DE GIORGI-NASH-MOSER THEORY FOR KINETIC EQUATIONS WITH NONLOCAL DIFFUSIONS

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Abstract. We extend the celebrated De Giorgi-Nash-Moser theory to a class of nonlocal hypoelliptic equations naturally arising in kinetic theory, which combine a first-order operator with an elliptic one involving fractional derivatives along only part of the coordinates. Provided that the nonlocal tail in velocity of weak solutions is just p-summable along the drift variables, we prove the first local L^2-L^∞ estimate for kinetic integral equations. Then, we establish the first strong Harnack inequality under the aforementioned tail summability assumption. The latter is in fact naturally implied in literature, e. g., from the usual mass density boundedness (as for the Boltzmann equation without cut-off), and it reveals to be in clear accordance with the very recent counterexample by Kaßmann and Weidner [46].

Armed with the aforementioned results, we are able to provide a geometric characterization of the Harnack inequality in the same spirit of the seminal paper by Aronson and Serrin [9] for the (local) parabolic counterpart.

Contents

1. Introduction

In the present paper we study a wide class of kinetic integro-differential equations of the form

$$(\partial_t + v \cdot \nabla_x) f = \mathcal{L}_v f + h \quad \text{in } \Omega \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \,, \tag{1.1}$$

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where h is a possibly unbounded scalar field, and the diffusion term $\mathcal{L}_v$ is given by

$$\mathcal{L}_v f(t,x,v) := \text{ p. v.}\int_{\mathbb{R}^n} \big(f(t,x,w)-f(t,x,v)\big)K(t,x,w,v)\,\mathrm{d}w. \tag{1.2}$$

The kernel $K:\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{2n}\to[0,\infty)$ is a measurable symmetric kernel of differentiability order $s\in(0,1)$; that is, there exists a constant $\Lambda\geq 1$ such that

$$\Lambda^{-1}|v-w|^{-n-2s}\leq\ K(t,x,v,w)\ \leq \Lambda|v-w|^{-n-2s}, \quad \text{for a. e. } v,w\in\mathbb{R}^n, \tag{1.3}$$

where we assume that the condition above holds for all t and x; throughout the following we omit the t and x dependence to clean up the notation.

As a prototype one can just keep in mind the classical fractional Laplacian operator $(-\Delta_v)^s$, with respect to the v-variables, given by

$$(-\Delta_v)^s f(t,x,v) := c_{n,s}\ \text{p. v.}\int_{\mathbb{R}^n} \frac{f(t,x,v)-f(t,x,w)}{|v-w|^{n+2s}}\,\mathrm{d}w\,. \tag{1.4}$$

In the display above, $c_{n,s}$ is a positive constant only depending on the dimension n and the differentiability exponent s; see [23, Section 2] for further details. Furthermore, the integrals in (1.2)-(1.4) may be singular at the origin, and hence must be interpreted in the appropriate sense. Since we are considering diffusion terms with possibly rough coefficients, the related equation has to have a natural weak formulation, for which we refer the reader to Section 2 below.

Such hypoelliptic equations arise as linearized models for the Boltzmann equation without cutoff. They also naturally show up in several different models, because of the special role played by the velocity variable and its interaction with both the drift and the diffusion terms – see e. g. the comprehensive introduction in [57] and the references therein; see also Section 1 in [65] – even in other unrelated disciplines as for instance in Finance in order to describe the evolution of Asian options, where the drift term is connected with risk-free interest rates.

Our main goal is to prove general quantitative estimates for weak solutions to (1.1), by finally completing the longly unaccomplished De Giorgi-Nash-Moser theory for integro-differential Kolmogorov-Fokker-Planck equations.

1.1. **The De-Giorgi-Nash-Moser theory: state of the art.** Roughly speaking, by "De Giorgi-Nash-Moser (*DGNM*) theory" we mean the following fundamental results for weak solutions to Partial Differential Equations: L^2-L^∞ estimate; Hölder regularity; Harnack inequalities.

In the local case $\big($when $s=1$; let us say, $\mathcal{L}_v\approx\partial_{v_i}(a_{i,k}(\cdot)\partial_{v_j}f)\big)$, apart from constituting the missing piece in solving Hilbert's 19^{th} Problem, the *DGNM* theory has revealed to be fundamental for uniformly *elliptic* and *parabolic* equations with rough coefficients in divergence form and, since the pioneering works by De Giorgi and Nash, together with the subsequent important contribution by Moser, its extension to more general equations involving various operators has been one of the main goals for entire generations of Mathematics communities. Both the refined estimates and the iterative methods presented in the aforementioned works finally found an exhaustive extension in the difficult case of *kinetic* only a few years ago, because of the increased difficulties in the very form of the involved equations itself, where ellipticity fails in some direction. The completeness of the *DGNM* theory is accomplished thanks to the results in [29] and [31, 33], where weak and strong Harnack inequalities can be found together with Hölder regularity; it is also worth referring to [22] for a constructive proof of the weak Harnack inequality for rough kinetic equations. For what concerns the L^2-L^∞ estimate, this was firstly done in [60]. See, also, [66] for a preliminary Hölder result via the extension of the original proof by Moser.

The situation becomes even more convoluted when the operator $\mathcal{L}_v$ is a general integro-differential operator, as the fractional Laplacian with rough coefficients. Indeed, the development of the *DGMN* theory for nonlocal equations underwent a substantial growth during the past decades. In particular, after the breakthrough results by Kaßmann [41, 43] on the validity of the classical Harnack inequality, a quite comprehensive nonlocal theory was presented in more general integro-differential *elliptic frameworks*, even for nonlinear fractional equations. Since the literature is really too wide to attempt any comprehensive list here, we only refer to [13, 18, 19, 25, 42, 58], and the references therein.

Further difficulties do appear in the integro-differential *parabolic framework* where the intrinsic scaling of the involved cylinders depends not only on the time variable t, but also on the differentiability order s. Despite such non-negligible technicalities, parabolic Harnack inequalities, Hölder continuity and L^2-L^∞ inequality are available for general fractional equations as finally shown in the very important paper [45] in part extending the results in the elliptic counterpart in [18]. All in all, both in the nonlocal elliptic and parabolic frameworks, as in the local ones, the *DGMN* theory is complete.

For what regards the case of nonlocal *kinetic equations* as in (1.1), it is enlightening to focus on an even wider class as the one modeling the non-cutoff Boltzmann equation for which important estimates and regularity results were recently proven via fine variational techniques and radically new approaches. An inspiring step in such an advance relies on the method proposed in the breakthrough paper [38], where Imbert and Silvestre are able to derive a weak Harnack inequality for a very large class of kinetic integro-differential equations as in (1.1) with very mild assumptions on the integral diffusion in velocity having degenerate kernels K in (1.2) which are not symmetric (not in the usual way), nor pointwise bounded by Gagliardo-type kernels; see Theorem 1.6 there. In the conditional regime, where the hydrodynamic quantities mass, energy and entropy are bounded above and the mass is bonded away from the vacuum, the aforementioned result is enough to derive as corollary the Hölder regularity for nonnegative solutions of the spatially inhomogeneous Boltzmann equation without cut-off. Further related regularity estimates in the conditional regime were subsequently proven in [39]. All in all, despite the fine estimates and the new techniques mentioned above − see also the related interesting papers [55, 65] − a strong Harnack-type inequality is still missing. More than this, despite some polynomial L^∞ bound such as

$$\|f(t)\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n)} \lesssim 1 + t^{-\beta}, \quad \beta > 0,$$

can be established for the Boltzmann equation without cut-off under pointwise bounds on some observables and condition on the solution f (see for example [26,59,64]), boundedness of solutions is usually assumed a priori in the nonlocal kinetic literature. Likewise, a quantitative control on the L^∞ norm of the solutions, i. e. L^2-L^∞ estimate, is missing.

Eventually, all the mentioned combined efforts in pushing forward the nonlocal theory of hypoelliptic equations culminated with the establishment of an ingenuous counterexample by Kaßmann and Weidner in [46], where they built a sequence of solutions $\{f_\varepsilon\}$ to

$$v \cdot \nabla_x f + (-\Delta_v)^s f = 0\,, \tag{1.5}$$

such that, for some points $z_{\rm o} \in \mathbb{R}^n$ the ratio $f_\varepsilon(0)/f_\varepsilon(z_{\rm o})$ blows-up as $\varepsilon \searrow 0$; see Theorem 1.1 there. This implies the failure of the Harnack inequality for (1.5), and, by extension, for (1.1) since the solutions $\{f_\varepsilon\}$ are time-independent. Moreover, a closer inspection reveals that a local L^2-L^∞ estimate for solutions to (1.1) generally fails too, even when an error term is added on its right-hand side, basically a tail-type contribution– see Formula (1.10) below – in striking contrast to all the parabolic and elliptic literature on fractional equations ([19, 44, 45]). Such peculiar feature of (1.5) is a pure effect originating from the combination of the nonlocality of the diffusion term with the anisotropy behavior of the drift, and it is in odd contrast with all the previous literature dealing with local kinetic equations. Furthermore, it is worth noticing that such a phenomenon is quite remarkable given that the degeneracy of (1.5) is no obstruction to C^∞-regularity; see for example [39]. Indeed, by velocity averaging techniques ([10]) it is possible to transfer regularity from the v-variable to the x one as it happens for purely local operators.

Within this framework, the result in [46] could have been the end of any hope to complete the *DGNM* theory for nonlocal kinetic equations, beyond any assumptions on the involved diffusion kernels. However, by proposing a refined version of the L^2-L^∞ estimate, as well as a revised nonlocal strong Harnack inequality, in clear accordance with the aforementioned counterexample, our forthcoming Theorem 1.1 and 1.5 serve as a final completion of the whole nonlocal theory for Kolmogorov equations, being concomitantly the integro-differential counterpart of the aforementioned recent results achieved for local kinetic equations with rough coefficient, along with the results already obtained in both the fractional elliptic and parabolic context. As an important addition, which will be clearer in the following, both our strategy and proofs seem very much adaptable to deal with more general nonlocal ultraparabolic equations.

1.2. **Main results.** The underlying geometry of equations (1.1) is determined by a homogeneous Lie group structure. Hence, to state our main results, which reflect this non-Euclidean background, we endow $\mathbb{R}^{1+2n}$ with the Galilean transformation

$$z_{\rm o} \circ z := (t + t_{\rm o},\, x + x_{\rm o} + t v_{\rm o},\, v + v_{\rm o}) \qquad \text{for any } z_{\rm o}, z \in \mathbb{R}^{1+2n}\,, \tag{1.6}$$

and the usual kinetic scaling $\delta_r : \mathbb{R}^{1+2n} \mapsto \mathbb{R}^{1+2n}$ defined by

$$\delta_r(z) := (r^{2s} t,\, r^{1+2s} x,\, r v) \qquad \text{for any } r > 0. \tag{1.7}$$

Also, note that the inverse of each element $z_0 = (t_0, x_0, v_0) \in \mathbb{R}^{1+2n}$ is defined and

$$z_{\rm o}^{-1} \circ z = (t - t_{\rm o}, x - x_{\rm o} - (t - t_{\rm o}) v_{\rm o}, v - v_{\rm o}) \qquad \text{for any } z = (t, x, v) \in \mathbb{R}^{1+2n}.$$

Then for any $r > 0$, we denote by Q_r a cylinder centered in the origin of radius r; that is,

$$Q_r \equiv Q_r(0) := U_r(0,0) \times B_r(0) = (-r^{2s}, 0] \times B_{r^{1+2s}}(0) \times B_r(0)\,.$$

For every $z_{\rm o} \in \mathbb{R}^{1+2n}$ and for every $r > 0$, the *slanted* cylinder $Q_r(z_{\rm o})$ is defined as follows,

$$Q_r(z_{\rm o}) := \Big\{ z := (t, x, v) \in \mathbb{R}^{1+2n} : -r^{2s} < t - t_{\rm o} \le 0, \\ |x - x_{\rm o} - (t - t_{\rm o}) v_{\rm o}| < r^{1+2s},\ |v - v_{\rm o}| < r \Big\}. \tag{1.8}$$

We denote with $\boldsymbol{d}$ the *homogeneous dimension* related to (1.7) defined as

$$\boldsymbol{d} := n(2 + 2s) + 2s. \tag{1.9}$$

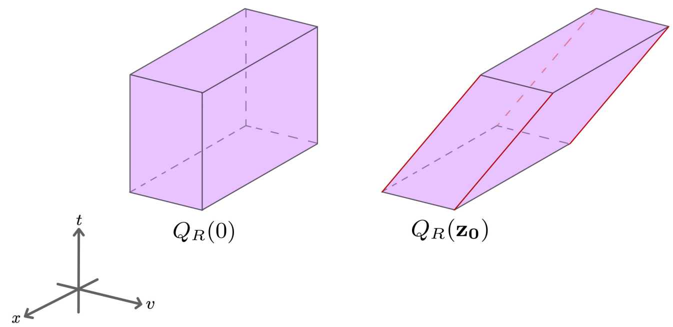

FIGURE 1. On the left the cylinder $Q_R(0)$ centered at the origin; on the right a *slanted* cylinder $Q_R(\mathbf{z}_\mathrm{o}) \equiv Q_R(t_\mathrm{o}, x_\mathrm{o}, v_\mathrm{o})$ according to the invariant transformation given in (1.8).

Such quantity encodes the scaling properties of the underlying kinetic scalings. Indeed, we have that $|Q_r| = r^{\boldsymbol{d}}|Q_1|$, and in general $|\delta_r(\Omega)| = r^{\boldsymbol{d}}|\Omega|$, for any Lebesgue measurable sets $\Omega \subset \mathbb{R}^{1+2n}$.

Moreover, as expected when dealing with nonlocal operators, to control the growth of solutions at infinity we consider "the *nonlocal tail* of a function f centred in $v_\mathrm{o} \in \Omega_v \subset \mathbb{R}^n$ of diffusion radius r", which is given by

$$\mathrm{Tail}(f; B_r(v_\mathrm{o})) := r^{2s} \int_{\mathbb{R}^n \setminus B_r(v_\mathrm{o})} |f(v)||v_\mathrm{o} - v|^{-n-2s}\,\mathrm{d}v. \tag{1.10}$$

The nonlocal tail was firstly defined in the purely p-fractional elliptic setting in [18, 19] and subsequently proven to be decisive in the analysis of many other nonlocal problems when a fine quantitative control of the naturally arising long-range interactions is needed; see, e. g., [11, 14, 52], and the references therein.

In order to overcome the nonlocality issues mentioned above (which also prevent a strong Harnack inequality from Hölder estimates), in the present paper we prove a totally new δ-interpolative L^∞-inequality with tail for weak subsolutions to (1.1); also, possibly unbounded source terms h are taken into account. The parameter $0 < \delta \leq 1$ in such a boundedness estimate can be suitably chosen in order to balance in a quantitative way the local contributions and the nonlocal ones; see in particular the right-side of inequality (1.12) in the theorem below; that is, the L^p-norm along the drift variables of the nonlocal Tail-quantity in velocity. Moreover, in order to keep track of the behavior of our estimates at large velocities, we denote with the bracket $\langle \cdot \rangle := (1 + |\cdot|^2)^{\frac{1}{2}}$. Here below it is our first main result and it constitutes a veritable novelty in the whole kinetic integral panorama.

Theorem 1.1 (The δ-interpolative L^2-L^∞ estimate)**.** *Let $\Omega := (t_1, t_2) \times \Omega_x \times \Omega_v \subset \mathbb{R}^{1+2n}$ be a domain, $s \in (0, 1)$ and let $\boldsymbol{d}$ be the homogeneous dimension in (1.9). Assume that $f \in \mathcal{W}$ is a weak subsolution to*

$$(\partial_t + v \cdot \nabla_x) f = \mathcal{L}_v f + h \quad \text{in } \Omega.$$

and let

$$p > \frac{2s}{\boldsymbol{d}}. \tag{1.11}$$

If $\mathrm{Tail}(f_+;B) \in L^p_{\rm loc}((t_1,t_2)\times\Omega_x)$ *for any* $B \Subset \Omega_v$ *and* $h \in L^p_{\rm loc}(\Omega)$*, for some* p *satisfying* (1.11)*, then, for any* $Q_r(z_{\rm o}) \Subset \Omega$ *and any* $\delta \in (0,1]$*, it holds*

$$
\sup_{Q_{\frac{r}{2}}(z_{\rm o})} f \quad \leq \quad c\left(\frac{\langle v_{\rm o}\rangle}{\delta r^{n+2s}}\right)^{\frac{dp}{2sp-d}} \|f_+\|_{L^2(Q_r(z_{\rm o}))} + \|h\|_{L^p(Q_r(z_{\rm o}))} + \delta\|\,\mathrm{Tail}(f_+;B_{\frac{r}{2}}(v_{\rm o}))\|_{L^p(U_r(t_{\rm o},x_{\rm o}))}\,, \tag{1.12}
$$

where $c \equiv c(n,s,\Lambda,p) > 0$

The finiteness of the L^p-energy of the tail term is a turning point in the local analysis of (1.1). This is in contrast with most of the parabolic literature, where nonlocal effects have been compensated via a supremum tail, which apparently does the trick coupled with further global assumptions on the solution, despite not natively arising from the scaling of the involved equations. Such a L^∞-Tail choice appears very strong and easily adaptable to obtain several estimates even for solutions to (1.1). Nevertheless, it is a concrete stumbling block to concretize our program to obtain also a strong Harnack inequality under light nonlocal assumptions. On the contrary, the L^1 boundedness of the Tail would have been a borderline result, being critical with respect to kinetic scalings; see [46]. Then, by working on the p-summability in transport of the Tail contribution we are able to find a balance for such a discrepancy, in turn also dealing with the combined effects due to the transport term of the equation. Accordingly, a couple of additional remarks are in order.

