HARNACK INEQUALITIES FOR KINETIC INTEGRAL EQUATIONS

FRANCESCA ANCESCHI, GIAMPIERO PALATUCCI, AND MIRCO PICCININI

Abstract. We deal with the following wide class of kinetic equations,

$$\Big[\partial_t + v\cdot\nabla_x\Big] f = \mathcal{L}_v f.$$

Above, the diffusion term $\mathcal{L}_v$ is an integro-differential operator, whose nonnegative kernel is of fractional order $s\in(0,1)$ having merely measurable coefficients.

Firstly, we obtain a general L^∞-interpolation inequality with a natural nonlocal tail term in velocity, in turn giving local boundedness even for weak subsolutions f without any sign assumption. This is a veritable novelty, being boundedness usually assumed apriori in such a setting. Then, provided that their nonlocal tail in velocity is $(2+\varepsilon)$-summable along the transport variables, we prove a general Strong Harnack inequality, which in the simpler case of globally nonnegative weak solutions f reads as follows

$$\sup_{Q^-} f \ \leqslant\ c\,\inf_{Q^+} f + c\,\|\,\mathrm{Tail}(f)\|_{L^{2+\varepsilon}_{t,x}}\,,$$

where $Q^\pm$ are suitable slanted cylinders. This is the first strong Harnack inequality for kinetic integral equations under the aforementioned tail summability assumption, which is in fact naturally implied in literature, e. g., from the usual mass density boundedness (as for the Boltzmann equation without cut-off), and in clear accordance with the very recent surprising counterexample by Kaßmann and Weidner; see [KW24c].

A new standalone result, a Besicovitch-type covering argument for very general kinetic geometries, independent on the involved equation is also needed, stated and proved.

Contents

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1. Introduction

In the present work we study the following wide class of kinetic integro-differential equations,

$$\partial_t f + v\cdot\nabla_x f = \mathcal{L}_v f + h \quad \text{in } \Omega\subset\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\,, \tag{1.1}$$

where h is an appropriate source term, and the diffusion term $\mathcal{L}_v$ is given by

$$\mathcal{L}_v f(t,x,v) = \text{ p. v.}\int_{\mathbb{R}^n}\big(f(t,x,w)-f(t,x,v)\big)K(t,x,w,v)\,\mathrm{d}w\,, \tag{1.2}$$

with $K=K(t,x,w,v)\approx|w-v|^{-n-2s}$ being a symmetric kernel of order $s\in(0,1)$ with merely measurable coefficients, whose prototype is the classical fractional Laplacian operator $(-\Delta_v)^s$, with respect to the v-variables, given by

$$(-\Delta_v)^s f(t,x,v) := c_{n,s}\ \text{p. v.}\int_{\mathbb{R}^n}\frac{f(t,x,v)-f(t,x,w)}{|v-w|^{n+2s}}\,\mathrm{d}w\,. \tag{1.3}$$

In the display above, $c_{n,s}$ is a positive constant only depending on the dimension n and the differentiability exponent s; see [DPV12, Section 2] for further details. Such hypoelliptic equations arise as linearized models for the Boltzmann equation without cutoff. We also notice that the integrals in (1.2)-(1.3) may be singular at the origin and they must be interpreted in the appropriate sense. Since we are considering diffusion terms with possibly rough coefficients, the related equation has to have a suitable weak formulation; we immediately refer the reader to Section 2 below for precise assumptions on the involved quantities.

One of our mail goals will be the proof of Harnack inequalities for weak solutions to (1.1).

During the last century the validity of the classical Harnack inequality has been an open problem in the nonlocal setting, and more in general for integro-differential operators. The first answer for the purely fractional Dirichlet equation had been eventually given by Kaßmann in his breakthrough papers [Kas07,Kas11], where the strong Harnack inequality is proven to be still valid by adding an extra term on the right-hand side which cannot be dropped nor relaxed even in the most simple case when $\mathcal{L}_v$ does coincide with the fractional Laplacian operator $(-\Delta_v)^s$ in (1.3); see Theorem 1.2 in [Kas07]. Such an extra term does completely disappear in the case of nonnegative weak solutions; see Theorem 3.1 in [Kas11], so that one falls in the classical strong Harnack formulation. We refer to [DKP14], where the needed nonlocal addendum had been firstly defined in a precise quantitative way and introduced as "nonlocal tail", even for a more general nonlinear fractional framework.

After the breakthrough results by Kaßmann, Harnack-type inequalities and a quite comprehensive nonlocal De Giorgi-Nash-Moser theory have been presented in more general *integro-differential elliptic frameworks*, even for nonlinear fractional equations. The literature is really too wide to attempt any comprehensive list here; we refer to [DKP14, ROS14, DKP16, BLS18, Fal20, Now21, BKO23, KL23, CKW23, FR24] and the references therein; it is worth presenting also the important Harnack inequalities in [KW24a], which deals with very irregular integro-differential kernels so that a link to Boltzmann-type collision kernels seems veritably close.

The situation becomes more convoluted in the integro-differential parabolic framework. Indeed, in order to prove Harnack-type results, the intrinsic scaling of the involving cylinders will depend not only on the time variable t, as in the classical pioneering work by DiBenedetto [Dib88, Urb08], but also on the differentiability order s. As one can imagine, this is not for free even for the purely p-fractional heat equation. However, few fractional parabolic Harnack inequalities are still available in the case when the kernel of the leading operator is a sort of $(s,2)$-Gagliardo-type one, as, e. g., in [Str19] in part extending the results in the elliptic counterpart in [DKP14]; see also [KS14], the very recent paper [KW24c], and the aforementioned paper [KW24a] dealing with very intricate irregular kernels. Nevertheless, notable differences in such a parabolic framework inevitably arise, and the validity of a (strong) Harnack inequality could fail depending on the specific assumptions on the involved kernels, even when starting from bounded solutions; see, e. g., [BS05, BS07].

As noticed above, already in the nonlocal parabolic framework – that to some extent should be seen as the space homogeneous version of (1.1) – one needs new strategies and ideas (as a concrete example, see the fine analysis in [Sil16]), and strong Harnack inequalities are still not assured (as in the case of the aforementioned counter-examples). More specifically, even for purely kinetic equations with fractional diffusion as in (1.1) the validity or not of a strong Harnack inequality has been an open problem. This is not a surprise because of the very form of the equations in (1.1) which also involves a transport term, and the nonlocality in velocity has to be dealt with keeping into account the involuted intrinsic scalings naturally arising. More specifically, even in the stationary case and considering the constant coefficients operator modeling (1.1), that is

$$\mathcal{K} := v\cdot\nabla_x + (-\Delta_v)^s\,, \tag{1.4}$$

one needs to carefully handle the drift term $v\cdot\nabla_x$. Indeed, in combination with the nonlocal behavior of the fractional Laplacian it makes the resulting operator $\mathcal{K}$ very sensitive to changes of the exterior data along the x-direction.

Eventually, this combined behavior led to the establishment of an ingenuous counterexample by Kaßmann and Weidner in [KW24c], where they built a sequence of solutions $\{f_j\}$ to the kinetic fractional Kolmogorov equation $\mathcal{K}f=0$ such that the ratio $f_j(0,0)/f_j(\frac{1}{2}e_n,0)\nearrow\infty$ as $j\searrow 0$; see Theorem 1.1 there. In particular, this implies the failure of the Harnack inequality for (1.1). As mentioned before, such a failure is a pure effect originating from the combination of the nonlocality of the diffusion term combined with the anisotropy of $\mathcal{K}$, and it is very surprising when compared to all the previous literature dealing with local kinetic equations, as for instance those where strong Harnack inequalities have been established; see the fundamental results in [GIMV19, GM22, GI23] and the survey [Mou18], alongside with [LP94, KP16, AT19, PP04a, ADGLMR, AR22] for other strictly related classes of hypoelliptic equations.

Furthermore, it is worth noticing that such a phenomenon is quite remarkable given that the degeneracy of $\mathcal{K}$ is no obstruction to C^∞-regularity for fractional equations as in (1.1); see for example [IS20b, IS22]. Indeed, by velocity averaging techniques ([Bou02]) it is possible to transfer regularity from the v-variable to the x-one as happens for purely local operators.

In order to better clarify the situation, it is enlightening to focus on the fundamental class of nonlocal kinetic equations modeling the Boltzmann problem without cut-off, for which very important estimates and regularity results were recently proven via fine variational techniques and radically new approaches. An inspiring step in such an advance in the regularity theory relies on the approach proposed in the breakthrough paper [IS20b], where the authors, amongst other results, are able to derive a weak Harnack inequality for essentially bounded solutions to a very large class of kinetic integro-differential equations as in (1.1) with very mild assumptions on the integral diffusion in velocity having degenerate kernels K in (1.2) which are not symmetric (not in the usual way) nor pointwise bounded by Gagliardo-type kernels; see Theorem 1.6 there. Under a coercivity condition on $\mathcal{L}_v$ and other natural assumptions (see Section 1.1 in [IS20b]), the same result for the Boltzmann equation mentioned above follows as a corollary. Further related regularity estimates under conditional assumptions on the solutions f have been subsequently proven in [IS22]. Despite the fine estimates and the new approach in [IS20b, IS22], a strong Harnack-type inequality is still missing. A very recent step in this direction is the following inequality

$$\sup_{\widetilde{Q}^-} f \;\leqslant\; c\left(\inf_{\widetilde{Q}^+} f\right)^{\beta}. \tag{1.5}$$

obtained in [Loh24a] via a quantitative De Giorgi-type approach based on (local) trajectories and where the solutions f are assumed globally bounded a priori.

This is a nontrivial result ([Loh24a, Theorem 1.3]), but the exponent $0<\beta\leqslant 2s/(n+2ns)<1$ in the estimate above is in fact a root, and thus a strong Harnack inequality cannot be deduced, in accordance with the aforementioned counterexample in [KW24c]. Notice that the cylinders $\widetilde{Q}^{\pm}$ in (1.5) above are naturally slanted in order to deal with the underlying kinetic geometry; see in particular Figure 2 in [Loh24a]); also compare with the slanted cylinders in [Sto19, IS20b, IMS20, IS20a, IS22] as well as with the sharp ones in our forthcoming Theorems 1.4 and 1.5 here, also pictured in forthcoming Figure 1.

Again, for integro-differential equations, the situation is different than for classical second order equations. For this, we take the liberty to quote the clarifying explanation by the authors in [IS20b, Pag. 548], «`It is not true that the maximum of a nonnegative subsolution can be bounded above by a multiple of its` L^2 `norm. One needs to impose an extra global restriction (in this case we assume` $0\leqslant f\leqslant 1$ `globally). This is because of nonlocal effects, since the positive values of the function outside the domain of the equation may pull the maximum upwards.`»

In order to overcome the nonlocality issues mentioned above (which will also prevent a strong Harnack inequality from Hölder estimates), in the present paper we prove a totally new δ-interpolation L^∞-inequality with tail for weak subsolutions to (1.1) which are not even required to be nonnegative. The parameter $0<\delta\leqslant 1$ in such a boundedness estimate can be suitably chosen in order to balance in a quantitative way the local contributions and the nonlocal ones; see in particular the right-side of the inequality (1.6) in the theorem below; that is, the $L^{2+\varepsilon}$-norm along the drift variables of the nonlocal Tail-quantity in velocity, for which we immediately refer to forthcoming Definition 2.2 in Section 2.3, where related observations on the fractional framework and the underlying hypoelliptic geometry are also presented. Now, we are ready to state our first main result, the local boundedness estimate, which constitutes a veritable novelty in the kinetic integral framework. We have the following

Theorem 1.1 (**The δ-interpolative L^∞-L^2 estimate).** *Let $Q_1\equiv Q_1(\mathbf{0})\subset\Omega$, let $f\in\mathcal{W}$ be a weak subsolution to (1.1) in Ω such that* $\mathrm{Tail}(f_+)\in L^{2+\varepsilon}_{\mathrm{loc}}((t_1,t_2)\times\Omega_x)$*, and let $h\in L^{2+\varepsilon}_{\mathrm{loc}}(\Omega)$ for some $\varepsilon>\varepsilon^\star$ be such that $h\leqslant 0$. Then, for* *any* $Q_R \equiv Q_R(\mathbf{0}) \subset Q_1$ *and any* $\delta \in (0,1]$*, it holds*

$$\text{(1.6)}\quad \sup_{Q_{\frac{R}{2}}} f \leqslant \frac{c}{\sqrt[\alpha]{\delta R^{n+5s}}}\|f_+\|_{L^2(Q_R)} + \delta \,\|\operatorname{Tail}(f_+;B_{R/2})\|_{L^{2+\varepsilon}(U_R)} + \|h\|_{L^{2+\varepsilon}(Q_R)}\,,$$

where $Q_R := U_R \times B_R \equiv (-R^{2s},0]\times B_{R^{1+2s}}\times B_R$*, and the quantities* $\alpha, \varepsilon^\star > 0$ *only depend on the dimension* n *and the exponent* s*, see forthcoming Formula* (3.32)*; whereas the positive constant* c *also depends on the kernel structural constant* Λ *in* (2.2)*.*

Remark 1.2. It is possible to drop the sign assumption on the source term h by assuming an L^∞-bound, as usually done in literature; see for instance the aforementioned [Mou18] and [IS20b]. Due to the forthcoming use of this result to prove our Harnack inequalities-type results, we prefer to state it in the most general framework allowing h to be possibly unbounded.

The backbone of the proof of Theorem 1.1 is an hypoelliptic gain of Sobolev regularity for weak subsolutions, which is proven by making use of the fundamental solution of the fractional Kolmogorov equation instead of velocity averaging techniques and it is where the $\varepsilon^\star(n,s)$ quantity arises. Then, the proof of Theorem 1.1 is obtained combining such gain of summability with a Caccioppoli inequality – see forthcoming Lemma 3.1 – as well as with a fine iterative argument taking into account both the $L^{2+\varepsilon}$-energy of the Tail term and the desired interpolative effect.

Remark 1.3. For this, a comment on our turning point to attack the whole problem via the energy in the transport variable of the nonlocal tail is in order. Basically, in most of the aforementioned parabolic literature the nonlocal effects have been compensated via a supremum tail, which apparently did the trick (sometimes under further global assumptions on the solution), despite not natively coming from the scaling of the involved parabolic equations. Such a L^∞-Tail choice appears very strong and easily adaptable to obtain several estimates even for solutions to (1.1). Nevertheless, it also reveals to be a concrete stumbling block to concretize our program in order to obtain the desired strong Harnack inequality under light nonlocal assumptions. On the contrary, a L^1 boundedness of the Tail would have been too weak, because unsuitable to control the deterioration for large velocities following the nonlocal diffusion term. Then, by working on the $(2+\varepsilon)$-summability in transport of the Tail contribution we are able to find a balance for such a discrepancy, in turn also dealing with the combined effects by the transport term in the equation.
Accordingly, a couple of additional remarks are in order.

- **1.3(a)** First of all, thanks to the specific construction of the sequence of solutions $\{f_j\}$ in the fine counterexample by Kaßmann and Weidner ([KW24c]) one can see that (1.6) does not contradict [KW24c, Theorem 1.1]; see the definition of ε^* together with the comments after forthcoming Theorem 1.5 on Page 7 for further details.
- **1.3(b)** By definition, weak solutions to (1.1) are not required to have finite $L^{2+\varepsilon}$-energy of their nonlocal tail in velocity. However, apart from the mathematical point of view (for which we also refer to the explanations at the end of [KW24c, Section 1.2], and to the introduction in [Sto19]), it is worth pointing out that the usual boundedness requirements of notable hydrodynamic quantities required in physical models for the Boltzmann

equation without cut-off and related kinetic equations plainly imply our requirement for the local finiteness of the $L^{2+\varepsilon}$-energy of the nonlocal tail, see for instance the condition on the mass as in [Sil16, Theorems 1.1-1.2], [Mou18, Formula (1.4)], [GIMV19, Formula (1.3)], [IMS20, Formula (1.3)], [IS20a, Section 1.4-Assumption (H)], [IS20b, Section 1.3], [IS22, Assumption 1.1], and so on.

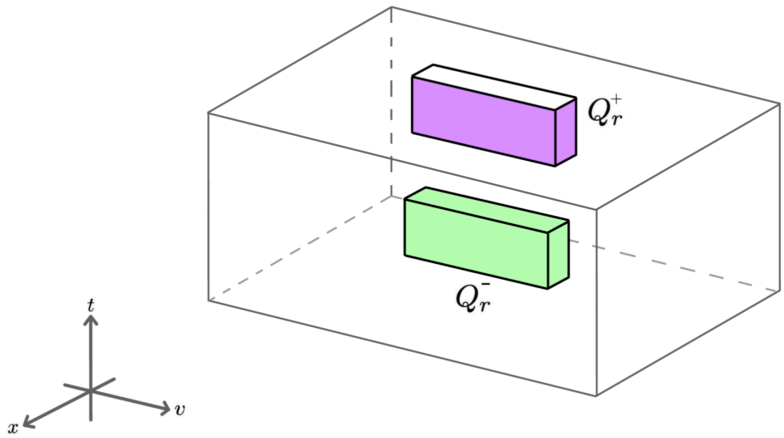

FIGURE 1. The geometry of the Harnack inequalities for kinetic equations with integro-differential diffusion.

As expected, the feasibility of the result in Theorem 1.1 above will allow us to bypass the global boundedness assumption on the solutions f usually assumed in the previous kinetic literature, in turn being fundamental in order to prove the main estimates for solutions to (1.1) presented right below. We start with a weak Harnack inequality in the case when the function f is merely a supersolution.

Theorem 1.4 (The weak Harnack inequality). *Let $Q_1 \equiv Q_1(\mathbf{0}) \subset \Omega$, let $f \in \mathcal{W}$ be a nonnegative weak supersolution to (1.1) in Ω, and let $h \in L^{2+\varepsilon}_{\rm loc}(\Omega)$ for some $\varepsilon > \varepsilon^\star$, where $\varepsilon^\star$ is given in* Theorem 1.1*, such that $h \leqslant 0$. Then, there exist r_0, c and ζ depending on s and the dimension n such that*

$$\left(\fint_{Q^-_{r_0}} f(t,x,v)^\zeta \,{\rm d}v\,{\rm d}x\,{\rm d}t \right)^{\frac{1}{\zeta}} \leqslant c \inf_{Q^+_{r_0}} f + \|h\|_{L^{2+\varepsilon}(Q_1)} \,, \tag{1.7}$$

where

$$Q^+_{r_0} := (-{r_0}^{2s}, 0] \times B_{{r_0}^{1+2s}} \times B_{r_0}$$
$$\text{and} \quad Q^-_{r_0} := (-1, -1 + {r_0}^{2s}] \times B_{{r_0}^{1+2s}} \times B_{r_0}. \tag{1.8}$$

The proof of Theorem 1.4 is basically obtained by extending the De Giorgi approach; that is, by proving both a new suitable Intermediate Value lemma and a Measure-to-Pointwise lemma. This is done in the same spirit of the pioneering work [CCV11], and the subsequent delicate extension in the kinetic framework in [Loh24a], as well as of recent parabolic results for fractional heat equations (as seen, e. g., in [Str19, Lia22]); see also [CF13] for related regularity estimates for the fractional parabolic obstacle problem. However, because of the difficulties naturally arising in the hypoelliptic framework here, we have to deal with the intrinsic peculiarities due to the natural scaling trichotomy: time, space and velocity. In addition, since we are looking for clean Harnack inequalities on optimal (local) slanted cylinders with optimal gap in time, we handle all the related nonlocal estimates in a possibly sharp way by taking into account the tail contributions in this sort of expansion of positivity of suitable subsolutions. The δ-interpolative L^∞-L^2 inequality in Theorem 1.1 is in fact applied to a suitable (not positive) sequence $g = g_k$ approximating an auxiliary subsolution to (1.1) which is built starting from a supersolution f. For this, in clear accordance with related weak Harnack proven in the kinetic integral literature, we do not need to require the integrability of the nonlocal tail. Also, even if we have to operate several modifications in usual fractional estimates, we still cannot apply the standard Krylov-Safonov covering lemma in the framework we are dealing with. For these reasons, we eventually complete the proof of Theorem 1.4 by means of the new Ink-spots Theorem by Imbert and Silvestre proved in [IS20b, Section 9], which allows us to deal with the naturally *slanted* geometry; see Section 2.2 below. Lastly, it is worth remarking that this is the first time a weak Harnack estimate is proved for kinetic integral equations possibly including unbounded source term h.

Eventually, considering null source term h and no a priori boundedness assumptions for solutions f to (1.1), we are able to prove a new (possibly sharp) formulation of the classical strong Harnack inequality for kinetic equations with nonlocal diffusion, provided only the local energy bounded assumption on the tail discussed in Remark 1.3. Hence, our third main result reads as follows,

Theorem 1.5 (**The strong Harnack inequality**). *Let $Q_2 \equiv Q_2(\mathbf{0}) \subset \Omega$, and let $f \in \mathcal{W}$ be a weak solution to (1.1) with $h = 0$ in Ω such that $f \geqslant 0$ in Ω and $\mathrm{Tail}(f) \in L^{2+\varepsilon}_{\mathrm{loc}}((t_1,t_2)\times\Omega_x)$ for some $\varepsilon > \varepsilon^\star$. Then, given r_0 the one in Theorem 1.4, for any $0 < r \leqslant r_0$ it holds*

$$
\begin{aligned}
\sup_{Q_r^-} f \quad &\leqslant \quad c \inf_{Q_r^+} f + c\,\|\,\mathrm{Tail}(f_+; B_{r/2})\|_{L^{2+\varepsilon}(U_r(-1+r^{2s},0))} \\
&\qquad + c\,\|\,\mathrm{Tail}(f_-; B_2)\|_{L^{2+\varepsilon}(U_2)}\ ,
\end{aligned} \tag{1.9}
$$

where $\varepsilon^\star$ only depends on n and s, see forthcoming Formula (3.32); whereas the constant $c > 0$ also depends on the kernel structural constant Λ in forthcoming formula (2.2).

