Strategic execution trajectories

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ABSTRACT

We obtain the optimal execution strategy for two sequential trades in the presence of a transient price impact. We first present a novel and general solution method for the case of a single trade (a *metaorder*) that is executed as a sequence of sub-trades (*child orders*). We then analyze the case of two sequential metaorders, including the case where the size and direction of the second metaorder are uncertain at the time the first metaorder is initiated. We obtain the optimal execution strategy under two different cost functions. First, we minimize the total cost when each metaorder is benchmarked to the price at its initiation, the *total separate costs* approach widely used by practitioners. Although simple, we show that optimizing total separate costs can lead to a significant understatement of the real costs of trading whilst also adversely impacting order scheduling. We overcome these issues by introducing a new cost function that splits the second metaorder into two parts, one that is predictable when the first metaorder is initiated and a residual that is not. The predictable and residual parts of the second metaorder are benchmarked using the initiation prices of the first and second metaorders, respectively. We prove existence of an optimal execution trajectory for linear instantaneous price impact and positive definite decay, and derive the explicit form of the minimizer in the special case of exponentially decaying impact, however uniqueness in general remains unproven. Various numerical examples are included for illustration.

KEYWORDS

Block trade, financial markets, market impact, market manipulation, optimal transport, optimal trade execution, optimization, quantitative finance, transactions costs.

AMS CLASSIFICATION

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C61, D53, D81.

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Contents

1. Introduction

Reducing transaction costs is an important goal for any institution that trades in financial markets, and optimizing trade execution strategies is a well-studied problem. Any trade generates an impact in the market and on average buy trades (or simply *buys*) push prices up whereas sells push prices down. Such shifts in price are termed *price impact*. Most approaches for optimizing trade execution focus on the problem of executing a single large trade (which we call a *metaorder*) within a given time horizon. To achieve this, the metaorder is split into smaller *child orders* that are executed sequentially, and each child order is potentially affected by the price impact of all previous child orders.

The literature on optimal execution and price impact largely divides along empirical and theoretical lines. The empirical papers report concave market impact as a function of metaorder size, and price reversion (often termed *relaxation*) after metaorder completion, see [3, 4, 9]. The market impact of metaorders may be further decomposed into temporary and permanent impact, however estimation of these components empirically is challenging due to the difficulty of distinguishing between impact that is truly permanent and impact that decays slowly, while the informational content of trades also complicates matters. The theoretical strand of the literature focuses on models of market impact under which execution schedules are derived that minimise some measure of costs, with some papers modelling the whole limit order book [23, 1] and others ignoring microstructure [14]. Recently propagator models of market impact, which ignore limit order book microstructure, have proved popular both in the academic literature and with practitioners due to their relative simplicity. Gatheral [18] derives no-arbitrage conditions for them. The widely studied optimal execution problem is usually framed in terms of minimizing the expected execution cost benchmarked against the pre-trade mid-price, however other approaches are possible. For example, [10, 16] use volume-weighted average price (VWAP) during the execution period as their benchmark, and in [2] both the expected execution cost and its volatility are taken into account. Price prediction can also be considered within the optimal execution problem [15], and the execution trajectories obtained range from simple constant rate trajectories (in volume time) to more complex solutions [11].

In practice, financial institutions both monitor and seek to minimize execution costs with respect to some benchmark averaged over many metaorders, and frequently use the classical cost definition given by the implementation shortfall in [24]. Definition of the benchmark is important and [26] argues that benchmarking to VWAP can lead to suboptimal execution because the VWAP price is affected by the trades. Best practice within the industry is often considered to be benchmarking to a pre-trade price, for example the metaorder arrival price, as this is not impacted by subsequent trading and is therefore not considered gameable. However this overlooks that the metaorder arrival price may have been impacted by previous metaorders, an effect which can be especially important when consecutive metaorders are correlated as discussed in [21]. See also [5] for an empirical paper which shows that consecutive metaorders may interact when they are close together in time. Thus optimizing expected or average individual metaorder execution costs with arrival price as a benchmark may lead to suboptimal outcomes once sequential effects are included.

1.1. *Two metaorders – the simplest sequential case*

Although most literature deals with optimizing the execution of a single trade over a single time horizon (STSH), here we extend this to a pair of sequential metaorders, and include the case where both the size of the second metaorder and its direction (buy or sell) are uncertain at the time the first metaorder is initiated. This setup is motivated by characteristics of the metaorder streams typical of large systematic investment managers where the optimal position to hold (e.g. long q shares) is updated whenever market or other data are sampled. Only the changes in optimal position which exceed some materiality threshold become metaorders, and small *noise trades* are ignored. Such managers often set their data sampling schedules in advance, and therefore know the times at which metaorders in any particular security may arise, however the size and direction of these metaorders is uncertain until the latest data has been sampled. Motivated by this, we assume throughout that the times at which trades arise are known in advance, and we explore the impact of uncertainty in both trade size and trade direction.[1]

Our intention is to keep the model setup as simple as possible in order to illustrate how optimizing individual metaorder costs can lead to suboptimal outcomes when a richer objective function is considered that captures sequential effects.[2] Thus for our analysis, throughout we assume transient linear market impact, no permanent impact, no alpha signals and no complex liquidity stucture. Under these assumptions, the price impact of a trade can be decomposed into two components: the *instantaneous price impact*, representing an immediate shock to market prices, and the *decay component* representing the gradual dissipation of this price shock over time. Instantaneous price impact tends to be higher for larger metaorders[3] so for simplicity we assume a linear function of order size. For the decay component, we first derive some theoretical results keeping its form general, and then focus on the important special case of exponential decay as this yields a tractable explicit solution. Our setup coincides with the so called *linear continuous propagator* model first defined by Bouchaud *et al.* for discrete time in [8], and for continuous time by Gatheral in [18]. Details are given in Section 2.

Existence and uniqueness of the minimum cost solution in the STSH case of the linear continuous propagator model were established by Gatheral *et al.* in [19]. Curato *et al.* in [11] assumed a nonlinear instantaneous price impact and positive definite decay function, and used homotopy analysis to continuously deform an initial strategy and thereby lower the expected execution cost. In Section 2.1 we provide a new proof of uniqueness, employing a more direct approach that uses optimal transport. The assumptions of our and their approaches are different: we assume an equivalent condition to positive definiteness for the decay component, but assume linear instantaneous price impact. Obizhaeva and Wang in [23] studied the STSH problem in a similar framework, however their approach used a limit order book (LOB) with price-time matching. We do not pursue the LOB extension of our results here, leaving that for separate analysis. There is a growing literature on applications of optimal transport, and in particular martingale optimal transport, in finance and econometrics, see [6], [12], [13], [17] and [20]. To the best of our knowledge this paper is the first to apply optimal transport in the context of optimal execution.

Sections 3.1 and 3.2 cover the main contributions of this study, focusing attention away from the standard STSH problem to optimizing the combined execution of two sequential metaorders.[4] Clearly, if the second metaorder commences before the impact of the first metaorder has completely dissipated then there is interaction between the two executions and, in general, there is no reason to expect the optimal strategy for the combined execution to coincide with overlaying two separately optimal STSH trajectories. Sections 3.1 and 3.2 explore two different ways of formulating the total cost over the two metaorders. In Section 3.1 each metaorder is benchmarked to the market price at which it is initiated. This criterion, which we refer to as the *total separate costs* approach, is widely used by practitioners in retrospective transaction cost monitoring and analysis. Whilst common in that context, it raises some issues when used for execution optimization because the benchmark price of the second metaorder depends on the first metaorder, and both the size and direction of the second metaorder may be unknown when the first metaorder initializes. We later show that optimizing total separate costs can lead to potentially undesirable effects including backloading in the case of correlated

[1] The discarding of noise trades is easily accommodated by allowing the trade size distribution to support trades of size zero.

[2] Although we seek model simplicity, we do expect the qualitative results obtained to hold for any reasonable price impact model.

[3] Empirical analysis typically suggests a mildly nonlinear monotonic relationship with order size. We keep the instantaneous impact component as straightforward as possible whilst maintaining realism.

[4] We restrict attention to two metaorders for the sake of simplicity and to emphasize the main ideas. However we expect our findings can be generalized to an arbitrary number of metaorders.

metaorders (i.e. where more is traded in the latter part of each execution period). This happens because backloading increases the impact on the next arrival price, making it easier for the next metaorder to perform well against its benchmark.

There exist several alternative criteria for analysing sequential transactions, for example the stitching together approach discussed by Harvey *et al.* in [21], however these too can have issues when used for forward planning. Our solution is to decompose the second metaorder into a predictable part known at the time the first metaorder initializes, plus a non-predictable residual that becomes known only when the second metaorder initializes. The predictable part of the second metaorder is benchmarked to the price when the first metaorder initializes, and the non-predictable part to the initialisation price of the second metaorder. We refer to this criterion as the *total hybrid costs* approach and explore it in Section 3.2. In Section 4 we analyze the two metaorder problem for both separate costs and hybrid costs in the special case of exponentially decaying impact, illustrating these results in Section 5 with a range of numerically derived optimal execution schedules. Finally, appendices are provided that include some proofs together with additional results about the separate costs case.

2. The Single Trade Single Horizon (STSH) problem

Letting T_0 and T denote the times at which trading commences and finishes, respectively, and V the inventory required to be held at time T, then without loss of generality we take $T_0 = 0$ and further assume $V > 0$, corresponding to a buy, since precisely analogous results hold in the sell case when $V < 0$. We denote by $G(t) > 0$ the price impact discussed in Section 1, so that $G(0)$ is the instantaneous price impact and $G(t)$ for $t > 0$ represents the decay component. We assume $G(t)$ is non-increasing for all $t \geq 0$ and additionally that $G(t)$ is convex.

We adopt a special case of the continuous-time price model given by Gatheral *et al.* in [19] and refer the reader there for a detailed discussion. We now introduce the concept of a *trading schedule* in the form of a stochastic process $X = (X_t)$ that describes the inventory held at each time, and therefore satisfies $\{X_{0^-} = 0, X_T = V\}$. In the absence of our trading, the so called *unimpacted price* is defined on a given filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and is assumed to be driven by a standard Wiener process W_t. Again, following the discussion in [19], every admissible execution schedule $X = (X_t)$ gives rise to a finite Borel measure $\mathrm{d}X_t$, and it follows that the price model depends on the execution schedule X only through this measure $\mathrm{d}X_t$. When the trading schedule is continuous,[5] then allowing for the impact of our trading, the *impacted price* S_t evolves according to

$$S_t = \underbrace{S_0 + \int_0^t \sigma \, \mathrm{d}W_t}_{\text{unimpacted price}} + \underbrace{\int_0^t G(t-s) \, \mathrm{d}X_s}_{\text{impact}} \tag{2.1}$$

where the constant $\sigma > 0$ denotes the volatility of the unimpacted price. More generally, the trading schedule can contain jumps corresponding to instantaneous buy or sell *block trades.*

We now consider the *slippage* or *implementation shortfall* cost of the execution schedule, which is defined as the total price paid for buying V units over time horizon $[0, T]$ minus the price that hypothetically could have been paid were all V units purchased at the initial unimpacted (or *arrival*) price S_0. With $S_t^0 = S_0 + \int_0^t \sigma \, \mathrm{d}W_t$ denoting the unimpacted price at time t, when X is continuous

[5]This includes the possibility X_t contains some continuous singular component.

the total purchase price is given by

$$\int S_t \,\mathrm{d}X_t = \int S_t^0 \,\mathrm{d}X_t + \int \int_{\{s<t\}} G(t-s)\,\mathrm{d}X_s \mathrm{d}X_t,$$

whereas if X has a jump of size ΔX_t at time t then the trade ΔX_t occurs at cost $G(0)(\Delta X_t)^2/2+\Delta X_t S_t$, see Gatheral *et al.* [19]. Combining these two cost expressions, the total purchase price for the general execution schedule X is therefore

$$\int S_t \,\mathrm{d}X_t + \frac{G(0)}{2}\sum (\Delta X_t)^2.$$

Hence, similar to Lemma 2.3 of [19], the expected slippage cost satisfies

$$\mathbb{E}\left[\int S_t \,\mathrm{d}X_t + \frac{G(0)}{2}\sum (\Delta X_t)^2\right] - V S_0 = \mathbb{E}[\mathcal{C}(X)] \tag{2.2}$$

where the expectations in equation (2.2) are with respect to the measure $\mathbb{P}$ and $\mathcal{C}(X)$ is defined by

$$\mathcal{C}(X) \equiv \frac{1}{2}\int_0^T \int_0^T G(|t-s|)\,\mathrm{d}X_s\,\mathrm{d}X_t. \tag{2.3}$$

As discussed in Predoiu *et al.* [25, Section 3], without loss of generality we may restrict the search for an optimal execution schedule X to nonrandom functions of time, as such a solution minimizes $\mathbb{E}[\mathcal{C}(X)]$ over all execution schedules including stochastic ones.[6]

We find it convenient to re-write the functional in equation (2.3) by defining $G(t) = G(-t)$ for all $t \in \mathbb{R}$, so that G becomes an even function, and also implicitly to regard X_t as identically zero for $t < 0$ so that X_{0^-} has a clear meaning as the limit from the left. Our goal is therefore to determine the trading schedule X that minimizes the slippage cost $\mathcal{C}(X)$ subject to completing the purchase of V units within time horizon $[0, T]$, or more succinctly:

$$\text{find } X \text{ that minimizes } \mathcal{C}(X) \text{ subject to } \{X_{0^-} = 0, X_T = V\}$$

where

$$\mathcal{C}(X) = \frac{1}{2}\int_0^T \int_0^T G(t-s)\,\mathrm{d}X_s\,\mathrm{d}X_t. \tag{2.4}$$

Our treatment ignores limit order book (LOB) microstructure features such as those discussed in Obizhaeva and Wang [23], so the roles played by bids, offers, spreads and depths remain open for separate study.

2.1. ***Uniqueness of the minimizer via optimal transport***

Without loss of generality, from hereon we take $S_0 = 0$. As noted in Curato *et al.* [11], the STSH case was completely solved by Gatheral *et al.* [19] who showed that optimal strategies always exist, are nonrandom functions of time, and are non-alternating between buy and sell trades when instantaneous price impact is linear in the trading rate and decays as a convex function of time. We do not re-derive

[6] See also pages 7-8 of `http://faculty.baruch.cuny.edu/jgatheral/JOIM2011.pdf`.

these results, but instead demonstrate how optimal transport may be employed to show uniqueness. Accordingly we assume non-negativity of the optimizer and then examine carefully the structure of the optimal solution as this proves propaedeutic to our later treatment of the adjacent metaorders case.

Theorem 2.1. *Let G in expression (2.4) denote an even continuous function, strictly convex on $(0, \infty)$. Then the problem*

$$\min_X \mathcal{C}(X) \qquad \textit{subject to} \qquad \mathrm{d}X_t \geq 0,\, X_{0^-} = 0,\, X_T = V > 0 \tag{2.5}$$

has a unique solution.

Proof. We first note existence by recalling that the space of nonnegative measures $\mathrm{d}X_t$ with constant total mass is compact for the weak* topology. Hence, since the functional that we are minimizing is continuous under weak* convergence, there exists a minimum.

To show uniqueness we use optimal transport (or equivalently, rearrangement) techniques. Let $\mathrm{d}X_t^{(1)}$ and $\mathrm{d}X_t^{(2)}$ denote two minimizers, and consider the optimal transport[7] problem:

$$\min_{\bar{\Gamma}} \int_{[0,T]\times[0,T]} |t_1 - t_2|^2 \mathrm{d}\bar{\Gamma}(t_1, t_2) \;\text{ subject to }\; (\pi_i)_\# \bar{\Gamma} = \mathrm{d}X^{(i)}$$

where $\pi_i(t_1, t_2) = t_i$ denotes the canonical projection onto the i-th variable, for $i = 1, 2$. Standard optimal transport results (see [27, Chapter 2]) give that there exists a minimizing measure $\bar{\Gamma} \geq 0$ which enjoys the following monotonicity property:

$$\text{if } (t_1, t_2), (s_1, s_2) \in \operatorname{supp}(\bar{\Gamma}) \quad \text{then} \quad s_1 \leq t_1 \text{ implies } s_2 \leq t_2. \tag{2.6}$$

We now define $\mathrm{d}\bar{X} = \{(\pi_1 + \pi_2)/2\}_\# \bar{\Gamma}$ and note, by the definition of push-forward, that

$$\int_0^T \mathrm{d}\bar{X}_t = \int_{[0,T]\times[0,T]} \mathrm{d}\bar{\Gamma}(t_1, t_2) = \int_{[0,T]} \mathrm{d}X_t^{(1)} = V.$$

Furthermore, we have

$$\begin{aligned}
\mathcal{C}(\bar{X}) &= \int_{[0,T]\times[0,T]} \int_{[0,T]\times[0,T]} G\left(\frac{t_1 + t_2}{2} - \frac{s_1 + s_2}{2}\right) \mathrm{d}\bar{\Gamma}(t_1, t_2)\, \mathrm{d}\bar{\Gamma}(s_1, s_2) \\
&= \int_{[0,T]\times[0,T]} \int_{[0,T]\times[0,T]} G\left(\frac{t_1 - s_1}{2} + \frac{t_2 - s_2}{2}\right) \mathrm{d}\bar{\Gamma}(t_1, t_2)\, \mathrm{d}\bar{\Gamma}(s_1, s_2).
\end{aligned}$$

Now, since G is strictly convex on both $(0, \infty)$ and $(-\infty, 0)$, a consequence of property (2.6) is that for all $(t_1, t_2), (s_1, s_2) \in \operatorname{supp}(\bar{\Gamma})$ with $s_1 \leq t_1$, we have that both $t_1 - s_1 \geq 0$ and $t_2 - s_2 \geq 0$, and

[7] Let $\mathcal{M}_V(Z)$ denote the set of nonnegative measures on a space Z with total mass V. In the terminology of optimal transport, a measure $\Gamma \in \mathcal{M}_V(X \times Y)$ that has marginals $\mu \in \mathcal{M}_V(X)$ and $\nu \in \mathcal{M}_V(Y)$ is called a transport plan between μ and ν. The marginal condition corresponds to saying that $(\pi_1)_\#\Gamma = \mu$ and $(\pi_2)_\#\Gamma = \nu$, where $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. We recall that, given two measures $\sigma_1 \in \mathcal{M}_V(Z_1)$ and $\sigma_2 \in \mathcal{M}_V(Z_2)$ and a measurable map $S : Z_1 \to Z_2$, we say that $S_\#\sigma_1 = \sigma_2$ if $\sigma_2(B) = \sigma_1(S^{-1}(B))$ for all σ_2-measureable sets B.

hence

$$G\left(\frac{t_1-s_1}{2}+\frac{t_2-s_2}{2}\right)\le\frac{G(t_1-s_1)+G(t_2-s_2)}{2}.$$

Note that the inequality above is strict unless $t_2-t_1=s_2-s_1$, and that an analogous result holds for $t_1\le s_1$. This proves that

$$\mathcal{C}(\bar{X})\le\frac{1}{2}\int_{[0,T]\times[0,T]}\int_{[0,T]\times[0,T]}\big[G(t_1-s_1)+G(t_2-s_2)\big]\,\mathrm{d}\bar{\Gamma}(t_1,t_2)\,\mathrm{d}\bar{\Gamma}(s_1,s_2)=\frac{\mathcal{C}\left(X^{(1)}\right)+\mathcal{C}\left(X^{(2)}\right)}{2},$$

with strict inequality unless $X^{(1)}\equiv X^{(2)}$. Since $X^{(1)}$ and $X^{(2)}$ are both minimizers, the inequality above must be an equality, in which case $X^{(1)}\equiv X^{(2)}$, as required. □

2.1.1. Optimality conditions

We now derive optimality conditions for $\mathrm{d}X_t$, the unique minimizer of the previous section. Since our minimization problem concerns processes that are monotonically increasing, our strategy for obtaining these conditions is based on considering perturbations of $\mathrm{d}X_t$ that preserve both this monotonicity and satisfaction of the total volume constraint. We do this in two distinct ways. The first way involves a function $\eta(t)$ that has zero mean with respect to $\mathrm{d}X_t$, while the second way scales $\mathrm{d}X_t$ by a constant less than 1 and then adds an arbitrary nonnegative finite measure $\mathrm{d}Y_t$. As we shall see, these two types of perturbation yield the desired optimality conditions.

