Conical points of convex surfaces: generalized unit normal and angle defect

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Abstract. Given a conical point of a convex surface, a robust notion of generalized unit normal is introduced. Its relationship with the polar to the tangent cone implies the BV regularity of our unit normal. We then show that the area of the geodesically convex hull of its image, called angle defect, describes the energy concentration of the Gauss curvature of smooth surfaces approximating the tangent cone at the conical point.

Keywords : Convex surfaces, conical points, unit normal, angle defect

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Introduction

In the study of extremum problems in many areas of applied mathematics, convexity plays an increasingly important role. For a self-contained account of the recent developments of the theory of convex sets and convex functions we refer to the monograph [22].

Second order invariants of smooth oriented surfaces in 3D are encoded by the graph of the Gauss map, the function defining the unit normal at any point of the surface, see [10]. Concerning non-parametric surfaces given by the graph of bounded and convex functions (that is, by pieces of the boundary of a 3D convex body), the unit normal is well defined at the full 2D-measure set of regular points. Moreover, as observed in [9], it is a function of *bounded variation*. The Jump component of its distributional derivative contains the relevant information about the set of ridge points: they are points where the tangent cone is the union of two half-planes meeting at a line, whose orientation is identified by the one-sided limits of the unit normal. Therefore, in an unspecified sense, convex functions may be viewed as functions with bounded hessian, that provide the functional framework in order to study e.g. plastic behavior without hardening of plates, compare [12].

Complementary to regular and ridge points is the at most countable set of *conical points*. In mechanics, a conical point is e.g. obtained when deforming a flat surface through a punching. However, analytical properties at conical points of convex surfaces cannot be analyzed by means of the previous functional setting. On the other hand, in all examples one has in mind, one checks the existence of a "normal" to the oriented convex surface that is defined around a conical point, whose image in the Gauss sphere $\mathbb{S}^2$ collects the information of the tangent cone.

The aim of this paper is to give a robust notion of *generalized normal* at conical points P of a convex surface. Our normal is a function $\theta \mapsto \nu_P(\theta)$ of the angle identifying the outcoming direction at a conical point, taking values into $\mathbb{S}^2$. Its definition clearly depends on the directional derivatives of a local parameterization of the convex surface around the point P, see Proposition 3.1. The BV regularity of the function ν_P is the first main result of this paper, and we now briefly describe how it is obtained.

Let $\partial\Omega$ be the convex surface given by the boundary of a convex body Ω in $\mathbb{R}^3$ with non-empty interior. For each point P at $\partial\Omega$, up to a rigid motion, we have $P = (0_{\mathbb{R}^2}, \lambda_P)$ for some $\lambda_P > 0$, and the tangent cone Σ_P is equal to the graph of a function $\sigma_P : \mathbb{R}^2 \to \mathbb{R}$ such that $\sigma_P(0_{\mathbb{R}^2}) = \lambda_P$. Denote by $\mathbf{c}_P$ the convex spherical curve identified by the intersection of the tangent cone Σ_P with the unit sphere centered at P, i.e.,

$$\mathbf{c}_P = \{z - P \mid z \in \Sigma_P\,,\ |z - P| = 1\} \subset \mathbb{S}^2\,.$$

Then, $\mathbf{c}_P$ is a rectifiable and closed curve with finite *geodesic total curvature*, say $\mathrm{TC}_{\mathbb{S}^2}(\mathbf{c}_P) < \infty$. Referring to Sec. 1 for some notation and background material, and using results taken from [20], in Theorem 2.2

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we show the existence of a convex and closed spherical curve γ_P, whose length is equal to $\mathrm{TC}_{\mathbb{S}^2}(\mathbf{c}_P)$, that agrees with the limit of the *polar curves* of any sequence of inscribed spherical polygonals converging to $\mathbf{c}_P$. Therefore, the curve γ_P will be called *polar* to $\mathbf{c}_P$, and we also denote by $\mathcal{N}_P$ the spherically convex region of the Gauss sphere enclosed by the polar γ_P.

In Theorem 3.3, we then prove that *the set $\mathcal{N}_P$ agrees with the geodesic convex hull of the image of the generalized unit normal ν_P*. Therefore, $\mathcal{N}_P$ may be called *generalized normal cone*. As a consequence, in Proposition 3.4 we obtain that ν_P is a function of bounded variation.

We now call *angle defect* of the convex surface at P the area $\mathcal{AD}(P)$ of $\mathcal{N}_P$. Note that $\mathcal{AD}(P) = 0$ if P is a regular or ridge point, whereas $\mathcal{AD}(P) > 0$ if P is a conical point. In that case, in fact, the polar γ_P is a non-degenerate, simple, convex curve.

If P is a conical point, one expects that (in a weak sense) the *Gauss curvature* of the tangent cone Σ_P is concentrated at the point P, where it is described by a Dirac measure δ with a positive weight equal to the angle defect $\mathcal{AD}(P)$. This is the content of our second main result.

Assume first that the spherical curve $\mathbf{c}_P$ is smooth, and recall that σ_P is the function whose graph is equal to the tangent cone Σ_P. If P is a conical point, then σ_P is not differentiable at the origin. However, in Proposition 4.1 we show the existence of a smooth sequence $\{\sigma_k\} \subset C^\infty(B^2)$ such that $\sigma_k \to \sigma_P$ uniformly in B^2, and

$$|\mathbf{K}(x, \sigma_k(x))| \sqrt{1 + |\nabla \sigma_k(x)|^2}\, \mathcal{L}^2 \llcorner B^2 \rightharpoonup \mathcal{AD}(P)\, \delta_{0_{\mathbb{R}^2}}$$

weakly-* in the sense of the measures, as $k \to \infty$, where $\mathbf{K} = \mathbf{K}(x, \sigma_k(x))$ is the Gauss curvature of the graph $\mathcal{G}_{\sigma_k}$ of σ_k at the point $(x, \sigma_k(x))$. Moreover,

$$\lim_{k\to\infty} \int_{\mathcal{G}_{\sigma_k}} |\mathbf{K}|\, d\mathcal{H}^2 = \mathcal{AD}(P)\,.$$

Furthermore, Theorem 4.2 extends the latter result to any point P of the convex surface $\partial\Omega$.

We conclude this introduction by giving a short (and not exhaustive) list of recent regularity results in this framework. We first recall that boundaries of convex bodies with non-empty interior are surfaces with curvature measures in the sense of Federer [15], compare e.g. [9], whereas the Radon-Nikodým derivatives of curvature and surface area measures of convex bodies (as Hessian measures) have been studied in [14]. Moreover, a thorough analysis of non differentiability points of convex functions is given in [3], see also the survey paper [7], and we refer to [1] for a study of the structure properties of the singular set. Also, the fine properties of subgradients of convex and lower semicontinuous functions obtained by Alexandrov [5], can be recovered by viewing them as a particular class of monotone functions, see [2]. Finally, Lusin-type properties of convex functions and convex bodies have been recently analyzed in [11].

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1 Preliminary results

We collect some well-known properties of curves in Euclidean spaces. We then focus on spherical curves, recalling some results from [21] concerning rectifiable curves with finite geodesic total curvature. We refer to Secs. 3.1-3.2 of [8] for the notation adopted on one-dimensional functions of bounded variations.

1.1 Length and total curvature

Consider a curve $\mathbf{c}$ in the Euclidean space $\mathbb{R}^3$ parameterized by the continuous map $\mathbf{c} : [a, b] \to \mathbb{R}^3$. Any polygonal curve $\mathbf{p}$ *inscribed* in $\mathbf{c}$, say $\mathbf{p} \ll \mathbf{c}$, is obtained by choosing a finite partition $\mathcal{D} := \{a = t_0 < t_1 < \ldots < t_{m-1} < t_m = b\}$ of $[a, b]$, say $\mathbf{p} = \mathbf{p}(\mathcal{D})$, and letting $\mathbf{p} : [a, b] \to \mathbb{R}^3$ such that $\mathbf{p}(t_i) = \mathbf{c}(t_i)$ for $i = 0, \ldots, m$, and $\mathbf{p}(t)$ affine on each interval $I_i := [t_{i-1}, t_i]$. We call $\operatorname{mesh} \mathbf{p}$ the maximum length of its edges. The *length* $\mathcal{L}(\mathbf{c})$ of $\mathbf{c}$ is defined by

$$\mathcal{L}(\mathbf{c}) := \sup\{\mathcal{L}(\mathbf{p}) \mid \mathbf{p} \ll \mathbf{c}\}$$

and $\mathbf{c}$ is said to be *rectifiable* if $\mathcal{L}(\mathbf{c}) < \infty$. By uniform continuity, for each $\varepsilon > 0$ we can find $\delta > 0$ such that $\operatorname{mesh} \mathbf{p} < \varepsilon$ if $\operatorname{mesh} \mathcal{D} < \delta$ and $\mathbf{p} = \mathbf{p}(\mathcal{D})$. As a consequence, taking $\mathbf{p}_n = \mathbf{p}(\mathcal{D}_n)$, where $\{\mathcal{D}_n\}$ is any sequence of partitions of I such that $\operatorname{mesh} \mathcal{D}_n \to 0$, we get $\operatorname{mesh} \mathbf{p}_n \to 0$ and hence the convergence $\mathcal{L}(\mathbf{p}_n) \to \mathcal{L}(\mathbf{c})$ of the length functional. Finally, the curve $\mathbf{c}$ is rectifiable if and only if the function $\mathbf{c}$ is of bounded variation, say $\mathbf{c} \in \mathrm{BV}((a,b),\mathbb{R}^3)$, and in that case

$$\mathcal{L}(\mathbf{c}) = \mathrm{Var}_{\mathbb{R}^3}(\mathbf{c}) = |D\mathbf{c}|(a,b)\,.$$

In particular, if $\mathbf{c} \in C^1([a,b],\mathbb{R}^3)$ we get $\mathcal{L}(\mathbf{c}) = \int_a^b \|\dot{\mathbf{c}}(t)\|\,dt < \infty$.