As natural when dealing with fractional problems, in order to compensate the possible negative interactions at large velocities of the solution which can pull the infimum down – in turn leading to the failure of the Harnack inequality in the elliptic case [Kas07, Kas11] – a tail contribution of the negative part of the solution does naturally appear in the right-hand side of our estimates. However, in striking contrast with its elliptic and parabolic counterparts ([DKP14, KW24c]), even when restricting to globally nonnegative solutions, a nonlocal reminder – given by the $(2+\varepsilon)$-norm of $\mathrm{Tail}(f_+)$ – still persists in the estimate. Thus, it has been fundamental our detection of such a precise quantity which will permit us to control the expected deterioration given by the aforementioned anisotropic combination of the drift with the nonlocal diffusion as seen in the model example (1.4), which in turn takes part to the failure of the classical Harnack estimate in the kinetic integral setting. Furthermore, our new tail formulation in both Theorem 1.1 and Theorem 1.5 is somehow sharp. Indeed, for the aforementioned sequence of stationary solutions $\{f_j\}$ in [KW24c] the quantity

$$
\sup f_j / (\inf f_j + \|\mathrm{Tail}(f_j)\|_{L^{2+\varepsilon}})\,,
$$

does blow up as $j \searrow 0$ when $\varepsilon < (n-2) + n/(2s) < \varepsilon^\star$.

As for the proof of Theorem 1.5, we rely on a splitting technique as in the parabolic case, see [KW24c], so that we are also able to deal with the more general case where possibly sign-changing solutions outside Ω are allowed. Indeed, we re-write the solution as $f = f_+ - f_-$, and we treat the resulting negative part appearing in the diffusion as a negative source term

$$\mathcal{L}_v f \;\approx\; \mathcal{L}_v f_+ - \int_{\mathbb{R}^n\setminus\Omega_v} f_-(w)K(v,w)\,\mathrm{d}w.$$

Then, by the $(2+\varepsilon)$-summability of f along the transport variables, it plainly follows that

$$\int_{\mathbb{R}^n\setminus\Omega_v} f_-(w)K(v,w)\,\mathrm{d}w \;\in\; L^{2+\varepsilon}_{\mathrm{loc}}(L^\infty_{\mathrm{loc}})\,,$$

so that it becomes crucial to prove Theorem 1.5 to actually establish both the L^∞-L^2 estimate and the weak Harnack inequality under more general integrability assumptions on the source term h. Subsequently, the proof will combine our weak Harnack inequality (1.7) in Theorem 1.4 with a precise application of the L^∞-L^2 estimate in Theorem 1.1. Finally, a new iterative argument taking into account the involved transport and diffusion radii is applied in order to complete the estimate in (1.9). This step relies on a new Besicovitch's covering lemma for slanted cylinders; see forthcoming Lemma 6.1 at Page 39, which reminds of the classical covering argument appearing in the last step of most regularity results for both local and/or nonlocal elliptic or parabolic problems via the usual variational approach. We believe that the latter is a very general argument that will find many applications in the kinetic frameworks.

Still in theme of Harnack-type inequalities for kinetic equations, it is worth mentioning the very recent paper [Loh24b], in which amongst other interesting results the author proves a strong Harnack inequality (with the same gap in time in the cylinders as in (1.5)) for kinetic integral equations for *global solutions*, a priori bounded, periodic in the space variable, and under an integral monotonicity-in-time assumption (see Definition 2.2 there). The usual nonlocality issues are partially annihilated by the peculiar global framework there, so that no tail contributions do appear.

1.1. **Novelty of the results: a brief summary.** Under a suitable local $L^{2+\varepsilon}$-boundedness of the nonlocal tail in velocity of the solutions – naturally implied by the related physical models as well as by all the pre-existent literature – we prove the validity of the very first strong Harnack inequality with tail formulation for kinetic integral equations, whose diffusion term in velocity is given by fractional Laplacian-type operators with merely measurable coefficients, in turn extending to the nonlocal framework Harnack inequalities for the classical Kolmogorov-Fokker-Planck equation, as, e. g., in [GIMV19], as well as extending to the kinetic framework Harnack inequalities for both the fractional elliptic equations and the parabolic one ([Kas11, Str19]). Also, no a priori boundedness is assumed on the solutions, which in fact is proven even for subsolutions without sign assumptions via the aforementioned $L^{2+\varepsilon}$-boundedness of the kinetic tail and suitable energy estimates. As a further addition, both our strategy and proofs are feasible to be used in very general hypoelliptic frameworks, and our final new slanted covering Lemma is basically untied to our equation (1.1) being in fact a purely geometric property and the kinetic counterpart of classical Besicovitch covering-type results.

1.2. **Further developments.** We believe our whole approach and new general independent results to be the starting point in order to attack several *open problems* related to nonlocal kinetic equations, as, e. g., those listed below.

• By replacing the linear diffusion class of fractional operators with nonlinear p-Laplacian-type operators, as firstly done in [AP23], and in the parabolic setting in [Lia22,Lia24]. The nonlinear growth p framework in those Gagliardo seminorms seems to be not so far from the framework presented here in the superquadratic case when $p > 2$; the singular case when $1 < p < 2$ being trickier. However, several "linear" fractional techniques are not disposable; it is no accident that Harnack inequalities are still not available even in the space homogeneous counterpart; say, in the parabolic setting. Nevertheless, our estimates together with the generalization in [AP23] – and the techniques employed in order to treat nonlinear fractional parabolic equations in [Lia22] – might be a first outset for dealing with the fractional counterpart of nonlinear subelliptic operators.

• Accordingly to the spirit of related results in literature, as for instance the Harnack inequalities in [AT19] and in [Jul15], one could consider to attack the problem in (1.1) via a viscosity approach, in the same flavor of the Krylov-Safonov approach presented in [Sil06, DFP19] for general integro-differential equations.

• Similar results can be expected for energy solutions to a family of kinetic equations strictly related to (1.1) which arises from different physical models, by replacing the drift with a more general term as $\partial_t + b(v) \cdot \nabla_x$, including more general physical models, as e. g. considering possibly relativistic effects when $b(v) := \frac{v}{\sqrt{1+|v|^2}}$. Classical regularity theory has been developed in the local case in [Zhu21]; see also [APR21] for Harnack inequality and lower bound of the fundamental solution for the relativistic Fokker-Planck operator.

• Coming back to purely kinetic equations, our strategy and techniques could be repeated in order to attack the very wide class of integro-differential kernels as those considered in [IS20b, OS23], and thus implying a strong Harnack-type inequality for the Boltzmann non-cutoff equation (under the usual assumptions on the hydroquantities, in turn also implying the validity of our local $L^{2+\varepsilon}$-assumption on the tail). Such a result appears to be very challenging, because of the weaker assumptions on the involved kernels in the diffusion term still enjoying some subtle cancellation property, but lacking a pointwise control as in the purely fractional framework. However, by following our strategy one can take advantage of the fact that a sharp weak Harnack inequality (Theorem 1.6 in [IS20b]) and several other important estimates are already available; see [IS20b, IS21, OS23]. Lastly, our Besicovitch covering result will be finally applied with no modifications at all.

• Our result in Theorem 1.5 could be of some feasibility even to apparently unrelated problems, as, a concrete example, in the mean fields game theory. It is known that under specific assumptions, mean field games can be seen as a coupled system of two equations, a Fokker-Planck-type equation evolving forward in time (governing the evolution of the density function of the agents), and a Hamilton-Jacobi-type equation evolving backward in time (governing the computation of the optimal path for the agents). Such a forward vs. backward propagation in time should lead to interesting phenomena in time which are present in nature but they have not been investigated in the nonlocal framework yet. Our contribution in the present manuscript together with other recent results and new techniques as the ones developed in [DQT19, Gof21, Dav22] could be unexpectedly helpful for such an intricate investigation.

• Finally, it is well known about the many direct consequences and applications of a strong Harnack inequality, as for instance, maximum principles, eigenvalues estimates, Liouville-type theorems, comparison principles, global integrability, and so on. For a discussion of certain of the aforementioned PDE aspects in the local framework counterpart we refer to [KPP16].

1.3. **Outline of the paper.** In Section 2 below we briefly fix the notation by also introducing the fractional kinetic framework together with important needed results. In Section 3 we prove fundamental kinetic energy estimates with tail and our δ-interpolative L^∞-L^2 estimate in Theorem 1.1. Section 4 is devoted to a nonlocal expansion of positivity (via a De Giorgi-type intermediate lemma and the measure-to-pointwise lemma) in order to accurately estimate the infimum of the subsolutions to (1.1), which precisely takes into account the nonlocality in the diffusion via the kinetic tail. In Section 5 we shall complete the proof of Theorem 1.4. In subsequent Section 6 we state and prove the new Besicovitch-type covering result which naturally can be applied in general kinetic geometries. Finally, in Section 7 we are able to prove the strong Harnack inequality given by Theorem 1.5.

2. Preliminaries

In this section we fix notation, and we briefly recall the necessary underlying framework in which one needs to work in order to deal with the class of nonlocal kinetic equation (1.1). For a more comprehensive analysis of Lie groups in the kinetic setting we refer the reader to the surveys [AP20, APR24] and the references therein; the interested reader could also refer to the recent papers [MPP23, PP24] where an intrinsic Taylor formula and Sobolev embeddings are presented for the class of operators in (1.2).

2.1. **Notation.** In this section we fix some of the notation we are going to used throughout the paper.

We denote with c a positive universal constant greater than one, which may change from line to line. For the sake of readability, dependencies of the constants will be often omitted within the chains of estimates, therefore stated after the estimate. Relevant dependencies on parameters will be emphasized by using parentheses.

As customary, for any $R > 0$ and any $y_0 \in \mathbb{R}^m$ we denote by

$$B_R(y_0) \equiv B(y_0; R) := \big\{ y \in \mathbb{R}^m \,:\, |y - y_0| < R \big\},$$

the open ball with radius R and center y_0. For any $\beta > 0$ we will denote with $\beta B_R(y_0)$ the rescaled ball by a factor of β, i. e. $\beta B_R(y_0) = B_{\beta R}(y_0)$.

For any set $\mathcal{O} \subset \mathbb{R}^m$ we will denote the Lebesgue measure of $\mathcal{O}$ with $|\mathcal{O}|$. Moreover, for any $f \in L^1(\mathcal{O})$, we let

$$(f)_{\mathcal{O}} := \fint_{\mathcal{O}} f \, {\rm d}y := \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} f \, {\rm d}y.$$

For any $k \in \mathbb{R}$, we denote the positive and negative part of f as

$$(f(y) - k)_+ := \max\{f(y) - k, 0\} \quad \text{and} \quad (f(y) - k)_- := \max\{k - f(y), 0\}.$$

Clearly $(f(y)-k)_+ \neq 0$ on the super-level set $\{y \in \mathbb{R}^m : f(y) > k\}$, whereas $(f(y)-k)_- \neq 0$ on $\{y \in \mathbb{R}^m : f(y) < k\}$.

2.2. **The underlying geometry.** We start by endowing $\mathbb{R}^{1+2n} = \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ with the following Galilean transformation

$$(t_0, x_0, v_0) \circ (t, x, v) := (t + t_0,\ x + x_0 + tv_0,\ v + v_0).$$

With respect to the group law $\circ$, the couple $(\mathbb{R}^{1+2n}, \circ)$ is a Lie group with identity element $\mathbf{0} \equiv (0,0,0)$ and inverse element for $(t,x,v) \in \mathbb{R}^{1+2n}$ given by $(-t,\ -x+tv,\ -v)$.

For any $r > 0$, consider the usual kinetic scaling $\delta_r : \mathbb{R}^{1+2n} \mapsto \mathbb{R}^{1+2n}$ defined by

$$\delta_r(t,x,v) := (r^{2s}t,\ r^{1+2s}x,\ rv).$$

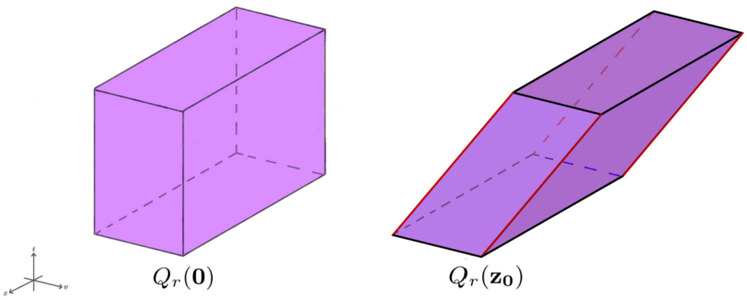

FIGURE 2. On the left the cylinder $Q_r(\mathbf{0})$ centred at the origin; on the right a *slanted* cylinder $Q_r(\mathbf{z_0}) \equiv Q_r(t_0, x_0, v_0)$ according to the invariant transformation given in (2.1).

As customary in the hypoelliptic setting, we need to introduce a family of fractional kinetic cylinders respecting the invariant transformations defined above. For any $r > 0$, we denote by Q_r a cylinder centred in $(0,0,0)$ of radius r; that is,

$$Q_r \equiv Q_r(\mathbf{0}) := U_r(0,0) \times B_r(0) = (-r^{2s}, 0] \times B_{r^{1+2s}}(0) \times B_r(0)\,.$$

For every $(t_0, x_0, v_0) \in \mathbb{R}^{1+2n}$ and for every $r > 0$, the *slanted* cylinder $Q_r(t_0, x_0, v_0)$ is defined as follows,

$$\begin{aligned} Q_r(t_0,x_0,v_0) &:= \{(t_0,x_0,v_0) \circ \delta_r(t,x,v) : (t,x,v) \in Q_1\} \\ &\equiv \{(t,x,v) \in \mathbb{R}^{1+2n} : t_0 - r^{2s} < t \leqslant t_0, \\ &\qquad |x - x_0 - (t-t_0)v_0| < r^{1+2s},\ |v - v_0| < r\}. \end{aligned} \tag{2.1}$$

Roughly speaking the integro-differential equation (1.1) is invariant under the kinetic scaling δ_r and left-invariant with respect to Galilean transform, namely, for any $(t_0, x_0, v_0) \in \mathbb{R}^{1+2n}$ and any $r > 0$, if f is a solution to (1.1) in $Q_r(t_0, x_0, v_0)$

then $f((t_0, x_0, v_0) \circ \delta_r(\cdot))$ solves an equation of the same ellipticity class as (1.1) in Q_1.

Now, we recall a suitable covering argument in the same flavour of the Krylov-Safonov Ink-spots theorem. Indeed, in our framework one cannot plainly apply the usual Calderón-Zygmund decomposition, because there is no space to tile slanted cylinders with varying slopes. This is a major difficulty in the nonlocal kinetic framework which had been firstly addressed in an original way by Imbert and Silvestre in [IS20b], who were able to state and prove a custom version of the Ink-spots theorem. Such a result, that we present right below in Theorem 2.1, allows us to conclude the proof of the weak Harnack inequality being the main ingredient of the proof of a new Besicovitch-type covering. Note that this covering is also suitable for very general kinetic-type frameworks when slanted cylinders do naturally lead the involved geometry; see forthcoming Section 6.

In order to state Imbert-Silvestre's Ink-Spots Theorem, we need to introduce the *stacked (and slanted) cylinders* $\bar{Q}_r^m$ for some given $m \in \mathbb{N}$. We have

$$\bar{Q}_r^m(t_0, x_0, v_0) := \{(t, x, v) \in \mathbb{R}^{1+2n} : 0 < t - t_0 \leqslant mr^{2s},$$
$$|x - x_0 - (t - t_0)v_0| < (m+2)r^{1+2s},\ |v - v_0| < r\}.$$

Notice that the cylinder $\bar{Q}_r^m$ starts at the end (in time) of Q_r and its duration (still in time) is exactly m-times the one of Q_r, whereas its spatial radius is $(m+2)$-times the one of Q_r; see Figure 2.2 below.

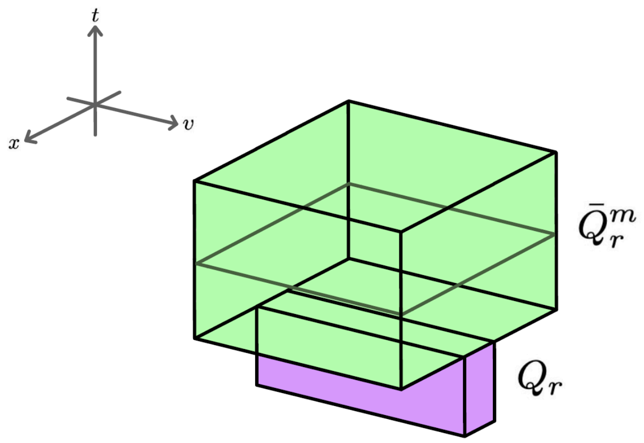

FIGURE 3. A stacked (and slanted) cylinder $\bar{Q}_r^m$. We refer to Section 10 in [IS20b] for a very detailed analysis and further related results.

Then we have the following

Theorem 2.1 (**the Ink-spots Theorem with leakage**; see [IS20b, Corollary 10.2]). *Let $E \subset F$ be bounded measurable sets. Assume that*

(i) $E \subset Q_1$,

(ii) *there exists two constants* $\mu, r_0 \in (0,1)$ *and an integer* $m \in \mathbb{N}$ *such that for any cylinder* $Q = Q_\sigma(t_0, x_0, v_0) \subset Q_1$ *satisfying* $|Q \cap E| \geqslant (1-\mu)|Q|$, *then* $\bar{Q}^m \subset F$ *and also* $\sigma < r_0$.

Then,

$$|E| \leqslant \frac{m+1}{m}(1-c\mu)\left(|F \cap Q_1| + Cmr_0^{2s}\right)$$

for some constants c *and* C *depending only on* n *and* s.

It is worth noticing that related Krylov-Safonov-type results both in the local and nonlocal kinetic framework can be also found in [PP04b, SS16]. In this respect, we also refer to [AT19, DY24] where a class of Kolmogorov-Fokker-Planck equations in non-divergence form is considered.

2.3. **The nonlocal energy setting.** We now introduce our fractional functional setting. Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$; for $s \in (0,1)$ we recall the definition of the classical fractional Sobolev spaces $H^s(\mathcal{O})$; i. e.,

$$H^s(\mathcal{O}) \equiv W^{s,2}(\mathcal{O}) := \left\{ f \in L^2(\mathcal{O}) \ : \ [f]_{H^s(\mathcal{O})} < +\infty \right\},$$

where the fractional seminorm $[f]_{H^s(\mathcal{O})}$ is the usual one via Gagliardo kernels,

$$[f]_{H^s(\mathcal{O})} := \left(\iint_{\mathcal{O}\times\mathcal{O}} \frac{|f(v)-f(w)|^2}{|v-w|^{n+2s}} \, \mathrm{d}v \, \mathrm{d}w \right)^{1/2}.$$

A norm of $H^s(\mathcal{O})$ is given by

$$\|f\|_{H^s(\mathcal{O})} := \|f\|_{L^2(\mathcal{O})} + [f]_{H^s(\mathcal{O})}.$$

A function f belongs to $H^s_{\mathrm{loc}}(\mathcal{O})$ if $f \in H^s(\mathcal{O}')$ whenever $\mathcal{O}' \Subset \mathcal{O}$.

As mentioned in the Introduction, the kernel $K : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{2n} \to [0,\infty)$ is a measurable kernel having s-differentiability for any $s \in (0,1)$; that is, there exists a positive constant Λ such that

$$\Lambda^{-1}|v-w|^{-n-2s} \leqslant K(v,w) \leqslant \Lambda|v-w|^{-n-2s}, \quad \text{for a. e. } v,w \in \mathbb{R}^n, \tag{2.2}$$

where we assume that the condition above hold for all t and x; we omit the t and x dependence to clean up the notation.

It is worth noticing that most of the estimates in the rest of the paper would still work by weakening such a pointwise control from above, and by assuming appropriate coercivity, local integral boundedness and cancelation properties. The pointwise control from below by a Gagliardo-type kernel, on the contrary, is strongly emplyed throughout this work. However, for the sake of simplicity, we prefer to present our results for the class of measurable kernel as in (2.2), to allow the reader to just keep in mind the case when the diffusion in velocity is a purely fractional Laplacian with coefficients. Consequently, no precise dependance on the constant Λ will be explicitly written when not needed.

As expected when dealing with nonlocal operators, long-range contributions must be taken into account, and this is done via the by-now classical tail quantity.

Definition 2.2. *Let* f *be a measurable function on* $(t_1,t_2) \times \Omega_x \times \mathbb{R}^n \subset \mathbb{R}^{1+2n}$. *The* "(kinetic) nonlocal tail of f centred in $v_0 \in \Omega_v \subset \mathbb{R}^n$ of diffusion radius r" *is the quantity* $\mathrm{Tail}(f; B_r(v_0))$ *given by*

$$\mathrm{Tail}(f; B_r(v_0)) := r^{2s} \int_{\mathbb{R}^n \setminus B_r(v_0)} \frac{|f(\cdot,\cdot,v)|}{|v_0-v|^{n+2s}} \, \mathrm{d}v$$

The kinetic nonlocal tail is the linear version of the nonlocal tail quantity firstly defined in the purely p-fractional elliptic setting in [DKP14, DKP16] and subsequently proven to be decisive in the analysis of many other nonlocal problems when a fine quantitative control of the naturally arising long-range interactions is needed; see, e. g., [MRT16,BLS18,PK18,KNS22,BKO23,DFM24,BDLMS] and the references therein. Several tail related properties of nonlocal harmonic functions are naturally expected – as for instance the fact that their tail is finite, and that their tail is controlled by that of their negative part, and so on – and they are proven in [KKP17]. However, in our kinetic framework we need to operate step by step in the forthcoming proofs dealing with suitable local L^p estimates in the transport variables; see for instance the precise estimates in the proof of the δ-interpolative L^∞ inequality and in that of the gain of integrability of subsolutions to (1.1); see Section 3.