We start by considering a function $\eta(t)$ that is continuous on $[0,T]$ and satisfies $\int_0^T\eta(t)\,\mathrm{d}X_t=0$ but is otherwise arbitrary, so that for all $\varepsilon>0$ sufficiently small we have that $\mathrm{d}X_{t,\varepsilon}\equiv\{1+\varepsilon\eta(t)\}\mathrm{d}X_t$ is nonnegative and satisfies $\int_0^T\mathrm{d}X_{t,\varepsilon}=V$. Hence, by the optimality of $\mathrm{d}X_t$ and the fact that G is even, we have

$$\left.\frac{\mathrm{d}\,\mathcal{C}(X_\varepsilon)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}=\int_0^T\left\{\int_0^T G(t-s)\,\mathrm{d}X_s\right\}\eta(t)\,\mathrm{d}X_t=0.$$

Now, since $\eta(t)$ is arbitrary apart from having zero mean with respect to $\mathrm{d}X_t$, and both $\mathrm{d}X_t\ge0$ and $G>0$, it must be that $F(t)\equiv\int_0^T G(t-s)\,\mathrm{d}X_s$ for $t\in[0,T]$ satisfies

$$F(t)=\Lambda \text{ for some constant } \Lambda>0 \text{ on } \operatorname{supp}(\mathrm{d}X_t). \tag{2.7}$$

Exploring the structure of $F(t)$ outside $\operatorname{supp}(\mathrm{d}X_t)$ requires our second type of perturbation. For $\mathrm{d}Y_t\ge0$ a nonnegative finite measure and $\alpha_Y=V^{-1}\int_0^T\mathrm{d}Y_t$, we define $\mathrm{d}X_{t,\varepsilon}\equiv(1-\varepsilon\alpha_Y)\mathrm{d}X_t+\varepsilon\mathrm{d}Y_t$. The measure $\mathrm{d}X_{t,\varepsilon}$ is admissible for the optimization problem (2.5) since $\mathrm{d}X_{t,\varepsilon}\ge0$ for $\varepsilon>0$ sufficiently small, and it is elementary to show $\int_0^T\mathrm{d}X_{t,\varepsilon}=V$. Thus, for $F(t)$ as defined above, it follows by optimality and property (2.7) that

$$
\begin{aligned}
0 \le \lim_{\varepsilon\to 0^+} \frac{\mathcal{C}(X_\varepsilon)-\mathcal{C}(X)}{\varepsilon} &= \int_0^T \left\{\int_0^T G(t-s)\,\mathrm{d}X_s\right\} (\mathrm{d}Y_t - \alpha_Y\,\mathrm{d}X_t) \\
&= \int_0^T F(t)\,\mathrm{d}Y_t - \alpha_Y \int_0^T F(t)\,\mathrm{d}X_t = \int_0^T F(t)\,\mathrm{d}Y_t - \alpha_Y \int_0^T \Lambda\,\mathrm{d}X_t \\
&= \int_0^T F(t)\,\mathrm{d}Y_t - \alpha_Y \Lambda V = \int_0^T F(t)\,\mathrm{d}Y_t - \Lambda \int_0^T \mathrm{d}Y_t = \int_0^T \{F(t)-\Lambda\}\,\mathrm{d}Y_t.
\end{aligned}
$$

Now, since $\mathrm{d}Y_t \ge 0$ is arbitrary, it must be that $F(t) \ge \Lambda$ for $t \in [0,T]$. In summary, we have proved:

Proposition 2.2. *Let $\mathrm{d}X_t$ be the unique minimizer of optimization problem (2.5) and $F(t) \equiv \int_0^T G(t-s)\,\mathrm{d}X_s$ for $t \in [0,T]$. Then there exists a constant $\Lambda > 0$ such that $F(t) = \Lambda$ on $\operatorname{supp}(\mathrm{d}X_t)$ and $F(t) \ge \Lambda$ for all $t \in [0,T]$.*

2.1.2. Structure of the minimizer

In Proposition (2.2), we expressed the optimality of $\mathrm{d}X_t$ via the behaviour of the function $F(t)$, which is the convolution of $\mathrm{d}X_t$ and the price impact kernel $G(t)$. In this section we unpack this integral quantity involving $\mathrm{d}X_t$ and $G(t)$ to expose the structure of the minimizer $\mathrm{d}X_t$. The approach is rather delicate, as it involves understanding the fine regularity properties of $F(t)$. With this in mind, we make some additional simplifying assumptions on G, namely that G is strictly convex on $t \in (0,\infty)$, is twice differentiable with continuous second derivative on $\mathbb{R}\backslash\{0\}$, and additionally satisfies $0 < G(t) \le C_0$, $0 > G'(t) \ge -C_0$ and $0 < G''(t) \le C_0$ for some constant $C_0 > 0$.[8,9] Our analysis exploits that the second derivative of $G(t)$ has a Dirac delta function at the origin, and provides a particularly flexible approach that can be applied also in the case of multiple trades (see Section 3.1.1).

We begin by noting that, by our assumptions, the second derivative of $G(t)$ has a singular part at $t = 0$ due to the jump in G'. More precisely, if we set $\gamma \equiv -G'(0^+) + G'(0^-) = 2G'(0^-)$ and denote by D^2G the distributional second derivative of G, with G'' the pointwise second derivative of $G(t)$ that exists everywhere apart from $t = 0$, then

$$
D^2G = G''\,\mathrm{d}t - \gamma\delta_0 \tag{2.8}
$$

where δ_0 denotes the Dirac delta function. Next, we define

$$
H(t) \equiv \int_0^T G''(t-s)\,\mathrm{d}X_s \quad \text{for } t \in [0,T]
$$

and note that $H > 0$ since both $G'' > 0$ and $\mathrm{d}X_t \ge 0$. Hence, using equation (2.8) and recalling the definition of F (see Proposition 2.2), we obtain

$$
D^2F = \int_0^T D^2G(t-s)\,\mathrm{d}X_s = H\,\mathrm{d}t - \gamma\,\mathrm{d}X_t \quad \text{on } (0,T). \tag{2.9}
$$

In particular, this implies that $D^2F = H\,\mathrm{d}t > 0$ on $(0,T)\backslash\operatorname{supp}(\mathrm{d}X_t)$. Now, since F attains its

[8] Although the assumptions here exclude power law cases such as $G(t) = |t|^{-1/2}$, they accommodate close approximations such as $G(t) = (a+|t|)^{-1/2}$ for any $a > 0$.

[9] Since the form of G outside the domain of integration plays no role, these bounds on G need to be satisfied only for $t \in (0,T)$.

minimum on $\operatorname{supp}(\mathrm{d}X_t)$, we have

$$0 \le D^2F = H\,\mathrm{d}t - \gamma\,\mathrm{d}X_t \le H\,\mathrm{d}t \le \|H\|_\infty\,\mathrm{d}t \text{ on } (0,T),$$

whereby it follows that F is convex on $(0,T)$ and $D^2F \in L^\infty((0,T))$.[10] Recalling equation (2.9), we obtain also that $\mathrm{d}X_t \in L^\infty((0,T))$, namely $\mathrm{d}X_t = x_t\,\mathrm{d}t$ with $x_t \in L^\infty((0,T))$.

We now note that $F'' = 0$ almost everywhere on the set $\operatorname{supp}(\mathrm{d}X_t)$ since F is constant there. This, combined with equation (2.9) and the fact that $\mathrm{d}X_t \in L^\infty((0,T))$, yields

$$\mathrm{d}X_t = \frac{1}{\gamma}H\,\mathrm{d}t \quad \text{on } \operatorname{supp}(\mathrm{d}X_t) \cap (0,T).$$

Also, since F is convex and $D^2F = H\,\mathrm{d}t > 0$ outside $\operatorname{supp}(\mathrm{d}X_t)$, we deduce that

$$\{F = \Lambda\} \cap (0,T) = \operatorname{supp}(\mathrm{d}X_t) \cap (0,T) \quad \text{is an interval.}$$

We now claim that $\operatorname{supp}(\mathrm{d}X_t) = [0,T]$. Indeed, assume for instance that $\operatorname{supp}(\mathrm{d}X_t) = [a,b]$ with $a > 0$. Then, since $F' = 0$ on $\operatorname{supp}(\mathrm{d}X_t)$, we obtain

$$0 = F'(a) = \int_a^b G'(a-s)\,\mathrm{d}X_s. \tag{2.10}$$

However, $G' > 0$ on $(-\infty, 0)$, so the above integral must be strictly positive, which is a contradiction, implying that $a = 0$. By a similar argument, $b = T$. Finally we note that $\mathrm{d}X_t$ cannot be a L^∞ function extending all the way to $t = 0$ (respectively, all the way to $t = T$), since if it were then we could apply equation (2.10) at $t = 0$ (respectively, at $t = T$) and thereby obtain a contradiction. Thus $\mathrm{d}X_t$ must be singular at both 0 and T, with Dirac delta functions at each location.

We now compute the masses μ_0 and μ_T associated with these Dirac delta functions and write

$$\mathrm{d}X_t = \mu_0\,\delta_0 + \mu_T\,\delta_T + \frac{1}{\gamma}H(t)\,\mathrm{d}t.$$

For small $\epsilon > 0$ we have

$$0 = F'(\epsilon) = \int_0^T G'(\epsilon - s)\,\mathrm{d}X_s = \mu_0\,G'(\epsilon) + \mu_T\,G'(\epsilon - T) + \frac{1}{\gamma}\int_0^T G'(\epsilon - s)H(s)\,\mathrm{d}s,$$

and similarly

$$0 = F'(T-\epsilon) = \int_0^T G'(T-\epsilon-s)\,\mathrm{d}X_s = \mu_0\,G'(T-\epsilon) + \mu_T\,G'(-\epsilon) + \frac{1}{\gamma}\int_0^T G'(T-\epsilon-s)H(s)\,\mathrm{d}s.$$

Thus letting $\epsilon \to 0^+$ and recalling that $G'(-t) = -G'(t)$, we obtain the pair of equations

$$\begin{aligned} G'(0^+)\,\mu_0 - G'(T)\,\mu_T - \frac{1}{\gamma}\int_0^T G'(s)H(s)\,\mathrm{d}s = 0 \quad \text{and} \\ G'(T)\,\mu_0 - G'(0^+)\,\mu_T + \frac{1}{\gamma}\int_0^T G'(T-s)H(s)\,\mathrm{d}s = 0, \end{aligned}$$

[10] In other words, the distributional Hessian of F is bounded and has no singular part.

and hence that

$$
\begin{aligned}
\mu_0 &= \frac{1}{G'(0^+)^2 - G'(T)^2} \frac{1}{\gamma} \int_0^T \left\{ G'(0^+) G'(t) + G'(T) G'(T-t) \right\} H(t) \, \mathrm{d}t, \\
\mu_T &= \frac{1}{G'(0^+)^2 - G'(T)^2} \frac{1}{\gamma} \int_0^T \left\{ G'(0^+) G'(T-t) + G'(T) G'(t) \right\} H(t) \, \mathrm{d}t.
\end{aligned} \tag{2.11}
$$

In conclusion, we have proved the following:[11]

Proposition 2.3. *Let $\mathrm{d}X_t$ be the unique minimizer of optimization problem* (2.5)*. Then*

$$
\mathrm{d}X_t = \mu_0 \, \delta_0 + \mu_T \, \delta_T + \frac{1}{\gamma} H(t) \, \mathrm{d}t \ \text{ on } [0,T] \tag{2.12}
$$

where $\gamma \equiv 2G'(0^-) > 0$, $H(t) \equiv \int_0^T G''(t-s) \, \mathrm{d}X_s$ for $t \in [0,T]$ and the constants $\mu_0, \mu_T > 0$ satisfy the pair of equations (2.11)*.*

Remark 2.4. *The pair of equations* (2.11) *is particularly useful in the exponential case $G(t) = e^{-\kappa|t|}$, as it allows us to recover immediately the classical formula for the minimizer (see [19]). Indeed, since $G'' = \kappa^2 G$ and $\gamma = 2\kappa$, using Propositions 2.2 and 2.3 we obtain*

$$
H(t) = \kappa^2 F(t) = \kappa^2 \Lambda \ \text{ and } \ \mathrm{d}X_t = \mu_0 \, \delta_0 + \mu_T \, \delta_T + \frac{\kappa}{2} \Lambda \, \mathrm{d}t \ \text{ on } [0,T],
$$

with

$$
\mu_0 = \frac{\kappa \Lambda}{2(1 - e^{-2\kappa T})} \int_0^T \left\{ e^{-\kappa t} + e^{\kappa(t - 2T)} \right\} \mathrm{d}t = \frac{\Lambda}{2}, \ \text{ and}
$$

$$
\mu_T = \frac{\kappa \Lambda}{2(1 - e^{-2\kappa T})} \int_0^T \left\{ e^{-\kappa(T-t)} + e^{-\kappa(T+t)} \right\} \mathrm{d}t = \frac{\Lambda}{2}.
$$

Thus the volume constraint $\int_0^T \mathrm{d}X_t = V$ gives

$$
V = \Lambda + \frac{\kappa \Lambda}{2} \int_0^T \mathrm{d}t = \Lambda \left(1 + \frac{\kappa T}{2} \right) \ \text{ which implies } \ \Lambda = \frac{2V}{2 + \kappa T},
$$

---

[11] Alternatively, using that F is constant and that $X_T = V$, one gets the following relations:

$$
G(0)\mu_0 + G(T)\mu_T + \frac{1}{\gamma} \int_0^T G(s) H(s) \, \mathrm{d}s = G(T)\mu_0 + G(0)\mu_T + \frac{1}{\gamma} \int_0^T G(T-s) H(s) \, \mathrm{d}s,
$$

$$
\mu_0 + \mu_T + \frac{1}{\gamma} \int_0^T H(s) \, \mathrm{d}s = V.
$$

Solving this linear system, one obtains the following expressions for μ_0 and μ_T:

$$
\begin{aligned}
\mu_0 &= \frac{1}{2} \left[V - \frac{1}{\gamma} \int_0^T H(s) \, \mathrm{d}s + \frac{1}{G(0) - G(T)} \left(\frac{1}{\gamma} \int_0^T G(s) H(s) \, \mathrm{d}s - \frac{1}{\gamma} \int_0^T G(T-s) H(s) \, \mathrm{d}s \right) \right], \\
\mu_T &= \frac{1}{2} \left[V - \frac{1}{\gamma} \int_0^T H(s) \, \mathrm{d}s - \frac{1}{G(0) - G(T)} \left(\frac{1}{\gamma} \int_0^T G(s) H(s) \, \mathrm{d}s - \frac{1}{\gamma} \int_0^T G(T-s) H(s) \, \mathrm{d}s \right) \right].
\end{aligned}
$$

and therefore the optimal execution strategy is given by

$$\mathrm{d}X_t = \frac{V}{2+\kappa T}\,\delta_0 + \frac{V}{2+\kappa T}\,\delta_T + \frac{\kappa V}{2+\kappa T}\,\mathrm{d}t \ \text{ on } [0,T].$$

3. Optimal Execution of Two Adjacent Metaorders

We now move away from the extensively studied STSH problem and begin exploration of optimal execution in the two metaorder case which, as far as we know, is a new research direction. For simplicity, we consider only the case where the first metaorder has to be completed by the time the second metaorder arrives. Similar to the STSH setup, we assume the first metaorder requires purchasing quantity $V_{[0,1]} > 0$ over the time interval $[0, T_1]$, where $V_{[0,1]}$ is assumed known at time $t = 0$. This is augmented with a second metaorder for quantity $V_{[1,2]} \in \mathbb{R}$ to be executed over period $[T_1, T_2]$, corresponding to a buy or a sell depending on the sign of $V_{[1,2]}$. In Section 3.1, full knowledge of $V_{[1,2]}$ is assumed available at time $t = 0$, and $V_{[1,2]}$ is therefore treated as a deterministic constant for all $t \in [0, T_2]$. In contrast, in Section 3.2 we assume that $V_{[1,2]}$ becomes known only at the instant when the first metaorder has completed and the second metaorder has yet to start, which is time $t = T_1$. In this case, we treat $V_{[1,2]}$ as an exogenous random variable for all times $t < T_1$, and additionally assume $V_{[1,2]}$ is independent of both the unimpacted and impacted price processes.[12] For clarity, we emphasize that the times T_1 and T_2 are assumed known here, and are therefore treated as fixed constants throughout, see Section (1.1).[13]

At this point we need to introduce the concepts of *round trip* trades and *price manipulation*. The trading schedule $X = \{X_s\}_{s\geq 0} \not\equiv 0$ is described as a *round trip* trade if $\int_0^T \mathrm{d}X_s = 0$, corresponding to some mixture of buys and sells that nets out to zero in total. If an execution problem accommodates *price manipulation* then it means there exists at least one such round trip trade that has negative expected cost, see [22]. In the STSH case with linear transient impact and exponential decay, the absence of price manipulation was proved by Gatheral in [18]. However no such result is available for two adjacent metaorders even in the case of linear transient impact and exponential decay. We exclude the possibility of price manipulation solutions by making these inadmissible within both metaorder horizons $[0, T_1]$ and $[T_1, T_2]$, although in our later numerical results we do explore relaxing this requirement. In particular, since $V_{[0,1]}$ is assumed positive, this means we exclude any selling over the first interval $[0, T_1]$, whereas over the second interval $[T_1, T_2]$ all trades must be of the same sign as $V_{[1,2]}$.[14] No additional assumptions are required.

Extending equation (2.3) to obtain a two-period slippage requires choosing an appropriate benchmark price for the second metaorder. Benchmarking this with the price just before the second metaorder starts (but after the first metaorder completes) corresponds to the total separate costs approach, whereas benchmarking using the price at $t = 0$ for both the first and second metaorders corresponds to the so called *stitching together* cost. Of these two possibilities, we examine in detail only the total separate costs case. Limitations of the stitching together approach are discussed in [21]. In Section 3.1 the optimal execution schedule under separate costs is derived, while in Section 3.2 we

[12] The assumption that the second metaorder is independent of both the unimpacted and impacted price processes is, of course, mathematically convenient. However the main motivation for it is that it reflects the majority of systematic strategies employed by fund managers, where trades are typically generated by observing prices and other data over histories that are disjoint from, or have only minimal overlap with, the execution horizon. Such an assumption may be unjustified in other cases, e.g. the strategies deployed by high-frequency traders (HFTs).

[13] Extending these results, for example to accommodate randomness in the arrival time and duration of the second metaorder, is left for a separate study.

[14] This mimics the industry practice of large buy-side firms where each metaorder is typically executed as a sequence of same-sign child-orders.

derive results under an alternative benchmarking scheme that decomposes the second metaorder into predictable and non-predictable components which are benchmarked separately.

3.1. *The Two Trade Separate Costs (TTSC) Problem*

Motivated by the separate costs criterion introduced in Section 1 and mentioned above, the first and second metaorders $V_{[0,1]} > 0$ and $V_{[1,2]} \in \mathbb{R}$ are both assumed known at time $t = 0$. As usual, the first metaorder is benchmarked to the price just before its execution begins, which is S_{0^-}. Some care is needed in how we benchmark the second metaorder, as what is intuitively required is the price just before its execution commences whilst factoring in all impact from the first metaorder. This price, which we denote by $S^*_{T_1}$, is given by

$$S^*_{T_1} = \underbrace{S_0 + \int_0^{T_1} \sigma \, \mathrm{d}W_t}_{\text{unimpacted price}} + \underbrace{\int_0^{T_1} G(T_1 - s) \, \mathrm{d}X_s^{[0,1]}}_{\text{total impact of first metaorder}} \tag{3.1}$$

similar to equation (2.1). For this two trade separate costs (TTSC) problem the analogue of the slippage cost (or implementation shortfall) is then

$$\int_0^{T_1} S_t \, \mathrm{d}X_t^{[0,1]} - V_{[0,1]} S_{0^-} + \int_{T_1}^{T_2} S_t \, \mathrm{d}X_t^{[1,2]} - V_{[1,2]} S^*_{T_1} \tag{3.2}$$

for $S^*_{T_1}$ as given by equation (3.1). We emphasise that $S^*_{T_1}$ in the last term of expression (3.2) is the price used to benchmark the second metaorder, and therefore *includes* the impact of any block trade at the end of the first metaorder but *excludes* the impact of any block trade at the start of the second. See Figure 1.

Remark 3.1. *If the second metaorder quantity $V_{[1,2]}$ to be traded within interval $[T_1, T_2]$ were known prior to T_1, then benchmarking its execution using the price at T_1 may be optimistic. For example, if the two trades are in the same direction, this would be likely to understate the total trading cost. More appropriate would be to benchmark the second trade using the prevailing price at the instant $V_{[1,2]}$ became known, or an approach like that discussed in Section 3.2.*

Without loss of generality we assume $S_{0^-} = 0$ and as usual take $V_{[0,1]} > 0$ so that the first trade is a buy. By taking expectations[15] and performing manipulations similar to those applied in obtaining equation (2.4) from equation (2.1), from expression (3.2) the expected slippage cost becomes

$$\begin{aligned} &\frac{1}{2} \int_0^{T_1} \int_0^{T_1} G(t-s) \, \mathrm{d}X_s^{[0,1]} \, \mathrm{d}X_t^{[0,1]} + \frac{1}{2} \int_{T_1}^{T_2} \int_{T_1}^{T_2} G(t-s) \, \mathrm{d}X_s^{[1,2]} \, \mathrm{d}X_t^{[1,2]} \\ &\qquad + \int_0^{T_1} \int_{T_1}^{T_2} G(t-s) \, \mathrm{d}X_t^{[1,2]} \, \mathrm{d}X_s^{[0,1]} - V_{[1,2]} \int_0^{T_1} G(T_1 - s) \, \mathrm{d}X_s^{[0,1]}. \end{aligned} \tag{3.3}$$

Recalling that the second trade can only commence once the first trade has completed, expression (3.3) has the following intuitive interpretation: trading from $X_t^{[0,1]}$ may affect the execution prices of trades within $X_t^{[1,2]}$, including at time T_1; however trading from $X_t^{[1,2]}$, including any trade that $X_t^{[1,2]}$ requires

[15]With respect to $\mathbb{P}$, as before.

