Asymptotic location and shape of the optimal favorable region in a Neumann spectral problem

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Abstract

We complete the study concerning the minimization of the positive principal eigenvalue associated with a weighted Neumann problem settled in a bounded regular domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, for the weight varying in a suitable class of sign-changing bounded functions. Denoting with u the optimal eigenfunction and with D its super-level set, corresponding to the positivity set of the optimal weight, we prove that, as the measure of D tends to zero, the unique maximum point of u, $P \in \partial\Omega$, tends to a point of maximal mean curvature of $\partial\Omega$. Furthermore, we show that D is the intersection with Ω of a $C^{1,1}$ nearly spherical set, and we provide a quantitative estimate of the spherical asymmetry, which decays like a power of the measure of D.

These results provide, in the small volume regime, a fully detailed answer to some long-standing questions in this framework.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a (open and connected) domain, $D \subset \Omega$ be a measurable subset of positive measure, $\beta > 0$ be a given constant, and let us consider the principal eigenvalue

$$\lambda(D) = \lambda(D, \Omega) := \inf \left\{ \int_\Omega |\nabla u|^2 \, dx : u \in H^1(\Omega), \ \int_D u^2 - \beta \int_{\Omega \setminus D} u^2 \, dx = 1 \right\}, \tag{1.1}$$

associated to the indefinite weighted Neumann eigenvalue problem

$$\begin{cases} -\Delta u = \lambda m_D u & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \qquad \text{where } m_D := \mathbb{1}_D - \beta \mathbb{1}_{\Omega \setminus D}. \tag{1.2}$$

Throughout this paper we mainly deal with a fixed domain Ω, bounded and with regular boundary (at least $C^{3,\theta}$, for some $\theta > 0$, although for a relevant part of our results $C^{2,1}$ is enough), and we omit the dependence of λ on Ω in case no confusion arises. For such an Ω, $\lambda(D, \Omega)$ is strictly positive, and achieved by a strictly positive non-constant eigenfunction, if and only if m_D has negative average (thus avoiding $\lambda(D) = 0$, with constant eigenfunction). This condition translates in terms of the Lebesgue measure of D as

$$0 < |D| =: \delta < \frac{\beta |\Omega|}{\beta + 1}. \tag{1.3}$$

For any such δ, let us consider the shape optimization problem

$$\Lambda(\delta) = \min\left\{\lambda(D) : D \subset \Omega,\ \text{measurable},\ |D| = \delta\right\}. \tag{1.4}$$

It has been proved in [18] that $\Lambda(\delta)$ is achieved for every δ enjoying (1.3) by an optimal shape D_δ, with associated positive eigenfunction $u_\delta \in C^{1,\alpha}(\overline{\Omega})$, normalized in $L^2(\Omega)$. In particular, u_δ satisfies (1.2) with $\lambda = \Lambda(\delta)$ and $D = D_\delta$, and moreover D_δ is a superlevel set of u_δ, for a suitable choice of the level.

In this paper we pursue the analysis started in [21], concerning the asymptotic location and shape of the optimal set D_δ, in the small volume regime $\delta \to 0$ (in which (1.3) is always satisfied). In particular, we provide a complete answer to some questions which were left open in [21].

Before recalling the results contained in [21] and outlining our contributions here, let us briefly describe one main motivation to investigate (1.4), which comes from population dynamics and is related to the optimal design of an habitat to enhance the chances of persistence of a species. Consider a population of density $u = u(x,t)$ which disperses in an insulated region Ω according to a Cauchy-Neumann reaction-diffusion model of logistic type:

$$\begin{cases} u_t - d\Delta u = m(x)u - u^2 & x \in \Omega,\ t > 0, \\ u = u_0 \geq 0 & x \in \overline{\Omega},\ t = 0 \\ \partial_\nu u = 0 & x \in \partial\Omega,\ t > 0 \end{cases}$$

(see [5, 2]). In such a model, favorable and hostile zones of the heterogeneous habitat respectively correspond to positivity and negativity sets of the sign-changing weight $m \in L^\infty(\Omega)$, while the constant motility rate $d > 0$ encodes the intensity of diffusion. It is well known (see e.g. [2]) that persistence of the population, for every nontrivial initial datum u_0, is equivalent to the existence of a positive steady state, which in turn is equivalent to the strict negativity of the (non-weighted) principal eigenvalue $\tilde{\lambda}_1 = \tilde{\lambda}_1(m,d)$ of the eigenvalue problem

$$\begin{cases} -d\Delta u - m(x)u = \tilde{\lambda} u & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

On the other hand, using the monotonicity of $\tilde{\lambda}_1(m,d)$ with respect to d, it is not difficult to see that, for every fixed habitat $m \in L^\infty(\Omega)$, there exists a threshold motility $d^* = d^*(m) \in [0, +\infty]$ such that

$$\tilde{\lambda}_1(m,d) < 0 \qquad \Longleftrightarrow \qquad d < d^*(m).$$

In particular, since d^* is defined by $\tilde{\lambda}_1(m, d^*(m)) = 0$, we obtain that the corresponding positive eigenfunction φ^* satisfies

$$\begin{cases} -\Delta\varphi^* = \dfrac{1}{d^*(m)} m\varphi^* & \text{in } \Omega, \\ \partial_\nu \varphi^* = 0 & \text{on } \partial\Omega, \end{cases}$$

so that $\lambda_1(m) := \dfrac{1}{d^*(m)}$ is the positive principal eigenvalue of (1.2) (with m instead of m_D).

Since the population persists if and only if $d < d^*(m)$, it is natural to be interested in maximizing such threshold, i.e. minimizing $\lambda_1(m)$, with respect to m or other relevant parameters of the problem. In this direction, there is a large variety of contributions in the literature (see e.g. [25, 3, 1, 20, 11, 24, 23] and references therein), for instance including other possible types of diffusion or addressing different but related optimization problems, also appearing in the context of composite membranes (see [7, 8] and the review [19]). On the other hand, one of the most considered problem is the minimization of $\lambda_1(m)$ for m in a class of weights $\mathcal{M}$ which fixes the total resources $\int_\Omega m$, as well as pointwise lower and upper bounds $-\beta \le m \le 1$, see e.g. [5, 18, 17]. In such a case, it has been shown in [18] that the infimum of $\lambda_1(m)$ is achieved by a bang-bang (piecewise constant) optimal weight $m = m_D$, as in (1.2), for some measurable set $D \subset \Omega$, which represents the favorable zone of the optimal habitat. Therefore, the optimal design problem for the survival threshold of the population reduces to (1.4), where δ is prescribed by the fixed average of m. Moreover, any optimal set D_δ is a superlevel set of the associated positive eigenfunction u_δ, so that the location of D_δ is somewhat related to that of maximum points of u_δ.

While the picture is nowadays completely clear in dimension $N = 1$, not very much is known about the properties of D_δ, u_δ, in dimension $N \ge 2$; several conjectures and open questions have been formulated in the literature [6, 2, 18, 20, 19] about the shape and location of D_δ, also in the case of other boundary conditions. Up to our knowledge, the only results for general domains Ω are contained in [21], in the small volume regime, inspired by techniques in the framework of concentration results for semilinear problems [22]. More precisely, the results in [21] are obtained through a blow-up procedure, in connection with the limit problem

$$\begin{aligned} I &:= \min\left\{\lambda(A, \mathbb{R}^N_+) : A \subset \mathbb{R}^N_+,\ \text{measurable},\ |A| = 1\right\} \\ &= \min\left\{\lambda(A, \mathbb{R}^N) : A \subset \mathbb{R}^N,\ \text{measurable},\ |A| = 2\right\}, \end{aligned} \tag{1.6}$$

where $\mathbb{R}^N_+ := \mathbb{R}^N \cap \{x : x_N > 0\}$ denotes the N-dimensional upper half-space. It has been shown in [21, Sec. 2], by reflection and symmetrization, that I is achieved by a half ball, centered at the boundary of $\mathbb{R}^N_+$, with an associated radial and radially decreasing eigenfunction. Precisely, we have that

$$I = \lambda(B^+_{r_2}, \mathbb{R}^N_+) = \lambda(B_{r_2}, \mathbb{R}^N), \tag{1.7}$$

where $B_{r_2} \subset \mathbb{R}^N$ denotes the ball of measure 2, centered at the origin, and r_2 denotes its radius. Moreover such minimizer is unique up to translations along $\partial\mathbb{R}^N_+$, and in turn $\lambda(B^+_{r_2}, \mathbb{R}^N_+)$ is achieved by (the restriction to $\mathbb{R}^N_+$ of) $w \in H^1_{\mathrm{rad}}(\mathbb{R}^N) \cap C^{1,1}(\mathbb{R}^N)$, solution of

$$-\Delta w = I m w \quad \text{in } \mathbb{R}^N, \qquad \text{where } m := \mathbb{1}_{B_{r_2}} - \beta \mathbb{1}_{\mathbb{R}^N \setminus B_{r_2}}. \tag{1.8}$$

Here $w > 0$ is radially symmetric and radially decreasing; as a matter of fact, w is explicit in terms of Bessel functions, and it decays exponentially at infinity:

$$|w(x)| + |\nabla w(x)| \sim C|x|^{-(N-1)/2} e^{-\sqrt{I\beta}|x|} \qquad \text{as } |x| \to +\infty. \tag{1.9}$$

In the next statement we collect the part of the results obtained in [21] which is relevant for the present discussion. To this aim, for any $P \in \partial\Omega$ we denote with H_P the mean curvature of $\partial\Omega$ at P.

Theorem 1.1 ([21, Thms. 1.2, 1.6])**.** *Let $\partial\Omega$ be of class $C^{2,1}$. There exists $\delta_0 > 0$ such that, for every $\delta \in (0, \delta_0)$:*

1. *u_δ has a unique local maximum point $P_\delta \in \partial\Omega$;*
2. *D_δ is connected.*

Moreover, as $\delta \to 0$,

3. *for any choice of* $r_{\pm}(\delta)$ *such that* $|B_{r_{\pm}(\delta)}| = 2\delta(1 \pm o_\delta(1))$,

$$B_{r_-(\delta)}(P_\delta) \cap \Omega \;\subset\; D_\delta \;\subset\; B_{r_+(\delta)}(P_\delta) \cap \Omega;$$

4. *for a universal constant* $\Gamma > 0$, *explicit in terms of* w *(see equation (3.1)),*

$$I\delta^{-2/N}(1 + o_\delta(1)) \le \Lambda(\delta) \le I\delta^{-2/N}\left(1 - \Gamma H_P \delta^{1/N} + o(\delta^{1/N})\right),$$

for any $P \in \partial\Omega$.

In particular, for δ small, the optimal set D_δ is connected, it intersects $\partial\Omega$, and it roughly looks as the intersection of Ω with a shrinking ball centered at moving points $P_\delta \in \partial\Omega$. In this respect, the main questions left open in [21] concern the asymptotic location of P_δ as $\delta \to 0$, as well as more quantitative information about the asymptotic shape of D_δ.

Our aim here is twofold. First of all, we are now able to obtain an exact expansion of the optimal eigenvalue with respect to δ, which allows us to detect the location of P_δ at the points of highest mean curvature of $\partial\Omega$, as stated in our first main result.

Theorem 1.2. *With the same assumptions and notations of Theorem 1.1, we have that, as* $\delta \to 0$,

1. $H_{P_\delta} \to \max\{H_P : P \in \partial\Omega\} =: \widehat{H}$;
2. $\Lambda(\delta) = I\delta^{-2/N}\left(1 - \Gamma \widehat{H} \delta^{1/N} + o(\delta^{1/N})\right)$.

The proof of Theorem 1.2 is based on the asymptotic expansion of a suitable Rayleigh quotient related to $\Lambda(\delta)$. In performing this expansion one cannot take advantage of any kind of linearization argument, due to the presence of the discontinuous weight. However, exploiting the analogies with concentration results for singularly perturbed semilinear elliptic equations, obtained in [9, 10], we can show a refined exponential decay of u_δ. This allows us to obtain the desired second order expansion of $\Lambda(\delta)$ yielding the information in Theorem 1.2.

Next, we focus on the asymptotic shape of the optimal set D_δ and we show that it is nearly spherical in a quantitative way, as in the following result.

Theorem 1.3. *Let* $\partial\Omega$ *be of class* $C^{3,\theta}$, *for some* $\theta > 0$. *There exists* $\overline{\delta} > 0$ *such that, for every* $\delta \in (0, \overline{\delta})$, *there exists* $Q_\delta \in \partial\Omega$, $\rho_\delta \in C^{1,1}(\mathbb{S}^{N-1})$, *such that*

$$D_\delta = \left\{ x \in \Omega : |x - Q_\delta| < \delta^{1/N}\left(r_2 + \rho_\delta\left(\frac{x - Q_\delta}{|x - Q_\delta|}\right)\right)\right\} \tag{1.10}$$

where r_2 *is the radius of the ball of measure* 2.

Moreover, as $\delta \to 0$, $\delta^{-1/N}|P_\delta - Q_\delta| \to 0$ *and*

1. $\|\rho_\delta\|_{C^{1,1}} \to 0$;
2. $\|\rho_\delta\|_{L^2} = o(\delta^{1/2N})$.