Definition 1.1 The *Fréchet distance* $d(\mathbf{c}_1, \mathbf{c}_2)$ between two rectifiable curves is the infimum, over all strictly monotonic reparameterizations, of the maximum pointwise distance.

Therefore, if $d(\mathbf{c}_1, \mathbf{c}_2) = 0$, the two curves are equivalent in the following sense: homeomorphic reparameterizations that approach the infimal value zero will limit to the more general reparameterization that might eliminate or introduce intervals of constancy, compare [23].

Moreover, if $\{\mathbf{c}_h\}$ is a sequence of rectifiable curves in $\mathbb{R}^3$ such that $d(\mathbf{c}_h, \mathbf{c}) \to 0$ as $h \to \infty$ for some rectifiable curve $\mathbf{c}$, then by lower semicontinuity

$$\mathcal{L}(\mathbf{c}) \leq \liminf_{h\to\infty} \mathcal{L}(\mathbf{c}_h)\,. \tag{1.1}$$

We call *rotation* $\mathbf{k}^*(\mathbf{p})$ of a polygonal curve $\mathbf{p}$ in $\mathbb{R}^3$ the sum of the exterior angles between consecutive segments. Milnor [18] defined the *total curvature* $\mathrm{TC}(\mathbf{c})$ of a curve $\mathbf{c}$ in $\mathbb{R}^3$ by

$$\mathrm{TC}(\mathbf{c}) := \sup\{\mathbf{k}^*(\mathbf{p}) \mid \mathbf{p} \ll \mathbf{c}\}\,.$$

Then $\mathrm{TC}(\mathbf{p}) = \mathbf{k}^*(\mathbf{p})$ for each polygonal $\mathbf{p}$. Moreover, if a curve $\mathbf{c}$ has compact support and finite total curvature, $\mathrm{TC}(\mathbf{c}) < \infty$, then it is a rectifiable curve.

Assume now that a rectifiable curve $\mathbf{c}$ is parameterized by arc-length, so that $\mathbf{c} = \mathbf{c}(s)$, with $s \in [0, L] = \overline{I}_L$, where $I_L := (0, L)$ and $L = \mathcal{L}(\mathbf{c})$. If $\mathbf{c}$ is smooth and regular, one has $\mathrm{TC}(\mathbf{c}) = \int_0^L |\mathbf{k}|\,ds$, where $\mathbf{k}(s) := \ddot{\mathbf{c}}(s)$ is the curvature vector. More generally, since $\mathbf{c}$ is a Lipschitz function, by Rademacher's theorem (cf. [8, Thm. 2.14]) it is differentiable $\mathcal{L}^1$-a.e. in I_L. Denoting by $\dot{f} := \frac{d}{ds} f$ the derivative w.r.t. arc-length parameter s, the tantrix $\mathbf{t} = \dot{\mathbf{c}}$ exists a.e., and actually $\mathbf{t} : I_L \to \mathbb{R}^3$ is a function of bounded variation. Since moreover $\mathbf{t}(s) \in \mathbb{S}^2$ for a.e. s, where $\mathbb{S}^2$ is the Gauss sphere

$$\mathbb{S}^2 := \{z \in \mathbb{R}^3 \,:\, |z| = 1\}$$

we shall write $\mathbf{t} \in \mathrm{BV}(I_L, \mathbb{S}^2)$. The essential variation $\mathrm{Var}_{\mathbb{S}^2}(\mathbf{t})$ of $\mathbf{t}$ in $\mathbb{S}^2$ differs from $\mathrm{Var}_{\mathbb{R}^3}(\mathbf{t})$, as its definition involves the geodesic distance $d_{\mathbb{S}^2}$ in $\mathbb{S}^2$ instead of the Euclidean distance in $\mathbb{R}^3$. Therefore, $\mathrm{Var}_{\mathbb{R}^3}(\mathbf{t}) \leq \mathrm{Var}_{\mathbb{S}^2}(\mathbf{t})$, and equality holds if and only if $\mathbf{t}$ has a continuous representative. More precisely,

$$\mathrm{Var}_{\mathbb{S}^2}(\mathbf{t}) = \int_0^L |\dot{\mathbf{t}}|\,ds + \sum_{s\in J_{\mathbf{t}}} d_{\mathbb{S}^2}(\mathbf{t}(s+), \mathbf{t}(s-)) + |D^C\mathbf{t}|(I_L) \tag{1.2}$$

whereas in the formula for $\mathrm{Var}_{\mathbb{R}^3}(\mathbf{t})$, that is equal to $|D\mathbf{t}|(I_L)$, one has to replace in (1.2) the geodesic distance $d_{\mathbb{S}^2}(\mathbf{t}(s+), \mathbf{t}(s-))$ with the Euclidean distance $|\mathbf{t}(s+) - \mathbf{t}(s-)|$ at each Jump point $s \in J_{\mathbf{t}}$.

The following facts hold:

i) if $\mathbf{p}$ and $\mathbf{p}'$ are inscribed polygonals and $\mathbf{p}'$ is obtained by adding a vertex in $\mathbf{c}$ to the vertices of $\mathbf{p}$, then $\mathbf{k}^*(\mathbf{p}) \leq \mathbf{k}^*(\mathbf{p}')$;

ii) if $\mathbf{c}$ has finite total curvature, for each point v in $\mathbf{c}$, small open arcs of $\mathbf{c}$ with an end point equal to v have small total curvature.

As a consequence, compare [23], it turns out that $\mathrm{TC}(\mathbf{c}) = \mathrm{Var}_{\mathbb{S}^2}(\mathbf{t})$, see (1.2), and the total curvature of $\mathbf{c}$ is equal to the limit of $\mathbf{k}^*(\mathbf{p}_n)$ for *any* sequence $\{\mathbf{p}_n\}$ of polygonals in $\mathbb{R}^3$ inscribed in $\mathbf{c}$ such that $\operatorname{mesh} \mathbf{p}_n \to 0$. More precisely, if $\mathbf{t}_n$ is the tantrix of $\mathbf{p}_n$, then $\mathrm{Var}_{\mathbb{S}^2}(\mathbf{t}_n) \to \mathrm{Var}_{\mathbb{S}^2}(\mathbf{t})$.

1.2 Geodesic total curvature of spherical curves

We now consider curves supported in the unit sphere $\mathbb{S}^2$ of $\mathbb{R}^3$.

If $\mathbf{c}$ is a smooth and regular curve in $\mathbb{S}^2$, parameterized by arc-length, the unit tangent vector $\mathbf{t}(s) := \dot{\mathbf{c}}(s)$ satisfies $\dot{\mathbf{t}} \bullet \mathbf{t} \equiv 0$, whence the curvature vector $\mathbf{k}(s) := \dot{\mathbf{t}}(s)$ is orthogonal to $\mathbf{t}(s)$. Since the unit normal $\mathbf{n}(s) = \mathbf{c}(s)$, denoting by $\mathbf{u}(s) := \mathbf{n}(s) \times \mathbf{t}(s)$ the unit conormal, we get

$$\mathbf{k}(s) = \mathfrak{K}_g(s)\,\mathbf{u}(s) - \mathbf{n}(s)\,.$$

The triad $(\mathbf{t}, \mathbf{n}, \mathbf{u})$ is called Darboux frame, whereas $\mathfrak{K}_g := \mathbf{k} \bullet \mathbf{u}$ is the *geodesic curvature* of $\mathbf{c}$, and the normal curvature is equal to -1. The geodesic torsion is equal to zero, and the Frenet-Serret formulas in $\mathbb{R}^3$ are equivalent to the Darboux system:

$$\dot{\mathbf{t}} = \mathfrak{K}_g \mathbf{u} - \mathbf{n}\,, \quad \dot{\mathbf{n}} = \mathbf{t}\,, \quad \dot{\mathbf{u}} = -\mathfrak{K}_g \mathbf{t}\,.$$

Let $\mathbf{p}$ be a polygonal in $\mathbb{S}^2$. The *geodesic rotation* $\mathbf{k}_{\mathbb{S}^2}(\mathbf{p})$ of $\mathbf{p}$ is the sum of the turning angles between the consecutive geodesic arcs of $\mathbf{p}$. The polygonal $\mathbf{p}$ is said to be inscribed in a curve $\mathbf{c} : [a, b] \to \mathbb{S}^2$ if it is obtained by choosing a partition $a \leq t_0 < t_1 < \cdots < t_m \leq b$ and connecting with geodesic segments the consecutive points $\mathbf{c}(t_i)$ of the curve. Note that one has $\mathrm{TC}(\mathbf{p}) = \mathbf{k}_{\mathbb{S}^2}(\mathbf{p}) + \mathcal{L}(\mathbf{p})$.

For a general curve $\mathbf{c}$ supported in $\mathbb{S}^2$, we denote by $\mathcal{P}_{\mathbb{S}^2}(\mathbf{c})$ the class of polygonals in $\mathbb{S}^2$ which are inscribed in $\mathbf{c}$. The following property has been proved in [13, Thm. 3.4].