It is also worth noticing that it is usually the nonnegativeness of solutions to interfere with the validity of Harnack inequalities in fractional settings and $\mathrm{Tail}((f)_-)$ is the decisive player in such a game, as it has been firstly showed by Kaßmann in [Kas07,Kas11] and then confirmed in the many subsequently related results. On the contrary, our strategy to make use of a nonlocal L^∞-L^2-type estimate does involve an auxiliary (possibly not positive) subsolution g whose error term will be controlled by the kinetic nonlocal tail of its positive part $(g)_+$; see the formulation in (1.6) and the details in the related proofs in the rest of the present paper; compare also with Remark 1.3 in the Introduction.

Now, we consider the following *tail space*

$$L^1_{2s}(\mathbb{R}^n) := \left\{ g \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \,:\, \|g\|_{L^1_{2s}(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \frac{|g(v)|}{(1+|v|)^{n+2s}}\,\mathrm{d}v < \infty \right\},$$

as firstly defined in [KKP16]; see Section 2 there for related properties.

Given $\Omega := (t_1,t_2)\times\Omega_x\times\Omega_v \subset \mathbb{R}^{1+2n}$, we denote by $\mathcal{W}$ the natural functions space to which weak solutions to (1.1) belong to. We have

$$\begin{aligned}\mathcal{W} := \big\{ f \in L^2_{\mathrm{loc}}((t_1,t_2)\times\Omega_x;\, H^s_{\mathrm{loc}}(\Omega_v)) \cap L^1_{\mathrm{loc}}((t_1,t_2)\times\Omega_x; L^1_{2s}(\mathbb{R}^n))\\ : f_t + v\cdot\nabla_x f \in L^2_{\mathrm{loc}}((t_1,t_2)\times\Omega_x;\, H^{-s}(\mathbb{R}^n)) \big\}.\end{aligned}$$

Furthermore, we denote by $\mathcal{E}(\cdot)$ the nonlocal energy associated with our diffusion term $\mathcal{L}_v$ in (1.3)

$$\mathcal{E}(f,\phi) := \iint_{\mathbb{R}^n\times\mathbb{R}^n} \big(f(t,x,v)-f(t,x,w)\big)\big(\phi(t,x,v)-\phi(t,x,w)\big)K(v,w)\,\mathrm{d}v\,\mathrm{d}w\,,$$

for any test function ϕ smooth enough. We are now in the position to recall the definition of weak sub- and supersolution.

Definition 2.3. *A function* $f\in\mathcal{W}$ *is a* weak subsolution (*resp.,* supersolution) *to* (1.1) *in* Ω *if*

$$\int_{t_1}^{t_2}\int_{\Omega_x} \mathcal{E}\,(f,\phi)\,\mathrm{d}x\,\mathrm{d}t + \int_{t_1}^{t_2}\int_{\Omega_x} \langle (f_t+v\cdot\nabla_x f)\,|\,\phi\rangle\,\mathrm{d}x\,\mathrm{d}t \overset{(\geqslant\ resp.)}{\leqslant} \int_{\Omega} h\,\phi\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t$$

for any nonnegative $\phi\in L^2_{\mathrm{loc}}((t_1,t_2)\times\Omega_x; H^s(\mathbb{R}^n))$ *such that* $\operatorname{supp}\phi(t,x,\cdot)\Subset\Omega_v$*; in the display above we denote by* $\langle\cdot\,|\,\cdot\rangle$ *the usual duality paring between* H^s *and* H^{-s}.

A function $f\in\mathcal{W}$ *is a* weak solution *to* (1.1) *if it is both a weak sub- and supersolution.*

3. Interpolative L^∞-L^2-type estimate

This section is devoted to the proof of the local boundedness estimate with tail for subsolutions to (1.1) with no a priori sign assumptions, as stated in Theorem 1.1.

3.1. **Kinetic energy estimates with tail.** Firstly, we need a precise energy estimate which will require to prove a Caccioppoli-type estimate with (kinetic) tail, and a Gehring-type one for subsolutions to (1.1). We have the following

Lemma 3.1. *Let $Q_1 \equiv Q_1(\mathbf{0}) \subset \Omega$. Let f be a weak subsolution in Ω according to Definition 2.3 with $h \in L^2_{\rm loc}(\Omega)$ such that $h \leqslant 0$. For any $Q_r \equiv Q_r(\mathbf{0}) \subset Q_1$, any $q \in [2, q^\star)$, where $q^\star = q^\star(n,s) > 2$ is the exponent introduced in (3.12) below, any $p > 2$, and any $0 < \theta < \sigma < \rho < r < 1$, the following estimate does hold,*

$$
\begin{aligned}
\|\omega\|^2_{L^q(Q_\theta)} \leqslant\ & c_0 \int_{U_r} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\omega(v)\omega(w)|\varphi(v)-\varphi(w)|^2}{|v-w|^{n+2s}}\,{\rm d}v\,{\rm d}w\,{\rm d}x\,{\rm d}t \\
& + c_1 \int_{Q_r} \omega^2\,{\rm d}v\,{\rm d}x\,{\rm d}t + c_2 \|{\rm Tail}(\omega; B_r)\|^2_{L^p(U_r)} |Q_r \cap \{\omega > 0\}|^{1-\frac{2}{p}} \\
& + c_0 \int_{Q_r} h\omega\,{\rm d}v\,{\rm d}x\,{\rm d}t
\end{aligned}
$$

where

$$
\begin{aligned}
&(3.1) && c_0 := c\|\nabla_v \psi\|^2_{L^\infty} + c\,(\rho-\sigma)^{-2(n+2s)}\,, \\
&(3.2) && c_1 := c\left(\|\nabla_v\psi\|^2_{L^\infty} + (\rho-\sigma)^{-2(n+2s)}\right)\left(\|v\cdot\nabla_x\varphi\|_{L^\infty} + r^{2s}\right) \\
& && \qquad + c\,\|v\cdot\nabla_x\psi\|^2_{L^\infty}\,, \\
&(3.3) && c_2 := c\,r^{-4s}(r-\rho)^{-2(n+2s)}
\end{aligned}
$$

and where c depends only on n, s and the kernel constant Λ in (2.2), $\omega := (f-k)_+$, for any $k \in \mathbb{R}$, and where $\varphi \in C^\infty_c(B_{r^{1+2s}} \times B_r)$ is a cut-off function such that $\varphi(x,v) \equiv 1$ in $B_{\rho^{1+2s}} \times B_\rho$ and $\psi \in C^\infty_c(B_{\sigma^{1+2s}} \times B_\sigma)$ is a cut-off function such that $\psi \equiv 1$ on $B_{\theta^{1+2s}} \times B_\theta$.

Proof. For the sake of the reader, it is convenient to divide the present proof in two separate steps.

Step 1: Kinetic Caccioppoli inequality with tail. Up to regularizing by mollification, for any fixed $t \in (-r^{2s}, 0]$ we can assume that $\omega\varphi^2$ is sufficiently regular in order to be an admissible test function compactly supported in the cylinder $(Q_r)^t := \{(v,x) \in \mathbb{R}^{2n} : (t,x,v) \in Q_r\}$. Consider now the weak formulation in Definition 2.3 by choosing as a test function $\phi \equiv \omega\varphi^2$ there; for a. e. $t \in (-r^{2s}, 0]$ it yields

$$
\begin{aligned}
\int_{B_r^{1+2s}\times B_r} h\omega\varphi^2\,{\rm d}x\,{\rm d}v \geqslant\ & \int_{(Q_r)^t} (f_t + v\cdot\nabla_x f)\omega\varphi^2\,{\rm d}x\,{\rm d}v \\
& + \int_{B_{r^{1+2s}}} \mathcal{E}(f, \omega\varphi^2)\,{\rm d}x =: I_1 + I_2. \qquad (3.4)
\end{aligned}
$$

We start by considering I_1. Using the fact that $\partial_t\varphi = 0$, we have that

$$
(3.5) \qquad I_1 \geqslant \frac{1}{2}\frac{\rm d}{{\rm d}t}\int_{(Q_r)^t} (\omega\varphi)^2\,{\rm d}x\,{\rm d}v - \frac{1}{2}\int_{(Q_r)^t} |v\cdot\nabla_x(\varphi^2)|\omega^2\,{\rm d}x\,{\rm d}v\,.
$$

For what concern I_2 we note that

$$\begin{aligned}\big(f(v)-f(w)\big)&\big(\omega\varphi^2(v)-\omega\varphi^2(w)\big)\\ &= \big((f(v)-k)-(f(w)-k)\big)\big(\omega\varphi^2(v)-\omega\varphi^2(w)\big)\\ &\geqslant \big(\omega\varphi(v)-\omega\varphi(w)\big)^2-\omega(v)\omega(w)|\varphi(v)-\varphi(w)|^2\,,\end{aligned}$$

which yields

$$\begin{aligned} I_2 \;\geqslant\; & \int_{B_{r^{1+2s}}}[\omega\varphi]^2_{H^s(\mathbb{R}^n)}\,\mathrm{d}x \\ & -\int_{B_{r^{1+2s}}}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\omega(v)\omega(w)|\varphi(v)-\varphi(w)|^2}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x\,,\end{aligned}\tag{3.6}$$

where we also used the definition of the kernel K in (2.2) by neglecting a constant depending on Λ there, for the sake of simplicity. Combining (3.5) and (3.6) with (3.4), it yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{(Q_r)^t}(\omega\varphi)^2\,\mathrm{d}x\,\mathrm{d}v+\int_{B_{r^{1+2s}}}[\omega\varphi]^2_{H^s(\mathbb{R}^n)}\,\mathrm{d}x\\ &\qquad\leqslant c\int_{B_{r^{1+2s}}}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\omega(v)\omega(w)|\varphi(v)-\varphi(w)|^2}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x\\ &\qquad\quad+c\int_{(Q_r)^t}|v\cdot\nabla_x(\varphi^2)|\omega^2\,\mathrm{d}v\,\mathrm{d}x+c\int_{(Q_r)^t}h\omega\,\mathrm{d}x\,\mathrm{d}v\end{aligned}\tag{3.7}$$

Then, by integrating (3.7) in $[\tau_1,\tau_2]$, for $-r^{2s}\leqslant\tau_1<\tau_2\leqslant 0$, we get

$$\begin{aligned}&\int_{B_{r^{1+2s}}\times B_r}(\omega\varphi)^2(\tau_2,x,v)\,\mathrm{d}x\,\mathrm{d}v+\int_{\tau_1}^{\tau_2}\int_{B_{r^{1+2s}}}[\omega\varphi]^2_{H^s(\mathbb{R}^n)}\,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant c\int_{U_r}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\omega(v)\omega(w)|\varphi(v)-\varphi(w)|^2}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+c\int_{Q_r}|v\cdot\nabla_x(\varphi^2)|\omega^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t+c\int_{B_{r^{1+2s}}\times B_r}(\omega\varphi)^2(\tau_1,x,v)\,\mathrm{d}x\,\mathrm{d}v\\ &\quad+c\int_{Q_r}h\omega\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t\end{aligned}\tag{3.8}$$

Taking the supremum over τ_2 on the left-hand side and the average integral over $\tau_1\in[-r^{2s},0]$ on both sides of the inequality, we get (recalling that $\varphi\leqslant 1$ begin a test function)

$$\begin{aligned}&\sup_{t\in[-r^{2s},0]}\int_{B_{r^{1+2s}}\times B_r}(\omega\varphi)^2\,\mathrm{d}v\,\mathrm{d}x\\ &\quad\leqslant c\int_{U_r}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\omega(v)\omega(w)|\varphi(v)-\varphi(w)|^2}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad+c\int_{Q_r}|v\cdot\nabla_x(\varphi^2)|\,\omega^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t+c\,r^{2s}\int_{Q_r}(\omega\varphi)^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad+c\int_{Q_r}h\omega\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t\end{aligned}\tag{3.9}$$

Reconsidering (3.8) and evaluating with $\tau_1 = -r^{2s}$ and $\tau_2 = 0$ there, we then have

$$\begin{aligned}
&\int_{U_r} [\omega\varphi]^2_{H^s(\mathbb{R}^n)} \,\mathrm{d}x\,\mathrm{d}t \\
&\quad\leqslant c\int_{U_r}\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\omega(v)\omega(w)|\varphi(v)-\varphi(w)|^2}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + c\int_{Q_r} |v\cdot\nabla_x(\varphi^2)|\omega^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t + c\int_{B_{r^{1+2s}}\times B_r} (\omega\varphi)^2(-r^{2s},x,v)\,\mathrm{d}x\,\mathrm{d}v \\
&\qquad + c\int_{Q_r} h\omega\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t.
\end{aligned}$$

Thus, combining the display above with (3.9) twice (we employ it the first time to estimate the sup on the left-hand side of (3.8) and the second time to estimate the last term of the above display), we obtain the following Caccioppoli-type estimate,

$$\begin{aligned}
&\sup_{t\in[-r^{2s},0]}\int_{B_{r^{1+2s}}\times B_r} (\omega\varphi)^2\,\mathrm{d}v\,\mathrm{d}x + \int_{U_r}[\omega\varphi]^2_{H^s(\mathbb{R}^n)}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\leqslant c\int_{U_r}\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\omega(v)\omega(w)|\varphi(v)-\varphi(w)|^2}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + c\int_{Q_r}|v\cdot\nabla_x(\varphi)^2|\omega^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t + cr^{2s}\int_{Q_r}(\omega\varphi)^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + c\int_{Q_r} h\omega\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t
\end{aligned} \tag{3.10}$$

Step 2: Local L^q-estimate for subsolutions. Now, an L^q-estimate whose proof is a refinement of the original result for solutions to the Boltzmann equation without cut-off by Imbert and Silvestre – see in particular Lemma 6.1 and Proposition 2.2 in [IS20b] – which in turn reminds of the strategy in [PP04b] and to classical Gehring-type results is presented. For fine Gehring lemmata in the nonlocal elliptic case we refer to [KMS15, Sch16]; a direct counterpart of the latter papers in the kinetic framework being challenging. It is worth noticing that the exponent $\varepsilon^\star$ in the statement of Theorem 1.1 will basically show up here, being linked to the maximal gain in summability in forthcoming Formula (3.12).

Fix, $0<\theta<\sigma<\rho<r<1$ and consider a smooth cut-off $\psi=\psi(x,v)\in C^\infty_c(B_{\sigma^{1+2s}}\times B_\sigma)$ such that $\psi\equiv 1$ on $B_{\theta^{1+2s}}\times B_\theta$ and $0\leqslant\psi\leqslant 1$. Then, the function $g:=\omega\psi$, where $\omega=(f-k)_+$ is a subsolution, satisfies the following chain of inequalities:

$$\begin{aligned}
[\partial_t + v\cdot\nabla_x]\; g - \mathcal{L}_v g \;&=\; ([\partial_t+v\cdot\nabla_x]\omega)\,\psi + \omega(v\cdot\nabla_x)\psi \\
&\quad - \psi\mathcal{L}_v\omega - \omega\mathcal{L}_v\psi + \mathcal{I}(\omega,\psi) + h,
\end{aligned}$$

where $\mathcal{I}(\omega,\psi)$ is a remainder term obtained when explicitly applying $-\mathcal{L}_v$ to the product $\omega\psi$ and defined as:

$$\mathcal{I}(\omega,\psi) := \int_{\mathbb{R}^n}(\omega(v)-\omega(w))(\psi(v)-\psi(w))K(v,w)\,\mathrm{d}w.$$

Then, recalling that ω is a subsolution of (1.1) and that $h \leqslant 0$ we get:

$$[\partial_t + v\cdot\nabla_x]\ g - \mathcal{L}_v g \leqslant \omega\,(v\cdot\nabla_x\psi) - \omega\mathcal{L}_v\psi + \mathcal{I}(\omega,\psi).$$

Then considering the definition of $\mathcal{L}_v$ and of $\mathcal{I}(\omega,\psi)$ we observe that two terms cancel out on the right-hand side leading to the following estimate:

$$[\partial_t + v\cdot\nabla_x]\ g - \mathcal{L}_v g \leqslant \omega\,(v\cdot\nabla_x\psi) - \int_{\mathbb{R}^n}\omega(w)(\psi(v)-\psi(w))K(v,w)\,\mathrm{d}w =: H.$$

Hence, by [IS20b, Lemma 6.10] for any $q > q^*$ only depending on n and s the following gain of integrability for g holds:

$$\|g\|^2_{L^q(Q_\sigma)} \leqslant c\left(\|g(-\sigma^{2s})\|^2_{L^2(B_{\sigma^{1+2s}}\times B_\sigma)} + \|H\|^2_{L^2([-\sigma^{2s},0]\times\mathbb{R}^{2n})}\right), \tag{3.11}$$

where q is such that $\frac{1}{q} > \frac{1}{p^*} - \frac{1}{2}$ and $p^* = \frac{2n(1+s)+2s}{2n(1+s)+s} \in (1,2)$, i.e.

$$q < \frac{2(n(1+s)+s)}{n(1+s)} = 2 + \frac{2s}{n(1+s)} =: q^\star. \tag{3.12}$$

Thus, we are left to prove that the right-hand side H is in L^2. First of all, we observe that

$$\begin{aligned}\|H\|^2_{L^2([-\sigma^{2s},0]\times\mathbb{R}^{2n})} \lesssim\ & \|\omega\,(v\cdot\nabla_x\psi)\,\|^2_{L^2([-\sigma^{2s},0]\times\mathbb{R}^{2n})}\\ &+\Big\|\int_{\mathbb{R}^n}\frac{|\omega(w)(\psi(w)-\psi(v))|}{|v-w|^{n+2s}}\,\mathrm{d}w\Big\|^2_{L^2([-\sigma^{2s},0]\times\mathbb{R}^{2n})}.\end{aligned}$$

The first term can easily be estimated recalling that ψ does not depend on t and is a cut-off function:

$$\begin{aligned}\|\omega\,(v\cdot\nabla_x\psi)\,\|^2_{L^2([-\sigma^{2s},0]\times\mathbb{R}^{2n})} &= \int_{Q_\sigma}\omega^2|v\cdot\nabla_x\psi|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant \|v\cdot\nabla_x\psi\|^2_{L^\infty}\int_{Q_r}\omega^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\,,\end{aligned} \tag{3.13}$$

where we used the fact that $\sigma < \rho < r$ and that ψ is supported in $B_{\sigma^{1+2s}}\times B_\sigma$.

Now, we only need to prove an estimate for the integral term. To do this, we split the proof depending on the order of integrability s of the involved kernel. Firstly, for any fixed v, we split the integral term in two:

$$\begin{aligned}\int_{\mathbb{R}^n}\frac{|\omega(w)(\psi(w)-\psi(v))|}{|v-w|^{n+2s}}\,\mathrm{d}w &= \int_{B_{(\rho-\sigma)/2}(v)}\frac{|\omega(w)(\psi(w)-\psi(v))|}{|v-w|^{n+2s}}\,\mathrm{d}w\\ &\quad+\int_{\mathbb{R}^n\setminus B_{(\rho-\sigma)/2}(v)}\frac{|\omega(w)(\psi(w)-\psi(v))|}{|v-w|^{n+2s}}\,\mathrm{d}w\\ &:= J_1 + J_2,\end{aligned}$$

so that, for any $v\in\operatorname{supp}\psi\subset B_\sigma$ we have that $B_{(\rho-\sigma)/2}(v)\subset B_\rho$. Indeed, for any $w\in B_{(\rho-\sigma)/2}(v)$ we have

$$|w| \leqslant |w-v| + |v| \leqslant \sigma + \frac{\rho-\sigma}{2} = \frac{\rho+\sigma}{2} \leqslant \rho, \tag{3.14}$$

since $\sigma < \rho$.

Now, we separately estimate the two integrals. We start with the term J_2.

$$
\begin{aligned}
&\|J_2\|^2_{L^2([-\sigma^{2s},0]\times\mathbb{R}^{2n})}\\
&\quad\leqslant c\int_{U_\sigma\times B_\rho}\left(\int_{\mathbb{R}^n\setminus B_{(\rho-\sigma)/2}(v)}\frac{\omega(w)|\psi(v)-\psi(w)|}{|v-w|^{n+2s}}\,\mathrm{d}w\right)^2\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+c\int_{U_\sigma\times(\mathbb{R}^n\setminus B_\rho)}\left(\int_{\mathbb{R}^n\setminus B_{(\rho-\sigma)/2}(v)}\frac{\omega(w)|\psi(v)-\psi(w)|}{|v-w|^{n+2s}}\,\mathrm{d}w\right)^2\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Then, applying on the second term the commutator estimate in [IS20b, Lemma 4.10, Lemma 4.11] (also tracking dow the explicit dependencies on the radius $(\rho-\sigma)/2$) we get

$$
\begin{aligned}
\|J_2\|^2_{L^2([-\sigma^{2s},0]\times\mathbb{R}^{2n})} \;\leqslant\;& c\,(\sigma-\rho)^{-4s}\int_{Q_r}\omega^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\\
&+c\int_{U_\sigma\times B_\rho}\left(\int_{\mathbb{R}^n\setminus B_{(\rho-\sigma)/2}(v)}\frac{\omega(w)|\psi(v)-\psi(w)|}{|v-w|^{n+2s}}\,\mathrm{d}w\right)^2\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\,,
\end{aligned}
$$

also using that $\sigma<\rho<r$.