Remark 1.2. Firstly, it is possible to check that (1.12) is not in contrast with the stationary situation presented in [46, Theorem 1.1]; see the comments after forthcoming Theorem 1.5 on Page 8 for further details. Moreover, even if by definition weak solutions to (1.1) are not required to have finite L^p-energy of their nonlocal tail in velocity, the usual constraints on the notable hydrodynamic observable required in physical models for the Boltzmann equation without cut-off and related kinetic equations plainly imply our requirements on the L^p-energy of the nonlocal tail, see for instance the condition on the mass as in [64, Theorems 1.1-1.2], [57, Formula (1.4)], [29, Formula (1.3)], [36, Formula (1.3)], [37, Section 1.4-Assumption (H)], [38, Section 1.3], [39, Assumption 1.1], [59, Formula (1.2)], [26, Formula (1.9)] and so on.

Remark 1.3. The lower bound on the integrability condition of tail in (1.11) is the expected one. Indeed, the $\mathrm{Tail}(\cdot)$ essentially behaves as the source term h. Hence, one can note that in complete analogy to the (local) ultraparabolic case, (1.11) is the correct integrability assumption on the source to guarantee boundedness of solutions; see [8, 66]. Moreover, if one restricts to fractional parabolic equations, than (1.11) becomes

$$
p > \frac{n+2s}{2s}\,,
$$

which is the analogous lower bound on the integrability of the source term to guarantee boundedness of solutions; see [46, Lemma 3.2]. Lastly, as proven in [46], for stationary solutions of (1.1), the estimate (1.12) is generally false whenever the Tail belongs to L^p, for $p < \frac{n(1+2s)}{2s}$.

The proof of Theorem 1.1 relies on a fine De Giorgi-type recursive argument taking into account both the L^p-energy of the Tail term and the desired interpolative effect. However, the starting point in our proof is far from the usual elliptic or parabolic strategy since the diffusion operator is localized in time and in space, and this precludes a plain application of Sobolev inequality. In fact, the backbone of the related iterative procedure is an hypoelliptic gain of Sobolev regularity whose proof extends and refines similar results in the Kolmogorov-Fokker-Planck framework. In this respect, it is worth recalling the original result for solutions to the Boltzmann equation without cut-off by Imbert and Silvestre – see in particular Lemma 6.1 and Proposition 2.2 in [38] – which in turn reminds of the strategy in [60] by making use of the so-called parametrix of (1.1); i. e., the fundamental solution of the fractional Kolmogorov equation.

Such an integrability gain result is obtained also by proving a suitable kinetic Cacciop-poli estimate with tail, and it is presented in Theorem 1.4 below. We stress that the maximal summability exponent that appears below is the expected one, as also anticipated in Remark 1.3. For this, we believe that our result could be of independent interest.

Theorem 1.4 (Local gain of integrability). *Let $\Omega := (t_1, t_2) \times \Omega_x \times \Omega_v \subset \mathbb{R}^{1+2n}$ be a domain and $s \in (0,1)$. Assume that $f \in \mathcal{W}$ is a weak subsolution to*

$$(\partial_t + v \cdot \nabla_x) f = \mathcal{L}_v f + h \quad \text{in } \Omega.$$

Then, begin $\boldsymbol{d}$ the homogeneous dimension in (1.9), it holds that $f \in L^q_{\rm loc}(\Omega)$ for all

$$2 \leq q \leq 2\left(1 + \frac{4s}{\boldsymbol{d} - 4s}\right). \tag{1.13}$$

Furthermore, for any $p > 2$ such that $\mathrm{Tail}(f_+; B) \in L^p_{\rm loc}((t_1,t_2) \times \Omega_x)$, for any $B \Subset \Omega_v$, $h \in L^p_{\rm loc}(\Omega)$, any $Q_r(z_{\rm o}) \Subset \Omega$, the following estimate does hold

$$\begin{aligned} \|(f-\kappa)_+\|_{L^q(Q_\varrho(z_{\rm o}))} \leq\ & \frac{c\,\langle v_{\rm o}\rangle}{(r-\varrho)^{2(n+2s)}} \|(f-\kappa)_+\|_{L^2(Q_r(z_{\rm o}))} \\ & + \frac{c\,|Q_r(z_{\rm o}) \cap \{f > \kappa\}|^{\frac12 - \frac1p}}{r-\varrho} \|h\|_{L^p(Q_r(z_{\rm o}))} \\ & + \frac{c\,|Q_r(z_{\rm o}) \cap \{f > \kappa\}|^{\frac12 - \frac1p}}{(r-\varrho)^{2(n+2s)}} \|\mathrm{Tail}((f-\kappa)_+; B_r(v_{\rm o}))\|_{L^p(U_r(t_{\rm o}, x_{\rm o}))}, \end{aligned}$$

for any $\kappa \in \mathbb{R}$, any $\varrho \in (0,r)$ and where the constants $c \equiv c(n,s,\Lambda) > 0$.

As expected, the feasibility of the result in Theorem 1.1 above will allow us to bypass the global boundedness assumption on the solutions f usually assumed in previous kinetic literature, in turn being fundamental in order to prove several estimates for solutions to (1.1) as those presented right below in Theorem 1.5. Eventually, considering null source term h and no a priori boundedness assumptions for solutions f to (1.1), we are able to prove a new (possibly sharp) formulation of the classical strong Harnack inequality for kinetic equations with nonlocal diffusion, provided only the local summability assumption on the tail discussed in Remarks 1.2 and 1.3. Hence, our third main result reads as follows,

Theorem 1.5 (The Strong Harnack inequality). *Let $\Omega := (t_1, t_2) \times \Omega_x \times \Omega_v \subset \mathbb{R}^{1+2n}$ be a domain, $Q_2(0) \Subset \Omega$, and $s \in (0,1)$. Assume that $f \in \mathcal{W}$ is a globally nonnegative weak solution to*

$$(\partial_t + v \cdot \nabla_x) f = \mathcal{L}_v f \quad \text{in } \Omega.$$

If $\mathrm{Tail}(f;B) \in L^p_{\mathrm{loc}}((t_1,t_2)\times\Omega_x)$*, for any* $B \Subset \Omega_v$*, for some* $p>2$ *satisfying* (1.11) *then there exists* $r_{\mathrm{o}} \in (0,1)$ *depending only on* n *and* s *such that*

$$\sup_{Q^-_{r_{\mathrm{o}}}} f \quad \leq \quad c \inf_{Q^+_{r_{\mathrm{o}}}} f + c\,\|\,\mathrm{Tail}(f;B_{r_{\mathrm{o}}/2}(0))\|_{L^p(U_{r_{\mathrm{o}}}(-1+r_{\mathrm{o}}^{2s},0))}\,, \tag{1.14}$$

where $c \equiv c(n,s,p,\Lambda)>0$*, and*

$$Q^-_{r_{\mathrm{o}}} := (-1,-1+r_{\mathrm{o}}^{2s}] \times B_{r_{\mathrm{o}}^{1+2s}} \times B_{r_{\mathrm{o}}} \quad \text{and} \quad Q^+_{r_{\mathrm{o}}} := (-r_{\mathrm{o}}^{2s},0] \times B_{r_{\mathrm{o}}^{1+2s}} \times B_{r_{\mathrm{o}}}. \tag{1.15}$$

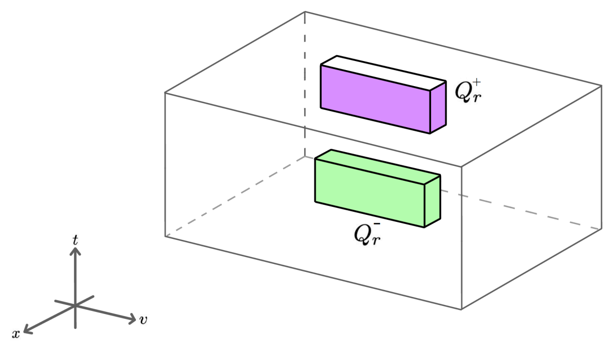

FIGURE 2. The geometry of the Harnack inequalities for kinetic equations.

As natural when dealing with fractional problems, it is usually the negativity of solutions which does interfere with the validity of Harnack inequalities, and $\mathrm{Tail}(f_-)$ is the decisive player in such a game, in order to compensate the possible negative interactions of the solution at infinity which can pull the infimum down, in turn leading to the failure of the Harnack inequality in the elliptic case [41,43]. However, in striking contrast with its elliptic and parabolic counterparts ([18,45]), even when restricting to globally nonnegative solutions, a nonlocal reminder still persists in the estimate. Thus, it has been fundamental our detection of such a precise quantity which controls the combined anisotropic and nonlocal behaviour of (1.1), as seen in the model example (1.5), which in turn takes part to the failure of the classical Harnack estimate. Furthermore, our new tail formulation in both Theorem 1.1 and Theorem 1.5 is somehow sharp, in the sense that for the aforementioned sequence of stationary solutions $\{f_\varepsilon\}$ in [46] the quantity

$$\frac{\sup f_\varepsilon}{\inf f_\varepsilon + \|\mathrm{Tail}(f_\varepsilon)\|_{L^p}} < \infty \qquad \text{as } \varepsilon \searrow 0\,.$$

Remark 1.6 (Hölder continuity as a corollary)**.** As expected, by combining our L^2-L^∞ estimate (1.12) with the weak Harnack inequality in [38], one can prove in a quantitative way the Hölder continuity of weak solutions to (1.1) by also dropping the usual a priori boundedness assumption common in previous kinetic literature; see [65, Theorem 1.1], [55, Theorem 1.2] and [38, Theorem 1.5]. This is easily done via the now classical Moser scheme by simply checking the validity of the so-called Growth Lemma.

It is worth stressing that the tail term in our formulation does not interfere with the expected applications, as already mentioned in the preceding remark in order to obtain the Hölder continuity. In this respect, as another concrete consequence, we are able to extend to our context the classical geometric characterization of the Harnack inequality in the same spirit of the seminal paper [9] for parabolic equations as well as in the important counterpart in the local ultraparabolic framewok given in the relevant paper [61]. Indeed, thanks to Chow's Lemma one can infer that $\mathbb{R}^{1+2n}$ is connected with respect to the group associated to the underlying Lie algebra, and we can consider suitable integral curves in order to state a geometric Harnack inequality characterizing the involved sets in such inequality. We refer the reader to Theorem 4.7 in Section 4.2.

1.3. **Some further developments.** We believe our whole approach and new general independent results to be the starting point in order to attack several *open problems* related to nonlocal kinetic equations, as, e. g., those listed below.

• By replacing the linear diffusion class of fractional operators with nonlinear p-Laplacian-type operators, done in the parabolic setting in [53, 54]. The nonlinear growth p framework in those Gagliardo seminorms seems to be not so far from that presented there in the superquadratic case when $p > 2$; the singular case when $1 < p < 2$ being trickier. However, several "linear" fractional techniques are not applicable; it is no accident that Harnack inequalities are still not available even in the space homogeneous counterpart; say, in the parabolic setting. Nevertheless, our estimates and the techniques employed in order to treat nonlinear fractional parabolic equations in [53] might be a first outset for dealing with the fractional counterpart of nonlinear Kolmogorov-type operators.

• In accordance with the spirit of related results, as for instance the Harnack inequalities in [1, 24] and in [40], one could consider to attack the problem in (1.1) via a viscosity approach, in the same flavor of the Krylov-Safonov approach presented in [17, 63] for general integro-differential equations. This is however a difficult problem even for the case of local diffusion for general hypoelliptic equations in non-divergenge form.

• Similar results can be expected for energy solutions to a family of kinetic equations strictly related to (1.1), which arises from different physical models by replacing the drift with a more general term as $\partial_t + b(v) \cdot \nabla_x$, including more general physical settings, as e. g. considering possibly relativistic effects. Classical regularity theory has been developed in the local case in [67]; see also [7] for Harnack inequality and lower bound of the fundamental solution for the relativistic Fokker-Planck operator.

• In the spirit of very recent advancement of gradient regularity estimates for Fokker-Planck equations, it would be natural to wonder whether the same results obtained for (local) nonlinear Fokker-Planck equations in [47] do hold in the nonlocal case as well. Also, comparing them to the recent development of nonlocal potential estimates; see the results in [20, 21] for elliptic and parabolic equations.

• Most of the forthcoming estimates in the present paper would be still valid by weakening the pointwise control in (1.3) from below, and by assuming appropriate coercivity, local integral boundedness and cancelation properties. On the contrary, the pointwise control from above by a Gagliardo-type kernel, is strongly employed throughout this work, and therefore not easily disposable. One can be interested in working with more general kernel as the ones employed in [44] for nonlocal parabolic equations.

• Our estimates could be the basis in order to prove a Gehring-type lemma for kinetic integral equations, which, as well as their counterpart in the nonlocal elliptic framework

([18, 19]) constitutes a fundamental tool in order to detect such self-improving property ([51]); see also the different approach in the very relevant paper [62] via a robust nonlocal nonlinear commutator estimate concerning the transfer of derivatives onto test functions. Such a result will be thus the natural nonlocal version of the very recent result for classical kinetic Fokker-Planck equations presented in [32].

• Our result in Theorem 1.5 could be of some feasibility even to apparently unrelated problems, as, a concrete example, in the mean fields game theory. It is known that under specific assumptions, mean field games can be seen as a coupled system of two equations, a Fokker-Planck-type equation evolving forward in time (governing the evolution of the density function of the agents), and a Hamilton-Jacobi-type equation evolving backward in time (governing the computation of the optimal path for the agents). Such a forward vs. backward propagation in time should lead to interesting phenomena in time which are present in nature, but they have not been investigated in the nonlocal context yet. Our contribution in the present manuscript together with other recent results and new techniques as the ones developed in [15, 16, 28] could be unexpectedly helpful for such an intricate investigation.

• Finally, it is well known about the many direct consequences and applications of a strong Harnack inequality, as for instance, maximum principles, eigenvalues estimates, Liouville-type theorems, comparison principles, global integrability, and so on. For a discussion of certain of the aforementioned PDE aspects in the local counterpart we refer to [48].

1.4. **Outline of the paper.** In Section 2 below we briefly fix the notation, we introduce the relevant function spaces, together with the related weak formulation, as well as recalling some preliminary results. In Section 3 we prove the gain of integrability for weak subsolutions to (1.1), see Theorem 1.4, and the L^2-L^∞ estimate in Theorem 1.1. Section 4 is devoted to the completion of the proof of Theorem 1.5 as well as of its geometric version

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2. Notation and preliminaries

In this section we fix notation and we briefly recall the necessary information on the underlying functional framework required to deal with (1.1).

We denote with c a positive universal constant greater than one, which may change from line to line. For the sake of readability, dependencies of the constants will be often omitted within the chains of estimates, therefore stated after the estimate. Relevant dependencies on parameters will be emphasized by using parentheses. For any $D \subset \mathbb{R}^n$ we define with χ_D the indicator function of D. As customary, for any $r > 0$ and any $y_o \in \mathbb{R}^n$ we denote by $B_r(y_o) \equiv B(y_o; r) := \big\{ y \in \mathbb{R}^n \,:\, |y - y_o| < r \big\}$, the open ball with radius r and center y_o. We shall often abbreviate $B_1 \equiv B_1(0)$, where we denote with $0_{\mathbb{R}^n} := 0$. For any measurable function g, we define the positive and negative part of g as $g_\pm(y) := \max\{\pm g(y), 0\}$.

For $s \in (0,1)$ we denote with $W^{s,2}(D)$ the classical fractional Sobolev space

$$W^{s,2}(D) := \Big\{ f \in L^2(D) \,:\, [f]_{s,2;D} < +\infty \Big\}\,,$$

where the fractional seminorm $[f]_{s,2;D}$ is the usual one via Gagliardo kernels

$$[f]_{s,2;D} := \left(\int_D \int_D \frac{|f(v) - f(w)|^2}{|v - w|^{n+2s}} \,\mathrm{d}v \,\mathrm{d}w \right)^{1/2},$$

and where we have equipped $W^{s,2}$ with the usual norm

$$\|f\|_{W^{s,2}(D)} := \|f\|_{L^2(D)} + [f]_{s,2;D}\,.$$

In order to lightened the notation we will often denote with $[g]_{s,2} \equiv [g]_{s,2;\mathbb{R}^n}$. A function f belongs to $W^{s,2}_{\mathrm{loc}}(D)$ if $f \in W^{s,2}(D')$ whenever $D' \Subset D$.

