OPTIMAL COEFFICIENTS FOR ELLIPTIC PDES

Dedicated to the memory of Hedy Attouch

G. Buttazzo†, J. Casado-Díaz††, F. Maestre††

† Dipartimento di Matematica, Università di Pisa,
Largo B. Pontecorvo, 5
56127 Pisa, ITALY

†† Dpto. de Ecuaciones Diferenciales y Análisis Numérico,
Facultad de Matemáticas, C. Tarfia s/n
41012 Sevilla, SPAIN

e-mail: giuseppe.buttazzo@unipi.it, jcasadod@us.es, fmaestre@us.es

Abstract. We consider an optimization problem related to elliptic PDEs of the form $-\operatorname{div}(a(x)\nabla u)=f$ with Dirichlet boundary condition on a given domain Ω. The coefficient $a(x)$ has to be determined, in a suitable given class of admissible choices, in order to optimize a given criterion. We first deal with the case when the cost is the so-called elastic compliance, and then we discuss the more general case when the problem is written as an optimal control problem.

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1. Introduction

Various quantities are involved in the study of elliptic PDEs, which we often refer to as *data*. In particular, in several situations the coefficients of an elliptic operator have to be determined in order to optimize a given cost functional. In all the paper Ω is a given bounded open subset of $\mathbb{R}^d$ and $H^1_0(\Omega)$ is the usual Sobolev space of functions with zero boundary trace.

We give here below a presentation of the optimization problem, and in Section 5 we provide some numerical simulation. Other kinds of optimization problems for elliptic PDEs, namely the optimal choice of lower order terms, together with their regularity, have been considered in [4], [18], [22].

1.1. Position of the problem.

We consider the problem of minimizing a cost functional of the form

$$J(u)=\int_\Omega j(x,u)\,dx,$$

where $j(x,s)$ is a suitable cost integrand with the appropriate growth conditions, and u is the solution of the elliptic equation

$$\begin{cases}-\operatorname{div}\big(a(x)\nabla u\big)=f & \text{in }\Omega\\ u\in H^1_0(\Omega).\end{cases}\tag{1.1}$$

Here the right-hand side $f \in L^2(\Omega)$ is prescribed, while the coefficient a has to be chosen in a suitable admissible class $\mathcal{A}$ in order to minimize the functional J above. The problem is then an optimal control problem, where u is the state variable, a the control variable, J the cost functional, and (1.1) is the state equation. This amounts then to the problem

$$\min\Big\{J(u)\ :\ u \text{ solves (1.1)},\ a \in \mathcal{A}\Big\}.$$

The admissible class $\mathcal{A}$ is usually given in the form

$$\mathcal{A} = \Big\{a \geq 0\ :\ \int_\Omega \psi(a)\,dx \leq 1\Big\}.$$

A simpler way to impose the constraint on a is to write the problem in the form

$$\min_{a\geq 0}\ \min_{u\in H_0^1(\Omega)} \Big\{\int_\Omega \big(j(x,u) + \lambda\psi(a)\big)\,dx\ :\ u \text{ solves (1.1)}\Big\}. \tag{1.2}$$

where $\lambda > 0$ plays the role of a Lagrange multiplier. Moreover, we will assume ψ convex and non-negative. Replacing ψ by $\lambda\psi$ we can also assume $\lambda = 1$.

2. The compliance and energy problems.

A particular case of the problem considered above occurs when the cost J is the so-called *compliance*, that is

$$j(x,u) = f(x)u.$$

In this case an easy integration by parts transforms the problem in the max/min problem

$$\max_{a\geq 0}\ \min_{u\in H_0^1(\Omega)} \int_\Omega \Big(\frac{1}{2}a(x)|\nabla u|^2 - f(x)u - \frac{1}{2}\psi(a)\Big)\,dx.$$

We first assume that ψ is superlinear, that is

$$\lim_{s\to+\infty} \frac{\psi(s)}{s} = +\infty, \tag{2.1}$$

and we set

$$\mathcal{E}(a) = \inf_{u\in C_0^1(\Omega)} \int_\Omega \Big(\frac{1}{2}a(x)|\nabla u|^2 - f(x)u - \frac{1}{2}\psi(a)\Big)\,dx.$$

Concerning the right-hand side f, in several problems some concentration phenomena for data occur, so we simply require that the right-hand side f is a signed measure.