The proof of this theorem is based on the expansion of $\Lambda(\delta)$ obtained in Theorem 1.2 and on sharp quantitative estimates about problem (1.6), adapted from [14], where the analogous problem is considered in the case of Dirichlet boundary conditions. Such adaptation is highly nontrivial and technically demanding: in the Neumann case concentration happens at the boundary, and quantitative estimates are strongly affected by the local regularity and geometry of $\partial\Omega$. In particular, the point Q_δ can be seen as a sort of projection on $\partial\Omega$ of the barycenter of

D_δ. Its precise definition is provided in Lemma 4.2 ahead, and its role is to prevent translations of optimizers associated with (1.6). Moreover, we remark that in principle the eigenfunctions u_δ, for which D_δ is a superlevel set, are naturally uniformly bounded only in $C^{1,\alpha}$, $\alpha < 1$; the $C^{1,1}$ control in the above theorem is more delicate to be obtained, and it requires the use of recent results about the regularity of transmission problems, see [4, 12]. Actually, this is the only part of the argument which requires the further regularity assumptions on $\partial\Omega$; on the other hand, such further regularity should reflect also on that of the free boundary, see Remark 4.9. Furthermore, by Gagliardo-Nirenberg inequality, it is possible to combine the estimates in Theorem 1.3 to obtain quantitative information concerning the rate of decay of the $C^{1,\alpha}$ norm of ρ_δ, for $\alpha < 1$ (see Remark 5.4):

$$\|\rho_\delta\|_{C^{1,\alpha}} = o\left(\delta^{\frac{(1-\alpha)}{N(4+N)}}\right) \quad \forall\, \alpha \in (0,1).$$

In view of Theorems 1.2, 1.3, we have a fairly complete picture about the shape and location of the optimal favorable shape D_δ, in the small volume regime, in the case of Neumann boundary conditions. As we mentioned, in the case of Dirichlet boundary conditions, the same issues have been recently faced in [14]: also in this case the optimal shape is $C^{1,1}$ nearly spherical, but asymptotically located at a point inside Ω with maximal distance from the boundary. As far as the asymptotic location and shape is concerned, these results completely agree with those in dimension $N = 1$: it is well known that, for any δ, the optimal shape D_δ is an interval positioned at the boundary of the interval Ω (for Neumann boundary conditions) or at its center (for Dirichlet ones). From this perspective, these results also provide positive answers to questions raised in the literature [6, 18, 19], which in turn were partially motivated by the above mentioned one-dimensional description.

On the other hand, the quantitative estimates of the spherical asymmetry show the emergence of a phenomenon which is peculiar to the case of dimension $N \geq 2$. Indeed, in dimension $N = 1$, both Ω and D_δ are intervals, i.e. balls, with no spherical asymmetry. In dimension $N \geq 2$, this is possible only if Ω is a ball and Dirichlet boundary conditions are assumed, see [17]. For general Ω, or for Neumann conditions, the spherical asymmetry is non-trivial, and it exhibits very different decay rates. Indeed, in the Dirichlet case, the decay fully inherits that of the solution of the limit problem, namely of exponential rate; in the case of Neumann boundary conditions, it is triggered by the geometry and the regularity of the boundary $\partial\Omega$. This phenomenon enlighten once again analogies with the semilinear case.

Structure of the paper. Section 2 is devoted to the proof of the exponential decay of the eigenfunction u_δ and of w_δ, its counterpart in the blow-up configuration. Thanks to this information, in Section 3 we can prove Theorem 1.2, in particular the bound from below in the expansion of $\Lambda(\delta)$. Section 4 is devoted to prove that D_δ is nearly spherical and to show that its parametrization is actually $C^{1,1}$, exploiting the regularity results for transmission problems. Finally, in Section 5 we use the quantitative estimates from [14] to conclude the proof of Theorem 1.3.

Notation.

- $|\cdot|$ denotes the Lebesgue N dimensional measure and $\mathcal{H}^{N-1}(\cdot)$ the Hausdorff $N-1$ dimensional measure.
- For a function f, its positive/negative parts are denoted as $f^{\pm}(x) = \max\{\pm f(x), 0\}$.
- The characteristic function of a set E is denoted by $\mathbb{1}_E$.

- $B_r(x)$ denotes the ball of radius $r > 0$ centered at $x \in \mathbb{R}^N$. If $x = 0$, we often write $B_r = B_r(0)$. We call $\omega_N = |B_1|$ the measure of a ball of radius 1.
- We denote with $r_2 > 0$ the radius of the ball B_{r_2}, such that $|B_{r_2}| = 2$.
- $\mathbb{R}^N_+ = \mathbb{R}^{N-1} \times \mathbb{R}_+$, $B_r^+ = B_r \cap (\mathbb{R}^N_+)$.
- H_P denotes the mean curvature of $\partial\Omega$ at $P \in \partial\Omega$, and $\widehat{H} = \max_{P\in\partial\Omega} H_P$.
- Given a set $A \subset \mathbb{R}^N$, we denote $\mathrm{bar}(A)$ its barycenter.
- $C, C_1, C', \ldots$ denote any (non-negative) universal constant, which may also change from line to line.

2 Uniform exponential decay of the eigenfunctions

Let us recall that, for $\delta > 0$ small, u_δ denotes the positive, L^2-normalized eigenfunction associated with $\Lambda(\delta) = \lambda(D_\delta, \Omega)$, with unique maximum point $P_\delta \in \partial D_\delta \cap \partial\Omega$.

The starting point of this section is the following version of a result which was obtained in [21].

Theorem 2.1 ([21, Proposition 4.11]). *With the notation above, for all $\eta > 0$ there exist $\delta_0 > 0$, $\widehat{C} > 0$ such that, for all $\delta \in (0, \delta_0)$, there is a subdomain $\Omega_\delta^{(i)} \subset \Omega$ satisfying:*

(i) $P_\delta \in \partial\Omega_\delta^{(i)}$ *and* $\mathrm{diam}(\Omega_\delta^{(i)}) \le \widehat{C}\delta^{1/N}$,

(ii) $\|u_\delta\|_{L^\infty(\Omega)} \le C^*\delta^{-1/2}$,

(iii) $|u_\delta(x)| \le C^*\delta^{-1/2}\eta e^{-\frac{\mu_1 d(x)}{\delta^{1/N}}}$, *for all* $x \in \Omega \setminus \Omega_\delta^{(i)}$,

where $d(x) := \min\{\mathrm{dist}(x, \partial\Omega_\delta^{(i)}), \eta_0\}$ *and* C^*, μ_1, η_0 *are positive constants depending only on* Ω.

With a careful look at the proof of [22, Theorem 2.3] and using the same approach of [10, Lemma 2.3], we can prove the following decay result for u_δ.

Theorem 2.2. *There exist universal constants $C_1, C_2 > 0$ and $\delta_0 > 0$ such that, for all $0 < \delta < \delta_0$,*

$$u_\delta(x) \le C_1\delta^{-1/2}e^{-C_2|x-P_\delta|\delta^{-1/N}}, \qquad \text{for all } x \in \Omega. \tag{2.1}$$

Proof. Step 1. For all $\varepsilon > 0$ there is $R = R(\varepsilon) > 0$ such that, for δ sufficiently small we have $\delta^{1/2}|u_\delta(x)| \le \varepsilon$ for $|x - P_\delta| > R\delta^{1/N}$.

This follows thanks to Theorem 2.1, choosing $\eta = \frac{\varepsilon}{C^*}$. In fact, thanks to (i), we deduce that if $|x - P_\delta| > R\delta^{1/N}$ for $R > 2\widehat{C}$ (independent of δ), then $x \in \Omega \setminus \Omega_\delta^{(i)}$. Eventually, thanks to (iii), we conclude the claim, since the exponential is for sure less than or equal to 1.

Step 2. There exist $R_0 > 0$ and $\nu_0 > 0$ such that for all $R > R_0$ and δ sufficiently small we have

$$\sup_{|x-P_\delta|>R\delta^{1/N}} \delta^{1/2}u_\delta(x) \ge 2 \sup_{|x-P_\delta|>(R+\nu_0)\delta^{1/N}} \delta^{1/2}u_\delta(x).$$

Let us assume, for the sake of contradiction, that there are sequences $R_n \to +\infty$, $\nu_n \to +\infty$, $\delta_n \to 0$ and $x_n \in \Omega$ such that $|x_n - P_n| \geq (R_n + \nu_n)\delta^{1/N}$, being $P_n \in \partial\Omega$ the maximum point for u_{δ_n} and

$$\delta_n^{1/2} u_{\delta_n}(x_n) = \mu_n > \frac{M_n}{2}, \qquad M_n = \sup_{|x-P_n|>R_n\delta_n^{1/N}} \delta_n^{1/2} u_{\delta_n}(x).$$

For the uniform decay proved in Step 1, we deduce that $\mu_n, M_n \to 0$ as $n \to +\infty$. We define the auxiliary function

$$v_n(y) = \delta_n^{1/2} \frac{u_{\delta_n}(\delta_n^{1/N} y + x_n)}{\mu_n},$$

then $v_n(0) = 1$, $0 < v_n < 2$ if $|y| < \nu_n$ (thus $|\delta_n^{1/N} y + x_n - P_n| > R_n\delta_n^{1/N}$) and v_n solves the equation

$$\begin{cases} -\Delta v_n(y) = \delta_n^{2/N} \Lambda(\delta_n) m_{\delta_n}(\delta_n^{1/N} y + x_n) v_n(y), & \text{for } y \in \frac{\Omega - x_n}{\delta_n^{1/N}}, \\ \partial_\nu v_n = 0, & \text{on } \partial \frac{\Omega - x_n}{\delta_n^{1/N}}. \end{cases}$$

For every compact set $K \subset \mathbb{R}^N$, $m_{\delta_n} = -\beta$ for n sufficiently big in $K \cap (\delta_n^{-1/N}(\Omega - x_n))$. Then we can use Theorem 1.1 to pass to the limit as $n \to +\infty$; we obtain that $v_n \to v$ locally uniformly (and locally $W^{2,p}$) to a positive solution to either

$$\begin{cases} -\Delta v = -I\beta v, & \text{in } \mathbb{R}^N_+, \\ \partial_\nu v = 0, & \text{on } \mathbb{R}^{N-1}, \end{cases} \qquad \text{or} \qquad -\Delta v = -I\beta v, \qquad \text{in } \mathbb{R}^N.$$

This gives a contradiction, because the only bounded nonnegative solution of both the above problems is the trivial one, while $v(0) = 1$.

Conclusion. Iterating Step 2 one obtains that, for all $k \in \mathbb{N}$ (taking $\nu_0 > R_0$, which is always possible),

$$\sup_{|x-P_\delta|>k\nu_0\delta^{1/N}} u_\delta(x) \leq 2^{-k} \sup_{|x-P_\delta|>\nu_0\delta^{1/N}} u_\delta(x) \leq 2^{-k} \|u_\delta\|_{L^\infty(\Omega)}.$$

For all $x \in \Omega$, we can find $k \in \mathbb{N}$ such that

$$k \leq \frac{|x - P_\delta|}{\nu_0 \delta^{1/N}} \leq k + 1;$$

recalling also conclusion (ii) of Theorem 2.1, we obtain

$$u_\delta(x) \leq 2^{-k} C^* \delta^{-1/2} \leq C\delta^{-1/2} e^{-\frac{|x-P_\delta|}{\nu_0\delta^{1/N}}},$$

which in turn yields (2.1). □

As a consequence, we have the following estimates.

Corollary 2.3. *There exist $\delta_0, R_0 > 0$ and universal constants C_1, C_2 such that, for all $R > R_0$ and $\delta \leq \delta_0$ we have*

$$\begin{aligned} \int_{\{x\in\Omega:\, |x-P_\delta|>R\delta^{1/N}\}} |u_\delta|^2 \leq C_1 \delta^{-1} e^{-2C_2 R}, \\ \int_{\{x\in\Omega:\, |x-P_\delta|>R\delta^{1/N}\}} |\nabla u_\delta|^2 \leq C_1 \delta^{-1} e^{-2C_2 R}. \end{aligned} \tag{2.2}$$

Proof. The first estimate is immediate. Indeed, in view of Theorem 2.2

$$\int_{\{x\in\Omega:\,|x-P_\delta|>R\delta^{1/N}\}} |u_\delta|^2 \le |\Omega| C_1 \delta^{-1} e^{-2C_2 R}.$$

We aim now to provide a decay information on the L^2 norm of the gradient of u_δ. It is clear that for all $R > 3$, $\{x \in \Omega : |x - P_\delta| > R\delta^{1/N}\} \subset D_\delta^c$, so that the equation satisfied by u_δ becomes

$$-\Delta u_\delta = -\beta\,\Lambda(\delta) u_\delta.$$

We take a smooth cutoff function so that

$$\begin{cases} \eta = 1, & \text{in } \Omega \setminus B_{2R\delta^{1/N}}, \\ |\nabla \eta| \le 1, & \text{in } \Omega \cap (B_{2R\delta^{1/N}} \setminus B_{R\delta^{1/N}}), \\ \eta = 0, & \text{in } B_{R\delta^{1/N}}, \end{cases}$$

which is possible up to take R_0 sufficiently big (here all the balls are centered at P_δ). Then, testing the equation with $\eta^2 u_\delta \in H^1(\Omega)$, we obtain

$$\int_{\Omega\setminus B_{R\delta^{1/N}}} \nabla u_\delta \cdot \nabla(u_\delta \eta^2) = -\beta\,\Lambda(\delta) \int_{\Omega\setminus B_{R\delta^{1/N}}} \eta^2 u_\delta^2 \le 0,$$

thus

$$\int_{\Omega\setminus B_{R\delta^{1/N}}} |\nabla(u_\delta\eta)|^2 - \int_{\Omega\setminus B_{R\delta^{1/N}}} |\nabla\eta|^2 u_\delta^2 = \int_{\Omega\setminus B_{R\delta^{1/N}}} |\nabla u_\delta|^2\eta^2 + 2\int_{\Omega\setminus B_{R\delta^{1/N}}} u_\delta \eta \nabla\eta\cdot\nabla u_\delta \le 0.$$

As a consequence, using (2.1), we have

$$\int_{\Omega\setminus B_{2R\delta^{1/N}}} |\nabla u_\delta|^2 \le \int_{\Omega\setminus B_{R\delta^{1/N}}} |\nabla(u_\delta\eta)|^2 \le \int_{B_{2R\delta^{1/N}}\setminus B_{R\delta^{1/N}}} |\nabla\eta|^2 u_\delta^2 \le C_1\delta^{-1}e^{-2C_2R},$$

so that also the second part of the claim is proved. □

3 Sharp bound from below with the curvature

This section is devoted to the proof of Theorem 1.2. We first recall that the constant Γ in the expansion has been calculated in [21], in terms of the eigenfunction w of the limit problem (1.6), see (1.8). With the present notation, it reads

$$\Gamma = \frac{2(N-1)\gamma}{\int_{\mathbb{R}^N_+} |\nabla w|^2\,dz}, \qquad \text{where } \gamma = \frac{1}{N+1}\int_{\mathbb{R}^N_+} |\nabla w|^2 z_N\,dz \tag{3.1}$$

(here, for $z \in \mathbb{R}^N_+$, we write $z = (z', z_N)$, with $z' \in \mathbb{R}^{N-1}$ and $z_N > 0$).