We will denote with $W^{-s,2}(\mathbb{R}^n)$ the dual of $W^{s,2}(\mathbb{R}^n)$ and denote with $\langle \cdot | \cdot \rangle$ the usual duality paring between $W^{-s,2}$ and $W^{s,2}$. For any $f \in W^{s,2}(\mathbb{R}^n)$ we define $\mathcal{L}_v f$ as an element of $W^{-s,2}(\mathbb{R}^n)$ that acts on $\phi \in W^{s,2}(\mathbb{R}^n)$ via

$$\langle \mathcal{L}_v f | \phi \rangle = \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (f(v) - f(w))(\phi(v) - \phi(w)) K(v,w) \,\mathrm{d}v \,\mathrm{d}w.$$

We shall often denote by $\mathcal{E}_D(\cdot)$ the nonlocal energy above

$$\mathcal{E}_D(f, \phi) := \frac{1}{2} \int\!\!\int_D \big(f(v) - f(w)\big)\big(\phi(v) - \phi(w)\big) K(v,w) \,\mathrm{d}v \,\mathrm{d}w\,,$$

for any $D \subseteq \mathbb{R}^n \times \mathbb{R}^n$. In the case when $D = \mathbb{R}^n \times \mathbb{R}^n$ or $D = B_r \times B_r$ we simply write $\mathcal{E}(f,\phi) := \mathcal{E}_{\mathbb{R}^n \times \mathbb{R}^n}(f, \phi)$ and $\mathcal{E}_{B_r}(f,\phi) := \mathcal{E}_{B_r \times B_r}(f,\phi)$.

Consider the following tail space

$$L^1_{2s}(\mathbb{R}^n) := \left\{ g \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \,:\, \int_{\mathbb{R}^n} \frac{|g(v)|}{(1 + |v|)^{n+2s}} \,\mathrm{d}v < \infty \right\},$$

as firstly defined in [49]; see Section 2 in [50] for related properties.

Given $\Omega := (t_1, t_2) \times \Omega_x \times \Omega_v \subset \mathbb{R}^{1+2n}$, we denote by $\mathcal{W}$ the natural functions space to which weak solutions to (1.1) belong to, and defined as

$$\begin{aligned} \mathcal{W} := \Big\{ f \in L^2_{\mathrm{loc}}((t_1,t_2) \times \Omega_x; W^{s,2}_{\mathrm{loc}}(\Omega_v)) \cap L^1_{\mathrm{loc}}((t_1,t_2) \times \Omega_x; L^1_{2s}(\mathbb{R}^n)) \\ : (\partial_t + v \cdot \nabla_x) f \in L^2_{\mathrm{loc}}((t_1,t_2) \times \Omega_x; W^{-s,2}(\mathbb{R}^n)) \Big\}. \end{aligned}$$

We are now in the position to recall the definition of weak sub- and supersolution.

Definition 2.1. *A function $f \in \mathcal{W}$ is a* weak subsolution (*resp.,* supersolution) *to* (1.1) *in Ω with $h \in L^2(\Omega)$ if*

$$\int_{t_1}^{t_2} \int_{\Omega_x} \mathcal{E}\left(f, \phi\right) \mathrm{d}x \,\mathrm{d}t + \int_{t_1}^{t_2} \int_{\Omega_x} \langle f_t + v \cdot \nabla_x f | \phi \rangle \,\mathrm{d}x \,\mathrm{d}t \overset{(\leq,\ resp.)}{\geq} \int_\Omega h \,\phi \,\mathrm{d}z\,,$$

for any nonnegative $\phi \in L^2_{\rm loc}((t_1,t_2)\times\Omega_x;W^{s,2}(\mathbb{R}^n))$ *such that* $\operatorname{supp}(\phi(t,x,\cdot)) \Subset \Omega_v$ *for a. e.* $(t,x)\in(t_1,t_2)\times\Omega_x$. *A function* $f\in\mathcal{W}$ *is a* weak solution *to* (1.1) *if it is both a weak sub- and supersolution.*

As already remarked in the introduction, in order to establish the gain of integrability in Theorem 1.4 we need to invoke the hypoelliptic nature of equation (1.1); see [35]. Indeed, we shall rely on the regularizing properties of the fundamental solution of the fractional Kolmogorov equation. For some $h\in L^2(\mathbb{R}^{1+2n})$, we consider

$$(\partial_t + v\cdot\nabla_x)f + (-\Delta_v)^s f = h.$$

This equation admits a fundamental solution

$$P(z) := \begin{cases} \dfrac{c(n)}{t^{n+\frac{n}{s}}}\mathcal{P}\left(\dfrac{x}{t^{1+\frac{1}{2s}}};\dfrac{v}{t^{\frac{1}{2s}}}\right) & \text{if } t>0\,,\\ 0 & \text{if } t\le 0\,, \end{cases} \tag{2.1}$$

where the kernel $\mathcal{P}$ is defined in Fourier variables

$$\mathscr{F}[\mathcal{P}](\eta,\theta) := \exp\left(-\int_0^1 |\eta+\tau\theta|^{2s}\,\mathrm{d}\tau\right).$$

We refer the reader to [30] for the derivation of the fundamental solution in (2.1) together with polynomial upper and lower bound; see also [34] for an alternative proof via probabilistic methods. Then, the following result holds true

Lemma 2.2 (Proposition 1.11 in [27])**.** *Let* $\beta\in(0,\boldsymbol{d})$, *with* $\boldsymbol{d}$ *being the homogeneous dimension in* (1.9) *and let* $P\in C(\mathbb{R}^{1+2n}\setminus\{0\})$ *be a* δ_r*-homogeneous kernel of degree* $\beta-\boldsymbol{d}$. *If* $h\in L^2(\mathbb{R}^{1+2n})$, *then*

$$f(z) = \int_{\mathbb{R}^{1+2n}} P(z_1^{-1}\circ z)h(z_1)\,\mathrm{d}z_1$$

is defined almost everywhere and there exists a (universal) constant $c>0$ *such that*

$$\|f\|_{L^q(\mathbb{R}^{1+2n})} \le c\,\|h\|_{L^2(\mathbb{R}^{1+2n})}\,,$$

where

$$\frac{1}{q} = \frac{1}{2} - \frac{\beta}{\boldsymbol{d}}.$$

Given δ_r as in (1.7) we have that

$$P(\delta_r(z)) = r^{-n(2+2s)}P(z).$$

So, using the same notation as in Lemma 2.2 above, we obtain that $\beta=2s$. So, given

$$f(z) = \int_{\mathbb{R}^{1+2n}} P(z_1^{-1}\circ z)h(z_1)\,\mathrm{d}z_1$$

we get that

$$\|f\|_{L^q(\mathbb{R}^{1+2n})} \le c\,\|h\|_{L^2(\mathbb{R}^{1+2n})} \quad \text{for } q = 2\left(1+\frac{4s}{\boldsymbol{d}-4s}\right). \tag{2.2}$$

Moreover, the following weak Harnack inequality holds true, even in the more intricate Boltzmann non-cutoff context. We took the liberty to adjust the statement below in view of our setting, and in accordance with our boundedness result in Theorem 1.1.

Theorem 2.3 (Theorem 1.6 in [38])**.** *Let $s \in (0,1)$. Then, there exist $r_{\rm o}$, $\bar R > 1$, ζ and c such that if $f \in \mathbb{W}$ is a globally nonnegative weak supersolution to*

$$(\partial_t + v \cdot \nabla_x) f = \mathcal{L}_v f \quad \text{in}\, [-1,0] \times B_{\bar R^{1+2s}} \times B_{\bar R}\,,$$

then,

$$\left(\int_{Q^-_{r_{\rm o}}} f^\zeta(z)\,{\rm d}z \right)^{\frac{1}{\zeta}} \leq c \inf_{Q^+_{r_{\rm o}}} f\,,$$

where $Q^\pm_{r_{\rm o}}$ are defined in (1.15) and the constant c and ζ depends only on n, s and Λ, while $r_{\rm o}$ and $\bar R$ depends only on n and s.

For an alternative proof of the weak Harnack estimate we refer the reader to [4].

We recall some results on the geometry of the slanted cylinders in (1.8). Firstly, we state a covering property. For a similar result in the classic kinetic framework we refer to [60, Lemma 4.2]; see also [6].

Lemma 2.4. *There exist two universal constants $c_* \equiv c_*(s) \in (0,1)$ and $\beta \equiv \beta(s) \geq 1$ such that, for any $1/2 \leq \varrho < r \leq 1$ and any $z_{\rm o} \in \mathbb{R}^{1+2n}$, it holds*

$$Q_{(c_*(r-\varrho))^\beta}(z_1) \subset Q_r(z_{\rm o}) \qquad \forall z_1 \in Q_\varrho(z_{\rm o}). \tag{2.3}$$

Proof. Define $c_*(s) := \min\{1, 2s\}$ and $\beta(s) := \max\{1, \frac{1}{2s}\}$. Let us note that for any $z_1 = (t_1, x_1, v_1) \in Q_\varrho(z_{\rm o})$ and any $z_2 = (t_2, x_2, v_2) \in Q_{(c_*(r-\varrho))^\beta}(z_1)$ we have

$$t_1 \in (t_{\rm o} - \varrho^{2s}, t_{\rm o}], \quad v_1 \in B_\varrho(v_{\rm o}) \quad \text{and} \quad |x_1 - x_{\rm o} - (t_1 - t_{\rm o})v_{\rm o}| < \varrho^{1+2s}. \tag{2.4}$$

Next, note that, when $s \in [1/2, 1)$ we have

$$\begin{aligned} r^{2s} - \varrho^{2s} &= 2s \left(\int_0^1 (\varrho + \sigma(r-\varrho))^{2s-1}\,{\rm d}\sigma \right)(r-\varrho) \\ &\geq 2s \left(\int_0^1 \sigma^{2s-1}\,{\rm d}\sigma \right)(r-\varrho)^{2s} = (r-\varrho)^{2s}\,, \end{aligned} \tag{2.5}$$

whereas, when $s \in (0, 1/2)$, since $\varrho + \sigma(r-\varrho) \leq r \leq 1$ for any $\sigma \in (0,1)$, we have

$$r^{2s} - \varrho^{2s} = 2s \left(\int_0^1 \frac{1}{(\varrho + \sigma(r-\varrho))^{1-2s}}\,{\rm d}\sigma \right)(r-\varrho) \geq 2s(r-\varrho). \tag{2.6}$$

Hence,

$$v_2 \in B_{(c_*(r-\varrho))^\beta}(v_1) \quad \subseteq \begin{cases} B_{(r-\varrho)}(v_1) & \text{if } s \in [1/2,1) \\ B_{(2s(r-\varrho))^{\frac{1}{2s}}}(v_1) & \text{if } s \in (0,1/2) \end{cases} \subseteq B_{(r-\varrho)}(v_1) \subseteq B_r(v_{\rm o})\,,$$

where in the case $s \in (0,1/2)$ we have used that $(2s(r-\varrho))^{\frac{1}{2s}} = (2s(r-\varrho))^{\frac{1}{2s}-1} 2s(r-\varrho) \leq (r-\varrho)$, given that $\frac{1}{2s} > 1$. Moreover, by combining (2.4), (2.5) and (2.6), we have for the time interval

$$\begin{cases} t_2 \in (t_1 - (r-\varrho)^{2s}, t_1] & \text{if } s \in [1/2,1) \\ t_2 \in (t_1 - 2s(r-\varrho), t_1] & \text{if } s \in (0,1/2) \end{cases} \quad \subset \quad (t_{\rm o} - r^{2s}, t_{\rm o}] \quad \forall s \in (0,1).\,,$$

whereas for the spatial variables

$$
\begin{aligned}
|x_2 - x_{\rm o} - (t_2 - t_{\rm o})v_{\rm o}| &\leq |x_2 - x_1 - (t_2 - t_1)v_1| + |x_1 - x_{\rm o} - (t_1 - t_{\rm o})v_{\rm o}| \\
&\quad + |(t_2 - t_1)(v_1 - v_{\rm o})| \\
&\leq \begin{cases} (r-\varrho)^{1+2s} + \varrho^{1+2s} + (r-\varrho)^{2s}\varrho & \text{if } s \in [1/2, 1) \\ \big(2s(r-\varrho)\big)^{\frac{1+2s}{2s}} + \varrho^{1+2s} + 2s(r-\varrho)\varrho & \text{if } s \in (0, 1/2) \end{cases} \\
&\leq \begin{cases} (r-\varrho)^{2s} r + \varrho^{1+2s} & \text{if } s \in [1/2, 1) \\ 2s(r-\varrho) r + \varrho^{1+2s} & \text{if } s \in (0, 1/2) \end{cases} \\
&\leq r^{1+2s} \quad \forall s \in (0,1)\,,
\end{aligned}
$$

since in a similar way $\big(2s(r-\varrho)\big)^{\frac{1}{2s}} = \big(2s(r-\varrho)\big)^{\frac{1}{2s}-1+1} \leq 2s(r-\varrho)$ given that $\frac{1}{2s} - 1 > 0$ when $s \in (0, \frac{1}{2})$. □

Lastly, we conclude by stating some classical iteration argument which will turn out to be useful in establishing our main results.

Lemma 2.5 (see, e. g., Lemma 2.7 in [18]). *Let $\alpha > 0$ and let $\{Y_j\}_{j\in\mathbb{N}}$ be a sequence of positive real numbers such that*

$$
Y_{j+1} \leq c_* \, b^j \, Y_j^{1+\alpha},
$$

with $c_* > 0$, $b > 1$. If

$$
Y_{\rm o} \leq c_*^{-\frac{1}{\alpha}} b^{-\frac{1}{\alpha^2}},
$$

then $\lim\limits_{j\to\infty} Y_j = 0$.

Lemma 2.6 (see, e. g., Lemma 4.11 in [14]). *Let $\Psi : [\varrho, r] \to [0, +\infty)$ be a bounded function, $\varepsilon \in (0,1)$, A_1, A_2, A_3, β_1, $\beta_2 \geq 0$ and $\varrho, r > 0$. Assume that*

$$
\Psi(\sigma') \leq \varepsilon \Psi(\sigma) + \frac{A_1}{(\sigma - \sigma')^{\beta_1}} + \frac{A_2}{(\sigma - \sigma')^{\beta_2}} + A_3\,,
$$

holds for every $\varrho \leq \sigma' < \sigma \leq r$. Then,

$$
\Psi(\varrho) \leq c \left[\frac{A_1}{(r-\varrho)^{\beta_1}} + \frac{A_2}{(r-\varrho)^{\beta_2}} + A_3 \right],
$$

where $c \equiv c(\varepsilon, \beta_1, \beta_2) > 0$.

3. Local gain of integrability and local boundedness estimates

This section is devoted to the proof of the gain of integrability for subsolutions to (1.1), as stated in Theorem 1.4, which constitutes an important step in the subsequent proof of the L^2-L^∞ (interpolative) estimate. Let us just remind that for the sake of readability we will simply denote with $f(t,x,v) = f(v)$, just to differentiate between the double variables appearing in the nonlocal energy $\mathcal{E}(\cdot)$ in the weak formulation of (1.1).

Firstly, we need a precise Caccioppoli-type estimates with tail for subsolutions to (1.1).

Lemma 3.1 (Energy estimates). *Let $Q_1 \equiv Q_1(0) \subset \mathbb{R}^{1+2n}$ and $s \in (0,1)$. Assume that $f \in \mathcal{W}$ is a weak subsolution to (1.1) in Q_1. Then, for any $0 < \varrho < r \leq 1$, any $p > 2$, any $\kappa \in \mathbb{R}$*

and any cut-off function $\phi \in C_c^\infty(B_{(\frac{\varrho+r}{2})^{1+2s}} \times B_{\frac{\varrho+r}{2}})$*, it holds*

$$
\begin{aligned}
&\sup_{t\in[-r^{2s},0]} \int_{B_{r^{1+2s}}\times B_r} (\phi F_+)^2 \,\mathrm{d}v\,\mathrm{d}x + \int_{U_r} \mathcal{E}(\phi F_+, \phi F_+)\,\mathrm{d}x\,\mathrm{d}t \\
&\leq \frac{c}{(r-\varrho)^{2(1+2s)}} \int_{Q_r} F_+^2 \,\mathrm{d}z + c\,|Q_r \cap \{f>k\}|^{1-\frac{2}{p}} \left(\int_{Q_r} h^p \,\mathrm{d}z\right)^{\frac{2}{p}} \\
&\quad + \frac{c\,|Q_r \cap \{f>k\}|^{1-\frac{2}{p}}}{(r-\varrho)^{2(n+2s)}} \left(\int_{U_r} \mathrm{Tail}(F_+;B_r)^p \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{2}{p}},
\end{aligned} \tag{3.1}
$$

and

$$
\begin{aligned}
&\sup_{t\in[-r^{2s},0]} \int_{B_{r^{1+2s}}\times B_r} (\phi F_+)^2 \,\mathrm{d}v\,\mathrm{d}x + \int_{U_r} \mathcal{E}_{B_r}(\phi F_+, \phi F_+)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + c\,r^{-n-2s} \int_{U_r}\int_{B_r}\int_{B_r} F_-(v)F_+(w)\,\mathrm{d}w\,\mathrm{d}z \\
&\leq \int_{B_r^{1+2s}\times B_r} F_+(-r^{2s})\,\mathrm{d}v\,\mathrm{d}x + \frac{c}{(r-\varrho)^{2(1+2s)}} \int_{Q_r} F_+^2\,\mathrm{d}z \\
&\quad + \frac{c\,|Q_r \cap \{f>k\}|^{1-\frac{2}{p}}}{(r-\varrho)^{2(n+2s)}} \left(\int_{U_r} \mathrm{Tail}(F_+;B_r)^p \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{2}{p}} \\
&\quad + c\,|Q_r \cap \{f>k\}|^{1-\frac{2}{p}} \left(\int_{Q_r} h^p\,\mathrm{d}z\right)^{\frac{2}{p}},
\end{aligned} \tag{3.2}
$$

where we denote with $F_\pm := (f-\kappa)_\pm$ *and* $c \equiv c(n,s,\Lambda) > 0$.