We estimate the second term. Given the fundamental observation that for any $v\in B_\rho$, the ball $B_{(\rho-\sigma)/2}(v)\subset B_r$ (see (3.14) and recall that $\sigma<\rho<r$), we have

$$
(\mathbb{R}^n\setminus B_r)\cup\big(B_r\setminus B_{(\rho-\sigma)/2}(v)\big)=\mathbb{R}^n\setminus B_{(\rho-\sigma)/2}(v).
$$

Thus, for any $w\in\mathbb{R}^n\setminus B_r$ and any $v\in B_\rho$, we re-center the Gagliardo kernel at the origin thanks to the chain of inequalities

$$
\frac{|w|}{|v-w|}\leqslant\frac{|v|}{|v-w|}+1\leqslant 1+\frac{|v|}{\big||w|-|v|\big|}\leqslant\frac{r}{r-\rho}+1\leqslant\frac{2}{r-\rho}\,, \tag{3.15}
$$

since $\sigma<\rho<r<1$. Thus, we are led to

$$
\begin{aligned}
&\int_{U_\sigma\times B_\rho}\left(\int_{\mathbb{R}^n\setminus B_{(\rho-\sigma)/2}(v)}\frac{\omega(w)|\psi(v)-\psi(w)|}{|v-w|^{n+2s}}\,\mathrm{d}w\right)^2\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\leqslant\int_{U_\sigma\times B_\rho}\left(\int_{\mathbb{R}^n\setminus B_r}\frac{\omega(w)|\psi(v)-\psi(w)|}{|v-w|^{n+2s}}\,\mathrm{d}w\right)^2\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+\int_{U_\sigma\times B_\rho}\left(\int_{B_r\setminus B_{(\rho-\sigma)/2}(v)}\frac{\omega(w)|\psi(v)-\psi(w)|}{|v-w|^{n+2s}}\,\mathrm{d}w\right)^2\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\overset{(3.15)}{\leqslant} c\,(r-\rho)^{-2(n+2s)}\int_{U_r}\left(\int_{\mathbb{R}^n\setminus B_r}\frac{\omega(w)}{|w|^{n+2s}}\,\mathrm{d}w\right)^2\mathbb{1}_{B_r\cap\{\omega(t,x)>0\}}(t,x)\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+c\,(\rho-\sigma)^{-2(n+2s)}\int_{U_r}\left(\int_{B_r\setminus B_{(\rho-\sigma)/2}(v)}\omega(w)\,\mathrm{d}w\right)^2\mathrm{d}x\,\mathrm{d}t\\
&\quad\leqslant c\,r^{-4s}(r-\rho)^{-2(n+2s)}\|\mathrm{Tail}(\omega;B_r)\|^2_{L^p(U_r)}|Q_r\cap\{\omega>0\}|^{1-\frac{2}{p}}+\frac{c\|\omega\|^2_{L^2(Q_r)}}{(\rho-\sigma)^{2(n+2s)}}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|J_2\|^2_{L^2([-\sigma^{2s},0]\times\mathbb{R}^{2n})} \;\overset{(3.3)}{\leqslant}\;& \frac{c\,\|\omega\|^2_{L^2(Q_r)}}{(\rho-\sigma)^{2(n+2s)}}\\
&+c_2\,\|\mathrm{Tail}(\omega;B_r)\|^2_{L^p(U_r)}\big|Q_r\cap\{\omega>0\}\big|^{1-\frac{2}{p}}\,.
\end{aligned} \tag{3.16}
$$

Now, let us estimate the L^2 norm of J_1. We differentiate the cases depending on the range of the differentiability exponent $s \in (0,1)$.

When $s \in (0,\frac{1}{2})$, we apply the commutator estimate in [IS20b, Lemma 4.10], obtaining that

$$\|J_1\|^2_{L^2([-\sigma^{2s},0]\times\mathbb{R}^{2n})} \leqslant c\,\|\nabla_v\psi\|^2_{L^\infty}\|\omega\|^2_{L^2(Q_r)}. \tag{3.17}$$

On the other hand, when $s \in [\frac{1}{2},1)$, we apply the commutator estimate in [IS20b, Lemma 4.10, Lemma 4.11] getting

$$\begin{aligned}\|J_1\|^2_{L^2(Q_\sigma)} &\leqslant c\|\nabla_v\psi\|^2_{L^\infty}\left(\|\omega\|^2_{L^2(Q_\rho)} + \int_{U_\rho}[\omega]^2_{H^s(B_\rho)}\,\mathrm{d}x\,\mathrm{d}t\right)\\ &\leqslant c\|\nabla_v\psi\|^2_{L^\infty}\left(\|\omega\|^2_{L^2(Q_r)} + \int_{U_r}[\omega\varphi]^2_{H^s(\mathbb{R}^n)}\,\mathrm{d}x\,\mathrm{d}t\right),\end{aligned} \tag{3.18}$$

where $\varphi \in C^\infty_c(B_{r^{1+2s}}\times B_r)$ is a cut-off function such that $\varphi \equiv 1$ on $B_{\rho^{1+2s}}\times B_\rho$.

Combining now, (3.16), (3.17), (3.18) and recalling that $\rho<r$, we get

$$\begin{aligned}&\|J_1\|^2_{L^2([-\sigma^{2s},0]\times\mathbb{R}^{2n})} + \|J_2\|^2_{L^2([-\sigma^{2s},0]\times\mathbb{R}^{2n})}\\ &\overset{(3.1)}{\leqslant} c_0\big(\|\omega\|^2_{L^2(Q_r)} + \int_{U_r}[\omega\varphi]^2_{H^s(\mathbb{R}^n)}\,\mathrm{d}x\,\mathrm{d}t\big)\\ &\qquad + c_2\,\|\mathrm{Tail}(\omega;B_r)\|^2_{L^p(U_r)}|Q_r\cap\{\omega>0\}|^{1-\frac{2}{p}}\,.\end{aligned} \tag{3.19}$$

Combining (3.11), (3.13) and (3.19), we obtain now

$$\begin{aligned}\|g\|^2_{L^q(Q_\sigma)} &\leqslant c\left(\|g(-\sigma^{2s})\|^2_{L^2(B_{\sigma^{1+2s}}\times B_\sigma)} + \|H\|^2_{L^2([-\sigma^{2s},0]\times\mathbb{R}^{2n})}\right)\\ &\leqslant c\|g(-\sigma^{2s})\|^2_{L^2(B_{\sigma^{1+2s}}\times B_\sigma)} + c\|v\cdot\nabla_x\psi\|^2_{L^\infty}\int_{Q_r}\omega^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + c_0\big(\|\omega\|^2_{L^2(Q_r)} + \int_{U_r}[\omega\varphi]^2_{H^s(\mathbb{R}^n)}\,\mathrm{d}x\,\mathrm{d}t\big)\\ &\quad + c_2\,\|\mathrm{Tail}(\omega;B_r)\|^2_{L^p(U_r)}|Q_r\cap\{\omega>0\}|^{1-\frac{2}{p}}\,.\end{aligned} \tag{3.20}$$

Conclusion. It suffices to combine (3.20) with (3.10) to get the desired result. □

3.2. **The first lemma of De Giorgi with $(2+\varepsilon)$-tail.** We begin by proving a simple iteration Lemma, which is a natural modification to the classical iteration argument in [GG82, Lemma 1.1]

Lemma 3.2. *Let $\alpha>0$ and let $\{A_j\}_{j\in 2\mathbb{N}}$ be a sequence of positive real numbers such that*

$$A_{j+2} \leqslant \bar{C}\,b^j\,A_j^{1+\alpha},$$

with $\bar{C}>0$, $b>1$. If

$$A_0 \leqslant \bar{C}^{-\frac{1}{\alpha}}b^{-\frac{2}{\alpha^2}}, \tag{3.21}$$

then $\lim_{j\to\infty} A_j = 0$.

Proof. Assume inductively that

$$A_j \;\leqslant\; b^{-\frac{\alpha j+2}{\alpha^2}}\bar{C}^{-\frac{1}{\alpha}}\,.$$

Indeed, certainly the induction assumption holds if $j=0$ by the assumption on A_0 in (3.21). By the recursive inequality and the induction assumption we then get

$$\begin{aligned} A_{j+2} &\leqslant \bar{C}\, b^j\, A_j^{1+\alpha} \\ &\leqslant \bar{C}^{1-\frac{1+\alpha}{\alpha}} b^{j-\frac{(\alpha j+2)(1+\alpha)}{\alpha^2}} \\ &= b^{-\frac{\alpha(j+2)+2}{\alpha^2}} \bar{C}^{-\frac{1}{\alpha}}\,, \end{aligned}$$

proving the induction step and finishing the proof. □

We are now in the position to prove the δ-interpolative L^∞-L^2 inequality of Theorem 1.1.

Proof of Theorem 1.1. Let $R>0$ and, for any $j\in\mathbb{N}$, define a decreasing family of positive radii $r_j := \frac{1}{2}(1+2^{-j})R$ and a family of slanted cylinders $Q_j \equiv Q_{r_j}(\mathbf{0})$ such that $Q_{j+1} \Subset Q_j$ for every $j\in\mathbb{N}$. We will denote with $U_j := (-r_j^{2s},0]\times B_{r_j^{1+2s}}$, so that $Q_j := U_j \times B_j$.

Consider a family $\{\varphi_j\}_{j\in\mathbb{N}}$ of test functions $\varphi_j \equiv \varphi_j(x,v)$, such that

$$0\leqslant \varphi_j \leqslant 1 \quad \varphi_j \equiv 1 \,\text{on}\, B_{r_{j+1}^{1+2s}} \times B_{r_{j+1}} \quad \text{and} \quad \varphi_j \in C_0^\infty\left(B_{(\frac{r_j+r_{j+1}}{2})^{1+2s}} \times B_{\frac{r_j+r_{j+1}}{2}}\right).$$

Moreover, we have that

$$|\nabla_v \varphi_j| \lesssim \frac{2^j}{R} \quad \text{and} \quad |v\cdot\nabla_x\varphi_j| \lesssim \frac{2^{j(1+2s)}}{R^{2s}}.$$

For any $j\in\mathbb{N}$, let $k_j := (1-2^{-j})k_0$, with $k_0>0$ which will be fixed later on, and define $\omega_j := (f-k_j)_+$.

With this bit of notation, apply Lemma 3.1, with ω_{j+1}, $\theta := r_{j+2}$, $\sigma := (r_{j+2}+r_{j+1})/2$, $\rho := r_{j+1}$, $r := r_j$, $\psi := \varphi_{j+1}$ and $\varphi := \varphi_j$.

Note that the constants in (3.1), (3.2) and (3.3) can bounded as follow

$$c_0, c_1, c_2 \leqslant \frac{c\, 2^{j(2n+4s)}}{R^{2(n+4s)}}. \tag{3.22}$$

An application of Lemma 3.1, for some $q\equiv q(n,s)\in(2,q^\star)$, recalling that $\omega_{j+2}\leqslant \omega_{j+1}$, yields that

$$\begin{aligned} &\int_{Q_{j+2}} \omega_{j+2}^2 \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ &\quad\leqslant \left(\int_{Q_{j+2}} \omega_{j+1}^q \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{2}{q}} |Q_{j+2}\cap\{f\geqslant k_{j+2}\}|^{1-\frac{2}{q}} \\ &\quad\leqslant \frac{c\,2^{j(2n+4s)}}{R^{2(n+4s)}} \left(\int_{U_j}\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\omega_{j+1}(v)\omega_{j+1}(w)|\varphi_j(v)-\varphi_j(w)|^2}{|v-w|^{n+2s}} \,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}t \right.\\ &\qquad + \int_{Q_j} \omega_{j+1}^2 \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t + \|\mathrm{Tail}(\omega_{j+1};B_j)\|^2_{L^p(U_j)} |Q_j\cap\{f>k_{j+1}\}|^{1-\frac{2}{p}} \\ &\qquad \left. + \int_{Q_j} h\omega_{j+1}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\right) |Q_{j+2}\cap\{f\geqslant k_{j+2}\}|^{1-\frac{2}{q}} \\ &\quad=: (I_1+I_2+I_3+I_4)|Q_{j+2}\cap\{f\geqslant k_{j+2}\}|^{1-\frac{2}{q}}. \end{aligned} \tag{3.23}$$

The choice of the exponent $p > 2+\varepsilon^\star$ in the display above will be clarified at the end of the proof. Starting from I_1, we get that

$$\begin{aligned} I_1 &= c\int_{Q_j}\int_{B_j} \frac{\omega_{j+1}(v)\omega_{j+1}(w)|\varphi_j(v)-\varphi_j(w)|^2}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + c\int_{Q_j}\int_{\mathbb{R}^n\setminus B_j} \frac{\omega_{j+1}(v)\omega_{j+1}(w)\varphi_j^2(v)}{|v-w|^{n+2s}}\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ &=: I_{1,1}+I_{1,2}\,. \end{aligned}$$

The first integral $I_{1,1}$ can be treated assuming $\omega_{j+1}(v)\geqslant \omega_{j+1}(w)$, noticing that the reverse inequality holds true when one exchanges the roles of v and w, as follows,

$$\begin{aligned} I_{1,1} &\leqslant c2^{2j}R^{-2}\int_{Q_j}\omega_{j+1}^2\left(\int_{2B_j(v)}\frac{\mathrm{d}w}{|v-w|^{n-2(1-s)}}\right)\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant c2^{2j}R^{-2}\int_{Q_j}\omega_{j+1}^2\left(\int_0^{2R}\rho^{2(1-s)-1}\,\mathrm{d}\rho\right)\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant \frac{c\,2^{2j}}{2(1-s)R^{2s}}\int_{Q_j}\omega_j^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\,. \end{aligned}$$

As for the second integral $I_{1,2}$, we have by Hölder's Inequality for some $p>2$ that

$$\begin{aligned} I_{1,2} &\leqslant c\int_{Q_j\cap\,\mathrm{supp}\ \varphi_j}\int_{\mathbb{R}^n\setminus B_j}\frac{\omega_{j+1}(v)\omega_{j+1}(w)}{|v-w|^{n+2s}}\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant \left(\int_{Q_j}\omega_{j+1}^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac12} \\ &\quad\times\Big[\int_{Q_j\cap\,\mathrm{supp}\ \varphi_j}\left(\int_{\mathbb{R}^n\setminus B_j}\frac{\omega_{j+1}(w)}{|v-w|^{n+2s}}\,\mathrm{d}w\right)^2\mathbb{1}_{\{f(v)>k_{j+1}\}}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\Big]^{\frac12} \\ &\leqslant \frac{c2^{j(n+2s)}}{R^{2s}}\left(\int_{Q_j}\omega_j^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac12}\|\,\mathrm{Tail}(f_+;B_{R/2})\|_{L^p(U_R)}|Q_j\cap\{f>k_{j+1}\}|^{\frac12-\frac1p} \\ &\leqslant \frac{c2^{j(n+2s+1-\frac2p)}k_0^2}{\delta R^{2s}}\left(\int_{Q_j}\frac{\omega_j^2}{k_0^2}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\right)^{1-\frac1p}, \end{aligned}$$

up to choosing

$$k_0 \geqslant \delta\|\,\mathrm{Tail}(f_+;B_{R/2})\|_{L^p(U_R)} \qquad \text{for } \delta\in(0,1]\,, \tag{3.24}$$

and where we have used that, for $w\in\mathbb{R}^n\setminus B_j$ and $v\in\mathrm{supp}\ \varphi_j$ (recalling that the support in the v-variable of φ_j is contained in $B_{(r_j+r_{j+1})/2}$)

$$\frac{|w|}{|v-w|}\leqslant 1+\frac{|v|}{|w|-|v|}\leqslant 1+\frac{r_j+r_{j+1}}{r_j-r_{j+1}}\leqslant 2^{j+4}\,,$$

as well as

$$
\begin{aligned}
|Q_j \cap \{f > k_{j+1}\}| &= |Q_j \cap \{f - k_j > k_{j+1} - k_j\}| \\
&\leqslant |Q_j \cap \{f - k_j > 2^{-j-1} k_0\}| \\
&\leqslant 2^{2j+2} \int_{Q_j} \frac{\omega_j^2}{k_0^2} \,\mathrm{d}v \,\mathrm{d}x \,\mathrm{d}t\,. \qquad (3.25)
\end{aligned}
$$

Combining together all the previous estimates for $I_{1,1}$ and $I_{1,2}$ we get that

$$
I_1 \leqslant \frac{c2^{j(n+2s+1-\frac{1}{p})}k_0^2}{\delta R^{2s}} \left\{ \int_{Q_j} \frac{\omega_j^2}{k_0^2} \,\mathrm{d}v \,\mathrm{d}x \,\mathrm{d}t + \left(\int_{Q_j} \frac{\omega_j^2}{k_0^2} \,\mathrm{d}v \,\mathrm{d}x \,\mathrm{d}t \right)^{1-\frac{1}{p}} \right\}. \qquad (3.26)
$$

In a similar way, recalling the particular choice of the test function φ_j, we have that

$$
\begin{aligned}
I_3 &= \|\mathrm{Tail}(\omega_{j+1}; B_j)\|^2_{L^p(U_j)} |Q_j \cap \{f > k_{j+1}\}|^{1-\frac{2}{p}} \\
&\leqslant \|\mathrm{Tail}(f_+; B_{R/2})\|^2_{L^p(U_R)} |Q_j \cap \{f > k_{j+1}\}|^{1-\frac{2}{p}} \\
&\overset{(3.24)}{\leqslant} C2^{j(2-\frac{4}{p})} \big(\frac{k_0}{\delta}\big)^2 \big(\int_{Q_j} \frac{\omega_j^2}{k_0^2} \,\mathrm{d}v \,\mathrm{d}x \,\mathrm{d}t \big)^{1-\frac{2}{p}}. \qquad (3.27)
\end{aligned}
$$

Lastly, I_4 can be treated as follows by applying Hölder's Inequality

$$
\begin{aligned}
I_4 &= \int_{Q_j} h \,\omega_{j+1} \,\mathrm{d}v \,\mathrm{d}x \,\mathrm{d}t \\
&\leqslant \left(\int_{Q_j \cap \{f > k_{j+1}\}} h^2 \,\mathrm{d}v \,\mathrm{d}x \,\mathrm{d}t \right)^{\frac{1}{2}} \left(\int_{Q_j} \omega_{j+1}^2 \,\mathrm{d}v \,\mathrm{d}x \,\mathrm{d}t \right)^{\frac{1}{2}} \\
&= k_0^2 \left(\int_{Q_j \cap \{f > k_{j+1}\}} \frac{h^2}{k_0^2} \,\mathrm{d}v \,\mathrm{d}x \,\mathrm{d}t \right)^{\frac{1}{2}} \left(\int_{Q_j} \frac{\omega_{j+1}^2}{k_0^2} \,\mathrm{d}v \,\mathrm{d}x \,\mathrm{d}t \right)^{\frac{1}{2}} \\
&\leqslant k_0^2 \left(\frac{\|h\|_{L^p(Q_R)}}{k_0} \right) \left(\int_{Q_j} \frac{\omega_j^2}{k_0^2} \,\mathrm{d}v \,\mathrm{d}x \,\mathrm{d}t \right)^{\frac{1}{2}} |Q_j \cap \{f > k_{j+1}\}|^{\frac{1}{2}-\frac{1}{p}} \\
&\overset{(3.25)}{\leqslant} k_0^2 \left(\int_{Q_j} \frac{\omega_j^2}{k_0^2} \,\mathrm{d}v \,\mathrm{d}x \,\mathrm{d}t \right)^{1-\frac{1}{p}}, \qquad (3.28)
\end{aligned}
$$

once chosen

$$
k_0 \geqslant \|h\|_{L^p(Q_R)}. \qquad (3.29)
$$

Moreover, the measure of the superlevel set in (3.23) can be estimated as follows, recalling that the sequence of $\{r_j\}$ is decreasing in j whereas the sequence of $\{k_j\}$ is increasing,

$$
\begin{aligned}
|Q_{j+2} \cap \{f > k_{j+2}\}|^{1-\frac{2}{q}} &\leqslant |Q_j \cap \{f - k_j > k_{j+1} - k_j\}|^{1-\frac{2}{q}} \\
&\leqslant |Q_j \cap \{f - k_j > 2^{-j-1} k_0\}|^{1-\frac{2}{q}} \\
&\leqslant c2^{j(2-\frac{4}{q})} \left(\int_{Q_j} \frac{\omega_j^2}{k_0^2} \,\mathrm{d}v \,\mathrm{d}x \,\mathrm{d}t \right)^{1-\frac{2}{q}}. \qquad (3.30)
\end{aligned}
$$

We now define A_j

$$A_j := \int_{Q_j} \frac{\omega_j^2}{k_0^2}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\,,$$

so that, by putting (3.26), (3.27), (3.28) and (3.30) in (3.23) and recalling (3.22), we get

$$A_{j+2} \leqslant \bar{C}\, b^j \left(A_j^{1+\frac{q-2}{q}} + A_j^{1+(1-\frac{1}{p}-\frac{2}{q})} + A_j^{1+(1-\frac{2}{p}-\frac{2}{q})} \right). \tag{3.31}$$

with

$$b := 2^{3n+6s+5} > 1 \qquad \text{and} \qquad \bar{C} := \frac{c(p,q)}{\delta^2 R^{2n+10s}} > 0.$$

Now, note that $1-2/q>0$, given that $q>2$, and that $\frac{1}{p} < \frac{1}{2} - \frac{1}{q}$, which is in fact possible for p large enough, say $p := 2+\varepsilon$, with $\varepsilon > \varepsilon^\star$ where

$$\varepsilon^\star = \varepsilon^\star(n,s) = \frac{2n(1+s)}{s}, \tag{3.32}$$

which derives from the growth gain $q^\star$ defined in (3.12).