We are going to prove the following result.

Theorem 3.1. *Let $P_\delta \in \partial\Omega$ be the unique maximum point of the function u_δ attaining $\Lambda(\delta)$, and let us define*

$$\alpha_\delta = (N-1)H_{P_\delta}, \tag{3.2}$$

where H_{P_δ} denotes the mean curvature of $\partial\Omega$ at P_δ.

We have, as $\delta \to 0$,

$$\Lambda(\delta) \ge I\,\delta^{-2/N}\left(1 - \frac{2\gamma\,\alpha_\delta}{\int_{\mathbb{R}^N_+}|\nabla w|^2}\,\delta^{1/N} + o(\delta^{1/N})\right).$$

As a matter of fact, Theorem 1.2 follows at once from the result above.

Proof of Theorem 1.2. Recalling Theorem 1.1, point 4, and since

$$\Gamma H_{P_\delta} = \frac{2\gamma\, \alpha_\delta}{\int_{\mathbb{R}^N_+} |\nabla w|^2}, \tag{3.3}$$

we infer that Theorem 3.1 implies

$$H_{P_\delta} + o_\delta(1) \geq H_P, \qquad \text{for every } P \in \partial\Omega,$$

as $\delta \to 0$, and both claims in Theorem 1.2 follow. $\square$

The remaining part of this section is devoted the proof of Theorem 3.1, which is rather long and needs many intermediate steps. Such proof is based on an improved analysis of the blow-up procedure used in [21, Section 4], which in turn was inspired by [22]. We summarize the key points of such procedure in the following.

Recall that the domain $\Omega \subset \mathbb{R}^N$ is at least $C^{2,1}$, and let $P \in \partial\Omega$ to be chosen below. We call $x = (x_1, \ldots, x_N)$ a set of coordinates centered at P, translated so that P is the origin and rotated so that the outer unit normal to the boundary of Ω at P is $-e_N$. Using the notation

$$x' = (x_1, \ldots, x_{N-1}),$$

there exist $d_0 > 0$, a $C^{2,1}$ function

$$\psi\colon \left\{ x' \in \mathbb{R}^{N-1} : |x'| < d_0 \right\} \to \mathbb{R}, \tag{3.4}$$

and a neighborhood of the origin $\mathcal{N}$ such that

i) $\psi(0) = 0$, $\nabla\psi(0) = 0$, $\Delta\psi(0) = (N-1)H_0 =: \alpha$,

ii) $\partial\Omega \cap \mathcal{N} = \left\{ (x', x_N) : x_N = \psi(x') \right\}, \qquad \Omega \cap \mathcal{N} = \left\{ (x', x_N) : x_N > \psi(x') \right\}.$

For a certain $d_1 > 0$, we define a diffeomorphism

$$\Phi\colon \left\{ y \in \mathbb{R}^N : |y| \leq d_1 \right\} \to \mathbb{R}^N, \qquad x = \Phi(y) = (\Phi_1(y), \ldots, \Phi_N(y)),$$

as

$$\Phi_j(y) = \begin{cases} y_j - y_N \frac{\partial\psi}{\partial y_j}(y'), & \text{for } j = 1, \ldots, N-1, \\ y_N + \psi(y'), & \text{for } j = N. \end{cases}$$

Remark 3.2. It is worth noticing that, in case $\partial\Omega$ is of class $C^{k,\alpha}$, then Φ is only $C^{k-1,\alpha}$. In particular, under our assumptions, Φ is always at least $C^{1,1}$.

We note that $D\Phi(0) = \mathrm{Id}$, due to the properties of ψ, and therefore Φ is locally invertible in, say, $B_{3\ell}$ for some $\ell > 0$. Then we can assume

$$\Phi(B^+_{2\ell}) \subset \Omega, \quad \text{and} \quad \Psi\colon \Phi(B^+_{3\ell}) \to B^+_{3\ell},\ \Psi(x) := \Phi^{-1}(x). \tag{3.5}$$

The map Ψ can be seen as a local diffeomorphism straightening the boundary around $0 \in \partial\Omega$. For future reference, we remark that

$$\det D\Phi(y) = 1 - \alpha y_N + O(|y|^2),$$
$$\left|\frac{y}{|y|} D\Psi(\Phi(y))\right|^2 = 1 + 2y_N \sum_{i,j=1}^{N-1} \psi_{ij}(0)\frac{y_i y_j}{|y|^2} + O(|y|^2), \qquad \text{as } y \to 0, \tag{3.6}$$

where $\psi_{ij} = \frac{\partial\psi}{\partial y_j \partial y_i}$ and we refer to [22, Lemma A.1] for further details.

In our case, we choose $P = P_\delta$, the maximum point of the optimal eigenfunction u_δ. As a consequence, the rotation and translation to set P_δ at the origin become $\delta-$dependent, and the Taylor expansions in (3.6) hold with $\alpha = \alpha_\delta$, $\psi_{ij} = \psi_{ij}^\delta$. Let us also observe that all the decay estimates of the previous section will be applied in this one taking $P_\delta = 0$.

The transformed eigenfunction is defined by

$$v_\delta(y) := u_\delta(\Phi_\delta(y)), \qquad y \in \overline{B_{2\kappa}^+}. \tag{3.7}$$

It is then easy to extend by symmetry in the whole $B_{2\kappa}$ the function v_k also where $y_N < 0$:

$$\widetilde{v}_\delta(y) := \begin{cases} v_\delta(y), & \text{if } y_N \geq 0, \\ v_\delta(y', -y_N), & \text{if } y_N < 0. \end{cases} \tag{3.8}$$

At this point, we can introduce the blow-up sequence, for $\delta_k > 0$,

$$w_\delta(z) = \delta^{1/2}\,\widetilde{v}_\delta\left(\delta^{1/N} z\right), \qquad z \in \overline{B_{\kappa\delta^{-1/N}}}. \tag{3.9}$$

We obtain that w_δ satisfies the following equation in divergence form:

$$\begin{cases} -\operatorname{div}\left(A^\delta \nabla w_\delta\right) = \delta^{2/N}\Lambda(\delta) J_\delta \cdot \widetilde{m}_\delta w_\delta & \text{in } B_{\kappa\delta^{-1/N}}, \\ \partial_N w_\delta = 0 = A^\delta \nabla w_\delta \cdot e_N & \text{on } \{z_N = 0\} \cap B_{\kappa\delta^{-1/N}}, \end{cases} \tag{3.10}$$

where the rescaled weight is, for $z \in B_{\kappa\delta^{-1/N}}$,

$$\widetilde{m}_\delta(z) = \begin{cases} m_\delta\left(\Phi_\delta(\delta^{1/N} z)\right), & \text{if } z_N \geq 0, \\ m_\delta\left(\Phi_\delta(\delta^{1/N} z', -\delta^{1/N} z_N\right), & \text{if } z_N < 0. \end{cases}$$

and the scalar J_δ and the matrix A^δ are defined as

$$J_\delta(z) = |\det D\Phi_\delta(\delta^{1/N} z)|, \qquad A^\delta(z) = J_\delta(z)[D\Psi_\delta(\Phi_\delta(\delta^{1/N} z))]^T D\Psi_\delta(\Phi_\delta(\delta^{1/N} z))$$

for $z_N \geq 0$, and extended in the natural way for $z_N < 0$.

Remark 3.3. Notice that the coefficients matrix A^δ is Lipschitz continuous. Moreover, in case $\partial\Omega \in C^{3,\theta}$, we have that A^δ is $C^{1,\theta}$ with respect to z', as the even reflection involves only z_N (recall Remark 3.2). In any case, it is standard to see that (3.10) can be written also in non-divergence form, with a drift term with L^∞ coefficients, see e.g. [16, Theorem 8.8]. In particular, since $\frac{\partial w_\delta}{\partial z_N} = 0$ on $\{z_N = 0\}$, we have that w_δ is a $W^{2,p}(B_{\kappa\delta^{-1/N}})$ solution of such non-divergence form equation. We refer to [21, Section 4] for further details.

Under the above construction, by the results in [21], Section 4 (in particular, Remark 4.4 therein), we obtain the following convergence properties for the blow-up sequences.

Proposition 3.4. *Under the above notation, and denoting with w, m the optimizers of the limit problem, as in (1.8), we have*

- $w_\delta \to w$ *strongly in* $H^1_{loc}(\mathbb{R}^N)$ *and in* $C^{1,\alpha}_{loc}$;
- $\widetilde{m}_\delta \overset{*}{\rightharpoonup} m$ *weakly$*$ in* L^∞_{loc}.

as $\delta \to 0$.

The exponential decay of u_δ, obtained in Corollary 2.3, entails an (uniform in δ) exponential decay for w_δ and its gradient.

Lemma 3.5. *There exist universal positive constants $C_3, C_4 > 0$ and there exist $R_0, \delta_0 > 0$ such that, for all $R > R_0$ and $\delta \le \delta_0$ we have that*

$$\begin{aligned} w_\delta(z) \le C_3 e^{-C_4|z|}, &\qquad \text{for all } z \in B_{\kappa\delta^{-1/N}}\,, \\ \int_{B_{\kappa\delta^{-1/N}} \setminus B_R} |w_\delta|^2 \le C_3 e^{-2C_4 R}, & \end{aligned} \tag{3.11}$$

Moreover, for $R_2 > R_1 > R_0$,

$$\begin{aligned} |\nabla w_\delta(z)| \le C_3 e^{-C_4|z|}, &\qquad \text{for all } z \in B_{\kappa\delta^{-1/N}}\,, \\ \int_{B_{\kappa\delta^{-1/N}} \setminus B_R} |\nabla w_\delta|^2 \le C_3 e^{-2C_4 R}, & \end{aligned} \tag{3.12}$$

where the balls are centered at the origin, maximum point for w_δ.

Proof. The pointwise exponential decay of w_δ (and of its L^2 norm) follows directly from Theorem 2.2 and the fact that the diffeomorphism is close to the identity and centered at P_δ (which is translated at 0), so that

$$0 < w_\delta(z) = \delta^{1/2} u_\delta(\Phi_\delta(\delta^{1/N} z)) \le C_1 e^{-C_2 \delta^{-1/N} |\Phi_\delta(\delta^{1/N} z)|} \le C_3 e^{-C_4|z|}.$$

Then the decay of its L^2 norm outside B_R is immediate.

Concerning the exponential decay of the gradient, we use elliptic regularity, see e.g. [16, Theorem 9.11] (recall Remark 3.3) together with the Morrey-Sobolev embedding of $W^{2,p} \subset C^{1,\alpha}$ for p sufficiently large. More precisely, let z be such that $B_4(z) \subset B_{\kappa\delta^{-1/N}}$. Then, using (3.11), we have that,

$$\|w_\delta\|_{C^{1,\alpha}(B_3(z)\setminus B_2(z))} \le C\|w_\delta\|_{W^{2,p}(B_3(z)\setminus B_2(z))} \le C\|w_\delta\|_{L^p(B_4(z)\setminus B_1(z))} \le Ce^{-C_4|z|}$$

for a universal constant $C > 0$, and where we have taken into account that the balls are centered at a generic point z. □

Remark 3.6. In view of Lemma 3.5, one can use a concentration-compactness kind of argument to obtain strong $H^1(\mathbb{R}^N)$ convergence of w_δ to w. We refer to [14, Lemma 2.3], [13, Section 4], where the argument was fully detailed in the case of Dirichlet boundary conditions.