Proof. Let $0<\varrho<r<1$ and let consider a cut-off function such that

$$
\begin{cases}
\phi \in C_c^\infty(B_{((\frac{\varrho+r}{2})^{1+2s}} \times B_{\frac{\varrho+r}{2}}), \\
0 \leq \phi \leq 1 \text{ and } \phi \equiv 1 \text{ on } B_{\varrho^{1+2s}} \times B_\varrho \\
|\nabla_v \phi| \leq c/(r-\varrho) \text{ and } |v\cdot\nabla_x\phi| \leq c/(r-\varrho)^{1+2s}.
\end{cases}
$$

Consider in the weak formulation a test function $F_+\phi^2$, up to mollification (see for instance [33, Section 2] and [5, Section 3]). Then, for a. e. $t \in (-r^{2s},0]$ it yields

$$
\begin{aligned}
\int_{B_{r^{1+2s}}\times B_r} h\phi^2 F_+\,\mathrm{d}x\,\mathrm{d}v \quad \geq \quad & \int_{B_{r^{1+2s}}\times B_r} (f_t + v\cdot\nabla_x f)\phi^2 F_+\,\mathrm{d}x\,\mathrm{d}v \\
& + \int_{B_{r^{1+2s}}} \mathcal{E}(f, \phi^2 F_+)\,\mathrm{d}x =: I_1 + I_2.
\end{aligned} \tag{3.3}
$$

We start by considering I_1. Using the fact that $\partial_t\phi = 0$, and that, by [38, Formula (A8)],

$$
(\partial_t + v\cdot\nabla_x)\,F_+ = (\partial_t + v\cdot\nabla_x)\,F_+\chi_{\{f>\kappa\}}\,,
$$

we have that

$$
\begin{aligned}
I_1 \quad &\geq \quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_{r^{1+2s}}\times B_r} (\phi F_+)^2\,\mathrm{d}x\,\mathrm{d}v - \int_{B_{r^{1+2s}}\times B_r} |v\cdot\nabla_x\phi| F_+^2\,\mathrm{d}x\,\mathrm{d}v \\
&\geq \quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_{r^{1+2s}}\times B_r} (\phi F_+)^2\,\mathrm{d}x\,\mathrm{d}v - \frac{c}{(r-\varrho)^{1+2s}}\int_{B_{r^{1+2s}}\times B_r} F_+^2\,\mathrm{d}x\,\mathrm{d}v.
\end{aligned} \tag{3.4}
$$

Now we focus on the term I_2, by adapting the same argument used for fractional parabolic equations in order to estimate the nonlocal energy $\mathcal{E}$; see in particular [44]. Start splitting

$$\mathcal{E}(f,\phi^2F_+) = \mathcal{E}_{B_r}(f,\phi^2F_+) + 2\mathcal{E}_{B_r\times(\mathbb{R}^n\setminus B_r)}(f,\phi^2F_+)\,. \tag{3.5}$$

We start recalling the following algebraic identities

$$\begin{aligned} f(v)-f(w) &= (F_+(v)-F_+(w)) - (F_-(v)-F_-(w))\,,\\ (F_+(v)-F_+(w))(\phi^2F_+(v)-\phi^2F_+(w)) &= (\phi F_+(v)-\phi F_+(w))^2\\ &\quad -F_+(v)F_+(w)(\phi(v)-\phi(w))^2\,, \end{aligned}$$

from which we actually obtain

$$\begin{aligned} &(f(v)-f(w))(\phi^2F_+(v)-\phi^2F_+(w))\\ &\qquad = -(F_-(v)-F_-(w))(\phi^2F_+(v)-\phi^2F_+(w))\\ &\qquad\quad + (\phi F_+(v)-\phi F_+(w))^2 - F_+(v)F_+(w)(\phi(v)-\phi(w))^2\,, \end{aligned}$$

which actually yields

$$\begin{aligned} \mathcal{E}_{B_r}(f,\phi^2F_+) &= \mathcal{E}_{B_r}(\phi F_+,\phi F_+) - \mathcal{E}_{B_r}(F_-,\phi^2F_+)\\ &\quad - c\int_{B_r}\int_{B_r} F_+(v)F_+(w)(\phi(v)-\phi(v))^2K(v,w)\,\mathrm{d}w\,\mathrm{d}v\\ &\geq \mathcal{E}_{B_r}(\phi F_+,\phi F_+) + cr^{-n-2s}\int_{B_r}\int_{B_r} F_-(v)F_+(w)\,\mathrm{d}w\,\mathrm{d}v\\ &\quad - \frac{c}{(r-\varrho)^2}\int_{B_r} F_+^2(v)\,\mathrm{d}v\,, \end{aligned}$$

where in the last line we have used that, since K is symmetric, up to exchange the roles of v and w we can assume that $F_+(v)\geq F_+(w)$, so that

$$\begin{aligned} &\int_{B_r}\int_{B_r} F_+(v)F_+(w)(\phi(v)-\phi(v))^2K(v,w)\,\mathrm{d}w\,\mathrm{d}v\\ &\qquad\leq \int_{B_r}\int_{2B_r(v)} F_+^2(v)\frac{\|\nabla_v\phi\|^2|v-w|^2}{|v-w|^{n+2s}}\,\mathrm{d}w\,\mathrm{d}v\\ &\qquad\leq \frac{c}{(r-\varrho)^2}\int_{B_r} F_+^2(v)\left(\int_{B_{2r}(v)}\frac{\mathrm{d}w}{|v-w|^{n-2(1-s)}}\right)\mathrm{d}v\\ &\qquad\leq \frac{c}{(r-\varrho)^2}\int_{B_r} F_+^2(v)\,\mathrm{d}v \end{aligned} \tag{3.6}$$

Now, we estimate the other part of the nonlocal energy. Since on the level set when $f(v)>\kappa$ it holds $f(w)-f(v)\leq f(w)-\kappa$ for a. e. $w\in\mathbb{R}^n\setminus B_r$, we have that

$$\begin{aligned} \mathcal{E}_{B_r\times(\mathbb{R}^n\setminus B_r)}(f,\phi^2F_+) &= -\int_{B_r}\int_{\mathbb{R}^n\setminus B_r}(f(w)-f(v))\phi^2F_+(v)K(v,w)\,\mathrm{d}w\,\mathrm{d}v\\ &\geq -\int_{B_r}\int_{\mathbb{R}^n\setminus B_r} F_+(w)\phi^2F_+(v)K(v,w)\,\mathrm{d}w\,\mathrm{d}v. \end{aligned} \tag{3.7}$$

Combining (3.4) and (3.5), (3.6) and (3.7) in (3.3) yields

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_{r^{1+2s}}\times B_r}(\phi F_+)^2\,\mathrm{d}x\,\mathrm{d}v &+ \int_{B_{r^{1+2s}}}\mathcal{E}_{B_r}(\phi F_+,\phi F_+)\,\mathrm{d}x\\
&+c\,r^{-n-2s}\int_{B_r^{1+2s}}\int_{B_r}\int_{B_r}F_-(v)F_+(w)\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x\\
&\leq c\int_{B_{r^{1+2s}}}\int_{B_r}\int_{\mathbb{R}^n\setminus B_r}F_+(w)\phi^2F_+(v)K(v,w)\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x\\
&+\left(\frac{c}{(r-\varrho)^{1+2s}}+\frac{c}{(r-\varrho)^2}\right)\int_{B_{r^{1+2s}}\times B_r}F_+^2\,\mathrm{d}v\,\mathrm{d}x\\
&+c\int_{B_{r^{1+2s}}\times B_r}hF_+\,\mathrm{d}x\,\mathrm{d}v.
\end{aligned}
$$

After integration in time and easy manipulations as in [65, Lemma 2.2], we arrive at

$$
\begin{aligned}
\sup_{t\in[-r^{2s},0]}\int_{B_{r^{1+2s}}\times B_r}(\phi F_+)^2\,\mathrm{d}z &+ \int_{U_r}\mathcal{E}_{B_r}(\phi F_+,\phi F_+)\,\mathrm{d}x\,\mathrm{d}t\\
&+c\,r^{-n-2s}\int_{U_r}\int_{B_r}\int_{B_r}F_-(v)F_+(w)\,\mathrm{d}w\,\mathrm{d}z\\
&\leq \int_{B_r^{1+2s}\times B_r}F_+(-r^{2s})\,\mathrm{d}v\,\mathrm{d}x\\
&+c\int_{U_r}\int_{B_r}\int_{\mathbb{R}^n\setminus B_r}F_+(w)\phi^2F_+(v)K(v,w)\,\mathrm{d}w\,\mathrm{d}z\\
&+\left(\frac{c}{(r-\varrho)^{1+2s}}+\frac{c}{(r-\varrho)^2}\right)\int_{Q_r}F_+^2\,\mathrm{d}z+c\int_{Q_r}hF_+\,\mathrm{d}z.
\end{aligned}
\tag{3.8}
$$

In order to conclude we just estimate the nonlocal contribution and the contribution given by the source term. Let us first note that since $\phi(v)=0$ on $\mathbb{R}^n\setminus B_{(r+\varrho)/2}$, we have that $I_2\neq 0$ only on $Q_r\cap\mathrm{supp}(\phi)$. Then, by applying Hölder's Inequality twice with $\left(\frac{p}{2},\frac{p}{p-2}\right)$, with $p>2$, we obtain

$$
\begin{aligned}
\int_{U_r}\int_{B_r}\int_{\mathbb{R}^n\setminus B_r}&F_+(w)\phi^2F_+(v)K(v,w)\,\mathrm{d}w\,\mathrm{d}z\\
&\leq\left(\int_{Q_r}F_+^2\,\mathrm{d}z\right)^{\frac12}\left[\int_{Q_r\cap\mathrm{supp}(\phi)}\left(\int_{\mathbb{R}^n\setminus B_r}\frac{F_+(w)}{|v-w|^{n+2s}}\,\mathrm{d}w\right)^2\chi_{\{f>\kappa\}}\,\mathrm{d}z\right]^{\frac12}\\
&\leq\frac{c\,|Q_r\cap\{f>\kappa\}|^{\frac12-\frac1p}}{(r-\varrho)^{n+2s}}\left(\int_{Q_r}F_+^2\,\mathrm{d}z\right)^{\frac12}\left(\int_{U_r}\mathrm{Tail}(F_+;B_r)^p\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac1p}\\
&\leq c\int_{Q_r}F_+^2\,\mathrm{d}z+\frac{c\,|Q_r\cap\{f>\kappa\}|^{1-\frac2p}}{(r-\varrho)^{2(n+2s)}}\left(\int_{U_r}\mathrm{Tail}(F_+;B_r)^p\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac2p},
\end{aligned}
\tag{3.9}
$$

where in the last display we have used Young's Inequality and we have centered the Gagliardo kernel since for any $v \in B_r \cap \operatorname{supp}(\phi) \subset B_{(r+\varrho)/2}$ and any $w \in \mathbb{R}^n \setminus B_r$, it holds

$$\frac{|w|}{|v-w|} \leq 1 + \frac{|v|}{||w|-|v||} \leq 1 + \frac{r+\varrho}{r-\varrho} = \frac{c\,r}{r-\varrho}.$$

As for the source contributions we estimate via Hölder's and Young's Inequality

$$\begin{aligned}
\int_{Q_r} hF_+ \,\mathrm{d}z &\leq \left(\int_{Q_r} h^2 \chi_{\{f>\kappa\}} \,\mathrm{d}z\right)^{\frac12} \left(\int_{Q_r} F_+^2 \,\mathrm{d}z\right)^{\frac12} \\
&\leq \left(\int_{Q_r} h^p \,\mathrm{d}z\right)^{\frac1p} \left(\int_{Q_r} F_+^2 \,\mathrm{d}z\right)^{\frac12} |Q_r \cap \{f>\kappa\}|^{\frac12-\frac1p} \\
&\leq c\,|Q_r \cap \{f>\kappa\}|^{1-\frac2p} \left(\int_{Q_r} h^p \,\mathrm{d}z\right)^{\frac2p} + \int_{Q_r} F_+^2 \,\mathrm{d}z. \qquad (3.10)
\end{aligned}$$

The energy estimate (3.2) follows by combining (3.8), (3.9) and (3.10). For the estimate in (3.1), we can proceed in a similar way: the only thing that differs is in the way we have to treat the nonlocal energy. Indeed, by symmetry of the involved kernel K, we can simply restrict on the subcase when $f(v) \geq f(w)$ (up to exchange the roles of v and w), and thus we simply use that

$$\begin{aligned}
&\big(f(v)-f(w)\big)\big(\phi^2F_+(v)-\phi^2F_+(w)\big) \\
&\quad = \big((f(v)-\kappa)-(f(w)-\kappa)\big)\big(\phi^2F_+(v)-\phi^2F_+(w)\big) \\
&\quad = \begin{cases} \big(F_+(v)-F_+(w)\big)\big(\phi^2F_+(v)-\phi^2F_+(w)\big) & \text{if } f(v)\geq f(w)>\kappa, \\ \big(f(v)-f(w)\big)\phi^2F_+(v) & \text{if } f(v)>\kappa\geq f(w), \\ 0 & \text{if } \kappa \geq f(v) \geq f(w), \end{cases} \\
&\quad \geq \begin{cases} \phi^2F_+^2(v)+\phi^2F_+^2(w)-F_+(w)\phi^2F_+(v)-F_+(v)\phi^2F_+(w) & \text{if } f(v)\geq f(w)>\kappa, \\ \phi^2F_+^2(v) & \text{if } f(v)>\kappa\geq f(w), \\ 0 & \text{if } \kappa \geq f(v) \geq f(w), \end{cases} \\
&\quad = \big(F_+\phi(v)-F_+\phi(w)\big)^2 - F_+(v)F_+(w)\big(\phi(v)-\phi(w)\big)^2,
\end{aligned}$$

which yields the bound

$$\begin{aligned}
I_2 &= \int_{B_{r^{1+2s}}} \mathcal{E}(\phi F_+,\phi F_+)\,\mathrm{d}x - \int_{B_{r^{1+2s}}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} F_+(v)F_+(w)(\phi(v)-\phi(w))^2 K(v,w)\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x \\
&= \int_{B_{r^{1+2s}}} \mathcal{E}(\phi F_+,\phi F_+)\,\mathrm{d}x - \int_{B_{r^{1+2s}}} \int_{B_r}\int_{B_r} F_+(v)F_+(w)(\phi(v)-\phi(w))^2 K(v,w)\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x \\
&\quad - \int_{B_{r^{1+2s}}} \int_{B_r}\int_{\mathbb{R}^n\setminus B_r} F_+(v)F_+(w)\phi^2(v) K(v,w)\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x\,,
\end{aligned}$$

and then treat the diagonal contributions as in (3.6) and the off-diagonal as in (3.9). □

3.1. **Proof of Theorem 1.4.** We apply the group law (1.6), in order to center the main cylinder in the origin. Indeed, note that $f_{z_o}(z) := f(z_o \circ z)$ satisfies

$$\partial_t f_{z_o} + (v+v_o)\cdot\nabla_x f_{z_o} = \mathcal{L}_v f_{z_o} + h_{z_o} \quad \text{in } z_o^{-1}\circ\Omega\,,$$

where $h_{z_o}(z) := h(z_o \circ z)$. Now, fix $0 < \varrho < r < 1$ such that $Q_r \equiv Q_r(0) \subset z_o^{-1} \circ \Omega$, and define $\sigma_1 := \varrho + (r-\varrho)2^{-3}$ and $\sigma_2 := \varrho + (r-\varrho)2^{-2}$, so that $0 < \varrho < \sigma_1 < \sigma_2 < r < 1$. Define now, two cut-off functions ψ and φ such that:

$$\begin{cases} \psi = \psi(x,v) \in C_c^\infty(B_{(\frac{\varrho+\sigma_1}{2})^{1+2s}} \times B_{\frac{\varrho+\sigma_1}{2}}) \\ \psi \equiv 1 \text{ on } B_{\varrho^{1+2s}} \times B_\varrho \text{ and } 0 \le \psi \le 1\,, \\ |\nabla_v \psi| \le c/(r-\varrho) \text{ and } |(v+v_o)\cdot\nabla_x \psi| \le c\,\langle v_o\rangle/(r-\varrho)^{1+2s} \end{cases}$$

and

$$\begin{cases} \varphi = \varphi(x,v) \in C_c^\infty(B_{(\frac{r+\sigma_2}{2})^{1+2s}} \times B_{\frac{r+\sigma_2}{2}}) \\ \varphi \equiv 1 \text{ on } B_{\sigma_2^{1+2s}} \times B_{\sigma_2} \text{ and } 0 \le \varphi \le 1\,, \\ |\nabla_v \varphi| \le c/(r-\varrho) \text{ and } |(v+v_o)\cdot\nabla_x \varphi| \le c\,\langle v_o\rangle/(r-\varrho)^{1+2s}. \end{cases}$$

Moreover, we consider a cut-off function η to restrict ourselves to a compact set in t, defined as

$$\begin{cases} \eta \in C_c^\infty(-(\frac{\varrho+\sigma_1}{2})^{2s}, 0]) \\ \eta \equiv 1 \text{ on } [-\varrho^{2s}, 0] \text{ and } 0 \le \eta \le 1\,, \\ |\partial_t \eta| \le c/(r-\varrho)^{2s}. \end{cases}$$

Define $F_+ := (f_{z_o} - \kappa)_+$ and let us apply now (3.1), with the choice above of φ, σ_2 and r, obtaining

$$\begin{aligned} &\sup_{t\in[-r^{2s},0]} \int_{B_{r^{1+2s}}\times B_r} (\varphi F_+)^2 \,\mathrm{d}v\,\mathrm{d}x + \int_{U_r} \mathcal{E}(\varphi F_+, \varphi F_+)\,\mathrm{d}x\,\mathrm{d}t \\ &\le \frac{c\,\langle v_o\rangle}{(r-\varrho)^{2(1+2s)}} \int_{Q_r} F_+^2 \,\mathrm{d}z + c\,|Q_r \cap \{f_{z_o} > \kappa\}|^{1-\frac{2}{p}} \left(\int_{Q_r} h_{z_o}^p \,\mathrm{d}z \right)^{\frac{2}{p}} \\ &\quad + \frac{c\,|Q_r \cap \{f_{z_o} > \kappa\}|^{1-\frac{2}{p}}}{(r-\varrho)^{2(n+2s)}} \left(\int_{U_r} \mathrm{Tail}(F_+; B_r)^p \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{2}{p}}. \end{aligned} \tag{3.11}$$

Moreover, applying once again (3.1) with ψ, ϱ and σ_1, we arrive at

$$\begin{aligned} \int_{U_{\sigma_1}} \mathcal{E}(\psi F_+, \psi F_+)\,\mathrm{d}x\,\mathrm{d}t \quad &\le \quad \frac{c\,\langle v_o\rangle^2}{(r-\varrho)^{2(1+2s)}} \int_{Q_r} F_+^2 \,\mathrm{d}z \\ &\quad + c\,|Q_r \cap \{f_{z_o} > \kappa\}|^{1-\frac{2}{p}} \left(\int_{Q_r} h_{z_o}^p \,\mathrm{d}z \right)^{\frac{2}{p}} \\ &\quad + \frac{c\,|Q_r \cap \{f_{z_o} > \kappa\}|^{1-\frac{2}{p}}}{(r-\varrho)^{2(n+2s)}} \left(\int_{U_r} \mathrm{Tail}(F_+; B_r)^p \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{2}{p}}\,, \end{aligned} \tag{3.12}$$

where $c \equiv c(n,s,\Lambda) > 0$.