Theorem 1.2 *Let $\mathbf{c}$ be a regular curve in $\mathbb{S}^2$ of class C^2, parameterized by arc-length. Then, for any sequence $\{\mathbf{p}_n\} \subset \mathcal{P}_{\mathbb{S}^2}(\mathbf{c})$ such that* $\operatorname{mesh} \mathbf{p}_n \to 0$*, one has*

$$\lim_{n\to\infty} \mathbf{k}_{\mathbb{S}^2}(\mathbf{p}_n) = \int_{\mathbf{c}} |\mathfrak{K}_g|\,ds = \int_0^L |\mathfrak{K}_g(s)|\,ds\,.$$

As a consequence, one is tempted to define the geodesic total curvature of a curve $\mathbf{c}$ in $\mathbb{S}^2$ as in the Euclidean case, i.e., by the supremum of the geodesic rotation $\mathbf{k}_{\mathbb{S}^2}(\mathbf{p})$ computed among all the polygonals $\mathbf{p}$ in $\mathcal{P}_{\mathbb{S}^2}(\mathbf{c})$. However, as observed in [13], since $\mathbb{S}^2$ has positive sectional curvature, the analogous to the monotonicity property i) fails to hold, and hence latter definition does not work.

Example 1.3 Let $\mathbf{c}$ be a parallel which is not a great circle. If $\mathbf{p}, \mathbf{p}' \in \mathcal{P}_{\mathbb{S}^2}(\mathbf{c})$, where $\mathbf{p}'$ is obtained by adding a vertex in $\mathbf{c}$ to the vertices of $\mathbf{p}$, then $\mathbf{k}_{\mathbb{S}^2}(\mathbf{p}) \geq \mathbf{k}_{\mathbb{S}^2}(\mathbf{p}') > \int_{\mathbf{c}} |\mathfrak{K}_g|\,ds$.

In order to overcome this drawback, the good intrinsic notion turns out to be the one proposed by Alexander-Bishop [4], that goes back to the one considered by Alexandrov-Reshetnyak [6].

For this purpose, compare e.g. [17], we recall that the *modulus* $\mu_{\mathbf{c}}(\mathbf{p})$ of a polygonal $\mathbf{p}$ in $\mathcal{P}_{\mathbb{S}^2}(\mathbf{c})$ is the maximum of the geodesic diameter of the arcs of $\mathbf{c}$ determined by the consecutive vertexes in $\mathbf{p}$. For $\varepsilon > 0$, we also let

$$\Sigma_\varepsilon(\mathbf{c}) := \{\mathbf{p} \in \mathcal{P}_{\mathbb{S}^2}(\mathbf{c}) \mid \mu_{\mathbf{c}}(\mathbf{p}) < \varepsilon\}\,.$$

Definition 1.4 The *total intrinsic curvature* of a curve $\mathbf{c}$ in $\mathbb{S}^2$ is

$$\mathrm{TC}_{\mathbb{S}^2}(\mathbf{c}) := \lim_{\varepsilon\to 0^+} \sup\{\mathbf{k}_{\mathbb{S}^2}(\mathbf{p}) \mid \mathbf{p} \in \Sigma_\varepsilon(\mathbf{c})\}\,.$$

Clearly, the above limit is equal to the infimum as $\varepsilon > 0$ of $\sup\{\mathbf{k}_{\mathbb{S}^2}(\mathbf{p}) \mid \mathbf{p} \in \Sigma_\varepsilon(\mathbf{c})\}$. Moreover, arguing as in [17, Prop. 2.1], for a spherical polygonal we always have $\mathrm{TC}_{\mathbb{S}^2}(\mathbf{p}) = \mathbf{k}_{\mathbb{S}^2}(\mathbf{p})$. If e.g. $\mathbf{c} = \mathbf{c}_{\theta_0}$ is the parallel with constant co-latitude $\theta_0 \in]0, \pi/2]$, one has $\mathrm{TC}_{\mathbb{S}^2}(\mathbf{c}_{\theta_0}) = 2\pi\,\cos\theta_0$. In particular, $\mathrm{TC}_{\mathbb{S}^2}(\mathbf{c}_{\theta_0}) = 0$ if and only if $\theta_0 = \pi/2$, so that $\mathbf{c}_{\pi/2}$ is a great circle, whence a geodesic in $\mathbb{S}^2$. Most importantly, compare [6, Thm. 6.3.2], one has:

Proposition 1.5 *The geodesic total curvature* $\mathrm{TC}_{\mathbb{S}^2}(\mathbf{c})$ *of any curve $\mathbf{c}$ in $\mathbb{S}^2$ is equal to the limit of the geodesic rotation $\mathbf{k}_{\mathbb{S}^2}(\mathbf{p}_h)$ of* any *sequence of polygonals $\{\mathbf{p}_h\} \subset \mathcal{P}_{\mathbb{S}^2}(\mathbf{c})$ such that $\mu_{\mathbf{c}}(\mathbf{p}_h) \to 0$.*

Proposition 1.5 fills the gap given by the lack of monotonicity observed in Example 1.3, yielding to the conclusion that Definition 1.4 involves a control on the modulus and not on the mesh.

As a consequence, by Theorem 1.2 one infers that for smooth curves $\mathbf{c}$ in $\mathbb{S}^2$ one has $\mathrm{TC}_{\mathbb{S}^2}(\mathbf{c}) = \int_{\mathbf{c}} |\mathfrak{K}_g|\, ds$. By [13, Cor. 3.6], for piecewise smooth curves $\mathbf{c}$ in $\mathbb{S}^2$ one similarly obtains that

$$\mathrm{TC}_{\mathbb{S}^2}(\mathbf{c}) = \int_0^L |\mathfrak{K}_g(s)|\, ds + \sum_i |\alpha_i|\,. \tag{1.3}$$

In this formula, the integral is computed separately outside the corner points of $\mathbf{c}$, where the geodesic curvature $\mathfrak{K}_g$ is well-defined, and the second addendum denotes the finite sum of the absolute value of the oriented turning angles α_i between the incoming and outcoming unit tangent vectors at each corner point of $\mathbf{c}$. Therefore, for piecewise smooth curves we can rewrite formula (1.3) as

$$\mathrm{TC}_{\mathbb{S}^2}(\mathbf{c}) = \int_0^L |\dot{\mathbf{t}} \bullet \mathbf{u}|\, ds + \sum_{s\in J_{\mathbf{t}}} d_{\mathbb{S}^2}(\mathbf{t}(s+), \mathbf{t}(s-))\,.$$

For a curve $\mathbf{c}$ in $\mathbb{S}^2$, we clearly have $\mathrm{TC}_{\mathbb{S}^2}(\mathbf{c}) \le \mathrm{TC}(\mathbf{c})$, but it is false in general that if $\mathrm{TC}_{\mathbb{S}^2}(\mathbf{c}) < \infty$, then also $\mathrm{TC}(\mathbf{c}) < \infty$. If one e.g. takes a curve in $\mathbb{S}^2$ that winds around an equator infinitely many times, its total intrinsic curvature is zero but its length and total curvature are both infinite. Dealing with rectifiable curves $\mathbf{c}$ in $\mathbb{S}^2$, one instead has

$$\mathrm{TC}_{\mathbb{S}^2}(\mathbf{c}) < \infty \iff \mathrm{TC}(\mathbf{c}) < \infty\,.$$

Assume now that $\mathbf{c} : \overline{I}_L \to \mathbb{S}^2 \subset \mathbb{R}^3$ is a rectifiable spherical curve parameterized in arc-length, where $I_L = (0, L)$ and $L = \mathcal{L}(\mathbf{c})$. If $\mathrm{TC}_{\mathbb{S}^2}(\mathbf{c}) < \infty$, then the curve is *one-sidedly smooth* in the sense of [6, Sec. 3.1], i.e., it has a left and a right tangent $\mathbf{T}_\pm(s)$ at all points $\mathbf{c}(s)$ in the so called "strong sense". This implies that for each $s \in I_L$ and $\delta > 0$, we can find $\varepsilon > 0$ such that any secant inscribed in the arc $\mathbf{c}_{|\,[s,s+\varepsilon]}$ forms with the straight line $\mathbf{T}_+(s)$ an angle less than δ, and similarly for the left tangent.

Furthermore, recalling that the tantrix $\mathbf{t}$ is a function of bounded variation, the weak conormal $\mathbf{u} \in \mathrm{BV}(I_L, \mathbb{S}^2)$ is well defined, $\mathbf{u}(s) \in T_{\mathbf{c}(s)}\mathbb{S}^2$ for a.e. $s \in I_L$, and one has

$$D^C\mathbf{t} = \mathbf{u}\,(\mathbf{u} \bullet D^C\mathbf{t})$$

i.e., the Cantor component $D^C\mathbf{t}$ of the distributional derivative of the tantrix is tangential to $\mathbb{S}^2$.

The previous discussion motivates the introduction of the *energy functional*:

$$\mathcal{F}(\mathbf{t}) := \int_0^L |\dot{\mathbf{t}} \bullet \mathbf{u}|\, ds + |D^C\mathbf{t}|(I_L) + \sum_{s\in J_{\mathbf{t}}} d_{\mathbb{S}^2}(\mathbf{t}(s+), \mathbf{t}(s-))\,, \tag{1.4}$$

where, we recall, $\dot{\mathbf{t}} \bullet \mathbf{u}$ is the tangential component of the differential of the tantrix $\mathbf{t} := \dot{\mathbf{c}}$, so that $|\dot{\mathbf{t}}| \ge |\dot{\mathbf{t}} \bullet \mathbf{u}|$. Therefore, by (1.2) we clearly have $\mathcal{F}(\mathbf{t}) \le \mathrm{Var}_{\mathbb{S}^2}(\mathbf{t})$, where strict inequality holds in general.