Going back to the blow-up procedure, we now introduce the optimal sets (nonrescaled and rescaled)

$$\begin{aligned} D_\delta =& \{x \in \Omega : m_\delta(x) = 1\}, \qquad |D_\delta| = \delta, \\ \widetilde{D}_\delta :=& \left\{ z \in B_{\kappa\delta^{-1/N}} : z \in \frac{\Psi_\delta(D_\delta)}{\delta^{1/N}} \text{ or } (z', -z_N) \in \frac{\Psi_\delta(D_\delta)}{\delta^{1/N}} \right\}, \end{aligned} \tag{3.13}$$

and we recall that

$$\widetilde{D}_\delta = \{z \in B_{\kappa\delta^{-1/N}} : \widetilde{m}_\delta(z) = 1\} = \{z \in B_{\kappa\delta^{-1/N}} : w_\delta(z) > t_\delta\}. \tag{3.14}$$

The core of the proof of Theorem 3.1 consists in bounding the optimal level I of the limit problem (1.8) in terms of a weighted Rayleigh quotient of w_δ. An issue in this direction is that

$$|\widetilde{D}_\delta| = 2 + o_\delta(1) \qquad \text{as } \delta \to 0,$$

but the error term cannot be discarded. This can be easily overcome using the reflection and scaling properties of the limit problem, namely

$$\min \left\{ \lambda(A, \mathbb{R}^N) : A \subset \mathbb{R}^N, \text{ measurable, } |A| = \ell \right\} = I \cdot \left(\frac{\ell}{2} \right)^{-2/N}, \qquad \text{for all } \ell > 0.$$

Lemma 3.7. *We have, as $\delta \to 0$,*

$$I \cdot \left(\frac{|\widetilde{D}_\delta|}{2} \right)^{-2/N} \le \lambda(\widetilde{D}_\delta, \mathbb{R}^N) \le \frac{\int_{B_{\kappa\delta^{-1/N}}} |\nabla w_\delta(z)|^2 \, dz}{\int_{B_{\kappa\delta^{-1/N}}} \widetilde{m}_\delta w_\delta^2(z) \, dz} + o(\delta^{1/N}). \tag{3.15}$$

Proof. We need to extend w_δ to the whole $\mathbb{R}^N$ to make it an admissible competitor for the limit problem. We extend $\widetilde{m}_\delta = -\beta$ in $\mathbb{R}^N \setminus B_{\kappa\delta^{-1/N}}$, as it is natural. We define the new function

$$\widetilde{w}_\delta(z) = \begin{cases} w_\delta(z), & \text{in } B_{\kappa\delta^{-1/N}}, \\ h(z), & \text{in } B_{(\kappa+1)\delta^{-1/N}} \setminus B_{\kappa\delta^{-1/N}}, \\ 0, & \text{in } \mathbb{R}^N \setminus B_{(\kappa+1)\delta^{-1/N}}, \end{cases}$$

where h is the unique harmonic extension:

$$\begin{cases} -\Delta h = 0, & \text{in } B_{(\kappa+1)\delta^{-1/N}} \setminus B_{\kappa\delta^{-1/N}}, \\ h = w_\delta, & \text{on } \partial B_{\kappa\delta^{-1/N}}, \\ h = 0, & \text{on } \partial B_{(\kappa+1)\delta^{-1/N}}. \end{cases}$$

Thanks to Lemma 3.5, we have that

$$w_\delta(z) \le C_3 e^{-C_4 \kappa \delta^{-1/N}}, \qquad \text{on } \partial B_{\kappa\delta^{-1/N}},$$

and the same also holds for the normal derivative, in view of (3.12). As a consequence, calling for the sake of simplicity $A = B_{(\kappa+1)\delta^{-1/N}} \setminus B_{\kappa\delta^{-1/N}}$, we deduce

$$\left| \int_A -\beta \widetilde{w}_\delta^2(z) \, dz \right| \le C|A| e^{-2C_4 \kappa \delta^{-1/N}} \le o(\delta^{1/N}).$$

Moreover, using the Divergence Theorem, we obtain

$$\int_A |\nabla \widetilde{w}_\delta(z)|^2 \, dz = \int_{\partial B_{\kappa\delta^{-1/N}}} \widetilde{w}_\delta \partial_\nu \widetilde{w}_\delta \, d\mathcal{H}^{N-1} \le o(\delta^{1/N}).$$

All in all, we have proved (3.15). $\square$

In the next three lemmas, we proceed with the estimates of all the unknown terms in (3.15), in terms of α_δ and u_δ. We start from the measure of $\widetilde{D}_\delta$.

Lemma 3.8. *We have, as $\delta \to 0$,*

$$\left(\frac{|\widetilde{D}_\delta|}{2}\right)^{-2/N} = 1 - \frac{2}{N}\alpha_\delta \delta^{1/N} \frac{\omega_{N-1}}{N+1} r_2^{N+1} + o(\delta^{1/N}). \tag{3.16}$$

Proof. We recall that, thanks to Theorem 1.1 (or to [21, Lemma 4.5 and Proposition 4.7]), we have, as $\delta \to 0$

$$(1 - o_\delta(1))B_{r_2} \subset \widetilde{D}_\delta \subset (1 + o_\delta(1))B_{r_2}, \tag{3.17}$$

recalling that $|B_{r_2}| = 2$. Then, we can compute

$$\delta = |D_\delta| = \int_\Omega \mathbb{1}_{D_\delta}(x)\,dx = \int_{\mathbb{R}^N_+} \delta \mathbb{1}_{D_\delta}\left(\Phi_\delta(\delta^{1/N} z)\right)\,\det\left(D\Phi_\delta(\delta^{1/N} z)\right)dz,$$

and using also [22, Lemma A.1] (see (3.6)), we obtain

$$\begin{aligned} 2 &= \int_{\mathbb{R}^N} \mathbb{1}_{\widetilde{D}_\delta}(z)\left(1 - \alpha_\delta \delta^{1/N} z_N + O(\delta^{2/N}|z|^2)\right)dz \\ &= |\widetilde{D}_\delta| - \alpha_\delta \delta^{1/N} \int_{\mathbb{R}^N} \mathbb{1}_{\widetilde{D}_\delta}(z)\left(z_N + O(\delta^{1/N}|z|^2)\right)dz. \end{aligned}$$

Using (3.17), it is easy to prove, as $\delta \to 0$,

$$2 + \alpha_\delta \delta^{1/N} \int_{(1-o_\delta(1))B_{r_2}} z_N\,dz + O(\delta^{2/N}) \le |\widetilde{D}_\delta| \le 2 + \alpha_\delta \delta^{1/N} \int_{(1+o_\delta(1))B_{r_2}} z_N\,dz + O(\delta^{2/N}).$$

We can now compute, by scaling, that

$$\int_{(1+o_\delta(1))B_{r_2}} z_N\,dz = (1 + o_\delta(1))^{\frac{N+1}{N}} \int_{B_{r_2}} y_N\,dy = (1 + o_\delta(1))\frac{2\omega_{N-1}}{N+1} r_2^{N+1},$$

where we made the change of variable $z = (1 + o_\delta(1))^{1/N} y$ and we used

$$\int_{(B_{r_2})^+} z_N\,dz = \frac{\omega_{N-1}}{N+1} r_2^{N+1},$$

denoting as usual r_2 the radius of the ball of measure 2. As a consequence, we have, for $\delta \to 0$,

$$|\widetilde{D}_\delta| = 2 + 2\alpha_\delta \delta^{1/N} \frac{\omega_{N-1}}{N+1} r_2^{N+1} + o(\delta^{1/N}). \tag{3.18}$$

Thanks to these estimates, we obtain, as $\delta \to 0$,

$$\frac{|\widetilde{D}_\delta|}{2} = 1 + \alpha_\delta \delta^{1/N} \frac{\omega_{N-1}}{N+1} r_2^{N+1} + o(\delta^{1/N})$$

yielding the conclusion. □

Next we proceed with the expansion of the numerator of the Rayleigh quotients of u_δ and w_δ.

Lemma 3.9. *Recalling that $\alpha_\delta = (N-1)H_{P_\delta}$ and that γ is defined in* (3.1)*, we have, as $\delta \to 0$,*

$$\int_\Omega |\nabla u_\delta(x)|^2 dx = \delta^{-2/N} \int_{B^+_{\kappa\delta^{-1/N}}} |\nabla w_\delta(z)|^2 \, dz \left(1 - \delta^{1/N} \frac{(N-1)\gamma \, \alpha_\delta}{\int_{\mathbb{R}^N_+} |\nabla w|^2} + o(\delta^{1/N}) \right) \tag{3.19}$$

Proof. Let $r > 0$ such that $B^+_r \subset \Phi(B^+_\kappa)$. From Corollary 2.3, we infer

$$\left| \int_\Omega |\nabla u_\delta|^2 - \int_{\Phi(B^+_\kappa)} |\nabla u_\delta|^2 \right| \le \int_{\Omega \setminus B^+_r} |\nabla u_\delta|^2 \le C_1 \delta^{-1} e^{-C_2 r \delta^{-1/N}} \le o(\delta^{1/N}).$$

On the other hand, exploiting the usual change of variables $y = \Psi(x)$ and $z = y\delta^{-1/N}$, recalling (3.9) and applying [22, Lemma A.1].

$$\begin{aligned} \int_{\Phi(B^+_\kappa)} |\nabla u_\delta(x)|^2 dx &= \int_{B^+_\kappa} |\nabla v_\delta(y)|^2 \left| \frac{y}{|y|} D\Psi(x) \right|^2 \det D\Phi(y) \, dy \\ &= \delta^{-2/N} \int_{B^+_{\kappa\delta^{-1/N}}} |\nabla w_\delta(z)|^2 \left[1 + \delta^{1/N} z_N \Big(2 \sum_{i,j=1}^{N-1} \psi_{ij} \frac{z_i z_j}{|z|^2} - \alpha_\delta \Big) + O(\delta^{2/N}|z|^2) \right] dz. \end{aligned} \tag{3.20}$$

At this point, one can check that there is a constant C_0, independent of δ, such that

$$\left(1 + \delta^{1/N} z_N \Big[2 \sum_{i,j=1}^{N-1} \psi_{ij} \frac{z_i z_j}{|z|^2} - \alpha_\delta \Big] + O(\delta^{2/N}|z|^2) \right) \le C_0, \qquad \text{for all } z \in B^+_{\kappa\delta^{-1/N}}.$$

We now want to show that

$$\int_{B^+_{\kappa\delta^{-1/N}}} \left(|\nabla w_\delta|^2 - |\nabla w|^2 \right) \left[\delta^{1/N} z_N \Big(2 \sum_{i,j=1}^{N-1} \psi_{ij} \frac{z_i z_j}{|z|^2} - \alpha_\delta \Big) + O(\delta^{2/N}|z|^2) \right] dz = o(\delta^{1/N}), \tag{3.21}$$

as $\delta \to 0$. To prove this, we use the H^1_{loc} convergence of w_δ to w (Proposition 3.4) and their exponential decay (Lemma 3.5, equation (1.9)). Precisely, let us fix $\varepsilon > 0$ and find $R = R(\varepsilon) > 0$ such that, for all $\delta > 0$ sufficiently small

$$\begin{aligned} \int_{B^+_{\kappa\delta^{-1/N}} \setminus B^+_R} |\nabla w_\delta(z)|^2 \left[z_N \Big(2 \sum_{i,j=1}^{N-1} \psi_{ij} \frac{z_i z_j}{|z|^2} - \alpha_\delta \Big) + O(\delta^{1/N}|z|^2) \right] dz &\le C \int_{B^+_{\kappa\delta^{-1/N}} \setminus B^+_R} e^{-C_4|z|} |z| dz \\ &\le C \left(\delta^{-1} e^{-k\delta^{-1/N}} - R^N e^{-C_4 R} \right) \le \varepsilon, \\ \int_{B^+_{\kappa\delta^{-1/N}} \setminus B^+_R} |\nabla w(z)|^2 \left[z_N \Big(2 \sum_{i,j=1}^{N-1} \psi_{ij} \frac{z_i z_j}{|z|^2} - \alpha_\delta \Big) + O(\delta^{1/N}|z|^2) \right] dz &\le \varepsilon. \end{aligned}$$

On the other hand, w_δ converges strongly in $H^1(B_R)$ to w, hence

$$\int_{B^+_R} \left(|\nabla w_\delta|^2 - |\nabla w|^2 \right) \left[z_N \Big(2 \sum_{i,j=1}^{N-1} \psi_{ij} \frac{z_i z_j}{|z|^2} - \alpha_\delta \Big) + O(\delta^{1/N}|z|^2) \right] dz \le C(R) o_\delta(1) \le \varepsilon,$$

up to take $\delta(\varepsilon)$ small enough. All in all, (3.21) follows.

We also recall that, thanks to the exponential decay of w, it is clear that

$$\int_{(\mathbb{R}^N_+)\setminus B^+_{\kappa\delta^{-1/N}}} |\nabla w|^2\, dz = o(\delta^{1/N}), \qquad \text{as } \delta \to 0. \tag{3.22}$$

We can now manage the higher order terms in (3.20), using (3.21) and (3.22)

$$\begin{aligned}
&\int_{B^+_{\kappa\delta^{-1/N}}} |\nabla w_\delta(z)|^2 \left[\delta^{1/N} z_N \Big(2\sum_{i,j=1}^{N-1} \psi_{ij}\frac{z_i z_j}{|z|^2} - \alpha_\delta\Big) + O(\delta^{2/N}|z|^2)\right] dz \\
&= \int_{B^+_{\kappa\delta^{-1/N}}} \Big(|\nabla w_\delta(z)|^2 - |\nabla w(z)|^2 + |\nabla w(z)|^2\Big) \left[\delta^{1/N} z_N \Big(2\sum_{i,j=1}^{N-1} \psi_{ij}\frac{z_i z_j}{|z|^2} - \alpha_\delta\Big) + O(\delta^{2/N}|z|^2)\right] dz \\
&= \int_{B^+_{\kappa\delta^{-1/N}}} |\nabla w(z)|^2 \left[\delta^{1/N} z_N \Big(2\sum_{i,j=1}^{N-1} \psi_{ij}\frac{z_i z_j}{|z|^2} - \alpha_\delta\Big) + O(\delta^{2/N}|z|^2)\right] dz + o(\delta^{1/N}) \\
&= \int_{\mathbb{R}^N_+} |\nabla w(z)|^2 \left[\delta^{1/N} z_N \Big(2\sum_{i,j=1}^{N-1} \psi_{ij}\frac{z_i z_j}{|z|^2} - \alpha_\delta\Big) + O(\delta^{2/N}|z|^2)\right] dz + o(\delta^{1/N}) \\
&= -\delta^{1/N}[(N-1)\alpha_\delta \gamma] + o(\delta^{1/N}).
\end{aligned}$$

We note that the last equality follows with computations similar to the ones of [21, Step 1 of the proof of Proposition 3.5], in particular using that

$$\sum_{i,j=1}^{N-1} \psi_{ij} \int_{\mathbb{R}^N_+} |\nabla w(z)|^2 \frac{z_i z_j}{|z|^2} z_N\, dz = \gamma\, \alpha_\delta.$$

Then, coming back to (3.20), we have

$$\int_\Omega |\nabla u_\delta(x)|^2 = \delta^{-2/N} \int_{B^+_{\kappa\delta^{-1/N}}} |\nabla w_\delta(z)|^2\, dz \left(1 - \delta^{1/N} \frac{(N-1)\gamma\, \alpha_\delta}{\int_{B^+_{\kappa\delta^{-1/N}}} |\nabla w_\delta(z)|^2\, dz} + o(\delta^{1/N})\right). \tag{3.23}$$

Moreover, using again the convergence and the exponential decay of w_δ and w, it is clear that

$$\int_{\mathbb{R}^N_+} |\nabla w|^2\, dz = \int_{B^+_{\kappa\delta^{-1/N}}} |\nabla w_\delta|^2\, dz + o_\delta(1), \qquad \text{as } \delta \to 0,$$

hence, (3.23) yields the conclusion. □

We use a similar strategy for the denominator of the Rayleigh quotient, with the key tools being again Proposition 3.4 and Lemma 3.5.