Thus, we look at what equation the function $g := \eta\psi F_+$ satisfies. Let us apply the transport operator to g. Then, distributionally,

$$\begin{aligned} (\partial_t + (v+v_o)\cdot\nabla_x) g &= F_+\big(\partial_t + (v+v_o)\cdot\nabla_x)(\eta\psi)\big) + \eta\psi(\partial_t + (v+v_o)\cdot\nabla_x) F_+ \\ &= F_+\big(\partial_t + (v+v_o)\cdot\nabla_x)(\eta\psi)\big) + \eta\psi\chi_{\{f_{z_o}>\kappa\}}(\partial_t + (v+v_o)\cdot\nabla_x)(f_{z_o}-\kappa) \\ &\le F_+\big(\partial_t + (v+v_o)\cdot\nabla_x)(\eta\psi)\big) + \eta\psi\chi_{\{f_{z_o}>\kappa\}}\mathcal{L}_v(f_{z_o}-\kappa) + \eta\psi\chi_{\{f_{z_o}>\kappa\}} h_{z_o} \end{aligned} \tag{3.13}$$

Now, we employ a proper integration by parts of $\mathcal{L}_v$,

$$\mathcal{L}_v[(f_{z_o}-\kappa)\psi] = \psi\mathcal{L}_v[(f_{z_o}-\kappa)] + (f_{z_o}-\kappa)\mathcal{L}_v\psi + \mathcal{I}((f_{z_o}-\kappa), \psi)\,, \tag{3.14}$$

where $\mathcal{I}((f_{z_o}-\kappa),\psi)$ is a remainder term defined as

$$\mathcal{I}((f_{z_o}-\kappa),\psi) := \int_{\mathbb{R}^n} \big((f_{z_o}(v)-\kappa)-(f_{z_o}(w)-\kappa)\big)\big(\psi(v)-\psi(w)\big)K(v,w)\,\mathrm{d}w,$$

together with the following pointwise inequality (see [12]),

$$\chi_{\{\Psi>0\}}\mathcal{L}_v\Psi \leq \mathcal{L}_v\Psi_+. \tag{3.15}$$

Indeed, by (3.14) together with (3.15) we have that the nonlocal diffusion in (3.13) can be bounded as follows (recalling that η depends only on time),

$$\begin{aligned}
\eta\psi\chi_{\{f_{z_o}>\kappa\}}\mathcal{L}_v(f_{z_o}-\kappa) &= \chi_{\{f_{z_o}>\kappa\}}\mathcal{L}_v[(f_{z_o}-\kappa)\eta\psi] - (\eta(f_{z_o}-\kappa)\mathcal{L}_v\psi)\chi_{\{f_{z_o}>\kappa\}} \\
&\quad -(\eta\mathcal{I}((f_{z_o}-\kappa),\psi))\chi_{\{f_{z_o}>\kappa\}} \\
&\leq \mathcal{L}_v g + \eta\left(\int_{\mathbb{R}^n} F_+(w)\big(\psi(v)-\psi(w)\big)K(v,w)\,\mathrm{d}w\right)\chi_{\{f_{z_o}>\kappa\}}\,.
\end{aligned}\tag{3.16}$$

Lastly, let us note that $\mathcal{L}_v g \in L^2(\mathbb{R}^{1+n}; W^{-s,2}(\mathbb{R}^n))$. Indeed, by Cauchy-Schwartz's Inequality and the kernel assumptions in (1.3), we have that for any $\xi \in W^{s,2}(\mathbb{R}^n)$

$$|\langle\mathcal{L}_v g|\xi\rangle| \leq \left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(g(w)-g(v))(\xi(w)-\xi(v))K(v,w)\,\mathrm{d}w\,\mathrm{d}v\right| \leq \Lambda[\xi]_{s,2}[g]_{s,2}\,,$$

which yields

$$\begin{aligned}
\|\mathcal{L}_v g\|^2_{L^2(\mathbb{R}^{1+n};W^{-s,2}(\mathbb{R}^n))} &= \int_{\mathbb{R}^{1+n}} \|\mathcal{L}_v g\|^2_{W^{-s,2}(\mathbb{R}^n)}\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{\mathbb{R}^{1+n}} \sup_{\substack{\xi\in W^{s,2}(\mathbb{R}^n)\\ \|\xi\|_{s,2}\leq 1}} |\langle\mathcal{L}_v g|\xi\rangle|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\leq c\int_{Q_r} g^2\,\mathrm{d}z + c\int_{U_r}[g]^2_{s,2}\,\mathrm{d}x\,\mathrm{d}t\,,
\end{aligned}\tag{3.17}$$

recalling the choice of the cut-off η and ψ.

Now, consider a translation in time $\tilde{g} := g\big(t+(\frac{\varrho+\sigma_1}{2})^{2s},x,v\big)$ so that at time $t = -(\frac{\varrho+\sigma_1}{2})^{2s}$ we are in the origin. Given the invariance of the equation with respect to time translations, combining also the previous chain of estimates (3.16) with (3.13) yields that $\tilde{g}$ satisfies distributionally

$$\begin{aligned}
(\partial_t+(v+v_o)\cdot\nabla_x)\tilde{g}+(-\Delta_v)^s\tilde{g} &\leq F_+\big((\partial_t+(v+v_o)\cdot\nabla_x)(\eta\psi)\big)+\eta\psi\chi_{\{f_{z_o}>\kappa\}}h_{z_o} \\
&\quad +(\mathcal{L}_v+(-\Delta_v)^s)\tilde{g} \\
&\quad +\eta\left(\int_{\mathbb{R}^n}F_+(w)\big(\psi(w)-\psi(v)\big)K(v,w)\,\mathrm{d}w\right)\chi_{\{f_{z_o}>\kappa\}} \\
&=: H
\end{aligned}$$

In order to apply Lemma 2.2, define G to be the solution to

$$\begin{cases}
(\partial_t+(v+v_o)\cdot\nabla_x)G+(-\Delta_v)^sG = H & \text{in } [0,\infty]\times\mathbb{R}^{2n}, \\
G(0,x,v)=\tilde{g}(0,x,v)=g\left(-(\frac{\varrho+\sigma_1}{2})^{2s},x,v\right)=0 & \text{in } \mathbb{R}^{2n}, \\
G(t,x,v)=0 & \text{in } (-\infty,0)\times\mathbb{R}^{2n}.
\end{cases}$$

Assuming $H \in L^2(\mathbb{R}^{1+2n})$, such a solution G does exist. Moreover, by maximum principle as in [38, Lemma A.12], applied there to the function $\tilde{g}-G$, we get $G \geq \tilde{g} \geq 0$. Hence, in order to derive integrability conditions on $\tilde{g}$ (and in turn on g) it is enough to apply Lemma 2.2 on G. By (2.2) we finally obtain

$$\|F_+\|^2_{L^q(Q_\varrho)} \ \le \ \|G\|^2_{L^q(Q_\varrho)} \ \le \ \|G\|^2_{L^q(\mathbb{R}^{1+2n})} \ \le \ c\,\|H\|^2_{L^2(\mathbb{R}^{1+2n})}\,,$$

where the exponent q is such that

$$\frac{1}{q}=\frac{1}{2}-\frac{2s}{\boldsymbol{d}}\,,$$

with $\boldsymbol{d}$ being the homogeneous dimension defined in (1.9).

Notice now that we can estimate the L^2-norm of H as follows,

$$\begin{aligned}
&\|H\|^2_{L^2(\mathbb{R}^{1+2n})}\\
&\qquad\le c\,\|F_+\left((\partial_t+(v+v_{\rm o})\cdot\nabla_x)(\eta\psi)\right)\|^2_{L^2(Q_r)}\\
&\qquad\quad+c\,\|(\eta\psi h_{z_{\rm o}})\chi_{\{f_{z_{\rm o}}>\kappa\}}\|^2_{L^2(Q_r)}+c\,\|\mathcal{L}_v g\|^2_{L^2(L^2(U_r\times\mathbb{R}^n))}+c\,\|(-\Delta_v)^s g\|^2_{L^2(U_r\times\mathbb{R}^n)}\\
&\qquad\quad+c\,\left\|\left(\int_{\mathbb{R}^n}\frac{F_+(w)(\psi(w)-\psi(v))}{|v-w|^{n+2s}}\,{\rm d}w\right)\chi_{\{f_{z_{\rm o}}>\kappa\}}\right\|^2_{L^2([-\sigma_1^{2s},0]\times\mathbb{R}^{2n})}\,,
\end{aligned}$$

where we have used the definition of η, and the fact that we are truncating the support in time and space. In the display above, $c>0$ is a universal constant. The first term can be plainly estimated by recalling the very definition of the cut-off functions ψ and η, so that

$$\begin{aligned}
\|F_+\left((\partial_t+(v+v_{\rm o})\cdot\nabla_x)(\eta\psi)\right)\|^2_{L^2(Q_r)} \ &= \ \int_{Q_r}F_+^2\left(|\partial_t\eta|^2+|(v+v_{\rm o})\cdot\nabla_x\psi|^2\right){\rm d}z\\
&\le \ \frac{c\,\langle v_{\rm o}\rangle^2}{(r-\varrho)^{2(1+2s)}}\int_{Q_r}F_+^2\,{\rm d}z. \qquad (3.18)
\end{aligned}$$

As for the source term contribution, we apply Hölder's Inequality with $p>2$

$$\begin{aligned}
\|(\eta\psi h_{z_{\rm o}})\chi_{\{f_{z_{\rm o}}>\kappa\}}\|^2_{L^2(Q_r)} \ &\le \ \int_{Q_r\cap\{f_{z_{\rm o}}>\kappa\}}h^2_{z_{\rm o}}\,{\rm d}z\\
&\le \ c\,|Q_r\cap\{f_{z_{\rm o}}>\kappa\}|^{1-\frac{2}{p}}\left(\int_{Q_r}h^p_{z_{\rm o}}\,{\rm d}z\right)^{\frac{2}{p}}\,, \qquad (3.19)
\end{aligned}$$

since $0\le\psi,\eta\le 1$.

In order to estimate the contribution given by $\mathcal{L}_v g$ and by $(-\Delta_v)^s g$ we proceed by duality relying on (3.17). Indeed, we get that

$$\begin{aligned}
&\|\mathcal{L}_v g\|^2_{L^2(U_r\times\mathbb{R}^n)}+\|(-\Delta_v)^2 g\|^2_{L^2(U_r\times\mathbb{R}^n)}\\
&\qquad\le c\,\|\mathcal{L}_v g\|^2_{L^2(U_r;W^{-s,2}(\mathbb{R}^n))}+c\,\|(-\Delta_v)^s g\|^2_{L^2(U_r;W^{-s,2}(\mathbb{R}^n))}\\
&\qquad\le c\int_{Q_r}F_+^2\,{\rm d}z+c\int_{U_r}[g]^2_{s,2}\,{\rm d}x\,{\rm d}t\\
&\qquad\le \frac{c\,\langle v_{\rm o}\rangle^2}{(r-\varrho)^{2(1+2s)}}\int_{Q_r}F_+^2\,{\rm d}z+c\,|Q_r\cap\{f_{z_{\rm o}}>\kappa\}|^{1-\frac{2}{p}}\left(\int_{Q_r}h^p_{z_{\rm o}}\,{\rm d}z\right)^{\frac{2}{p}}\\
&\qquad\quad+\frac{c\,|Q_r\cap\{f_{z_{\rm o}}>\kappa\}|^{1-\frac{2}{p}}}{(r-\varrho)^{2(n+2s)}}\left(\int_{U_r}{\rm Tail}(F_+;B_r)^p\,{\rm d}x\,{\rm d}t\right)^{\frac{2}{p}}.
\end{aligned}$$

where ψ is compactly supported in $B_{\sigma_1^{1+2s}}$, $|\psi|\le 1$ and (3.12).

Finally, it only remains to prove an estimate for the integral term. Firstly, for any fixed v, we split the integral term as follows,

$$
\begin{aligned}
\Big(\int_{\mathbb{R}^n} \frac{F_+(w)|\psi(w)-\psi(v)|}{|v-w|^{n+2s}}\,\mathrm{d}w\Big)\chi_{\{f_{z_o}>\kappa\}}(v) \\
&= \left(\int_{B_{\frac{r-\varrho}{16}}(v)} \frac{F_+(w)|\psi(w)-\psi(v)|}{|v-w|^{n+2s}}\,\mathrm{d}w\right)\chi_{\{f_{z_o}>\kappa\}}(v) \\
&\quad + \left(\int_{\mathbb{R}^n\setminus B_{\frac{r-\varrho}{16}}(v)} \frac{F_+(w)|\psi(w)-\psi(v)|}{|v-w|^{n+2s}}\,\mathrm{d}w\right)\chi_{\{f_{z_o}>\kappa\}}(v) \\
&:= J_1 + J_2\,,
\end{aligned}
$$

so that, for any $v \in \operatorname{supp}(\psi(x,\cdot)) \subset B_{(\varrho+\sigma_1)/2}$ we have that $B_{(r-\varrho)/16}(v) \subset B_{\sigma_2}$. Indeed, for any $w \in B_{(r-\varrho)/2}(v)$, since $\varrho < \sigma_1 < \sigma_2$, we have $|w| \leq |w-v| + |v| \leq \frac{\varrho+\sigma_1}{2} + \frac{r-\varrho}{16} = \varrho + \frac{r-\varrho}{8} = \sigma_2$.

