomposition

$$F_h(u_h^\varepsilon) = \frac{1}{2}\int_{U^\varepsilon} A_h(x)\nabla^s u_h^\varepsilon(x)\cdot\nabla^s u_h^\varepsilon(x)\,\mathrm{d}x + \frac{1}{2}\int_{\Omega\setminus U^\varepsilon} A_h(x)\nabla^s u_h^\varepsilon(x)\cdot\nabla^s u_h^\varepsilon(x)\,\mathrm{d}x \\ + \frac{1}{2}\int_{\mathbb{R}^n\setminus\Omega} A_0(x)\nabla^s u_h^\varepsilon(x)\cdot\nabla^s u_h^\varepsilon(x)\,\mathrm{d}x. \tag{3.39}$$

By (3.37) and (3.38), the last integral in (3.39) satisfies

$$\begin{aligned} &\lim_{h\to\infty}\frac{1}{2}\int_{\mathbb{R}^n\setminus\Omega} A_0(x)\nabla^s u_h^\varepsilon(x)\cdot\nabla^s u_h^\varepsilon(x)\,\mathrm{d}x \\ &= \lim_{h\to\infty}\frac{1}{2}\int_{\mathbb{R}^n\setminus\Omega} A_0(x)(\nabla^s u(x) + R_h^\varepsilon(x))\cdot(\nabla^s u(x) + R_h^\varepsilon(x))\,\mathrm{d}x \\ &= \frac{1}{2}\int_{\mathbb{R}^n\setminus\Omega} A_0(x)\nabla^s u(x)\cdot\nabla^s u(x)\,\mathrm{d}x. \end{aligned} \tag{3.40}$$

Concerning the second integral in (3.39), we note that, by (3.30) and (3.38),

$$\nabla^s u_h^\varepsilon = \nabla^s u + R_h^\varepsilon \quad \text{a.e. in } \Omega\setminus U^\varepsilon.$$

Then, by (2.21), (3.27), and (3.29),

$$\begin{aligned} &\limsup_{h\to\infty}\frac{1}{2}\int_{\Omega\setminus U^\varepsilon} A_h(x)\nabla^s u_h^\varepsilon(x)\cdot\nabla^s u_h^\varepsilon(x)\,\mathrm{d}x \\ &\le \lim_{h\to\infty}\frac{\Lambda}{2}\|\nabla^s u + R_h^\varepsilon\|^2_{L^2(\Omega\setminus U^\varepsilon;\mathbb{R}^n)} = \frac{\Lambda}{2}\|\nabla^s u\|^2_{L^2(\Omega\setminus U^\varepsilon;\mathbb{R}^n)} \le \frac{\Lambda}{2}\|\nabla v\|^2_{L^2(\Omega\setminus K^\varepsilon;\mathbb{R}^n)} \le \frac{\Lambda}{2}\varepsilon. \end{aligned} \tag{3.41}$$

Finally, for what concerns the first integral in (3.39), we observe that, since $\chi^\varepsilon = 1$ in U^ε, by (3.27) and (3.38) we have

$$\nabla^s u_h^\varepsilon = \nabla v + \nabla(v_h^\varepsilon - v) + R_h^\varepsilon = \nabla v_h^\varepsilon + R_h^\varepsilon \quad \text{a.e. in } U^\varepsilon.$$

Thus, (2.21) and (3.27) imply that

$$\begin{aligned} &\frac{1}{2}\int_{U^\varepsilon} A_h(x)\nabla^s u_h^\varepsilon(x)\cdot\nabla^s u_h^\varepsilon(x)\,\mathrm{d}x \\ &= \frac{1}{2}\int_{U^\varepsilon} A_h(x)(\nabla v_h^\varepsilon(x) + R_h^\varepsilon(x))\cdot(\nabla v_h^\varepsilon(x) + R_h^\varepsilon(x))\,\mathrm{d}x \\ &= \frac{1}{2}\int_{U^\varepsilon} A_h(x)\nabla v_h^\varepsilon(x)\cdot\nabla v_h^\varepsilon(x)\,\mathrm{d}x + \int_{U^\varepsilon} A_h\nabla v_h^\varepsilon(x)\cdot R_h^\varepsilon(x)\,\mathrm{d}x \\ &\quad + \frac{1}{2}\int_{U^\varepsilon} A_h R_h^\varepsilon(x)\cdot R_h^\varepsilon(x)\,\mathrm{d}x \\ &\le G_h(v_h^\varepsilon) + \int_{U^\varepsilon} A_h\nabla v_h^\varepsilon(x)\cdot R_h^\varepsilon(x)\,\mathrm{d}x + \frac{1}{2}\int_{U^\varepsilon} A_h R_h^\varepsilon(x)\cdot R_h^\varepsilon(x)\,\mathrm{d}x \end{aligned} \tag{3.42}$$