Let us note that, with respect to $j \in \mathbb{N}$, given that $\{\omega_j\}_{j\in\mathbb{N}}$ is a sequence of nonnegative decreasing function (since $\{k_j\}_{j\in\mathbb{N}}$ is increasing in $j \in \mathbb{N}$) and that $\{Q_j\}_{j\in\mathbb{N}}$ is a sequence of decreasing cylinders, it holds that $\{A_j\}_{j\in\mathbb{N}}$ is decreasing too with respect to $j \in \mathbb{N}$. Hence, we have that $A_j \leqslant A_0 \leqslant 1$, up to choosing k_0 such that

$$k_0 \geqslant \|f_+\|_{L^2(Q_R)}. \tag{3.33}$$

Hence, we can rewrite the inequality in (3.31) as follows,

$$A_{j+2} \leqslant \bar{C} b^j A_j^{1+\alpha}\,,$$

for some positive $\alpha \equiv \alpha(n,s) := 1 - \frac{2}{p} - \frac{2}{q} > 0$ and $b > 1$. Then, up to choosing

$$k_0 := (\delta R^{n+5s})^{-\frac{1}{\alpha}} c^{\frac{1}{2\alpha}} b^{\frac{1}{\alpha^2}} \|f_+\|_{L^2(Q_R)} + \delta\, \|\operatorname{Tail}(f_+; B_{R/2})\|_{L^{2+\varepsilon}(U_R)} + \|h\|_{L^{2+\varepsilon}(Q_R)}\,,$$

which is in clear accordance with (3.24) and (3.33) and (3.29). The iteration argument of Lemma 3.2 yields that $A_j \to 0$ as $j \to \infty$, which gives the desired result. □

4. Towards a Harnack inequality

This section is devoted to the proof of some of the main ingredients required to obtain the Harnack inequalities in (1.7) and (1.9); i. e., the De Giorgi Intermediate Values lemma and the Measure-to-pointwise one, which in turn does also rely on a suitable application of the δ-interpolative L^∞ inequality in Theorem 1.1 by carefully estimating the tail contributions; see forthcoming Section 4.2.

4.1. De Giorgi's Intermediate Values Lemma.

Our strategy will extend that in the pioneering paper [CCV11], which will help in some of the estimates on the nonlocal energy terms arising from the diffusion in velocity. However, some decisive modifications need to be carried out because of our kinetic framework; that is, the novel presence of the transport term in (1.1). Also, it is worth noticing that our methods are feasible of further generalizations when more spatial commutators are involved.

Consider $\mu < 1$, $r_3 > r_2 > 0$ and a cut-off function $\varphi \equiv \varphi(x) \in C_0^\infty(\mathbb{R}^n)$ such that $0 \leqslant \varphi \leqslant 1$, $\varphi \equiv 1$ in $B_{r_2^{1+2s}}$ and $\varphi \equiv 0$ outside $B_{r_3^{1+2s}}$. Then define the following three auxiliary functions $F_i = F_i(v)$,

$$
\begin{aligned}
&F_0(v) := \frac{1}{r_3}\max\left\{-r_3,\ \frac{\min\left\{0,\ |v|^2 - 2r_3^2\right\}}{r_3}\right\},\\
&F_1(v) := \frac{1}{r_3}\max\left\{-r_3,\ \frac{\min\left\{0,\ |v|^2 - r_3^2\right\}}{r_3}\right\},\\
&F_2(v) := \frac{1}{r_2}\max\left\{-r_2,\ \frac{\min\left\{0,\ |v|^2 - r_2^2\right\}}{r_2}\right\}.
\end{aligned}
\tag{4.1}
$$

The underling kinetic geometry is coming up. Indeed, instead of a single Lipschitz function F which does the job in the purely fractional parabolic setting in [CCV11, Section 4], here we need three consecutive functions φ_i depending on F_i

$$
\varphi_i = \varphi_i(x,v) := 2 - \varphi(x) + \mu^i F_i(v)\,, \qquad \text{for } i = 0,1,2, \tag{4.2}
$$

in clear accordance with the very fine extension in the kinetic framework in [Loh24a].

Then, we state the following

Theorem 4.1. *Let f be a weak subsolution to* (1.1) *in Ω according to* Definition 2.3 *such that $f \leqslant 1$ and let $Q_1 \equiv Q_1(\mathbf{0}) \subset \Omega$. Assume that $h \in L^{2+\varepsilon}_{\rm loc}(\Omega)$, with $\|h\|_{L^{2+\varepsilon}(Q_1)} \leqslant 1$, for some $\varepsilon > \varepsilon^\star(n,s) > 0$; see Formula* (3.32). *Consider $0 < r_1, r_2 < r_3 < 1$ and $0 > t_2 > t_1 > -1$. Define now*

$$
Q^{(1)} := (-1,t_1] \times B_{r_1^{1+2s}} \times B_{r_1}, \qquad Q^{(2)} := (t_2,0] \times B_{r_2^{1+2s}} \times B_{r_2}
$$

$$
Q^{(3)} := (-1,0] \times B_{r_3^{1+2s}} \times B_{r_3}.
$$

Given $\delta_1, \delta_2 \in (0,1)$ there exist $\nu, \mu \equiv \nu, \mu(\delta_1,\delta_2,r_1,r_2,r_3,s,n)$ such that if it holds

$$
|\{f \leqslant \varphi_0\} \cap Q^{(1)}| \geqslant \delta_1 |Q^{(1)}| \quad \textit{and} \quad |\{f \geqslant \varphi_2\} \cap Q^{(2)}| \geqslant \delta_2 |Q^{(2)}|\,, \tag{4.3}
$$

then f satisfies

$$
|\{\varphi_0 < f < \varphi_2\} \cap Q^{(3)}| \geqslant \nu |Q^{(3)}|\,,
$$

where φ_i are defined in (4.2) *for $i = 1,2,3$.*

Proof. The proof requires a sort of both nonlocal and kinetic approach based on suitable choices in order to estimate all the energy contributions by tracking explicit dependencies on the involved quantities so that the forthcoming Harnack inequalities do depend on local slanted cylinders.

Step 1: The energy estimate. Up to regularizing by mollification, for any fixed $t \in (-1,0]$ we assume that $(f-\varphi_1)_+$ is sufficiently regular in order to be an admissible test function compactly supported in the cylinder $(Q^{(3)})^t := \{(v,x) \in \mathbb{R}^{2n} : (t,x,v) \in Q^{(3)}\}$. Consider now the weak formulation in Definition 2.3 by choosing as a test function $\phi \equiv (f-\varphi_1)_+$ there for a. e. $t \in (-1,0]$ it yields

$$
\begin{aligned}
\int_{(Q^{(3)})^t} h(f-\varphi_1)_+ \,\mathrm{d}x\,\mathrm{d}v \quad &\geqslant \quad \int_{(Q^{(3)})^t} (f_t + v\cdot\nabla_x f)(f-\varphi_1)_+ \,\mathrm{d}x\,\mathrm{d}v \\
&\quad + \int_{B_{r_3^{1+2s}}} \mathcal{E}(f,(f-\varphi_1)_+)\,\mathrm{d}x \\
&=: \quad I_1 + I_2. \qquad (4.4)
\end{aligned}
$$

We start by considering I_1. Using the fact that $\partial_t\varphi = 0$ and that $\nabla_x\varphi_1 = \nabla_x\varphi \neq 0$ only on $B_{r_3^{1+2s}} \setminus B_{r_2^{1+2s}}$ by (4.2), we have that

$$
\begin{aligned}
I_1 \quad &= \quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{(Q^{(3)})^t} (f-\varphi_1)_+^2 \,\mathrm{d}x\,\mathrm{d}v \\
&\quad + \int_{(Q^{(3)})^t} v\cdot\nabla_x\varphi_1 (f-\varphi_1)_+ \,\mathrm{d}x\,\mathrm{d}v \\
&= \quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{(Q^{(3)})^t} (f-\varphi_1)_+^2 \,\mathrm{d}x\,\mathrm{d}v - \mu\int_{(Q^{(3)})^t} |v\cdot\nabla\varphi|\,\mathrm{d}v\,\mathrm{d}x\,, \qquad (4.5)
\end{aligned}
$$

where we have used the fact that $(f-\varphi_1)_+ \leqslant \mu$. Moreover, we get

$$
I_1 \geqslant \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{(Q^{(3)})^t} (f-\varphi_1)_+^2 \,\mathrm{d}x\,\mathrm{d}v - c\,\mu\,. \qquad (4.6)
$$

We now consider the integral I_2. We start noticing that by the linearity of the involved energy $\mathcal{E}(\cdot)$, we have that

$$
\begin{aligned}
&\mathcal{E}(f,(f-\varphi_1)_+) \\
&\qquad = [(f-\varphi_1)_+]_{H^s}^2 - \mathcal{E}((f-\varphi_1)_-,(f-\varphi_1)_+) + \mathcal{E}(\varphi_1,(f-\varphi_1)_+). \qquad (4.7)
\end{aligned}
$$

Moreover, we note that

$$
\begin{aligned}
\mathcal{E}(\varphi_1,(f-\varphi_1)_+) \quad &\leqslant \quad \frac{1}{2}[(f-\varphi_1)_+]_{H^s}^2 \\
&\quad + c\,\mu^2 \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|F_1(v)-F_1(w)|^2}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w \\
&\leqslant \quad \frac{1}{2}[(f-\varphi_1)_+]_{H^s}^2 + c\,\mu^2 \iint_{2B_{r_3}\times 2B_{r_3}} \frac{|F_1(v)-F_1(w)|^2}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w \\
&\quad + c\,\mu^2 \iint_{(\mathbb{R}^n\setminus 2B_{r_3})\times 2B_{r_3}} \frac{|F_1(v)-F_1(w)|^2}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w \\
&= \quad \frac{1}{2}[(f-\varphi_1)_+]_{H^s}^2 + c\,\mu^2 \iint_{B_{r_3}\times B_{r_3}} \frac{\mathrm{d}v\,\mathrm{d}w}{|v-w|^{n-2(1-s)}} \\
&\quad + c\,\mu^2 \iint_{(\mathbb{R}^n\setminus 2B_{r_3})\times B_{r_3}} \frac{\mathrm{d}v\,\mathrm{d}w}{|v-w|^{n+2s}} \\
&\leqslant \quad \frac{1}{2}[(f-\varphi_1)_+]_{H^s}^2 + c\,\mu^2 + c\,\mu^2 \int_{\mathbb{R}^n\setminus 2B_{r_3}} \frac{\mathrm{d}w}{|w|^{n+2s}} \\
&\leqslant \quad \frac{1}{2}[(f-\varphi_1)_+]_{H^s}^2 + c\mu^2\,, \qquad (4.8)
\end{aligned}
$$

where we used the Lipschitz continuity of F_1 and the fact that it is null outside B_{r_3} – see its definition in (4.1) – alongside with the definition of $\mathcal{E}$ and the following estimate for $w \in \mathbb{R}^n \setminus 2B_{r_3}$ and $v \in B_{r_3}$

$$\frac{|w|}{|v-w|} \leqslant 1 + \frac{|v|}{|w|-|v|} \leqslant 1 + \frac{r_3}{2r_3 - r_3} = 2.$$

Recalling (4.6) and the fact that $-I_1 \geqslant I_2 - \int_{(Q^{(3)})^t} h(f-\varphi_1)_+ \,\mathrm{d}x\,\mathrm{d}v$ by (4.4), it yields (recalling that $\mu^2 < \mu < 1$)

$$c\,\mu + \int_{(Q^{(3)})^t} h(f-\varphi_1)_+ \,\mathrm{d}x\,\mathrm{d}v \geqslant I_2 + \frac{\mathrm{d}}{\mathrm{d}t}\int_{(Q^{(3)})^t} (f-\varphi_1)_+^2 \,\mathrm{d}x\,\mathrm{d}v. \tag{4.9}$$

Moreover, by combining (4.7) with (4.8), we obtain that

$$\begin{aligned} I_2 \quad \leqslant \quad & \frac{3}{2}\int_{B_{r_3^{1+2s}}} [(f-\varphi_1)_+]^2_{H^s}\,\mathrm{d}x \\ & - \int_{B_{r_3^{1+2s}}} \mathcal{E}((f-\varphi_1)_-, (f-\varphi_1)_+)\,\mathrm{d}x + c\,\mu^2. \end{aligned} \tag{4.10}$$

Then, by summing (4.10) with (4.9), and recalling that $\mu < 1$, it follows

$$\begin{aligned} c\,\mu + \int_{(Q^{(3)})^t} h(f-\varphi_1)_+ \,\mathrm{d}x\,\mathrm{d}v - 2I_2 \quad \geqslant \quad & \frac{\mathrm{d}}{\mathrm{d}t}\int_{(Q^{(3)})^t} (f-\varphi_1)_+^2 \,\mathrm{d}x\,\mathrm{d}v \\ & -\frac{3}{2}\int_{B_{r_3^{1+2s}}} [(f-\varphi_1)_+]^2_{H^s}\,\mathrm{d}x \\ & + \int_{B_{r_3^{1+2s}}} \mathcal{E}((f-\varphi_1)_-, (f-\varphi_1)_+)\,\mathrm{d}x\,; \end{aligned}$$

so that, also in view of (4.7), we finally arrive at

$$\begin{aligned} c\,\mu \; + \int_{(Q^{(3)})^t} h(f-\varphi_1)_+ \,\mathrm{d}x\,\mathrm{d}v \quad \geqslant \quad & \frac{\mathrm{d}}{\mathrm{d}t}\int_{(Q^{(3)})^t} (f-\varphi_1)_+^2 \,\mathrm{d}x\,\mathrm{d}v \\ & + \frac{1}{2}\int_{B_{r_3^{1+2s}}} [(f-\varphi_1)_+]^2_{H^s}\,\mathrm{d}x \\ & - \int_{B_{r_3^{1+2s}}} \mathcal{E}((f-\varphi_1)_-, (f-\varphi_1)_+)\,\mathrm{d}x \\ & +2\int_{B_{r_3^{1+2s}}} \mathcal{E}(\varphi_1, (f-\varphi_1)_+)\,\mathrm{d}x. \end{aligned}$$

We now estimate the energy contribution $\mathcal{E}(\varphi_1, (f-\varphi_1)_+)$. We firstly split the contribution given by the nonlocal term as follows,

$$\begin{aligned} & \mathcal{E}(\varphi_1, (f-\varphi_1)_+) \\ & = \iint_{|v-w|\geqslant 1} \frac{(\varphi_1(v)-\varphi_1(w))((f-\varphi_1)_+(v)-(f-\varphi_1)_+(w))}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w \\ & \quad + \iint_{|v-w|< 1} \frac{(\varphi_1(v)-\varphi_1(w))((f-\varphi_1)_+(v)-(f-\varphi_1)_+(w))}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w. \end{aligned} \tag{4.11}$$

Using the very definition of φ_1 and the boundedness of the auxiliary function F_1 in (4.1), we obtain that

$$\begin{aligned}\left|\iint_{|v-w|\geqslant 1} \frac{(\varphi_1(v)-\varphi_1(w))(f-\varphi_1)_+(v)}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w\right| \\ &\leqslant c\mu \int_{|v-w|\geqslant 1} \frac{\mathrm{d}w}{|v-w|^{n+2s}} \int_{\mathbb{R}^n} (f-\varphi_1)_+(v)\,\mathrm{d}v \\ &\leqslant c\,\mu^2 \int_1^\infty \sigma^{-1-2s}\,\mathrm{d}\sigma \;=\; c\,\mu^2\,,\end{aligned}$$

where we have also used that $(f-\varphi_1)_+ \leqslant \mu$ and it is compactly supported. Then, the first integral in the right-hand side of (4.11) can be estimated as follows

$$\left|\iint_{|v-w|\geqslant 1} \frac{(\varphi_1(v)-\varphi_1(w))((f-\varphi_1)_+(v)-(f-\varphi_1)_+(w))}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w\right| \leqslant c\,\mu^2.$$

Let us consider now the second integral in the right-hand side of (4.11). By suitable applying the Hölder inequality and the Young inequality, we can deduce that

$$\begin{aligned}\left|\iint_{|v-w|<1} \frac{(\varphi_1(v)-\varphi_1(w))((f-\varphi_1)_+(v)-(f-\varphi_1)_+(w))}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w\right| \\ &\leqslant \delta \iint_{|v-w|<1} \frac{|(f-\varphi_1)_+(v)-(f-\varphi_1)_+(w)|^2}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w \\ &\quad + \frac{1}{\delta} \iint_{|v-w|<1} \frac{|\varphi_1(v)-\varphi_1(w)|^2}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w\,,\end{aligned}$$

for some $\delta > 0$ which will be fixed later on. Now, the second integral in the right-hand side of the display above can be estimated via the definition of φ_1 as well as done in previous estimate (4.10), so that

$$\begin{aligned}&\iint_{|v-w|<1} \frac{|\varphi_1(v)-\varphi_1(w)|^2\,\mathrm{d}v\,\mathrm{d}w}{|v-w|^{n+2s}} \\ &\leqslant \mu^2 \iint_{|v-w|<1} \frac{|F_1(v)-F_1(w)|^2}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w \\ &\leqslant \mu^2 \iint_{2B_{r_3}\times(2B_{r_3}\cap\{|v-w|<1\})} \frac{|F_1(v)-F_1(w)|^2}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w \\ &\quad + 2\,\mu^2 \iint_{(\mathbb{R}^n\setminus 2B_{r_3})\times(2B_{r_3}\cap\{|v-w|<1\})} \frac{|F_1(v)-F_1(w)|^2}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w \\ &= c\,\mu^2 \iint_{B_{r_3}\times(B_{r_3}\cap\{|v-w|<1\})} \frac{\mathrm{d}v\,\mathrm{d}w}{|v-w|^{n-2(1-s)}} \\ &\quad + c\,\mu^2 \iint_{(\mathbb{R}^n\setminus 2B_{r_3})\times(B_{r_3}\cap\{|v-w|<1\})} \frac{\mathrm{d}v\,\mathrm{d}w}{|w|^{n+2s}} \\ &\leqslant c\,\mu^2 \int_0^1 \sigma^{1+2s}\,\mathrm{d}\sigma \int_{B_{r_3}} \mathrm{d}v + c\,\mu^2 \int_{2r_3}^\infty \frac{\mathrm{d}\sigma}{\sigma^{2s+1}} \int_{B_{r_3}} \mathrm{d}v \leqslant c\,\mu^2|B_{r_3}|\,.\end{aligned}$$

All in all, from (4.11) the following estimate has been obtained,

$$|\mathcal{E}(\varphi_1,(f-\varphi_1)_+)| \;\leqslant\; c(\delta)\,\mu^2 + \delta\,[(f-\varphi_1)_+]^2_{H^s}\,,$$

for a suitable positive constant $c = c(\delta)$. Choosing δ sufficiently small in order to absorb the seminorm $[(f-\varphi_1)_+]_{H^s}$ in the right-hand side of the display above, it yields

$$
\begin{aligned}
c\mu + \int_{(Q^{(3)})^t} h(f-\varphi_1)_+ \,\mathrm{d}x\,\mathrm{d}v \ \geqslant\ & \frac{\mathrm{d}}{\mathrm{d}t}\int_{(Q^{(3)})^t} (f-\varphi_1)_+^2 \,\mathrm{d}x\,\mathrm{d}v \\
& + \int_{B_{r_3^{1+2s}}} [(f-\varphi_1)_+]^2_{H^s}\,\mathrm{d}x \\
& - \int_{B_{r_3^{1+2s}}} \mathcal{E}((f-\varphi_1)_-,(f-\varphi_1)_+)\,\mathrm{d}x. \qquad (4.12)
\end{aligned}
$$

Moreover, recalling that $(f-\varphi_1)_+(f-\varphi_1)_- = 0$, we obtain that

$$
\mathcal{E}((f-\varphi_1)_-,(f-\varphi_1)_+) = -2\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(f-\varphi_1)_+(v)(f-\varphi_1)_-(w)}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w.
$$

Hence, the inequality in (4.12) can be written as follows,

$$
\begin{aligned}
c\mu + \int_{(Q^{(3)})^t} h(f-\varphi_1)_+ \,\mathrm{d}x\,\mathrm{d}v \ \geqslant\ & \frac{\mathrm{d}}{\mathrm{d}t}\int_{(Q^{(3)})^t} (f-\varphi_1)_+^2 \,\mathrm{d}x\,\mathrm{d}v \\
& + \int_{B_{r_3^{1+2s}}} [(f-\varphi_1)_+]^2_{H^s}\,\mathrm{d}x \\
& + \int_{B_{r_3^{1+2s}}} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(f-\varphi_1)_+(v)(f-\varphi_1)_-(w)}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x. \qquad (4.13)
\end{aligned}
$$