Theorem 2.1. *Under assumption* (2.1)*, the functional $\mathcal{E}(a)$ admits a maximizer $a_{opt} \in L^1(\Omega)$, provided the right-hand side f is such that $\mathcal{E}(a) > -\infty$ for at least a coefficient $a \in L^1(\Omega)$.*

Proof. Since for every $u \in C_0^1(\Omega)$ the map

$$a \mapsto \int_\Omega \Big(\frac{1}{2}a(x)|\nabla u|^2 - \frac{1}{2}\psi(a)\Big)\,dx - \int u\,df$$

is weakly $L^1(\Omega)$ upper semicontinuous, the functional $\mathcal{E}(a)$ is weakly $L^1(\Omega)$ upper semicontinuous, being the infimum of a family of upper semicontinuous functions. In addition, testing with $u = 0$, we have

$$\mathcal{E}(a) \leq -\frac{1}{2}\int_\Omega \psi(a)\,dx.$$

Hence, by the superlinearity of ψ and by the well-known weak $L^1(\Omega)$ compactness theorem, the existence of an optimal coefficient a_{opt} is easily established by means of the direct methods of the calculus of variations. □

The case when ψ has only a linear growth:

$$\lim_{s\to+\infty}\frac{\psi(s)}{s}=k>0 \tag{2.2}$$

is more delicate. In fact, in this case a more careful definition of the integrals

$$\int_\Omega |\nabla u|^2\,da \qquad\text{and}\qquad \int_\Omega \psi(a)$$

is needed. We refer to [5] for more details about this case, which has strong links with the theory of optimal transportation, as first shown in [3] and [2]. However, by an argument similar to the one of Theorem 2.1, an optimal coefficient a_{opt} still exists, but in the larger class $\mathscr{M}(\Omega)$ of nonnegative measures on Ω, as stated below.

Theorem 2.2. *Under assumption* (2.2)*, the functional $\mathcal{E}(a)$ admits a maximizer a_{opt} in the class $\mathscr{M}(\Omega)$, provided the right-hand side f is such that $\mathcal{E}(a)>-\infty$ for at least a coefficient $a\in\mathscr{M}(\Omega)$.*

It is interesting to characterize the optimal coefficient a_{opt} in terms of some suitable auxiliary variational problem. Thanks to a well-known result from min/max theory that allows to exchange the order of inf and sup, due to the convexity with respect to the variable u and the concavity with respect to the variable a (see for instance [10] and [14]), the initial problem becomes

$$\inf_{u\in C_0^1(\Omega)}\sup_{a\ge 0}\int_\Omega\Big(\frac{1}{2}a(x)|\nabla u|^2-\frac{1}{2}\psi(a)\Big)\,dx-\int u\,df. \tag{2.3}$$

The supremum with respect to a can be now easily computed:

$$\sup_{a\ge 0}\int_\Omega\Big(\frac{1}{2}a(x)|\nabla u|^2-\frac{1}{2}\psi(a)\Big)\,dx-\int u\,df=\int_\Omega\frac{1}{2}\psi^*\big(|\nabla u|^2\big)\,dx-\int u\,df,$$

where ψ^* is the Legendre-Fenchel conjugate function of ψ. The auxiliary variational problem is then

$$\inf_{u\in C_0^1(\Omega)}\int_\Omega\frac{1}{2}\psi^*\big(|\nabla u|^2\big)\,dx-\int u\,df. \tag{2.4}$$

Since $\psi^*(t)\ge t-\psi(1)$ it is easy to see that, at least when $f\in H^{-1}(\Omega)$, the auxiliary variational problem admits a solution $\bar u\in H_0^1(\Omega)$. Moreover, if ψ is strictly increasing, then the function $s\mapsto\psi^*(s^2)$ is strictly convex and therefore $\bar u$ is unique. The optimal coefficient a_{opt} can now be recovered through the optimality condition

$$a_{opt}|\nabla\bar u|^2=\psi(a_{opt})+\psi^*(|\nabla\bar u|^2). \tag{2.5}$$

Remark 2.3. In some situations it is important to allow the right-hand side f to be singular, for instance with concentrations on regions of lower dimensions. In general we can assume that $f\in\mathscr{M}$, the class of measures with finite mass; even if for some choice of the coefficient a we may have $\mathcal{E}(a)=-\infty$, the optimal compliance problem is still meaningful, because these "bad" coefficients are ruled out by the optimization criterion, consisting in maximizing $\mathcal{E}(a)$. Moreover, all the arguments also apply to the case when, instead of having a boundary Dirichlet condition, we have the Neumann one, assuming as usual that the right-hand side f has zero average.