Let us separately estimate the two integrals. We start with the term J_2,

$$
\begin{aligned}
\|J_2\|^2_{L^2([-\sigma_1^{2s},0]\times\mathbb{R}^{2n})} &\leq c\int_{U_{\sigma_1}\times B_{\frac{\varrho+\sigma_1}{2}}} \chi\{f_{z_o}>\kappa\} \left(\int_{\mathbb{R}^n\setminus B_{\frac{r-\varrho}{16}}(v)} \frac{F_+(w)|\psi(v)-\psi(w)|}{|v-w|^{n+2s}}\,\mathrm{d}w\right)^2 \mathrm{d}z \\
&\quad + c\int_{U_{\sigma_1}\times(\mathbb{R}^n\setminus B_{\frac{\varrho+\sigma_1}{2}})} \left(\int_{\mathbb{R}^n\setminus B_{\frac{r-\varrho}{16}}(v)} \frac{F_+(w)|\psi(v)-\psi(w)|}{|v-w|^{n+2s}}\,\mathrm{d}w\right)^2 \mathrm{d}z.
\end{aligned}
$$

Consider the second one in the right-hand side in the preceding inequality. Note that for any $v \in \mathbb{R}^n \setminus B_{(\varrho+\sigma_1)/2}$ the test $\psi(v) = 0$, hence the integral is actually non-zero only when $w \in B_{(\varrho+\sigma_1)/2}$, where ψ is supported. Then, by Hölder's Inequality, we obtain that

$$
\begin{aligned}
\int_{U_{\sigma_1}\times(\mathbb{R}^n\setminus B_{\frac{\varrho+\sigma_1}{2}})} \left(\int_{\mathbb{R}^n\setminus B_{\frac{r-\varrho}{16}}(v)} \frac{F_+(w)\psi(w)}{|v-w|^{n+2s}}\,\mathrm{d}w\right)^2 \mathrm{d}z \\
&\leq \int_{U_{\sigma_1}\times(\mathbb{R}^n\setminus B_{\frac{\varrho+\sigma_1}{2}})} \left(\int_{B_{\frac{\varrho+\sigma_1}{2}}\setminus B_{\frac{r-\varrho}{16}}(v)} \frac{F_+^2(w)\psi^2(w)}{|v-w|^{n+2s}}\,\mathrm{d}w\right) \\
&\quad \times \left(\int_{B_{\frac{\varrho+\sigma_1}{2}}\setminus B_{\frac{r-\varrho}{16}}(v)} \frac{\mathrm{d}w}{|v-w|^{n+2s}}\right)\mathrm{d}z \\
&\leq \frac{c}{(r-\varrho)^{n+2s}} \left(\int_{Q_{\sigma_1}} F_+^2\,\mathrm{d}z\right)\left(\int_{\mathbb{R}^n\setminus B_{\frac{\varrho+\sigma_1}{2}}} \frac{\mathrm{d}v}{|v|^{n+2s}}\right) \\
&\leq \frac{c}{(r-\varrho)^{n+4s}} \int_{Q_r} F_+^2\,\mathrm{d}z\,,
\end{aligned}
$$

where we have used the fact that for any $v \in \mathbb{R}^n \setminus B_{(\varrho+\sigma_1)/2}$ and any $w \in B_{(\varrho+\sigma_1)/2} \setminus B_{(r-\varrho)/16}(v)$ it holds $|v|/|v-w| \leq c/(r-\varrho)$. Then, also using that $\sigma_1 < \sigma_2 < r$, we get

$$
\begin{aligned}
\|J_2\|^2_{L^2([-\sigma_1^{2s},0]\times\mathbb{R}^{2n})} \leq \; & \frac{c}{(r-\varrho)^{n+4s}} \int_{Q_r} F_+^2 \,\mathrm{d}z \\
& + c\int_{U_{\sigma_1}\times B_{\frac{\varrho+\sigma_1}{2}}} \chi_{\{f_{z_o}>\kappa\}} \left(\int_{\mathbb{R}^n\setminus B_{\frac{r-\varrho}{16}}(v)} \frac{F_+(w)|\psi(v)-\psi(w)|}{|v-w|^{n+2s}} \,\mathrm{d}w \right)^2 \mathrm{d}z \,.
\end{aligned}
$$

We now estimate the second term. Since for any $v \in B_{(\varrho+\sigma_1)/2}$, the ball $B_{(r-\varrho)/16}(v) \subset B_r$, so we can split $(\mathbb{R}^n \setminus B_r) \cup \big(B_r \setminus B_{\frac{r-\varrho}{16}}(v)\big) = \mathbb{R}^n \setminus B_{\frac{r-\varrho}{16}}(v)$. Thus, for any $w \in \mathbb{R}^n \setminus B_r$ and any $v \in B_{(\varrho+\sigma_1)/2}$, we re-center the Gagliardo kernel at the origin since $|w|/|v-w| \leq c\, r/(r-\varrho)$ Thus, for any $p>2$, by applying Hölder's Inequality we get

$$
\begin{aligned}
& \int_{U_{\sigma_1}\times B_{\frac{\varrho+\sigma_1}{2}}} \chi_{\{f_{z_o}>\kappa\}} \left(\int_{\mathbb{R}^n\setminus B_{\frac{r-\varrho}{16}}(v)} \frac{F_+(w)|\psi(v)-\psi(w)|}{|v-w|^{n+2s}} \,\mathrm{d}w \right)^2 \mathrm{d}z \\
& \quad \leq c \int_{U_{\sigma_1}\times B_{\frac{\varrho+\sigma_1}{2}}} \chi_{\{f_{z_o}>\kappa\}} \left(\int_{\mathbb{R}^n\setminus B_r} \frac{F_+(w)}{|v-w|^{n+2s}} \,\mathrm{d}w \right)^2 \mathrm{d}z \\
& \qquad + c \int_{U_{\sigma_1}\times B_{\frac{\varrho+\sigma_1}{2}}} \chi_{\{f_{z_o}>\kappa\}} \left(\int_{B_r\setminus B_{\frac{r-\varrho}{16}}(v)} \frac{F_+(w)}{|v-w|^{n+2s}} \,\mathrm{d}w \right)^2 \mathrm{d}z \\
& \quad \leq \frac{c\,|Q_r \cap \{f_{z_o}>\kappa\}|^{1-\frac{2}{p}}}{(r-\varrho)^{2(n+2s)}} \left(\int_{U_r} \mathrm{Tail}(F_+;B_r)^p \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{2}{p}} + \frac{c}{(r-\varrho)^{2(n+2s)}} \int_{Q_r} F_+^2 \,\mathrm{d}z \,.
\end{aligned}
$$

Hence, combining all estimates for J_2 yields that

$$
\begin{aligned}
\|J_2\|^2_{L^2([-\sigma_1^{2s},0]\times\mathbb{R}^{2n})} \leq \; & \frac{c}{(r-\varrho)^{2(n+2s)}} \int_{Q_r} F_+^2 \,\mathrm{d}z \\
& + \frac{c\,|Q_r \cap \{f_{z_o}>\kappa\}|^{1-\frac{2}{p}}}{(r-\varrho)^{2(n+2s)}} \left(\int_{U_r} \mathrm{Tail}(F_+;B_r)^p \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{2}{p}} .
\end{aligned} \tag{3.20}
$$

Let us consider the contributions given by J_1. First let us note that when $v \in \mathbb{R}^n \setminus B_{\sigma_1}$ then $B_{(r-\varrho)/16}(v) \cap \mathrm{supp}(\psi(x,\cdot)) = \emptyset$. Indeed, for any $v \in \mathbb{R}^n \setminus B_{\sigma_1}$ and any $w \in B_{(r-\varrho)/16}(v)$ it holds $|w| \geq |v| - |v-w| \geq \sigma_1 - (r-\varrho)/16 = \varrho + (r-\varrho)/16 = (\varrho+\sigma_1)/2$. Then, for $(v,w) \in (\mathbb{R}^n \setminus B_{\sigma_1}) \times B_{(r-\varrho)/16}(v)$, we have $\psi(w) = \psi(v) = 0$, since $B_{(\varrho+\sigma_1)/2} \subset B_{\sigma_1}$, given that $0 < \varrho < \sigma_1$. Hence, we just restrict J_1 on Q_{σ_1}.

In order to proceed estimating further, we differentiate the cases depending on the range of the differentiability exponent $s \in (0,1)$. Assume that $s \in (0,1/2)$. By Hölder's Inequality we obtain

$$
\begin{aligned}
&\|J_1\|^2_{L^2([-\sigma_1^{2s},0]\times\mathbb{R}^{2n})}\\
&\quad\leq \int_{Q_{\sigma_1}}\left(\int_{B_{\frac{r-\varrho}{16}}(v)}\frac{F_+(w)|\psi(w)-\psi(v)|}{|v-w|^{n+2s}}\,\mathrm{d}w\right)^2\mathrm{d}z\\
&\quad\leq c\int_{Q_{\sigma_1}}\left(\int_{B_{\frac{r-\varrho}{16}}(v)}\frac{F_+^2(w)|\psi(w)-\psi(v)|}{|v-w|^{n+2s}}\,\mathrm{d}w\right)\left(\int_{B_{\frac{r-\varrho}{16}}(v)}\frac{|\psi(w)-\psi(v)|}{|v-w|^{n+2s}}\,\mathrm{d}w\right)\mathrm{d}z\\
&\quad\leq c\int_{Q_{\sigma_1}}\left(\int_{B_{\frac{r-\varrho}{16}}(v)}\frac{F_+^2(w)|\psi(w)-\psi(v)|}{|v-w|^{n+2s}}\,\mathrm{d}w\right)\left(\int_{B_{\frac{r-\varrho}{16}}(v)}\frac{\|\nabla_v\psi\|_{L^\infty}\,\mathrm{d}w}{|v-w|^{n-2(\frac12-s)}}\right)\mathrm{d}z\\
&\quad\leq \frac{c}{r-\varrho}\int_{Q_{\sigma_1}}\int_{B_{\frac{r-\varrho}{16}}(v)}\frac{F_+^2(w)|\psi(w)-\psi(v)|}{|v-w|^{n+2s}}\,\mathrm{d}w\,\mathrm{d}z.
\end{aligned}
$$

Note that for $v\in B_{\sigma_1}\setminus B_{(\varrho+\sigma_1)/2}$, it holds $\psi(v)=0$, hence the integral is non-zero only when $w\in B_{(\varrho+\sigma_1)/2}$. Moreover, for any $v\in B_{\sigma_1}\setminus B_{(\varrho+\sigma_1)/2}$ and any $w\in B_{(\varrho+\sigma_1)/2}$ we have that $|v-w|\leq|v|+|w|\leq 2\sigma_1$, which yields, by Fubini's Theorem recalling that $s\in(0,\frac12)$, that

$$
\begin{aligned}
&\int_{U_{\sigma_1}\times(B_{\sigma_1}\setminus B_{\frac{\varrho+\sigma_1}{2}})}\int_{B_{\frac{r-\varrho}{16}}(v)}\frac{F_+^2(w)|\psi(w)-\psi(v)|}{|v-w|^{n+2s}}\,\mathrm{d}w\,\mathrm{d}z\\
&\qquad\leq \int_{Q_{\sigma_2}}F_+^2\left(\int_{B_{2\sigma_1}(w)}\frac{|\psi(w)|}{|v-w|^{n+2s}}\,\mathrm{d}v\right)\mathrm{d}z\\
&\qquad\leq \int_{Q_{\sigma_2}}F_+^2\left(\int_{B_{2\sigma_1}(w)}\frac{\|\nabla_v\psi\|_{L^\infty}\,\mathrm{d}v}{|v-w|^{n-2(\frac12-s)}}\right)\mathrm{d}z\\
&\qquad\leq \frac{c}{r-\varrho}\int_{Q_r}F_+^2\,\mathrm{d}z.
\end{aligned}
$$

On the other hand, when $v\in B_{(\varrho+\sigma_1)/2}$ we have that $B_{(r-\varrho)/16}(v)\subset B_{\sigma_1}$, and for $w\in B_{\sigma_1}$ the ball $B_{(\varrho+\sigma_1)/2}\subset B_{2\sigma_1}(w)$. Indeed, for $w\in B_{(r-\varrho)/16}(v)$ we have that $|w|\leq \frac{\varrho+\sigma_1}{2}+\frac{r-\varrho}{16}\leq\sigma_2$ and $|v-w|\leq 2\sigma_2$. Then, by Fubini's Theorem, we obtain

$$
\begin{aligned}
&\int_{U_{\sigma_1}\times B_{\frac{\varrho+\sigma_1}{2}}}\int_{B_{\frac{r-\varrho}{16}}(v)}\frac{F_+^2(w)|\psi(w)-\psi(v)|}{|v-w|^{n+2s}}\,\mathrm{d}w\,\mathrm{d}z\\
&\qquad\leq \int_{Q_{\sigma_2}}F_+^2\left(\int_{B_{2\sigma_2}(w)}\frac{|\psi(v)-\psi(w)|}{|v-w|^{n+2s}}\,\mathrm{d}v\right)\mathrm{d}z\\
&\qquad\leq \int_{Q_{\sigma_2}}F_+^2\left(\int_{B_{2\sigma_2}(w)}\frac{\|\nabla_v\psi\|_{L^\infty}\,\mathrm{d}v}{|v-w|^{n-2(\frac12-s)}}\right)\mathrm{d}z\\
&\qquad\leq \frac{c}{r-\varrho}\int_{Q_r}F_+^2\,\mathrm{d}z.
\end{aligned}
$$

All in all, we obtain

$$\|J_1\|^2_{L^2([-\sigma_1^{2s},0]\times\mathbb{R}^{2n})} \leq \frac{c}{(r-\varrho)^2}\int_{Q_r} F_+^2\,\mathrm{d}z. \tag{3.21}$$

Let us focus on the case when $s \in [1/2, 1)$. We estimate the contribution given by J_1 via duality as in [38, Lemma 4.11]. Since $J_1(v)$ is supported in B_{σ_1}, for any $\xi \in W^{s,2}(\mathbb{R}^n)$, we have

$$\begin{aligned}
\int_{B_{\sigma_1}} J_1(v)\xi(v)\,\mathrm{d}v &= \int_{B_{\sigma_1}}\left(\int_{B_{\frac{r-\varrho}{16}}(v)}\frac{\xi(v)F_+(w)(\psi(w)-\psi(v))}{|v-w|^{n+2s}}\,\mathrm{d}w\right)\mathrm{d}v\\
&= \frac{1}{2}\int_{B_{\sigma_1}}\left[F_+(v)\left(\int_{B_{\frac{r-\varrho}{16}}(v)}\frac{(\xi(v)-\xi(w))(\psi(w)-\psi(v))}{|v-w|^{n+2s}}\,\mathrm{d}w\right)\right.\\
&\quad\left.+\xi(v)\left(\int_{B_{\frac{r-\varrho}{16}}(v)}\frac{(F_+(w)-F_+(v))(\psi(w)-\psi(v))}{|v-w|^{n+2s}}\,\mathrm{d}w\right)\right]\mathrm{d}v\\
&= \frac{1}{2}\int_{B_{\sigma_1}}F_+(v)\left(\int_{B_{\frac{r-\varrho}{16}}(v)}\frac{(\xi(v)-\xi(w))(\psi(w)-\psi(v))}{|v-w|^{n+2s}}\,\mathrm{d}w\right)\mathrm{d}v\\
&\quad+\frac{1}{2}\int_{B_{\sigma_1}\setminus B_{\frac{\varrho+\sigma_1}{2}}}\xi(v)\left(\int_{B_{\frac{\varrho+\sigma_1}{2}}\cap B_{\frac{r-\varrho}{16}}(v)}\frac{(F_+(w)-F_+(v))\psi(w)}{|v-w|^{n+2s}}\,\mathrm{d}w\right)\mathrm{d}v\\
&\quad+\frac{1}{2}\int_{B_{\frac{\varrho+\sigma_1}{2}}}\xi(v)\left(\int_{B_{\frac{r-\varrho}{16}}(v)}\frac{(F_+(w)-F_+(v))(\psi(w)-\psi(v))}{|v-w|^{n+2s}}\,\mathrm{d}w\right)\mathrm{d}v\\
&=: J_{1,1}+J_{1,2}+J_{1,3}\,,
\end{aligned} \tag{3.22}$$

since the second integral is non-zero only when $w \in B_{(\varrho+\sigma_1)/2}$ where ψ is supported.

We separately estimate the integrals above. Starting from $J_{1,1}$, by applying Cauchy-Schwartz's Inequality we have that

$$\begin{aligned}
|J_{1,1}| &\leq \frac{c\,[\xi]_{s,2}\|\nabla_v\psi\|_{L^\infty}\|F_+\|_{L^2(B_{\sigma_1})}}{2(1-s)}(r-\varrho)^{1-s}\\
&\leq \frac{c\,[\xi]_{s,2}\|F_+\|_{L^2(B_{\sigma_1})}}{r-\varrho}.
\end{aligned} \tag{3.23}$$

In a similar fashion we can estimate $J_{1,2}$ and $J_{1,3}$. Indeed, for $J_{1,2}$ we have

$$\begin{aligned}
|J_{1,2}| &\leq \frac{c\,\|\xi\|_{L^2(\mathbb{R}^n)}[F_+]_{s,2;B_{\sigma_1}}\|\nabla_v\psi\|_{L^\infty}}{2(1-s)}(r-\varrho)^{1-s}\\
&\leq \frac{c\,\|\xi\|_{L^2(\mathbb{R}^n)}[F_+]_{s,2;B_{\sigma_1}}}{r-\varrho}.
\end{aligned} \tag{3.24}$$

Lastly, noting that for any $v \in B_{(\varrho+\sigma_1)/2}$ it holds $B_{(r-\varrho)/16}(v) \subset B_{\sigma_1}$, we have

$$
\begin{aligned}
|J_{1,3}| &\leq \frac{c\,\|\xi\|_{L^2(\mathbb{R}^n)}[F_+]_{s,2;B_{\sigma_1}}\|\nabla_v\psi\|_{L^\infty}}{2(1-s)}(r-\varrho)^{1-s} \\
&\leq \frac{c\,\|\xi\|_{L^2(\mathbb{R}^n)}[F_+]_{s,2;B_{\sigma_1}}}{r-\varrho}. \qquad (3.25)
\end{aligned}
$$

Then, combining (3.23), (3.24) and (3.25) into (3.22), yields

$$
\sup_{\substack{\xi\in W^{s,2}(\mathbb{R}^n)\\ \|\xi\|_{s,2}\leq 1}} \left| \int_{B_{\sigma_1}} J_1(v)\xi(v)\,\mathrm{d}v \right| \leq \frac{c}{r-\varrho}\Big(\|F_+\|_{L^2(B_{\sigma_1})} + [F_+]_{s,2;B_{\sigma_1}}\Big),
$$

so that, by definition of the test function φ, we obtain

$$
\begin{aligned}
\|J_1\|^2_{L^2([-\sigma_1^{2s},0]\times\mathbb{R}^{2n})} &\leq \frac{c}{(r-\varrho)^2}\left(\int_{Q_{\sigma_1}} F_+^2\,\mathrm{d}z + \int_{U_{\sigma_1}} [F_+]^2_{s,2;B_{\sigma_1}}\,\mathrm{d}x\,\mathrm{d}t\right) \\
&\leq \frac{c}{(r-\varrho)^2}\left(\int_{Q_{\sigma_1}} F_+^2\,\mathrm{d}z + \int_{U_{\sigma_1}} [\varphi F_+]^2_{s,2}\,\mathrm{d}x\,\mathrm{d}t\right). \qquad (3.26)
\end{aligned}
$$

Combining (3.20), (3.21), (3.26) yields

$$
\begin{aligned}
&\|J_1\|^2_{L^2([-\sigma_1^{2s},0]\times\mathbb{R}^{2n})} + \|J_2\|^2_{L^2([-\sigma_1^{2s},0]\times\mathbb{R}^{2n})} \\
&\leq \frac{c\,\langle v_{\mathrm{o}}\rangle^2}{(r-\varrho)^{2(n+2s)}}\int_{Q_r} F_+^2\,\mathrm{d}z + \frac{c}{(r-\varrho)^2}\int_{U_r}[\varphi F_+]^2_{s,2}\,\mathrm{d}x\,\mathrm{d}t \qquad (3.27) \\
&\quad + \frac{c\,|Q_r\cap\{f_{z_{\mathrm{o}}}>\kappa\}|^{1-\frac{2}{p}}}{(r-\varrho)^{2(n+2s)}}\left(\int_{U_r}\mathrm{Tail}(F_+;B_r)^p\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{2}{p}},
\end{aligned}
$$

where $c \equiv c(n,s,\Lambda) > 0$.