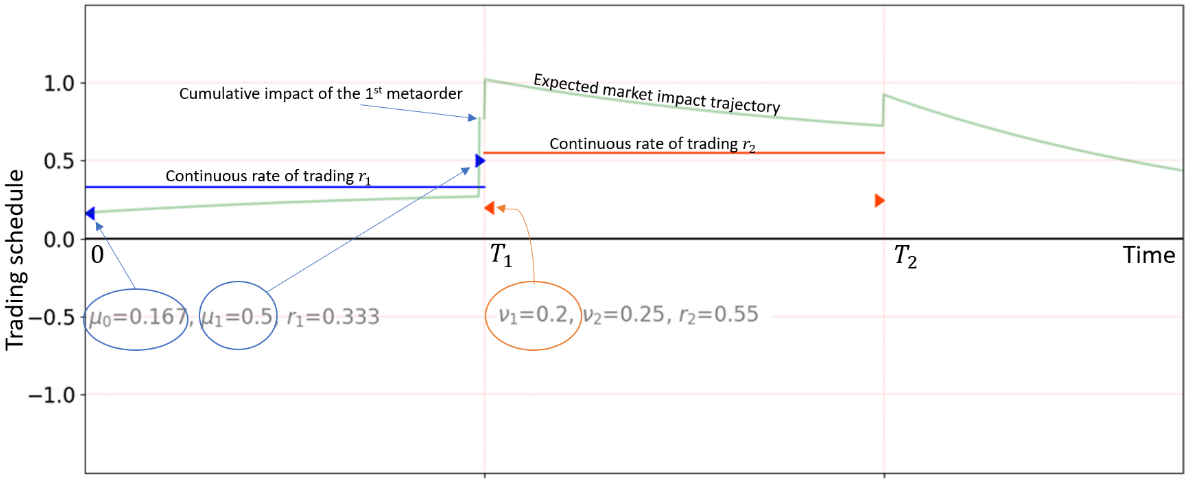

Figure 1. Schematic illustrating the component parts of the two metaorders and execution intervals. The coloured triangles positioned at the start and end of the first and second intervals represent block trades of sizes (μ_0, μ_1) and (ν_1, ν_2) respectively. The blue and orange horizontal lines, which here extend over the full time-range of the first and second intervals (but in later examples may only partially cover these intervals) indicate the extent of continuous trading within each period, with vertical height indicating trading rate. Thus the volume executed by continuous trading in the second period is $r_2(T_2 - T_1)$ where r_2 denotes the height of the orange line. The green line shows the combined price impact from all trading activity and reversion over the first and second periods up to each time t. For this example, the impact is shown increasing in the first period but decreasing in the second, indicating that the reversion induced by the combined first period trades and the block trade at the beginning of the second period is stronger than the additional impact generated by the continuous trading within the second interval. Beyond time T_2, which is when trading finishes, the impact decays monotonically. The price impact noted at time T_1 is used for benchmarking the second metaorder, see equation (3.1), so includes the impact of the block trade at the end of the first period (μ_1) but excludes the impact of the block trade at the start of the second period (ν_1).

at time T_1, can have no effect on the execution prices of trades within $X_t^{[0,1]}$. We start by re-writing expression (3.3) as

$$\begin{aligned}\mathcal{C}\left(X^{[0,1]}, X^{[1,2]}\right) \equiv & \frac{1}{2}\int_0^{T_1}\int_0^{T_1} G(t-s)\,\mathrm{d}X_s^{[0,1]}\,\mathrm{d}X_t^{[0,1]} + \frac{1}{2}\int_{T_1}^{T_2}\int_{T_1}^{T_2} G(t-s)\,\mathrm{d}X_s^{[1,2]}\,\mathrm{d}X_t^{[1,2]} \\ & + \int_0^{T_1}\int_{T_1}^{T_2} [G(t-s) - G(T_1 - s)]\,\mathrm{d}X_t^{[1,2]}\,\mathrm{d}X_s^{[0,1]}\end{aligned} \tag{3.4}$$

which follows because $\int_{T_1}^{T_2} \mathrm{d}X_t^{[1,2]} = V_{[1,2]}$. Expressions (3.3) and (3.4) are both deterministic functions of time, so in the TTSC cost minimization problem which follows we restrict the search for an optimal execution schedule $(X^{[0,1]}, X^{[1,2]})$ to nonrandom functions of time. Our analysis of the TTSC case uses similar arguments to the STSH discussion of Section 2, beginning with the existence of minimizers.

Theorem 3.2. *Let G be an even continuous function, strictly convex on $(0, \infty)$ with $\mathcal{C}\left(X^{[0,1]}, X^{[1,2]}\right)$ as defined in equation (3.4). Then the TTSC problem*

$$\min_{X^{[0,1]}, X^{[1,2]}} \mathcal{C}\left(X^{[0,1]}, X^{[1,2]}\right) \text{ subject to } \begin{cases} X_{0-}^{[0,1]} = 0,\ X_{T_1}^{[0,1]} = V_{[0,1]} > 0,\ \mathrm{d}X_t^{[0,1]} \geq 0, \\ X_{T_1-}^{[1,2]} = 0,\ X_{T_2}^{[1,2]} = V_{[1,2]} \in \mathbb{R},\ V_{[1,2]}\,\mathrm{d}X_t^{[1,2]} \geq 0 \end{cases} \tag{3.5}$$

has a solution (which is a nonrandom function of time).

Proof. Similar to Theorem 2.1, this is an immediate consequence of the weak* compactness of non-negative measures with constant total mass. $\square$

Remark 3.3. *In this general setting, uniqueness is not clear.*

3.1.1. Optimality conditions

Let $(\mathrm{d}X_t^{[0,1]}, \mathrm{d}X_t^{[1,2]})$ denote a minimizer. To simplify the notation, let

$$F_{0,1}(t) \equiv \int_0^{T_1} G(t-s)\,\mathrm{d}X_s^{[0,1]} + \int_{T_1}^{T_2} [G(t-s) - G(t-T_1)]\,\mathrm{d}X_s^{[1,2]} \tag{3.6}$$

and

$$F_{1,2}(t) \equiv \int_0^{T_1} [G(t-s) - G(T_1-s)]\,\mathrm{d}X_s^{[0,1]} + \int_{T_1}^{T_2} G(t-s)\,\mathrm{d}X_s^{[1,2]} \tag{3.7}$$

so that

$$\mathcal{C}\left(X^{[0,1]}, X^{[1,2]}\right) = \frac{1}{2}\int_0^{T_1} F_{0,1}(t)\,\mathrm{d}X_t^{[0,1]} + \frac{1}{2}\int_{T_1}^{T_2} F_{1,2}(t)\,\mathrm{d}X_t^{[1,2]}. \tag{3.8}$$

As in Section 2.1.1, we derive the optimality conditions using two different kinds of perturbation. For the first, let η_1 and η_2 denote continuous functions on $[0,T_1]$ and $[T_1,T_2]$, respectively, that satisfy

$$\int_0^{T_1} \eta_1(t)\,\mathrm{d}X_t^{[0,1]} = \int_{T_1}^{T_2} \eta_2(t)\,\mathrm{d}X_t^{[1,2]} = 0$$

but are otherwise arbitrary. Then, for any $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$ sufficiently small, the execution schedules $\mathrm{d}X_{t,\varepsilon_1}^{[0,1]} \equiv \{1+\varepsilon_1\eta_1(t)\}\mathrm{d}X_t^{[0,1]}$ and $\mathrm{d}X_{t,\varepsilon_2}^{[1,2]} \equiv \{1+\varepsilon_2\eta_2(t)\}\mathrm{d}X_t^{[1,2]}$ are admissible for the TTSC problem in Theorem 3.2. Hence, by the minimality of $(\mathrm{d}X_t^{[0,1]}, \mathrm{d}X_t^{[1,2]})$ and the fact that G is even, we have

$$0 = \left.\frac{\mathrm{d}\,\mathcal{C}(X_{\varepsilon_1}^{[0,1]}, X^{[1,2]})}{\mathrm{d}\varepsilon_1}\right|_{\varepsilon_1=0} = \int_0^{T_1} F_{0,1}(t)\,\eta_1(t)\,\mathrm{d}X_t^{[0,1]}$$

and

$$0 = \left.\frac{\mathrm{d}\,\mathcal{C}(X^{[0,1]}, X_{\varepsilon_2}^{[1,2]})}{\mathrm{d}\varepsilon_2}\right|_{\varepsilon_2=0} = \int_{T_1}^{T_2} F_{1,2}(t)\,\eta_2(t)\,\mathrm{d}X_t^{[1,2]},$$

and hence that

$$F_{0,1}(t) = \Lambda_{0,1} \in \mathbb{R} \text{ on } \operatorname{supp}(\mathrm{d}X_t^{[0,1]}) \quad\text{and}\quad F_{1,2}(t) = \Lambda_{1,2} \in \mathbb{R} \text{ on } \operatorname{supp}(\mathrm{d}X_t^{[1,2]}). \tag{3.9}$$

Combining equations (3.8) and (3.9) we therefore obtain

$$\mathcal{C}\left(X^{[0,1]}, X^{[1,2]}\right) = \frac{1}{2}\left(\Lambda_{0,1}V_{[0,1]} + \Lambda_{1,2}V_{[1,2]}\right).$$

We deduce structure beyond $\operatorname{supp}(\mathrm{d}X_t^{[0,1]})$ and $\operatorname{supp}(\mathrm{d}X_t^{[1,2]})$ by means of a second kind of perturbation. Letting $\mathrm{d}Y_{t,1}$ denote a nonnegative arbitrary measure on $[0,T_1]$, and $\mathrm{d}Y_{t,2}$ an arbitrary measure on

$[T_1,T_2]$ with the same sign as $V_{[1,2]}$, we set $(c_{Y,1},c_{Y,2}) \equiv (\int_0^{T_1} \mathrm{d}Y_{t,1}/V_{[0,1]}, \int_{T_1}^{T_2} \mathrm{d}Y_{t,2}/V_{[1,2]})$ and define the perturbations $\mathrm{d}X_{t,\varepsilon_1}^{[0,1]} \equiv (1-\varepsilon_1 c_{Y,1})\mathrm{d}X_t^{[0,1]} + \varepsilon_1 \mathrm{d}Y_{t,1}$ and $\mathrm{d}X_{t,\varepsilon_2}^{[1,2]} \equiv (1-\varepsilon_2 c_{Y,2})\mathrm{d}X_t^{[1,2]} + \varepsilon_2 \mathrm{d}Y_{t,2}$. Hence, applying the argument utilized just before Proposition 2.2, the assumed optimality of $(\mathrm{d}X_t^{[0,1]}, \mathrm{d}X_t^{[1,2]})$ together with equation (3.9) give that

$$0 \le \lim_{\varepsilon_1 \to 0^+} \frac{\mathcal{C}(X_{\varepsilon_1}^{[0,1]}, X^{[1,2]}) - \mathcal{C}(X^{[0,1]}, X^{[1,2]})}{\varepsilon_1} = \int_0^{T_1} \{F_{0,1}(t) - \Lambda_{0,1}\}\, \mathrm{d}Y_{t,1}$$

and

$$0 \le \lim_{\varepsilon_2 \to 0^+} \frac{\mathcal{C}(X^{[0,1]}, X_{\varepsilon_2}^{[1,2]}) - \mathcal{C}(X^{[0,1]}, X^{[1,2]})}{\varepsilon_2} = \int_{T_1}^{T_2} \{F_{1,2}(t) - \Lambda_{1,2}\}\, \mathrm{d}Y_{t,2}.$$

Now, since $\mathrm{d}Y_{t,1}$ and $\mathrm{d}Y_{t,2}$ are arbitrary, we conclude that

$$F_{0,1}(t) - \Lambda_{0,1} \ge 0 \text{ on } [0,T_1] \text{ and } V_{[1,2]}\{F_{1,2}(t) - \Lambda_{1,2}\} \ge 0 \text{ on } [T_1,T_2].$$

Summarizing the above, we have proved the following:

Proposition 3.4. *Let $(\mathrm{d}X_t^{[0,1]}, \mathrm{d}X_t^{[1,2]})$ denote a minimizer of the TTSC problem in Theorem 3.2, with $F_{0,1}(t)$ and $F_{1,2}(t)$ as defined in equations (3.6) and (3.7) respectively. Then there exist real constants $\Lambda_{0,1}$ and $\Lambda_{1,2}$ such that*

(1) $F_{0,1}(t) = \Lambda_{0,1}$ on $\mathrm{supp}(\mathrm{d}X_t^{[0,1]})$ with $F_{0,1}(t) - \Lambda_{0,1} \ge 0$ on $[0,T_1]$, and
(2) $F_{1,2}(t) = \Lambda_{1,2}$ on $\mathrm{supp}(\mathrm{d}X_t^{[1,2]})$ with $V_{[1,2]} \times (F_{1,2}(t) - \Lambda_{1,2}) \ge 0$ on $[T_1,T_2]$.

3.1.2. Structure of the minimizer

Thanks to Proposition 3.4, we can now analyze the structure of minimizers of the TTSC problem. We do so under the same additional assumptions about G, G' and G'' as listed at the start of Section 2.1.2. For the analysis which follows it is convenient to define the component parts of $F_{0,1}$ and $F_{1,2}$ in equations (3.6) and (3.7) as follows:

$$\begin{aligned} \Phi_{0,1}(t) &\equiv \int_0^{T_1} G(t-s)\, \mathrm{d}X_s^{[0,1]}, & \Psi_{0,1}(t) &\equiv \int_{T_1}^{T_2} [G(t-s) - G(t-T_1)]\, \mathrm{d}X_s^{[1,2]}, \\ \Phi_{1,2}(t) &\equiv \int_{T_1}^{T_2} G(t-s)\, \mathrm{d}X_s^{[1,2]}, & \Psi_{1,2}(t) &\equiv \int_0^{T_1} [G(t-s) - G(T_1-s)]\, \mathrm{d}X_s^{[0,1]}. \end{aligned} \tag{3.10}$$

Additionally, we set

$$\begin{aligned} H_{0,1}(t) &\equiv \int_0^{T_1} G''(t-s)\, \mathrm{d}X_s^{[0,1]} + \int_{T_1}^{T_2} [G''(t-s) - G''(t-T_1)]\, \mathrm{d}X_s^{[1,2]} \quad \text{and} \\ H_{1,2}(t) &\equiv \int_0^{T_1} G''(t-s)\, \mathrm{d}X_s^{[0,1]} + \int_{T_1}^{T_2} G''(t-s)\, \mathrm{d}X_s^{[1,2]}. \end{aligned} \tag{3.11}$$

We now separately examine the two cases $V_{[1,2]} > 0$ and $V_{[1,2]} < 0$.

Case 1: The second trade is a buy, so $V_{[1,2]} > 0$

The main result has the following interpretation:[16]

(1) $\mathrm{d}X^{[0,1]}$ always has a nonzero atom (block trade) at T_1 plus the possibility of continuous trading over some interval $[a, T_1]$, with an additional atom at $t = 0$ only when $a = 0$; and
(2) $\mathrm{d}X^{[1,2]}$ always has a nonzero atom (block trade) at T_2 plus the possibility of continuous trading over some interval $[c, T_2]$, with an additional atom at T_1 only when $c = T_1$.

The above is made precise in Proposition 3.5 and proved in Appendix A.

Proposition 3.5. *Let $(\mathrm{d}X^{[0,1]}, \mathrm{d}X^{[1,2]})$ denote a minimizer of the the TTSC problem in Theorem 3.2 with $V_{[1,2]} > 0$, and let $H_{0,1}$ and $H_{1,2}$ be as defined by the pair of equations (3.11). Then:*
either

$$\mathrm{d}X_t^{[0,1]} = \mu_{T_1}\,\delta_{T_1} + \frac{1}{\gamma}H_{0,1}(t)\,\mathrm{d}t \textit{ with } \mathrm{supp}(\mathrm{d}X_t^{[0,1]}) = [a, T_1] \textit{ for some } a > 0,\ \mu_{T_1} > 0$$

or

$$\mathrm{d}X_t^{[0,1]} = \mu_0\,\delta_0 + \mu_{T_1}\,\delta_{T_1} + \frac{1}{\gamma}H_{0,1}(t)\,\mathrm{d}t \textit{ with } \mathrm{supp}(\mathrm{d}X_t^{[0,1]}) = [0, T_1] \textit{ for some } \mu_0 \geq 0,\ \mu_{T_1} > 0.$$

Also:
either

$$\mathrm{d}X_t^{[1,2]} = \nu_{T_2}\,\delta_{T_2} + \frac{1}{\gamma}H_{1,2}(t)\,\mathrm{d}t \textit{ with } \mathrm{supp}(\mathrm{d}X_t^{[1,2]}) = [c, T_2] \textit{ for some } c > T_1,\ \nu_{T_2} > 0$$

or

$$\mathrm{d}X_t^{[1,2]} = \nu_{T_1}\,\delta_{T_1} + \nu_{T_2}\,\delta_{T_2} + \frac{1}{\gamma}H_{1,2}(t)\,\mathrm{d}t \textit{ with } \mathrm{supp}(\mathrm{d}X_t^{[1,2]}) = [T_1, T_2] \textit{ for some } \nu_{T_1} \geq 0,\ \nu_{T_2} > 0.$$

Case 2: The second trade is a sell, so $V_{[1,2]} < 0$

Similar to above, the main result is given by Proposition 3.6 and is proved in Appendix A.

Proposition 3.6. *Let $(\mathrm{d}X^{[0,1]}, \mathrm{d}X^{[1,2]})$ denote a minimizer of the the TTSC problem in Theorem 3.2 with $V_{[1,2]} < 0$, and let $H_{0,1}$ and $H_{1,2}$ be as defined by the pair of equations (3.11). Then:*
either

$$\mathrm{d}X_t^{[0,1]} = \mu_0\,\delta_0 + \frac{1}{\gamma}H_{0,1}(t)\,\mathrm{d}t \textit{ on } [0, b] \textit{ for some } b < T_1,\ \mu_0 > 0$$

or

$$\mathrm{d}X_t^{[0,1]} = \mu_0\,\delta_0 + \mu_{T_1}\,\delta_{T_1} + \frac{1}{\gamma}H_{0,1}(t)\,\mathrm{d}t \textit{ on } [0, T_1] \textit{ for some } \mu_0 \geq 0,\ \mu_{T_1} > 0.$$

[16]Let $I(\cdot)$ denote the indicator function. Throughout our numerical experiments for the TTSC problem with exponentially decaying impact, the first period optimal schedules were either of the form $\mu_{T_1}\delta_{T_1}$ or $\mu_0\delta_0 + r_1 I(0 < t < T_1) + \mu_{T_1}\delta_{T_1}$ with $\mu_0 > 0$, $\mu_{T_1} > 0$ and $r_1 > 0$. In particular, we never observed a solution of the form $r_1 I(a \leq t < T_1) + \delta_{T_1}$ for nonzero r_1 and $a > 0$.

Also:
either

$$\mathrm{d}X_t^{[1,2]} = \nu_{T_1}\,\delta_{T_1} + \frac{1}{\gamma} H_{1,2}(t)\,\mathrm{d}t \ \text{ on } [T_1, d] \text{ for some } d < T_2,\ \nu_{T_1} < 0$$

or

$$\mathrm{d}X_t^{[1,2]} = \nu_{T_1}\,\delta_{T_1} + \nu_{T_2}\,\delta_{T_2} + \frac{1}{\gamma} H_{1,2}(t)\,\mathrm{d}t \ \text{ on } [T_1, T_2] \text{ for some } \nu_{T_1} < 0,\ \nu_{T_2} \leq 0.$$

3.2. ***The Two Trade Hybrid Costs (TTHC) Problem***

Our treatment of the TTSC problem in the previous section benchmarks the second metaorder using the price $S^*_{T_1}$ defined in equation (3.1) where T_1 denotes the time at which the second trade becomes known. In this section we consider the case where some information about the second trade may be known in advance of time T_1. Specifically, we again consider two trades (metaorders) of sizes $V_{[0,1]} > 0$ and $V_{[1,2]} \in \mathbb{R}$ to be executed over periods $[0, T_1]$ and $[T_1, T_2]$, but we now decompose the second trade into two parts: a predictable part of size $V_{[1,2],p} = \mathbb{E}V_{[1,2]}$ that is assumed known at time $T = 0$, and a surprise (or *trade amend*) part of size $V_{[1,2],a} = V_{[1,2]} - V_{[1,2],p}$ that becomes known only at time T_1. These predictable and surprise parts of the second trade are benchmarked using the price S_{0^-} for the predictable part $V_{[1,2],p}$ and $S^*_{T_1}$ for the surprise part $V_{[1,2],a}$. As in equation (3.2) we reiterate that $S^*_{T_1}$ includes the impact of any block trade at the end of the first period but excludes the impact of any block trade at the start of the second period. Thus at time $t = 0$, we have that $V_{[1,2],p}$ is a constant whereas $V_{[1,2],a}$ is a random variable satisfying $\mathbb{E}V_{[1,2],a} = 0$. As before, the first trade remains benchmarked using the initial price S_{0^-}. We refer to this Two Trade Hybrid Costs setup as the TTHC problem.