By exploiting a suitable notion of *weak parallel transport*, and a *generalized Gauss-Bonnet theorem*, in [21] we proved that *for any rectifiable curve $\mathbf{c}$ in $\mathbb{S}^2$ with finite geodesic total curvature, one has:*

$$\mathrm{TC}_{\mathbb{S}^2}(\mathbf{c}) = \mathcal{F}(\mathbf{t})\,.$$

2 Polar to spherical curves

Fenchel [16] in the 1950's exploited the *spherical polarity* of the tangent and binormal indicatrix in order to analyze differential geometric properties of smooth curves in $\mathbb{R}^3$. In his survey, Fenchel proposed a general method that gathers several results on curves in a unified scheme.

In this section, we introduce a notion of *polar* of any rectifiable spherical curve with finite geodesic total curvature, by exploiting polarity along inscribed approximating polygonals. For the sake of simplicity, we only consider closed curves, the case of open curves being treated in a similar way.

Let $\mathbf{p}$ be a closed and oriented spherical polygonal in $\mathbb{S}^2$. Then, the support of $\mathbf{p}$ is the union of m non-trivial geodesic arcs γ_i, where γ_i has initial point z_{i-1} and end point z_i, for $i = 1, \dots, m$, and we let $z_m := z_0$. We denote by $\mathbf{b}_i \in \mathbb{S}^2$ the "north pole" corresponding to the great circle passing through γ_i and with the same orientation as γ_i, and we let $\mathbf{b}_{m+1} := \mathbf{b}_1$. For $i = 1, \dots, m$, we also denote by Γ_i a minimal geodesic arc in $\mathbb{S}^2$ with initial point $\mathbf{b}_i$ and end point $\mathbf{b}_{i+1}$. Note that Γ_i is uniquely determined if $\mathbf{b}_{i+1} \ne -\mathbf{b}_i$.

Definition 2.1 We call *polar* of the spherical polygonal $\mathbf{p}$ the closed oriented curve $\gamma_{\mathbf{p}}$ in $\mathbb{S}^2$ obtained by connecting the consecutive geodesic arcs Γ_i, for $i = 1, \ldots, m$.

The polar of $\mathbf{p}$ is a spherical polygonal. Moreover, since the length of each Γ_i is equal to the geodesic distance between $\mathbf{b}_i$ and $\mathbf{b}_{i+1}$, and hence to the turning angle of $\mathbf{p}$ at the vertex z_i, we have:

$$\mathcal{L}(\gamma_{\mathbf{p}}) = \mathbf{k}_{\mathbb{S}^2}(\mathbf{p}) < \infty \,.$$

Furthermore, since polarity is an involutive transformation, we also have

$$\mathbf{k}_{\mathbb{S}^2}(\gamma_{\mathbf{p}}) = \mathcal{L}(\mathbf{p}) < \infty \,.$$

Using arguments taken from [20], we prove the following

Theorem 2.2 *Let $\mathbf{c}$ be a rectifiable and closed spherical curve with finite (and non-zero) geodesic total curvature, $T := \mathrm{TC}_{\mathbb{S}^2}(\mathbf{c}) < \infty$. There exists a rectifiable curve $\gamma_{\mathbf{c}} : [0, T] \to \mathbb{S}^2$ parameterized by arc-length, so that*

$$\mathcal{L}(\gamma_{\mathbf{c}}) = \mathrm{TC}_{\mathbb{S}^2}(\mathbf{c}) \,,$$

satisfying the following property. For any sequence $\{\mathbf{p}_h\}$ of inscribed closed polygonals, let $\gamma_h : [0, T] \to \mathbb{S}^2$ denote for each h the parameterization with constant velocity of the polar $\gamma_{\mathbf{p}_h}$ of $\mathbf{p}_h$, whence

$$\mathcal{L}(\gamma_h) = \mathcal{L}(\gamma_{\mathbf{p}_h}) = \mathbf{k}_{\mathbb{S}^2}(\gamma_{\mathbf{p}_h}) \quad \forall\, h \,.$$

If $\mu_{\mathbf{c}}(\mathbf{p}_h) \to 0$, then $\gamma_h \to \gamma$ uniformly on $[0, T]$ and $\mathcal{L}(\gamma_h) \to \mathcal{L}(\gamma_{\mathbf{c}})$.

Proof: It is divided into four steps.

Step 1. Choose an optimal sequence $\{\mathbf{p}_h\}$ of polygonal curves inscribed in $\mathbf{c}$ such that $\mu_{\mathbf{c}}(\mathbf{p}_h) \to 0$ and $T_h \to T$, where $T_h := \mathbf{k}_{\mathbb{S}^2}(\mathbf{p}_h)$ and $T = \mathrm{TC}_{\mathbb{S}^2}(\mathbf{c})$. For h large enough so that $T_h > 0$, we have $\mathcal{L}(\gamma_{\mathbf{p}_h}) = T_h$, and we may and do assume that $\gamma_{\mathbf{p}_h} : [0, T_h] \to \mathbb{S}^2$ is parameterized in arc-length.

Define $\gamma_h : [0, T] \to \mathbb{S}^2$ by $\gamma_h(s) := \gamma_{\mathbf{p}_h}((T_h/T)s)$, so that $\|\dot\gamma_h(s)\| = T_h/T$ a.e., where $T_h/T \to 1$. By Ascoli-Arzela's theorem, we can find a (not relabeled) subsequence of $\{\gamma_h\}$ that uniformly converges in $[0, T]$ to some Lipschitz continuous function $\gamma : [0, T] \to \mathbb{S}^2$, and we denote $\gamma = \gamma_{\mathbf{c}}$.

Step 2. We claim that a subsequence of $\{\dot\gamma_h\}$ converges to $\dot\gamma = \dot\gamma_{\mathbf{c}}$ strongly in L^1. As a consequence, we deduce that $\|\dot\gamma_{\mathbf{c}}\| = 1$ a.e. on $[0, T]$, and hence that

$$\mathcal{L}(\gamma_{\mathbf{c}}) = \int_0^T \|\dot\gamma_{\mathbf{c}}(s)\|\, ds = T = \mathrm{TC}_{\mathbb{S}^2}(\mathbf{c}) \,.$$

In order to prove the claim, we let τ_h denote the tantrix of the curve γ_h, so that $\tau_h(s) = \dot\gamma_h(s)/\|\dot\gamma_h(s)\|$ for a.e. s. Then, τ_h has essential total variation in $\mathbb{S}^2$ lower than the sum $\mathcal{L}(\gamma_{\mathbf{p}_h}) + \mathbf{k}_{\mathbb{S}^2}(\gamma_{\mathbf{p}_h})$, that we already know to be equal to the sum $\mathcal{L}(\mathbf{p}_h) + \mathbf{k}_{\mathbb{S}^2}(\mathbf{p}_h)$. Therefore, there exists $\bar h$ such that

$$\mathrm{Var}_{\mathbb{S}^2}(\tau_h) \le 2\big(\mathcal{L}(\mathbf{c}) + \mathrm{TC}_{\mathbb{S}^2}(\mathbf{c})\big) < \infty \,, \quad \forall\, h \ge \bar h \,.$$

As a consequence, recalling that $\|\dot\gamma_h(s)\| \to 1$ for a.e. s, by compactness, a (not relabeled) subsequence of $\{\dot\gamma_h\}$ converges weakly-* in the BV-sense to some BV-function $v : [0, T] \to \mathbb{S}^2$.

We show that $v(s) = \dot\gamma_{\mathbf{c}}(s)$ for a.e. $s \in [0, T]$. This yields that the sequence $\{\dot\gamma_h\}$ converges strongly in L^1 (and hence a.e. on $[0, T]$) to the function $\dot\gamma_{\mathbf{c}}$.

In fact, using that by Lipschitz-continuity

$$\gamma_h(s) = \gamma_h(0) + \int_0^s \dot\gamma_h(\lambda)\, d\lambda \qquad \forall\, s \in [0, T]$$

and setting

$$V(s) := \gamma_{\mathbf{c}}(0) + \int_0^s v(\lambda)\, d\lambda \,, \qquad s \in [0, T]$$

by the weak-* BV convergence $\dot\gamma_h \rightharpoonup v$, which implies the strong L^1 convergence, we have $\gamma_h \to V$ in L^∞, hence $\dot\gamma_h \to \dot V = v$ a.e. on $[0,T]$. But we already know that $\gamma_h \to \gamma_{\mathbf{c}}$ in L^∞, thus we get $v = \dot\gamma_{\mathbf{c}}$.

Step 3. Let now $\{\widetilde{\mathbf{p}}_h\}$ denote any sequence of closed polygonal curves inscribed in $\mathbf{c}$ such that $\mu_{\mathbf{c}}(\widetilde{\mathbf{p}}_h) \to 0$. We claim that possibly passing to a subsequence, the polar curves $\gamma_{\widetilde{\mathbf{p}}_h}$ converge uniformly (up to reparameterizations) to the curve $\gamma_{\mathbf{c}}$.

In fact, we recall that the polar to a polygonal spherical curve $\mathbf{p}$ is defined in terms of vector products of couples of consecutive points of its geodesic segments, the vector product being continuous. Moreover, the Frechét distance between the two sequences $\{\mathbf{p}_h\}$ and $\{\widetilde{\mathbf{p}}_h\}$ goes to zero, since

$$d(\mathbf{p}_h, \widetilde{\mathbf{p}}_h) \le d(\mathbf{p}_h, \mathbf{c}) + d(\widetilde{\mathbf{p}}_h, \mathbf{c})\,.$$

Whence, the polars of $\mathbf{p}_h$ and of $\widetilde{\mathbf{p}}_h$ must converge uniformly (up to reparameterizations) to the same limit function. Therefore, the sequence $\gamma_{\widetilde{\mathbf{p}}_h}$ converges in the Frechét distance to the curve $\gamma_{\mathbf{c}}$ obtained in Step 1.