Remark 2.4. When $\psi(s) = \gamma s$, using the equivalence between coefficient optimization and optimal transport problem, pointed out in [2], the following summability properties for the optimal coefficient a_{opt} have been obtained (see [12]):

$$f \in \mathscr{M} \quad \Longrightarrow \quad \mu_{opt} \in \mathscr{M} \text{ possibly not unique;}$$
$$f \in L^1(\Omega) \quad \Longrightarrow \quad \mu_{opt} \in L^1(\Omega) \text{ and is unique;}$$
$$f \in L^p(\Omega) \quad \Longrightarrow \quad \mu_{opt} \in L^p(\Omega) \text{ for every } p \in [1, +\infty];$$
$$\operatorname{spt}(\mu_{opt}) \subset \text{convex envelope of } \begin{cases} \operatorname{spt}(f) & \text{in the Neumann case} \\ \operatorname{spt}(f) \cup \partial\Omega & \text{in the Dirichlet case.} \end{cases}$$

In addition, a mild BV and $W^{1,1}$ regularity for μ_{opt} is available in some cases in dimension two. More precisely, when $d = 2$ and under some additional assumptions on the regularity of Ω and on the behavior of the datum f, we have (see [13]):

$$f \in BV(\Omega) \cap L^\infty(\Omega) \quad \Longrightarrow \quad \mu_{opt} \in BV(\Omega),$$
$$f \in W^{1,1}(\Omega) \cap L^\infty(\Omega) \quad \Longrightarrow \quad \mu_{opt} \in W^{1,1}(\Omega).$$

In higher dimension. Furthermore, the correspondence between the optimization and transport problems is unclear when the function ψ is nonlinear.

Example 2.5. Taking $\psi(s) = s^2/2$ and $f \in W^{-1,4/3}(\Omega)$, we obtain the auxiliary variational problem

$$\min\Big\{\int_\Omega \Big(\frac{1}{4}|\nabla u|^4 - f(x)u\Big)\,dx \ :\ u \in W_0^{1,4}(\Omega)\Big\},$$

or equivalently the nonlinear PDE

$$-\Delta_4 u = f, \qquad u \in W_0^{1,4}(\Omega),$$

whose unique solution $\bar u$ provides the optimal coefficient $a_{opt}(x) = |\nabla \bar u(x)|^2$. For instance, if Ω is the unit ball, and $f = 1$ we obtain

$$\bar u(x) = \frac{3}{4d^{1/3}}\big(1 - |x|^{4/3}\big), \qquad a_{opt}(x) = \frac{|x|^{2/3}}{d^{2/3}}.$$

Conversely, taking Ω the unit disc in $\mathbb{R}^2$ and $f = \delta_0$ the unit Dirac mass at the origin, gives

$$\bar u(x) = \frac{3}{(16\pi)^{1/3}}(1 - |x|^{2/3}), \qquad a_{opt}(x) = (2\pi|x|)^{-2/3}.$$

Example 2.6. Taking

$$\psi(s) = \begin{cases} \gamma s & \text{if } \alpha \le s \le \beta \\ +\infty & \text{otherwise,} \end{cases} \tag{2.6}$$

with $0 < \alpha < \beta$, $\gamma > 0$, we have the auxiliary variational problem

$$\min\Big\{\int_\Omega \frac{|\nabla u|^2 - \gamma}{2}\Big(\beta 1_{|\nabla u|^2 \ge \gamma} + \alpha 1_{|\nabla u|^2 \le \gamma}\Big) - f(x)u\,dx \ :\ u \in H_0^1(\Omega)\Big\},$$

whose unique solution $\bar u$ provides the optimal coefficient $a_{opt} \in L^\infty(\Omega)$. It has been proved in [6] (see also [7]) that, when Ω is of class $C^{1,1}$ and $f \in L^2(\Omega)$, then $\bar u$ is in $H^2(\Omega)$ and $\nabla a_{opt} \cdot \nabla \bar u$ belongs to $L^2(\Omega)$.

