Homogenization of changing-type evolution equations

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Abstract

In this paper we study the homogenization of the linear equation

$$R(\varepsilon^{-1}x)\frac{\partial u_\varepsilon}{\partial t} - \mathrm{div}(a(\varepsilon^{-1}x)\cdot\nabla u_\varepsilon) = f\ ,$$

with appropriate initial/final conditions, where R is a measurable bounded periodic function and a is a bounded uniformly elliptic matrix, whose coefficients a_{ij} are measurable periodic functions.

Since we admit that R may vanish and change sign, the usual compactness of the solutions in L^2 may not hold if the mean value of R is zero.

1 Introduction

In this paper we will state a homogenization result for the problem

$$\begin{cases} R_\varepsilon(x)\dfrac{\partial u_\varepsilon}{\partial t}(x,t) - \mathrm{div}\big(a_\varepsilon(x)\nabla u_\varepsilon(x,t)\big) = f(x,t) & \text{in } \Omega\times(0,T)\,, \\ u_\varepsilon(x,t) = 0 & \text{on } \partial\Omega\times(0,T)\,, \\ u_\varepsilon(x,0) = \varphi(x) & \text{on } \Omega_{+,\varepsilon}\,, \\ u_\varepsilon(x,T) = \psi(x) & \text{on } \Omega_{-,\varepsilon}\,; \end{cases} \tag{1.1}$$

i.e., we will characterize the asymptotic limit, for $\varepsilon \to 0^+$, of the sequence of the solutions $\{u_\varepsilon\}$. Here, Ω is an open bounded subset of $\mathbf{R}^N$ with smooth boundary, $T > 0$, $f \in L^2(0,T;H^{-1}(\Omega))$, $\varphi,\psi \in L^2(\Omega)$, $a_\varepsilon(x) = a(\varepsilon^{-1}x)$, where $a = a_{ij}$ is a measurable bounded periodic matrix which is uniformly elliptic, $R_\varepsilon(x) = R(\varepsilon^{-1}x)$, where $R : \mathbf{R}^n \to \mathbf{R}$ is a measurable bounded periodic function which may vanish and change sign, $\Omega_{+,\varepsilon}$ (respectively, $\Omega_{-,\varepsilon}$) is the subset of Ω where $R_\varepsilon > 0$ (respectively, $R_\varepsilon < 0$).

In the case $R \geqslant 0$, the homogenization of problem (1.1) has been studied in [14], [16]. Some recent existence results can be found in [17] and, for an interesting survey of physical applications, see also [5].

In the case where R changes sign, particular cases of the equation in (1.1), for fixed $\varepsilon = 1$, arise in the kinetic theory (see, for instance, [6]) and have been already considered in [3] and [11]. Problem (1.1), in its general setting, is studied in [13] and [15], in connection with existence and uniqueness results and, in some particular situations, in [1], in connection with the homogenization.

We point out that in (1.1) the initial datum is only prescribed in the region where $R_\varepsilon > 0$, i.e. where the equation is "forward parabolic", the final datum only where $R_\varepsilon < 0$, i.e. where the equation is "backward parabolic", while no datum is given in the region where $R_\varepsilon = 0$, i.e. where the equation is "elliptic" in the variable x, with t as a parameter.

Setting $\overline{R}$ the mean value of the function R (which can be positive, negative or null), by standard a-priori estimates, it is not difficult to see that, up to a subsequence, the solutions u_ε converge, weakly in $L^2(0,T;H_0^1(\Omega))$, to a function u_0 which we will characterize as the weak solution, in the sense of distributions, of the equation

$$\overline{R}\frac{\partial u_0}{\partial t}(x,t) - \operatorname{div}\big(a^*\nabla u_0(x,t)\big) = f(x,t) \qquad \text{in } \Omega \times (0,T)\,, \tag{1.2}$$

with $u_0(x,t) = 0$ on $\partial\Omega$, in the sense of traces, for a.e. $t \in (0,T)$ and proper initial/final conditions. Here, a^* is the homogenized matrix given in (3.13) below. Note that in the case $\overline{R} = 0$, we have a parametric elliptic equation.

The main difference with respect to the results in [16] is the lack of compactness of the sequence $\{u_\varepsilon\}$ in the space of continuous functions, which is crucial to pass to the limit in the initial/final data.

In the case $\overline{R} \neq 0$, the previous problem was solved by the authors in [1] for the case of Laplace operator, even with an appropriate nonlinear reaction term in the righthandside.

In that paper, we obtained that, in the case $\overline{R} > 0$, the limit $u_0 \in L^2(0,T;H_0^1(\Omega))$ is uniquely determined by (1.2), with the appropriate limit data given by the original initial datum and no final datum is prescribed; in the case $\overline{R} < 0$, the limit u_0 satisfies the final condition, while the initial condition is lost. The crucial tool is a compactness result in $L^2(\Omega \times (0,T))$ for the sequence $\{u_\varepsilon\}$, which does not hold true, in general, if we admit that $\overline{R}$ can also be null.

In the present paper, we overcome this difficulty, obtaining the homogenization result for any $\overline{R}$ (and for a general operator), using a different compactness property (see lemma 3.4), which leads to a new and (at least for the linear case) more general proof. Neverthless, the starting point is the usual asymptotic expansion of the operator and of the solutions, introduced in [4] (see Section 3).

In the case $\overline{R} > 0$, our main result is that the whole sequence $\{u_\varepsilon\}$ converges strongly in $L^2(\Omega \times (0,T))$ to the solution $u_0 \in L^2(0,T;H_0^1(\Omega))$ of the problem

$$\begin{cases} \overline{R}\dfrac{\partial u_0}{\partial t}(x,t) - \operatorname{div}\big(a^*\nabla u_0(x,t)\big) = f(x,t) & \text{in } \Omega \times (0,T)\,, \\ u_0(x,t) = 0 & \text{on } \partial\Omega \times (0,T)\,, \\ u_0(x,0) = \varphi(x) & \text{on } \Omega\,; \end{cases} \tag{1.3}$$

where a^* is the usual constant homogenized matrix (see theorem 3.3). Analogous results are obtained for $\overline{R} < 0$ and for $\overline{R} = 0$ (see theorems 3.5 and 3.6): in the first case only the final condition passes to the limit; in the second one no initial/final condition passes to the limit. We remark that in all these cases there is no interaction between the homogenization of the two operators R_ε and a_ε.

Finally, we recall that our result solves an open problem suggested by A. Pankov in one of his books (see [12], open problems - 10).

The paper is organized as follows: in Section 2 we set our notations and recall some preliminary results on the existence and uniqueness of the solution of (1.1). In Section 3 the homogenization thoerems, i.e. the main results of the paper, are stated and proved (see theorems 3.3, 3.5 and 3.6). Moreover, in that section we obtain a crucial compactness result (see lemma 3.4) for an appropriate temporal mean average of the sequence of the solutions $\{u_\varepsilon\}$.

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2 Notations and preliminary results

2.1 Notations

Let $A \subseteq \mathbf{R}^n$, $n \geqslant 1$, be a given open set, with smooth boundary, for simplicity. We denote by $\overline{A}$ the closure of A and by ∂A the boundary of A.

For any integer k, $\mathcal{C}^k(A)$ (resp. $\mathcal{C}^\infty(A)$) is the set of all real functions defined on A, which admit continuous partial derivatives up to order k (resp. having continuous partial derivatives of any order). In particular, $\mathcal{C}^k_c(A)$, is the subset of those functions belonging to $\mathcal{C}^k(A)$, with compact support in A. For simplicity, $\mathcal{C}^0(A)$ (resp. $\mathcal{C}^0_c(A)$) is also denoted by $\mathcal{C}(A)$ (resp. $\mathcal{C}_c(A)$).