and, since $(v_h^\varepsilon)_h$ is uniformly bounded in $H^1(\Omega)$, by (3.32), (3.37) and (3.42), we get

$$\limsup_{h\to\infty}\frac{1}{2}\int_{U^\varepsilon} A_h(x)\nabla^s u_h^\varepsilon(x)\cdot\nabla^s u_h^\varepsilon(x)\,\mathrm{d}x \le G_\infty(v) + \frac{\Lambda}{2}\varepsilon. \tag{3.43}$$

Therefore, by (3.40), (3.41) and (3.43), we obtain that for all $\varepsilon > 0$

$$\limsup_{h\to\infty} F_h(u_h^\varepsilon) \le F_\infty(u) + \Lambda\varepsilon. \tag{3.44}$$

To conclude, we use the following diagonal argument. In view of [15, Definition 4.1 and Remark 4.3], by (3.36) and (3.44), we have that for all $\varepsilon > 0$

$$\Gamma\text{-}\limsup_{h\to\infty} F_h(u) := \sup_{k\in\mathbb{N}}\limsup_{h\to\infty}\inf_{z\in B_{\frac{1}{k}}(u)} F_h(z) \le F_\infty(u) + \Lambda\varepsilon.$$

Hence, by letting $\varepsilon \to 0$, we conclude that

$$\Gamma\text{-}\limsup_{h\to\infty} F_h(u) \le F_\infty(u)$$

and, by the properties of the Γ-lim sup (see e.g. [15, Proposition 8.1]), there exists a sequence $(u_h)_h \subset L^2(\mathbb{R}^n)$ such that $u_h \to u$ strongly in $L^2(\mathbb{R}^n)$ as $h \to \infty$ and

$$\limsup_{h\to\infty} F_h(u_h) = \Gamma\text{-}\limsup_{h\to\infty} F_h(u) \le F_\infty(u).$$

This implies the validity of (3.26), and concludes the proof of the Theorem. □

3.3. ***H*-compactness in the symmetric case via Γ-convergence.** In this last subsection, we provide an alternative proof of Theorem 3.2, purely based on variational techniques.

Let $A_h, A_\infty \in \mathcal{M}^{\mathrm{sym}}(\lambda, \Lambda, \Omega, A_0)$, $h \in \mathbb{N}$, for a given $A_0 \in \mathcal{M}^{\mathrm{sym}}(\lambda, \Lambda, \mathbb{R}^n)$. We first show that the Γ-convergence of the local energies $(G_h)_h$ to G_∞, respectively defined in (2.19) and (2.20), associated with $B_h \coloneqq A_h|_\Omega$ and $B_\infty \coloneqq A_\infty|_\Omega$, implies the convergence of the nonlocal momenta. To this aim, following the strategies adopted in [16, Lemma 4.11] and [3, Theorem 4.5], we define the functionals $\mathcal{F}_h, \mathcal{F}_\infty \colon L^2(\mathbb{R}^n;\mathbb{R}^n) \to \mathbb{R}$ as

$$\mathcal{F}_h(\Phi) \coloneqq \frac{1}{2}\int_{\mathbb{R}^n} A_h(x)\Phi(x)\cdot\Phi(x)\,\mathrm{d}x \quad \text{for all } \Phi \in L^2(\mathbb{R}^n;\mathbb{R}^n),$$
$$\mathcal{F}_\infty(\Phi) \coloneqq \frac{1}{2}\int_{\mathbb{R}^n} A_\infty(x)\Phi(x)\cdot\Phi(x)\,\mathrm{d}x \quad \text{for all } \Phi \in L^2(\mathbb{R}^n;\mathbb{R}^n),$$