Another case where (1.2) reduces to a variational problem is the minimization of the energy, corresponding to

$$j(x,u) = -f(x)u.$$

Similarly to (2.3), the problem becomes

$$\min_{a\geq 0}\ \min_{u\in H_0^1(\Omega)} \int_\Omega \Big(\frac{1}{2}a(x)|\nabla u|^2 - f(x)u + \frac{1}{2}\psi(a)\Big)\,dx. \tag{2.7}$$

which, computing the minimum in a, can be written as

$$\min_{u\in H_0^1(\Omega)} \int_\Omega \Big(-\frac{1}{2}\psi^*\big(-|\nabla u|^2\big) - fu\Big)\,dx. \tag{2.8}$$

Imposing that

$$\lim_{s\to 0^+} s\psi(s) = \infty,$$

the functional becomes coercive over $W_0^{1,1}(\Omega)$. Assuming also that ψ is decreasing and that $\psi(1/s)$ is convex, we have that the functional in (2.8) is convex. Therefore, under these assumptions and taking f smooth enough, problem (2.8) has a solution in $W_0^{1,1}(\Omega)$ (it is not necessarily in $H_0^1(\Omega)$). However, in other situations this functional is not convex and then (2.8) may not have a solution. To avoid this difficulty it is necessary to deal with a relaxed problem formulation consisting in replacing the function $\xi\in\mathbb{R}^d\mapsto -\psi(-|\xi|^2)$ by its convex hull.

Example 2.7. Related to Example 2.5, we take $\psi(s) = 1/(2s^2)$. Then problem (2.8) becomes

$$\min_{u\in W_0^{1,\frac{4}{3}}(\Omega)} \int_\Omega \Big(\frac{3}{4}|\nabla u|^{\frac{4}{3}} - fu\Big)\,dx,$$

which has a unique solution $\overline{u}$ if f is in $W^{-1,4}(\Omega)$. The optimal control is given by $a_{opt} = |\nabla\overline{u}|^{-2/3}$. In this way, if Ω is the unit ball in $\mathbb{R}^d$ and $f=1$, we get

$$\overline{u}(x) = \frac{1-|x|^4}{4d^3}, \qquad a_{opt}(x) = \frac{d^2}{|x|^2}.$$

Example 2.8. Taking

$$\psi(s) = \begin{cases}\gamma(\beta - s) & \text{if } s\in[\alpha,\beta]\\ +\infty & \text{otherwise,}\end{cases} \tag{2.9}$$

with $0<\alpha<\beta$, $\gamma>0$, we have

$$-\psi^*(-|\xi|^2) = \begin{cases}\beta|\xi|^2 & \text{if } |\xi|^2\leq\gamma\\ \alpha|\xi|^2 + \gamma(\beta-\alpha) & \text{if } |\xi|^2>\gamma,\end{cases}$$

which is not convex. Computing its convex hull (see e.g. [15]) we get the relaxed formulation

$$\min_{u\in H_0^1(\Omega)} \int_\Omega \Big(\frac{1}{2}\phi\big(|\nabla u|\big) - fu\Big)\,dx, \tag{2.10}$$

with

$$\phi(s)=\begin{cases}\beta s^2 & \text{if } s^2\le\dfrac{\alpha}{\beta}\gamma\\ 2\sqrt{\alpha\beta\gamma}\,s-\alpha\gamma & \text{if } \dfrac{\alpha}{\beta}\gamma\le s^2\le\dfrac{\beta}{\alpha}\gamma\\ \alpha s^2+\gamma(\beta-\alpha) & \text{if } \dfrac{\beta}{\alpha}\gamma\le s^2.\end{cases}$$

The Euler-Lagrange equation for (2.10) proves that for a given solution $\overline{u}$, the associated optimal "relaxed control" is given by

$$a_{opt}=\frac{\phi'\big(|\nabla\overline{u}|\big)}{2|\nabla\overline{u}|}.$$

It can be proved that this optimal relaxed control coincides with the optimal relaxed control defined in the following section. Moreover, although a_{opt} and $\nabla\overline{u}$ may not be unique, the function $\overline{\sigma}=a_{opt}\nabla\overline{u}$ is unique.