Lemma 3.10. *It results*

$$\begin{aligned}
&\int_\Omega m_\delta(x) u_\delta(x)^2\, dx = \int_{B_{\kappa\delta^{-1/N}}} \widetilde{m}_\delta w_\delta^2\, dz \Big(1 - \delta^{1/N} \frac{\gamma_1\, \alpha_\delta}{\int_{\mathbb{R}^N_+} m w^2} + o(\delta^{1/N})\Big), \\
&\textit{where } \gamma_1 = \int_{\mathbb{R}^N_+} m(z) w(z)^2 z_N\, dz.
\end{aligned} \tag{3.24}$$

Proof. First, as $\Phi(B_\kappa^+) \supset B_r^+$ for some $r > 0$, the exponential decay of u_δ (see Corollary 2.3) and the fact that $-\beta \le m_\delta \le 1$, imply

$$\left|\int_\Omega m_\delta u_\delta^2 - \int_{\Phi(B_\kappa^+)} m_\delta u_\delta^2\right| \le \max\{\beta, 1\} \int_{\Omega\setminus B_R^+} u_\delta^2 \le C_1 e^{-C_2 R\delta^{-1/N}} \le o(\delta^{1/N}), \text{ as } \delta \to 0.$$

Then we have,

$$\begin{aligned}
&\int_\Omega m_\delta(x)u_\delta^2(x)\,dx = \int_{B_\kappa^+} \widetilde{m}_\delta(y\delta^{-1/N})v_\delta^2(y)\det D\Phi(y)\,dy \\
&= \int_{B^+_{\kappa\delta^{-1/N}}} \widetilde{m}_\delta(z)w_\delta^2(z)\det D\Phi(y)\,dz \\
&= \int_{B^+_{\kappa\delta^{-1/N}}} \widetilde{m}_\delta(z)w_\delta^2(z)\left(1 - \alpha_\delta z_N\delta^{1/N} + O(\delta^{2/N}|z|^2)\right)dz.
\end{aligned}$$

As above, we have that there is a constant C_0, independent of δ, such that

$$\left|1 - \alpha_\delta\delta^{1/N}z_N + O(\delta^{2/N}|z|^2)\right| \le C_0, \qquad \text{for all } z \in B^+_{\kappa\delta^{-1/N}}.$$

With the same strategy as the one used for the gradient, we first prove that, as $\delta \to 0$,

$$\int_{B^+_{\kappa\delta^{-1/N}}} |\widetilde{m}_\delta w_\delta^2 - mw^2|\left(\alpha_\delta z_N\delta^{1/N} + O(\delta^{2/N}|z|^2)\right)dz = o(\delta^{1/N}), \tag{3.25}$$

which we split in two easier steps,

$$\begin{aligned}
&\int_{B^+_{\kappa\delta^{-1/N}}} |\widetilde{m}_\delta w_\delta^2 - \widetilde{m}_\delta w^2|\left(\alpha_\delta z_N\delta^{1/N} + O(\delta^{2/N}|z|^2)\right)dz = o(\delta^{1/N}), \\
&\int_{B^+_{\kappa\delta^{-1/N}}} |\widetilde{m}_\delta w^2 - mw^2|\left(\alpha_\delta z_N\delta^{1/N} + O(\delta^{2/N}|z|^2)\right)dz = o(\delta^{1/N}).
\end{aligned} \tag{3.26}$$

To prove the estimates in (3.26), we use the H^1_{loc} convergence of w_δ to w and their exponential decay. We start from the first. More precisely, let us fix $\varepsilon > 0$ and find $R = R(\varepsilon) > 0$ such that, for all $\delta > 0$ sufficiently small

$$\begin{aligned}
&\int_{B^+_{\kappa\delta^{-1/N}}\setminus B_R^+} w_\delta(z)^2\left[\alpha_\delta z_N + O(\delta^{1/N}|z|^2)\right]dz \le \varepsilon, \\
&\int_{B^+_{\kappa\delta^{-1/N}}\setminus B_R^+} w(z)^2\left[\alpha_\delta z_N + O(\delta^{1/N}|z|^2)\right]dz \le \varepsilon,
\end{aligned}$$

On the other hand, in B_R, w_δ converges strongly in H^1 to w (and $-\beta \le \widetilde{m}_\delta \le 1$), hence

$$\int_{B_R^+} \widetilde{m}_\delta(w_\delta^2 - w^2)\left[\alpha_\delta z_N + O(\delta^{1/N}|z|^2)\right]dz \le C(R)o_\delta(1) \le \varepsilon,$$

so that the first estimate in (3.26) follows.

The second estimate in (3.26) can be proved in a similar way. We fix again $\varepsilon > 0$ and find $R = R(\varepsilon) > 0$ such that, for all $\delta > 0$ sufficiently small

$$\int_{B^+_{\kappa\delta^{-1/N}}\setminus B_R^+} (\widetilde{m}_\delta - m)w(z)^2\left[\alpha_\delta z_N + O(\delta^{1/N}|z|^2)\right]dz \le \varepsilon,$$

which is possible thanks to the exponential decay of w and the boundedness of $\widetilde{m}_\delta$ and m. In B_R, $\widetilde{m}_\delta$ converges weakly $*$ in L^∞ to m (and the other terms in the integral are clearly L^1), hence

$$\int_{B_R^+} (\widetilde{m}_\delta - m) w^2 \left[\alpha_\delta z_N + O(\delta^{1/N}|z|^2)\right] dz \le C(R) o_\delta(1) \le \varepsilon,$$

up to take $\delta(\varepsilon)$ small enough. In conclusion we have proved (3.25).

We also recall that, thanks to the exponential decay of w (and the fact that $-\beta \le m \le 1$), it is clear that

$$\int_{\mathbb{R}^N_+ \setminus B^+_{\kappa\delta^{-1/N}}} m w^2 \, dz = o(\delta^{1/N}), \qquad \text{as } \delta \to 0. \tag{3.27}$$

From (3.25), adding and subtracting the suitable terms, we obtain

$$\begin{aligned}
&\int_{B^+_{\kappa\delta^{-1/N}}} \widetilde{m}_\delta(z) w_\delta(z)^2 \Big(-\alpha_\delta z_N \delta^{1/N} + O(\delta^{2/N}|z|^2)\Big) dz \\
&= \int_{B^+_{\kappa\delta^{-1/N}}} (\widetilde{m}_\delta w_\delta^2 - m w^2) + m(z) w(z)^2 \Big(-\alpha_\delta z_N \delta^{1/N} + O(\delta^{2/N}|z|^2)\Big) dz \\
&= \int_{B^+_{\kappa\delta^{-1/N}}} m(z) w(z)^2 \Big(-\alpha_\delta z_N \delta^{1/N} + O(\delta^{2/N}|z|^2)\Big) dz + o(\delta^{1/N}) = -\alpha_\delta \delta^{1/N} \gamma_1 + o(\delta^{1/N}).
\end{aligned}$$

All in all, we have

$$\int_{B^+_{\kappa\delta^{-1/N}}} \widetilde{m}_\delta w_\delta^2 = \int_{\mathbb{R}^N_+} m w^2 + o_\delta(1)\,,$$

showing the conclusion. □

Proof of Theorem 3.1. Putting together (3.19) and (3.24), we have

$$\Lambda(\delta) = \delta^{-2/N} \frac{\int_{B_{\kappa\delta^{-1/N}}} |\nabla w_\delta|^2}{\int_{B_{\kappa\delta^{-1/N}}} \widetilde{m}_\delta w_\delta^2} \left(1 - \delta^{1/N} \frac{(N-1)\gamma\, \alpha_\delta}{\int_{\mathbb{R}^N_+} |\nabla w|^2} + o(\delta^{1/N})\right) \left(1 + \delta^{1/N} \frac{\gamma_1\, \alpha_\delta}{\int_{\mathbb{R}^N_+} m w^2} + o(\delta^{1/N})\right), \tag{3.28}$$

which, in view of (3.15), implies

$$\Lambda(\delta) \ge \delta^{-2/N} \left(1 - \frac{\delta^{1/N}\, \alpha_\delta}{I \int_{\mathbb{R}^N_+} m w^2} \left[(N-1)\gamma - I\gamma_1\right] + o(\delta^{1/N})\right) \cdot I \left(\frac{|\widetilde{D}_\delta|}{2}\right)^{-2/N} \tag{3.29}$$

In turn, (3.16) yields

$$\begin{aligned}
\Lambda(\delta) \ge \delta^{-2/N} I \left(1 - \frac{\delta^{1/N}\, \alpha_\delta}{I \int_{\mathbb{R}^N_+} m w^2} [(N-1)\gamma - I\gamma_1] + o(\delta^{1/N})\right) \times \\
\times \left(1 - 2/N \delta^{1/N}\, \alpha_\delta \frac{\omega_{N-1}}{N+1} r_2^{N+1} + o(\delta^{1/N})\right).
\end{aligned}$$

As

$$(N-1)\gamma - I\gamma_1 = 2\gamma - 2I r_2^{N+1} \frac{\omega_{N-1}}{N(N+1)} \int_{\mathbb{R}^N_+} m w^2, \tag{3.30}$$

the theorem follows. □

Remark 3.11. With a closer look at the previous proof, and in particular at the role of (3.15) in estimate (3.29), we notice that also the following inequality holds true:

$$\begin{aligned}\Lambda(\delta) &\geq \delta^{-2/N}\left(1-\Gamma\widehat{H}\,\delta^{1/N}+o(\delta^{1/N})\right)\cdot\left(\frac{|\widetilde{D}_\delta|}{2}\right)^{2/N}\lambda(\widetilde{D}_\delta,\mathbb{R}^N),\\ &\geq \delta^{-2/N}\left(1-\Gamma\widehat{H}\,\delta^{1/N}+o(\delta^{1/N})\right)\cdot I\end{aligned}$$

where we used also (3.3) and Theorem 1.2. As a consequence,

$$\delta^{2/N}\Lambda(\delta)=\left(\frac{|\widetilde{D}_\delta|}{2}\right)^{2/N}\lambda(\widetilde{D}_\delta,\mathbb{R}^N)+o_\delta(1)=I+o_\delta(1)$$

as $\delta\to 0$, and finally

$$\delta^{2/N}\Lambda(\delta)\geq\left(\frac{|\widetilde{D}_\delta|}{2}\right)^{2/N}\lambda(\widetilde{D}_\delta,\mathbb{R}^N)-\Gamma\widehat{H}\;I\,\delta^{1/N}+o(\delta^{1/N}),$$

which establishes a lower bound for $\Lambda(\delta)$ on terms of an eigenvalue of the set $\widetilde{D}_\delta$ related to the limit problem.

4 Polar parameterization of the optimal sets

In the last part of the paper we are going to exploit the blow up analysis performed in the previous section, and in particular the asymptotic radial symmetry of the rescaled eigenfunctions w_δ and optimal sets $\widetilde{D}_\delta$, see (3.17), to obtain a polar parametrization of $\partial\widetilde{D}_\delta$ and to investigate its finer regularity properties, also from a quantitative point of view. Of course, such information can be translated to D_δ using the diffeomorphisms introduced at the beginning of Section 3.

To this aim, let us first recall the concept of nearly spherical set (used in the proof of a quantitative isoperimetric inequality first in [15]). For our aims, although $\widetilde{D}_\delta$ has measure 2 only in the limit, it is convenient to normalize the reference radius to r_2, where as usual $|B_{r_2}|=2$.

Definition 4.1. A (bounded) set $A\subset\mathbb{R}^N$ is a nearly spherical set of class $C^{k,\alpha}$, centered at Q, if there exists $\varphi=\varphi_A\in C^{k,\alpha}(\mathbb{S}^{N-1})$ with $\|\varphi\|_{L^\infty}\leq r_2/2$ such that

$$\partial A=\left\{x\in\mathbb{R}^N : x=Q+(r_2+\varphi(\theta))\theta,\text{for }\theta\in\mathbb{S}^{N-1}\right\}.$$

In such case, we say that A is parametrized by φ.