and consider their Fréchet derivatives $\mathcal{F}'_h$ and $\mathcal{F}'_\infty$, which are given by

$$\mathcal{F}'_h(\Phi)[\Psi] = \int_{\mathbb{R}^n} A_h(x)\Phi(x)\cdot\Psi(x)\,\mathrm{d}x \quad \text{and} \quad \mathcal{F}'_\infty(\Phi)[\Psi] = \int_{\mathbb{R}^n} A_\infty(x)\Phi(x)\cdot\Psi(x)\,\mathrm{d}x$$

for all $\Phi, \Psi \in L^2(\mathbb{R}^n;\mathbb{R}^n)$.

We note that $\mathcal{F}'_h$ and $\mathcal{F}'_\infty$ can be used to derive the convergence of the nonlocal momenta. Indeed, given a sequence $(u_h)_h \subset H^s_0(\Omega)$ and $u_\infty \in H^s_0(\Omega)$, then the convergence

$$\mathcal{F}'_h(\nabla^s u_h)[\Psi] \to \mathcal{F}'_\infty(\nabla^s u_\infty)[\Psi] \quad \text{for all } \Psi \in L^2(\mathbb{R}^n;\mathbb{R}^n)$$

is equivalent to

$$A_h\nabla^s u_h \to A_\infty \nabla^s u_\infty \quad \text{weakly in } L^2(\mathbb{R}^n;\mathbb{R}^n).$$

Proposition 3.4. *Fix $A_0 \in \mathcal{M}^{\mathrm{sym}}(\lambda, \Lambda, \mathbb{R}^n)$ and $A_h, A_\infty \in \mathcal{M}^{\mathrm{sym}}(\lambda, \Lambda, \Omega, A_0)$, $h \in \mathbb{N}$. Let $G_h, G_\infty \colon L^2(\Omega) \to [0,\infty]$ be the local energies, respectively defined in (2.19) and (2.20), and associated with $B_h \coloneqq A_h|_\Omega$, and $B_\infty \coloneqq A_\infty|_\Omega$. Assume that*

$$(G_h)_h \ \Gamma\text{-converges to } G_\infty \text{ strongly in } L^2(\Omega). \tag{3.45}$$

Let $F_h, F_\infty \colon L^2(\mathbb{R}^n) \to [0,\infty]$ be the nonlocal energies, respectively defined in (2.17) and (2.18), and let $(u_h)_h \subset H^s_0(\Omega)$ and $u_\infty \in H^s_0(\Omega)$ satisfy

$$u_h \to u_\infty \quad \text{strongly in } L^2(\mathbb{R}^n) \quad \text{and} \quad F_h(u_h) \to F_\infty(u_\infty) \quad \text{as } h \to \infty. \tag{3.46}$$

Then, the convergence of the nonlocal momenta holds, i.e.

$$\mathcal{F}'_h(\nabla^s u_h)[\Psi] \to \mathcal{F}'_\infty(\nabla^s u_\infty)[\Psi] \quad \text{for all } \Psi \in L^2(\mathbb{R}^n;\mathbb{R}^n) \text{ as } h \to \infty. \tag{3.47}$$

Proof. To prove (3.47), it is sufficient to show the following inequality

$$\mathcal{F}'(\nabla^s u_\infty)[\Psi] \le \liminf_{h\to\infty} \mathcal{F}'_h(\nabla^s u_h)[\Psi] \quad \text{for all } \Psi \in L^2(\mathbb{R}^n;\mathbb{R}^n). \tag{3.48}$$

Indeed, by replacing Ψ with $-\Psi$, and by the properties of the limit inferior, one get the desired condition (3.47).