Now, note that

$$
\begin{aligned}
\int_{Q^{(3)}} h(f-\varphi_1)_+ \,\mathrm{d}x\,\mathrm{d}v \ &\leqslant\ \left(\int_{Q^{(3)}} h^2 \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac12} \left(\int_{Q^{(3)}} (f-\varphi_1)_+^2 \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac12} \\
&\leqslant\ c\,\mu \left(\int_{Q^{(3)}} h^2 \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac12} \\
&\leqslant\ c\,\mu \|h\|_{L^{2+\varepsilon}(Q^{(3)})} |Q^{(3)}|^{1-\frac{2+\varepsilon}{2}} \leqslant c\,\mu\,,
\end{aligned}
$$

where we have used that $(f-\varphi_1)_+ \leqslant \mu$ and that $\|h\|_{L^{2+\varepsilon}(Q_1)} \leqslant 1$. Moreover, notice that both the second and the third term in the inequality above in (4.13) are nonnegative, once we define

$$
\mathscr{H}(t) := \int_{(Q^{(3)})^t} (f-\varphi_1)_+^2(t,x,v)\,\mathrm{d}x\,\mathrm{d}v\,,
$$

it yields that for $-1 < t \leqslant 0$ we have (collecting also the estimate for the source term h)

$$
\mathscr{H}'(t) \leqslant c\mu\,.
$$

Moreover, let us note that $(f-\varphi_1)_+ \leqslant \mu \mathbb{1}_{(Q^{(3)})^t}$, for a. e. $t \in (-1, 0]$, hence $\mathscr{H}(t) \leqslant c\,\mu$. Then, integrating inequality (4.13) in time for $-1 < \tau_1 < \tau_2 \leqslant 0$, we finally get

$$
\begin{aligned}
\int_{\tau_1}^{\tau_2} \int_{B_{r_3^{1+2s}}} \iint_{\mathbb{R}^n\times\mathbb{R}^n} & \frac{(f-\varphi_1)_+(v)(f-\varphi_1)_-(w)}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}t \\
& \leqslant c\,\mu(\tau_2-\tau_1) + |\mathscr{H}(\tau_2) - \mathscr{H}(\tau_1)| \\
& \leqslant c\,\mu(\tau_2-\tau_1) \ \leqslant\ c\,\mu\,. \qquad (4.14)
\end{aligned}
$$

Step 2: Estimating time and space slices. Starting from the assumption in (4.3), let us call Σ the set of times in $(-1,t_1]$ defined as follows,

$$\Sigma := \left\{ t \in (-1,t_1] \, : \, \Big|\{f(t,\cdot,\cdot) \leqslant \varphi_0\} \cap (Q^{(3)})^t \Big| \geqslant \frac{\delta_1 |Q^{(1)}|}{4} \right\}.$$

Such a set Σ satisfies the following estimate,

$$|\Sigma| \geqslant \frac{\delta_1}{2} \big(\frac{r_1}{r_3}\big)^{2n(1+s)} (t_1+1)\,.$$

Indeed, a plain computation leads to

$$\begin{aligned}
&\Big|\{f \leqslant \varphi_0\} \cap \Big((Q^{(3)})^t \times (-1,t_1] \Big) \Big| \\
&\quad = \Big|\{f(t,\cdot,\cdot) \leqslant \varphi_0\} \cap (Q^{(3)})^t \Big| \, |\Sigma| \, + \, \Big|\{f(t,\cdot,\cdot) \leqslant \varphi_0\} \cap (Q^{(3)})^t \Big| \, \big|\mathscr{C}_{(-1,t_1]}(\Sigma)\big|\,,
\end{aligned}$$

where, as customary, by $\mathscr{C}_{(-1,t_1]}(\Sigma)$ we denoted the complementary set of Σ in $(-1,t_1]$. Thus, we get

$$\begin{aligned}
&|\{f(t,\cdot,\cdot) \leqslant \varphi_0\} \cap (Q^{(3)})^t||\Sigma| \\
&\qquad \geqslant |\{f \leqslant \varphi_0\} \cap Q^{(1)}| - |\{f(t,\cdot,\cdot) \leqslant \varphi_0\} \cap (Q^{(3)})^t||\mathscr{C}_{(-1,t_1]}(\Sigma)| \\
&\qquad \overset{(4.3)}{\geqslant} \delta_1 |Q^{(1)}| - \frac{\delta_1 |Q^{(1)}|}{4} |\mathscr{C}_{(-1,t_1]}(\Sigma)| \\
&\qquad \geqslant \delta_1 |Q^{(1)}| - \frac{\delta_1 |Q^{(1)}|}{4} \\
&\qquad \geqslant \frac{3}{4} \delta_1 |Q^{(1)}| \, > \, \frac{1}{2} \delta_1 |Q^{(1)}|\,.
\end{aligned}$$

Then, dividing the previous inequality on both sides by $|\{f(t,\cdot,\cdot) \leqslant \varphi_0\} \cap (Q^{(3)})^t|$ yields

$$\begin{aligned}
|\Sigma| \; &\geqslant \; \frac{\delta_1}{2} \frac{|Q^{(1)}|}{|\{f(t,\cdot,\cdot) \leqslant \varphi_0\} \cap (Q^{(3)})^t|} \\
&\geqslant \; \frac{\delta_1}{2} \frac{|Q^{(1)}|}{|Q^{(3)}|} \; \geqslant \; \frac{\delta_1}{2} \big(\frac{r_1}{r_3}\big)^{2n(1+s)} (t_1+1)\,,
\end{aligned}$$

as claimed.

Starting again from estimate (4.14), we have

$$
\begin{aligned}
c\,\mu \;&\overset{(4.14)}{\geqslant}\; \int_{-1}^{0}\int_{B_{r_3^{1+2s}}}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{(f-\varphi_1)_+(v)(f-\varphi_1)_-(w)}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}t\\
&\geqslant\; c\int_{\Sigma}\int_{B_{r_3^{1+2s}}}\iint_{\mathbb{R}^n\times B_{r_3}}(f-\varphi_1)_+(v)(f-\varphi_1)_-(w)\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}t\\
&\geqslant\; c\int_{\Sigma}\int_{B_{r_3^{1+2s}}}\int_{(\{f(\cdot,x,t)\leqslant\varphi_0\}\cap B_{r_3})\times B_{r_3}}(f-\varphi_1)_+(v)(\varphi_1-\varphi_0)_+(w)\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}t\\
&\geqslant\; c\int_{\Sigma}\int_{B_{r_3^{1+2s}}}\iint_{(\{f(\cdot,x,t)\leqslant\varphi_0\}\cap B_{r_3})\times B_{r_3}}(f-\varphi_1)_+(v)(\mu F_1-F_0)_+(w)\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}t\\
&\geqslant\; c(1-\mu)\int_{\Sigma}\int_{B_{r_3^{1+2s}}}\left(\int_{B_{r_3}}(f-\varphi_1)_+(v)\,\mathrm{d}v\right)|\{f(\cdot,x,t)\leqslant\varphi_0\}\cap B_{r_3}|\,\mathrm{d}x\,\mathrm{d}t\\
&\geqslant\; \frac{c(1-\mu)|Q^{(1)}|\delta_1}{4\mu}\int_{\Sigma}\int_{(Q^{(3)})^t}(f-\varphi_1)_+^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\,, \qquad (4.15)
\end{aligned}
$$

where we have also used the following estimates,

$$
\begin{aligned}
&|\{f(\cdot,x,t)\leqslant\varphi_0\}\cap B_{r_3}| \;\geqslant\; \frac{\delta_1|Q^{(1)}|}{4} \quad \text{a. e. on } B_{r_3^{1+2s}} \text{ and for } t\in\Sigma\,,\\
&(f-\varphi_1)_+ \;\leqslant\; \mu\,,\\
&(\varphi_1-\varphi_0)_- \;=\; (\mu F_1-F_0)_+ \;\geqslant\; 1-\mu \quad \text{on } B_{r_3}\,,
\end{aligned}
$$

and

$$
\inf_{B_{r_3}\times B_{r_3}}|v-w|^{-n-2s} \;\geqslant\; c>0.
$$

Hence, we eventually get

$$
\int_{\Sigma}\int_{(Q^{(3)})^t}(f-\varphi_1)_+^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \;\leqslant\; \frac{c\mu^2}{(1-\mu)\delta_1} \;\leqslant\; \mu^{2-1/8}\,,
$$

if μ is sufficiently small. In particular, we have that

$$
\int_{(Q^{(3)})^t}(f-\varphi_1)_+^2\,\mathrm{d}v\,\mathrm{d}x \;\leqslant\; \mu^{2-1/4} \qquad (4.16)
$$

does hold for any $t\in(-1,t_1]$ except on a set Υ for which we have that, by Chebychev's Inequality,

$$
|\Upsilon| \;:=\; \left|\left\{t\in(-1,t_1]\,:\,\|(f-\varphi_1)_+(t,\cdot)\|^2_{L^2((Q^{(3)})^t)}>\mu^{2-1/4}\right\}\right| \leqslant \mu^{1/8}.
$$

Taking a smaller μ such that

$$
\mu\leqslant\left(\frac{\delta_1}{4}\right)^8, \qquad (4.17)
$$

we can finally have that (4.16) holds on a set $t\in(-1,t_1]$ of measure greater than $\frac{3}{4}\delta_1$.

Step 3: The intermediate set for f. Assume now that there exists a time $\tau_0 \in (t_2, 0)$ such that

$$|\{(v,x)|(f-\varphi_2)_+(\tau_0,\cdot,\cdot)>0\}\cap Q^{(2)}| > \frac{\delta_2}{2}|Q^{(2)}|.$$

Thus, at time τ_0 we have

$$\begin{aligned}
\mathcal{H}(\tau_0) &= \int_{(Q^{(3)})^{\tau_0}} (f-\varphi_1)_+^2(v,x,\tau_0)\,\mathrm{d}x\,\mathrm{d}v \\
&\geqslant \int_{(Q^{(3)})^{\tau_0}} (\varphi_2-\varphi_1)^2(v,x,\tau_0)\mathbb{1}_{\{(f-\varphi_2)_+(\cdot,\tau_0)>0\}}\,\mathrm{d}x\,\mathrm{d}v \\
&\geqslant \int_{(Q^{(3)})^{\tau_0}} (\mu^2F_2-\mu F_1)^2(v,x,\tau_0)\mathbb{1}_{\{(f-\varphi_2)_+(\cdot,\tau_0)>0\}}\,\mathrm{d}x\,\mathrm{d}v \\
&\geqslant \int_{(Q^{(2)})^{\tau_0}} \mu^2(\mu F_2-F_1)^2(v)\,\mathbb{1}_{\{(f-\varphi_2)_+(\tau_0,\cdot,\cdot)>0\}}\,\mathrm{d}x\,\mathrm{d}v \\
&\overset{(4.17)}{\geqslant} \frac{\mu^2}{2}\min_{\substack{v\in B_{r_2}\\ \mu\leqslant(\frac{\delta_1}{4})^8}} (\mu F_2-F_1)^2\,||\{(x,v)\,:\,(f-\varphi_2)_+(\tau_0,\cdot,\cdot)>0\}\cap Q^{(2)}|| \\
&\geqslant \overline{C}\frac{\mu^2}{4}\delta_2|Q^{(2)}|\,, \qquad (4.18)
\end{aligned}$$

where the positive constant $\overline{C}$ depends only on F_1, F_2 and δ_1.

Moreover, consider a time $\overline{\tau} \leqslant \tau_0$ such that $\overline{\tau} \in (-1, t_1]$ such that

$$\mathcal{H}(\overline{\tau}) = \int_{(Q^{(3)})^{\overline{\tau}}} (f-\varphi_1)_+^2(v,x,\overline{\tau})\,\mathrm{d}v\,\mathrm{d}x \,\leqslant\, \mu^{2-1/4}.$$

In this way, we choose μ sufficiently small (up to shrink a smaller δ_2 if needed) such that

$$\mu^{-1/4} \geqslant \overline{C}\frac{|Q^{(2)}|\delta_2}{16}.$$

and thus the energy $\mathcal{H}(\cdot)$ of $(f-\varphi_1)_+^2(t,\cdot,\cdot)$ passes through the range of times

$$D := \left\{\tau \in (\overline{\tau},\tau_0)\,:\,\overline{C}\frac{|Q^{(2)}|\mu^2}{16}\delta_2 < \mathcal{H}(\tau) < \overline{C}\frac{|Q^{(2)}|\mu^2}{4}\delta_2\right\}.$$

In such a range of times we have that

$$\left|\{(f-\varphi_2)_+(\tau,\cdot,\cdot)>0\}\cap(Q^{(3)})^{\tau}\right| \leqslant \frac{\delta_2}{2}|Q^{(2)}|. \tag{4.19}$$

Indeed, by contradiction assume that the reverse inequality holds true for some $\tau \in D$. Hence, at such a time slice τ, going through the same computation as in (4.18), we will arrive at $\mathcal{H}(\tau) \geqslant \overline{C}\frac{|Q^{(2)}|\mu^2}{4}\delta_2$ which is in contradiction with the fact that $\tau \in D$.

Thus, up to choose δ_2 sufficiently small, we have that the measure of the set appearing in (4.19) is negligible.

Now, we estimate the size of the set U of times slice of D for which

$$\left|\{(f-\varphi_0)_+(\tau,\cdot,\cdot)\leqslant 0\}\cap(Q^{(3)})^{t}\right| \,\geqslant\, \delta_1|Q^{(2)}|. \tag{4.20}$$

By (4.14) we have

$$
\begin{aligned}
c\,\mu &\overset{(4.14)}{\geqslant} \int_{-1}^{0}\int_{B_{r_3^{1+2s}}}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{(f-\varphi_1)_+(v)(f-\varphi_1)_-(w)}{|v-w|^{n+2s}}\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}t\\
&\geqslant \frac{c\delta_1|Q^{(2)}|}{\mu}\int_U\int_{B_{r_3^{1+2s}}}\int_{\mathbb{R}^n}(f-\varphi_1)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\\
&\geqslant \frac{c\delta_1|Q^{(2)}|}{\mu^2}\int_U \mathscr{H}(t)\,\mathrm{d}t \geqslant \frac{c\delta_1\delta_2|U||Q^{(2)}|}{16\mu^2}\,,
\end{aligned}
$$

where we have followed a similar reasoning as in (4.15)–(4.18), and in the last line we have also used the fact that $\tau\in U\subset D$. Therefore, we obtain that

$$
|U| \leqslant \frac{c\,\mu^3}{\delta_1\delta_2|Q^{(2)}|}.
$$

Then, by choosing

$$
\mu \leqslant \left(\frac{\delta_1\delta_2|D||Q^{(2)}|}{2c}\right)^{\frac{1}{3}}\,,
$$

we plainly deduce that

$$
|U| \leqslant \frac{|D|}{2}.
$$

Consider now those times $\tau\in D$ which are not in U. Hence, we have

$$
\Big|\{\varphi_0<f(\tau,\cdot,\cdot)<\varphi_2\}\cap(Q^{(3)})^\tau\Big| \geqslant \frac{|Q^{(3)}|}{2}\,, \tag{4.21}
$$

Indeed,

$$
\begin{aligned}
&\Big|\{\varphi_0<f(\tau,\cdot,\cdot)<\varphi_2\}\cap(Q^{(3)})^\tau\Big|\\
&\qquad\geqslant |(Q^{(3)})^\tau| - \Big|\{\varphi_0>f(\tau,\cdot,\cdot)\}\cap(Q^{(3)})^\tau\Big| - \Big|\{f(\tau,\cdot,\cdot)>\varphi_2\}\cap(Q^{(3)})^\tau\Big|\\
&\qquad\overset{(4.19),(4.20)}{\geqslant} |(Q^{(3)})^\tau| - \delta_1|Q^{(2)}| - \frac{\delta_2}{2}|Q^{(2)}|\\
&\qquad\geqslant \frac{|Q^{(3)}|}{2}\,,
\end{aligned}
$$

up to choose δ_1 and δ_2 small enough.

Hence, by (4.21), we finally deduce

$$
\begin{aligned}
|\{\varphi_0<f<\varphi_2\}\cap Q^{(3)}| &= \int_{-1}^{0}|\{\varphi_0<f(\tau,\cdot,\cdot)<\varphi_2\}\cap(Q^{(3)})^\tau|\,\mathrm{d}\tau\\
&\geqslant \int_{D\setminus U}|\{\varphi_0<f(\tau,\cdot,\cdot)<\varphi_2\}\cap(Q^{(3)})^\tau|\,\mathrm{d}\tau\\
&\geqslant \frac{|D||Q^{(3)}|}{4} \geqslant \nu|Q^{(3)}|\,,
\end{aligned}
$$

up to choose a constant $\nu\equiv\nu(\delta_1,\delta_2)$ sufficiently small, as desired. □

4.2. **The Measure-to-pointwise Lemma.** In order to prove a Measure-to-pointwise-type lemma, we will employ Theorem 4.1 established in the previous section, together with the nonlocal L^∞-L^2-type estimate with tail (1.6) stated in the Introduction.

Theorem 4.2. *Let $\mathring{\delta} \in (0,1)$ and for $0 < r_1, r_2 < 1$ and $0 > t_2 > t_1 > -1$ consider*

$$Q^{(1)} := (-1, t_1] \times B_{r_1^{1+2s}} \times B_{r_1}, \qquad Q^{(2)} := (t_2, 0] \times B_{r_2^{1+2s}} \times B_{r_2},.$$

Assume that $h \in L^{2+\varepsilon}_{\rm loc}(\Omega)$, with $h \leqslant 0$ and $\|h\|_{L^{2+\varepsilon}(Q_1)} \leqslant 1/4$, for some $\varepsilon > \varepsilon^\star(n,s) > 0$; see Formula (3.32). Let g be a weak subsolution to (1.1) in Ω such that $g \leqslant 1$ in Ω and

$$|\{g \leqslant 0\} \cap Q^{(1)}| \geqslant \mathring{\delta} |Q^{(1)}|. \tag{4.22}$$

Then there exists a real number $\vartheta \equiv \vartheta(\mathring{\delta}, \mu, \nu) \in (0,1)$ such that

$$g \leqslant 1 - \vartheta \qquad \text{in } \ Q_\rho(t_0, x_0, v_0), \tag{4.23}$$

for any $Q_\rho(t_0,x_0,v_0) \subseteq Q^{(2)}$, with $\rho \leqslant \min\{|t_2|, r_2\}/2$, where ν and μ are the constants introduced in Theorem 4.1.

Proof. Let us consider $0 < \mu < 1$ as in (4.2) and let us define a sequence of functions

$$g_k := \frac{1}{\mu^{2k}} \left(g - (1 - \mu^{2k}) \right) = 1 - \frac{1-g}{\mu^{2k}}, \qquad k \geqslant 0.$$

Given g is a subsolution, then also g_k is a subsolution to (1.1) for every $k \geqslant 0$. Additionally, since $g \leqslant 1$, then $g_k \leqslant 1$ for every $k \geqslant 0$, but notice that it may also be negative.

Moreover, it is true that

$$|\{g_k \leqslant 0\} \cap Q^{(1)}| \geqslant \mathring{\delta} |Q^{(1)}|. \tag{4.24}$$

Indeed, for every $k \geqslant 0$ the set $\{g_k \leqslant 0\}$ is equivalent to the set $\{g \leqslant 1 - \mu^{2k}\}$. Then, considering that $1 - \mu^{2k} \geqslant 0$ together with (4.22), the claim is proved for every $k \geqslant 0$.

Now, we apply the boundedness estimate (1.6) to every g_k, with $k \geqslant 0$ in $Q^{(2)}$, and we obtain

$$\begin{aligned} \sup_{Q_\rho(t_0,x_0,v_0)} g_k \ \leqslant \ & c\, \delta^{-\frac{1}{2\alpha}} \Big(\int_{Q^{(2)}} (g_k)_+^2 \, {\rm d}v \, {\rm d}x \, {\rm d}t \Big)^{\frac12} \\ & + \delta \| \, {\rm Tail}((g_k)_+; B_\rho(v_0)) \|_{L^{2+\varepsilon}(U_{2\rho}(t_0,x_0))} \\ & + \|h\|_{L^{2+\varepsilon}(Q^{(2)})} \,, \end{aligned}$$

for any $Q_\rho(t_0,x_0,v_0) \subset Q^{(2)}$, with $\rho \leqslant \min\{|t_2|, r_2\}/2$.

We now observe that if for some $\bar{k}$ the following inequality does hold true,

$$\begin{aligned} c\, \delta^{-\frac{1}{2\alpha}} \Big(\int_{Q^{(2)}} (g_{\bar{k}})_+^2 \, {\rm d}v \, {\rm d}x \, {\rm d}t \Big)^{\frac12} + \delta \| \, {\rm Tail}((g_k)_+; B_\rho(v_0)) \|_{L^{2+\varepsilon}(U_{2\rho}(t_0,x_0))} & \\ + \|h\|_{L^{2+\varepsilon}(Q^{(2)})} < \frac{1}{2} \,, & \end{aligned}$$

then $g_{\bar{k}} \leqslant 1/2$, and hence $g \leqslant 1 - \mu^{2\bar{k}}/2$ implying the thesis with $\vartheta := \mu^{2\bar{k}}/2$.