We denote by $L^p(A)$ and $W^{k,p}(A)$, $1 \leqslant p \leqslant \infty$, $k \in \mathbf{N}$, (resp. $L^p_{\text{loc}}(A)$ and $W^{k,p}_{\text{loc}}(A)$) the standard Lebesgue and Sobolev spaces. In particular, we set $H^1(A) := W^{1,2}(A)$ and denote by $H^1_0(A)$ the subset of $H^1(A)$ of those functions having null trace on ∂A. As usual, $H^{-1}(A)$ is the topological dual space of $H^1_0(A)$.

Let $Y = (0,1)^n$ be the unit cell in $\mathbf{R}^n$. A function defined on $\mathbf{R}^n$ is said to be Y-periodic if it is periodic of period 1 with respect to each variable x_i, with $1 \leqslant i \leqslant n$. We denote by $L^p_\#(Y)$ and $W^{k,p}_\#(Y)$, $1 \leqslant p \leqslant \infty$, $k \in \mathbf{N}$, the space of functions in $L^p_{\text{loc}}(\mathbf{R}^n)$ or $W^{k,p}_{\text{loc}}(\mathbf{R}^n)$, respectively, which are Y-periodic. As usual, $H^1_\#(Y) := W^{1,2}_\#(Y)$.

Let I be a real interval and X a topological vector space. We denote by $L^p(I;X)$, $1 \leqslant p \leqslant \infty$, the space of measurable functions $h : I \to X$, such that

$$\int_I \|h(t)\|^p_X \, dt < +\infty \quad \text{if } 1 \leqslant p < +\infty \ , \qquad \operatorname*{ess-sup}_{t \in I} \|h(t)\|_X < +\infty \quad \text{if } p = +\infty \ .$$

Throughout this paper, Ω is an open bounded subset of $\mathbf{R}^n$ with smooth boundary (for simplicity assume $\partial\Omega$ of class $\mathcal{C}^\infty$) and T is a positive number; we set $\Omega_T = \Omega \times (0,T)$. If it is not otherwise specified, we adopt the convention that repeated indices indicate summation. Finally, the letter C denotes a strictly positive constant which may vary from line to line.

2.2 Mixed type evolution equations

In the rest of this section we want to present an existence result for evolution equations of mixed type, i.e. which may be partially elliptic and partially parabolic, both forward and backward. To this purpose, let us introduce the following class of matrices.

Definition 2.1 *Fix* $\lambda, \Lambda \in \mathbf{R}$ *with* $0 < \lambda \leqslant \Lambda$. *We denote by* $\mathcal{M}_\Omega(\lambda, \Lambda)$ *the set of* $n \times n$ *matrices* $a = (a_{ij}(x))_{i,j=1,\dots n} \in L^\infty(\Omega; \mathbf{R}^{n^2})$ *such that*

$$\begin{cases} a_{ij}(x)\ \xi_i\ \xi_j \geqslant \lambda|\xi|^2 \\ |a(x)\cdot\xi| \leqslant \Lambda|\xi| \end{cases} \tag{2.1}$$

for every $\xi \in \mathbf{R}^n$, *for a.e.* $x \in \Omega$.

Given a matrix $a \in \mathcal{M}_\Omega(\lambda, \Lambda)$, we consider the operator $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ defined by $Au = -\mathrm{div}\big(a\nabla u\big)$. Let $R(x)$ be a given function in $L^\infty(\Omega)$. We set

$$\Omega_+ = \{x \in \Omega \mid R(x) > 0\}\,, \quad \Omega_0 = \{x \in \Omega \mid R(x) = 0\}\,, \quad \Omega_- = \{x \in \Omega \mid R(x) < 0\}\,, \tag{2.2}$$

and assume that $\Omega_\pm, \Omega_0$ have Lipschitz boundaries.

Our first step is to define a solution for the following problem

$$\begin{cases} R(x)\dfrac{\partial u}{\partial t}(x,t) - \mathrm{div}\big(a(x)\nabla u(x,t)\big) = f(x,t) & \text{in } \Omega\times(0,T)\,, \\ u(x,t) = 0 & \text{on } \partial\Omega\times(0,T)\,, \\ u(x,0) = \varphi(x) & \text{on } \Omega_+\,, \\ u(x,T) = \psi(x) & \text{on } \Omega_-\,, \end{cases} \tag{2.3}$$

where $f : \Omega_T \to \mathbf{R}$, $\varphi : \Omega_+ \to \mathbf{R}$, $\psi : \Omega_- \to \mathbf{R}$ are the data of the problem, for which appropriate regularity assumptions will be required.
In order to achieve this goal, we consider the space

$$\mathcal{W} = \big\{u \in L^2(0,T; H_0^1(\Omega)) \;\big|\; (Ru)' \in L^2(0,T; H^{-1}(\Omega))\big\} \tag{2.4}$$

(where $(Ru)'$ is the distributional derivative of Ru with respect to t), endowed with the natural norm

$$\|u\|_{\mathcal{W}} = \|u\|_{L^2(0,T;H_0^1(\Omega))} + \|(Ru)'\|_{L^2(0,T;H^{-1}(\Omega))}\,. \tag{2.5}$$

Following [13] and [15], we give the definition below.

Definition 2.2 *Let* $f \in L^2(0,T; H^{-1}(\Omega))$, $\varphi \in L^2(\Omega_+)$, $\psi \in L^2(\Omega_-)$. *We say that a function* $u \in \mathcal{W}$ *is a solution of the problem* (2.3), *if*

$$(Ru)'(t) + Au(t) = f(t) \qquad \text{for almost every } t \in (0,T)$$

and

$$u(x,0) = \varphi(x) \quad \text{for a.e. } x \in \Omega_+\,, \quad u(x,T) = \psi(x) \quad \text{for a.e. } x \in \Omega_-\,.$$

Note that $R^+u^2,\ R^-u^2 \in \mathcal{C}^0([0,T]; L^1(\Omega))$ (see [13] and [15]); therefore, the initial/final conditions make sense. Observe also that, denoting by $\langle\langle\cdot,\cdot\rangle\rangle$ the duality pairing between $L^2(0,T;H^{-1}(\Omega))$ and $L^2(0,T;H_0^1(\Omega))$ and by $((\cdot,\cdot))$ the scalar product in $L^2(0,T;L^2(\Omega))$, we have

$$\langle\langle (Ru)', \theta\rangle\rangle = -((Ru, \theta')) = -\int_0^T\int_\Omega R(x)u(x,t)\frac{\partial\theta}{\partial t}(x,t)dxdt \qquad \forall\theta \in C_c^1(\Omega_T)\,.$$

We have the following result (see [15]).

Theorem 2.3 *Let* f, φ, ψ *and* R *as before. Then, problem* (2.3) *admits a unique solution* (*in the sense of definition* 2.2). *Moreover, the following estimate holds:*

$$\|u\|_{\mathcal{W}} \leqslant C\Big[\|f\|_{L^2(0,T;H^{-1}(\Omega))} + \|\varphi|R|^{1/2}\|_{L^2(\Omega_+)} + \|\psi|R|^{1/2}\|_{L^2(\Omega_-)}\Big]\,,$$

where $C = C(\lambda, \Lambda, n, \Omega)$.

3 Homogenization

The aim of this paper is to study the homogenization of the problem (2.3), in the case when R and a_{ij} are periodic functions. To this purpose, let $a \in \mathcal{M}_Y(\lambda, \Lambda)$ be a Y-periodic matrix (i.e. a_{ij} are Y-periodic functions), and $R \in L^\infty_\#(Y)$ be a given function. We assume that the regions $\{x \in \mathbf{R}^n \ : \ R(x) > 0\}$, $\{x \in \mathbf{R}^n \ : \ R(x) < 0\}$ and $\{x \in \mathbf{R}^n \ : \ R(x) = 0\}$ have Lipschitz boundaries. For every $\varepsilon > 0$, we set $R_\varepsilon(x) = R(\varepsilon^{-1}x)$ and $a_\varepsilon(x) = a(\varepsilon^{-1}x)$. As done in (2.2), we denote by $\Omega_{+,\varepsilon}$ (resp. $\Omega_{-,\varepsilon}$ or $\Omega_{0,\varepsilon}$) the subset of Ω where $R_\varepsilon > 0$ (resp. $R_\varepsilon < 0$ or $R_\varepsilon = 0$).