For all $\Phi \in L^2(\Omega;\mathbb{R}^n)$, we define $\mathcal{G}_h^\Phi, \mathcal{G}_\infty^\Phi \colon L^2(\Omega) \to [0,\infty]$ as

$$\mathcal{G}_h^\Phi(v) := \begin{cases} \frac{1}{2}\int_\Omega A_h(x)(\nabla v(x) + \Phi(x)) \cdot (\nabla v(x) + \Phi(x))\,\mathrm{d}x & \text{if } v \in H^1(\Omega), \\ \infty & \text{if } v \in L^2(\Omega) \setminus H^1(\Omega), \end{cases}$$

$$\mathcal{G}_\infty^\Phi(v) := \begin{cases} \frac{1}{2}\int_\Omega A_\infty(x)(\nabla v(x) + \Phi(x)) \cdot (\nabla v(x) + \Phi(x))\,\mathrm{d}x & \text{if } v \in H^1(\Omega), \\ \infty & \text{if } v \in L^2(\Omega) \setminus H^1(\Omega). \end{cases}$$

By [15, Theorem 22.4] and [3, Theorem 4.2], (3.45) implies that for all $\Phi \in L^2(\Omega;\mathbb{R}^n)$

$$(\mathcal{G}_h^\Phi)_h \ \Gamma\text{-converges to } \mathcal{G}_\infty^\Phi \text{ strongly in } L^2(\Omega). \tag{3.49}$$

Let $\Psi \in L^2(\mathbb{R}^n;\mathbb{R}^n)$ and $(t_i)_i$ be a sequence of positive numbers, infinitesimal as $i \to \infty$. Moreover, let $u_h, u_\infty \in H_0^s(\Omega)$, $h \in \mathbb{N}$, satisfy (3.46), and define

$$v_h := I_{1-s}u_h \in H^1(\Omega) \quad \text{and} \quad v_\infty := I_{1-s}u_\infty \in H^1(\Omega).$$

By the continuity of $I_{1-s} \colon L^2(\mathbb{R}^n) \to L^2(\Omega)$, it holds that

$$v_h \to v_\infty \quad \text{strongly in } L^2(\Omega) \text{ as } h \to \infty$$

and, by (3.49) with $\Phi := t_i\Psi|_\Omega$, we get

$$\mathcal{G}_\infty^{t_i\Psi|_\Omega}(v_\infty) \le \liminf_{h\to\infty} \mathcal{G}_h^{t_i\Psi|_\Omega}(v_h) \quad \text{for all } i \in \mathbb{N}. \tag{3.50}$$

Since, by Proposition 2.6,

$$\nabla v_h = \nabla^s u_h \quad \text{and} \quad \nabla v_\infty = \nabla^s u_\infty \quad \text{a.e. in } \mathbb{R}^n,$$

we can then rephrase (3.50) as

$$\begin{aligned} &\frac{1}{2}\int_\Omega A_\infty(x)(\nabla^s u_\infty(x) + t_i\Psi(x)) \cdot (\nabla^s u_\infty(x) + t_i\Psi(x))\,\mathrm{d}x \\ &\le \liminf_{h\to\infty} \frac{1}{2}\int_\Omega A_h(x)(\nabla^s u_h(x) + t_i\Psi(x)) \cdot (\nabla^s u_h(x) + t_i\Psi(x))\,\mathrm{d}x. \end{aligned} \tag{3.51}$$

In addition, by (2.5) and (3.46), the sequence $(u_h)_h$ is uniformly bounded in $H_0^s(\Omega)$ and, by the strong convergence of $(u_h)_h$ to u in $L^2(\mathbb{R}^n)$, we have

$$\nabla^s u_h + t_i\Psi \to \nabla^s u_\infty + t_i\Psi \quad \text{weakly in } L^2(\mathbb{R}^n;\mathbb{R}^n) \text{ as } h \to \infty.$$

Then,

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^n\setminus\Omega} A_0(x)(\nabla^s u_\infty(x) + t_i\Psi(x)) \cdot (\nabla^s u_\infty(x) + t_i\Psi(x))\,\mathrm{d}x \\ &\le \liminf_{h\to\infty} \frac{1}{2}\int_{\mathbb{R}^n\setminus\Omega} A_0(x)(\nabla^s u_h(x) + t_i\Psi(x)) \cdot (\nabla^s u_h(x) + t_i\Psi(x))\,\mathrm{d}x. \end{aligned} \tag{3.52}$$