Step 4. Now, if $\{\widetilde{\mathbf{p}}_h\}$ is the (not relabeled) subsequence obtained in Step 3, by repeating the argument in Step 1 we infer that the limit function $\gamma = \gamma_{\mathbf{c}}$ is unique. As a consequence, a contradiction argument yields that all the sequence $\{\gamma_h\}$ uniformly converges to $\gamma_{\mathbf{c}}$ and that the limit curve $\gamma_{\mathbf{c}}$ does not depend on the choice of the sequence $\{\mathbf{p}_h\}$ of inscribed polygonals satisfying $\mu_{\mathbf{c}}(\mathbf{p}_h) \to 0$. Therefore, the curve $\gamma_{\mathbf{c}}$ is identified by $\mathbf{c}$. Arguing as in Step 2, we finally infer that $\mathcal{L}(\gamma_h) \to \mathcal{L}(\gamma_{\mathbf{c}})$, as required. □

The curve $\gamma_{\mathbf{c}}$ obtained in Theorem 2.2 may be called *polar* to the curve $\mathbf{c}$.

For future use, we finally note that if $\mathbf{c}$ is a geodesically convex, simple, and closed spherical curve, then $\mathcal{L}(\mathbf{c}) + \mathrm{TC}_{\mathbb{S}^2}(\mathbf{c}) < \infty$, and the support of $\mathbf{c}$ is contained in a half sphere of $\mathbb{S}^2$. Each inscribed polygonals $\mathbf{p}_h$ is convex, and hence the polar curves $\gamma_{\mathbf{p}_h}$ are geodesically convex, too. Since moreover $d(\mathbf{p}_h, \mathbf{c}) \to 0$, and the convergence in distance preserves the convexity property, we obtain:

Corollary 2.3 *If* $\mathbf{c}$ *is a closed, simple, and geodesically convex spherical curve, the polar* $\gamma_{\mathbf{c}}$ *is closed and geodesically convex, too.*

3 The generalized normal at conical points

Let Ω be a bounded convex set in $\mathbb{R}^{n+1}$, where $n \ge 2$, with non-empty interior (i.e., a *convex body* in $\mathbb{R}^{n+1}$), and let $O \in \mathrm{int}\,\Omega$. For any point P in the boundary $\partial\Omega$ of Ω, there exists a neighborhood $\mathcal{U}$ of P such that $\mathcal{U} \cap \partial\Omega$ can be viewed as a non-parametric hypersurface.

We choose an orthonormal frame $(x_1, \ldots, x_n, y)$ centered at the point O in such a way that $P = \lambda_P\, e_{n+1}$ for some $\lambda_P > 0$, where $(e_1, \ldots, e_n, e_{n+1})$ is the canonical basis defining the frame. Denote by $x = (x_1, \ldots, x_n)$ the coordinates on $\mathbb{R}^n := \mathrm{span}\{e_1, \ldots, e_n\}$, and by y the coordinate on the "vertical" direction e_{n+1}. Then, letting $B^n_r := \{x \in \mathbb{R}^n \,:\, |x| < r\}$, there exists $r = r_P > 0$ such that

$$\partial\Omega \cap (B^n_r \times]0, +\infty[) = \{(x, y) \in \mathbb{R}^{n+1} \mid x \in B^n_r\,,\ y = u_P(x)\}$$

for some bounded and concave function $u_P : B^n_r \to \mathbb{R}^+$. For each direction v in $\partial B^n_1 := \{v \in \mathbb{R}^n \,:\, |v| = 1\}$, since the differential quotient along radial directions is monotone, the limit

$$\partial_v u_P := \lim_{h \to 0^+} \frac{u_P(h\,v) - u_P(0_{\mathbb{R}^n})}{h} \in \mathbb{R} \tag{3.1}$$

is a real number, and the *tangent cone* Σ_P to $\partial\Omega$ at P is determined by the directional derivatives $\partial_v u_P$.

Precisely, Σ_P is given by the graph of a concave function $\sigma_P : \mathbb{R}^n \to \mathbb{R}$ such that $\sigma_P(0_{\mathbb{R}^n}) = \lambda_P$ and

$$\sigma_P(x) = \lambda_P + \partial_{v_x} u_P\, |x|\,, \quad x \ne 0_{\mathbb{R}^n}\,, \quad v_x := \frac{x}{|x|}\,, \tag{3.2}$$

so that we have:

$$\Sigma_P = \{(0_{\mathbb{R}^n}, \lambda_P) + h\,(v, \partial_v u_P) \mid h \ge 0\,,\ v \in \partial B^n_1\}\,. \tag{3.3}$$

Denote by $\mathcal{H}^k$ the k-dimensional Hausdorff measure in $\mathbb{R}^{n+1}$. Then, for $\mathcal{H}^n$-almost every point P in $\partial\Omega$, the tangent cone Σ_P is an affine n-space, with equation $y = \lambda_P + a_P \bullet x$ in the given frame coordinates, where $a_P \in \mathbb{R}^N$ and $\bullet$ denotes the scalar product. In that case, P is called a regular point, and one has $a_P = \nabla u_P(0_{\mathbb{R}^n})$ and $\partial_v u_P = a_P \bullet v$ for each $v \in \partial B_1^n$.

If P is a ridge point, after a rotation around the e_{n+1}-axis, and using the two alternatives $\pm$ by the sign, we find two vectors $a_P^\pm \in \mathbb{R}^n$ satisfying the compatibility conditions $a_P^- \bullet e_1 \neq a_P^+ \bullet e_1$ and $a_P^- \bullet e_i = a_P^+ \bullet e_i$, $\forall\, i = 2, \ldots, n$, such that $\partial_v u_P = a_P^\pm \bullet v$ for each $v \in \partial^\pm B_1^n$, where

$$\partial^\pm B_1^n := \{v \in \partial B_1^n \mid \pm v \bullet e_1 \geq 0\}\,.$$

Therefore, this time Σ_P is the union of the two half-spaces $\Sigma_P^\pm$ given by

$$\Sigma_P^\pm := \{(x, y) \in \mathbb{R}^{n+1} \mid \pm x \bullet e_1 \geq 0\,,\ y = \lambda_P + a_P^\pm \bullet x\}\,.$$

At regular points P, the outward unit normal to $\partial\Omega$ is given by $\mathbf{n}_P := (-a_P, 1)/\sqrt{|a_P|^2 + 1}$, and it identifies a point in the Gauss hypersphere

$$\mathbb{S}^n := \{z \in \mathbb{R}_z^{n+1} \,:\, |z| = 1\}\,.$$

Instead, at ridge points, two unit "normals" $\mathbf{n}_P^\pm := (-a_P^\pm, 1)/\sqrt{|a_P^\pm|^2 + 1}$ appear, the *normal cone* is degenerate, and it is identified by the geodesic arc connecting the points $\mathbf{n}_P^\pm$ in $\mathbb{S}^n$.

In general, there exists an at most countable set of points P in $\partial\Omega$ that are neither regular nor ridge points: they are called *conical points*.

We wish to describe the normal cone at a conical point P through a *generalized outward unit normal* in the Gauss hyper-sphere $\mathbb{S}^n$, in such a way that the *angle defect* at P is described by the n-dimensional area of the geodesic convex hull in $\mathbb{S}^n$ of the image of our unit normal. We only consider the case of dimension $n = 2$.

3.1 The generalized unit normal

Assume now $n = 2$. With the previous notation, we identify directions in the unit circle ∂B_1^2 with the angle $\theta \in [0, 2\pi[$, so that in polar coordinates $\partial B_1^2 = \{v_\theta := (\cos\theta, \sin\theta) \mid \theta \in [0, 2\pi[\}$. We also denote:

$$v_\theta^\perp := (-\sin\theta, \cos\theta)\,,$$

and note that since $\sin\varepsilon = \varepsilon + o(\varepsilon)$ and $\cos\varepsilon = 1 + o(\varepsilon)$, we have:

$$v_{\theta+\varepsilon} - v_\theta = \varepsilon\, v_\theta^\perp + (o(\varepsilon), o(\varepsilon))\,. \tag{3.4}$$

Proposition 3.1 *For every $\theta \in [0, 2\pi[$, there exists $\alpha_P(\theta) \in \mathbb{R}$ such that*

$$\partial_{v_{\theta+\varepsilon}} u_P = \partial_{v_\theta} u_P + \alpha_P(\theta)\,\varepsilon + o(\varepsilon)\,, \quad \forall\, \varepsilon > 0\,. \tag{3.5}$$

Therefore,

$$\lim_{\varepsilon\to 0^+} \frac{(v_\theta, \partial_{v_\theta} u_P) \times (v_{\theta+\varepsilon}, \partial_{v_{\theta+\varepsilon}} u_P)}{|(v_\theta, \partial_{v_\theta} u_P) \times (v_{\theta+\varepsilon}, \partial_{v_{\theta+\varepsilon}} u_P)|} = \frac{\big(\alpha_P(\theta)\, \sin\theta - \partial_{v_\theta} u_P\, \cos\theta, -\alpha_P(\theta)\, \cos\theta - \partial_{v_\theta} u_P\, \sin\theta, 1\big)}{\sqrt{1 + \partial_{v_\theta} u_P{}^2 + \alpha_P(\theta)^2}}\,.$$