Let us fix $f \in L^2(0,T;H^{-1}(\Omega))$, $\varphi, \psi \in L^2(\Omega)$, and, for $\varepsilon > 0$, consider the family of problems

$$\begin{cases} R_\varepsilon(x)\dfrac{\partial u_\varepsilon}{\partial t}(x,t) - \operatorname{div}\big(a_\varepsilon(x)\nabla u_\varepsilon(x,t)\big) = f(x,t) & \text{in } \Omega \times (0,T)\,, \\ u_\varepsilon(x,t) = 0 & \text{on } \partial\Omega \times (0,T)\,, \\ u_\varepsilon(x,0) = \varphi(x) & \text{on } \Omega_{+,\varepsilon}\,, \\ u_\varepsilon(x,T) = \psi(x) & \text{on } \Omega_{-,\varepsilon}\,. \end{cases} \tag{3.1}$$

Note that we consider as Cauchy conditions the restrictions of the functions φ and ψ respectively to $\Omega_{+,\varepsilon}$ and $\Omega_{-,\varepsilon}$.
By theorem 2.3, for every $\varepsilon > 0$, problem (3.1) has a unique solution

$$u_\varepsilon \in \mathcal{W}_\varepsilon = \big\{u \in L^2(0,T;H^1_0(\Omega)) \ \big|\ (R_\varepsilon u)' \in L^2(0,T;H^{-1}(\Omega))\big\}\,.$$

Moreover,

$$\|u_\varepsilon\|_{\mathcal{W}_\varepsilon} \leqslant C\Big[\|f\|_{L^2(0,T;H^{-1}(\Omega))} + \|\varphi\|_{L^2(\Omega)} + \|\psi\|_{L^2(\Omega)}\Big]\,, \tag{3.2}$$

where $C > 0$ does not depend on ε. Hence, we can assume that there exist a function $u \in L^2(0,T;H^1_0(\Omega))$ and a subsequence, which we still denote by $\{u_\varepsilon\}$, such that

$$u_\varepsilon \rightharpoonup u \qquad \text{weakly in } \ L^2(0,T;H^1_0(\Omega))\,. \tag{3.3}$$

It is our purpose to characterize the asymptotic limit u of the solutions u_ε, when $\varepsilon \to 0^+$.

The homogenization of problem (3.1) in the case $R \equiv 0$ or $R \geqslant C > 0$ (equivalently, $R \leqslant C < 0$), i.e. in the elliptic case or the standard parabolic case, is by now a classical matter (see e.g. [2], [4], [8]). A non classical homogenization result, in the case $R \geqslant 0$, is given in [14] and [16]. In the case of a coefficient R with non constant sign, a first homogenization result, for the Laplace operator and under the constraint $\int_Y R(y)\,dy \neq 0$, can be found in [1].

When we deal with a more general coercive operator and with no constraint on the mean value of R, as in the present situation, the homogenization of (3.1) can formally be done as usual, but the main difference will be in the error estimate (see theorems 3.3, 3.5, 3.6 and lemma 3.4, below).

In this approach, the solution u_ε is assumed to admit the following ansatz (or asymptotic expansion)

$$u_\varepsilon(x) = u_0(x,\frac{x}{\varepsilon},t) + \varepsilon u_1(x,\frac{x}{\varepsilon},t) + \varepsilon^2 u_2(x,\frac{x}{\varepsilon},t) + \varepsilon^3 u_3(x,\frac{x}{\varepsilon},t) + \dots \tag{3.4}$$

where each function $u_i(x,y,t)$ is Y-periodic with respect to the fast variable $y = x/\varepsilon$. Plugging this ansatz in the first equation of (3.1) and identifying different powers of ε, we obtain a cascade of equations. Defining the operator A_ε by $A_\varepsilon u = -\mathrm{div}(a_\varepsilon \nabla u)$, we may write $A_\varepsilon = \varepsilon^{-2} A_0 + \varepsilon^{-1} A_1 + A_2$, where

$$\begin{aligned}
A_0 &= -\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial}{\partial y_j}\right)\\
A_1 &= -\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial}{\partial x_j}\right) - \frac{\partial}{\partial x_i}\left(a_{ij}(y)\frac{\partial}{\partial y_j}\right)\\
A_2 &= -\frac{\partial}{\partial x_i}\left(a_{ij}(y)\frac{\partial}{\partial x_j}\right).
\end{aligned} \tag{3.5}$$

The two space variables x and y are taken as independent, and only at the end of the computation y is replaced by $\frac{x}{\varepsilon}$. The first equation in (3.1) is therefore equivalent to the following system

$$\begin{cases}
A_0 u_0 = 0\\[1ex]
A_0 u_1 + A_1 u_0 = 0\\[1ex]
R\dfrac{\partial u_0}{\partial t} + A_0 u_2 + A_1 u_1 + A_2 u_0 = f\\
R\dfrac{\partial u_1}{\partial t} + A_0 u_3 + A_1 u_2 + A_2 u_1 = 0\\[1ex]
\ldots\ldots
\end{cases} \tag{3.6}$$

the solutions of which are easily computed. To this aim, the following result well known result will be useful (see, for instance, [4] and [7]).

Proposition 3.1 *Consider the periodic problem*

$$\begin{cases}
A_0 v = g - \mathrm{div}\ G & \textit{in } Y\ ,\\[1ex]
v \in H^1_\#(Y)\,,
\end{cases} \tag{3.7}$$

where A_0 *is the operator defined by the first equality in* (3.5), $g \in L^2_\#(Y)$ *and* $G \in L^2_\#(Y; \mathbf{R}^N)$. *Then problem* (3.7) *admits a weak solution if and only if*

$$\int_Y g(y)\ dy = 0\,.$$

Moreover, in this case, the solution is unique up to an additive constant.

The first equation in (3.6) implies that $u_0(x, y, t) \equiv u_0(x, t)$ does not depend on y. The second equation in (3.6) gives the value of u_1 in terms of u_0, i.e.

$$u_1(x, \frac{x}{\varepsilon}, t) = -\chi^j(\frac{x}{\varepsilon})\frac{\partial u_0}{\partial x_j}(x, t) + \tilde{u}_1(x, t) \tag{3.8}$$

where $\chi^j(y)$, $j = 1, \ldots, n$, are the unique solutions in $H^1_\#(Y)$, with zero average, of the cell problem

$$\begin{cases}
A_0 \chi^j = -\dfrac{\partial a_{ij}}{\partial y_i} & \text{in } Y\ ;\\[1ex]
\displaystyle\int_Y \chi^j(y)dy = 0 & y \mapsto \chi^j(y)\ Y\text{-periodic}\ ;
\end{cases} \tag{3.9}$$

and $\tilde{u}_1$ is a non-oscillating function, which is by now not determined.