Hence, we are left with the proof of our statement when there exists $k_0 \geqslant 2$ such that

$$c\,\delta^{-\frac{1}{2\alpha}}\Big(\int_{Q^{(2)}}(g_k)_+^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac{1}{2}}+\delta\|\operatorname{Tail}((g_k)_+;B_{r_2/2})\|_{L^{2+\varepsilon}(U_{2\rho}(t_0,x_0))}$$
$$+\|h\|_{L^{2+\varepsilon}(Q^{(2)})}>\frac{1}{2}\qquad\forall k\ \text{s. t.}\ 0\leqslant k\leqslant k_0-1\,, \tag{4.25}$$

Then, for every $k\in\mathbb{R}$ such that $0\leqslant k\leqslant k_0-1$ it holds

$$|\{g_k\leqslant\varphi_0\}\cap Q^{(1)}|\;=\;|\{g\leqslant 1-\mu^{2k}\varphi(x)\}\cap Q^{(1)}|\;\geqslant\;\mathring{\delta}|Q^{(1)}|\,,$$

because $0<\mu<1$, $F_0(v)=-1$ in B_{r_1}, $\varphi(x)\in[0,1]$, and thus $1-\mu^{2k}\varphi(x)\geqslant 0$, allowing us to employ (4.24). Now, we also have that, choosing δ sufficiently small,

$$\begin{aligned}
\frac{\big|\{g_k>\varphi_2\}\cap Q^{(2)}\big|}{|Q^{(2)}|}\quad&\geqslant\quad\frac{|\{g_k\geqslant 0\}\cap Q_r^+|}{|Q_1(\mathbf{0})|}\\
&\geqslant\quad c\int_{Q^{(2)}}(g_k)_+^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\\
&>\quad c\,\delta^{\frac{1}{\alpha}}\Bigg(\frac{1}{2}-\delta\|\operatorname{Tail}((g_k)_+;B_\rho(v_0))\|_{L^{2+\varepsilon}(U_{2\rho}(t_0,x_0))}\\
&\qquad\qquad-\|h\|_{L^{2+\varepsilon}(Q^{(2)})}\Bigg)^2\\
&>\quad c\,\delta^{\frac{1}{\alpha}}\Bigg(\frac{1}{4}-\frac{\delta\omega_n|U_1(0,0)|^{\frac{1}{2+\varepsilon}}}{2s}\Bigg)^2\;=:\;\delta_2\;>\;0\,,
\end{aligned}$$

where $\omega_n=|\mathbb{S}^{n-1}|$ as usual and where we have used the fact that $\|h\|_{L^{2+\varepsilon}(Q^{(2)})}\leqslant\|h\|_{L^{2+\varepsilon}(Q_1(\mathbf{0}))}\leqslant 1/4$. The estimate above comes from the fact that $g_k\leqslant 1$, implying $0\leqslant(g_k)_+\leqslant 1$, combined with the definition of the indicator function $\mathbb{1}_{\{g_k\geqslant 0\}}$, the estimate in (4.25).

Finally, thanks to Theorem 4.1 applied to every g_k, with $0\leqslant k\leqslant k_0-1$, with the choice $\delta_1=\mathring{\delta}$ there, we can deduce the existence of a constant $\nu\equiv\nu(\mathring{\delta},n,s)>0$ (recalling the dependencies of δ_2) such that

$$|\,\{\varphi_0<g_k<\varphi_2\}\cap Q^{(3)}|\;\geqslant\;\nu|Q^{(3)}|\qquad\forall\ 0\leqslant k\leqslant k_0-1. \tag{4.26}$$

Now, note that

$$\frac{\nu(1-\mu)}{\mu}\leqslant\big(\frac{1}{\mu}-\mu^{k_0-1}\big).$$

since $0<\nu,\mu<1$. Hence,

$$\mu^{k_0-1}\leqslant\frac{1-\nu(1-\mu)}{\mu}\,,$$

which yields

$$k_0\leqslant 1+\frac{\log\big(\frac{1-\nu(1-\mu)}{\mu}\big)}{\log\mu}=\frac{\log(1-\nu(1-\mu))}{\log\mu}.$$

Hence, recalling that $g_{k_0+1}\leqslant 1/2$ in $Q_\rho(t_0,x_0,v_0)$ by definition of k_0 (see (4.25)), it yields

$$g\;\leqslant\;1-\frac{\mu^{2k_0+2}}{2}\;\leqslant\;1-\frac{\mu^{\frac{2\log(1-\nu(1-\mu))}{\log\mu}+2}}{2}\quad\text{in } Q_\rho(t_0,x_0,v_0).$$

Eventually, the claim follows by taking $\vartheta := \mu^{\frac{2\log(1-\nu(1-\mu))}{\log\mu}+2}/2$. □

5. Proof of the weak Harnack inequality

In view of the results of previous Section, it suffices to apply the final strategy in [IS20b] with no fundamental modifications except the fact that we rely on our Measure-to-point Lemma, in turn relying in Theorem 1.1. Thus, we do not need the whole architecture running the propagation of minima argument there (see, in particular, [IS20b, Sections 6 and 9]). For the sake of the reader, we will present the whole proof in a few steps right below.

now on, we assume $\|h\|_{L^{2+\varepsilon}(Q_1)} \leqslant 1/4$. Indeed, as shown in [IS20b], if this was not the case we could replace f and h with cf and ch, where

$$c := \big(2 \inf_{Q^+_{r_0}} f + 4\|h\|_{L^{2+\varepsilon}(Q_1(\mathbf{0}))}\big)^{-1},$$

such that the condition on the smallness of the $(2+\varepsilon)$-norm of the source term h is satisfied in order to apply the Measure-to-pointwise Lemma in Theorem 4.2 as well as the Intermediate value Lemma in Theorem 4.1.

Step 1: Propagation in measure. As a consequence of the result in Theorem 4.2 applied to $g = 1 - f/M$, we can prove that there exist two constants $M > 1$ and $\delta > 0$ such that if

$$|\{f \geqslant M\} \cap Q_1(\mathbf{0})| \geqslant (1-\delta)|Q_1(\mathbf{0})|,$$

then $f \geqslant 1$ on $\mathcal{Q} := [0, 2^{2s}] \times B_{2^{1+2s}} \times B_2$.

For this, we apply the measure-to-pointwise lemma to the function $f(t + 2^{2s}, x, v)$ in appropriated cylinders shifted in time, so that we can deduce

$$f \geqslant 1 \quad \text{in } \mathcal{Q},$$

by choosing $M = 1/\vartheta$, where ϑ is the one in Formula (4.23).

Step 2: Stacked propagation. By the same argument as in [IS20b, Corollary 9.2], for $k \geqslant 1$, $T_k = \sum_{i=1}^{k} 2^{2si}$, if f satisfies

$$|\{f \geqslant M^k\} \cap Q_1(\mathbf{0})| \geqslant (1-\delta)|Q_1(\mathbf{0})|,$$

for M and δ given by Step 1 above, then $f \geqslant 1$ in $Q[k] := [T_{k-1}, T_k] \times B_{2^{(1+2s)k}} \times B_{2^k}$; see Figure 5.

Step 3: Proof of the weak Harnack inequality (1.7). We prove that, for any $k \geqslant 1$ and for some fixed $r_0 \in (0,1)$ which will be chosen later on, it holds

$$|\{f > \bar{M}^k\} \cap Q^-_{r_0}| \leqslant \bar{c}(1-\bar{\delta})^k, \tag{5.1}$$

for some constants $\bar{M}$, $\bar{c}$ and $\bar{\delta} \in (0,1)$.

We start by induction. For $k=1$, we simply choose $\bar{c}$ and $\bar{\delta}$ so that

$$|Q^-_{r_0}| \leqslant \bar{c}(1-\delta) \quad \text{and} \quad \bar{\delta} \leqslant \delta.$$

Assume now that (5.1) holds true up to rank k and prove it for $k+1$. We want to apply the Ink-spot Theorem 2.1 with $\mu = \delta$, with some integer m (which will

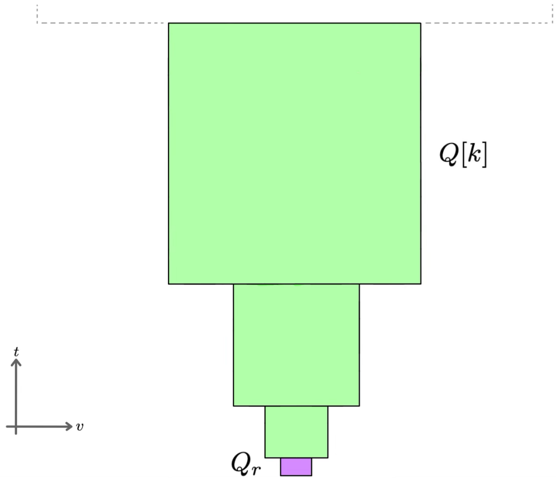

FIGURE 4. The stacked propagation geometry in the proof of the weak Harnack inequality in Theorem 1.4, as introduced in [IS20b, Section 9].

be fixed later on) and with $\bar{M} := M^m$, with M and δ being the constant give by Step 1.

Let us consider the following sets,

$$E := \{f \geqslant \bar{M}^{k+1}\} \cap Q^-_{r_0} \quad \text{and} \quad F := \{f > \bar{M}^k\} \cap Q_1(\mathbf{0}).$$

Clearly, by recalling the definition (1.8), we infer the sets $E \subset F \subset Q_1(\mathbf{0})$ are bounded and measurable. Let us assume that for any cylinder $Q_\sigma(t_0, x_0, v_0) \subset Q^-_{r_0}$, for some $\sigma \in (0, r_0)$, it holds

$$|Q_\sigma(t_0, x_0, v_0) \cap E| > (1-\delta)|Q_\sigma(t_0, x_0, v_0)|.$$

Hence,

$$|\{f \geqslant \bar{M}^{k+1}\} \cap Q_\sigma(t_0, x_0, v_0)| > (1-\mu)|Q_\sigma(t_0, x_0, v_0)|.$$

We apply now the Measure-to-pointwise Lemma, Theorem 4.2, to the subsolution g defined as follows,

$$g := 1 - f\left((t_0 + 1 - \sigma^{2s}, x_0, v_0) \circ \cdot\right) / \bar{M}^{k+1},$$

and we get that

$$f\left((t_0 + 1 - \sigma^{2s}, x_0, v_0) \circ \cdot\right) \geqslant \vartheta \bar{M}^{k+1} \qquad \text{on } Q^+_{\sigma/2}.$$

Thus, choosing $\mathfrak{q} := -\log \vartheta / \log(2^{2s}) > 0$ we get (recalling that we can assume f having infimum less or equal than 1)

$$1 \geqslant \vartheta \bar{M}^{k+1} = \left(\frac{1}{2}\right)^{2s\mathfrak{q}} \bar{M}^{k+1} \geqslant \left(\frac{\sigma}{2}\right)^{2s\mathfrak{q}} \bar{M}^{k+1}.$$

Then, we get that

$$r_0 := 2\bar{M}^{-\frac{k}{2s\mathfrak{q}}}.$$

Now to prove that $\bar{Q}^m_\sigma(t_0, x_0, v_0) \subset F$, that is $\bar{Q}^m_\sigma(t_0, x_0, v_0) \subset \{f \geqslant \bar{M}^k\}$, we apply the result of Step 2 with $k = m$ to $\bar{M}^{-k} f((t_0, x_0, v_0) \circ \cdot)$.

Thus, by the Ink-spot Theorem 2.1 (with $\mu \equiv \delta$ and $r_0 \equiv 2\bar{M}^{-\frac{k}{2sq}}$ there) we get

$$
\begin{aligned}
&|\{f \geqslant \bar{M}^{k+1}\} \cap Q^-_{r_0}| \\
&\quad \leqslant \frac{1+m}{m}(1-c\delta)\big(|\{f > \bar{M}^k\} \cap Q^-_{r_0}| + Cmr_0^{2s}\big) \\
&\quad \leqslant \frac{1+m}{m}(1-c\delta)\big(\bar{c}(1-\bar{\delta})^k + cm\bar{M}^{-\frac{k}{q}}\big) \quad \big(\text{by the induction step (5.1)}\big) \\
&\quad \leqslant \bar{c}\frac{1+m}{m}(1-c\delta)\left(1+\frac{cm}{\bar{c}}\right)(1-\bar{\delta})^k \quad \big(\text{choosing } 1-\bar{\delta} > \bar{M}^{-1/q}\big) \\
&\quad \leqslant \bar{c}(1-\bar{\delta})^{k+1}\,,
\end{aligned}
$$

up to choose m and consequently $\bar{c}$ large enough. This proves the desired induction step. The proof of estimate (1.7) will then follow by a standard argument via the layer-cake formula.

6. A new Besicovitch-type covering for slanted cylinders

As mentioned in the Introduction, the proof of Theorem 1.5 will rely on a new covering argument for the involved slanted cylinders. Such a general Besicovitch-type result will be presented right below.

The following properties for the slanted cylinders in (2.1) do hold true:

(1) **(Monotonicity)** Given a slanted cylinder $Q_\sigma(t,x,v)$, and $\rho > 0$, there exist a point (t',x',v') and two constant $\varkappa, \bar{\varepsilon} \in (0,1)$ such that

$$
\begin{aligned}
&\|(t,x,v)^{-1} \circ (t',x',v')\|_{\text{kin}} \leqslant \frac{\bar{\varepsilon}\varkappa\sigma}{\rho}\,, \\
\text{and} \quad &Q_{\frac{\varkappa\sigma}{\rho}}(t',x',v') \subset Q_\sigma(t,x,v) \subset Q_{\frac{\sigma}{\varkappa\rho}}(t',x',v')\,.
\end{aligned}
$$

(2) **(Exclusion)** There exists $\beta > 0$ such that for any $Q_\rho(t_0,x_0,v_0)$ and $(t,x,v) \notin Q_\rho(t_0,x_0,v_0)$ it holds

$$
Q_{\epsilon^\beta}(t,x,v) \cap Q_{(1-\epsilon)\rho}(t_0,x_0,v_0) = \emptyset \quad \text{for any } 0 < \epsilon < 1.
$$

(3) **(Inclusion)** There exists $\wp > 1$ such that for $0 < \sigma < \rho < 1$ and $(t,x,v) \in Q_\sigma(t_0,x_0,v_0)$ it holds

$$
Q_{(\rho-\sigma)^\wp}(t,x,v) \subset Q_\rho(t_0,x_0,v_0).
$$

(4) **(Engulfment)** There exists a constant $\kappa \equiv \kappa(s)$ such that for any $Q_\rho(t_0,x_0,v_0)$ and $Q_\sigma(t,x,v)$ with

$$
Q_\rho(t_0,x_0,v_0) \cap Q_\sigma(t,x,v) \neq \emptyset \quad \text{and} \quad 2\rho \geqslant \sigma\,,
$$

it holds that

$$
Q_\sigma(t,x,v) \subset \kappa Q_\rho(t_0,x_0,v_0)\,,
$$

with

$$
\begin{aligned}
\kappa Q_\rho(t_0,x_0,v_0) := \Big\{(t,x,v) \ : \ &-\frac{\kappa^{2s}+1}{2}\rho^{2s} < t-t_0 \leqslant \frac{\kappa^{2s}-1}{2}\rho^{2s} \\
&|v-v_0| < \kappa\rho, \ |x-x_0-(t-t_0)v_0| < (\kappa\rho)^{1+2s}\Big\}.
\end{aligned}
$$

The quantity $\|\cdot\|_{\text{kin}}$ is that obtained via the customary *kinetic distance*, firstly seen in [IS21] for proving Schauder estimates for Boltzmann equations; that is,

$$\|(t,x,v)\|_{\text{kin}} := \max\big\{|t|^{\frac{1}{2s}},\, |x|^{\frac{1}{1+2s}},\, |v|\big\}. \tag{6.1}$$

Compare, also, our *Engulfment* with Lemma 10.4 there.

We can now state and prove the following

Lemma 6.1 (**Besicovitch's covering Lemma for slanted cylinders).** *Let $\Omega \subset \mathbb{R}^{2n+1}$ be a bounded set. Assume that for any $(t,x,v) \in \Omega$ there exists a family of slanted cylinders $Q_r(t,x,v)$ with $r \leqslant R$, for some $R > 0$. Then, there exists a countable family $\mathcal{F} := \{Q_{r_k}(t_k,x_k,v_k)\}_{k\in\mathbb{N}}$ with the following properties:*

(i) $\Omega \subset \displaystyle\bigcup_{k=1}^{\infty} Q_{r_k}(t_k,x_k,v_k)$.

(ii) $(t_k,x_k,v_k) \notin \displaystyle\bigcup_{j<k} Q_{r_k}(t_j,x_j,v_j)$*, for any $k \geqslant 2$.*

(iii) *For $\epsilon \in (0,1)$, the family $\mathcal{F}_\epsilon := \big\{Q_{(1-\epsilon)r_k}(t_k,x_k,v_k)\big\}_{k\in\mathbb{N}}$ has bounded overlaps. Moreover,*

$$\sum_{k=1}^{\infty} \mathbb{1}_{Q_{(1-\epsilon)r_k}(t_k,x_k,v_k)}(t,x,v) \;\leqslant\; c\log\big(\frac{1}{\epsilon}\big)\,,$$

where the constant c depends on the Monotonicity *constants $\varkappa$ and $\bar{\varepsilon}$, and the* Exclusion *constant β.*

Proof. Let us assume with no loss of generality that $R := \sup\{r : Q_r(t,x,v) \in \mathcal{F}\}$. We set

$$\mathcal{F}_0 := \big\{Q_r(t,x,v) : \frac{R}{2} < r \leqslant R,\ Q_r(t,x,v) \in \mathcal{F}\big\}\,,$$

and

$$\mathcal{O}_0 := \big\{(t,x,v) : Q_r(t,x,v) \in \mathcal{F}_0\big\}.$$

Let us choose $Q_{r_1}(t_1,x_1,v_1) \in \mathcal{F}_0$. If $\mathcal{O}_0 \subset Q_{r_1}(t_1,x_1,v_1)$, then we stop. Otherwise, let us choose $Q_{r_2}(t_2,x_2,v_2)$ so that

(i) $Q_{r_2}(t_2,x_2,v_2) \in \mathcal{F}_0$;

(ii) $(t_2,x_2,v_2) \in \mathcal{O}_0 \setminus Q_{r_1}(t_1,x_1,v_1)$.

Now, if $\mathcal{O}_0 \subset Q_{r_1}(t_1,x_1,v_1) \cup Q_{r_2}(t_2,x_2,v_2)$, then we stop, otherwise we continue to iterate such a process. In such a way, we build a subfamily $\mathcal{F}_0' := \{Q_{r_j^0}(t_j^0,x_j^0,v_j^0)\}_{j\in\mathbb{N}}$ such that $(t_k^0,x_k^0,v_k^0) \in \mathcal{O}_0 \setminus \displaystyle\bigcup_{j<k} Q_{r_j^0}(t_j^0,x_j^0,v_j^0)$.

Now, we consider the following families,

$$\mathcal{F}_1 := \big\{Q_r(t,x,v) : \frac{R}{4} < r \leqslant \frac{R}{2},\ Q_r(t,x,v) \in \mathcal{F}\big\}\,,$$

and

$$\mathcal{O}_1 := \Big\{(t,x,v)\ :\ Q_r(t,x,v) \in \mathcal{F}_1 \text{ and } (t,x,v) \notin \bigcup_{j=1}^{\infty} Q_{r_j^0}(t_j^0,x_j^0,v_j^0)\Big\}.$$

In a similar fashion as above we build a family $\mathcal{F}_1' := \{Q_{r_j^1}(t_j^1,x_j^1,v_j^1)\}_{j\in\mathbb{N}}$ such that $(t_k^1,x_k^1,v_k^1) \in \mathcal{O}_1 \setminus \displaystyle\bigcup_{j<k} Q_{r_j^1}(t_j^1,x_j^1,v_j^1)$.

By iterating this process up to the k^{th}-stage, we obtain the following two families,

$$\mathcal{F}_k := \left\{ Q_r(t,x,v) : \frac{R}{2^{k+1}} < r \leqslant \frac{R}{2^k},\ Q_r(t,x,v) \in \mathcal{F} \right\},$$

and

$$\mathcal{O}_k := \left\{ (t,x,v) : Q_r(t,x,v) \in \mathcal{F}_k \text{ and } (t,x,v) \notin \bigcup_{i=0}^{k-1} \bigcup_{j=1}^{\infty} Q_{r^i_j}(t^i_j, x^i_j, v^i_j) \right\}.$$

From this, we get a family of cylinders $\mathcal{F}'_k := \{Q_{r^k_j}(t^k_j, x^k_j, v^k_j)\}_{j\in\mathbb{N}}$ so that $(t^k_\ell, x^k_\ell, v^k_\ell) \in \mathcal{O}_k \setminus \bigcup_{j<\ell} Q_{r^k_j}(t^k_j, x^k_j, v^k_j)$.

We now are in the position to prove that the collection of all slanted cylinders in all $\mathcal{F}'_k$ do satisfy the conditions of Lemma 6.1.

We start by proving that each family $\mathcal{F}'_i$ has bounded overlapping. For this, suppose that

$$(t,x,v) \in Q_{r^i_{j_1}}(t^i_{j_1}, x^i_{j_1}, v^i_{j_1}) \cap \dots \cap Q_{r^i_{j_m}}(t^i_{j_m}, x^i_{j_m}, v^i_{j_m}),$$

with $Q_{r^i_{j_\ell}}(t^i_{j_\ell}, x^i_{j_\ell}, v^i_{j_\ell}) \in \mathcal{F}'_i$. Now, let $Q_{r^i_0}(t^i_0, x^i_0, v^i_0)$ be the cylinder with $r^i_0 := \max\{r^i_{j_\ell} : 1 \leqslant \ell \leqslant m\}$. Note that, by construction, we can also assume that $(t^i_{j_N}, x^i_{j_N}, v^i_{j_N}) \notin Q_{r^i_{j_\ell}}(t^i_{j_\ell}, x^i_{j_\ell}, v^i_{j_\ell})$, for $\ell < N$.