Proof: The function $x \mapsto \sigma_p(x) - \lambda_P = \partial_{v_x} u_P\, |x|$ being concave, by computing the differential quotient at the point $x = 0_{\mathbb{R}^2} + v_\theta$ in the direction of $v_\theta^\perp$, it turns out that for each $\theta \in [0, 2\pi[$, and for $\varepsilon > 0$ small,

$$\varepsilon \mapsto \frac{\partial_{v_{\theta+\varepsilon}} u_P - \cos\varepsilon\, \partial_{v_\theta} u_P}{\sin\varepsilon}$$

is a bounded and monotone function, whence

$$\alpha_P(\theta) := \lim_{\varepsilon\to 0^+} \frac{\partial_{v_{\theta+\varepsilon}} u_P - \cos\varepsilon\, \partial_{v_\theta} u_P}{\sin\varepsilon} \in \mathbb{R}\,,$$

and the first formula readily follows. Let now $(\bar e_1,\bar e_2,\bar e_3)$ be the canonical basis in $\mathbb{R}^3_z$, and denote for simplicity $w(0):=\partial_{v_\theta}u_P$ and $w(\varepsilon):=\partial_{v_{\theta+\varepsilon}}u_P$. For $\varepsilon>0$, we first compute

$$[(v_\theta,w(0))\times(v_{\theta+\varepsilon},w(\varepsilon))]\bullet\bar e_1=\sin\theta(w(\varepsilon)-w(0)\cos\varepsilon)-\cos\theta w(0)\sin\varepsilon\,,$$

$$[(v_\theta,w(0))\times(v_{\theta+\varepsilon},w(\varepsilon))]\bullet\bar e_2=-\cos\theta(w(\varepsilon)-w(0)\cos\varepsilon)-\sin\theta w(0)\sin\varepsilon\,,$$

$$[(v_\theta,w(0))\times(v_{\theta+\varepsilon},w(\varepsilon))]\bullet\bar e_3=\sin\varepsilon\,.$$

Using that $w(\varepsilon)=w(0)+\alpha_P(\theta)\,\varepsilon+o(\varepsilon)$, we get:

$$(v_{\theta+\varepsilon},w(\varepsilon))\times(v_\theta,w(0))=\big((\alpha_P(\theta)\,\sin\theta-\partial_{v_\theta}u_P\,\cos\theta)\varepsilon+o(\varepsilon),(-\alpha_P(\theta)\,\cos\theta-\partial_{v_\theta}u_P)\,\varepsilon+o(\varepsilon),\varepsilon+o(\varepsilon)\big)\,,$$

and hence

$$\lim_{\varepsilon\to0^+}\frac{(v_{\theta+\varepsilon},w(\varepsilon))\times(v_\theta,w(0))}{\varepsilon}=\big(\alpha_P(\theta)\,\sin\theta-\partial_{v_\theta}u_P\,\cos\theta,-\alpha_P(\theta)\,\cos\theta-\partial_{v_\theta}u_P\,\sin\theta,1\big)\,,$$

where

$$|\big(\alpha_P(\theta)\,\sin\theta-\partial_{v_\theta}u_P\,\cos\theta,-\alpha_P(\theta)\,\cos\theta-\partial_{v_\theta}u_P\,\sin\theta,1\big)|=\sqrt{1+\partial_{v_\theta}u_P{}^2+\alpha_P(\theta)^2}\,.$$

The limit readily follows. □

Let now $\varphi_P:\partial B^2_1\to\mathbb{R}^2$ be the function given by

$$\varphi_P(v_\theta):=\big((\alpha_P(\theta),\partial_{v_\theta}u_P)\bullet v_\theta^\perp,\,(\alpha_P(\theta),\partial_{v_\theta}u_P)\bullet v_\theta\big)\,,\quad\theta\in[0,2\pi[\,,\tag{3.6}$$

so that

$$\varphi_P(v_\theta)\bullet v_\theta=\partial_{v_\theta}u_P\,,\quad\varphi_P(v_\theta)\bullet v_\theta^\perp=\alpha_P(\theta)\,.$$

We can re-write equation (3.2) as:

$$\sigma_P(x)=\lambda_P+\varphi_P\Big(\frac{x}{|x|}\Big)\bullet x\,,\quad x\neq0_{\mathbb{R}^2}\,,\tag{3.7}$$

whereas in Proposition 3.1 we have obtained:

$$\lim_{\varepsilon\to0^+}\frac{(v_\theta,\partial_{v_\theta}u_P)\times(v_{\theta+\varepsilon},\partial_{v_{\theta+\varepsilon}}u_P)}{|(v_\theta,\partial_{v_\theta}u_P)\times(v_{\theta+\varepsilon},\partial_{v_{\theta+\varepsilon}}u_P)|}=\frac{(-\varphi_P(v_\theta),1)}{\sqrt{1+|\varphi_P(v_\theta)|^2}}\,.\tag{3.8}$$

Finally, note that we actually have:

$$(v_\theta,\partial_{v_\theta}u_P)\times(v_\theta^\perp,\alpha_P(\theta))=(-\varphi_P(v_\theta),1)\,,\quad\forall\,\theta\in[0,2\pi[\,.\tag{3.9}$$

Definition 3.2 We call *generalized unit normal* the function $\nu_P:[0,2\pi[\to\mathbb{S}^2$ given by:

$$\nu_P(\theta):=\frac{(-\varphi_P(v_\theta),1)}{\sqrt{1+|\varphi_P(v_\theta)|^2}}\,,$$

where $\varphi_P(v_\theta)\in\mathbb{R}^2$ is given by (3.6), with $\partial_{v_\theta}u_P$ the directional derivative in (3.1), and $\alpha_P(\theta)\in\mathbb{R}$ the real number in (3.5).

If P is a ridge point, when $\theta\in[0,\pi[$ we have $\partial_{v_\theta}u_P=a_P^+\bullet v_\theta$, so that by (3.4) we obtain that $\alpha_P(\theta)=a_P^+\bullet v_\theta^\perp$, and hence $\varphi_P(v_\theta)=a_P^+$. Similarly, $\varphi_P(v_\theta)=a_P^-$ when $\theta\in[\pi,2\pi[$, and definitely

$$\nu_P(\theta)=\mathbf{n}_P^\pm\quad\text{if }\pm\cos\theta>0\,.$$

3.2 The polar curve of the normal

Let $P \in \partial\Omega$, where $\Omega \subset \mathbb{R}^3$ is a convex body. Consider the function $\mathbf{c}_P : [0, 2\pi] \to \mathbb{S}^2$ given by

$$\mathbf{c}_P(\theta) := \frac{(v_\theta, \partial_{v_\theta} u_P)}{\sqrt{1 + \partial_{v_\theta} u_P{}^2}}$$

for $\theta \in [0, 2\pi[$, and $\mathbf{c}_P(2\pi) := \mathbf{c}_P(0)$. By the previous construction, it defines a closed, simple, and geodesically convex spherical curve. According to Theorem 2.2, we denote by $\gamma_P = \gamma_{\mathbf{c}_P}$ the *polar of* $\mathbf{c}_P$.

By Corollary 2.3, we already know that γ_P is a closed and geodesically convex spherical curve. In the smooth case, since by (3.5) we get $\partial_\theta[\partial_{v_\theta} u_P] = \alpha_P(\theta)$, using (3.9) we infer that γ_P agrees with the spherical curve $\theta \mapsto \nu_P(\theta)$. More generally, we have:

Theorem 3.3 *Let P be a conical point. Then, the region of the Gauss sphere $\mathbb{S}^2$ enclosed by the polar γ_P of the curve $\mathbf{c}_P$ agrees with the geodesic convex hull of the image*

$$\{\nu_P(\theta) \,:\, \theta \in [0, 2\pi[\} \subset \mathbb{S}^2 \tag{3.10}$$

of the generalized unit normal ν_P.

Proof: Since P is a conical point, the polar γ_P is a simple curve. In fact, if γ_P is not simple (and non-degenerate), being convex its support is equal to a geodesic arc that is parameterized twice. Since polarity is involutive, the support of $\mathbf{c}_P$ has the same structure, and hence P must be a ridge point. We also denote by $\gamma_P : [0, L] \to \mathbb{S}^2$ an arc-length parameterization, where L is the length of γ_P.

For a fixed $\theta \in [0, 2\pi[$, we choose a sequence $\{\mathbf{p}_h\} \subset \mathcal{P}_{\mathbb{S}^2}(\mathbf{c}_P)$ of inscribed closed spherical polygonals with $\mu_{\mathbf{c}_P}(\mathbf{p}_h) \to 0$ and such that $\mathbf{c}_P(\theta)$ is a vertex of each $\mathbf{p}_h$. For any h, we choose $\varepsilon_h > 0$ in such a way that $\mathbf{c}_P(\theta + \varepsilon_h)$ is the vertex following $\mathbf{c}_P(\theta)$ of the polygonal $\mathbf{p}_h$. Since $\mu_{\mathbf{c}_P}(\mathbf{p}_h) \to 0$, by possibly taking a subsequence, we have that $\varepsilon_h \searrow 0$. Therefore, the limit (3.8) and Definition 3.2 give:

$$\lim_{h\to\infty} \frac{(v_\theta, \partial_{v_\theta} u_P) \times (v_{\theta+\varepsilon_h}, \partial_{v_{\theta+\varepsilon_h}} u_P)}{|(v_\theta, \partial_{v_\theta} u_P) \times (v_{\theta+\varepsilon_h}, \partial_{v_{\theta+\varepsilon_h}} u_P)|} = \nu_P(\theta)\,. \tag{3.11}$$