The third equation in (3.6) gives u_2 in terms of u_0, i.e.

$$u_2(x, \frac{x}{\varepsilon}, t) = \chi^0(\frac{x}{\varepsilon})\frac{\partial u_0}{\partial t}(x,t) + \chi^{ij}(\frac{x}{\varepsilon})\frac{\partial^2 u_0}{\partial x_i \partial x_j}(x,t) - \chi^j(\frac{x}{\varepsilon})\frac{\partial \tilde{u}_1}{\partial x_j}(x,t) + \tilde{u}_2(x,t) \tag{3.10}$$

where $\chi^0 \in H^1_\#(Y)$ is the unique solution, with zero average, of the cell problem

$$\begin{cases} A_0\chi^0 = \displaystyle\int_Y R(y)\ dy - R & \text{in } Y; \\ \displaystyle\int_Y \chi^0(y)dy = 0 & y \to \chi^0(y)\ Y\text{-periodic}; \end{cases} \tag{3.11}$$

and $\chi^{ij} \in H^1_\#(Y)$, for $i, j = 1, \dots, n$, are the unique solutions, with zero average, of another family of cell problems (see [4], (2.42) and (2.39))

$$\begin{cases} A_0\chi^{ij} = b_{ij} - \displaystyle\int_Y \tilde{b}_{ij}(y)\ dy & \text{in } Y; \\ \displaystyle\int_Y \chi^{ij}(y)dy = 0 & y \to \chi^{ij}(y)\ Y\text{-periodic}; \end{cases} \tag{3.12}$$

with

$$b_{ij}(y) = a_{ij}(y) - a_{ik}(y)\frac{\partial \chi^j}{\partial y_k} - \frac{\partial}{\partial y_k}(a_{ki}(y)\chi^j)\,,$$

$$\tilde{b}_{ij}(y) = a_{ij}(y) - a_{ik}(y)\frac{\partial \chi^j}{\partial y_k}\,,$$

and $\tilde{u}_2$ is another non-oscillating function, which is by now not determined.

The homogenized equation for u_0 is obtained by writing the compatibility condition (or Fredholm alternative) for the third equation in (3.6). If we define

$$\overline{R} = \int_Y R(y)\ dy\,,$$

this gives

$$\overline{R}\frac{\partial u_0}{\partial t}(x,t) - \operatorname{div}\big(a^*\nabla u_0(x,t)\big) = f(x,t)\,,$$

where the homogenized matrix a^* is defined by its constant entries a^*_{ij} given by

$$a^*_{ij} = \int_Y \tilde{b}_{ij}(y)\ dy = \int_Y [a_{ij}(y) - a_{ik}(y)\frac{\partial \chi^j}{\partial y_k}(y)]\ dy. \tag{3.13}$$

Remark 3.2 - Note that, so far, the functions $\tilde{u}_1$ in (3.8) and $\tilde{u}_2$ in (3.10) are non-oscillating functions that are not determined. This implies, as pointed out in [4], that if we stop expansion (3.4) at the first order (i.e. if we do not look at higher order equations in (3.6)), the function $\tilde{u}_1$ (and a fortiori $\tilde{u}_2$) does not play any role, and so we may choose $\tilde{u}_1 = \tilde{u}_2 \equiv 0$.

As far as the boundary conditions are concerned, we will show that three different situations may occur, depending on the sign of the average $\overline{R}$. We first examine the case where $\overline{R} > 0$. The following theorem will prove that the limit function u in (3.3), actually coincides with the function u_0, solution of (3.14) below, and that the whole sequence (not only a subsequence) converges strongly to u_0 in $L^2(\Omega \times (0,T))$.

Theorem 3.3 *For each $\varepsilon > 0$, let u_ε be the unique solutions of* (3.1). *Assume that $\overline{R} > 0$, and let u_0 be the unique solution of problem*

$$\begin{cases} \overline{R}\dfrac{\partial u_0}{\partial t}(x,t) - \operatorname{div}\big(a^*\nabla u_0(x,t)\big) = f(x,t) & \text{in } \Omega \times (0,T)\,, \\ u_0(x,t) = 0 & \text{on } \partial\Omega \times (0,T)\,, \\ u_0(x,0) = \varphi(x) & \text{on } \Omega\,. \end{cases} \tag{3.14}$$

Assume, in addition, that u_0 satisfies the following regularity assumptions:

$$u_0 \in L^\infty(0,T;W^{3,\infty}(\Omega))\,, \qquad \frac{\partial u_0}{\partial t} \in L^\infty(0,T;W^{2,\infty}(\Omega))\,, \qquad \frac{\partial^2 u_0}{\partial t^2} \in L^\infty(\Omega_T)\,. \tag{3.15}$$

Let u_1 be defined by (3.8), *with $\tilde{u}_1 = 0$. Then, for every $\tilde{T} \in (0,T)$, we have*

$$\|u_\varepsilon - u_0 - \varepsilon u_1\|_{L^2(0,\tilde{T};H^1(\Omega))} \to 0 \qquad \textit{for } \varepsilon \to 0^+\,. \tag{3.16}$$

Moreover, $\|u_\varepsilon - u_0\|_{L^2(\Omega_T)} \to 0$, for $\varepsilon \to 0^+$.

Note that problem (3.14) is well-posed in $L^2(0,T;H^1_0(\Omega))$ since it is easily seen that a^* is bounded and coercive (see [4], Remark 2.6).

In order to prove this theorem, we need the following lemma.

Lemma 3.4 *For every $\varepsilon > 0$, let $u_\varepsilon \in L^2(0,T;H^1_0(\Omega))$ be the unique solution of* (3.1) *and let $v \in L^2(0,T;H^1_0(\Omega))$ be a given function. Then, for every $s,t \in [0,T]$, $s < t$, there exists a function $\psi \in H^1_0(\Omega)$ such that*

$$\int_s^t |u_\varepsilon - v|^2\, d\tau \to \psi^2 \qquad \textit{strongly in } \; L^1(\Omega)\,,$$

up to a subsequence.

Proof - By a-priori estimates (3.2), it follows that, for every $s,t \in [0,T]$, $s < t$,

$$\int_s^t \left(\int_\Omega |\nabla u_\varepsilon|^2\, dx \right) d\tau \leqslant \|u_\varepsilon\|_{L^2(0,T;H^1_0(\Omega))} \leqslant C\,, \tag{3.17}$$

for a constant C independent of ε. Set

$$\psi_\varepsilon(x) = \left(\int_s^t |u_\varepsilon(x,\tau) - v(x,\tau)|^2\, d\tau \right)^{1/2} \qquad \text{for a.e. } x \in \Omega\,.$$

Clearly, $\{\psi_\varepsilon\} \subseteq H^1_0(\Omega)$ and, for a constant C independent of ε, we have

$$\begin{aligned} \int_\Omega |\nabla\psi_\varepsilon|^2\, dx &= \int_\Omega \left(\frac{1}{\int_s^t |u_\varepsilon - v|^2\, d\tau} \left| \int_s^t (u_\varepsilon - v)\nabla(u_\varepsilon - v)\, d\tau \right|^2 \right) dx \\ &\leqslant \int_\Omega \left(\int_s^t |\nabla(u_\varepsilon - v)|^2\, d\tau \right) dx \leqslant C\,, \end{aligned}$$

where the last inequality is due to (3.17). Hence, $\{\psi_\varepsilon\}$ is bounded in $H^1_0(\Omega)$ and then it is compact in $L^2(\Omega)$; i.e., up to a subsequence, $|\psi_\varepsilon|^2 \to |\psi|^2$, strongly in $L^1(\Omega)$, for a proper function $\psi \in L^2(\Omega)$ (actually, $\psi \in H^1_0(\Omega)$). $\square$

In particular, the previous lemma implies that, for $\varepsilon \to 0^+$,

$$\int_s^t \int_\Omega R\Big(\frac{x}{\varepsilon}\Big) |u_\varepsilon - v|^2 \; dx \; d\tau = \int_\Omega R\Big(\frac{x}{\varepsilon}\Big) \left[\int_s^t |u_\varepsilon - v|^2 \; d\tau\right] \; dx \to \overline{R} \int_\Omega \psi^2(x) \; dx \,,$$

since, by periodicity, $R_\varepsilon \rightharpoonup \overline{R}$ $*$-weakly in $L^\infty(\Omega)$.