Analogous to equation (3.2), the TTHC implementation shortfall is given by

$$\left[\int_0^{T_1} S_t\,\mathrm{d}X_t^{[0,1]} - V_{[0,1]}S_{0^-}\right] + \left[\int_{T_1}^{T_2} S_t\,\mathrm{d}X_{p,t}^{[1,2]} - V_{[1,2],p}S_{0^-}\right] + \left[\int_{T_1}^{T_2} S_t\,\mathrm{d}X_{a,t}^{[1,2]} - V_{[1,2],a}S_{T_1^*}\right]. \quad (3.12)$$

The first square-bracketed term of expression (3.12) corresponds to execution of the first trade $V_{[0,1]}$ benchmarked using price S_{0^-}. The second square-bracketed term corresponds to the predictable part of $V_{[1,2]}$ benchmarked using price S_{0^-}, while the third square-bracketed term corresponds to the unpredictable amend part of the second trade benchmarked using price $S^*_{T_1}$. For the avoidance of doubt, we reiterate that both the predictable and surprise parts of the second trade $V_{[1,2]}$ are executed within the second period $[T_1, T_2]$, and there is no reallocation of inventory between the two execution periods. As before, whilst our setup accommodates the case of a buy in the first period and a sell in second, it excludes the possibility of price manipulation because trading is restricted to be non-alternating within each period.[17]

Writing $\mathrm{d}X_t^{[1,2]} \equiv \mathrm{d}X_{p,t}^{[1,2]} + \mathrm{d}X_{a,t}^{[1,2]}$ for the execution schedule of the combined predictable and amend parts of $V_{[1,2]}$, and assuming $S_{0^-} = 0$, expression (3.12) becomes

$$\int_0^{T_1} S_t\,\mathrm{d}X_t^{[0,1]} + \int_{T_1}^{T_2} S_t\,\mathrm{d}X_t^{[1,2]} - V_{[1,2],a}S^*_{T_1} \quad (3.13)$$

for $S^*_{T_1}$ as given by equation (3.1). Expression (3.13) contains two distinct sources of randomness:

[17] In the numerical results which follow later we explore the impact of relaxing this restriction.

the process S_t corresponding to the impacted price, and the random variable $V_{[1,2],a}$ corresponding to the amend part of the second metaorder. We deal with these separately by first conditioning on $V_{[1,2],a} = v$ in expression (3.13), and then taking expectation with respect to $\mathbb{P}$, as before. Applying the same manipulations used to derive equation (3.4), the TTHC problem conditional on $V_{[1,2],a} = v$ thereby becomes to minimize

$$\begin{aligned}\mathcal{C}\left(X^{[0,1]}, X^{[1,2]} | V_{[1,2],a} = v\right) \;\equiv\; & \frac{1}{2}\int_0^{T_1}\int_0^{T_1} G(t-s)\,\mathrm{d}X_s^{[0,1]}\,\mathrm{d}X_t^{[0,1]} + \frac{1}{2}\int_{T_1}^{T_2}\int_{T_1}^{T_2} G(t-s)\,\mathrm{d}X_s^{[1,2]}\,\mathrm{d}X_t^{[1,2]} \\ & + \int_0^{T_1}\int_{T_1}^{T_2} G(t-s)\,\mathrm{d}X_t^{[1,2]}\,\mathrm{d}X_s^{[0,1]} - v\int_0^{T_1} G(T_1-s)\,\mathrm{d}X_s^{[0,1]} \qquad (3.14)\end{aligned}$$

subject to the constraints

$$X_{0-}^{[0,1]} = 0, \quad X_{T_1}^{[0,1]} = V_{[0,1]} > 0, \quad \mathrm{d}X_t^{[0,1]} \geq 0, \quad X_{T_{1-}}^{[1,2]} = 0, \quad X_{T_2}^{[1,2]} = V_{[1,2]} \in \mathbb{R},$$

$$V_{[1,2]}\,\mathrm{d}X_t^{[1,2]} \geq 0 \text{ and } V_{[1,2]} = V_{[1,2],p} + v.$$

For the deterministic case, that is when $V_{[1,2],a} = 0$ with probability 1, we note that existence of minimizers for the optimization problem (3.14) is an immediate consequence of the weak* compactness of nonnegative measures with constant total mass (see the proof of Theorem 2.1). If additionally we have that $G(t) = e^{-\kappa|t|}$, an analysis using essentially the same arguments as for the TTSC case shows that minimizers again have constant absolutely continuous parts inside both $[0, T_1]$ and $[T_1, T_2]$, and the possibility of Dirac deltas at the upper and lower bounds of these intervals.

Remark 3.7. *In the deterministic case when $V_{[1,2],a} = 0$ with probability 1 so that $V_{[1,2]}$ is known at time $t = 0$, and additionally the two execution intervals $[0, T_1]$ and $[T_1, T_2]$ are of the same duration, then the TTHC problem is symmetric under the transformation $t' = T_2 - t$ providing $V_{[0,1]}$ and $V_{[1,2]}$ are also swapped. In this case the minimizer which arises when $V_{[0,1]}$ and $V_{[1,2]}$ are swapped is just the time-reversed version of the minimizer obtained with them in their original order.*

Remark 3.8. *When $V_{[1,2],p} = 0$ so that the second trade is of random size with mean zero, then no part of $V_{[1,2]}$ is benchmarked against price S_{0^-}. In this case, and conditional on each $V_{[1,2]} \equiv V_{[1,2],a} = v$, then the optimization problem (3.14) coincides with the deterministic TTSC problem discussed in Section 3.2 (that is, their objective functions and constraints coincide), so the minimizers for the two problems coincide also.*

We do not further discuss the conditional TTHC problem given in equation (3.14), as rather than being of primary interest it is simply an intermediate construct that gets used in the next section.

3.2.1. Optimizing over uncertainty in the second trade $V_{[1,2]}$

Decomposing the second trade $V_{[1,2]}$ into its predictable mean and surprise components $V_{[1,2],p} = \mathbb{E}V_{[1,2]}$ and $V_{[1,2],a} = V_{[1,2]} - V_{[1,2],p}$, respectively, our focus now becomes unconditionally optimizing the TTHC cost allowing for the uncertainty in the random variable $V_{[1,2],a}$. It is important to note that this is the *only* source of randomness within this stochastic optimization, as the operation of taking expectation with respect to $\mathbb{P}$ in constructing the objective function (3.14) removes the randomness associated with the price process S_t.

We start with the time T_1 problem of optimizing TTHC costs over $X^{[1,2]}$ conditional on a given first period strategy $X^{[0,1]}$ and assumed known $V_{[1,2],a} = v$. More precisely, let $X^{[0,1]}$ denote an admissible strategy for the initial trade $V_{[0,1]}$ so that

$$X^{[0,1]}_{0^-} = 0,\ X^{[0,1]}_{T_1} = V_{[0,1]} > 0,\ \mathrm{d}X^{[0,1]}_t \geq 0,$$

and consider minimizing $\mathcal{C}(X^{[0,1]}, X^{[1,2]}|V_{[1,2],a} = v)$ as given in equation (3.14) over $X^{[1,2]}$ subject to

$$X^{[1,2]}_{T_{1-}} = 0, \quad X^{[1,2]}_{T_2} = V_{[1,2]} \in \mathbb{R}, \quad V_{[1,2]}\, \mathrm{d}X^{[1,2]}_t \geq 0, \quad V_{[1,2]} = V_{[1,2],p} + v. \tag{3.15}$$

Denoting by $m(X^{[0,1]}|V_{[1,2],a} = v)$ the minimum cost expression obtained for each $X^{[0,1]}$ and v, we define $\hat{m}[X^{[0,1]}]$ as the expectation of this minimum cost over the probability distribution of $V_{[1,2],a}$, that is

$$\hat{m}[X^{[0,1]}] \equiv \mathbb{E}_{V_{[1,2],a}}\left[m(X^{[0,1]}, V_{[1,2],a})\right] = \int m(X^{[0,1]}|V_{[1,2],a} = v)\, \mathrm{d}V_{[1,2],a}(v).$$

To solve the overall problem we now seek the admissible strategy $X^{[0,1]}$ that minimizes $\hat{m}[X^{[0,1]}]$. More precisely, we solve

$$\min_{X^{[0,1]}} \hat{m}[X^{[0,1]}] \quad \text{subject to} \quad X^{[0,1]}_{0^-} = 0,\ X^{[0,1]}_{T_1} = V_{[0,1]} > 0,\ \mathrm{d}X^{[0,1]}_t \geq 0. \tag{3.16}$$

As before, weak* compactness of nonnegative measures with constant total mass immediately gives that a minimizer of the optimization (3.16) exists.

3.2.2. Extension: The Stochastic TTSC Problem

If instead of taking $V_{[1,2],p} = \mathbb{E}V_{[1,2]}$ in the above we were to define $V_{[1,2],p} \equiv 0$ (even when $\mathbb{E}V_{[1,2]} \neq 0$) so that the random variable $V_{[1,2],a} = V_{[1,2]}$ comprises the entire second metaorder, then the middle term of equation (3.12) vanishes (since $S_{0^-} = 0$) and the second trade is wholly benchmarked using price $S^*_{T_1}$. We refer to this as the *Stochastic TTSC* problem and note it arises as the $\{V_{[1,2],p} \equiv 0, V_{[1,2],a} = V_{[1,2]}\}$ special case of the TTHC problem discussed above. More generally, even when $V_{[1,2],p} \neq 0$ the Stochastic TTSC objective function may be obtained from the TTHC objective function (when $S_{0^-} = 0$) simply by substracting from it the quantity $V_{[1,2],p}S^*_{T_1}$, see equation (3.13). This enables straightforward formulation of the Stochastic TTSC problem for numerical solution in terms of the TTHC objective function. Later we show optimal execution schedules obtained numerically for both the Stochastic TTSC and TTHC problems.

4. Two Adjacent Metaorders: exponentially decaying impact

To illustrate our results, for simplicity throughout this section we focus on the case of exponentially decaying impact with $G(t) = e^{-\kappa|t|}$ so that $G'' = \kappa^2 G$, while from equation (2.8) we have $\gamma \equiv 2G'(0^-) = 2\kappa$. Within this setup we derive the structure of optimal solutions under the TTSC and TTHC objectives discussed in Section 3. Analysis under more general assumptions about $G(t)$ remains open for separate study.

4.1. *The deterministic TTSC problem with exponentially decaying impact*

From Proposition 3.4 and equation (A2), respectively, we have that $F_{1,2} = \Lambda_{1,2}$ and $\mathrm{d}X_t^{[1,2]} = (2\kappa)^{-1} H_{1,2}\,\mathrm{d}t$ on $\mathrm{supp}(\mathrm{d}X_t^{[1,2]}) \cap (T_1, T_2)$. Hence, since $G'' = \kappa^2 G$, recalling the definitions of $F_{1,2}$ and $H_{1,2}$ from equations (3.7) and (3.11), we obtain

$$\begin{aligned} H_{1,2}(t) &= \int_0^{T_1} G''(t-s)\,\mathrm{d}X_s^{[0,1]} + \int_{T_1}^{T_2} G''(t-s)\,\mathrm{d}X_s^{[1,2]} = \kappa^2 \int_0^{T_1} G(t-s)\,\mathrm{d}X_s^{[0,1]} + \kappa^2 \int_{T_1}^{T_2} G(t-s)\,\mathrm{d}X_s^{[1,2]} \\ &= \kappa^2 F_{1,2}(t) + \kappa^2 \int_0^{T_1} G(T_1-s)\,\mathrm{d}X_s^{[0,1]} = \kappa^2 \Lambda_{1,2} + \kappa^2 \int_0^{T_1} G(T_1-s)\,\mathrm{d}X_s^{[0,1]}, \end{aligned}$$

so $H_{1,2}(t) = \hat{\Lambda}_{1,2}$ is constant on $\mathrm{supp}(\mathrm{d}X_t^{[1,2]}) \cap (T_1, T_2)$ where

$$\hat{\Lambda}_{1,2} \equiv \kappa^2 \Lambda_{1,2} + \kappa^2 \int_0^{T_1} e^{-\kappa(T_1-s)}\,\mathrm{d}X_s^{[0,1]}. \tag{4.1}$$

In particular, $\mathrm{d}X_t^{[1,2]}$ is constant on its support inside (T_1, T_2). Analogously, $H_{0,1} = \hat{\Lambda}_{0,1}$ is also constant, with

$$\hat{\Lambda}_{0,1} = \kappa^2 \Lambda_{0,1}. \tag{4.2}$$

As before, we consider two cases depending on the sign of $V_{[1,2]}$.

4.1.1. Case 1: The second trade is a buy so $V_{[1,2]} > 0$

From Proposition 3.5, we have that

$$\begin{aligned} \mathrm{d}X_t^{[0,1]} &= \mu_0\,\delta_0 + \mu_{T_1}\,\delta_{T_1} + I(a \le t \le T_1)\frac{\hat{\Lambda}_{0,1}}{2\kappa}\,\mathrm{d}t \text{ for } a \in [0, T_1) \text{ and} \\ \mathrm{d}X_t^{[1,2]} &= \nu_{T_1}\,\delta_{T_1} + \nu_{T_2}\,\delta_{T_2} + I(c \le t \le T_2)\frac{\hat{\Lambda}_{1,2}}{2\kappa}\,\mathrm{d}t \text{ for } c \in [T_1, T_2) \end{aligned}$$

where $I(\cdot)$ denotes the indicator function, μ_0, μ_{T_1}, ν_{T_1}, ν_{T_2}, $\hat{\Lambda}_{0,1}$ and $\hat{\Lambda}_{1,2}$ are all nonnegative constants, with $\mu_0 \cdot a = 0$ (that is, one of the two numbers vanishes) and likewise $\nu_{T_1} \cdot (c - T_1) = 0$.

To progress further we impose that the derivatives of $F_{0,1}$ and $F_{1,2}$ vanish on the support of our minimizer, similar to the approach of Section 2.1.2. The sign of the derivative of G' changes at the origin, so care is needed near the Dirac deltas. We thereby obtain the following system of equations:

$$\begin{aligned} 0 &= \lim_{\epsilon \to 0^+} F'_{0,1}(T_1 - \epsilon) = -\kappa\Big(\int_0^{T_1^-} e^{-\kappa(T_1-s)}\mathrm{d}X_s^{[0,1]} - \mu_{T_1} - \int_{T_1}^{T_2} [e^{-\kappa(s-T_1)} - 1]\mathrm{d}X_s^{[1,2]}\Big), \\ 0 &= \lim_{\epsilon \to 0^+} F'_{1,2}(T_2 - \epsilon) = -\kappa\Big(\int_0^{T_1} e^{-\kappa(T_2-s)}\mathrm{d}X_s^{[0,1]} + \int_{T_1}^{T_2^-} e^{-\kappa(T_2-s)}\mathrm{d}X_s^{[1,2]} - \nu_{T_2}\Big), \\ 0 &= \lim_{\epsilon \to 0^+} F'_{0,1}(a + \epsilon) = -\kappa\Big(\int_{a^+}^{T_1} e^{-\kappa(s-a)}\mathrm{d}X_s^{[0,1]} - \mu_0 + \int_{T_1}^{T_2} [e^{-\kappa(s-a)} - e^{-\kappa(T_1-a)}]\mathrm{d}X_s^{[1,2]}\Big), \\ 0 &= \lim_{\epsilon \to 0^+} F'_{1,2}(c + \epsilon) = -\kappa\Big(\int_0^{T_1} e^{-\kappa(c-s)}\mathrm{d}X_s^{[0,1]} + \nu_{T_1} - \int_{c^+}^{T_2} e^{-\kappa(s-c)}\mathrm{d}X_s^{[1,2]}\Big). \end{aligned} \tag{4.3}$$

Recalling that $\mu_0 \cdot a = 0$ so that $\mu_0 e^{-\kappa a} = \mu_0$, and $\nu_{T_1} \cdot (c - T_1) = 0$ which implies $\nu_{T_1} e^{-\kappa(c-T_1)} = \nu_{T_1}$, this system becomes

$$\begin{aligned}
0 &= e^{-\kappa T_1}\mu_0 + \frac{1-e^{-\kappa(T_1-a)}}{2\kappa^2}\hat{\Lambda}_{0,1} - \mu_{T_1} \\
&\quad - [e^{-\kappa(T_2-T_1)} - 1]\nu_{T_2} - \frac{e^{-\kappa(c-T_1)} - e^{-\kappa(T_2-T_1)} - \kappa(T_2-c)}{2\kappa^2}\hat{\Lambda}_{1,2}, \\
0 &= e^{-\kappa T_2}\mu_0 + \frac{e^{-\kappa(T_2-T_1)} - e^{-\kappa(T_2-a)}}{2\kappa^2}\hat{\Lambda}_{0,1} + e^{-\kappa(T_2-T_1)}\mu_{T_1} + e^{-\kappa(T_2-T_1)}\nu_{T_1} + \frac{1-e^{-\kappa(T_2-c)}}{2\kappa^2}\hat{\Lambda}_{1,2} - \nu_{T_2}, \\
0 &= \frac{1-e^{-\kappa(T_1-a)}}{2\kappa^2}\hat{\Lambda}_{0,1} + e^{-\kappa(T_1-a)}\mu_{T_1} - \mu_0 \\
&\quad + [e^{-\kappa(T_2-a)} - e^{-\kappa(T_1-a)}]\nu_{T_2} + \frac{e^{-\kappa(c-a)} - e^{-\kappa(T_2-a)} - \kappa(T_2-c)e^{-\kappa(T_1-a)}}{2\kappa^2}\hat{\Lambda}_{1,2}, \\
0 &= e^{-\kappa c}\mu_0 + \frac{e^{-\kappa(c-T_1)} - e^{-\kappa(c-a)}}{2\kappa^2}\hat{\Lambda}_{0,1} + e^{-\kappa(c-T_1)}\mu_{T_1} + \nu_{T_1} - \frac{1-e^{-\kappa(T_2-c)}}{2\kappa^2}\hat{\Lambda}_{1,2} - e^{-\kappa(T_2-c)}\nu_{T_2}.
\end{aligned} \tag{4.4}$$

Coupling these 4 equations with the volume constraints

$$V_{[0,1]} = \mu_0 + \mu_{T_1} + (T_1 - a)\frac{\hat{\Lambda}_{0,1}}{2\kappa} \quad \text{and} \quad V_{[1,2]} = \nu_{T_1} + \nu_{T_2} + (T_2 - c)\frac{\hat{\Lambda}_{1,2}}{2\kappa} \tag{4.5}$$

gives an overall system of 6 equations involving 8 unknowns. In order to solve this system, we distinguish 4 separate cases each of which involves 6 equations in 6 unknowns:

Case 1: $a = 0$, $c = T_1$, **Case 2:** $a > 0$, $c = T_1$ (hence $\mu_0 = 0$),
Case 3: $a = 0$, $c > T_1$ (hence $\nu_{T_1} = 0$) and **Case 4:** $a > 0$, $c > T_1$ (hence $\mu_0 = \nu_{T_1} = 0$).

Solution of the original TTSC problem proceeds by solving each of the above cases (if a case has no critical point then the global minimizer does not correspond to that case), evaluating the expected cost for each and then selecting the most favourable.

4.1.2. Case 2: The second trade is a sell so $V_{[1,2]} < 0$

Arguing exactly as above, thanks to Proposition 3.6 we have

$$\begin{aligned}
\mathrm{d}X_t^{[0,1]} &= \mu_0\,\delta_0 + \mu_{T_1}\,\delta_{T_1} + I(0 \le t \le b)\frac{\hat{\Lambda}_{0,1}}{2\kappa}\,\mathrm{d}t \text{ for } b \in (0, T_1] \text{ and} \\
\mathrm{d}X_t^{[1,2]} &= \nu_{T_1}\,\delta_{T_1} + \nu_{T_2}\,\delta_{T_2} + I(T_1 \le t \le d)\frac{\hat{\Lambda}_{1,2}}{2\kappa}\,\mathrm{d}t \text{ for } d \in (T_1, T_2]
\end{aligned}$$

where $I(\cdot)$ denotes the indicator function, μ_0, μ_{T_1}, ν_{T_1}, ν_{T_2}, $\hat{\Lambda}_{0,1}$ and $\hat{\Lambda}_{1,2}$ are all nonnegative constants $\mu_{T_1} \cdot (T_1 - b) = 0$ (that is, one of the two numbers vanishes) and $\nu_{T_2} \cdot (T_2 - d) = 0$. Repeating the same analysis as the last section, one obtains a system of equations the solution of which provides a minimizer.