Therefore, by (3.51) and (3.52),

$$\begin{aligned} \mathcal{F}_\infty(\nabla^s u_\infty + t_i\Psi) &= \frac{1}{2}\int_\Omega A_\infty(x)(\nabla^s u_\infty(x) + t_i\Psi(x)) \cdot (\nabla^s u_\infty(x) + t_i\Psi(x))\,\mathrm{d}x \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^n\setminus\Omega} A_0(x)(\nabla^s u_\infty(x) + t_i\Psi(x)) \cdot (\nabla^s u_\infty(x) + t_i\Psi(x))\,\mathrm{d}x \\ &\le \liminf_{h\to\infty} \frac{1}{2}\int_\Omega A_h(x)(\nabla^s u_h(x) + t_i\Psi(x)) \cdot (\nabla^s u_h(x) + t_i\Psi(x))\,\mathrm{d}x \\ &\quad + \liminf_{h\to\infty} \frac{1}{2}\int_{\mathbb{R}^n\setminus\Omega} A_0(x)(\nabla^s u_h(x) + t_i\Psi(x)) \cdot (\nabla^s u_h(x) + t_i\Psi(x))\,\mathrm{d}x \\ &\le \liminf_{h\to\infty} \mathcal{F}_h(\nabla^s u_h + t_i\Psi). \end{aligned}$$

Moreover, since by definition

$$\mathcal{F}_h(\nabla^s u_h) = F_h(u_h) \quad \text{and} \quad \mathcal{F}_\infty(\nabla^s u_\infty) = F_\infty(u_\infty),$$

then, by (3.46), it holds that

$$\frac{\mathcal{F}_\infty(\nabla^s u_\infty + t_i\Psi) - \mathcal{F}_\infty(\nabla^s u_\infty)}{t_i} \le \liminf_{h\to\infty} \frac{\mathcal{F}_h(\nabla^s u_h + t_i\Psi) - \mathcal{F}_h(\nabla^s u_h)}{t_i} \quad \text{for all } i \in \mathbb{N}.$$

Hence, there exists an increasing sequence of integers $(h_i)_i \subset \mathbb{N}$ such that

$$\frac{\mathcal{F}_\infty(\nabla^s u_\infty + t_i\Psi) - \mathcal{F}_\infty(\nabla^s u_\infty)}{t_i} - \frac{1}{i} \le \frac{\mathcal{F}_h(\nabla^s u_h + t_i\Psi) - \mathcal{F}_h(\nabla^s u_h)}{t_i} \quad \text{for all } h \ge h_i. \tag{3.53}$$

If we set $\varepsilon_h := t_i$ for $h_i \le h < h_{i+1}$ and $i \in \mathbb{N}$, then, by (3.53)

$$\liminf_{h\to\infty} \frac{\mathcal{F}_\infty(\nabla^s u_\infty + \varepsilon_h\Psi) - \mathcal{F}_\infty(\nabla^s u_\infty)}{\varepsilon_h} \le \liminf_{h\to\infty} \frac{\mathcal{F}_h(\nabla^s u_h + \varepsilon_h\Psi) - \mathcal{F}_h(\nabla^s u_h)}{\varepsilon_h}. \tag{3.54}$$

Note that the limit inferior on the left-hand side of (3.54) is actually achieved and coincides with the Fréchet derivative of the functional $\mathcal{F}_\infty$, i.e.

$$\mathcal{F}'_\infty(\nabla^s u_\infty)[\Psi] = \lim_{h\to\infty} \frac{\mathcal{F}_\infty(\nabla^s u_\infty + \varepsilon_h\Psi) - \mathcal{F}_\infty(\nabla^s u_\infty)}{\varepsilon_h}. \tag{3.55}$$

For what concerns the right-hand side of (3.54), we have

$$\frac{\mathcal{F}_h(\nabla^s u_h + \varepsilon_h\Psi) - \mathcal{F}_h(\nabla^s u_h)}{\varepsilon_h} = \mathcal{F}'_h(\nabla^s u_h)[\Psi] + \varepsilon_h \mathcal{F}_h(\Psi). \tag{3.56}$$

Since the last term on the right-hand side of (3.56) converges to 0 as $h \to \infty$, from (3.54)–(3.56) we get (3.48). □

We can finally provide an alternative proof of Theorem 3.2, purely based on variational techniques.