Proof of Theorem 3.3 - Let us introduce a cut-off function $\theta_\varepsilon : \Omega \to \mathbf{R}$, such that $\theta_\varepsilon \in \mathcal{C}^\infty(\Omega)$, $0 \leqslant \theta_\varepsilon(x) \leqslant 1 \quad \forall x \in \Omega$, $\theta_\varepsilon(x) = 0$ on the set $\Omega_\varepsilon = \{x \in \Omega \ : \ \mathrm{dist}(x, \partial\Omega) \geqslant \varepsilon\}$, $\theta_\varepsilon(x) = 1$ on $\partial\Omega$, $|\nabla \theta_\varepsilon| \leqslant C/\varepsilon$. Note that, by the regularity assumption on $\partial\Omega$, the region where $\theta_\varepsilon \neq 0$ has Lebesgue measure smaller than $C\,\varepsilon$.
Let us define the error function r_ε of the asymptotic expansion by

$$r_\varepsilon(x,y,t) = u_\varepsilon(x,t) - [u_0(x,t) + \varepsilon u_1(x,y,t)(1-\theta_\varepsilon(x)) + \varepsilon^2 u_2(x,y,t)(1-\theta_\varepsilon(x))]$$

where u_1 and u_2 are defined respectively in (3.8) and (3.10) and we choose $\tilde{u}_1 = \tilde{u}_2 = 0$ as explained in Remark 3.2.

It follows that $r_\varepsilon(x, \frac{x}{\varepsilon}, t)$ solves

$$\begin{cases} R_\varepsilon \dfrac{\partial r_\varepsilon}{\partial t} - \mathrm{div}\big(a_\varepsilon \nabla r_\varepsilon\big) = f_\varepsilon & \text{in } \Omega \times (0,T) \ , \\ r_\varepsilon = 0 & \text{on } \partial\Omega \times (0,T) \ , \\ r_\varepsilon\Big(x, \dfrac{x}{\varepsilon}, 0\Big) = \varphi_\varepsilon\Big(x, \dfrac{x}{\varepsilon}\Big) & \text{on } \Omega_{+,\varepsilon} \ , \\ r_\varepsilon\Big(x, \dfrac{x}{\varepsilon}, T\Big) = \psi_\varepsilon\Big(x, \dfrac{x}{\varepsilon}\Big) & \text{on } \Omega_{-,\varepsilon} \ , \end{cases} \tag{3.18}$$

where

$$\begin{aligned} \varphi_\varepsilon(x,y) \quad &= \varphi(x) - [u_0(x,0) + \varepsilon u_1(x,y,0)(1-\theta_\varepsilon(x)) + \varepsilon^2 u_2(x,y,0)(1-\theta_\varepsilon(x))] \\ &= -[\varepsilon u_1(x,y,0)(1-\theta_\varepsilon(x)) + \varepsilon^2 u_2(x,y,0)(1-\theta_\varepsilon(x))] \\ \psi_\varepsilon(x,y) \quad &= \psi(x) - [u_0(x,T) + \varepsilon u_1(x,y,T)(1-\theta_\varepsilon(x)) + \varepsilon^2 u_2(x,y,T)(1-\theta_\varepsilon(x))] \end{aligned}$$

and

$$\begin{aligned}
f_\varepsilon(x,y,t) \quad &= R(y)\frac{\partial u_\varepsilon}{\partial t}(x,t) - \operatorname{div}\big(a(y)\nabla u_\varepsilon(x,t)\big) - \left[R(y)\frac{\partial u_0}{\partial t}(x,t) - \operatorname{div}\big(a(y)\nabla u_0(x,t)\big)\right] \\
&-\varepsilon\left[R(y)\frac{\partial u_1}{\partial t}(x,y,t) - \operatorname{div}\big(a(y)\nabla u_1(x,y,t)\big)\right] \\
&-\varepsilon^2\left[R(y)\frac{\partial u_2}{\partial t}(x,y,t) - \operatorname{div}\big(a(y)\nabla u_2(x,y,t)\big)\right] \\
&+\varepsilon\left[R(y)\frac{\partial (u_1\theta_\varepsilon)}{\partial t}(x,y,t) - \operatorname{div}\big(a(y)\nabla (u_1\theta_\varepsilon)(x,y,t)\big)\right] \\
&+\varepsilon^2\left[R(y)\frac{\partial (u_2\theta_\varepsilon)}{\partial t}(x,y,t) - \operatorname{div}\big(a(y)\nabla (u_2\theta_\varepsilon)(x,y,t)\big)\right] \\
&= f(x,t) - R(y)\frac{\partial u_0}{\partial t}(x,t) - \frac{1}{\varepsilon}A_1u_0 - A_2u_0 \\
&-\varepsilon R(y)\frac{\partial u_1}{\partial t}(x,y,t) - \frac{1}{\varepsilon}A_0u_1 - A_1u_1 - \varepsilon A_2u_1 \\
&-\varepsilon^2 R(y)\frac{\partial u_2}{\partial t}(x,y,t) - A_0u_2 - \varepsilon A_1u_2 - \varepsilon^2 A_2u_2 \\
&+\varepsilon R(y)\frac{\partial u_1}{\partial t}(x,y,t)\theta_\varepsilon(x) + \varepsilon A_\varepsilon(u_1\theta_\varepsilon) \\
&+\varepsilon^2 R(y)\frac{\partial u_2}{\partial t}(x,y,t)\theta_\varepsilon(x) + \varepsilon^2 A_\varepsilon(u_2\theta_\varepsilon) \\
&= f(x,t) - \left[R(y)\frac{\partial u_0}{\partial t}(x,t) + A_0u_2 + A_1u_1 + A_2u_0\right] \qquad\qquad (= 0 \text{ by } (3.6)) \\
&-\frac{1}{\varepsilon}(A_1u_0 + A_0u_1) \qquad\qquad (= 0 \text{ by } (3.6)) \\
&-\varepsilon\Big[R(y)\big(1-\theta_\varepsilon(x)\big)\frac{\partial u_1}{\partial t}(x,y,t) + A_2u_1 + A_1u_2 \\
&+\varepsilon R(y)\big(1-\theta_\varepsilon(x)\big)\frac{\partial u_2}{\partial t}(x,y,t) + \varepsilon A_2u_2\Big] \\
&+\varepsilon A_\varepsilon(u_1\theta_\varepsilon) + \varepsilon^2 A_\varepsilon(u_2\theta_\varepsilon)\,.
\end{aligned}$$

Then it is easy to check that

$$\begin{aligned}
f_\varepsilon\left(x,\tfrac{x}{\varepsilon},t\right) \quad &= \varepsilon\, g_\varepsilon(x,t) - \varepsilon^2 \operatorname{div} G_\varepsilon(x,t) \\
&\qquad -\varepsilon \operatorname{div}\left(a\left(\tfrac{x}{\varepsilon}\right)\nabla(u_1\theta_\varepsilon)\left(x,\tfrac{x}{\varepsilon},t\right)\right) - \varepsilon^2 \operatorname{div}\left(a\left(\tfrac{x}{\varepsilon}\right)\nabla(u_2\theta_\varepsilon)\left(x,\tfrac{x}{\varepsilon},t\right)\right),
\end{aligned}$$

where

$$g_\varepsilon(x,t) \quad = \quad R\left(\frac{x}{\varepsilon}\right)(1-\theta_\varepsilon(x))\chi^j\left(\frac{x}{\varepsilon}\right)\frac{\partial^2 u_0}{\partial x_j \partial t}(x,t)$$

$$
\begin{aligned}
& - a_{ij}\left(\frac{x}{\varepsilon}\right)\chi^{k}\left(\frac{x}{\varepsilon}\right)\frac{\partial^3 u_0}{\partial x_i\partial x_j\partial x_k}(x,t) \\
& + a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial\chi^{0}}{\partial y_j}\left(\frac{x}{\varepsilon}\right)\frac{\partial^2 u_0}{\partial x_i\partial t}(x,t) \\
& + a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial\chi^{hk}}{\partial y_j}\left(\frac{x}{\varepsilon}\right)\frac{\partial^3 u_0}{\partial x_i\partial x_h\partial x_k}(x,t) \\
& - \varepsilon R\left(\frac{x}{\varepsilon}\right)(1-\theta_\varepsilon(x))\chi^{0}\left(\frac{x}{\varepsilon}\right)\frac{\partial^2 u_0}{\partial t^2}(x,t) \\
& - \varepsilon R\left(\frac{x}{\varepsilon}\right)(1-\theta_\varepsilon(x))\chi^{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial^3 u_0}{\partial x_i\partial x_j\partial t}(x,t)\,,
\end{aligned}
$$

$$
\begin{aligned}
(G_\varepsilon)_i(x,t) \quad = \quad & - a_{ij}\left(\frac{x}{\varepsilon}\right)\chi^{0}\left(\frac{x}{\varepsilon}\right)\frac{\partial^2 u_0}{\partial x_j\partial t}(x,t) \\
& - a_{ij}\left(\frac{x}{\varepsilon}\right)\chi^{hk}\left(\frac{x}{\varepsilon}\right)\frac{\partial^3 u_0}{\partial x_j\partial x_h\partial x_k}(x,t)\,.
\end{aligned}
$$