Putting together (3.18), (3.20), (3.27) and (3.19), up to relabeling the constant $c \equiv c(n,s,\Lambda) > 0$, we finally obtain

$$
\begin{aligned}
\|F_+\|^2_{L^q(Q_\varrho)} &\leq \frac{c}{(r-\varrho)^2}\left(\sup_{t\in[-r^{2s},0]}\int_{Q_r}(\varphi F_+)^2\,\mathrm{d}x\,\mathrm{d}t + \int_{U_r}[\varphi F_+]^2_{s,2}\,\mathrm{d}x\,\mathrm{d}t\right) \\
&\quad + \frac{c\,|Q_r\cap\{f_{z_{\mathrm{o}}}>\kappa\}|^{1-\frac{2}{p}}}{(r-\varrho)^{2(n+2s)}}\left(\int_{U_r}\mathrm{Tail}(F_+;B_r)^p\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{2}{p}} \\
&\quad + c\,|Q_r\cap\{f_{z_{\mathrm{o}}}>\kappa\}|^{1-\frac{2}{p}}\left(\int_{Q_r} h^p_{z_{\mathrm{o}}}\,\mathrm{d}z\right)^{\frac{2}{p}} + \frac{c\,\langle v_{\mathrm{o}}\rangle^2}{(r-\varrho)^{2(n+2s)}}\int_{Q_r}F_+^2\,\mathrm{d}z.
\end{aligned}
$$

So that, by combining the estimate above with (3.11), in view of the previous choice of test function φ, it yields the desired result up to translating back to z_{o}. □

3.2. **Proof of Theorem 1.1.** Our proof will generalize the one based on the classical iterative scheme with tail as firstly seen in [19], and this will be basically thanks to our integrability gain result in Theorem 1.4, since the standard starting process based on the Sobolev/Poincaré inequality can not be applied.

For any $j \in \mathbb{N}$, define $r_j := \frac{1}{2}(1+2^{-j})r$ and $\kappa_j := (1-2^{-j})\kappa$, where $\kappa > 0$ will be fixed later on. We apply Theorem 1.4 to f_{j+1}, with $\frac{1}{q} = \frac{1}{2} - \frac{2s}{\boldsymbol{d}}$ and with radii r_{j+1} and r_j. Let us begin noticing that

$$
\begin{aligned}
&\|\mathrm{Tail}((f-\kappa_{j+1})_+;B_{r_j}(v_{\rm o}))\|^2_{L^p(U_{r_j}(t_{\rm o},x_{\rm o}))}|Q_{r_j}(z_{\rm o})\cap\{f>\kappa_{j+1}\}|^{1-\frac{2}{p}}\\
&\quad\le \|\mathrm{Tail}(f_+;B_{r/2}(v_{\rm o}))\|^2_{L^p(U_r(t_{\rm o},x_{\rm o}))}|Q_{r_j}(z_{\rm o})\cap\{f-\kappa_j>\kappa_{j+1}-\kappa_j\}|^{1-\frac{2}{p}}\\
&\quad\le |\mathrm{Tail}(f_+;B_{r/2}(v_{\rm o}))\|^2_{L^p(U_r(t_{\rm o},x_{\rm o}))}|Q_{r_j}(z_{\rm o})\cap\{f-\kappa_j>2^{-j-1}\kappa\}|^{1-\frac{2}{p}}\\
&\quad\le c\,2^{2j}\left(\frac{\kappa}{\delta}\right)^2\left(\kappa^{-2}\|(f-\kappa_j)_+\|^2_{L^2(Q_{r_j}(z_{\rm o}))}\right)^{1-\frac{2}{p}}, \qquad (3.28)
\end{aligned}
$$

by applying Chebyschev's Inequality and choosing for any $\delta\in(0,1]$,

$$
\kappa\;\ge\;\delta\|\,\mathrm{Tail}(f_+;B_{r/2}(v_{\rm o}))\|_{L^p(U_r(t_{\rm o},x_{\rm o}))}\qquad\text{for }\delta\in(0,1]\,, \tag{3.29}
$$

In a similar way by also choosing

$$
\kappa\ge\|h\|_{L^p(Q_r(z_{\rm o}))}. \tag{3.30}
$$

we get

$$
\begin{aligned}
&\|h\|^2_{L^p(Q_{r_j}(z_{\rm o}))}|Q_{r_j}(z_{\rm o})\cap\{f>\kappa_{j+1}\}|^{1-\frac{2}{p}}\\
&\quad\le c\,2^{2j}\kappa^2\left(\kappa^{-2}\|(f-\kappa_j)_+\|^2_{L^2(Q_{r_j}(z_{\rm o}))}\right)^{1-\frac{2}{p}}. \qquad (3.31)
\end{aligned}
$$

Then, combining (3.28), (3.31) together with Theorem 1.4 yields that

$$
\begin{aligned}
&\|(f-\kappa_{j+1})_+\|^2_{L^2(Q_{r_{j+1}}(z_{\rm o}))}\\
&\quad\le\|(f-\kappa_j)_+\|^2_{L^q(Q_{r_{j+1}}(z_{\rm o}))}|Q_{r_j}(z_{\rm o})\cap\{f>\kappa_{j+1}\}|^{1-\frac{2}{q}}\\
&\quad\le c_*\,b^j\kappa^2\Big(\kappa^{-2}\|(f-\kappa_j)_+\|^2_{L^2(Q_{r_j}(z_{\rm o}))}\\
&\qquad\qquad+\left(\kappa^{-2}\|(f-\kappa_j)_+\|^2_{L^2(Q_{r_j}(z_{\rm o}))}\right)^{1-\frac{2}{p}}\Big)|Q_{r_j}(z_{\rm o})\cap\{f>\kappa_{j+1}\}|^{1-\frac{2}{q}},
\end{aligned}
$$

for

$$
b\equiv b(n,s)>1\quad\text{and}\quad c_*:=\left(\frac{c\,\langle v_{\rm o}\rangle}{\delta r^{2(n+2s)}}\right)^2>0.
$$

Thus, once defined Y_j

$$
Y_j:=\int_{Q_{r_j}(z_{\rm o})}\frac{(f-\kappa_j)_+^2}{\kappa^2}\,\mathrm{d}z
$$

and applying once again Chebychev's Inequality, we get

$$
Y_{j+1}\;\le c_*\,b^j\left(Y_j^{1+\frac{q-2}{q}}+Y_j^{1+(1-\frac{2}{p}-\frac{2}{q})}\right). \tag{3.32}
$$

Now, note that $1-2/q>0$, given that $q>2$, and that $1/p<1/2-1/q$, which is in fact possible for p large enough, say $p>p^*$, where

$$
\frac{1}{p^*}=\frac{1}{2}-\frac{1}{q}\quad=\quad\frac{1}{2}-\left(\frac{1}{2}-\frac{2s}{\boldsymbol{d}}\right)=\frac{2s}{\boldsymbol{d}}\,,
$$

which derives from the growth gain given by (1.13).

Hence, up to choosing κ such that

$$
\kappa\ge\|f_+\|_{L^2(Q_r(z_{\rm o}))}\,, \tag{3.33}
$$

we can rewrite (3.32) as follows

$$Y_{j+1} \le c_* b^j Y_j^{1+\alpha},$$

for some positive $\alpha \equiv \alpha(n,s) := 1-2/p-2/q = 2(\frac{2s}{\boldsymbol{d}}-\frac{1}{p}) > 0$ and $b>1$. Then, up to choosing

$$\begin{aligned}\kappa := \quad & b^{\frac{1}{2\alpha^2}} c^{\frac{1}{2\alpha}} \left(\frac{\langle v_{\rm o}\rangle}{\delta r^{2(n+2s)}}\right)^{\frac{1}{\alpha}} \|f_+\|_{L^2(Q_r(z_{\rm o}))} + \|h\|_{L^p(Q_r(z_{\rm o}))} \\ & + \delta \,\|\,\mathrm{Tail}(f_+; B_{r/2}(v_{\rm o}))\|_{L^p(U_r(t_{\rm o},x_{\rm o}))},\end{aligned}$$

in clear accordance with (3.29) and (3.33) and (3.30), the iteration argument of Lemma 2.5 yields that $Y_j \to 0$ as $j\to\infty$, which gives the desired result. □

4. Strong Harnack inequality and propagation

This section is devoted to the completion of the proof of the strong Harnack inequality in Theorem 1.5 as well as its geometric version in Theorem 4.7.

4.1. **Proof of Theorem 1.5.** Let $r_{\rm o}$ and ζ be given by the weak Harnack inequality in Theorem 2.3. Set $1/2 \le \sigma' < \sigma \le 1$ and $z_{\rm o} := (-1+r_{\rm o}^{2s},0,0)$ and for any $z_1 \in Q_{\sigma' r_{\rm o}}(z_{\rm o})$ it holds by (2.3) in Lemma 2.4 that $Q_{(c_*(\sigma-\sigma')r_{\rm o})^\beta}(z_1) \subset Q_{\sigma r_{\rm o}}(z_{\rm o})$. Hence, applying (1.12) we get

$$\begin{aligned} f(z_1) \quad \le \quad & \frac{c(\delta)}{[(\sigma-\sigma')r_{\rm o}]^{\frac{\boldsymbol{d}\beta p}{2sp-\boldsymbol{d}}}} \left(\int_{Q_{\sigma r_{\rm o}}(z_{\rm o})} f^2 \,{\rm d}z\right)^{\frac{1}{2}} \\ & + \delta \|\,\mathrm{Tail}(f; B_{(c_*(\sigma-\sigma')r_{\rm o})^\beta}(v_1)\|_{L^p(U_{(c_*(\sigma-\sigma')r_{\rm o})^\beta}(t_1,x_1))}. \end{aligned} \tag{4.1}$$

We estimate the nonlocal term. For a. e. $(t,x) \in U_{(c_*(\sigma-\sigma')r_{\rm o})^\beta}(t_1,x_1) \subset U_{\sigma r_{\rm o}}(-1+r_{\rm o}^{2s},0)$ we have that

$$\begin{aligned} & \big[(c_*(\sigma-\sigma')r_{\rm o})^\beta\big]^{2s} \int_{\mathbb{R}^n \setminus B_{(c_*(\sigma-\sigma')r_{\rm o})^\beta}(v_1)} \frac{f(t,x,w)}{|w-v_1|^{n+2s}}\,{\rm d}v \\ & \quad = \big[(c_*(\sigma-\sigma')r_{\rm o})^\beta\big]^{2s} \int_{B_{\sigma r_{\rm o}} \setminus B_{(c_*(\sigma-\sigma')r_{\rm o})^\beta}(v_1)} \frac{f(t,x,w)}{|w-v_1|^{n+2s}}\,{\rm d}w \\ & \qquad + \big[(c_*(\sigma-\sigma')r_{\rm o})^\beta\big]^{2s} \int_{\mathbb{R}^n \setminus B_{\sigma r_{\rm o}}} \frac{f(t,x,w)}{|w-v_1|^{n+2s}}\,{\rm d}w \\ & \quad \le c \sup_{Q_{\sigma r_{\rm o}}(z_{\rm o})} f + \frac{c\, r_{\rm o}^{2s}}{(\sigma-\sigma')^{n+2s}} \int_{\mathbb{R}^n \setminus B_{r_{\rm o}/2}} \frac{f(t,x,w)}{|w|^{n+2s}}\,{\rm d}w, \end{aligned}$$

where we have used the fact that $Q_{(c_*(\sigma-\sigma')r_{\rm o})^\beta}(z_1) \subset Q_{\sigma r_{\rm o}}(z_{\rm o})$ for any $z_1 \in Q_{\sigma' r_{\rm o}}(z_{\rm o})$, $\sigma > \frac{1}{2}$, that

$$\frac{|w|}{|w-v_1|} \le 1 + \frac{|v_1|}{|w|-|v_1|} \le 1 + \frac{\sigma'}{\sigma-\sigma'} \le \frac{1}{\sigma-\sigma'},$$

for any $w \in \mathbb{R}^n \setminus B_{\sigma r_{\rm o}}$ and that $\beta > 1$.

Thus, we can estimate the p-contribution of the tail in velocity as follows,

$$
\begin{aligned}
&\|\operatorname{Tail}(f; B_{(c_*(\sigma-\sigma')r_o)^\beta}(v_1))\|_{L^p(U_{(c_*(\sigma-\sigma')r_o)^\beta}(t_1,x_1))} \\
&\qquad \le c \sup_{Q_{\sigma r_o}(z_o)} f + \frac{c}{(\sigma-\sigma')^{n+2s}} \|\operatorname{Tail}(f; B_{r_o/2})\|_{L^p(U_{r_o}(-1+r_o^{2s},0))}\,,
\end{aligned}
\tag{4.2}
$$

where we have used that $Q_{(c_*(\sigma-\sigma')r_o)^\beta}(z_1) \subset Q_{\sigma r_o}(z_o) \subset Q_{r_o}(z_o)$.

Then, combining (4.2) and (4.1) we arrive at

$$
\begin{aligned}
\sup_{Q_{\sigma' r_o}(z_o)} f \;\le\; & \frac{c(\delta)\|f\|_{L^\zeta(Q^-_{r_o})}}{[(\sigma-\sigma')r_o]^{\beta_1}} + \left(c\delta + \frac{2-\zeta}{2}\right) \sup_{Q_{\sigma r_o}(z_o)} f \\
& + \frac{c(\delta)}{(\sigma-\sigma')^{\beta_2}} \|\operatorname{Tail}(f; B_{r_o/2})\|_{L^p(U_{r_o}(-1+r_o^{2s},0))}\,,
\end{aligned}
\tag{4.3}
$$

by also making use of an application of Young's Inequality (with exponents $2/\zeta$ and $2/(2-\zeta)$), where $\beta_1 \equiv \beta_1(n,p,s) > 0$ and $\beta_2 \equiv \beta_2(n,s) > 0$.

Choose $\delta \in (0,1)$ such that

$$
c\delta + \frac{2-\zeta}{2} =: \varepsilon < 1\,,
$$

which together with (4.3) yields

$$
\sup_{Q_{\sigma' r_o}(z_o)} f \le \frac{c\,\|f\|_{L^\zeta(Q_{r_o}(z_o))}}{[(\sigma-\sigma')r_o]^{\beta_1}} + \varepsilon \sup_{Q_{\sigma r_o}(z_o)} f + \frac{c}{(\sigma-\sigma')^{\beta_2}} \|\operatorname{Tail}(f; B_{r_o/2})\|_{L^p(U_{r_o}(-1+r_o^{2s},0))}.
$$

Hence, a final application of Lemma 2.6 together with the weak Harnack inequality in Theorem 2.3 yields the desired (1.14). □

Remark 4.1. Still in theme of Harnack-type inequalities for kinetic equations, it is worth mentioning the very recent paper [56], in which amongst other interesting results the author proves a strong Harnack inequality for kinetic integral equations for *global solutions*, a priori bounded, periodic in the space variable, and under an integral monotonicity-in-time assumption (see Definition 2.2 there). The usual nonlocality issues are partially annihilated by the peculiar global framework there, so that no tail contributions do appear.

4.2. **Geometric Harnack inequality.** By Chow's Lemma, we observe that $\mathbb{R}^{1+2n}$ is connected with respect to the group of translations introduced in (1.6), and hence given any two points $z_o, z_1 \in \mathbb{R}^{1+2n}$ we are able to connect them through an absolutely continuous integral curve of the vector fields generating the algebra. We are thus allowed to consider integral curves already employed in the study of the geometrical properties of the local Kolmogorov equation; see for instance [3], and/or already done by many authors in the nonlocal framework in order to obtain other types of results; see for instance [2, 56].