On the other hand, for every h there exists $\lambda_h > 0$ such that

$$(v_\theta, \partial_{v_\theta} u_P) \times (v_{\theta+\varepsilon_h}, \partial_{v_{\theta+\varepsilon_h}} u_P) = \lambda_h\, \mathbf{b}(h)\,,$$

where $\mathbf{b}(h) \in \mathbb{S}^2$ is the polar point of the oriented geodesic arc between $\mathbf{c}_P(\theta)$ and $\mathbf{c}_P(\theta + \varepsilon_h)$. Therefore, on account of Definition 2.1 of polar of $\mathbf{p}_h$, by Theorem 2.2 we infer that the sequence $\{\mathbf{b}(h)\}$ converges to a point in the support of the polar of the spherical curve $\mathbf{c}_P$. Using (3.11), we can thus find $t(\theta) \in [0, L[$ such that

$$\nu_P(\theta) = \gamma_P(t(\theta))\,. \tag{3.12}$$

Up to a translation of the arc-length parameter of γ_P, we may and do assume that $t(0) = 0$. Then, by the convexity of the curve γ_P, it turns out that the parameter function $\theta \mapsto t(\theta)$ is non-decreasing, i.e., for any $0 \le \theta_0 < \theta_1 < 2\pi$, we have $t(\theta_0) \le t(\theta_1)$. Finally, the thesis readily follows from the previous facts. □

As a consequence, we obtain:

Proposition 3.4 *Let P be a conical point. Then, the function $\theta \mapsto \nu_P(\theta)$ has bounded variation, $\nu_P \in \mathrm{BV}((0, 2\pi), \mathbb{S}^2)$. Moreover, if $\nu_P(\theta) \to \nu_P(0)$ as $\theta \to 2\pi^-$, its total variation in $\mathbb{S}^2$ is equal to the length of the spherical polar curve γ_P, i.e.,*

$$\mathrm{Var}_{\mathbb{S}^2}(\nu_P) = \mathcal{L}(\gamma_P)\,. \tag{3.13}$$

Proof: By the monotonicity property of the function $\theta \mapsto t(\theta)$ yielding to equation (3.12), and since γ_P is a convex curve, it turns out that ν_P is a function of bounded variation. If ν_P is not continuous, we choose $\theta_0 \in]0, 2\pi[$ in the at most countable set of discontinuity points of ν_P. Then, denoting by $\nu_P(\theta_0\pm) \in \mathbb{S}^2$ the left and right limits of ν_P at θ_0, it turns out that the geodesic distance between the two points is equal to the length of a geodesic arc lying in the support of γ_P and connecting the points

$$\lim_{t\to t(\theta_0)^-} \gamma_P(t)\,, \qquad \lim_{t\to t(\theta_0)^+} \gamma_P(t)\,.$$

On the other hand, if ν_P is continuous at some closed interval $[\theta_0,\theta_1]$, the total variation $|D\nu_P|(\theta_0,\theta_1)$ is equal to the length of the arc of γ_P with end points $\gamma_P(t(\theta_0))$ and $\gamma_P(t(\theta_1))$. Therefore, equation (3.13) follows from definition (1.2), where $\mathbf{t}=\nu_P$. $\square$

Remark 3.5 If $\nu_P(0)\neq\lim_{\theta\to 2\pi^-}\nu_P(\theta)$, we similarly obtain that the length of γ_P is equal to $\mathrm{Var}_{\mathbb{S}^2}(\nu_P)$ plus the length of the minimal geodesic arc between the points $\lim_{\theta\to 2\pi^-}\nu_P(\theta)$ and $\nu_P(0)$.

Our previous result implies that the polar curve γ_P can be obtained from the generalized unit normal ν_P by means of the same argument leading to the notion of *complete tantrix*, in the sense of Alexandrov-Reshetnyak [6]. To this purpose, we recall that if e.g. $\mathbf{c}$ is a rectifiable open curve in $\mathbb{R}^3$ with finite total curvature, parameterized in arc-length, the approximate derivative $\dot{\mathbf{c}}$ is a function of bounded variation in $BV(I_L,\mathbb{S}^2)$. Then, the complete tantrix $\mathbf{t_c}$ is obtained by connecting with geodesic arcs the points $\dot{\mathbf{c}}(s\pm)\in\mathbb{S}^2$, where $s\in I_L$ is any discontinuity point of $\dot{\mathbf{c}}$, and actually

$$\mathrm{TC}(\mathbf{c})=\mathrm{Var}_{\mathbb{S}^2}(\dot{\mathbf{c}})=\mathcal{L}(\mathbf{t_c})\,.$$

Motivated by Theorem 3.3, we finally give for any $P\in\partial\Omega$ the following

Definition 3.6 We call *generalized normal cone* $\mathcal{N}_P$ the geodesic convex hull of the set (3.10), i.e., the region of the Gauss sphere $\mathbb{S}^2$ enclosed by the polar γ_P of the curve $\mathbf{c}_P$.

4 The angle defect

Let Ω be a convex body in $\mathbb{R}^3$. For any point $P\in\partial\Omega$, we call *angle defect* of $\partial\Omega$ at P the area $\mathcal{AD}(P)$ of the generalized normal cone $\mathcal{N}_P$ given by Definition 3.6, i.e.,

$$\mathcal{AD}(P):=\mathcal{H}^2(\mathcal{N}_P)\,.$$

Note that $\mathcal{AD}(P)=0$ if P is a regular or ridge point, whereas $\mathcal{AD}(P)>0$ if P is a conical point. In that case, in fact, the polar γ_P is a non-degenerate simple curve. More precisely, Theorem 3.3 says that the angle defect is equal to the area of the spherically convex region enclosed by the polar γ_P of the spherical curve $\mathbf{c}_P$ generated by the tangent cone Σ_P to $\partial\Omega$ at P.

Assume now that the function φ_P in (3.7) is smooth. Then, the graph of σ_P is a non-parametric conical surface with zero Gauss curvature outside the point $(0_{\mathbb{R}^2},\lambda_P)$, and the angle defect $\mathcal{AD}(P)$ agrees with the area of the region in $\mathbb{S}^2$ enclosed by the spherical curve ν_P. If e.g. $\sigma_P(x)=\lambda_P-m\,|x|$, where $m>0$, we have $\varphi_P(v_\theta)=-m\,v_\theta$, and hence $\nu_P(\theta)=(m\,v_\theta,1)/\sqrt{1+m^2}$ for each $\theta\in[0,2\pi[$, so that

$$\mathcal{AD}(P)=2\pi\left(1-\frac{1}{\sqrt{1+m^2}}\right)\,.$$

In the smooth case, moreover, if σ_P is not differentiable at the origin, in a weak sense we can say that the Gauss curvature of the graph surface is concentrated at the singular point $(0_{\mathbb{R}^2},\lambda_P)$, where it is described by a Dirac measure δ with a positive weight equal to the angle defect $\mathcal{AD}(P)$.

To this purpose, we recall that if $\sigma:B^2\to\mathbb{R}$ is a smooth function, denoting by $\mathbf{K}$ the Gauss curvature of the graph surface $\mathcal{G}_\sigma$, since $\sqrt{1+|\nabla\sigma(x)|^2}\,dx$ is the area element of the parameterization $x\mapsto(x,\sigma(x))$, by the area formula we have:

$$\int_{\mathcal{G}_\sigma}|\mathbf{K}|\,d\mathcal{H}^2=\int_{B^2}|\mathbf{K}(x,\sigma(x))|\sqrt{1+|\nabla\sigma(x)|^2}\,dx\,,$$

where (compare e.g. [19])

$$\mathbf{K}(x,\sigma(x)) := \frac{1}{(1+|\nabla\sigma(x)|^2)^2}\,\det\begin{pmatrix}\partial_{1,1}\sigma(x) & \partial_{1,2}\sigma(x)\\ \partial_{1,2}\sigma(x) & \partial_{2,2}\sigma(x)\end{pmatrix}, \quad x\in B^2 .$$

With the previous notation, we in fact have:

Proposition 4.1 *Assume that the function φ_P in (3.7) is smooth. Then, there exists a smooth sequence $\{\sigma_k\}\subset C^\infty(B^2)$ such that $\sigma_k\to\sigma_P$ uniformly in B^2, and*

$$|\mathbf{K}(x,\sigma_k(x))|\sqrt{1+|\nabla\sigma_k(x)|^2}\,\mathcal{L}^2 \llcorner B^2 \rightharpoonup \mathcal{AD}(P)\,\delta_{0_{\mathbb{R}^2}}$$

weakly- in the sense of the measures, as $k\to\infty$. Moreover,*

$$\lim_{k\to\infty}\int_{\mathcal{G}_{\sigma_k}} |\mathbf{K}|\,d\mathcal{H}^2 = \mathcal{AD}(P)\,. \tag{4.1}$$

Proof: Let $\psi:[0,+\infty)\to\mathbb{R}$ be a weakly increasing, smooth and convex function such that $\psi(\rho)=0$ if $0\le\rho\le 1/4$, and $\psi(\rho)=\rho-1/2$ if $\rho\ge 1$. For $k\in\mathbb{N}^+$, define $\sigma_k(0_{\mathbb{R}^2})=\lambda_P$ and

$$\sigma_k(x) = \lambda_P + \frac{1}{k}\,\psi(k|x|)\,\varphi_P\Big(\frac{x}{|x|}\Big)\bullet\frac{x}{|x|}\,, \quad x\in B^2\setminus\{0_{\mathbb{R}^2}\}\,.$$

Each σ_k is a smooth function, and by equation (3.7) we readily estimate

$$\|\sigma_k-\sigma_P\|_{\infty,B^2} \le \frac{1}{2k}\,\max\{|\varphi_P(v_\theta)\bullet v_\theta| \,:\, \theta\in[0,2\pi[\}\quad \forall\,k\,,$$

so that uniform convergence follows from the smoothness of the function φ_P.