Taking Ω the unit ball in $\mathbb{R}^d$, $f=1$, $\gamma<1/(d^2\alpha\beta)$, and denoting $\tau=d\sqrt{\alpha\beta\gamma}$, we have

$$\overline{u}(x)=\begin{cases}\dfrac{(1-\tau^2)(\beta-\alpha)}{2d\alpha\beta}+\dfrac{1-|x|^2}{2d\beta} & \text{if } |x|<\tau\\ \dfrac{1-|x|^2}{2d\alpha} & \text{if } |x|>\tau,\end{cases}\qquad a_{opt}(x)=\begin{cases}\beta & \text{if } |x|<\tau\\ \alpha & \text{if } |x|>\tau.\end{cases}$$

Remark 2.9. For ψ given by (2.9), it has been proved in [6] that $\Omega\in C^{1,1}$ and $f\in L^2(\Omega)$ imply that $\overline{u}$ is in $H^2(\Omega)^d$ and that the derivatives of a_{opt} in the orthogonal directions to σ are in $L^2(\Omega)$.

3. The general problem.

In the case of a general optimal control problem of the form

$$\min_{a\ge 0}\ \min_{u\in H_0^1(\Omega)}\Big\{\int_\Omega\big(j(x,u)+\psi(a)\big)\,dx\ :\ u\text{ solves (1.1)}\Big\},\tag{3.1}$$

the existence of an optimal coefficient a_{opt} may fail, and a solution exists only in a *relaxed* sense. For $0<\alpha<\beta$, a counterexample to the existence of a solution a_{opt} can be found in [20], where

$$\psi(s)=\begin{cases}0 & \text{if } \alpha\le s\le\beta\\ +\infty & \text{otherwise,}\end{cases}$$

and

$$j(x,s)=|s-u_0(x)|^2$$

for a suitable function u_0. A different counterexample is illustrated in Section 4.

In order to understand what relaxed solutions are, we have to recall the notion of G-convergence, introduced by De Giorgi and Spagnolo in [11]: a sequence $a_n(x)$ of functions between α and β is said to G-converge to a symmetric $d\times d$ matrix $A(x)$ if for every $f\in L^2(\Omega)$ the solutions u_n of the PDEs

$$-\operatorname{div}\big(a_n\nabla u_n\big)=f,\qquad u_n\in H_0^1(\Omega)$$

converge in $L^2(\Omega)$ to the solution u of the PDE

$$-\operatorname{div}\big(A\nabla u\big)=f,\qquad u\in H_0^1(\Omega).\tag{3.2}$$

The question becomes now to characterize the G-closure $\overline{\mathcal{A}}$ of the set of coefficients a_n. A complete answer has been given by Murat and Tartar in [21], [23] (see also [17] for the two-dimensional case). They proved that the G-closure $\overline{\mathcal{A}}$ above consists of all symmetric $d\times d$ matrices $A(x)$ whose eigenvalues $\lambda_1(x)\le\lambda_2(x)\le\cdots\le\lambda_d(x)$ are between α and β and satisfy for a suitable $t\in[0,1]$ (depending on x) the following $d+2$ inequalities:

$$\begin{cases}\displaystyle\sum_{1\le i\le d}\frac{1}{\lambda_i-\alpha}\le\frac{1}{\nu_t-\alpha}+\frac{d-1}{\mu_t-\alpha}\\ \displaystyle\sum_{1\le i\le d}\frac{1}{\beta-\lambda_i}\le\frac{1}{\beta-\nu_t}+\frac{d-1}{\beta-\mu_t}\\ \nu_t\le\lambda_i\le\mu_t\qquad i=1,\dots,d,\end{cases}$$

being μ_t and ν_t respectively the arithmetic and the harmonic mean of α and β, namely

$$\mu_t=t\alpha+(1-t)\beta,\qquad\nu_t=\Big(\frac{t}{\alpha}+\frac{1-t}{\beta}\Big)^{-1}.$$

For instance, when $d=2$, the set above is given by the symmetric 2×2 matrices $A(x)$ whose eigenvalues $\lambda_1(x)$ and $\lambda_2(x)$ are between α and β and

$$\frac{\alpha\beta}{\alpha+\beta-\lambda_1(x)}\le\lambda_2(x)\le\alpha+\beta-\frac{\alpha\beta}{\lambda_1(x)}\ .$$