4.1.3. Solution by numerical optimization

The structure of the minimizers described above is more complicated than the STSH solution encountered previously, making solution of the TTSC problem via the four separate cases somewhat cumbersome. We avoid this complication by using numerical optimization and noting that in all four cases the optimal solution falls within the parametric family

$$
\begin{aligned}
\mathrm{d}X_t^{[0,1]} &= \mu_0\,\delta_0 + \mu_{T_1}\,\delta_{T_1} + I(a \le t \le b)\frac{V_{[0,1]} - \mu_0 - \mu_{T_1}}{b-a}\,\mathrm{d}t \text{ and} && (4.6)\\
\mathrm{d}X_t^{[1,2]} &= \nu_{T_1}\,\delta_{T_1} + \nu_{T_2}\,\delta_{T_2} + I(c \le t \le d)\frac{V_{[1,2]} - \nu_{T_1} - \nu_{T_2}}{d-c}\,\mathrm{d}t && (4.7)
\end{aligned}
$$

where $\mu_0 \ge 0$ and $\mu_{T_1} \ge 0$ satisfy $\mu_0 + \mu_{T_1} \le V_{[0,1]}$, ν_{T_1} and ν_{T_2} both have the same sign as $V_{[1,2]}$ and satisfy $|\nu_{T_1} + \nu_{T_2}| \le |V_{[1,2]}|$, and the constants (a, b, c, d) satisfy $0 \le a < b \le T_1$ and $T_1 \le c < d \le T_2$. Thus the problem becomes to minimize $\mathcal{C}(X_t^{[0,1]}, X_t^{[1,2]})$ given by

$$
\begin{aligned}
&\frac{1}{2}\mu_0^2 + \frac{1}{2}\mu_{T_1}^2 + \mu_0\mu_{T_1}e^{-\kappa T_1} + \left\{\mu_0\frac{e^{-\kappa a} - e^{-\kappa b}}{\kappa} + \mu_{T_1}\frac{e^{-\kappa(T_1-b)} - e^{-\kappa(T_1-a)}}{\kappa}\right\}\cdot\frac{V_{[0,1]} - \mu_0 - \mu_{T_1}}{b-a}\\
&+ \frac{\kappa(b-a) - 1 + e^{-\kappa(b-a)}}{\kappa^2}\cdot\left(\frac{V_{[0,1]} - \mu_0 - \mu_{T_1}}{b-a}\right)^2 + \frac{1}{2}\nu_{T_1}^2 + \frac{1}{2}\nu_{T_2}^2 + \nu_{T_1}\nu_{T_2}e^{-\kappa(T_2-T_1)}\\
&+ \left\{\nu_{T_1}\frac{e^{-\kappa(c-T_1)} - e^{-\kappa(d-T_1)}}{\kappa} + \nu_{T_2}\frac{e^{-\kappa(T_2-d)} - e^{-\kappa(T_2-c)}}{\kappa}\right\}\cdot\frac{V_{[1,2]} - \nu_{T_1} - \nu_{T_2}}{d-c}\\
&+ \frac{\kappa(d-c) - 1 + e^{-\kappa(d-c)}}{\kappa^2}\cdot\left(\frac{V_{[1,2]} - \nu_{T_1} - \nu_{T_2}}{d-c}\right)^2\\
&+ \left(\mu_0 + \mu_{T_1}e^{\kappa T_1} + \frac{e^{\kappa b} - e^{\kappa a}}{\kappa}\cdot\frac{V_{[0,1]} - \mu_0 - \mu_{T_1}}{b-a}\right)\cdot\left(\nu_{T_1}e^{-\kappa T_1} + \nu_{T_2}e^{-\kappa T_2} + \frac{e^{-\kappa c} - e^{-\kappa d}}{\kappa}\cdot\frac{V_{[1,2]} - \nu_{T_1} - \nu_{T_2}}{d-c}\right)
\end{aligned}
$$

over the above 8-dimensional space of decision variables. Some additional observations relating to the deterministic TTSC problem with exponentially decaying impact are provided in Appendix B.

4.2. *The TTHC problem with exponentially decaying impact*

As usual we take $V_{[0,1]} > 0$ so the first trade is a buy. Then following the steps of Section 3.2.1, for $G(t) = e^{-\kappa|t|}$ and fixed $X^{[0,1]}$, the optimization problem (3.14) conditionally on $V_{[1,2],a} = v$ becomes

$$
\min_{X^{[1,2]}}\left(\begin{array}{c}\frac{1}{2}\int_0^{T_1}\int_0^{T_1} e^{-\kappa|t-s|}\,\mathrm{d}X_s^{[0,1]}\,\mathrm{d}X_t^{[0,1]} + \frac{1}{2}\int_{T_1}^{T_2}\int_{T_1}^{T_2} e^{-\kappa|t-s|}\,\mathrm{d}X_s^{[1,2]}\,\mathrm{d}X_t^{[1,2]}\\ + \int_0^{T_1} e^{\kappa s}\,\mathrm{d}X_s^{[0,1]}\int_{T_1}^{T_2} e^{-\kappa t}\,\mathrm{d}X_t^{[1,2]} - v\int_0^{T_1} e^{-\kappa(T_1-s)}\,\mathrm{d}X_s^{[0,1]}\end{array}\right). \tag{4.8}
$$

Holding $X^{[0,1]}$ fixed in the above, the integrals over $[0, T_1]$ remain constant, so the above problem is of the form

$$
\min_{X^{[1,2]}}\left(\text{constant} + \frac{1}{2}\int_{T_1}^{T_2}\int_{T_1}^{T_2} e^{-\kappa|t-s|}\,\mathrm{d}X_s^{[1,2]}\,\mathrm{d}X_t^{[1,2]} + A\int_{T_1}^{T_2} e^{-\kappa t}\,\mathrm{d}X_t^{[1,2]}\right) \tag{4.9}
$$

where $A = A[X^{[0,1]}] \equiv \int_0^{T_1} e^{\kappa s}\,\mathrm{d}X_s^{[0,1]} > 0$ and $V_{[1,2]} = V_{[1,2],p} + v$.

Let $X^{[1,2]}$ denote a minimizer of the above. By considering perturbations as in Section 3.1.1 we deduce that the function

$$F_{1,2}(t) \equiv \int_{T_1}^{T_2} e^{-\kappa|t-s|}\,\mathrm{d}X_s^{[1,2]} + Ae^{-\kappa t}$$

is equal to a constant $\Lambda_{1,2}$ on the support of $X_t^{[1,2]}$, and is either greater than or equal to $\Lambda_{1,2}$, or less than or equal to $\Lambda_{1,2}$, depending on whether $V_{[1,2]} = V_{[1,2],p} + v$ is positive or negative, respectively. Now, taking the second derivative of $F_{1,2}$ on the support of $X^{[1,2]}$ and arguing as in Section 3.1.2, we obtain

$$0 = \kappa^2 F_{1,2}(t) - 2\kappa \mathrm{d}X_t^{[1,2]} \text{ on } \operatorname{supp}(\mathrm{d}X_t^{[1,2]}) \cap (T_1, T_2),$$

which on rearranging yields

$$\mathrm{d}X_t^{[1,2]} = \frac{\kappa}{2}\Lambda_{1,2} \text{ on } \operatorname{supp}(\mathrm{d}X_t^{[1,2]}) \cap (T_1, T_2).$$

Also, the support of $X_t^{[1,2]}$ is an interval.

Now, for each value of $V_{[1,2]} = V_{[1,2],p} + v$, we could apply Propositions 3.5 or 3.6 (depending on the sign of $V_{[1,2]}$) to deduce the structure of each minimizer. Whichever of these applies, the minimizer will be of the form

$$\mathrm{d}X_t^{[1,2]} = \nu_{T_1}\,\delta_{T_1} + \nu_{T_2}\,\delta_{T_2} + I(c \le t \le d)\frac{V_{[1,2]} - \nu_{T_1} - \nu_{T_2}}{d-c}\,\mathrm{d}t \tag{4.10}$$

where the interval (c, d) denotes $\operatorname{supp}(X^{[1,2]}) \cap (T_1, T_2)$ and $T_1 \le c < d \le T_2$. Discarding the constant term, the problem (4.9) then becomes to minimize

$$\begin{aligned}
&\frac{1}{2}\nu_{T_1}^2 + \frac{1}{2}\nu_{T_2}^2 + \nu_{T_1}\nu_{T_2}e^{-\kappa(T_2-T_1)} \\
&+ \left\{\nu_{T_1}\frac{e^{-\kappa(c-T_1)} - e^{-\kappa(d-T_1)}}{\kappa} + \nu_{T_2}\frac{e^{-\kappa(T_2-d)} - e^{-\kappa(T_2-c)}}{\kappa}\right\} \cdot \frac{V_{[1,2]} - \nu_{T_1} - \nu_{T_2}}{d-c} \\
&+ \frac{\kappa(d-c) - 1 + e^{-\kappa(d-c)}}{\kappa^2} \cdot \left(\frac{V_{[1,2]} - \nu_{T_1} - \nu_{T_2}}{d-c}\right)^2 \\
&+ A\left(\nu_{T_1}e^{-\kappa T_1} + \nu_{T_2}e^{-\kappa T_2} + \frac{e^{-\kappa c} - e^{-\kappa d}}{\kappa} \cdot \frac{V_{[1,2]} - \nu_{T_1} - \nu_{T_2}}{d-c}\right)
\end{aligned}$$

over the 4-dimensional space $(\nu_{T_1}, \nu_{T_2}, c, d)$ such that both ν_{T_1} and ν_{T_2} have the same sign as $V_{[1,2]}$ and $|\nu_{T_1} + \nu_{T_2}| \le |V_{[1,2]}|$. We denote the minimal value of the above expression by $Q(A, V_{[1,2]})$.

From expression (4.8), the cost of $(X^{[0,1]}, X^{[1,2]})$ where $X^{[1,2]}$ is a minimizer as above is given by

$$\frac{1}{2}\int_0^{T_1}\int_0^{T_1} e^{-\kappa|t-s|}\,\mathrm{d}X_s^{[0,1]}\,\mathrm{d}X_t^{[0,1]} - v\int_0^{T_1} e^{-\kappa(T_1-s)}\,\mathrm{d}X_s^{[0,1]} + Q\left(\int_0^{T_1} e^{\kappa s}\,\mathrm{d}X_s^{[0,1]}, V_{[1,2]}\right).$$

Taking the expectation of this by integrating with respect to $\mathrm{d}V_{[1,2],a}(v)$, the middle term vanishes because $\mathbb{E}V_{[1,2],a} = 0$ and we obtain

$$\frac{1}{2}\int_0^{T_1}\int_0^{T_1} e^{-\kappa|t-s|}\,\mathrm{d}X_s^{[0,1]}\,\mathrm{d}X_t^{[0,1]} + \mathbb{E}\Big[Q\Big(\int_0^{T_1} e^{\kappa s}\,\mathrm{d}X_s^{[0,1]}, V_{[1,2]}\Big)\Big]. \tag{4.11}$$

So, we are now left with minimizing this expression over admissible $X^{[0,1]}$.

Again, a minimizer is easily seen to exist. Hence, by considering first perturbations as in Section 3.1.1, and writing $\partial_1 Q$ for the partial derivative of Q with respect to its first argument, we deduce that the function

$$F_{0,1}(t) \equiv \int_0^{T_1} e^{-\kappa|t-s|}\,\mathrm{d}X_s^{[0,1]} + \mathbb{E}\Big[\partial_1 Q\Big(\int_0^{T_1} e^{\kappa s}\,\mathrm{d}X_s^{[0,1]}, V_{[1,2]}\Big)\Big]e^{\kappa t}$$

is equal to a constant $\Lambda_{0,1}$ on the support of $X_t^{[0,1]}$, and it is greater than or equal to $\Lambda_{0,1}$ everywhere else. Taking the second derivatives of $F_{0,1}$ on the support of $X^{[0,1]}$, and arguing as in Section 3.1.2, we obtain

$$0 = \kappa^2 F_{0,1}(t) - 2\kappa\mathrm{d}X_t^{[0,1]} \text{ on } \mathrm{supp}(\mathrm{d}X_t^{[0,1]}) \cap (0, T_1),$$

which rearranges to give

$$\mathrm{d}X_t^{[0,1]} = \frac{\kappa}{2}\Lambda_{0,1} \text{ on } \mathrm{supp}(\mathrm{d}X_t^{[0,1]}) \cap (0, T_1).$$

The support of $X_t^{[0,1]}$ is again an interval, so similar to equation (4.10), we have

$$\mathrm{d}X_t^{[0,1]} = \mu_0\,\delta_0 + \mu_{T_1}\,\delta_{T_1} + I(a \le t \le b)\frac{V_{[0,1]} - \mu_0 - \mu_{T_1}}{b-a}\,\mathrm{d}t$$

where (a, b) denotes $\mathrm{supp}(X^{[0,1]}) \cap (0, T_1)$ and $0 \le a < b \le T_1$. Substituting this into expression (4.11), the problem becomes to minimize

$$\begin{aligned}&\frac{1}{2}\mu_0^2 + \frac{1}{2}\mu_{T_1}^2 + \mu_0\mu_{T_1}e^{-\kappa T_1} + \Big\{\mu_0\frac{e^{-\kappa a} - e^{-\kappa b}}{\kappa} + \mu_{T_1}\frac{e^{-\kappa(T_1-b)} - e^{-\kappa(T_1-a)}}{\kappa}\Big\}\cdot\frac{V_{[0,1]} - \mu_0 - \mu_{T_1}}{b-a} \\ &+ \frac{\kappa(b-a) - 1 + e^{-\kappa(b-a)}}{\kappa^2}\cdot\Big(\frac{V_{[0,1]} - \mu_0 - \mu_{T_1}}{b-a}\Big)^2 + \mathbb{E}\Big[Q\Big(\mu_0 + \mu_{T_1}e^{\kappa T_1} + \frac{e^{\kappa b} - e^{\kappa a}}{\kappa}\cdot\frac{V_{[0,1]} - \mu_0 - \mu_{T_1}}{b-a}, V_{[1,2]}\Big)\Big]\end{aligned} \tag{4.12}$$

over the 4-dimensional space (μ_0, μ_{T_1}, a, b) where $\mu_0 \ge 0$, $\mu_{T_1} \ge 0$ and $\mu_0 + \mu_{T_1} \le V_{[0,1]}$.

Remark 4.1. *The roles of μ_0 and μ_{T_1} are not symmetric in the above, e.g. in the last term μ_{T_1} is multiplied by $e^{\kappa T_1} > 1$. This is consistent with intuition, as we would expect the strength of influence between the second trade and μ_{T_1} to be stronger than the strength of influence between the second trade and μ_0.*

4.2.1. Illustrative example: TTHC when the second trade has zero mean

Consider the special case where $V_{[1,2],p} = 0$, for example when $V_{[1,2],a}$ takes the values $\pm v$ with probability 1/2 for some $v > 0$, so that the second trade has zero mean and is equally likely to be a buy or a sell. Thus $\mathbb{P}(V_{[1,2],a} = +v) = \mathbb{P}(V_{[1,2],a} = -v) = 1/2$. In this case problem (4.12) becomes to minimize

$$\frac{1}{2}\mu_0^2 + \frac{1}{2}\mu_{T_1}^2 + \mu_0\mu_{T_1}e^{-\kappa T_1} + \left\{\mu_0\frac{e^{-\kappa a} - e^{-\kappa b}}{\kappa} + \mu_{T_1}\frac{e^{-\kappa(T_1-b)} - e^{-\kappa(T_1-a)}}{\kappa}\right\} \cdot \frac{V_{[0,1]} - \mu_0 - \mu_{T_1}}{b-a} \tag{4.13}$$
$$+ \frac{\kappa(b-a) - 1 + e^{-\kappa(b-a)}}{\kappa^2} \cdot \left(\frac{V_{[0,1]} - \mu_0 - \mu_{T_1}}{b-a}\right)^2 + \frac{1}{2}Q\left(\mu_0 + \mu_{T_1}e^{\kappa T_1} + \frac{e^{\kappa b} - e^{\kappa a}}{\kappa} \cdot \frac{V_{[0,1]} - \mu_0 - \mu_{T_1}}{b-a}, +v\right)$$
$$+ \frac{1}{2}Q\left(\mu_0 + \mu_{T_1}e^{\kappa T_1} + \frac{e^{\kappa b} - e^{\kappa a}}{\kappa} \cdot \frac{V_{[0,1]} - \mu_0 - \mu_{T_1}}{b-a}, -v\right)$$

over (μ_0, μ_{T_1}, a, b) where both $\mu_0 \geq 0$, $\mu_{T_1} \geq 0$ and $\mu_0 + \mu_{T_1} \leq V_{[0,1]}$. It is important to note that both the $V_{[1,2],a} = +v$ and $V_{[1,2],a} = -v$ branches of the expectation appear in the objective function. We return to this example later in Section 5.3.

5. Numerical exploration of the exponential case

In this section we examine numerically derived solutions for the TTSC and TTHC problems described in Sections 3.1 and 3.2 for $G(t) = e^{-\kappa|t|}$. We focus on qualitative and quantitative differences between the optimal TTSC and TTHC execution schedules, also evaluating performance of the optimal TTSC solution under the TTHC objective function, and *vice versa*. Alongside these solutions we provide a suboptimal benchmark that follows the STSH solution (see Remark 2.4) in both trading periods. We refer to this suboptimal benchmark, which deals with the first and second trades independently by solving each STSH problem in isolation, as the *Myopic* solution. All the examples below have $\kappa = 1$, $T_0 = 0$, $T_1 = 1$, $T_2 = 2$, and $S_0 = 0$, with the remaining parameters taking the values shown in Figures 2–6 and repeated in Tables 1 and 2. All calculations were performed using the SLSQP method of the `optimize` numerical minimization routine in the SciPy library.

5.1. *The deterministic case for two buy trades*

We consider three examples where the first and second trades are buys of equal, decreasing and increasing size by taking $(V_{[0,1]}, V_{[1,2]}) = (1, 1)$, $(3, 1)$, and $(1, 3)$ respectively. We accommodate these cases, where $V_{[1,2]}$ is assumed known with certainty, in the TTHC problem by taking $V_{[1,2],a} = 0$ with probability 1 so that $V_{[1,2]}$ is benchmarked wholly against price $S_0 = 0$. Situations where adjacent trades are of different sizes arise frequently, for example as a result of emerging alpha opportunities, changes in risk appetite or constraints on maximum holdings becoming binding.

The optimal Myopic, TTSC and TTHC execution schedules are shown in Figure 2 for the three examples above, and the optimal parameter and objective function values are given in Table 1. The Myopic solutions always have full support and include deltas at both ends of the $[0, T_1]$ and $[T_1, T_2]$ intervals, consistent with applying the STSH solution to both $V_{[0,1]}$ and $V_{[1,2]}$. Compared to the Myopic solutions, the first major departure we observe is that in some of the TTSC and TTHC panels the periods of continuous trading do not extend to the boundaries of the $[0, T_1]$ and $[T_1, T_2]$ intervals, corresponding to optimal schedules that contain trading gaps. The second departure we observe is that the optimal TTSC and TTHC schedules do not necessarily have deltas at both ends of the $[0, T_1]$ and $[T_1, T_2]$ intervals. For example, the optimal TTHC schedule for $(V_{[0,1]}, V_{[1,2]}) = (1, 1)$ has no delta at T_1, and coincides with the STSH solution for a single trade of size $(V_{[0,1]} + V_{[1,2]})$ executed over period $[0, T_2]$.

Also shown are values of the TTSC and TTHC objective functions for the depicted execution schedules. Clearly the optimal TTSC schedule evaluated under the TTHC objective function always yields a worse value than the optimal TTHC schedule under the TTHC objective, but the level of underperformance can be considerable, e.g. above 25% for $(V_{[0,1]}, V_{[1,2]}) = (1, 3)$. The optimal TTSC schedule sometimes even underperforms the Myopic solution under the TTHC objective, e.g. in the $(V_{[0,1]}, V_{[1,2]}) = (1, 1)$ and $(1, 3)$ cases. Conversely, the optimal TTHC schedule can considerably underperform both the TTSC and Myopic solutions under the TTSC objective, e.g. by more than 30% and 15%, respectively, for $(V_{[0,1]}, V_{[1,2]}) = (1, 3)$. Differences in the value of the TTSC and TTHC objective functions evaluated on the same solution can be even higher, e.g. the TTHC cost is more than three times the TTSC cost for the optimal TTSC solution for $(V_{[0,1]}, V_{[1,2]}) = (1, 1)$.