Let us now remark that, in view of Stampacchia's and Meyers' regularity theorems (see [10] and, for instance, [9], Chap. 8), the functions $\chi^j(y)$, $\chi^0(y)$ and $\chi^{ij}(y)$ defined by (3.9), (3.11) and (3.12), respectively, satisfy the following properties:

$$
\chi^j(y) \in L^\infty_\#(Y)\,, \quad \nabla_y\chi^j(y) \in L^{2+\sigma}_\#(Y)\,, \qquad \text{for every } j = 0,\dots,n, \tag{3.19}
$$

$$
\chi^{ij}(y) \in W^{1,2+\sigma}_\#(Y)\,, \qquad \text{for every } i,j = 1,\dots,n, \tag{3.20}
$$

for some $\sigma > 0$. It follows in particular that, under the regularity assumptions (3.15) on u_0,

$$
g_\varepsilon(x,t) \quad \text{is bounded in } L^\infty(0,T;L^{2+\sigma}(\Omega))\,, \tag{3.21}
$$

$$
G_\varepsilon(x,t) \quad \text{is bounded in } L^\infty(0,T;L^{2+\sigma}(\Omega;\mathbf{R}^n))\,. \tag{3.22}
$$

Fix δ such that $0 < \delta < T$, and let $\eta_\delta(t) : [0,T] \to \mathbf{R}$ be the function defined by

$$
\eta_\delta(t) = \begin{cases} 1 & \text{if } t \in [0, T-\delta]\ , \\[2mm] \dfrac{1}{\delta}(T-t) & \text{if } t \in (T-\delta, T]\ . \end{cases} \tag{3.23}
$$

We multiply the first equation in (3.18) by $r_\varepsilon(x,\frac{x}{\varepsilon},t)\eta_\delta(t)$ and integrate over Ω_T. It follows that

$$
\begin{aligned}
& \int_0^T \langle R_\varepsilon \frac{\partial r_\varepsilon}{\partial t}(t)\,,\ r_\varepsilon(t)\eta_\delta(t)\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}\,dt + \int_{\Omega_T} a_\varepsilon \nabla r_\varepsilon \nabla r_\varepsilon\,\eta_\delta\,dx\,dt \\
& \quad = \quad \varepsilon \int_{\Omega_T} g_\varepsilon\, r_\varepsilon\,\eta_\delta\,dx\,dt + \varepsilon^2 \int_{\Omega_T} G_\varepsilon \cdot \nabla r_\varepsilon\,\eta_\delta\,dx\,dt \\
& \qquad\quad + \varepsilon \int_{\Omega_T} a_\varepsilon \nabla(u_1\theta_\varepsilon)\nabla r_\varepsilon\,\eta_\delta\,dx\,dt + \varepsilon^2 \int_{\Omega_T} a_\varepsilon \nabla(u_2\theta_\varepsilon)\nabla r_\varepsilon\,\eta_\delta\,dx\,dt\,.
\end{aligned}
$$

Since

$$
\langle R_\varepsilon \frac{\partial r_\varepsilon}{\partial t}(t)\,,\ r_\varepsilon(t)\eta_\delta(t)\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} = \frac{d}{dt}\left[\frac{1}{2}\int_\Omega R_\varepsilon r_\varepsilon^2 \eta_\delta\,dx\right] - \frac{1}{2}\int_\Omega R_\varepsilon r_\varepsilon^2 \eta'_\delta\,dx
$$

for a.e. $t \in [0,T]$, using the ellipticity and boundedness condition (2.1), Poincaré's and Young's inequalities, and recalling that $\eta_\delta(0)=1$, $\eta_\delta(T)=0$, one obtains

$$\begin{aligned}
\lambda \int_{\Omega_T} |\nabla r_\varepsilon|^2 \eta_\delta \, dx\, dt \;&\leqslant\; \frac{\lambda}{2} \int_{\Omega_T} |\nabla r_\varepsilon|^2 \eta_\delta \, dx\, dt + C(\lambda, n, \Omega)\; \varepsilon^2 \int_{\Omega_T} g_\varepsilon^2 \eta_\delta \, dx\, dt \\
&\quad + C(\lambda)\; \varepsilon^4 \int_{\Omega_T} |G_\varepsilon|^2 \eta_\delta \, dx\, dt + C(\lambda, \Lambda)\; \varepsilon^2 \int_{\Omega_T} |\nabla(u_1\theta_\varepsilon)|^2 \eta_\delta \, dx\, dt \\
&\quad + C(\lambda, \Lambda)\; \varepsilon^4 \int_{\Omega_T} |\nabla(u_2\theta_\varepsilon)|^2 \eta_\delta \, dx\, dt + \frac{1}{2} \int_{\Omega_T} R_\varepsilon r_\varepsilon^2 \eta_\delta' \, dx\, dt \\
&\quad + \frac{1}{2} \int_\Omega R_\varepsilon(x) r_\varepsilon^2 \left(x, \frac{x}{\varepsilon}, 0\right)\, dx\,.
\end{aligned}$$

Using the definition of u_1, u_2, the regularity statements (3.19) and (3.20), recalling that $0 \leqslant \theta_\varepsilon \leqslant 1$, $|\nabla\theta_\varepsilon| \leqslant C/\varepsilon$, and that θ_ε is different from zero on a set which has measure of order ε, it is easy to check that

$$\begin{aligned}
&\varepsilon^2 \int_{\Omega_T} |\nabla(u_1\theta_\varepsilon)|^2 \, dx\, dt \leqslant C\, \varepsilon^{\frac{\sigma}{2+\sigma}}\,, \\
&\varepsilon^4 \int_{\Omega_T} |\nabla(u_2\theta_\varepsilon)|^2 \, dx\, dt \leqslant C\, \varepsilon^{\frac{\sigma}{2+\sigma}+2}\,,
\end{aligned} \tag{3.24}$$

with $\sigma > 0$. Moreover

$$\begin{aligned}
\frac{1}{2} \int_\Omega R_\varepsilon(x) r_\varepsilon^2 \left(x, \frac{x}{\varepsilon}, 0\right)\, dx \;&\leqslant\; \frac{1}{2} \int_{\Omega_{+,\varepsilon}} R_\varepsilon(x) r_\varepsilon^2 \left(x, \frac{x}{\varepsilon}, 0\right)\, dx \\
&=\; \frac{\varepsilon^2}{2} \int_{\Omega_{+,\varepsilon}} |R_\varepsilon| [u_1(1-\theta_\varepsilon) + \varepsilon u_2(1-\theta_\varepsilon)]^2 \Big(x, \frac{x}{\varepsilon}, 0\Big)\, dx \\
&\leqslant\; C\, \varepsilon^2\,.
\end{aligned}$$