Proof of Theorem 3.2. Let $(A_h)_h \subset \mathcal{M}^{\rm sym}(\lambda, \Lambda, \Omega, A_0)$ and let $(F_h)_h$ be the associated nonlocal energies, defined in (2.17). By Theorem 3.3, there exist a not relabeled subsequence of $(A_h)_h$ and $A_\infty \in \mathcal{M}^{\rm sym}(\lambda, \Lambda, \Omega, A_0)$ such that, if F_∞ denotes the nonlocal energy associated with A_∞, as in (2.18), then

$$(F_h)_h \ \Gamma\text{-converges to } F_\infty \text{ strongly in } L^2(\mathbb{R}^n).$$

Moreover, it holds that

$$(G_h)_h \ \Gamma\text{-converges to } G_\infty \text{ strongly in } L^2(\Omega), \tag{3.57}$$

where the local energies G_h and G_∞ are defined as usual.

We now show that

$$(A_h)_h \ H\text{-converges to } A_\infty \text{ in } H^s_0(\Omega),$$

by first proving that (2.13) and (2.14) hold for every $f \in L^2(\mathbb{R}^n)$. Later, by a density argument, we extend the validity of (2.13) and (2.14) to every $f \in H^{-s}(\Omega)$.

Step 1. We fix $f \in L^2(\mathbb{R}^n)$ and, for any $h \in \mathbb{N}$, we define $F^f_h, F^f_\infty \colon L^2(\mathbb{R}^n) \to [0,\infty]$ as

$$F^f_h(u) := F_h(u) + \int_{\mathbb{R}^n} f(x)u(x)\,\mathrm{d}x, \quad F^f_\infty(u) := F_\infty(u) + \int_{\mathbb{R}^n} f(x)u(x)\,\mathrm{d}x \quad \text{for all } u \in L^2(\mathbb{R}^n).$$

Since we have perturbed with continuity F_h and F_∞, then

$$(F^f_h)_h \ \Gamma\text{-converges to } F^f_\infty \text{ strongly in } L^2(\mathbb{R}^n),$$

in virtue of [15, Proposition 6.21].

We notice that the solutions $u_h \in H^s_0(\Omega)$ and $u_\infty \in H^s_0(\Omega)$ of problems (P^f_h) and (P^f_∞), whose existence is guaranteed by Lemma 2.12, minimise the energies F^f_h and F^f_∞, respectively, i.e.

$$F^f_h(u_h) = \min_{u \in L^2(\mathbb{R}^n)} F^f_h(u) \quad \text{and} \quad F^f_\infty(u_\infty) = \min_{u \in L^2(\mathbb{R}^n)} F^f_\infty(u).$$

Therefore, by the Fundamental Theorem of Γ-convergence (see e.g. [15, Theorem 7.8]), we get

$$u_h \to u_\infty \text{ strongly in } L^2(\mathbb{R}^n) \quad \text{and} \quad F^f_h(u_h) \to F^f_\infty(u_\infty) \text{ as } h \to \infty. \tag{3.58}$$

In particular, since $(u_h)_h \subset H^s_0(\Omega)$ is uniformly bounded, we conclude that

$$u_h \to u_\infty \text{ weakly in } H^s_0(\Omega) \text{ as } h \to \infty, \tag{3.59}$$

which is the convergence of the solutions (2.13).

We finally observe that (3.57) and (3.58) allow us to apply Proposition 3.4, which gives the convergence of the momenta (2.14).

Step 2. We fix now $f \in H^{-s}(\Omega)$ and denote $u_h, u_\infty \in H^s_0(\Omega)$, $h \in \mathbb{N}$, the solutions of the problems (P^f_h) and (P^f_∞), respectively.

Since the embedding $H^s_0(\Omega) \subset L^2(\mathbb{R}^n)$ is continuous and dense, so is the embedding $L^2(\mathbb{R}^n) \subset H^{-s}(\Omega)$. Therefore, we can find a sequence $(f_j)_j \subset L^2(\mathbb{R}^n)$ such that

$$f_j \to f \text{ strongly in } H^{-s}(\Omega) \text{ as } j \to \infty.$$

For all $j \in \mathbb{N}$, let $u^j_h, u^j_\infty \in H^s_0(\Omega)$ be the solutions of the problems $(P^{f_j}_h)$ and $(P^{f_j}_\infty)$, respectively. Fixed $g \in H^{-s}(\Omega)$, by Proposition 2.8 and Lemma 2.12, we obtain

$$|\langle g, u_h - u_\infty \rangle_{H^{-s}(\Omega)\times H^s_0(\Omega)}| \le |\langle g, u^j_h - u^j_\infty \rangle_{H^{-s}(\Omega)\times H^s_0(\Omega)}| + C\|g\|_{H^{-s}(\Omega)}\|f_j - f\|_{H^{-s}(\Omega)}.$$

Hence, in view of (3.59) in Step 1, by letting first $h \to \infty$ and then $j \to \infty$, we obtain (2.13).