Since moreover $\psi(k\rho)=0$ if $\rho\le 1/(4k)$, the Gauss curvature of the graph surface $\mathcal{G}_{\sigma_k}$ is possibly non zero only at points $(x,\sigma_k(x))$ such that $1/(4k)\le|x|\le 1$. By the area formula we thus obtain

$$\int_{\mathcal{G}_{\sigma_k}} |\mathbf{K}|\,d\mathcal{H}^2 = I_k + II_k\,,$$

where, letting $A_k := B^2_{1/k}\setminus B^2_{1/(4k)}$ and $B_k := B^2\setminus B^2_{1/k}$, we have denoted

$$I_k := \int_{A_k} |\mathbf{K}(x,\sigma_k(x))|\sqrt{1+|\nabla\sigma_k(x)|^2}\,dx\,,\quad II_k := \int_{B_k} |\mathbf{K}(x,\sigma_k(x))|\sqrt{1+|\nabla\sigma_k(x)|^2}\,dx\,.$$

Recalling that σ_P is concave and ψ convex, it turns out that σ_k is a concave function on A_k. As a consequence, the integral I_k is equal to the area of the geodesically convex set enclosed by the unit normal $\mathbf{n}$ to the graph $\mathcal{G}_{\sigma_k}$ computed at points $(x,\sigma_k(x))$, where $|x|=1/k$. On the other hand, using polar coordinates we infer that

$$\mathbf{n}(x,\sigma_k(x)) = \frac{(-\varphi_P(v_\theta),1)}{\sqrt{1+|\varphi_P(v_\theta)|^2}}\,,\quad \text{if}\quad x=\frac{1}{k}\,v_\theta\,,\quad \theta\in[0,2\pi[\,.$$

Therefore, by Definition 3.2 of generalized unit normal, the latter geodesically convex set is equal to the normal cone $\mathcal{N}_P$, see Definition 3.6, and hence its area agrees with the angle defect $\mathcal{AD}(P)$, so that

$$I_k = \mathcal{AD}(P)\quad\forall\,k\,.$$

Moreover, using that $k^{-1}\psi(k\rho)=\rho-1/(2k)$ for $1/k\le\rho\le 1$, we obtain

$$II_k = \frac{1}{4k}\int_{1/k}^1\left(\int_0^{2\pi}\frac{f'(\theta)^2/(k\rho)}{\big(1+f(\theta)^2+\big(1-1/(2k)\big)^2 f'(\theta)^2\big)^{3/2}}\,d\theta\right)d\rho$$

and hence, by the smoothness of f and by dominated convergence we infer that $II_k\to 0$ as $k\to\infty$, so that the limit (4.1) holds true. Since the weak-* convergence follows from standard arguments of measure theory (see [8]), the proof is complete. □

More generally, we finally obtain:

Theorem 4.2 *For every $P \in \partial\Omega$, there exists a smooth sequence $\{\sigma_k\} \subset C^\infty(B^2)$ such that $\sigma_k \to \sigma_P$ uniformly in B^2 and the limit (4.1) holds.*

Proof: Since $\partial_{v_\theta} u_P \to \partial_{v_0} u_P$ as $\theta \nearrow 2\pi$, we can extend $\theta \mapsto \partial_{v_\theta} u_P$ to a continuous and 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$ such that $f(\theta) = \partial_{v_\theta} u_P$ if $\theta \in [0, 2\pi[$. By the proof of Proposition 3.1, we infer that f has bounded variation on each bounded open interval, and that f has a bounded and 2π-periodic right derivative $f'_+(\theta)$ for each $\theta \in \mathbb{R}$, with $f'_+(\theta) = \alpha(\theta)$ for $\theta \in [0, 2\pi[$. In a similar way, we obtain that f has a bounded and 2π-periodic left derivative $f'_-(\theta)$ for each $\theta \in \mathbb{R}$. Therefore, f is a Sobolev map in $W^{1,1}_{\text{loc}}(\mathbb{R})$, with derivative $f'(\theta)$ equal to $\alpha_P(\theta)$ for $\mathcal{L}^1$-almost every $\theta \in]0, 2\pi[$.

Let $\phi_h(\alpha) := h\,\phi(h\alpha)$ for $h \in \mathbb{N}^+$, where $\phi : \mathbb{R} \to \mathbb{R}$ is a standard symmetric convolution kernel, i.e., $\phi(-\alpha) = \phi(\alpha) \geq 0$ for each α, $\phi(\alpha) = 0$ if $|\alpha| \geq 1$, and $\int_{-1}^1 \phi(\alpha)\, d\alpha = 1$. Setting

$$f_h(\theta) := \int_{\mathbb{R}} \phi_h(\alpha)\, f(\theta - \alpha)\, d\alpha\,,$$

we find a sequence $\{f_h\} \subset C^\infty(\mathbb{R})$ of 2π-periodic smooth functions converging to f uniformly in $\mathbb{R}$ and such that

$$\lim_{h\to\infty} \int_0^{2\pi} |f'_h(\theta)|\, d\theta = \int_0^{2\pi} |\alpha_P(\theta)|\, d\theta\,,$$

where by the Sobolev regularity of f

$$f'_h(\theta) = \int_{\mathbb{R}} \phi_h(\alpha)\, f'(\theta - \alpha)\, d\alpha\,, \quad \forall\, \theta \in \mathbb{R}\,. \tag{4.2}$$

According to (3.7), we then define

$$\sigma_{P,h}(x) := \lambda_P + \varphi_{P,h}\Big(\frac{x}{|x|}\Big) \bullet x\,, \quad x \neq 0_{\mathbb{R}^2}\,,$$

where as in (3.6) we have set

$$\varphi_{P,h}(v_\theta) := \big((f_h(\theta), f'_h(\theta)) \bullet v_\theta^\perp,\, (f_h(\theta), f'_h(\theta)) \bullet v_\theta\big)\,, \quad \theta \in [0, 2\pi]\,,$$

so that $\varphi_{P,h}(v_\theta) \bullet v_\theta = f_h(\theta)$ and $\varphi_{P,h}(v_\theta) \bullet v_\theta^\perp = f'_h(\theta)$ for each θ, whereas according to (3.9)

$$(v_\theta, f_h(\theta)) \times (v_\theta^\perp, f'_h(\theta)) = (-\varphi_{P,h}(v_\theta), 1)\,, \quad \forall\, \theta \in [0, 2\pi]\,.$$

Therefore, the previous argument yields to the polar curve parameterized by the smooth function

$$\nu_{P,h}(\theta) := \frac{(-\varphi_{P,h}(v_\theta), 1)}{\sqrt{1 + |\varphi_{P,h}(v_\theta)|^2}}\,, \quad \theta \in [0, 2\pi]\,.$$

Moreover, by the $W^{1,1}_{\text{loc}}$ convergence we infer that the sequence $\{\nu_{P,h}\}$ converges in the sense of the Frechét distance to the polar curve γ_P. As a consequence, denoting by $\mathcal{N}_{P,h}$ the region enclosed by the curve $\nu_{P,h}$, and by $\mathcal{AD}(P, h)$ its area, we obtain that $\mathcal{AD}(P, h) \to \mathcal{AD}(P)$ as $h \to \infty$.

We thus have to show that $\sigma_{P,h}$ is a concave function for every $h \in \mathbb{N}^+$. In fact, by applying Proposition 4.1 to the smooth function $\varphi_{P,h}$, the assertion follows through a diagonal argument.

Now, by the concavity and Sobolev regularity of the function σ_P, the graph of σ_P lies below the tangent space T_{x_0} to σ_P at $(x_0, \sigma_P(x_0))$, for $\mathcal{L}^2$-almost every $x_0 \in \mathbb{R}^2$. In polar coordinates, for every $\rho > 0$ and a.e. θ, with $x_0 = \rho\, v_\theta$, the tangent space T_{x_0} is given by the graph of the function

$$\widetilde{\rho}\, v_{\widetilde{\theta}} \mapsto \rho\, f(\theta) + \big(f(\theta)\, v_\theta + f'(\theta)\, v_\theta^\perp\big) \bullet (\widetilde{\rho}\, v_{\widetilde{\theta}} - \rho\, v_\theta) = \big(f(\theta)\, v_\theta + f'(\theta)\, v_\theta^\perp\big) \bullet \widetilde{\rho}\, v_{\widetilde{\theta}}\,.$$

Therefore, property $\sigma_P(x) \leq T_{x_0}(x)$ is equivalent to

$$f(\theta + \varepsilon) \leq \big(f(\theta)\, v_\theta + f'(\theta)\, v_\theta^\perp\big) \bullet v_{\theta+\varepsilon}\,,$$

where we have taken $\varepsilon = \widetilde{\theta} - \theta$. We thus obtain the validity of inequality

$$f(\theta + \varepsilon) - f(\theta)\cos\varepsilon - f'(\theta)\sin\varepsilon \leq 0$$

for a.e. $\theta \in \mathbb{R}$ and for all $\varepsilon \in \mathbb{R}$. Now, using (4.2), for each h we have

$$f_h(\theta + \varepsilon) - f_h(\theta)\cos\varepsilon - f_h'(\theta)\sin\varepsilon = \int_{\mathbb{R}} \phi_h(\alpha)\big(f(\theta - \alpha + \varepsilon) - f(\theta - \alpha)\cos\varepsilon - f'(\theta - \alpha)\sin\varepsilon\big)\, d\alpha \leq 0$$

for every θ, ε, and h, which implies that the graph of $\sigma_{P,h}$ lies below the tangent space to $\sigma_{P,h}$ at $(x_0, \sigma_{P,h}(x_0))$, for every $x_0 \in \mathbb{R}^2$, and hence the required concavity of the functions $\sigma_{P,h}$. □

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