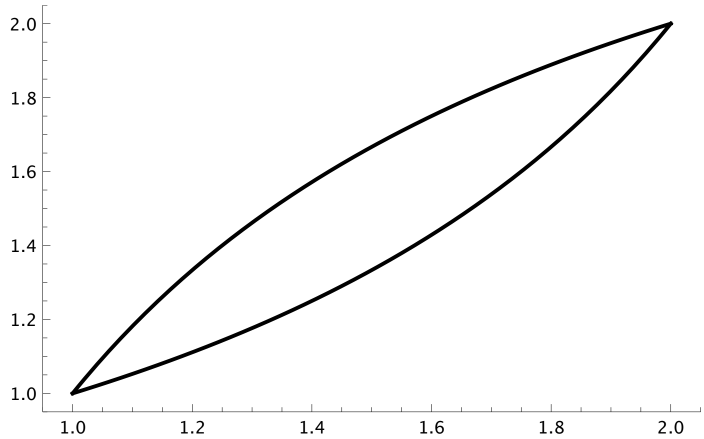

FIGURE 1. Attainable matrices, in the plane (λ_1,λ_2), for $d=2$, $\alpha=1$, $\beta=2$.

As far as we know, an explicit form of the relaxation

$$\Psi(A)=\inf_{a_n\to_G A}\liminf_n\int_\Omega\psi(x,a_n)\,dx;$$

with a general function $\psi(x,a)$ is not known. The case $\psi(x,a)=g(x)a$ has been considered in [8] and [9]. For instance, if ψ is given by (2.6), denoting by $\overline{\mathcal{A}}$ the G-closure described above, the relaxation $\Psi(A)$ is given by

$$\Psi(A)=\begin{cases}\displaystyle\int_\Omega\gamma\lambda_{max}\big(A(x)\big)\,dx & \text{if }A\in\overline{\mathcal{A}}\\ +\infty & \text{otherwise,}\end{cases}$$

being $\lambda_{max}(A)$ the largest eigenvalue of the $d\times d$ symmetric matrix A. Namely, taking into account that the solution of the state equation

$$-\operatorname{div}\big(A(x)\nabla u\big)=f,\qquad u\in H^1_0(\Omega),$$

does not vary if we replace $A(x)$ by another matrix function $B(x)$ such that

$$A(x)\nabla u=B(x)\nabla u\qquad\text{a.e. in }\Omega,$$

and that for every $\xi\in\mathbb{R}^d$ one has

$$\big\{A\xi:\ A\in\overline{\mathcal{A}}\big\}=\big\{\eta\in\mathbb{R}^d:\ (\eta-\nu_t\xi)\cdot(\eta-\mu_t\xi)\le 0\big\},$$

we have that a relaxation of (3.1) with $\psi(x,a)=g(x)a$ is given by

$$\min_{\substack{\nu_t I\le A\le\mu_t I\\ 0\le t\le 1}}\ \min_{u\in H^1_0(\Omega)}\Big\{\int_\Omega\big(j(x,u)+g(x)\mu_t\big)\,dx\ :\ u\text{ solves (3.2)}\Big\},\tag{3.3}$$

where I denotes the identity matrix.

As an example, we can consider the energy problem $j(x,u)=-f(x)u$ with ψ given by (2.9). Then, taking into account (2.7) and (3.3) the relaxed problem can be written as

$$\min_{0\le t\le 1}\ \min_{u\in H^1_0(\Omega)}\Big\{\int_\Omega\big(\frac{\nu_t}{2}|\nabla u|^2-f(x)u-\gamma\mu_t\big)\,dx\Big\}.\tag{3.4}$$

Using the minimum in t this proves again that u solves (2.10).

Thanks to (3.3) we obtain a system of optimality conditions. For this purpose, we assume the function $j(x,s)$ derivable with respect to s with appropriate growth conditions.