Finally, fixed $\Phi \in L^2(\mathbb{R}^n;\mathbb{R}^n)$, by Proposition 2.8 and Lemma 2.12, we get

$$\begin{aligned}
&\left|\int_\Omega (A_h(x)\nabla^s u_h(x) - A_\infty(x)\nabla^s u_\infty(x)) \cdot \Phi(x)\,\mathrm{d}x\right| \\
&\le \left|\int_\Omega (A_h(x)\nabla^s u^j_h(x) - A_\infty(x)\nabla^s u^j_\infty(x)) \cdot \Phi(x)\,\mathrm{d}x\right| + C\|\Phi\|_{L^2(\mathbb{R}^n;\mathbb{R}^n)}\|f_j - f\|_{H^{-s}(\Omega)}.
\end{aligned}$$

Then, in virtue of Step 1, by first letting $h \to \infty$ and then $j \to \infty$, we obtain (2.14), leading to the H-convergence of the sequence $(A_h)_h$ to A_∞ in $H^s_0(\Omega)$. □

4. Conclusions and open problems

Our distributional approach leads the H-convergence theory to cover linear operators in fractional divergence form as well. In what follows, we list some possible future research directions stemming from our results that we believe may be of particular interest to the community.

(1) As explained in the Introduction, one of the goals of the H-convergence is to obtain the uniqueness of the limit matrix. As for the nonlocal problem, even if we are able to prove the existence of a H-limit, its uniqueness is still unknown. Classical proofs strongly rely on local arguments, which fail in the nonlocal scenario. It would be very interesting to investigate this issue.

(2) A first direction that we are planning to investigate concerns the study of the asymptotic behaviour of *monotone operators* in fractional divergence form with superlinear growth, whose local counterpart is studied in [41, Chapter 11]. A key tool useful to characterise the H-limit still as a monotone operator is the Div-Curl Lemma [25, 26], and the lack of an analogous one in the fractional case precludes to prove the H-compactness for this class of operators by standard techniques.

(3) In Subsection 3.3, we prove that the H-compactness in the symmetric case can be equivalently obtained through the Γ-compactness of the associated energies. In [2], the authors show that an analogous result can be obtained also in the case of not necessarily symmetric matrices for which, a priori, there is no natural energy associated with the problem. We conjecture that the techniques used in the aforementioned paper can be adapted in the fractional scenario as well in order to obtain an alternative proof of Theorem 3.1, purely based on variational techniques.

(4) Once the H-convergence for elliptic operators has been characterised, it is natural to ask whether this can provide information about the asymptotic behaviour of sequences of parabolic nonlocal operators of the form

$$\partial_t - \operatorname{div}^s(B_h(x)\nabla^s).$$

In [22], the authors show that, whenever the sequence of matrix-valued functions $(A_h)_h$ is independent of time, then the parabolic H-limit $B_\infty(x,t)$ coincides with the elliptic H-limit $A_\infty(x)$, meaning that B_∞ is constant in time. Again, the authors conjecture that a similar discussion can be extended to the nonlocal scenario.

(5) The most famous application of the H-convergence relies in the periodic homogenisation of operators of the type

$$-\operatorname{div}(a(hx)\nabla u(x)), \quad h \in \mathbb{N},$$

where a is 1-periodic. We conjecture that an extension to the distributional fractional setting through techniques similar to those of this paper may be possible. In particular, we think that in this case the hypothesis of fixing a matrix A_0 outside the reference domain, used in the proof of Theorem 3.1, can be relaxed.

(6) Recently, the H-convergence has been extended also to the sub-Riemannian framework and, more generally, to operators depending on vector fields, see e.g. [21, 22, 23, 24]. Since the definition of fractional operators is more involved in a general sub-Riemannian setting, we plan to extend Theorem 3.1 at least to the case of Carnot groups.

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