Therefore, using also (3.21), (3.22) and possibly passing to a subsequence, we conclude that

$$\begin{aligned}
\limsup_{\varepsilon\to 0^+} \int_{\Omega_T} |\nabla r_\varepsilon|^2 \eta_\delta \; dx\; dt \;&\leqslant\; C \limsup_{\varepsilon\to 0^+} \int_{\Omega_T} R_\varepsilon r_\varepsilon^2 \eta_\delta' \; dx\; dt \\
&=\; -\frac{C}{\delta} \lim_{\varepsilon\to 0^+} \int_\Omega R_\varepsilon \left[\int_{T-\delta}^T r_\varepsilon^2 \; dt\right] \; dx \\
&=\; -\frac{C}{\delta} \overline{R} \int_\Omega \psi^2(x) \; dx \leqslant 0\,,
\end{aligned}$$

where, by lemma 3.4, with v replaced by u_0, we have

$$\psi^2(x) = L^1\!-\lim_{\varepsilon\to 0^+} \int_{T-\delta}^T |u_\varepsilon - u_0|^2 \; dt = L^1\!-\lim_{\varepsilon\to 0^+} \int_{T-\delta}^T r_\varepsilon^2 \; dt\,.$$

This implies that

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} \int_{\Omega_T} |\nabla r_\varepsilon|^2 \eta_\delta \; dx\; dt = 0\,, \\
&\int_\Omega \psi^2(x)\; dx = \lim_{\varepsilon\to 0^+} \int_{T-\delta}^T \int_\Omega |u_\varepsilon - u_0|^2 \; dx\; dt = 0\,.
\end{aligned} \tag{3.25}$$

Since δ is arbitrary, it follows that, for every $\tilde{T} \in (0,T)$,

$$\lim_{\varepsilon\to 0^+} \int_{\Omega\times(0,\tilde{T})} |\nabla r_\varepsilon|^2 \; dx \; dt = 0 \,,$$

and, taking into account (3.24) and the estimate

$$\int_{\Omega\times(0,\tilde{T})} \varepsilon^4 |\nabla\big(u_2(1-\theta_\varepsilon)\big)|^2 \; dx \; dt \leqslant C\varepsilon^2 \,,$$

we also have

$$\|u_\varepsilon - u_0 - \varepsilon u_1\|_{L^2(0,\tilde{T};H^1(\Omega))} \to 0 \,,$$

and therefore

$$\lim_{\varepsilon\to 0^+} \int_{\Omega\times(0,\tilde{T})} |u_\varepsilon - u_0|^2 \; dx \; dt = 0 \,. \tag{3.26}$$

By (3.26) and (3.25), it follows

$$\lim_{\varepsilon\to 0^+} \int_{\Omega_T} |u_\varepsilon - u_0|^2 \; dx \; dt = 0 \,.$$

Finally, since u_0 is uniquely determined by (3.14), it follows that the result holds for the whole sequence, and not only for a subsequence. □

We obtain analogous results in the other two cases, i.e. $\overline{R} < 0$ or $\overline{R} = 0$. If $\overline{R} < 0$, the limit equation is backward-parabolic, and it is the final condition for $t = T$ which passes to the limit, as stated in the following theorem.

Theorem 3.5 *Assume the same hypotheses of theorem* 3.3*, but suppose that* $\overline{R} < 0$*. Let* u_0 *be the unique solution of*

$$\begin{cases} \overline{R}\dfrac{\partial u_0}{\partial t}(x,t) - \mathrm{div}\big(a^*\nabla u_0(x,t)\big) = f(x,t) & \text{in } \Omega\times(0,T) \;, \\ u_0(x,t) = 0 & \text{on } \partial\Omega\times(0,T) \;, \\ u_0(x,T) = \psi(x) & \text{on } \Omega \;, \end{cases} \tag{3.27}$$

and assume that it satisfies (3.15)*. Then, for every* $\tilde{T} \in (0,T)$,

$$\|u_\varepsilon - u_0 - \varepsilon u_1\|_{L^2(\tilde{T},T;H^1(\Omega))} \to 0 \qquad \text{for } \; \varepsilon \to 0^+ \,.$$

Moreover, $\|u_\varepsilon - u_0\|_{L^2(\Omega_T)} \to 0$*, for* $\varepsilon \to 0^+$*.*

The proof is not very different from the one of theorem 3.3, if we replace the function η_δ defined in (3.23) by

$$\eta_\delta(t) = \begin{cases} t/\delta & \text{if } t \in [0,\delta] \;, \\ 1 & \text{if } t \in (\delta,T] \;. \end{cases}$$

Similarly, taking

$$\eta_\delta(t) = \begin{cases} t/\delta & \text{if } t \in [0,\delta] \;, \\ 1 & \text{if } t \in (\delta,T-\delta) \;, \\ (T-t)/\delta & \text{if } t \in [T-\delta,T] \;, \end{cases}$$

one can prove the borderline case:

Theorem 3.6 *Assume the same hypotheses of theorem* 3.3*, but suppose that* $\overline{R} = 0$*. Let* $u_0(\cdot, t)$ *be the unique solution of the following family of elliptic problems*

$$\begin{cases} -\mathrm{div}\big(a^* \nabla u_0(x,t)\big) = f(x,t) & \text{in } \Omega, \text{ for a.e. } t \in (0,T) \ , \\ u_0(x,t) = 0 & \text{on } \partial\Omega, \text{ for a.e. } t \in (0,T) \ , \end{cases} \tag{3.28}$$

and assume that it satisfies (3.15)*. Then, for every* $\tilde{T}_1, \tilde{T}_2 \in (0,T)$*,* $\tilde{T}_1 < \tilde{T}_2$*, we have*

$$\|u_\varepsilon - u_0 - \varepsilon u_1\|_{L^2(\tilde{T}_1, \tilde{T}_2; H^1(\Omega))} \to 0 \qquad \text{for } \ \varepsilon \to 0^+ \, .$$

Moreover, $\|u_\varepsilon - u_0\|_{L^2(\Omega_T)} \to 0$*, for* $\varepsilon \to 0^+$*.*

Remark 3.7 - In the particular case of constant matrix a (the model case is the Laplace operator), the cell functions χ^j and χ^{ij}, $i,j = 1, \ldots, n$, are identically equal to zero, so that the first corrector u_1 can be choosen equal to zero. This implies that, when $\overline{R} > 0$, for every $\tilde{T} \in (0,T)$,

$$\|\nabla u_\varepsilon - \nabla u_0\|_{L^2(\Omega \times (0,\tilde{T}))} \to 0 \qquad \text{for } \varepsilon \to 0^+ \ ,$$

i.e. we have the strong convergence of the gradients in $L^2(\Omega \times (0,\tilde{T}))$ (clearly, the same property holds in the other two cases, if we replace $L^2(\Omega \times (0,\tilde{T}))$ by $L^2(\Omega \times (\tilde{T},T))$ or $L^2(\Omega \times (\tilde{T}_1,\tilde{T}_2))$, respectively). Note that, for $\overline{R} \neq 0$, this result was previously obtained in [1], by means of a different technique.

If u_0 does not satisfy the regularity assumptions (3.15), one can proceed by approximation with smoother data. We will only state the result in the case where $\overline{R} > 0$, since the analogous results for $\overline{R} \leqslant 0$ can be stated and proved with almost no difference.