As a matter of fact, it is not difficult to exploit (3.17) and the implicit function theorem to see that $\widetilde{D}_\delta$ is nearly spherical, centered at the maximum point $0=\Psi_\delta(P_\delta)$, at least when δ is sufficiently small. On the other hand, in order to employ suitable quantitative estimates obtained in [14], it is necessary to adjust the center of the parametrization, choosing instead the barycenter $\mathrm{bar}(\widetilde{D}_\delta)$ (and the corresponding preimage on $\partial\Omega$). This is possible thanks to the following lemma, where we keep track of $\Psi_\delta(P_\delta)=0$ for the sake of clarity.

Lemma 4.2. *Under the notation of Section 3 we have, as $\delta \to 0$:*

1. $\mathrm{bar}(\widetilde{D}_\delta) \in \{z : z_N = 0\}$ *and* $\left|\mathrm{bar}(\widetilde{D}_\delta) - \Psi_\delta(P_\delta)\right| = o_\delta(1)$*;*
2. $Q_\delta := \Phi_\delta(\mathrm{bar}(\widetilde{D}_\delta)) \in \partial D_\delta$ *and* $|Q_\delta - P_\delta| = o(\delta^{1/N})$*.*

Proof. First, by symmetry of $\widetilde{D}_\delta$, we obtain that $\mathrm{bar}(\widetilde{D}_\delta) \in \{z : z_N = 0\}$ and thus $Q_\delta \in \partial D_\delta$ (recall equation (3.13)).

Next, for every $\varepsilon > 0$, let $B_{r_-}(\Psi_\delta(P_\delta))$, and $B_{r_+}(\Psi_\delta(P_\delta))$ be such that $|B_{r_\pm}| = 2(1 \pm \varepsilon)$ and, for δ sufficiently small,

$$B_{r_-} \subset \widetilde{D}_\delta \subset B_{r_+}.$$

We infer

$$\begin{aligned}\left|\mathrm{bar}(\widetilde{D}_\delta) - \Psi_\delta(P_\delta)\right| &\le \frac{1}{|\widetilde{D}_\delta|}\left|\int_{\widetilde{D}_\delta}(x - \Psi_\delta(P_\delta))\,dx\right| = \frac{1}{|\widetilde{D}_\delta|}\left|\int_{\widetilde{D}_\delta \setminus B_{r_-}(\Psi_\delta(P_\delta))}(x - \Psi_\delta(P_\delta))\,dx\right| \\ &\le \frac{1}{|\widetilde{D}_\delta|}\int_{B_{r_+}(\Psi_\delta(P_\delta))\setminus B_{r_-}(\Psi_\delta(P_\delta))}|x - \Psi_\delta(P_\delta)|dx \le C\varepsilon(1 + o_\delta(1))\end{aligned}$$

and the lemma follows. □

Taking into account the lemma above, we change reference system in the blow-up analysis by a vanishing translation, and from now on we assume that

$$\Psi_\delta(Q_\delta) = \mathrm{bar}(\widetilde{D}_\delta) = 0, \qquad \Psi_\delta(P_\delta) = -\,\mathrm{bar}(\widetilde{D}_\delta). \tag{4.1}$$

Although in principle this changes the definitions of $\widetilde{D}_\delta$, w_δ, Ψ_δ and so on, Lemma 4.2 implies that all the results in Section 3 hold true also in the new reference system; in particular, equation (3.17) holds with balls centered at $0 = \mathrm{bar}(\widetilde{D}_\delta)$.

Proposition 4.3. *For δ sufficiently small, $\widetilde{D}_\delta$ is nearly spherical of class $C^{1,\alpha}$, $0 < \alpha < 1$, centered at $0 = \mathrm{bar}(\widetilde{D}_\delta)$ and parametrized by φ_δ. In addition*

$$\|\varphi_\delta\|_{C^{1,\alpha}(\mathbb{S}^{N-1})} \to 0, \qquad \text{as } \delta \to 0.$$

Proof. The proof can be obtained (with obvious changes) as in [14, Proposition 3.11]. Indeed, using polar coordinates and recalling (3.14) we can write

$$\widetilde{D}_\delta = \{z \in B_{k\delta^{-1/N}} : w_\delta(z) > t_\delta\} = \{w_\delta(\rho\theta) > t_\delta\}$$

where $\rho > 0$ and $\theta = \frac{x}{|x|} \in \mathbb{S}^{N-1}$. Moreover, in view of (3.17) one has

$$\partial\widetilde{D}_\delta \subset (1 + o_\delta(1))B_{r_2} \setminus (1 - o_\delta(1))B_{r_2} \subset B_{\overline{r}} \setminus B_{\underline{r}},$$

for some $0 < \underline{r} < r_2 < \overline{r}$. Let us consider $F(\varphi, \theta) := w_\delta((r_2 + \varphi)\theta)$ for $\varphi = \rho - r_2$ and $\rho \in [\underline{r}, \overline{r}]$. As w_δ converges to w in $C^{1,\alpha}(\overline{B}_{\overline{r}})$, so that

$$\max_{B_{\overline{r}}\setminus B_{\underline{r}}} \partial_r w_\delta < \frac{1}{2}\max_{B_{\overline{r}}\setminus B_{\underline{r}}} \partial_r w < 0, \quad \text{for every } \theta \in \mathbb{S}^{N-1}.$$

Then we can apply the Implicit Function Theorem to the function $F(\rho, \theta)$, obtaining $F(\rho, \theta) = t_\delta$ if and only if $\rho = \rho(\theta)$. This argument can be implemented for every $\theta \in S^{N-1}$, so that by compactness we obtain a globally defined $\varphi_\delta(\theta) = \rho(\theta) - r_2$. Since w_δ is of class $C^{1,\alpha}$, the same regularity holds for φ_δ. Furthermore,

$$\nabla\varphi_\delta = -\frac{(r_2+\varphi_\delta(\theta))}{\partial_\rho F(\rho,\theta)}\nabla_T w_\delta = -\frac{(r_2+\varphi_\delta(\theta))}{\partial_\rho F(\rho,\theta)}\left(\nabla w_\delta - (\nabla w_\delta\cdot\theta)\theta\right).$$

where $\nabla_T w_\delta$ denotes the tangential component of the gradient of w_δ, and $\theta = \frac{x}{|x|}$ Then the conclusion is a direct consequence of the $C^{1,\alpha}$ convergence of w_δ. □

In order to improve the asymptotic information provided in the last proposition, from now on we assume that, for some $\theta > 0$,

$$\partial\Omega \text{ is of class } C^{3,\theta},$$

so that Φ_δ is of class $C^{2,\theta}$; we are going to prove a $C^{1,1}$ global regularity result (up to the boundary) for the eigenfunctions w_δ. From this, we will deduce the $C^{1,1}$ regularity for the nearly spherical representation φ_δ. This yields a $C^{1,1}$ convergence result of $\widetilde{D}_\delta$ to the ball with measure two, thus improving Proposition 4.3.

To this aim, we follow the strategy of [14, Section 5], where analogous results were derived for a similar problem with Dirichlet boundary conditions. In particular, we deduce the desired regularity as a corollary of the regularity for transmission problems (see [4, 12]). The main difficulty in extending the regularity results in [14, Section 5] to the problem we consider here is the positioning of the optimal favorable regions: in the case of homogeneous Dirichlet boundary conditions the favorable regions asymptotically concentrate in the interior of Ω ([14, Theorem 1.1]), while for Neumann boundary conditions the concentration occurs at $\partial\Omega$ (Theorem 1.1). Hence, [14, Section 5] essentially deals with interior regularity results, and we need to extend all results up to the boundary of Ω.

Proposition 4.4. *Assume that $\partial\Omega$ is of class $C^{3,\theta}$. Then, for all $i = 1,\dots,N-1$, the functions $\partial_i w_\delta$ are $H^1\left(B^+_{\kappa\delta^{-1/N}}\right)$-solutions of the transmission problem*

$$\begin{cases} -\operatorname{div}(A^\delta\nabla\partial_i w_\delta) = \operatorname{div}((\partial_i A^\delta)\nabla w_\delta) & \\ \qquad\qquad + \delta^{2/N}\Lambda(\delta)\widetilde{m}_\delta(J_\delta\partial_i w_\delta + w_\delta\partial_i J_\delta) & \text{in } \left(\widetilde{D}_\delta\cap B^+_{\kappa\delta^{-1/N}}\right)\cup\left(B^+_{\kappa\delta^{-1/N}}\setminus\overline{\widetilde{D}_\delta}\right), \\ [\partial_i w_\delta] = 0, & \text{on } \partial\widetilde{D}_\delta\cap B^+_{\kappa\delta^{-1/N}}, \\ [A^\delta\nabla(\partial_i w_\delta)\cdot\nu] = -\delta^{2/N}\Lambda(\delta)(1+\beta)w_\delta J_\delta\nu_i & \text{on } \partial\widetilde{D}_\delta\cap B^+_{\kappa\delta^{-1/N}}, \\ \partial_N\partial_i w_\delta = 0 = A^\delta\nabla\partial_i w_\delta\cdot e_N & \text{on } \{z_N = 0\}\cap B_{\kappa\delta^{-1/N}}, \end{cases} \tag{4.2}$$

where $[\cdot]$ denotes the jump across $\partial\widetilde{D}_\delta\cap B^+_{\kappa\delta^{-1/N}}$, ν_i denotes the i-th component of the outer unit normal to $\widetilde{D}_\delta\cap B^+_{\kappa\delta^{-1/N}}$, and e_N denotes the normal to the hyperplane $\{z_N = 0\}$. Moreover, there exists a constant $C > 0$ such that

$$\|w_\delta\|_{C^{2,\theta}\left(\overline{\widetilde{D}_\delta\cap B^+_{\kappa\delta^{-1/N}}}\right)} + \|w_\delta\|_{C^{2,\theta}\left(\overline{B^+_{\kappa\delta^{-1/N}}\setminus\widetilde{D}_\delta}\right)} \le C, \tag{4.3}$$

uniformly as $\delta\to 0$.

Proof. Let us start with (4.2). To begin with, notice that $\partial_N\partial_i w_\delta = 0 = A^\delta\nabla\partial_i w_\delta\cdot e_N$ a.e. on $\{z_N = 0\}\cap\overline{B^+_{\kappa\delta^{-1/N}}}$, since we are considering only derivatives of w_δ tangential to $\{z_N = 0\}$.

Using the integration by parts formula, for any $\varphi \in C_c^\infty\left(B^+_{\kappa\delta^{-1/N}}\right)$ we have

$$\begin{aligned}
&\delta^{2/N}\Lambda(\delta)\int_{B^+_{\kappa\delta^{-1/N}}}\widetilde{m}_\delta\varphi J_\delta\partial_i w_\delta = \delta^{2/N}\Lambda(\delta)\left[\int_{\widetilde{D}_\delta\cap B^+_{\kappa\delta^{-1/N}}}\varphi J_\delta\partial_i w_\delta - \beta\int_{B^+_{\kappa\delta^{-1/N}}\setminus\widetilde{D}_\delta}\varphi J_\delta\partial_i w_\delta\right] = \\
&= \delta^{2/N}\Lambda(\delta)(1+\beta)\int_{\partial\widetilde{D}_\delta\cap B^+_{\kappa\delta^{-1/N}}} w_\delta J_\delta\varphi\nu_i - \delta^{2/N}\Lambda(\delta)\int_{B^+_{\kappa\delta^{-1/N}}}\widetilde{m}_\delta\left(w_\delta J_\delta\partial_i\varphi + w_\delta\varphi\partial_i J_\delta\right),
\end{aligned} \tag{4.4}$$

Now, using (3.10) and integrating again by parts we have that

$$\begin{aligned}
&\delta^{2/N}\Lambda(\delta)\int_{B^+_{\kappa\delta^{-1/N}}}\widetilde{m}_\delta w_\delta J_\delta\partial_i\varphi = \delta^{2/N}\Lambda(\delta)\left[\int_{\widetilde{D}_\delta\cap B^+_{\kappa\delta^{-1/N}}} w_\delta J_\delta\partial_i\varphi - \beta\int_{B^+_{\kappa\delta^{-1/N}}\setminus\widetilde{D}_\delta} w_\delta J_\delta\partial_i\varphi\right] = \\
&= -\left[\int_{\widetilde{D}_\delta\cap B^+_{\kappa\delta^{-1/N}}}\operatorname{div}\left(A^\delta\nabla w_\delta\right)\partial_i\varphi + \int_{B^+_{\kappa\delta^{-1/N}}\setminus\widetilde{D}_\delta}\operatorname{div}\left(A^\delta\nabla w_\delta\right)\partial_i\varphi\right] = \\
&= \int_{B^+_{\kappa\delta^{-1/N}}} A^\delta\nabla w_\delta\cdot\partial_i\nabla\varphi = -\int_{B^+_{\kappa\delta^{-1/N}}} A^\delta\nabla\partial_i w_\delta\cdot\nabla\varphi - \int_{B^+_{\kappa\delta^{-1/N}}}\partial_i\left(A^\delta\right)\nabla w_\delta\cdot\nabla\varphi.
\end{aligned} \tag{4.5}$$

Combining (4.4) and (4.5) we obtain (4.2).