Definition 4.2. *A curve* $\gamma : [0,T] \to \mathbb{R}^{1+2n}$ *is* admissible *if*

$$
\dot\gamma(\tau) = \sum_{k=1}^{n} \omega_k(\tau)\partial_{v_k}(\gamma(\tau)) + (v\cdot\nabla_x - \partial_t)(\gamma(\tau)) \qquad \text{a.e. in } [0,T],
\tag{4.4}
$$

where $\omega_1,\dots,\omega_n \in L^1([0,T])$, *and it is absolutely continuous.*

Given $z_o = (t_o, x_o, v_o), z = (t,x,v) \in \mathbb{R}^{1+2n}$, γ steers z_o *in* z, *for* $t < t_o$, *if* $\gamma(0) = z_o$ *and* $\gamma(T) = z$.

Since the problem at hand (1.1) is backward, we always need to consider a Cauchy problem with final datum. Hence, in order to have a positive parameter τ governing the curve γ, we apply a change of variables and replace the vector field $v\cdot\nabla_x+\partial_t$ with $v\cdot\nabla_x-\partial_t$ in the definition of above. Furthermore, in (4.4) we identify each vector field with a vector of $\mathbb{R}^{1+2n}$ as follows

$$\begin{aligned} v\cdot\nabla_x-\partial_t &\sim \big(-1,v_1,\cdots,v_n,0,\cdots,0\big)^{\mathrm{T}}\,, \\ \text{and}\quad \partial_{v_j} &\sim \big(0,\cdots,0,1,0,\cdots 0\big)^{\mathrm{T}} \quad \text{for } j=1,\dots,n, \end{aligned}$$

where in the last vector 1 occupies the $(1+n+j)^{th}$-position.

Now, to find a curve γ defined as in (4.4) we need to solve the following problem

$$\begin{cases} \dot t(\tau)=-1 \\ \dot x_j(\tau)=v_j(\tau) & \text{for } j=1,\dots,n, \\ \dot v_j(\tau)=\omega_j(\tau) & \text{for } j=1,\dots,n, \end{cases}$$

where ω is a suitable control $\omega=(\omega_1,\dots,\omega_n)\in\big(L^2([0,T])\big)^n$. Thus, an admissible curve γ steering $z_{\rm o}$ in z is defined for a. e. $\tau\in[0,T]$ as

$$\begin{cases} \gamma_1(\tau)=t_{\rm o}-\tau, \\ \gamma_{j+1}(\tau)=x_{\rm o,j}+v_{\rm o,j}\tau+\displaystyle\int_0^\tau\int_0^r\omega_j(r')\,{\rm d}r'\,{\rm d}r & \text{for } j=1,\dots,n, \\ \gamma_{j+n+1}(\tau)=v_{\rm o,j}+\displaystyle\int_0^\tau\omega_j(r)\,{\rm d}r & \text{for } j=1,\dots,n, \end{cases}$$

where $x_{\rm o,j}$ and $v_{\rm o,j}$ denote the j-th component of the vector $x_{\rm o}$ and $v_{\rm o}$, respectively. In particular, when $n=1$ we get

$$\gamma(\tau)=\left(t_{\rm o}-\tau,x_{\rm o}+v_{\rm o}\tau+\int_0^\tau\int_0^r\omega(r')\,{\rm d}r'\,{\rm d}r,v_{\rm o}+\int_0^\tau\omega(r)\,{\rm d}r\right).$$

Definition 4.3. *Let Ω be an open subset of $\mathbb{R}^{1+2n}$ and $z_{\rm o}=(t_{\rm o},x_{\rm o},v_{\rm o})\in\Omega$. The* attainable *set $\mathscr{A}_{z_{\rm o}}(\Omega)$ is defined as*

$$\mathscr{A}_{z_{\rm o}}(\Omega):=\big\{z\in\Omega:\ \exists\gamma:[0,T]\to\mathbb{R}^{1+2n}\ \textit{admissible curve s. t.}\ \gamma(0)=z_{\rm o},\gamma(T)=z\big\}.$$

Then, to give a geometric characterization of the set on which our Harnack inequality holds true we rely on a fundamental tool firstly developed in the local uniformly parabolic case by Aronson and Serrin [9], and later on extended to the local Kolmogorov setting by Polidoro in [61].

Definition 4.4. *A set $\{z_{\rm o},\dots,z_\ell\}\subset\Omega$ is a* Harnack chain *connecting $z_{\rm o}$ to z_ℓ if there exist ℓ positive constants $c_1,\dots,c_\ell$ such that*

$$f(z_j)\le c_jf(z_{j-1})\quad\forall j=1,\dots,\ell,$$

for every solution f of (1.1), with $h=0$.

Now, for any radius $r>0$, we define

$$D_r:=\left\{-1+\frac{1}{2}r^{2s}\right\}\times B_{r^{1+2s}}\times B_r,$$

and for every $z_{\rm o}=(t_{\rm o},x_{\rm o},v_{\rm o})\in\mathbb{R}^{1+2n}$ we have

$$D_r(z_{\rm o}):=(t_{\rm o},x_{\rm o},v_{\rm o})\circ D_r.$$

We observe that by its definition D_r is a subset of $Q_r^-(z_o)$.

Lemma 4.5. *Let $\gamma : [0,T] \to \mathbb{R}^{1+2n}$ be an admissible curve such that $\gamma(0) = z_0 = (t_0, x_0, v_0)$. For any $b \in [0,T]$, such that $b < 1$, for which there exists a positive constant η such that*

$$\int_0^b |\omega(\varrho)|^2 \,\mathrm{d}\varrho \le \eta\,,$$

then

$$\gamma(b) \in D_{\overline{r}}(\gamma(0)) \;\; \textit{with } \overline{r} = (2(1-b))^{\frac{1}{2s}}\,.$$

Proof. Firstly, we consider the case where $\gamma(0) = (0,0,0)$, given that by the translation invariance of the vector fields in (4.4), we can infer every other possible case. Our aim is to show that there exists $\eta > 0$ such that $\gamma(b) \in D_{\overline{r}}(0)$, for some appropriate $\overline{r} > 0$. Note that $\overline{r} > 0$ needs to be chosen in such a way that

$$-1 + \frac{1}{2}\overline{r}^{2s} = -b \qquad \Longrightarrow \qquad \overline{r} = (2(1-b))^{\frac{1}{2s}}\,.$$

Now, we will show that for $j = 1, \dots, n$

$$\left|\int_0^b \omega_j(\varrho)\,\mathrm{d}\varrho\right| \le \overline{r} \quad \text{and} \quad \left|\int_0^b \int_0^\varrho \omega_j(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\varrho\right|^{\frac{1}{1+2s}} \le \overline{r}\,. \tag{4.5}$$

For this, we apply Hölder's Inequality, for every $j = 1, \dots, n$, to get

$$\left|\int_0^b \omega_j(\varrho)\,\mathrm{d}\varrho\right| \le \int_0^b |\omega_j(\varrho)|\,\mathrm{d}\varrho \le \|\omega_j\|_{L^2([0,b])}\sqrt{b} \le \sqrt{\eta b}\,.$$

For what concerns the second estimate in (4.5), again by Hölder's Inequality, we have

$$\begin{aligned}
\left|\int_0^b \int_0^\varrho \omega_j(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\varrho\right| &\le \int_0^b \|\omega_j\|_{L^2([0,\varrho])}\sqrt{\varrho}\,\mathrm{d}\varrho \\
&\le \|\omega_j\|_{L^2([0,b])}\left[\frac{2}{3}\varrho^{\frac{3}{2}}\right]_{\varrho=0}^{\varrho=b} = \frac{2}{3}b^{\frac{3}{2}}\|\omega_j\|_{L^2([0,b])},
\end{aligned}$$

and this implies

$$\left|\int_a^b \int_a^\varrho \omega_j(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\varrho\right|^{\frac{1}{1+2s}} \le \left(\frac{2}{3}\sqrt{\eta b^3}\right)^{\frac{1}{1+2s}}\,.$$

The proof is finally complete by choosing η such that

$$\eta \le \min\left\{\frac{\overline{r}^2}{b},\, \frac{9}{4}\frac{\overline{r}^{2(1+2s)}}{b^3}\right\}. \tag{4.6}$$

□

Now, we are in a position to prove an intermediate result which will easily lead to the proof of the desired Geometric Harnack inequality. We have the following

Proposition 4.6. *Under the assumptions of Theorem 1.5, if $z_o \in \Omega$, then for every $z \in \mathring{\mathscr{A}}_{z_o}$ there exists an open neighborhood $U(z)$ and a constant $c_z > 0$ such that*

$$\sup_{U(z)} f \le c_z \left(f(z_o) + \sum_{i=0}^{\ell} \|\,\mathrm{Tail}(f; B_{r_o/2}(v_i))\|_{L^p(U_{r_o}(-1+r_o^{2s}-t_i, x_i))} \right).$$

Proof. In view of the result in the preceding Lemma, the proof below can now go in a similar fashion as in [3]; we just have to take care of the intrinsic substrate and the tail term. For any given $z = (t, x, v) \in \mathring{\mathscr{A}}_{z_o}$ we construct a finite Harnack chain connecting z with z_o. By Chow's Lemma we know that there exists an admissible curve $\gamma : [0, T] \to \mathbb{R}^{1+2n}$ steering z_o in z. Without loss of generality we assume $T \geq 1$.

Some further notation is now required. Denote by $\mathcal{C} := (-1, 1)^{1+2n}$; that is, an open neighborhood of the origin of $\mathbb{R}^{1+2n}$. Thus, thanks to the continuity of the Galilean change of variable in (1.6) and of the dilation $\{\delta_r\}_{r>0}$ in (1.7), for every $z_1 \in \mathbb{R}^{1+2n}$, the family $\big(\mathcal{C}_r(z_1)\big)_{r>0}$ given by

$$\mathcal{C}_r(z_1) := z_1 \circ \delta_r(\mathcal{C})$$

is a neighborhood basis of the point z_1. Then, again in view of the continuity of the group law and dilation, for every $\tau \in [0, T]$ there exists a positive r such that $\mathcal{C}_r(\gamma(\tau)) \subseteq \Omega$. Thus we can define

$$r(\tau) := \sup\big\{r > 0 \,:\, \mathcal{C}_r(\gamma(\tau)) \subseteq \Omega\big\}. \tag{4.7}$$

Note that the function in (4.7) is continuous $\big($as function of $\tau \in [0, T]\big)$, and thus it is well defined the positive number $r_{\min}$ given by

$$r_{\min} := \min_{\tau \in [0,T]} r(\tau). \tag{4.8}$$

Since $Q_r(\gamma(\tau)) \subset \mathcal{C}_r(\gamma(\tau))$ we actually have that, by definition the very definition of (4.8),

$$Q_r(\gamma(\tau)) \subseteq \Omega \quad \text{for every } \tau \in [0, T] \quad \text{and } r \in (0, r_{\min}).$$

On the other hand, notice that the function $\mathscr{G}(\tau)$ defined by

$$\mathscr{G}(\tau) := \int_0^\tau |\omega(\varrho)|^2 \,\mathrm{d}\varrho$$

is (uniformly) continuous in $[0, T]$. Then, there exists a positive constant $\bar{\eta}$ such that

$$\mathscr{G}(\tau) \leq \bar{\eta} \qquad \text{for every } \tau \in [0, T].$$

Now, let us consider

$$\widetilde{r} := \min\left\{r_o, \frac{1}{2} r_{\min}\right\},$$

which is such that $\widetilde{r} < 1$, since we recall that $r_o \in (0, 1)$ is the radius appearing in Theorem 1.5, and we consider $\nu_o = 1 - \frac{1}{2}\widetilde{r}^{2s}$. If $\bar{\eta} \leq \eta$, where η is the constant computed in (4.6) with $\bar{r} = \widetilde{r}$, then we proceed to work on the full interval $[0, T]$. Otherwise, there exists $\bar{T} \in (0, T)$ such that the desired inequality holds, and the proof will work in the same fashion.

For the sake of the reader, we place ourselves in the first case; i. e., when $\bar{\eta} \leq \eta$ with our choice for $\widetilde{r}$, and we construct our Harnack chain of step ν_0. Let ℓ be the unique positive integer such that $(\ell - 1)\nu_o < T$, and $\ell\nu_o \geq T$. We define $\{\tau_j\}_{j\in\{0,1,\dots,\ell\}} \in [0, T]$ as follows,

$$\tau_j = j\nu_o \text{ for } j = 0, 1, \dots, \ell - 1, \quad \text{and } \tau_\ell = T.$$

Now we apply Lemma 4.5, up to traslating the initial point, to any portion of the curve γ originating from τ_j and ending in τ_{j+1}, and we obtain

$$\gamma(\tau_{j+1}) \in D_{\tilde{r}}(\gamma(\tau_j)), \qquad \text{for } j = 0, \dots, \ell - 2,$$

and also to the couple $\tau_{\ell-1}$ and ending in τ_ℓ, and we obtain

$$\gamma(\tau_{\ell-1}) \in D_{\tilde{r}_1}(\gamma(\tau_\ell)),$$

for a possibly different $\widetilde{r}_1$ (or up to moving $\gamma(\tau_\ell)$ further). Now, by its very definition, for every $j=1,\dots,\ell-1$ we have $D_{\tilde{r}}(\gamma(\tau_j)) \subset Q^-_{r_\mathrm{o}}(\gamma(\tau_j))$, and also $Q_{2\tilde{r}}(\gamma(\tau_j)) \subseteq \Omega$.

Lastly, for some $r_1 \in (0,\widetilde{r}]$, we obtain

$$\gamma(\tau_\ell) \in Q^-_{r_1}(\gamma(\tau_{\ell-1})).$$

It remains to show that $(\gamma(\tau_j))_{j=0,1,\dots,\ell}$ is a Harnack chain. By Theorem 1.5, for every $j=1,\dots,\ell-2$ we get

$$\begin{aligned}
f(\gamma(\tau_{j+1})) &\le \sup_{Q^-_{\tilde{r}}(\gamma(\tau_j))} f \\
&\le \sup_{Q^-_{r_\mathrm{o}}(\gamma(\tau_j))} f \\
&\le c \inf_{Q^+_{r_\mathrm{o}}(\gamma(\tau_j))} f + c\,\|\operatorname{Tail}(f;B_{r_\mathrm{o}/2}(v(\tau_j))\|_{L^p(U_{r_\mathrm{o}}(-1+r_\mathrm{o}^{2s}-t(\tau_j),x(\tau_j)))} \\
&\le c\,f(\gamma(\tau_j)) + c\,\|\operatorname{Tail}(f;B_{r_\mathrm{o}/2}(v(\tau_j))\|_{L^p(U_{r_\mathrm{o}}(-1+r_\mathrm{o}^{2s}-t(\tau_j),x(\tau_j)))},
\end{aligned}$$

Eventually, we apply Theorem 1.5 to the set $Q_{r_1}(\gamma(\tau_{\ell-1})) \subseteq \Omega$, and we obtain

$$\sup_{U(z)} f \le c_z \left(f(z_\mathrm{o}) + \sum_{i=0}^{\ell} \|\operatorname{Tail}(f;B_{r_\mathrm{o}/2}(v_i))\|_{L^p(U_{r_\mathrm{o}}(-1+r_\mathrm{o}^{2s}-t_i,x_i))} \right),$$

where $c_z = \sum_{i=1}^{\ell} c^{j+1-i}$ and $U(z) = Q^-_{r_1}(\gamma(\tau_{\ell-1}))$. This completes the proof. □

We are ready to complete the proof of the geometric Harnack result.

Theorem 4.7 (Geometric Harnack inequality)**.** *Under the assumptions of* Theorem 1.5, *for every $z_\mathrm{o} \in \Omega$ and for any compact subset $D \Subset \mathscr{A}_{z_\mathrm{o}}$, there exists a positive constant $c > 0$, depending only on n, s p Λ and D, such that*

$$\sup_{D} f \le c \left(f(z_\mathrm{o}) + \sum_{i=0}^{\ell} \|\operatorname{Tail}(f;B_{r_\mathrm{o}/2}(v_i))\|_{L^p(U_{r_\mathrm{o}}(-1+r_\mathrm{o}^{2s}-t_i,x_i))} \right).$$

Proof. Let D be any compact subset of $\mathscr{A}_{z_\mathrm{o}}$. Hence, if $U(z)$ denotes a neighborhood of $z=(v,x,t) \in K$, then

$$D \subseteq \bigcup_{z\in D} U(z).$$

Since D is compact, then we can extract a finite covering $\{U(z_j)\}_{j=1,\dots,m}$ of it. Then we apply Proposition 4.6 to every $U(z_j)$, with $j=1,\dots,m$, obtaining

$$\sup_{U(z_j)} f \le c(z_j) \left(f(z_\mathrm{o}) + \sum_{i=0}^{\ell_j} \|\operatorname{Tail}(f;B_{R_{\overline{r}}/2}(v_i))\|_{L^p(U_{R_{\overline{r}}}(-1+r_\mathrm{o}^{2s}-t_i,x_i))} \right)$$

where ℓ_j is the number of points belonging to the Harnack chain for the specific set $U(z_j)$, each of which begins at the point z_o. By choosing $c = \max\{c(z_j) :\ j=1,\dots,m\}$ and $\ell := \sum_{j=1}^{m} \ell_j$ the proof is complete. □

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