All of the TTSC schedules depicted in Figure 2 backload execution in the first interval. This is because backloading the first trade drives-up the benchmark price of the second trade, and this quantity appears in the TTSC objective function with a negative sign. So backloading the first trade benefits the TTSC problem. In contrast, all the TTHC solutions in Figure 2 are frontloaded in the first period. This frontloading reduces the upward impact on both the benchmark price for the second period and the subsequent prices, which in turn benefits the buy trade undertertaken in the second interval. Thus frontloading the first trade benefits the TTHC problem. These effects are particularly clear in the $(V_{[0,1]}, V_{[1,2]}) = (1, 3)$ case, where trading in the first interval is almost maximally back- and frontloaded in the optimal TTSC and TTHC schedules, respectively. The second buy trade is backloaded in all the TTSC and TTHC schedules shown in Figure 2. This backloading allows beneficial capture of the downwards price drift that arises from the decaying impact of the first trade. Although backloading arises in the second period for both the TTSC and TTHC cases, the effect is typically milder in the TTHC schedules. This is because in the TTHC schedule there is less beneficial reversion to capture because the impact experienced by the benchmark price for the second period is smaller. The effect is particularly clear in the $(V_{[0,1]}, V_{[1,2]}) = (3, 1)$ case, where the TTSC and TTHC solutions exhibit pronounced backloading in the second period. This case also demonstrates that the TTSC solution can outperform the Myopic solution under the TTHC objective, whilst simultaneously the TTHC solution can outperform the Myopic solution under the TTSC objective. This situation arises because the opportunity provided in the second period by the impact of the first trade can prove advantageous overall even when execution in the first period is adversely loaded.

In Figure 3 solutions are provided for these same problems, but now under the additional *Full Domain (FD)* constraint $\{(a, b) = (0, 1), (c, d) = (1, 2)\}$. This ensures that any periods of nonzero rate continuous trading within $[0, T_1]$ or $[T_1, T_2]$ extend to the boundaries of these intervals. The corresponding optimal parameter and objective function values are given in Table 1. All the Myopic solutions, and the subset of TTSC and TTHC schedules that already satisfy the FD constraint, do not change between Figures 2 and 3. However, several of the TTSC and TTHC schedules shown in Figure 2 contain both trading gaps and nonzero rate continuous execution in the same interval, rendering them inadmissible. Applying the FD constraint to these either results in a period of nonzero rate continuous execution over the corresponding interval, e.g. the second period of the TTSC FD schedule for $(V_{[0,1]}, V_{[1,2]}) = (1, 1)$, or results in the continuous rate execution for that period vanishing altogether, e.g. the second period of the TTHC FD schedule for $(V_{[0,1]}, V_{[1,2]}) = (3, 1)$. Enforcing the FD constraint on an inadmissible solution always results in a less favourable value of the relevant objective function, although in practice, this change may be small, e.g. the TTHC objective function changes from 4.212 to 4.213 between the optimal TTHC and TTHC FD schedules when $(V_{[0,1]}, V_{[1,2]}) = (3, 1)$. Finally, recalling Remark 3.7, we note that both the TTHC and TTHC FD execution schedules in Figures 2 and 3 exhibit the expected time reversibility.

5.2. *Randomness and mixed trading*

In this section we retain $V_{[0,1]} = 1$ for the first trade, corresponding to a buy, and examine how optimal solutions behave when the second trade is not fully known at time $t = 0$ but contains some element

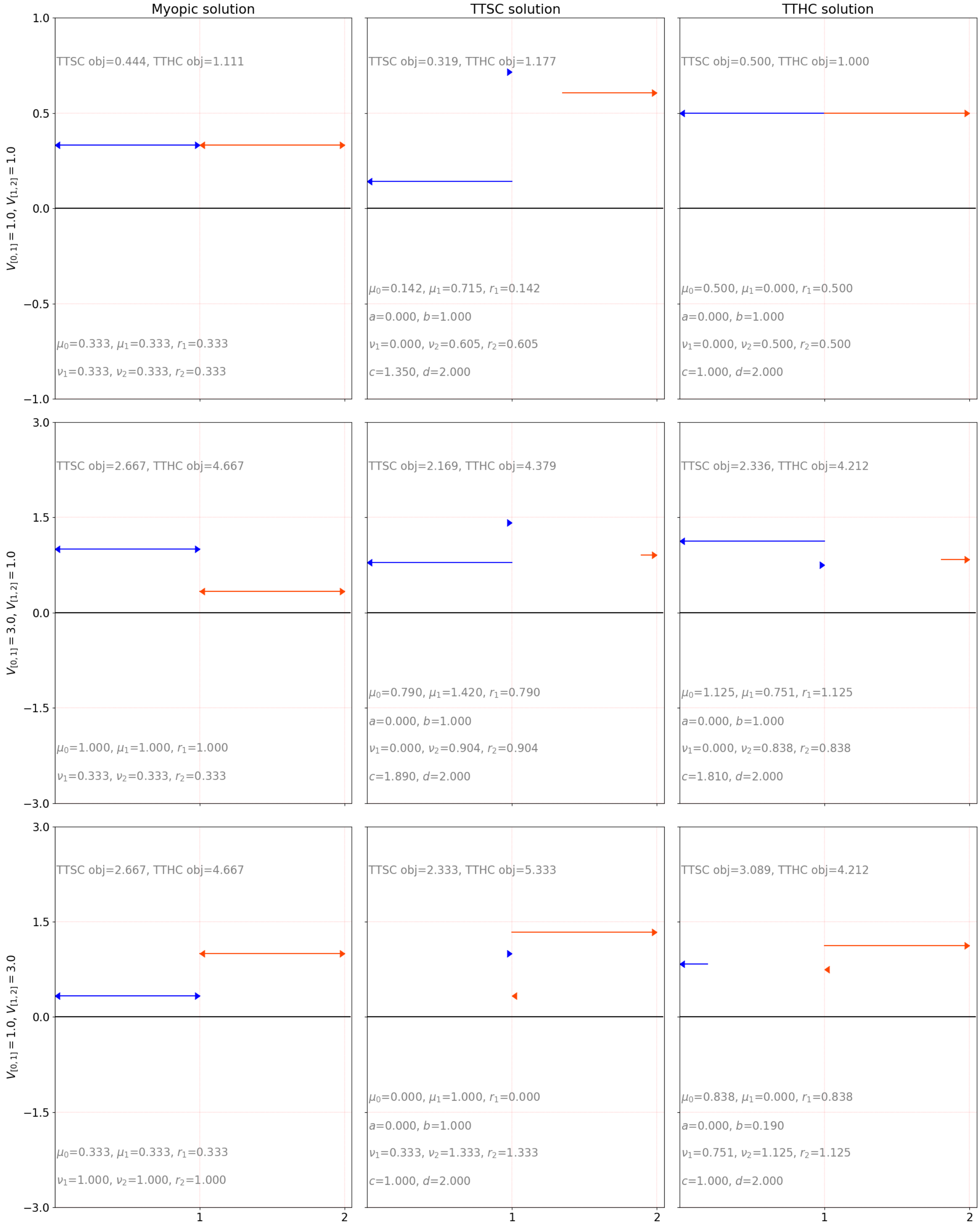

Figure 2. Optimal execution schedules for two buy orders that are known deterministically at time $t=0$. The top row depicts the case $(V_{[0,1]}, V_{[1,2]}) = (1,1)$, whereas the middle and bottom rows show the cases $(V_{[0,1]}, V_{[1,2]}) = (3,1)$ and $(1,3)$ respectively. The benchmark Myopic solution (discussed early in Section 5) is shown in the left column. The middle column depicts optimal execution schedules under the TTSC objective discussed in Section 3.1. The right column shows solutions under the TTHC objective when $V_{[1,2],a} = 0$ with probability 1, see Remark 3.7. The optimal parameter and objective function values are also shown in Table 1.

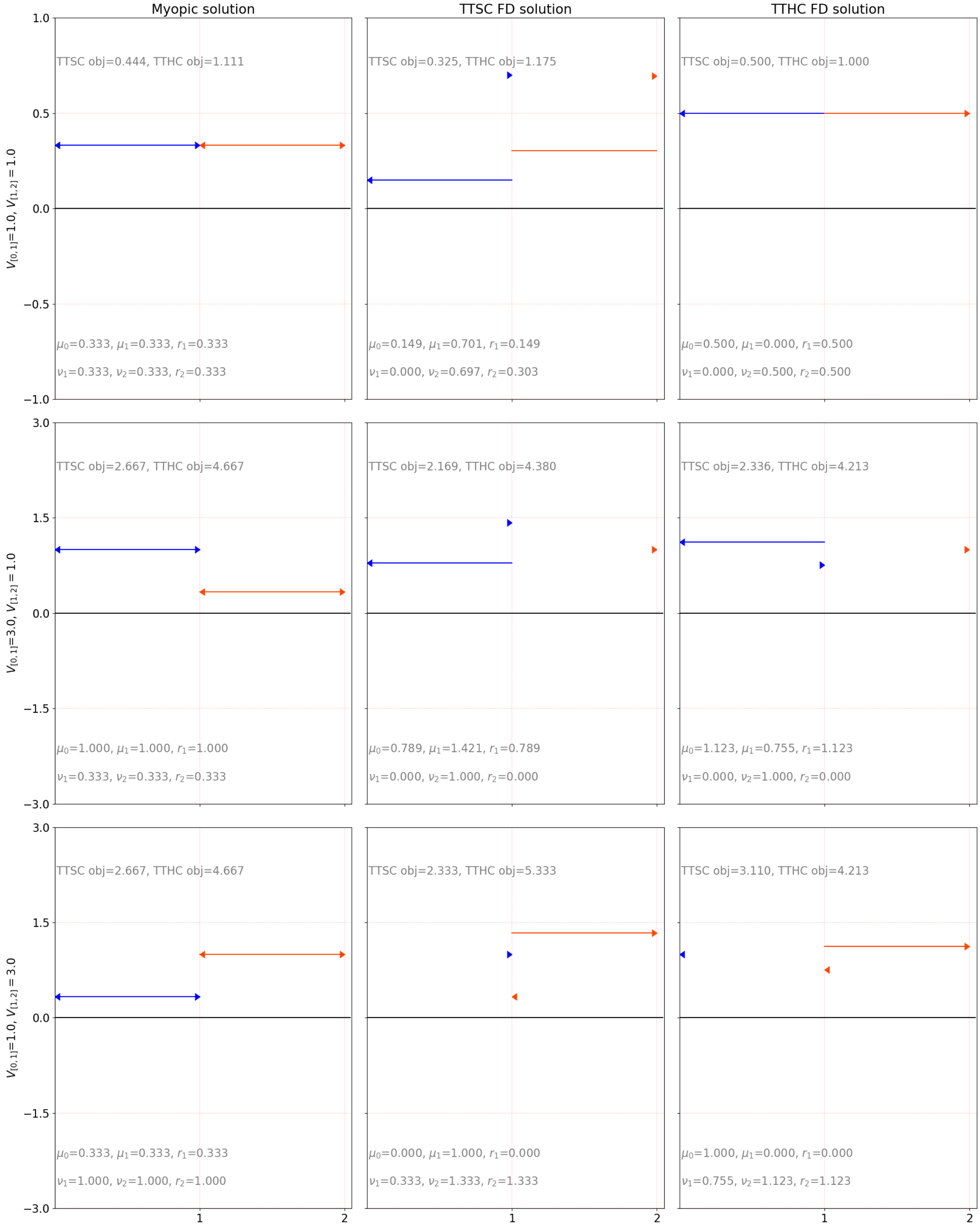

Figure 3. Optimal execution schedules when the Full Domain (FD) constraint introduced at the end of Section 5.1 is applied to the same trade examples as Figure 2, so that the periods of continuous trading throughout the first and second time intervals either vanish or occur at constant nonzero rate. Close comparison with Figure 2 shows that imposing the FD constraint has no impact for the bottom-middle and top-right panels, together with all the Myopic solutions in the left-hand column. The optimal parameter and objective function values are also shown in Table 1.

of uncertainty. Our coverage also examines the case $V_{[1,2]} < 0$ where the second trade is a sell. We reflect the uncertainty in $V_{[1,2]} \equiv V_{[1,2],p} + V_{[1,2],a}$ by considering several fixed values for $V_{[1,2],p} = \mathbb{E}V_{[1,2]}$ whilst assuming the random variable $V_{[1,2],a}$ takes values $\pm v$ each with probability 0.5. Note that the deterministic solutions discussed in Section 5.1 coincide with the $v = 0$ special case of this setup. We also provide solutions where the constraint $C^* \equiv \{\mathrm{d}X_t^{[0,1]} \geq 0, V_{[1,2]}\mathrm{d}X_t^{[1,2]} \geq 0\}$ is relaxed, thus admitting optimal trading schedules of alternating sign (i.e. both buys and sells) within each period.

Solutions for the $(V_{[0,1]}, V_{[1,2],p}, V_{[1,2],a}) = (1, 1, \pm 0.5)$ cases are provided in Figure 4, corresponding to two buys of unequal but similar magnitude. The corresponding optimal parameter and objective function values are given in Table 2. Also investigated, but not reported herein, were the $(V_{[0,1]}, V_{[1,2],p}, V_{[1,2],a}) = (1, 1, \pm 2)$ cases, corresponding to the second trade being a buy three times the size of the first, or a sell of the same magnitude. The resulting optimal TTSC and TTHC costs were higher for $V_{[1,2],a} = \pm 2$ than for $V_{[1,2],a} = \pm 0.5$ even though $\mathbb{E}V_{[1,2]}$ remains the same. From Figures 4 and 5 and Table 1 we observe that optimal values of the TTHC cost can be 2-3 times larger than the optimal TTSC cost. The optimal TTSC and TTHC costs were of closer relative size when $V_{[1,2],a} = \pm 2$, with the TTHC cost remaining the larger of the two. This behaviour is as expected, since increasing $V_{[1,2],a}$ reduces the correlation between the first and second trades[18] and also reduces the proportion of the second trade that is predictable, resulting in a greater proportion of $V_{[1,2]}$ being benchmarked against $S^*_{T_1}$. Additional numerical investigations demonstrate that optimal execution schedules and their corresponding objective values remain close to those of the deterministic case discussed in Section 5.1 when $V_{[1,2],a}$ is small compared to $V_{[0,1]}$ and $V_{[1,2]}$.

Most of the observations made in Section 5.1 about deterministic solutions also hold here, e.g. the backloading of optimal TTSC schedules relative to TTHC solutions. We therefore focus discussion on new phenomena, and start with the optimal C^*-unrestricted TTHC schedules depicted in the right-hand column of Figure 4. Within the first interval, that is over $[0, T_1]$, these agree with the deterministic $(V_{[0,1]}, V_{[1,2],p}, V_{[1,2],a}) = (1, 1, 0)$ solution depicted in the top-right panel of Figure 2, and also with the (not shown) optimal C^*-unrestricted TTHC solutions for $(V_{[0,1]}, V_{[1,2],p}, V_{[1,2],a}) = (1, 1, \pm 2)$ again over period $[0, T_1]$. This agreement with the deterministic solution over $[0, T_1]$ may not hold when the C^*-constraint is enforced, e.g. the third column of Figure 4 has $b = 0.97$ leading to a trading gap in the first period that corresponds to slight frontloading compared to the optimal deterministic and C^*-unrestricted TTHC schedules.[19] This trading gap arises because both the $V_{[1,2],a} = +v$ and $V_{[1,2],a} = -v$ branches of the expectation appear in the TTHC objective given by expression (4.12), and the C^*-constraint makes inadmissible the alternating trading (both selling and buying) shown in the second period of the bottom-right panel in Figure 4. Holding $V_{[1,2],p} = 1$ fixed and $V_{[1,2],a} = \pm v$ each with probability 0.5, then compared to the deterministic TTHC solution, on $[0, T_1]$ the C^*-constrained TTHC schedule displays zero backloading when $v = 0$, backloading that increases and ultimately peaks as v increases, and then diminishing backloading as v further increases until eventually the deterministic and C^*-constrained TTHC schedules again coincide on $[0, T_1]$.[20]

Focusing on the bottom right-hand panel of Figure 4, the C^*-unrestricted solution places a block sell at the start of the second period in order to offset the price impact created by buying during the first period. Although this is mathematically optimal, such an execution schedule may be inconsistent with acceptable market practice and governing legislation, as it could be interpreted as issuing a sell with the specific intention of impacting market prices favourably for a subsequent buy.[21] Comparing the bottom panels in the third and fourth columns of Figure 4, although the optimal trading schedules

[18]For example, suppose $V_{[0,1]} = \pm 1$ with probability 0.5, $V_{[1,2],p} = \mathrm{sgn}(V_{[0,1]})$ and $V_{[1,2],a} = \pm v$ also with probability 0.5. Then the correlation between the first and second trades is $(1 + v^2)^{-1/2}$.

[19]Alternative choices of $(V_{[0,1]}, V_{[1,2],p}, V_{[1,2],a})$ can make this trading gap much larger, e.g. $(0.8, 1.0, 0.5)$ yields $b = 0.79$.

[20]Our results for $v = 2$ produced deterministic and C^*-constrained TTHC schedules that coincided on $[0, T_1]$.

[21]See `https://www.handbook.fca.org.uk/handbook/MAR/1/6.html` for regulations on manipulating transactions and wash trades.

Figure 2, top	**First trade (buy of size 1.0)**				**Second trade (buy of size 1.0)**				**TTSC**	**TTHC**
	μ_0	μ_1	$[a,b]$	r_1	ν_1	ν_2	$[c,d]$	r_2	**Obj.**	**Obj.**
Myopic	0.333	0.333	$[0.000, 1.000]$	0.333	0.333	0.333	$[1.000, 2.000]$	0.333	0.444	1.111
Optimal TTSC	0.142	0.715	$[0.000, 1.000]$	0.142	0.000	0.605	$[1.350, 2.000]$	0.605	0.319	1.177
Optimal TTHC	0.500	0.000	$[0.000, 1.000]$	0.500	0.000	0.500	$[1.000, 2.000]$	0.500	0.500	1.000
Figure 2, middle	**First trade (buy of size 3.0)**				**Second trade (buy of size 1.0)**				**TTSC**	**TTHC**
	μ_0	μ_1	$[a,b]$	r_1	ν_1	ν_2	$[c,d]$	r_2	**Obj.**	**Obj.**
Myopic	1.000	1.000	$[0.000, 1.000]$	1.000	0.333	0.333	$[1.000, 2.000]$	0.333	2.667	4.667
Optimal TTSC	0.790	1.420	$[0.000, 1.000]$	0.790	0.000	0.904	$[1.890, 2.000]$	0.904	2.169	4.379
Optimal TTHC	1.125	0.751	$[0.000, 1.000]$	1.125	0.000	0.838	$[1.810, 2.000]$	0.838	2.336	4.212
Figure 2, bottom	**First trade (buy of size 1.0)**				**Second trade (buy of size 3.0)**				**TTSC**	**TTHC**
	μ_0	μ_1	$[a,b]$	r_1	ν_1	ν_2	$[c,d]$	r_2	**Obj.**	**Obj.**
Myopic	0.333	0.333	$[0.000, 1.000]$	0.333	1.000	1.000	$[1.000, 2.000]$	1.000	2.667	4.667
Optimal TTSC	0.000	1.000	NA	0.000	0.333	1.333	$[1.000, 2.000]$	1.333	2.333	5.333
Optimal TTHC	0.838	0.000	$[0.000, 0.190]$	0.838	0.751	1.125	$[1.000, 2.000]$	1.125	3.089	4.212
Figure 3, top	**First trade (buy of size 1.0)**				**Second trade (buy of size 1.0)**				**TTSC**	**TTHC**
	μ_0	μ_1	$[a,b]$	r_1	ν_1	ν_2	$[c,d]$	r_2	**Obj.**	**Obj.**
Myopic	0.333	0.333	$[0.000, 1.000]$	0.333	0.333	0.333	$[1.000, 2.000]$	0.333	0.444	1.111
Optimal FD TTSC	0.149	0.701	$[0.000, 1.000]$	0.149	0.000	0.697	$[1.000, 2.000]$	0.303	0.325	1.175
Optimal FD TTHC	0.500	0.000	$[0.000, 1.000]$	0.500	0.000	0.500	$[1.000, 2.000]$	0.500	0.500	1.000
Figure 3, middle	**First trade (buy of size 3.0)**				**Second trade (buy of size 1.0)**				**TTSC**	**TTHC**
	μ_0	μ_1	$[a,b]$	r_1	ν_1	ν_2	$[c,d]$	r_2	**Obj.**	**Obj.**
Myopic	1.000	1.000	$[0.000, 1.000]$	1.000	0.333	0.333	$[1.000, 2.000]$	0.333	2.667	4.667
Optimal FD TTSC	0.789	1.421	$[0.000, 1.000]$	0.789	0.000	1.000	NA	0.000	2.169	4.380
Optimal FD TTHC	1.123	0.755	$[0.000, 1.000]$	1.123	0.000	1.000	NA	0.000	2.336	4.213
Figure 3, bottom	**First trade (buy of size 1.0)**				**Second trade (buy of size 3.0)**				**TTSC**	**TTHC**
	μ_0	μ_1	$[a,b]$	r_1	ν_1	ν_2	$[c,d]$	r_2	**Obj.**	**Obj.**
Myopic	0.333	0.333	$[0.000, 1.000]$	0.333	1.000	1.000	$[1.000, 2.000]$	1.000	2.667	4.667
Optimal FD TTSC	0.000	1.000	NA	0.000	0.333	1.333	$[1.000, 2.000]$	1.333	2.333	5.333
Optimal FD TTHC	1.000	0.000	NA	0.000	0.755	1.123	$[1.000, 2.000]$	1.123	3.110	4.213

Table 1. Optimal parameter and objective function values for the solutions shown in Figures 2 and 3.

are significantly different, the corresponding TTHC objective values are very similar, suggesting that any advantage from relaxing the C^*-constraint is minimal.