Now we turn to (4.3). Notice that the condition $\partial_N\partial_i w_\delta = 0$ on $\{z_N = 0\}\cap\overline{B^+_{\kappa\delta^{-1/N}}}$ and the properties of the diffeomorphism Φ_δ (see Section 3) allow to extend $\partial_i w_\delta$ by reflection on the whole $B_{\kappa\delta^{-1/N}}$, together with its equation. Again, this reflected function solves a transmission problem, but this time on the whole $B_{\kappa\delta^{-1/N}}$. More precisely,

$$\begin{cases} -\operatorname{div}\left(A^\delta\nabla\partial_i w_\delta\right) = \delta^{2/N}\Lambda(\delta)\widetilde{m}_\delta\left(J_\delta\partial_i w_\delta + w_\delta\partial_i J_\delta\right) + \operatorname{div}\left(G_i^\delta\right) & \text{in } \widetilde{D}_\delta\cup\left(B_{\kappa\delta^{-1/N}}\setminus\overline{\widetilde{D}_\delta}\right), \\ [\partial_i w_\delta] = 0, \quad [A^\delta\nabla(\partial_i w_\delta)\cdot\nu] = -\delta^{2/N}\Lambda(\delta)(1+\beta)w_\delta J_\delta\nu_i & \text{on } \partial\widetilde{D}_\delta, \end{cases} \tag{4.6}$$

where G_i^δ denotes the even extension of the function $\left(\partial_i A^\delta\right)\nabla w_\delta$ with respect to $\{z_N = 0\}$. Since $i \neq N$ and recalling Remark 3.3, we have that this function is of class $C^{0,\theta}$. Hence, as a consequence of [12, Theorem 1.2] we have, for all $i = 1,\dots,N-1$

$$\|\partial_i w_\delta\|_{C^{1,\theta}\left(\overline{\widetilde{D}_\delta\cap B^+_{\kappa\delta^{-1/N}}}\right)} + \|\partial_i w_\delta\|_{C^{1,\theta}\left(\overline{B^+_{\kappa\delta^{-1/N}}\setminus\widetilde{D}_\delta}\right)} \le C. \tag{4.7}$$

Strictly speaking, the application of [12, Theorem 1.2] to (4.6) requires two adjustments: indeed, such result applies to solutions to transmission problems having fixed interface, while here $\partial\widetilde{D}_\delta$ depends on δ, and homogeneous Dirichlet boundary conditions. Actually, by small perturbations, we can modify (4.6) to meet both these conditions. We describe such arguments in full details in the proof of Proposition 4.6 ahead, where the same issues have to be faced for a related transmission problem, see (4.11) and (4.13), to obtain a more delicate estimate.

We are left to prove that

$$\left\|\partial_N^2 w_\delta\right\|_{C^{0,\theta}\left(\overline{\widetilde{D}_\delta\cap B^+_{\kappa\delta^{-1/N}}}\right)} + \left\|\partial_N^2 w_\delta\right\|_{C^{0,\theta}\left(\overline{B^+_{\kappa\delta^{-1/N}}\setminus\widetilde{D}_\delta}\right)} \le C. \tag{4.8}$$

Condition (4.8) follows isolating $\partial_N^2 w_\delta$ in the equation (3.10) restricted to $B^+_{\kappa\delta^{-1/N}}$, and using (4.7). Hence, the proof is concluded. □

A direct consequence of Proposition 4.4 is the following

Corollary 4.5. *There exists a constant* $C > 0$ *such that*

$$\|w_\delta\|_{C^{1,1}\left(\overline{B_{\kappa\delta^{-1/N}}}\right)} \leq C,$$

uniformly as $\delta \to 0$.

Proof. It is sufficient to notice that Proposition 4.4 gives the $C^{1,1}$ regularity of w_δ on the closed half ball $\overline{B^+_{\kappa\delta^{-1/N}}}$, and that, since $\partial_N w_\delta = 0$ on $\{z_N = 0\}$, such regularity is preserved by reflection with respect to $\{z_N = 0\}$. Moreover,

$$\|w_\delta\|_{C^{1,1}\left(\overline{B_{\kappa\delta^{-1/N}}}\right)} \leq 2\|w_\delta\|_{C^{1,1}\left(\overline{B^+_{\kappa\delta^{-1/N}}}\right)}$$

□

Our next aim is to prove the following

Proposition 4.6. *For any* $r \in (0, r_2/4)$,

$$\begin{aligned} &\|\nabla w_\delta - (\nabla w_\delta \cdot n)\, n\|_{C^{1,\theta}\left(\overline{\widetilde{D}_\delta \cap \left(B^+_{\kappa\delta^{-1/N}} \setminus B_r\right)}\right)} + \\ &\qquad + \|\nabla w_\delta - (\nabla w_\delta \cdot n)\, n\|_{C^{1,\theta}\left(\overline{B^+_{\kappa\delta^{-1/N}} \setminus \left(\widetilde{D}_\delta \cup B_r\right)}\right)} \to 0 \end{aligned} \tag{4.9}$$

as $\delta \to 0$, *where* $n = z/|z|$.

Proof. Let I, w and m be respectively the principal positive eigenvalue, the optimal eigenfunction and the corresponding weight of the design problem in $\mathbb{R}^N$, as introduced in (1.8). Recall that w is radially symmetric. Let us define $h_i = \partial_i w$ the partial derivatives of w in the directions tangent to $\{z_N = 0\}$, i.e. $i = 1, \ldots, N-1$. Notice that each h_i is even with respect to $\{z_N = 0\}$. The functions h_i solve the transmission problem

$$\begin{cases} -\Delta h_i = I m h_i & \text{in } B_{r_2} \cup (B_{\kappa\delta^{-1/N}} \setminus B_{r_2}), \\ [h_i] = 0, \quad [\partial_n h_i] = -I(1+\beta) w n_i & \text{on } \partial B_{r_2}, \end{cases} \tag{4.10}$$

where $n = z/|z|$.

Now consider the optimal (reflected) eigenfunctions w_δ in the blow-up scale, and denote $v_{\delta,i} := \partial_i w_\delta$, for $i = 1, \ldots, N-1$. From (4.6), we know that they solve the transmission problem

$$\begin{cases} -\operatorname{div}\left(A^\delta \nabla v_{\delta,i}\right) = \delta^{2/N} \Lambda(\delta) \widetilde{m}_\delta \left(v_{\delta,i} J_\delta + w_\delta \partial_i J_\delta\right) + \operatorname{div}\left(G^\delta_i\right) & \text{in } \widetilde{D}_\delta \cup \left(B_{\kappa\delta^{-1/N}} \setminus \overline{\widetilde{D}_\delta}\right), \\ [v_{\delta,i}] = 0, \quad [A^\delta \nabla v_{\delta,i} \cdot \nu] = -\delta^{2/N} \Lambda(\delta)(1+\beta) w_\delta J_\delta \nu_i & \text{on } \partial\widetilde{D}_\delta. \end{cases} \tag{4.11}$$

In order to compare the functions $v_{\delta,i}$ and h_i in $B_{\kappa\delta^{-1/N}}$, for δ sufficiently small we introduce a one-parameter family of radial diffeomorphisms $\Theta_\delta : \mathbb{R}^N \to \mathbb{R}^N$, with the following properties:

$$\begin{aligned} &\|\Theta_\delta - \mathrm{Id}\|_{C^{1,\alpha}(\mathbb{R}^N)} = o_\delta(1) \text{ for every } 0 < \alpha < 1, \text{ as } \delta \to 0, \\ &\Theta_\delta = \mathrm{Id} \text{ in } \mathbb{R}^N \setminus B_{\frac{3}{2}r_2}, \quad \Theta_\delta(B_{r_2}) = \widetilde{D}_\delta, \quad \Theta_\delta(\partial B_{r_2}) = \partial\widetilde{D}_\delta, \end{aligned} \tag{4.12}$$

where $|B_{r_2}| = 2$. The existence of such family of diffeomorphisms follows from Proposition 4.3 (actually Θ_δ depends on φ_δ there); we refer to [14, Section 6.1] for more details on such diffeomorphism. In particular, (4.12) holds true with $\alpha = \theta$.

Let the functions $\phi_{\delta,i}$ be defined as

$$\phi_{\delta,i} := v_{\delta,i} \circ \Theta_\delta - h_i.$$

From (4.10) and (4.12) we deduce that the functions $\phi_{\delta,i}$ solve, in $H^1\left(B_{\kappa\delta^{-1/N}}\right)$, the transmission problem

$$\begin{cases} -\operatorname{div}\left(M^\delta \nabla \phi_{\delta,i}\right) = f_i^\delta + \operatorname{div}\left(F_i^\delta\right) & \text{in } B_{r_2} \bigcup \left(B_{\kappa\delta^{-1/N}} \setminus B_{r_2}\right), \\ [\phi_{\delta,i}] = 0, \quad [M^\delta \nabla \phi_{\delta,i} \cdot n] = g_i^\delta & \text{on } \partial B_{r_2}, \end{cases} \tag{4.13}$$

where we have denoted

$$Y_\delta := |\det(D\Theta_\delta)|, \quad M^\delta := Y_\delta D\Theta_\delta^{-1} A^\delta D\Theta_\delta^{-T} \quad \text{in } B_{\kappa\delta^{-1/N}}, \quad Y_{\delta,T} := |D\Theta_\delta^{-T} n| Y_\delta \quad \text{on } \partial B_{r_2},$$
$$f_i^\delta := \delta^{2/N} \Lambda(\delta) \left(\widetilde{m}_\delta v_{\delta,i} J_\delta + \widetilde{m}_\delta w_\delta \partial_i J_\delta\right) \circ \Theta_\delta - Imh_i, \quad F_i^\delta := Y_\delta D\Theta_\delta^{-1} G_i^\delta + \left(M^\delta - \mathrm{Id}\right) \nabla h_i \quad \text{in } B_{\kappa\delta^{-1/N}},$$
$$g_i^\delta := (1+\beta)\left(Iwn_i - \delta^{2/N} \Lambda(\delta) \left(w_\delta J_\delta \nu_i\right) \circ \Theta_\delta Y_{\delta,T}\right) - \left[\left(M^\delta - Id\right) \nabla h_i \cdot n\right] \quad \text{on } \partial B_{r_2}.$$

Without loss of generality, we can also assume that

$$\phi_{\delta,i} = 0 \text{ on } \partial B_{\kappa\delta^{-1/N}}.$$

Indeed, for δ sufficiently small, it is sufficient to subtract from $\phi_{\delta,i}$ the function $z_{\delta,i}$ solution to

$$\begin{cases} -\operatorname{div}\left(M^\delta \nabla z_{\delta,i}\right) = 0 & \text{in } B_{\kappa\delta^{-1/N}}, \\ z_{\delta,i} = \phi_{\delta,i} & \text{on } \partial B_{\kappa\delta^{-1/N}}, \end{cases}$$

and to notice that, by elliptic regularity and [21, Remark 4.5],

$$\|z_{\delta,i}\|_{C^{1,\theta}\left(\overline{B_{\kappa\delta^{-1/N}}}\right)} \to 0, \quad \text{as } \delta \to 0.$$

We are in the position to apply the results in [12, Theorem 1.2], and since [14, Lemma 5.4], Proposition 3.4 and (3.6) imply that

$$\|f_i^\delta\|_{L^\infty\left(B_{\kappa\delta^{-1/N}}\right)} \to 0, \quad \|F_i^\delta\|_{C^{0,\theta}\left(B_{\kappa\delta^{-1/N}}\right)} \to 0 \quad \text{and} \quad \|g_i^\delta\|_{C^{0,\theta}\left(\partial B_{r_2}\right)} \to 0$$

as $\delta \to 0$, [12, Theorem 1.2] gives that, for all $i = 1, \dots, N-1$

$$\|\phi_{\delta,i}\|_{C^{1,\theta}\left(\overline{B_{r_2}}\right)} + \|\phi_{\delta,i}\|_{C^{1,\theta}\left(\overline{B_{\kappa\delta^{-1/N}} \setminus B_{r_2}}\right)} \to 0, \quad \text{as } \delta \to 0. \tag{4.14}$$

Moreover, as usual isolating $\partial_N^2 w_\delta$ in equation (3.10) and composing it with the diffeomorphism Θ_δ, it can be seen using (4.14) that

$$\|\partial_N\left(v_{\delta,N} \circ \Theta_\delta\right) - \partial_N^2 w\|_{C^{0,\theta}\left(\overline{B_{r_2} \cap B^+_{\kappa\delta^{-1/N}}}\right)} + \|\partial_N\left(v_{\delta,N} \circ \Theta_\delta\right) - \partial_N^2 w\|_{C^{0,\theta}\left(\overline{B^+_{\kappa\delta^{-1/N}} \setminus B_{r_2}}\right)} \to 0, \quad \text{as } \delta \to 0.$$

Hence, we have just shown that

$$\|\phi_{\delta,i}\|_{C^{1,\theta}\left(\overline{B_{r_2} \cap B^+_{\kappa\delta^{-1/N}}}\right)} + \|\phi_{\delta,i}\|_{C^{1,\theta}\left(\overline{B^+_{\kappa\delta^{-1/N}} \setminus B_{r_2}}\right)} \to 0, \quad \text{as } \delta \to 0, \quad \forall\, i = 1, \dots, N. \tag{4.15}$$

Now, in order to prove the proposition, it is sufficient to notice that, by (4.15) and (4.12) we have

$$\begin{aligned} &\| (\nabla w_\delta - (\nabla w_\delta \cdot n)\, n) \circ \Theta_\delta - (\nabla w - (\nabla w \cdot n)\, n) \|_{C^{1,\theta}\left(\overline{\widetilde{D}_\delta \cap \left(B^+_{\kappa\delta^{-1/N}} \setminus B_r\right)}\right)} + \\ &+ \| (\nabla w_\delta - (\nabla w_\delta \cdot n)\, n) \circ \Theta_\delta - (\nabla w - (\nabla w \cdot n)\, n) \|_{C^{1,\theta}\left(\overline{B^+_{\kappa\delta^{-1/N}} \setminus \left(\widetilde{D}_\delta \cup B_r\right)}\right)} \to 0. \end{aligned} \tag{4.16}$$

as $\delta \to 0$, but since the function w is radial, in $B^+_{\kappa\delta^{-1/N}}$

$$\nabla w - (\nabla w \cdot n)\, n = 0,$$

so that (4.16) can be rewritten as

$$\begin{aligned} &\| (\nabla w_\delta - (\nabla w_\delta \cdot n)\, n) \circ \Theta_\delta \|_{C^{1,\theta}\left(\overline{\widetilde{D}_\delta \cap \left(B^+_{\kappa\delta^{-1/N}} \setminus B_r\right)}\right)} + \\ &\qquad + \| (\nabla w_\delta - (\nabla w_\delta \cdot n)\, n) \circ \Theta_\delta \|_{C^{1,\theta}\left(\overline{B^+_{\kappa\delta^{-1/N}} \setminus \left(\widetilde{D}_\delta \cup B_r\right)}\right)} \to 0. \end{aligned} \tag{4.17}$$

To conclude, the claim (4.9) follows from (4.17) just rewriting

$$\nabla w_\delta - (\nabla w_\delta \cdot n)\, n = (\nabla w_\delta - (\nabla w_\delta \cdot n)\, n) \circ \Theta_\delta \circ \Theta_\delta^{-1}$$

and using (4.12) once again. □

As a consequence of Proposition 4.6, we have the following corollary.