Take (t_{opt},A_{opt}) an optimal solution of the relaxed problem, then for any admissible control (t,A) and $0\le\varepsilon\le 1$ the control

$$\big(t_{opt}+\varepsilon(t-t_{opt}),A_{opt}+\varepsilon(A-A_{opt})\big)$$

is also admissible. Using it and deriving with respect to ε we conclude that

$$A_{opt}\nabla\bar u\cdot\nabla\bar p+g(\beta-\alpha)t_{opt}=\max_{\substack{\nu_t I\le A\le\mu_t I\\ 0\le t\le 1}}\Big\{A_{opt}\nabla\bar u\cdot\nabla\bar p-g(\beta-\alpha)t_{opt}\Big\},\tag{3.5}$$

with $\bar u,\bar p$ the state and adjoint state functions, solutions of

$$-\operatorname{div}\big(A_{opt}\nabla\bar u\big)=f,\quad -\operatorname{div}\big(A_{opt}\nabla\bar p\big)=\partial_s j(x,\bar u),\qquad \bar u,\bar p\in H^1_0(\Omega).\tag{3.6}$$

Computing the maximum in (3.5), we obtain the optimality conditions (see for instance [1], [21])

$$\begin{cases}A_{opt}\nabla\bar u=\dfrac{\mu_{t_{opt}}+\nu_{t_{opt}}}{2}\nabla\bar u+\dfrac{\mu_{t_{opt}}-\nu_{t_{opt}}}{2}\dfrac{|\nabla\bar u|}{|\nabla\bar p|}\nabla\bar p & \text{a.e. in }\{\nabla\bar p\ne 0\}\\[2mm] A_{opt}\nabla\bar p=\dfrac{\mu_{t_{opt}}+\nu_{t_{opt}}}{2}\nabla\bar p+\dfrac{\mu_{t_{opt}}-\nu_{t_{opt}}}{2}\dfrac{|\nabla\bar p|}{|\nabla\bar u|}\nabla\bar u & \text{a.e. in }\{\nabla\bar u\ne 0\},\end{cases}\tag{3.7}$$

$$t_{opt}=\begin{cases}0 & \text{if } g<N^+-\dfrac{\beta}{\alpha}N^-\\[2mm] \dfrac{1}{\beta-\alpha}\Big(\sqrt{\dfrac{\alpha\beta N^-}{N^+-g}}-\alpha\Big) & \text{if } N^+-\dfrac{\beta}{\alpha}N^-\le g\le N^+-\dfrac{\alpha}{\beta}N^-\\[2mm] 1 & \text{if } N^+-\dfrac{\alpha}{\beta}N^-<g,\end{cases}\tag{3.8}$$

with

$$N^+ = \frac{|\nabla u||\nabla p| + \nabla u\cdot\nabla p}{2}, \qquad N^- = \frac{|\nabla u||\nabla p| - \nabla u\cdot\nabla p}{2}. \tag{3.9}$$

4. Nonexistence of an optimal coefficient

In this section we provide a counterexample to the existence of an optimal coefficient a_{opt} for (1.2). We take $\Omega = B(0,1)$ the unit ball in $\mathbb{R}^d$, and

$$\begin{cases} f = 1 \text{ the right-hand side,} \\ j(x_1,...,x_d,s) = (1+\varepsilon x_1)s \text{ with } \varepsilon > 0, \\ \psi(s) = \begin{cases} \tau^2 s & \text{if } s\in[1,2] \\ +\infty & \text{otherwise} \end{cases} \quad \text{with } 0<\tau<\dfrac{1}{d}. \end{cases} \tag{4.1}$$

Let us prove the existence of $\varepsilon_0 > 0$ such that (1.2) has no solution for $0 < \varepsilon < \varepsilon_0$.

First we observe that for $\varepsilon = 0$ problem (1.2) is a particular case of the compliance problem considered in Section 2. By (2.4) we get that the state function u_0 associated to an optimal control a_0 is the unique solution of (2.4) with

$$\psi^*(s) = \begin{cases} s-\tau^2 & \text{if } s<\tau^2 \\ 2(s-\tau^2) & \text{if } s>\tau^2. \end{cases}$$

By uniqueness u_0 is invariant by rotations and then is a radial function $u_0(r)$. Combined with (2.5), this implies

$$u_0'(r) = \begin{cases} -\dfrac{r}{d} & \text{if } r<d\tau \\ -\tau & \text{if } d\tau\le r\le 2d\tau \\ -\dfrac{r}{2d} & \text{if } 2\tau d<r, \end{cases} \qquad a_0(r) = \begin{cases} 1 & \text{if } r<d\tau \\ \dfrac{r}{d\tau} & \text{if } d\tau<r<2d\tau \\ 2 & \text{if } r>2d\tau. \end{cases} \tag{4.2}$$