Optimal execution schedules for these same examples under the previously used Full Domain (FD) constraint $\{(a,b) = (0,1), (c,d) = (1,2)\}$ are depicted in Figure 5, with corresponding optimal parameter and objective function values shown in Table 2. Compared with the results in Figure 4, only solutions depicted in the second and third columns are impacted by the FD constraint. Trading gaps like those arising in the third column of Figure 4, where $b = 0.97$, are not available under the FD constraint, but the TTHC FD solution nonetheless achieves frontloading in the first period by making the block trade at time $t = 0$ larger than the constant continuous trading rate over $[0, 1]$. The changes in optimal objective function values between corresponding panels of Figures 4 and 5 are small. This is potentially important in practice, as it suggests restricting attention to FD solutions incurs little underperformance even though the computational burden of the resulting optimization is significantly reduced.

5.3. *Interaction between the two trading intervals*

We now illustrate some key aspects of how trading within the first and second periods can interact even when the direction of the second trade is unpredictable. Recalling the setup of Example 4.2.1, we consider the case $(V_{[0,1]}, V_{[1,2],p}, V_{[1,2],a}) = (1, 0, \pm 1)$ so that the second trade is always of unit size but is equiprobably a buy or sell, with this information becoming known only at time $T_1 = 1$. Even under this

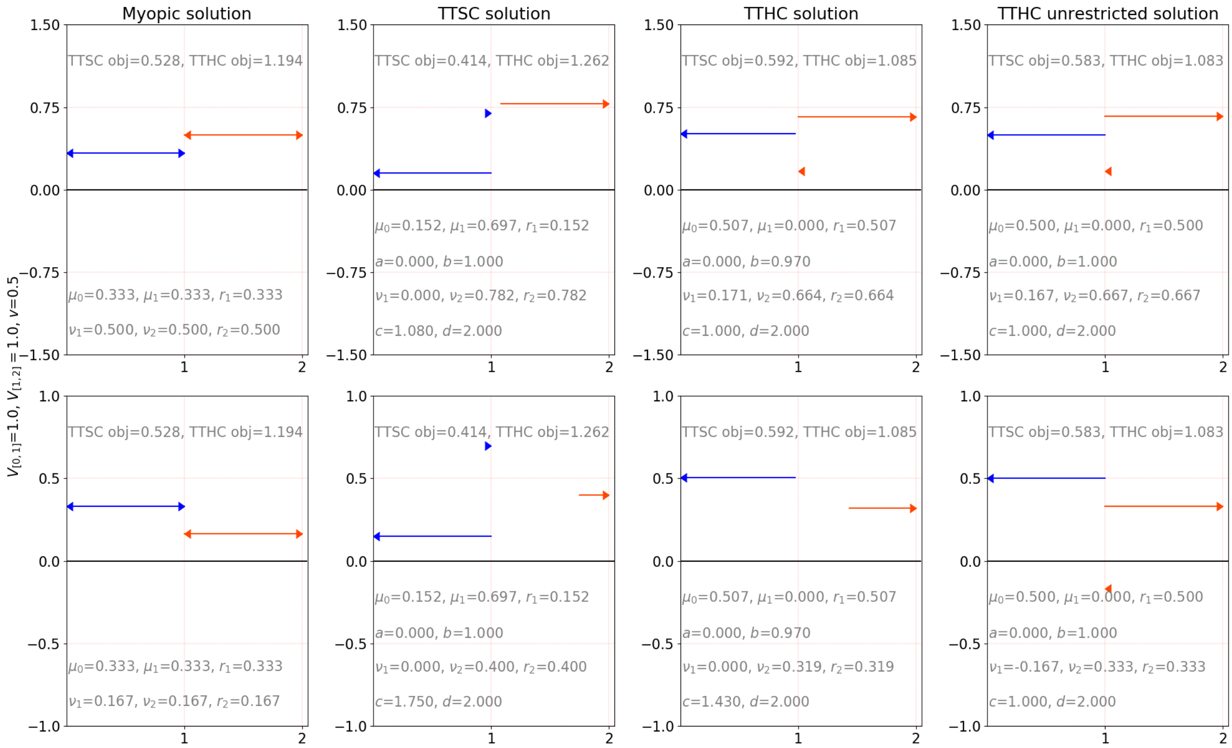

Figure 4. Optimal execution schedules when the first order is a buy with $V_{[0,1]} = 1$ and the second order $V_{[1,2]}$ has predicted part $V_{[1,2],p}$ that is known at time $t = 0$ and stochastic adjust component $V_{[1,2],a}$ that becomes known only at time T_1. The two examples considered have $V_{[0,1]} = V_{[1,2],p} = 1$, with $V_{[1,2],a} = 0.5$ (top row) and $V_{[1,2],a} = -0.5$ (bottom row). The benchmark Myopic solution (discussed early in Section 5) is shown in the left column. The second column depicts optimal execution schedules under the Stochastic TTSC objective discussed in Section 3.2.2, while the third column shows solutions for the Stochastic TTHC objective described in Section 4.2. The right column provides optimal schedules for the Stochastic TTHC objective when the C^*-constraint introduced in Section 5.2 is relaxed to accommodate alternating solutions within each period. Note that in the third column, parameter b has optimal value 0.97 so that the period of constant rate trading finishes before time $T_1 = 1$. The optimal parameter and objective function values are also shown in Table 2.

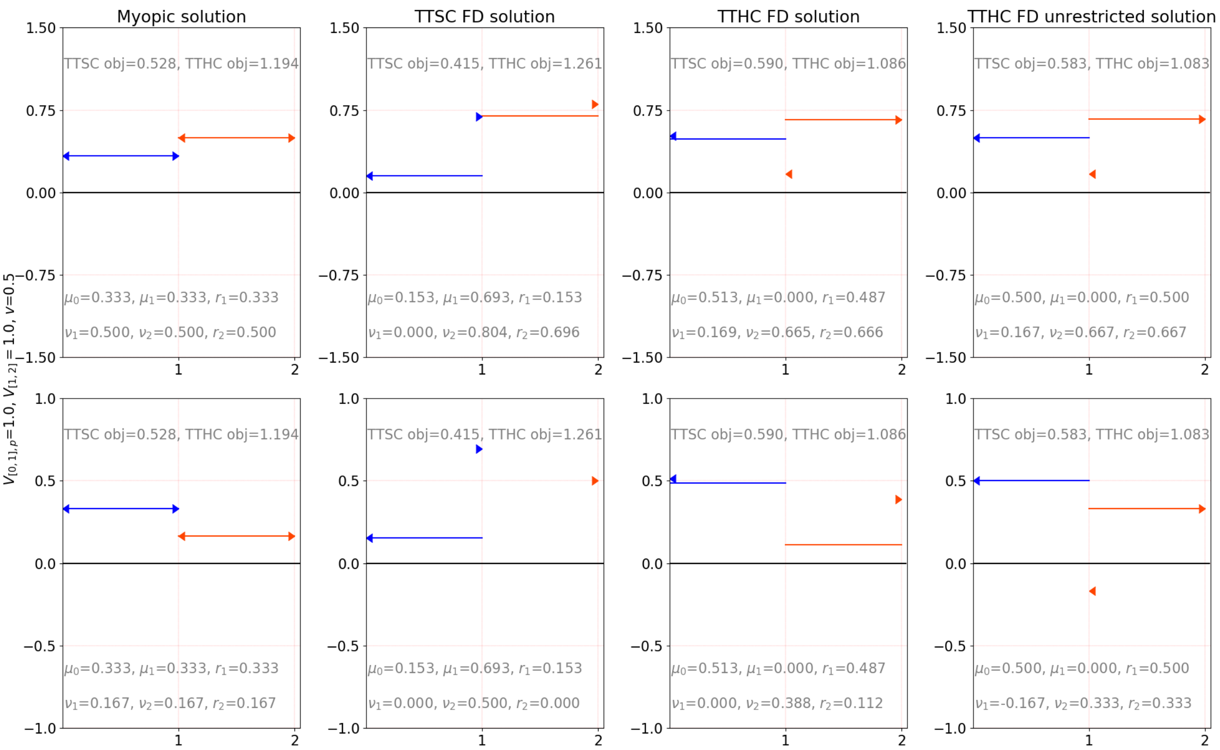

Figure 5. Optimal execution schedules when the Full Domain (FD) constraint introduced at the end of Section 5.1 is applied to the same trade examples as Figure 4, so that the periods of continuous trading throughout the first and second time intervals either vanish or occur at constant nonzero rate. Close comparison with Figure 4 shows that imposing the FD constraint only impacts the solutions in the second and third columns. The optimal parameter and objective function values are also shown in Table 2.

Figure 4, top	**1^{st} trade (buy of size 1.0)**				**2^{nd} trade $(V_{[1,2],p}, V_{[1,2],a}) = (1.0, 0.5)$**				**TTSC**	**TTHC**
	μ_0	μ_1	$[a, b]$	r_1	ν_1	ν_2	$[c, d]$	r_2	**Obj.**	**Obj.**
Myopic	0.333	0.333	$[0.000, 1.000]$	0.333	0.500	0.500	$[1.000, 2.000]$	0.500	0.528	1.194
Optimal TTSC	0.152	0.697	$[0.000, 1.000]$	0.152	0.000	0.782	$[1.080, 2.000]$	0.782	0.414	1.262
Optimal TTHC	0.507	0.000	$[0.000, 0.970]$	0.507	0.171	0.664	$[1.000, 2.000]$	0.664	0.592	1.085
Unrestricted TTHC	0.500	0.000	$[0.000, 1.000]$	0.500	0.167	0.667	$[1.000, 2.000]$	0.667	0.583	1.083
Figure 4, bottom	**1^{st} trade (buy of size 1.0)**				**2^{nd} trade $(V_{[1,2],p}, V_{[1,2],a}) = (1.0, -0.5)$**				**TTSC**	**TTHC**
	μ_0	μ_1	$[a, b]$	r_1	ν_1	ν_2	$[c, d]$	r_2	**Obj.**	**Obj.**
Myopic	0.333	0.333	$[0.000, 1.000]$	0.333	0.167	0.167	$[1.000, 2.000]$	0.167	0.528	1.194
Optimal TTSC	0.152	0.697	$[0.000, 1.000]$	0.152	0.000	0.400	$[1.750, 2.000]$	0.400	0.414	1.262
Optimal TTHC	0.507	0.000	$[0.000, 0.970]$	0.507	0.000	0.319	$[1.430, 2.000]$	0.319	0.592	1.085
Unrestricted TTHC	0.500	0.000	$[0.000, 1.000]$	0.500	-0.167	0.333	$[1.000, 2.000]$	0.333	0.583	1.083
Figure 5, top	**1^{st} trade (buy of size 1.0)**				**2^{nd} trade $(V_{[1,2],p}, V_{[1,2],a}) = (1.0, 0.5)$**				**TTSC**	**TTHC**
	μ_0	μ_1	$[a, b]$	r_1	ν_1	ν_2	$[c, d]$	r_2	**Obj.**	**Obj.**
Myopic	0.333	0.333	$[0.000, 1.000]$	0.333	0.500	0.500	$[1.000, 2.000]$	0.500	0.528	1.194
Optimal FD TTSC	0.153	0.693	$[0.000, 1.000]$	0.153	0.000	0.804	$[1.000, 2.000]$	0.696	0.415	1.261
Optimal FD TTHC	0.513	0.000	$[0.000, 1.000]$	0.487	0.169	0.665	$[1.000, 2.000]$	0.666	0.590	1.086
Unrestricted FD TTHC	0.500	0.000	$[0.000, 1.000]$	0.500	0.167	0.667	$[1.000, 2.000]$	0.667	0.583	1.083
Figure 5, bottom	**1^{st} trade (buy of size 1.0)**				**2^{nd} trade $(V_{[1,2],p}, V_{[1,2],a}) = (1.0, -0.5)$**				**TTSC**	**TTHC**
	μ_0	μ_1	$[a, b]$	r_1	ν_1	ν_2	$[c, d]$	r_2	**Obj.**	**Obj.**
Myopic	0.333	0.333	$[0.000, 1.000]$	0.333	0.167	0.167	$[1.000, 2.000]$	0.167	0.528	1.194
Optimal FD TTSC	0.153	0.693	$[0.000, 1.000]$	0.153	0.000	0.500	NA	0.000	0.415	1.261
Optimal FD TTHC	0.513	0.000	$[0.000, 1.000]$	0.487	0.000	0.388	$[1.000, 2.000]$	0.112	0.590	1.086
Unrestricted FD TTHC	0.500	0.000	$[0.000, 1.000]$	0.500	-0.167	0.333	$[1.000, 2.000]$	0.333	0.583	1.083
Figure 6, top	**1^{st} trade (buy of size 1.0)**				**2^{nd} trade $(V_{[1,2],p}, V_{[1,2],a}) = (0.0, 1.0)$**				**TTSC**	**TTHC**
	μ_0	μ_1	$[a, b]$	r_1	ν_1	ν_2	$[c, d]$	r_2	**Obj.**	**Obj.**
Myopic	0.333	0.333	$[0.000, 1.000]$	0.333	0.333	0.333	$[1.000, 2.000]$	0.333	0.667	0.667
Optimal TTSC	0.255	0.491	$[0.000, 1.000]$	0.255	0.000	0.575	$[1.260, 2.000]$	0.575	0.584	0.584
Optimal TTHC	0.255	0.491	$[0.000, 1.000]$	0.255	0.000	0.575	$[1.260, 2.000]$	0.575	0.584	0.584
Unrestricted TTHC	0.250	0.500	$[0.000, 1.000]$	0.250	-0.167	0.583	$[1.000, 2.000]$	0.583	0.583	0.583
Figure 6, bottom	**1^{st} trade (buy of size 1.0)**				**2^{nd} trade $(V_{[1,2],p}, V_{[1,2],a}) = (0.0, -1.0)$**				**TTSC**	**TTHC**
	μ_0	μ_1	$[a, b]$	r_1	ν_1	ν_2	$[c, d]$	r_2	**Obj.**	**Obj.**
Myopic	0.333	0.333	$[0.000, 1.000]$	0.333	-0.333	-0.333	$[1.000, 2.000]$	-0.333	0.667	0.667
Optimal TTSC	0.255	0.491	$[0.000, 1.000]$	0.255	-0.830	-0.085	$[1.000, 2.000]$	-0.085	0.584	0.584
Optimal TTHC	0.255	0.491	$[0.000, 1.000]$	0.255	-0.830	-0.085	$[1.000, 2.000]$	-0.085	0.584	0.584
Unrestricted TTHC	0.250	0.500	$[0.000, 1.000]$	0.250	-0.833	-0.083	$[1.000, 2.000]$	-0.083	0.583	0.583

Table 2. Optimal parameter and objective function values for the solutions shown in Figures 4, 5 and 6.

setup, where the predictable part of the second trade is zero and consecutive trades are uncorrelated, it turns out that price impact from trading in the first period is exploitable. Figure 6 depicts the optimal Myopic, Stochastic TTSC, Stochastic TTHC and C^*-unrestricted TTHC solutions, and the corresponding optimal parameter and objective function values are shown in Table 2. As expected, the graphs in the second and third columns are identical since the corresponding optimization problems coincide when $V_{[1,2],p} = 0$. The Myopic objective value, which is exactly twice that of the STSH solution for a unit buy, is outperformed by the TTSC and TTHC (identical) solutions which provide a cost improvement of over 10%. This improvement is achieved by backloading execution in the $[0, T_1]$ period compared to the Myopic solution, and either backloading or frontloading execution in the second period depending on whether the second trade is a buy or sell, respectively. To understand how this effect arises, recall that since the first trade is a buy it will inflate the price used for benchmarking $V_{[1,2],a}$. In the absence of further trading, prices would decline over $[T_1, T_2]$ as the impact of the first trade decays. When the second trade is a buy, this decaying price is advantageous so backloading arises. In contrast, if $V_{[1,2]} < 0$, corresponding to a sell, this decaying price is detrimental so frontloading arises. For an equiprobable mix (average) of the buy-buy and buy-sell cases, the beneficial effect for buy-buy more than offsets the detrimental impact for buy-sell, so backloading execution of the first trade provides advantage overall and arises in both cases.

Finally, solutions showing the impact of relaxing the C^*-constraint are depicted in the last column of Figure 6. The most obvious difference is the block trade sell at the start of $[T_1, T_2]$ when the second trade is a buy, corresponding to an alternating solution. However there are other subtle changes, e.g. for both buy and sell cases of the second trade, the (identical) second and third columns have TTSC and TTHC solutions with $(\mu_0, \mu_1, r_1) \approx (0.2546, 0.4907, 0.2546)$ whereas for the optimal C^*-unrestricted TTHC solution these parameters are $(0.25, 0.50, 0.25)$. Recalling that the expectation in the TTHC objective function given in expression (4.13) includes both buy and sell branches for the second trade, these parameter differences arise precisely because the alternating solution depicted in the top-right panel is inadmissible. The resulting TTHC solution therefore slightly frontloads in the first period compared to the C^*-unrestricted solution, with both solutions exhibiting clear backloading over $[0, T_1]$ compared to the Myopic solution. Indeed the proximity of the resulting optimal TTHC and C^*-unrestricted TTHC objective values suggests there is little underperformance for restricting attention to non-alternating solutions.

6. Discussion, extensions and conclusion

We have extended the standard single trade single horizon (STSH) optimal execution problem to the case of two adjacent metaorders, and explored properties of the class of optimal solutions under several ways of benchmarking implementation shortfall. To the best of our knowledge, this study is the first to consider optimal execution within such a framework, and is intended to provide steps towards the general case of multiple trades and multiple execution horizons. We find that even for the case of two metaorders, the currently widely followed practice of adopting the STSH solution myopically over each metaorder can significantly underperform compared to optimizing over a shortfall metric that accounts for both periods. Our results demonstrate that problems can still arise even when both periods are accounted for, as there is flexibility in how the shortfall metric is constructed. For example, separate cost benchmarking may result in significant understatement of transaction costs and adverse backloading when prior knowledge of future order flow is available. Our remedy for this is to optimize over the hybrid cost function used within our TTHC analysis. This leads to optimal execution schedules that typically frontload their trajectories compared to optimal TTSC solutions.

For the practitioner, our results also show that good solutions may prove adequate even if they are not globally optimal, e.g. those obtained under the full domain (FD) constraint, and that execution schedules under the separate cost criterion typically exhibit greater backloading than those under the hybrid cost criterion. Investors whose real objective function is the hybrid criterion, but who for simplicity use the separate cost criterion for scheduling or monitoring their execution, risk significantly understating the true cost of their trading. This is problematic even if explicit order optimization is not used, as heuristic experimentation (e.g. involving backloading) that happens to benefit the separate costs criterion may well lead to apparent improvements in execution quality whilst actually making things worse. It is unfortunate that alternative benchmarking metrics are not more widely used within the industry, as the separate costs approach can lead to detrimental outcomes.

A further extension of the two trade case (not examined here) is when inventory from the second period may be migrated for execution into the first period. Consider the case where $V_{[0,1]}$ is known but $V_{[1,2]}$ is uncertain with $V_{[1,2]} = V_{[1,2],p} + V_{[1,2],a}$ as before. An heuristic solution may prove adequate as a starting point, such as initialising with the STSH solution for quantity $V^* = V_{[0,1]} + V_{[1,2],p}$ over period $[0, T_2]$, but then at time T_1 updating this so that the total of $V_{[1,2],a}$ and the remaining part of V^* is completed over period $[T_1, T_2]$. Such a scheme requires discretion to be given to execution systems in anticipation of future unconfirmed orders, however such flexibility is not standard practice within the industry. We do not discuss this further, but note it is another optimal execution problem that falls outside the scope of standard one-period solutions.

Finally, a comment to aid intuition. When $V_{[1,2],p} = 0$, we observed that both the TTHC and TTSC solutions are backloaded compared to the Myopic solution. One interpretation is that the impact created by trading in the first period leads to some subsequent predictability in prices, and knowledge of this provides a source of *alpha* which can be exploited to advantage in the second period. Injecting knowledge of such an alpha (the expected impact) into an execution optimization cannot make things worse, even if it is correlated with the direction of subsequent trading.