In view of the *Monotonicity* property of the slanted cylinder, we have that there exist $\varkappa$, $\bar\varepsilon > 0$ such that

$$\begin{aligned} &\|(t^i_{j_\ell}, x^i_{j_\ell}, v^i_{j_\ell})^{-1} \circ (t'_\ell, x'_\ell, v'_\ell)\|_{\text{kin}} \leqslant \frac{\bar\varepsilon \varkappa r^i_{j_\ell}}{r^i_0}, \\ \text{and} \quad & Q_{\frac{\varkappa r^i_{j_\ell}}{r^i_0}}(t'_\ell, x'_\ell, v'_\ell) \subset Q_{r^i_{j_\ell}}(t^i_{j_\ell}, x^i_{j_\ell}, v^i_{j_\ell}) \subset Q_{\frac{r^i_{j_\ell}}{\varkappa r^i_0}}(t'_\ell, x'_\ell, v'_\ell), \end{aligned} \tag{6.2}$$

for any $1 \leqslant \ell \leqslant m$. Recalling that $(t^i_{j_N}, x^i_{j_N}, v^i_{j_N}) \notin Q_{r^i_{j_\ell}}(t^i_{j_\ell}, x^i_{j_\ell}, v^i_{j_\ell})$, we get by (6.2) that, for $N > \ell$,

$$(t^i_{j_N}, x^i_{j_N}, v^i_{j_N}) \notin Q_{\frac{\varkappa r^i_{j_\ell}}{r^i_0}}(t'_\ell, x'_\ell, v'_\ell). \tag{6.3}$$

Then, by combining (6.2) with (6.3) we have that

$$\begin{aligned} &\|(t^i_{j_N}, x^i_{j_N}, v^i_{j_N})^{-1} \circ (t^i_{j_\ell}, x^i_{j_\ell}, v^i_{j_\ell})\|_{\text{kin}} \\ &\quad \geqslant \frac{1}{c}\|(t^i_{j_N}, x^i_{j_N}, v^i_{j_N})^{-1} \circ (t'_\ell, x'_\ell, v'_\ell)\|_{\text{kin}} - c\,\|(t^i_{j_\ell}, x^i_{j_\ell}, v^i_{j_\ell})^{-1} \circ (t'_\ell, x'_\ell, v'_\ell)\|_{\text{kin}} \\ &\quad > \frac{\varkappa r^i_{j_\ell}}{r^i_0}\Big(\frac{1}{c} - c\,\bar\varepsilon\Big) > c(\varkappa, \bar\varepsilon) > 0, \end{aligned} \tag{6.4}$$

since $R2^{-(i+1)} < r^i_{j_\ell} \leqslant r^i_0 \leqslant R2^{-i}$, up to further restricting $\bar\varepsilon > 0$ and applying Proposition 5.1.7 in [BLU07] to the kinetic norm (6.1). Moreover, by taking into account (6.2), we have that $Q_{r^i_{j_\ell}}(t^i_{j_\ell}, x^i_{j_\ell}, v^i_{j_\ell})$ is contained in a slanted cylinder $Q_{\bar R}(\mathbf{0})$, with the radius $\bar R$ depending only on $\varkappa$ and $\bar\varepsilon$. By (6.4), proceeding as in [CG96, Lemma 1], we obtain that the overlapping in each family $\mathcal{F}'_i$ is at most α, with α depending only on $\kappa, \bar\varepsilon$ and the dimension n only.

Now, we prove that the family $\mathcal{F}'_i$ is finite. Since Ω is bounded and $R2^{-(i+1)} < r^i_j \leqslant R2^{-i}$, there exists a constant $C > 0$ such that $\mathcal{O}_i \subset Q_{Cr^i_1}(t^i_1, x^i_1, v^i_1)$ and $Cr^i_1 \geqslant R2^{-i}$. Then, for any $Q_{r^i_j}(t^i_j, x^i_j, v^i_j) \in \mathcal{F}'_i$ we get

$$
\begin{aligned}
Q_{\frac{\varkappa}{2C}}(t'_j, x'_j, v'_j) &\subset Q_{r^i_j}(t^i_j, x^i_j, v^i_j)\\
&\subset Q_{\frac{r^i_j}{C\varkappa r^i_1}}(t'_j, x'_j, v'_j)\\
&\subset Q_{\frac{2}{C\varkappa}}(t'_j, x'_j, v'_j) \subset Q_{\bar{R}}(\mathbf{0})\,, \qquad (6.5)
\end{aligned}
$$

with $\bar{R}$ depending only on C and $\varkappa$.

Since $\mathcal{F}'_i$ has overlapping bounded by α, we get

$$\sum_{j=1}^{\infty} \mathbb{1}_{Q_{r^i_j}(t^i_j,x^i_j,v^i_j)}(t,x,v) \leqslant \alpha\,,$$

which, in view of (6.5), implies

$$\sum_{j=1}^{\infty} \mathbb{1}_{Q_{\frac{\varkappa}{2C}}(t'_j,x'_j,v'_j)}(t,x,v) \leqslant \alpha \mathbb{1}_{Q_{\bar{R}}(\mathbf{0})}\,.$$

Hence, integrating the sums above we deduce that $\mathcal{F}'_i$ has a finite number of cylinders.

We now estimate the boundedness of overlapping between different generators of the families $\mathcal{F}'_i$. We start by shrinking the selected cylinders

$$(t_0, x_0, v_0) \in \bigcap_{i=1}^{\infty} Q_{(1-\epsilon)r^{e_i}_{j_i}}(t^{e_i}_{j_i}, x^{e_i}_{j_i}, v^{e_i}_{j_i})\,, \qquad (6.6)$$

with $e_1 < e_2 < \dots$, $R2^{-(e_i+1)} < r^{e_i}_{j_i} \leqslant R2^{-e_i}$. Fix now, i and $\ell > i$, let us measure the gap between e_i and e_ℓ. Since $r^{e_\ell}_{j_\ell} < r^{e_i}_{j_i}$ we have that

$$Q_{\frac{\varkappa(1-\epsilon)r^{e_\ell}_{j_\ell}}{r^{e_i}_{j_i}}}(t', x', v') \subset Q_{r^{e_\ell}_{j_\ell}}(t^{e_\ell}_{j_\ell}, x^{e_\ell}_{j_\ell}, v^{e_\ell}_{j_\ell}) \subset Q_{\frac{(1-\epsilon)r^{e_\ell}_{j_\ell}}{\varkappa r^{e_i}_{j_i}}}(t', x', v').$$

Moreover, by the *Exclusion* property we have that

$$Q_{\epsilon^\beta}(t^{e_\ell}_{j_\ell}, x^{e_\ell}_{j_\ell}, v^{e_\ell}_{j_\ell}) \cap Q_{(1-\epsilon)r^{e_i}_{j_i}}(t^{e_i}_{j_i}, x^{e_i}_{j_i}, v^{e_i}_{j_i}) = \emptyset.$$

Thus,

$$
\begin{aligned}
0 &< \epsilon^\beta\\
&< \|(t^{e_\ell}_{j_\ell}, x^{e_\ell}_{j_\ell}, v^{e_\ell}_{j_\ell})^{-1} \circ (t_0, x_0, v_0)\|_{\text{kin}}\\
&\leqslant c\,\|(t', x', v')^{-1} \circ (t_0, x_0, v_0)\|_{\text{kin}} + c\,\|(t^{e_\ell}_{j_\ell}, x^{e_\ell}_{j_\ell}, v^{e_\ell}_{j_\ell})^{-1} \circ (t', x', v')\|_{\text{kin}}\\
&\leqslant \frac{c\,(1-\epsilon)r^{e_\ell}_{j_\ell}}{\varkappa r^{e_i}_{j_i}} + c\,\bar{\varepsilon}\frac{\varkappa(1-\epsilon)r^{e_\ell}_{j_\ell}}{r^{e_i}_{j_i}} \leqslant c\,2^{e_i-e_\ell}\,,
\end{aligned}
$$

which yields

$$e_\ell - e_i \lesssim \log_2\Big(\frac{1}{\epsilon}\Big)\,,$$

where c depends only on $\bar{\varepsilon}$, $\varkappa$, and the *Exclusion* constant β and where we have always used Proposition 5. 1. 7 in [BLU07] to treat the kinetic norm (6.1).

All in all, the number of cylinders in (6.6) is bounded by a multiple of $\log_2\big(\frac{1}{\epsilon}\big)$, up to a multiplicative constant which – we recall – will depend only on $\bar{\varepsilon}$, $\varkappa$ and β.

Now consider the family $\mathcal{F}' := \{\mathcal{F}'_i\}_{i=1}^{\infty}$. Since any family $\mathcal{F}'_i$ covers $\mathcal{O}_i$, the family $\mathcal{F}'$ cover Ω, so (i) follows. Moreover, up to relabel the cylinders, one can deduce (ii). Finally, by the argument above, also (iii) is satisfied up to enlarge the constant. □

7. Proof of the Harnack inequality

This section is devoted to the completion of the proof of the strong Harnack inequality in Theorem 1.5. Armed with the weak Harnack estimate in (1.7) obtained in the preceding section, as well as with the L^∞-L^2-estimate in Theorem 1.1, in order to concretize our final strategy we will also need to go into the Besicovitch-type covering argument presented in Section 6.

Let us split $f = f_+ - f_-$. Note that $f_- = 0$ on Ω since $f \geqslant 0$. Hence, for a. e. $(t,x,v) \in \Omega$ we have that

$$\begin{aligned} 0 = \partial_t f + v\cdot\nabla_x f - \mathcal{L}_v f &= \partial_t f_+ + v\cdot\nabla_x f_+ - \mathcal{L}_v f_+ + \mathcal{L}_v f_- \\ &= \partial_t f_+ + v\cdot\nabla_x f_+ - \mathcal{L}_v f_+ \\ &\quad + \int_{\mathbb{R}^n\setminus\Omega_v} f_-(t,x,w)K(v,w)\,\mathrm{d}w. \end{aligned}$$

Thus, f_+ is a weak solution in Ω to the following equation

$$\partial_t f_+ + v\cdot\nabla_x f_+ - \mathcal{L}_v f_+ = -\int_{\mathbb{R}^n\setminus\Omega_v} f_-(t,x,w)K(v,w)\,\mathrm{d}w =: h. \tag{7.1}$$

Now, note that, by its own definition $h \leqslant 0$. Moreover, for any $w \in \mathbb{R}^n \setminus B_2$ and for any $v \in B_1$, we have that

$$\frac{|w|}{|v-w|} \leqslant 1 + \frac{|v|}{|v-w|} \leqslant 1 + \frac{|v|}{|w|-|v|} \leqslant 2\,, \tag{7.2}$$

which yields that, for any $Q_{2\rho}(t_0,x_0,v_0) \subset Q_1$, it holds

$$\begin{aligned} &\|h\|_{L^{2+\varepsilon}(Q_{2\rho}(t_0,x_0,v_0))} \\ &\quad\leqslant \left\| \int_{\mathbb{R}^n\setminus B_2} f_-(t,x,w)K(v,w)\,\mathrm{d}w \right\|_{L^{2+\varepsilon}(Q_{2\rho}(t_0,x_0,v_0))} \\ &\quad\overset{(7.2)}{\leqslant} c\, \left\| \int_{\mathbb{R}^n\setminus B_2} \frac{f_-(t,x,w)}{|w|^{n+2s}}\,\mathrm{d}w \right\|_{L^{2+\varepsilon}(U_{2\rho}(t_0,x_0))} \leqslant c\, \|\operatorname{Tail}(f_-;B_2)\|_{L^{2+\varepsilon}(U_2)}. \end{aligned} \tag{7.3}$$

Step 1: Covering argument. Let us set $1/2 \leqslant \sigma' < \sigma \leqslant 1$, $\rho := (1-\epsilon)[(\sigma-\sigma')r_0]^{\wp}$, with $\wp$ being the *Inclusion* exponent in Section 6, r_0 being the radius given by the weak Harnack inequality in Theorem 1.4 and ϵ given by Lemma 6.1 (iii), depending only on the *Monotonicity* constants.

Consider the cylinders $Q_\rho(t_0,x_0,v_0)$, for any $(t_0,x_0,v_0) \in Q_{\sigma' r_0}(-1+r_0^{2s},0,0)$, with ρ given as above. By taking into account the *Inclusion* property, any cylinder of this family satisfies $Q_\rho(t_0,x_0,v_0) \subset Q_{\sigma r_0}(-1+r_0^{2s},0,0)$.

We now apply Theorem 1.1 and, thanks to our Besicovitch-type covering presented in Section 6, up to renumbering the family, we can cover $Q_{\sigma' r_0}(-1+r_0^{2s},0,0)$ by a countable family of slanted cylinders

$$\mathcal{F} := \big\{Q_{\rho_k}(t_k,x_k,v_k)\big\}_{k\in\mathbb{N}} \qquad \text{with } \rho_k \approx \frac{(1-\epsilon)[(\sigma-\sigma')r_0]^{\wp}}{2^k}.$$

Moreover, since the covering has bounded overlaps we get, by Lemma 6.1 (iii), that, for a. e. $(t_0, x_0, v_0) \in Q_{\sigma' r_0}(-1 + r_0^{2s}, 0, 0)$ it holds that

$$\mathfrak{N} := \# \{k \in \mathbb{N} : (t_0, x_0, v_0) \in Q_{\rho_k}(t_k, x_k, v_k)\} \leqslant c \log\Big(\frac{1}{\epsilon}\Big),$$

with a slight abuse of notation, where c depends only on *Monotonicity* constants $\varkappa$, ε and the *Exclusion* constant β.

Step 2: Application of the L^∞-L^2 estimate (1.6). Now, we want to apply the boundedness estimate to f_+ which solves (7.1)

For a. e. $(t_0, x_0, v_0) \in Q_{\sigma' r_0}(-1 + r_0^{2s}, 0, 0)$, we have by Theorem 1.1

$$\begin{aligned} f(t_0, x_0, v_0) &\leqslant \sum_{k=1}^{\mathfrak{N}} \sup_{Q_{\rho_k}(t_k, x_k, v_k)} f \\ &\leqslant \sum_{k=1}^{\mathfrak{N}} \Big(\frac{\|f\|_{L^2(Q_{2\rho_k}(t_k,x_k,v_k))}}{\delta^{1/\alpha}\, \rho_k^{(n+5s)/\alpha}} + \delta \| \operatorname{Tail}(f_+; B_{\rho_k}(v_k))\|_{L^{2+\varepsilon}(U_{2\rho_k}(t_k,x_k))} \\ &\qquad + \|h\|_{L^{2+\varepsilon}(Q_{2\rho_k}(t_k,x_k,v_k))}\Big). \end{aligned} \tag{7.4}$$

Now, let us estimate the nonlocal tail in velocity. For a. e. $(t, x) \in U_{2\rho_k}(t_k, x_k) \subset U_{\sigma r_0}(-1 + r_0^{2s}, 0)$ we have that

$$\begin{aligned} \rho_k^{2s} \int_{\mathbb{R}^n \setminus B_{\rho_k}(v_k)} \frac{f_+(t,x,v)}{|v - v_k|^{n+2s}}\,\mathrm{d}v &= \rho_k^{2s} \int_{B_{\sigma r} \setminus B_{\rho_k}(v_k)} \frac{f_+(t,x,v)}{|v - v_k|^{n+2s}}\,\mathrm{d}v \\ &\quad + \rho_k^{2s} \int_{\mathbb{R}^n \setminus B_{\sigma r}} \frac{f_+(t,x,v)}{|v - v_k|^{n+2s}}\,\mathrm{d}v \\ &\leqslant c \sup_{Q_{\sigma r}(-1+r_0^{2s},0,0)} f \\ &\quad + \frac{r_0^{2s}}{(\sigma - \sigma')^{n+2s}} \int_{\mathbb{R}^n \setminus B_{r_0/2}} \frac{f_+(t,x,v)}{|v|^{n+2s}}\,\mathrm{d}v, \end{aligned}$$

where we have used the fact that $Q_{\rho_k}(t_k, x_k, v_k) \subset Q_{\sigma r_0}(-1 + r_0^{2s}, 0, 0)$ for any $(t_k, x_k, v_k) \in Q_{\sigma' r_0}(-1 + r_0^{2s}, 0, 0)$, $\sigma > \frac{1}{2}$ and that

$$\frac{|v|}{|v - v_k|} \leqslant 1 + \frac{|v_k|}{|v| - |v_k|} \leqslant 1 + \frac{\sigma'}{\sigma - \sigma'} \leqslant \frac{1}{\sigma - \sigma'},$$

for any $v \in \mathbb{R}^n \setminus B_{\sigma r_0}$.

Thus, we can estimate the $(2+\varepsilon)$-contribution of the tail in velocity as follows,

$$\begin{aligned} &\Big(\int_{U_{2\rho_k}(t_k,x_k)} \operatorname{Tail}(f_+; B_{\rho_k}(v_k))^{2+\varepsilon}\,\mathrm{d}x\,\mathrm{d}t \Big)^{\frac{1}{2+\varepsilon}} \\ &\leqslant c \sup_{Q_{\sigma r_0}(-1+r_0^{2s},0,0)} f \\ &\quad + \frac{c}{(\sigma - \sigma')^{n+2s}} \Big(\int_{U_{r_0}(-1+r_0^{2s},0))} \operatorname{Tail}(f_+; B_{r_0/2})^{2+\varepsilon}\,\mathrm{d}x\,\mathrm{d}t \Big)^{\frac{1}{2+\varepsilon}}, \end{aligned} \tag{7.5}$$

where we have used that $Q_{\rho_k}(t_k, x_k, v_k) \subset Q_{\sigma r_0}(-1+r_0^{2s}, 0, 0) \subset Q_{r_0}(-1+r_0^{2s}, 0, 0)$.

Hence, combining (7.4) with (7.5) and (7.3), recalling that any $Q_{\rho_k}(t_k,x_k,v_k) \subset Q_{\sigma r_0}(-1+r_0^{2s},0,0) \subset Q_{r_0}(-1+r_0^{2s},0,0)$, yields

$$\begin{aligned}
f(t_0,x_0,v_0) \;&\leqslant\; \frac{c(\delta)}{[(\sigma-\sigma')r_0]^{\frac{\wp(n+5s)}{\alpha}}}\|f\|_{L^2(Q_{\sigma r_0}(-1+r_0^{2s},0,0))} + c\delta \sup_{Q_{\sigma r_0}(-1+r_0^{2s},0,0)} f \\
&\quad + \frac{c(\delta)\,\|\,\mathrm{Tail}(f_+;B_{r_0/2})\|_{L^{2+\varepsilon}(U_{r_0}(-1+r_0^{2s},0))}}{(\sigma-\sigma')^{n+2s}} + \|\,\mathrm{Tail}(f_-;B_2)\|_{L^{2+\varepsilon}(U_2)} \\
&\leqslant\; \frac{c(\delta)}{[(\sigma-\sigma')r_0]^{\frac{\wp(n+5s)}{\alpha}}}\Big(\sup_{Q_{\sigma r_0}(-1+r_0^{2s},0,0)} f\Big)^{\frac{2-\zeta}{2}}\|f^\zeta\|^{\frac12}_{L^1(Q_{\sigma r_0}(-1+r_0^{2s},0,0))} \\
&\quad + c\delta \sup_{Q_{\sigma r_0}(-1+r_0^{2s},0,0)} f + \frac{c(\delta)\,\|\,\mathrm{Tail}(f_+;B_{r_0/2})\|_{L^{2+\varepsilon}(U_{r_0}(-1+r_0^{2s},0))}}{(\sigma-\sigma')^{n+2s}} \\
&\quad + \|\,\mathrm{Tail}(f_-;B_2)\|_{L^{2+\varepsilon}(U_2)} \\
&\leqslant\; \frac{c(\delta)\|f\|_{L^\zeta(Q_{r_0}(-1+r_0^{2s},0,0))}}{[(\sigma-\sigma')r_0]^{\frac{2\wp(n+5s)}{\zeta\alpha}}} + \Big(c\delta+\frac{2-\zeta}{2}\Big)\sup_{Q_{\sigma r_0}(-1+r_0^{2s},0,0)} f \\
&\quad + \frac{c(\delta)\,\|\,\mathrm{Tail}(f_+;B_{r_0/2})\|_{L^{2+\varepsilon}(U_{r_0}(-1+r_0^{2s},0))}}{(\sigma-\sigma')^{n+2s}} + \|\,\mathrm{Tail}(f_-;B_2)\|_{L^{2+\varepsilon}(U_2)}
\end{aligned} \tag{7.6}$$

by also making use of an application of Young's Inequality (with exponents $2/\zeta$ and $2/(2-\zeta)$).

Step 3: Iteration and conclusion. Choose $\delta \in (0,1)$ such that

$$c\delta + \frac{2-\zeta}{2} =: \xi < 1\,,$$

which together with (7.6) (passing on the supremum on the left-hand side for $(t_0,x_0,v_0) \in Q^-_{\sigma' r_0}$) yields

$$\begin{aligned}
\sup_{Q_{\sigma' r_0}(-1+r_0^{2s},0,0)} f \;&\leqslant\; \xi \sup_{Q_{\sigma r_0}(-1+r_0^{2s},0,0)} f + \frac{c}{(\sigma-\sigma')^{\frac{2\wp(n+5s)}{\zeta\alpha}}}\|f\|_{L^\zeta(Q_{r_0}(-1+r_0^{2s},0,0))} \\
&\quad + \frac{c(\delta)}{(\sigma-\sigma')^{n+2s}}\|\,\mathrm{Tail}(f_+;B_{r_0/2})\|_{L^{2+\varepsilon}(U_{r_0}(-1+r_0^{2s},0))} \\
&\quad + \|\,\mathrm{Tail}(f_-;B_2)\|_{L^{2+\varepsilon}(U_2)}.
\end{aligned}$$

Hence, a final application of the usual lemma (see, e. g., [GG82, Lemma 1.1]) together with the weak Harnack inequality in Theorem 1.4 and the estimate of the source term (7.3) yields the desired estimate (1.9). To conclude, we just notice that in the case when $\zeta \geqslant 2$ there is no need to apply Young's Inequality as in Step 2, being actually enough to choose $c\delta < 1$ in order to apply the aforementioned iteration lemma. □

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