Corollary 4.7. *For any $r \in (0, r_2/4)$,*

$$\|\nabla w_\delta - (\nabla w_\delta \cdot n)\, n\|_{C^{0,1}\left(\overline{B_{\kappa\delta^{-1/N}} \setminus B_r}\right)} \to 0, \qquad \text{as } \delta \to 0,$$

where $n = z/|z|$.

Finally, we can conclude this section with the following improvement of Proposition 4.3.

Proposition 4.8. *For δ sufficiently small $\widetilde{D}_\delta$ is nearly spherical of class $C^{1,1}$, parametrized by φ_δ (see Definition 4.1). In addition*

$$\|\varphi_\delta\|_{C^{1,1}(\mathbb{S}^{N-1})} \to 0, \qquad \text{as } \delta \to 0.$$

Proof. In view of Proposition 4.4 and Corollary 4.7, the result can be proved by arguing as in [14, Proposition 5.10]. □

Remark 4.9. The above $C^{1,1}$ decay is sharp, on $\mathbb{S}^{N-1}$, because of the reflection of the N-th derivative. Actually, since $\partial\Omega \in C^{3,\theta}$, one may expect further regularity of the free boundary. As a matter of fact, the estimates in $C^{2,\theta}$ of w_δ hold up to the free boundary, from inside and outside; therefore a continuation argument should allow to improve the above $C^{1,1}(\mathbb{S}^{N-1})$ decay to a $C^{2,\theta}(\mathbb{S}^{N-1}_+)$ one, but we do not pursue this argument here.

5 Quantitative estimates

In this section we conclude the proof of Theorem 1.3 by obtaining the quantitative estimates for the nearly spherical parametrizations of the optimal sets $\widetilde{D}_\delta$ (in the blow-up scale, as introduced in Sections 3, 4) and D_δ (in the original reference). These estimates are based on Proposition 4.8 and on the application of the following result.

Theorem 5.1 ([14, Theorem 1.4])**.** *There exist positive constants C, ε such that, for all $C^{1,1}$ nearly spherical sets $A \subset \mathbb{R}^N$, centered at the origin and parametrized by φ satisfying*

1. $\mathrm{bar}(A) = 0$,
2. $|A| = 2$,
3. $\|\varphi\|_{C^{1,1}(\mathbb{S}^{N-1})} \le \varepsilon$,

it holds

$$\lambda(A, \mathbb{R}^N) - \lambda(B_{r_2}, \mathbb{R}^N) \ge C\|\varphi\|^2_{L^2(\mathbb{S}^{N-1})}.$$

To apply this result to $\widetilde{D}_\delta$ we notice that its first assumption is verified by centering the blow-up analysis at $Q_\delta := \Phi_\delta(\mathrm{bar}(\widetilde{D}_\delta))$, rather than P_δ (recall that we are working in this setting since (4.1)), thanks to Lemma 4.2). On the other hand, $|\widetilde{D}_\delta|$ is equal to 2 only in the limit, thus we need to take into account an error term.

Lemma 5.2. *There exists $C > 0$ such that, for δ sufficiently small,*

$$\left(\frac{|\widetilde{D}_\delta|}{2}\right)^{2/N} \lambda(\widetilde{D}_\delta, \mathbb{R}^N) - I \ge C\|\varphi_\delta\|^2_{L^2(\mathbb{S}^{N-1})} + o(\delta^{1/N}).$$

Proof. First, recall that $\widetilde{D}_\delta$ is $C^{1,1}$ nearly spherical, with parametrization φ_δ satisfying Proposition 4.8. We infer that the set

$$A_\delta = \frac{2^{1/N}\widetilde{D}_\delta}{|\widetilde{D}_\delta|^{1/N}}$$

is nearly spherical, too: indeed, for every $\eta \in \partial A_\delta$ there exists $z \in \partial\widetilde{D}_\delta$ such that $\eta = 2^{1/N} z |\widetilde{D}_\delta|^{-1/N}$, so that

$$\partial A_\delta = \{\eta = (r_2 + \sigma_\delta(\theta))\,\theta\}, \quad \text{with } \sigma_\delta = \frac{2^{1/N} r_2}{|\widetilde{D}_\delta|^{1/N}} - r_2 + \frac{2^{1/N}}{|\widetilde{D}_\delta|^{1/N}}\varphi_\delta. \tag{5.1}$$

We are in position to apply Theorem 5.1 to A_δ, obtaining

$$\left(\frac{|\widetilde{D}_\delta|}{2}\right)^{2/N} \lambda(\widetilde{D}_\delta, \mathbb{R}^N) - I \ge C\|\sigma_\delta\|^2_{L^2}.$$

In turn, using Lemma 3.8,

$$\begin{aligned}\|\sigma_\delta\|^2_{L^2} &= \left\| \frac{2^{1/N} r_2}{|\widetilde{D}_\delta|^{1/N}} - r_2 + \frac{2^{1/N}}{|\widetilde{D}_\delta|^{1/N}}\varphi_\delta \right\|^2_{L^2} \ge \frac{1}{2}\left(\frac{2}{|\widetilde{D}_\delta|}\right)^{2/N}\|\varphi_\delta\|^2_{L^2} - \left\|\left(\frac{2}{|\widetilde{D}_\delta|}\right)^{1/N} - 1\right\|^2_{L^2} r_2^2 \\ &\ge C(1 - \delta^{1/N})\,\|\varphi_\delta\|^2_{L^2} + o(\delta^{1/N}) = C\,\|\varphi_\delta\|^2_{L^2} + o(\delta^{1/N}),\end{aligned}$$

where we used also Proposition 4.8, and the claim follows. □

Proposition 5.3. *As $\delta \to 0$,*

$$\|\varphi_\delta\|^2_{L^2(\mathbb{S}^{N-1})} = o(\delta^{1/N}). \tag{5.2}$$

Proof. We apply the second conclusion of Theorem 1.2, Remark 3.11 and Lemma 5.2 to obtain

$$\begin{aligned}
-I\Gamma\widehat{H}\delta^{1/N} + o(\delta^{1/N}) &= \delta^{2/N}\Lambda(\delta) - I \\
&= \delta^{2/N}\Lambda(\delta) - \left(\frac{|\widetilde{D}_\delta|}{2}\right)^{2/N}\lambda(\widetilde{D}_\delta, \mathbb{R}^N) + \left(\frac{|\widetilde{D}_\delta|}{2}\right)^{2/N}\lambda(\widetilde{D}_\delta, \mathbb{R}^N) - I \\
&\geq -I\Gamma\widehat{H}\delta^{1/N} + C\|\varphi_\delta\|^2_{L^2} + o(\delta^{1/N}),
\end{aligned} \tag{5.3}$$

implying the claim. □

Finally, we bring back the quantitative information (5.2) to D_δ.

Proof of Theorem 1.3. By now we have obtained

$$\widetilde{D}_\delta = \left\{ z \in B_{k\delta^{-1/N}} : |z| < r_2 + \varphi_\delta\left(\frac{z}{|z|}\right)\right\},$$

so that

$$\widetilde{D}^+_\delta := \left\{ z \in B^+_{k\delta^{-1/N}} : |z| < r_2 + \varphi_\delta\left(\frac{z}{|z|}\right)\right\} = \left\{ z \in B^+_{2r_2} : |z| < r_2 + \varphi_\delta\left(\frac{z}{|z|}\right)\right\} \tag{5.4}$$

for δ small enough.

On the other hand, recalling (3.13),

$$\widetilde{D}^+_\delta = \left\{ z \in B^+_{2r_2} : z = \tfrac{\Psi_\delta(x)}{\delta^{1/N}}, \text{ for some } x \in D_\delta \right\}.$$

Then $x \in \partial D_\delta$ if and only if $z \in \partial\widetilde{D}^+_\delta$, and the theorem will follow by a change of variable.

Thus, in the following, we consider a generic $z \in B^+_{2r_2}$ and $x = \Phi_\delta(\delta^{1/N}z)$, so that $|x| = O(\delta^{1/N})$ as $\delta \to 0$. We obtain

$$x = \Phi_\delta(\delta^{1/N}z) = \delta^{1/N}z + \delta^{2/N}R_\delta(z), \tag{5.5}$$

where the reminder R_δ is uniformly bounded in $C^{1,1}$: notice that this only requires $\Phi_\delta \in C^{1,1}$ uniformly in δ, i.e. $\partial\Omega \in C^{2,1}$ (recall Remark 3.2). Indeed, we have

$$\begin{aligned}
\delta^{2/N}R_\delta(z) &= \Phi_\delta(\delta^{1/N}z) - \delta^{1/N}z = \Phi_\delta(\delta^{1/N}z) - \Phi_\delta(0) - D\Phi_\delta(0)\delta^{1/N}z, \\
\delta^{1/N}DR_\delta(z) &= D\Phi_\delta(\delta^{1/N}z) - D\Phi_\delta(0),
\end{aligned}$$

which yield

$$\begin{aligned}
\delta^{2/N}|R_\delta(z)| &= \left| \int_0^1 \left[\frac{d}{dt}\Phi_\delta(t\delta^{1/N}z) - D\Phi_\delta(0)\delta^{1/N}z \right] dt \right| \\
&\leq \delta^{1/N}|z| \int_0^1 \left| D\Phi_\delta(t\delta^{1/N}z) - D\Phi_\delta(0) \right| dt \\
&\leq \delta^{2/N}|z|^2 \cdot \frac{1}{2}\|D\Phi_\delta\|_{C^{0,1}} \leq \delta^{2/N} \cdot 2\|D\Phi_\delta\|_{C^{0,1}} r_2^2
\end{aligned}$$

and

$$\delta^{1/N}|DR_\delta(z_1)-DR_\delta(z_2)| = \left|D\Phi_\delta(\delta^{1/N}z_1)-D\Phi_\delta(\delta^{1/N}z_2)\right| \le \delta^{1/N}\|D\Phi_\delta\|_{C^{0,1}}|z_1-z_2|.$$

From (5.5) we infer

$$|x|^2 = \delta^{2/N}|z|^2 + \delta^{\frac{3}{N}}\tilde{R}_\delta(z), \tag{5.6}$$

where $\tilde{R}_\delta$ is uniformly in $C^{1,1}$, too.

Let us introduce the polar coordinates

$$\rho = |x|,\ \vartheta = \frac{x}{|x|}, \qquad r = |z|,\ \xi = \frac{z}{|z|}.$$

Taking $x \in \partial D_\delta \cap \Omega$, and the corresponding $z = r\xi \in \partial\widetilde{D}_\delta^+$, then $r = r_2 + \varphi_\delta(\xi)$. By the properties of Φ_δ, Ψ_δ, φ_δ we have that the map

$$\mathbb{S}^{N-1} \ni \xi \mapsto \frac{\Phi_\delta(\delta^{1/N}(r_2+\varphi_\delta(\xi))\xi)}{|\Phi_\delta(\delta^{1/N}(r_2+\varphi_\delta(\xi))\xi)|} = \vartheta_\delta(\xi) \in \mathbb{S}^{N-1}$$

is uniformly bounded in $C^{1,1}$, with inverse $\vartheta \mapsto \xi = \xi_\delta(\vartheta)$, for δ small enough. Substituting into (5.6) we infer that (1.10) holds true with $\rho_\delta = \rho_\delta(\vartheta)$ implicitly defined by

$$2r_2\rho_\delta + \rho_\delta^2 = 2r_2\varphi_\delta(\xi_\delta(\vartheta)) + \varphi_\delta^2(\xi_\delta(\vartheta)) + \delta^{1/N}\tilde{R}_\delta\Big((r_2+\varphi_\delta(\xi_\delta(\vartheta)))\xi_\delta(\vartheta)\Big).$$

Then

$$\rho_\delta(\vartheta) = \varphi_\delta(\xi_\delta(\vartheta)) + \delta^{1/N}Z_\delta(\vartheta),$$

where Z_δ is uniformly bounded in $C^{1,1}$. Recalling (5.2) and Proposition 4.8 we obtain the desired estimates. $\square$

Remark 5.4. From Theorem 1.3, thanks to the Gagliardo Nirenberg interpolation inequalities, one can deduce as in [14, Corollary 1.5] estimates on different norms of ρ_δ, the most interesting one being

$$\|\rho_\delta\|_{C^{1,\alpha}} = o\left(\delta^{\frac{(1-\alpha)}{N(4+N)}}\right) \qquad \forall\,\alpha \in (0,1).$$

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