Acknowledgement

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Appendix A. Proofs of optimal TTSC execution schedules

A.1. *Proof of Proposition 3.5*

In this case $\mathrm{d}X_t^{[1,2]} \geq 0$. Similar to equation (2.9), we note that

$$D^2 F_{1,2} = H_{1,2}\,\mathrm{d}t - \gamma\,\mathrm{d}X_t^{[1,2]} \text{ on } (T_1, T_2). \tag{A1}$$

Since $\mathrm{d}X_t^{[0,1]} \geq 0$, $\mathrm{d}X_t^{[1,2]} \geq 0$ and $G'' > 0$, it follows that

$$D^2 F_{1,2} = H_{1,2}\,\mathrm{d}t > 0 \text{ on } (T_1, T_2) \setminus \operatorname{supp}(\mathrm{d}X_t^{[1,2]}).$$

Hence, since $F_{1,2}$ attains its minimum on $\operatorname{supp}(\mathrm{d}X_t^{[1,2]})$, it follows that $F_{1,2}$ is convex on (T_1, T_2), so as in Section 2.1.2 we obtain that $D^2 F_{1,2} \in L^\infty((0,T))$. Now, since $D^2 F_{1,2} = 0$ dt-almost everywhere on $\operatorname{supp}(\mathrm{d}X_t^{[1,2]})$, it follows from equation (A1) that

$$\mathrm{d}X_t^{[1,2]} = \frac{1}{\gamma} H_{1,2}\,\mathrm{d}t \text{ on } \operatorname{supp}(\mathrm{d}X_t^{[1,2]}) \cap (T_1, T_2). \tag{A2}$$

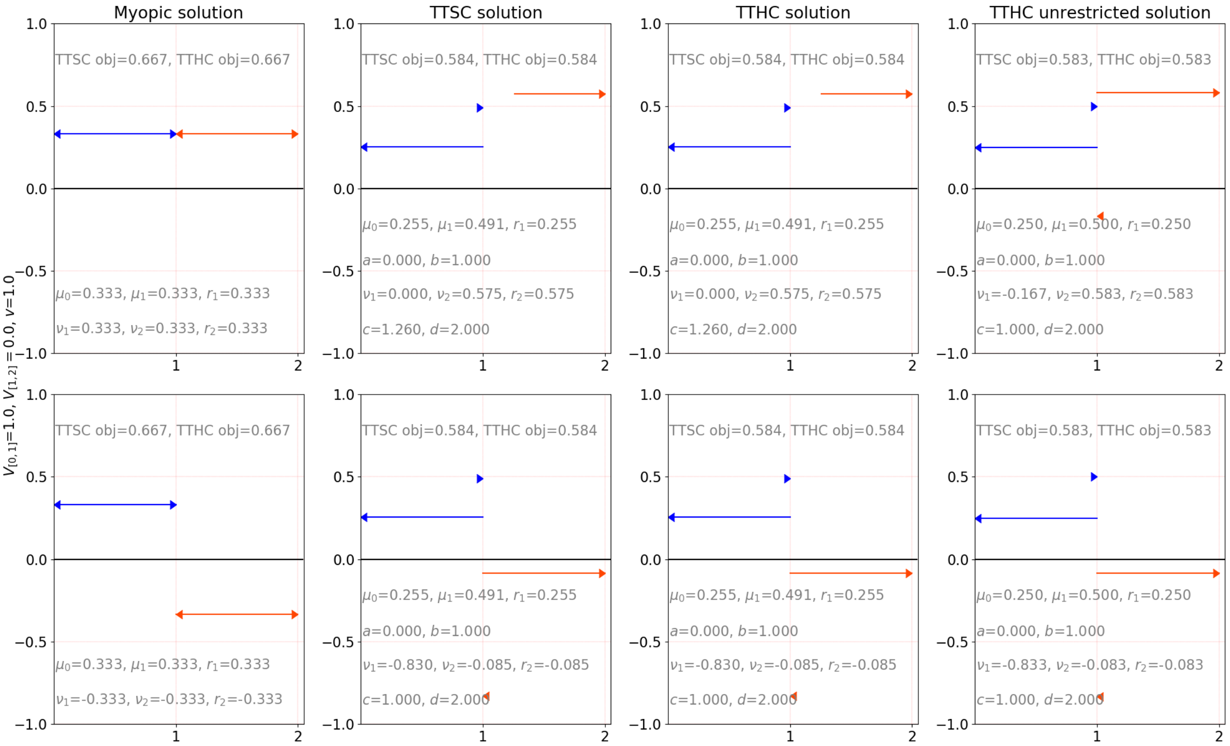

Figure 6. Optimal execution schedules when the first order is a buy with $V_{[0,1]} = 1$ and the second order has predicted part $V_{[1,2],p} = 0$ so that $V_{[1,2]} = V_{[1,2],a}$ is wholly stochastic and becomes known only at time $T_1 = 1$. The two examples considered have $V_{[1,2],a} = 1$ (top row) and $V_{[1,2],a} = -1$ (bottom row), corresponding to the $v = 1$ case of Example 4.2.1. The benchmark Myopic solution (discussed early in Section 5) is shown in the left column. The second column depicts optimal execution schedules under the Stochastic TTSC objective discussed in Section 3.2.2, while the third column shows solutions for the Stochastic TTHC objective described in Section 4.2. As expected, these two columns are identical since the corresponding optimization problems coincide when $V_{[1,2],p} = 0$. The right column provides optimal schedules for the Stochastic TTHC objective when the C^*-constraint introduced in Section 5.2 is relaxed to accommodate alternating solutions within each period. The optimal parameter and objective function values are also shown in Table 2.

Furthermore, since $F_{1,2}$ is convex and $D^2 F_{1,2} = H_{1,2}\,\mathrm{d}t > 0$ outside $\mathrm{supp}(\mathrm{d}X_t^{[1,2]})$, we have that

$$\{F_{1,2} = \Lambda_{1,2}\} \cap (T_1, T_2) = \mathrm{supp}(\mathrm{d}X_t^{[1,2]}) \cap (T_1, T_2) \text{ is an interval.}$$

We now deduce that $T_2 \in \mathrm{supp}(\mathrm{d}X_t^{[1,2]})$. For if not, then $\mathrm{supp}(\mathrm{d}X_t^{[1,2]}) = [a, b]$ for some $b < T_2$, and since $F'_{1,2} = 0$ on $\mathrm{supp}(\mathrm{d}X_t^{[1,2]}) \cap (T_1, T_2)$ we obtain

$$0 = F'_{1,2}(b) = \int_0^b G'(b-s)\,\mathrm{d}(X_s^{[0,1]} + X_s^{[1,2]}). \tag{A3}$$

Now, since $G' < 0$ on $(0, \infty)$ and $\mathrm{d}(X_s^{[0,1]} + X_s^{[1,2]}) \geq 0$, the above integral must be strictly negative, which is a contradiction. We note also that $\mathrm{d}X_t^{[1,2]}$ cannot be a L^∞ function all the way up to T_2, because if it were then applying equation (A3) at T_2 would lead to a contradiction. This establishes the second part of Proposition 3.5.

For the $[0,T_1]$ part we need a different analysis. We start by noting

$$D^2F_{0,1} = H_{0,1}\,\mathrm{d}t - \gamma\,\mathrm{d}X_t^{[0,1]} \text{ on } (0,T_1), \tag{A4}$$

however now $H_{0,1}$ does not have a definite sign, and we cannot say that $F_{0,1}$ is convex. However, we can say that $D^2F_{0,1} = H_{0,1}\mathrm{d}t$ is bounded outside the support of $\mathrm{d}X_t^{[0,1]}$, and on the support it is bounded from above. Hence, since $F_{0,1}$ has a minimum on $\mathrm{supp}(\mathrm{d}X_t^{[0,1]})$, as in Section 2.1.2 we obtain that $D^2F_{0,1} \in L^\infty((0,T))$. Thus, analogous to our previous results, we obtain

$$\mathrm{d}X_t^{[0,1]} = \frac{1}{\gamma}H_{0,1}\,\mathrm{d}t \text{ on } \mathrm{supp}(\mathrm{d}X_t^{[0,1]}) \cap (0,T_1).$$

Note that $F_{0,1} = \Phi_{0,1} + \Psi_{0,1}$ (see equation (3.10)), and that $\Phi_{0,1}$ is convex, while

$$\Psi_{0,1}'(t) = \int_{T_1}^{T_2} [G'(t-s) - G'(t-T_1)]\,\mathrm{d}X_s^{[1,2]} = -\int_{T_1}^{T_2}\int_{T_1}^{s} G''(t-\tau)\,\mathrm{d}\tau\,\mathrm{d}X_s^{[1,2]} < 0 \quad \text{for } t \in (0,T_1).$$

We now deduce that

$$\{F_{0,1} = \Lambda_{0,1}\} \cap (0,T_1) = \mathrm{supp}(\mathrm{d}X_t^{[0,1]}) \cap (0,T_1) \text{ is an interval.} \tag{A5}$$

We start by writing $(0,T_1)\setminus\mathrm{supp}(\mathrm{d}X_t^{[0,1]}) = \cup_{i\geq 1}(a_i,b_i)$, that is as a countable union of open intervals, and assume that $b_i \neq T_1$. Then $b_i \in \mathrm{supp}(\mathrm{d}X_t^{[0,1]})$ which implies $F_{0,1}(b_i) = \Lambda_{0,1}$. Now, by convexity, it follows that $\Phi_{0,1}$ is decreasing on (a_i,b_i), and so $F_{0,1} = \Phi_{0,1} + \Psi_{0,1}$ is strictly decreasing on (a_i,b_i), which implies that $F_{0,1}(a_i) > \Lambda_{0,1}$, and hence that $a_i \notin \mathrm{supp}(\mathrm{d}X_t^{[0,1]})$. This is possible only if $a_i = 0$. This proves $(0,T_1)\setminus\mathrm{supp}(\mathrm{d}X_t^{[0,1]})$ can consist of at most two intervals, and that these intervals must be of the form $(0,b_0)$ or (a_0,T_1). In particular, this implies the validity of claim (A5). Finally, if $\mathrm{supp}(\mathrm{d}X_t^{[0,1]}) = [a,b]$ with $b < T_1$, since $F_{0,1}' = 0$ on $\mathrm{supp}(\mathrm{d}X_t^{[0,1]}) \cap (0,T_1)$ we have

$$0 = F_{0,1}'(b) = \int_0^b G'(b-s)\,\mathrm{d}X_s^{[0,1]} - \int_{T_1}^{T_2}\int_{T_1}^{s} G''(b-\tau)\,\mathrm{d}\tau\,\mathrm{d}X_s^{[1,2]} < 0, \tag{A6}$$

which is a contradiction. We therefore conclude that $\mathrm{d}X_t^{[0,1]}$ cannot be a L^∞ function up to T_1. This establishes the remaining first part of Proposition 3.5. □

A.2. *Proof of Proposition 3.6*

In this case $\mathrm{d}X_t^{[1,2]} \leq 0$ and it follows from equation (A1) that

$$|D^2F_{1,2}| = |H_{1,2}|\,\mathrm{d}t \leq \|H_{1,2}\|_\infty\,\mathrm{d}t \text{ on } (T_1,T_2)\setminus\mathrm{supp}(\mathrm{d}X_t^{[1,2]})$$

and hence that $D^2F_{1,2} \geq -\|H_{1,2}\|_\infty\,\mathrm{d}t$ on (T_1,T_2). Thus, since $F_{1,2}$ attains now a maximum on $\mathrm{supp}(\mathrm{d}X_t^{[1,2]})$, as in Section 2.1.2 we deduce that $D^2F_{1,2} \in L^\infty((0,T))$. Now, since $D^2F_{1,2} = 0$ $\mathrm{d}t$-almost everywhere on $\operatorname{supp}(\mathrm{d}X_t^{[1,2]})$, it follows from equation (A1) that

$$\mathrm{d}X_t^{[1,2]} = \frac{1}{\gamma} H_{1,2}\,\mathrm{d}t \text{ on } \operatorname{supp}(\mathrm{d}X_t^{[1,2]}) \cap (T_1, T_2).$$

Noting now that $F_{1,2} = \Phi_{1,2} + \Psi_{1,2}$ (see equation (3.10)) and that $\Phi_{1,2}$ is convex while

$$\Psi'_{1,2}(t) = \int_0^{T_1} G'(t-s)\,\mathrm{d}X_s^{[0,1]} < 0 \text{ for } t \in (T_1, T_2),$$

then arguing as we did for equation (A7), it follows that

$$\{F_{1,2} = \Lambda_{1,2}\} \cap (T_1, T_2) = \operatorname{supp}(\mathrm{d}X_t^{[0,1]}) \cap (T_1, T_2) \text{ is an interval.} \tag{A7}$$

We now deduce that $T_1 \in \operatorname{supp}(\mathrm{d}X_t^{[1,2]})$. For if not, then $\operatorname{supp}(\mathrm{d}X_t^{[1,2]}) = [a, b]$ with $a > T_1$, and since $F'_{1,2} = 0$ on $\operatorname{supp}(\mathrm{d}X_t^{[1,2]}) \cap (T_1, T_2)$, we obtain

$$0 = F'_{1,2}(a) = \int_0^{T_1} G'(a-s)\,\mathrm{d}X_s^{[0,1]} + \int_a^{T_2} G'(a-s)\,\mathrm{d}X_s^{[1,2]} < 0 \tag{A8}$$

which is a contradiction. Furthermore $\mathrm{d}X_t^{[1,2]}$ cannot be a L^∞ function all the way up to T_1, since if it were then we could apply equation (A8) at T_1 and obtain another contradiction. This establishes the second part of Proposition 3.6.

For the first part, on $[0, T_1]$ we again have that $D^2 F_{0,1} \in L^\infty((0,T))$. Thus, analogous to before, we obtain

$$\mathrm{d}X_t^{[0,1]} = \frac{1}{\gamma} H_{0,1}\,\mathrm{d}t \text{ on } \operatorname{supp}(\mathrm{d}X_t^{[0,1]}) \cap (0, T_1).$$

Recalling $F_{0,1} = \Phi_{0,1} + \Psi_{0,1}$ with $\Phi_{0,1}$ a convex function, and

$$\Psi'_{0,1}(t) = \int_{T_1}^{T_2} [G'(t-s) - G'(t-T_1)]\,\mathrm{d}X_s^{[1,2]} = -\int_{T_1}^{T_2}\int_{T_1}^{s} G''(t-\tau)\,\mathrm{d}\tau\,\mathrm{d}X_s^{[1,2]} > 0 \text{ for } t \in (0, T_1),$$

we deduce that

$$\{F_{0,1} = \Lambda_{0,1}\} \cap (0, T_1) = \operatorname{supp}(\mathrm{d}X_t^{[0,1]}) \cap (0, T_1) \text{ is an interval.}$$

Finally, if $\operatorname{supp}(\mathrm{d}X_t^{[0,1]}) = [a, b]$ with $a > 0$, since $F'_{0,1} = 0$ on $\operatorname{supp}(\mathrm{d}X_t^{[0,1]}) \cap (0, T_1)$ we obtain

$$0 = F'_{0,1}(a) = \int_a^{T_1} G'(a-s)\,\mathrm{d}X_s^{[0,1]} - \int_{T_1}^{T_2}\int_{T_1}^{s} G''(a-\tau)\,\mathrm{d}\tau\,\mathrm{d}X_s^{[1,2]} > 0, \tag{A9}$$

which is a contradiction. Furthermore, $\mathrm{d}X_t^{[0,1]}$ cannot be a L^∞ function all the way up to T_1. This establishes the remaining first part of Proposition 3.6. $\square$

Appendix B. Additional observations on TTSC with exponential decay

Remark B.1. *In the case when $c < T_2$ we may derive a simple relation between the size of the last trade ν_{T_2} and the rate of trading inside the second interval $r_2 \equiv \hat{\Lambda}_{1,2}/(2\kappa)$. Let $\epsilon > 0$ be a small constant so that $T_2 - \epsilon > c$. Then, since $F'_{1,2}(T_2 - \epsilon) = 0$ by optimality, we obtain*

$$\begin{aligned}0 &= \int_0^{T_1} e^{-\kappa(T_2-\epsilon-s)} \mathrm{d}X_s^{[0,1]} + \int_{T_1}^{T_2-\epsilon} e^{-\kappa(T_2-\epsilon-s)} \mathrm{d}X_s^{[1,2]} - \int_{T_2-\epsilon}^{T_2^-} e^{\kappa(T_2-\epsilon-s)} \mathrm{d}X_s^{[1,2]} - e^{-\kappa\epsilon}\nu_{T_2} \\ &= e^{\kappa\epsilon}\Big(\int_0^{T_1} e^{-\kappa(T_2-s)} \mathrm{d}X_s^{[0,1]} + \int_{T_1}^{T_2^-} e^{-\kappa(T_2-s)} \mathrm{d}X_s^{[1,2]} - \nu_{T_2} \\ &\qquad - \int_{T_2-\epsilon}^{T_2^-} [e^{\kappa(T_2-2\epsilon-s)} + e^{-\kappa(T_2-s)}] \mathrm{d}X_s^{[1,2]} + [1 - e^{-2\kappa\epsilon}]\nu_{T_2} \Big).\end{aligned}$$

The first line in the last equation above vanishes as a consequence of the second equation in the system (4.4)*, so by a Taylor expansion we deduce that*

$$0 = -\int_{T_2-\epsilon}^{T_2^-} [e^{\kappa(T_2-2\epsilon-s)} + e^{-\kappa(T_2-s)}] \mathrm{d}X_s^{[1,2]} + [1 - e^{-2\kappa\epsilon}]\nu_{T_2} = -2\epsilon\, r_2 + 2\kappa\epsilon\, \nu_{T_2} + o(\epsilon)$$

whereby we obtain

$$r_2 = \kappa\, \nu_{T_2}. \tag{B1}$$

A similar analysis may be performed for the first interval.

Remark B.2. *Note that unless the full domain (FD) solution happens to be also a global minimizer, then Remark B.1 cannot be applied. In other words, the relation*

$$\frac{V_{[1,2]} - \nu_{T_1} - \nu_{T_2}}{T_2 - T_1} = \kappa\, \nu_{T_2}$$

(see equation (B1)*) holds whenever optimization without the FD constraint being imposed produces a full domain solution, but otherwise may fail to hold, as our numerical simulations show.*

B.1. *Parametric subfamily solutions*

It is also interesting to estimate the cost of solutions which impose particular constraints, for instance full domain support in the first period and $\nu_{T_1} = 0$ in the second. In this case the optimization problem becomes to minimize $\mathbb{E}\mathcal{C}[X_t^{[0,1]}, X_t^{[1,2]}]$ over the 4-dimensional space of decision variables $(\mu_0, \mu_{T_1}, \nu_{T_2}, c)$ where

$$\mathrm{d}X_t^{[0,1]} = \mu_0\, \delta_0 + \mu_{T_1}\, \delta_{T_1} + \frac{V_{[0,1]} - \mu_0 - \mu_{T_1}}{T_1}\, \mathrm{d}t$$

and

$$\mathrm{d}X_t^{[1,2]} = \nu_{T_2}\, \delta_{T_2} + I(c \le t \le T_2) \frac{V_{[1,2]} - \nu_{T_2}}{T_2 - c}\, \mathrm{d}t.$$

The objective function is no longer a quadratic expression but is now given by

$$
\begin{aligned}
&\frac{1}{2}\mu_0^2 + \frac{1}{2}\mu_{T_1}^2 + \mu_0\mu_{T_1}e^{-\kappa T_1} + \frac{1-e^{-\kappa T_1}}{\kappa}\cdot(\mu_0+\mu_{T_1})\cdot\frac{V_{[0,1]}-\mu_0-\mu_{T_1}}{T_1} + \\
&\frac{\kappa T_1 - 1 + e^{-\kappa T_1}}{\kappa^2}\cdot\left(\frac{V_{[0,1]}-\mu_0-\mu_{T_1}}{T_1}\right)^2 + \frac{1}{2}\nu_{T_2}^2 + \frac{1-e^{-\kappa(T_2-c)}}{\kappa}\cdot\nu_{T_2}\cdot\frac{V_{[1,2]}-\nu_{T_2}}{T_2-c} + \\
&\frac{\kappa(T_2-c)-1+e^{-\kappa(T_2-c)}}{\kappa^2}\cdot\left(\frac{V_{[1,2]}-\nu_{T_2}}{T_2-c}\right)^2 + \\
&\left\{\nu_{T_2}(e^{-\kappa T_2}-e^{-\kappa T_1}) + \frac{V_{[1,2]}-\nu_{T_2}}{T_2-c}\cdot\frac{e^{-\kappa c}-e^{-\kappa T_2}-\kappa e^{-\kappa T_1}(T_2-c)}{\kappa}\right\}\Bigg(\mu_0+\mu_{T_1}e^{\kappa T_1}+ \\
&\frac{e^{-\kappa T_1}-1}{\kappa}\cdot\frac{V_{[0,1]}-\mu_0-\mu_{T_1}}{T_1}\Bigg).
\end{aligned}
$$

Remark B.3. *To apply Remark B.1 to the above problem we need to know that the minimizer of the formula above is actually the global minimizer of the TTSC problem. Also, recall that Remark B.1 requires the assumption $c < T_2$. When this additionally holds then we have*

$$\frac{V_{[1,2]}-\nu_{T_2}}{T_2-c} = \kappa\,\nu_{T_2}$$

(see equation (B1)*), but otherwise this relation may fail, as our numerical results demonstrate.*

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