Corollary 3.8 *Assume that* $\overline{R} > 0$*, and that* u_ε *and* u_0 *are the solutions of problems* (3.1) *and* (3.14)*, respectively. Then*

$$\|u_\varepsilon - u_0\|_{L^2(\Omega_T)} \to 0 \qquad \text{for } \ \varepsilon \to 0^+ \, , \tag{3.29}$$

and, for every $\tilde{T} \in (0,T)$*,*

$$\left\| \nabla u_\varepsilon - \nabla u_0 - \nabla \chi^j \left(\frac{x}{\varepsilon} \right) \frac{\partial u_0}{\partial x_j} \right\|_{L^1(\Omega \times (0,\tilde{T}); \mathbf{R}^n)} \to 0 \qquad \text{for } \ \varepsilon \to 0^+ \, , \tag{3.30}$$

where $\chi^j(y)$*,* $j = 1, \ldots, n$*, are the functions defined in* (3.9)*. Moreover, if* $\nabla \chi^j(y)$ *belongs to* $L^q(Y)$*, for some* $q \in [2, \infty]$*, then one can replace the norm* L^1 *with the norm* L^r *in* (3.30)*, where*

$$\frac{1}{r} = \frac{1}{2} + \frac{1}{q} \, .$$

Proof - Let $\{f^{(\delta)}\} \subseteq \mathcal{C}^\infty(\overline{\Omega}_T)$ and $\{\varphi^{(\delta)}\} \subseteq \mathcal{C}_0^\infty(\Omega)$ be sequences of smooth functions such that

$$\begin{array}{lll} f^{(\delta)} \to f & \text{strongly in } L^2(0,T;H^{-1}(\Omega)) & \text{for } \ \delta \to 0^+ \, , \\ \varphi^{(\delta)} \to \varphi & \qquad \text{strongly in } L^2(\Omega) & \text{for } \ \delta \to 0^+ \, . \end{array}$$

Then, we define $u_\varepsilon^{(\delta)}$, $u_0^{(\delta)}$ to be the solutions of problems (3.1) and (3.14), respectively, where the data f and φ have been replaced by $f^{(\delta)}$, $\varphi^{(\delta)}$. It is clear that $u_0^{(\delta)} \in C^\infty(\overline{\Omega}_T)$. Then, by theorem 3.3, for every fixed $\delta > 0$, one has

$$\|u_\varepsilon^{(\delta)} - u_0^{(\delta)}\|_{L^2(\Omega_T)} \to 0 \qquad \text{for } \ \varepsilon \to 0^+ \, , \tag{3.31}$$

$$\|u_\varepsilon^{(\delta)} - u_0^{(\delta)} - \varepsilon u_1^{(\delta)}\|_{L^2(0,\tilde{T};H^1(\Omega))} \to 0 \qquad \text{for } \varepsilon \to 0^+ \,, \tag{3.32}$$

where

$$u_1^{(\delta)}\left(x, \frac{x}{\varepsilon}, t\right) = -\chi^j\left(\frac{x}{\varepsilon}\right)\frac{\partial u_0^{(\delta)}}{\partial x_j}(x,t)\,. \tag{3.33}$$

Moreover, by (3.2) and the linearity of the problem, it follows

$$\sup_\varepsilon \|u_\varepsilon^{(\delta)} - u_\varepsilon\|_{L^2(0,T;H_0^1(\Omega))} \to 0 \qquad \text{for } \delta \to 0^+ \,, \tag{3.34}$$

$$\|u_0^{(\delta)} - u_0\|_{L^2(0,T;H_0^1(\Omega))} \to 0 \qquad \text{for } \delta \to 0^+ \,. \tag{3.35}$$

Then, writing

$$\|u_\varepsilon - u_0\|_{L^2(\Omega_T)} \leqslant \|u_\varepsilon - u_\varepsilon^{(\delta)}\|_{L^2(\Omega_T)} + \|u_\varepsilon^{(\delta)} - u_0^{(\delta)}\|_{L^2(\Omega_T)} + \|u_0^{(\delta)} - u_0\|_{L^2(\Omega_T)} \,,$$

one immediately obtains (3.29). Similarly,

$$\begin{aligned}\left\|\nabla u_\varepsilon - \nabla u_0 - \nabla\chi^j\left(\frac{x}{\varepsilon}\right)\frac{\partial u_0}{\partial x_j}\right\|_{L^1(\Omega\times(0,\tilde{T});\mathbf{R}^n)} &\leqslant \left\|\nabla u_\varepsilon - \nabla u_\varepsilon^{(\delta)}\right\| + \left\|\nabla u_0 - \nabla u_0^{(\delta)}\right\| \\ + \left\|\nabla\chi^j\left(\frac{x}{\varepsilon}\right)\left(\frac{\partial u_0}{\partial x_j} - \frac{\partial u_0^{(\delta)}}{\partial x_j}\right)\right\| &+ \left\|\nabla u_\varepsilon^{(\delta)} - \nabla u_0^{(\delta)} - \varepsilon\nabla u_1^{(\delta)}\right\| + \varepsilon\left\|\chi^j\left(\frac{x}{\varepsilon}\right)\nabla\frac{\partial u_0^{(\delta)}}{\partial x_j}\right\| \,.\end{aligned}$$

By (3.34) and (3.35), the first three terms of the right-hand side are small, uniformly with respect to ε, if δ is small. Once δ has been fixed, the other terms go to zero as $\varepsilon \to 0^+$. This proves (3.30). Note that the restriction to the L^1-norm comes from the term

$$\left\|\nabla\chi^j\left(\frac{x}{\varepsilon}\right)\left(\frac{\partial u_0}{\partial x_j} - \frac{\partial u_0^{(\delta)}}{\partial x_j}\right)\right\|_{L^1(\Omega\times(0,\tilde{T});\mathbf{R}^n)} \,.$$

Therefore, if $\nabla\chi^j$ is more regular, the last assertion of the corollary follows from Hölder's inequality applied to this term. ☐

References

[1] M. Amar - A. Dall'Aglio - F. Paronetto, *Homogenization of forward-backward parabolic equations.* To appear.

[2] I. Babuška, *Homogeneization and its application. Mathematical and computational problems*, in Numerical Solution of Part. Diff. Eq., Proc. Third Symp. (SYNSPADE), B. Hubbard, ed., New York - London, 1976, Acad. Press, pp. 89–116.

[3] M. S. Baouendi - P. Grisvard, *Sur une équation d'evolution changeant de type*, J. Funct. Anal., 2 (1968), pp. 352–367.

[4] A. Bensoussan - J. L. Lions - G. Papanicolaou, *Asymptotic analysis for periodic structures*, North-Holland, Amsterdam, 1978.

[5] R. W. Carroll - R. E. Showalter, *Singular and Degenerate Cauchy Problems*, Academic Press, New York, 1976.

[6] C. Cercignani, *The Boltzmann Equation and Its Applications*, Springer-Verlag, New York, 1988.

[7] D. Cioranescu - P. Donato, *An introduction to homogenization*, Oxford University Press, New York, 1999.

[8] F. Colombini - S. Spagnolo, *Sur la convergence de solutions d'équations paraboliques*, J. Math. Pur. Appl., 56 (1977), pp. 263–306.

[9] D. Gilbarg - N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer - Verlag, Berlin, 1977.

[10] N. G. Meyers, *An L^p estimate for the gradient of solutions of second order elliptic divergence equations.*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., (3) 17 (1963), pp. 189–206.

[11] C. D. Pagani - G. Talenti, *On a forward-backward parabolic equation*, Ann. Mat. Pura Appl., 90 (1971), pp. 1–57.

[12] A. Pankov, *G-convergence and Homogenization of Nonlinear Partial Differential Operators*, Kluwer, 1997.

[13] A. A. Pankov - T. E. Pankova, *Nonlinear evolution equations with non-invertible operator coefficient at the derivative (in russian)*, Doklady Akad. Nauk Ukr., 9 (1993), pp. 18–20.

[14] F. Paronetto, *Homogenization of a class of degenerate parabolic equations*, Asymptotic Anal., 21 (3,4) (1999), pp. 275–302.

[15] F. Paronetto, *Existence results for a class of evolution equations of mixed type*, Preprint, (2002).

[16] F. Paronetto, *Homogenization of degenerate elliptic-parabolic equations*, To appear on Asymptotic Anal., (2003).

[17] R. E. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, American Mathematical Society, 1997.