On the other hand, recalling that the solution u of the state equation in (1.1) satisfies $a\nabla u = \sigma$, with σ the solution of

$$\min\Big\{\int_\Omega \frac{|\zeta|^2}{a}\,dx \;:\; -\operatorname{div}\zeta = 1\Big\},$$

and that

$$\int_\Omega u\,dx = \int_\Omega \frac{|\sigma|^2}{a}\,dx,$$

we get that (1.2) with $\varepsilon = 0$ is also equivalent to

$$\min_{1\le a\le 2}\ \min_{-\operatorname{div}\sigma=1}\int_\Omega\Big(\frac{|\sigma|^2}{a} + \tau^2 a\Big)\,dx.$$

Taking the minimum with respect to a, this provides

$$a(x) = \begin{cases} 1 & \text{if } |\sigma(x)|<\tau \\ \dfrac{|\sigma(x)|}{\tau} & \text{if } \tau\le|\sigma(x)|\le 2\tau \\ 2 & \text{if } |\sigma(x)|>2\tau, \end{cases}$$

with σ a solution of

$$\min_{-\operatorname{div}\sigma=1}\int_\Omega \Upsilon(|\sigma|)\,dx,\qquad \Upsilon(s)=\begin{cases} s^2+\tau^2 & \text{if } 0\le s<\tau\\ 2\tau s & \text{if } \tau\le s\le 2\tau\\ \dfrac{s^2}{2}+2\tau^2 & \text{if } s>2\tau.\end{cases} \tag{4.3}$$

By what proved above, this problem has a unique solution $\sigma_0(x)=-x/d$.

Let us now prove the non-existence result. Arguing by contradiction, we assume there exist $\varepsilon_k>0$ tending to zero and a_{ε_k} solution of (1.2). We denote by u_{ε_k} the corresponding state function. Since the state equation does not depend on ε, we have

$$\int_\Omega \big((1+\varepsilon_k x_1)u_{\varepsilon_k}+\tau^2 a_{\varepsilon_k}\big)\,dx\le \int_\Omega \big((1+\varepsilon_k x_1)u_0+\tau^2 a_0\big)\,dx. \tag{4.4}$$

Using also

$$\int_\Omega \big(u_0+\tau^2 a_0\big)\,dx=\int_\Omega \Upsilon(|\sigma_0|)\,dx,$$

and that σ_0 solves (4.3), we deduce from (4.4)

$$0\le \int_\Omega \big(\Upsilon(|\sigma_\varepsilon|)-\Upsilon(|\sigma_0|)\big)dx\le \varepsilon\int_\Omega x_1(u_0-u_\varepsilon)\,dx. \tag{4.5}$$

Taking into account that u_{ε_k} solves (1.1) with $a=a_{\varepsilon_k}$, we know ([11]) that there exist a subsequence, still denoted by ε_k, and $A\in L^\infty(\Omega)^{d\times d}$ symmetric such that

$$I\le A\le 2I \text{ a.e. in } \Omega,\qquad a_{\varepsilon_k}I\overset{G}{\rightharpoonup} A,$$

$$u_{\varepsilon_k}\rightharpoonup u \text{ in } H^1_0(\Omega),\qquad \sigma_{\varepsilon_k}\rightharpoonup A\nabla u \text{ in } L^2(\Omega)^d,\qquad -\operatorname{div}(A\nabla u)=1 \text{ in } \Omega.$$

Coming back to (4.5) and taking into account the convexity of Υ, we have that

$$\int_\Omega \Upsilon(|A\nabla u|)\,dx=\min_{-\operatorname{div}\zeta=1}\int_\Omega \Upsilon(|\zeta|)\,dx.$$

Thus, $A\nabla u$ is a solution of (4.3) and then u is a solution of (1.2). By uniqueness this proves that

$$u=u_0,\qquad A=a_0 I,\qquad \sigma=\sigma_0.$$

Now, we consider p_ε the adjoint state defined as the solution of

$$\begin{cases} -\operatorname{div}(a_{\varepsilon_k}\nabla p_{\varepsilon_k})=1+\varepsilon_k x_1 & \text{in } \Omega\\ p_{\varepsilon_k}=0 & \text{on } \partial\Omega.\end{cases}$$

Then $z_{\varepsilon_k}=(p_{\varepsilon_k}-u_{\varepsilon_k})/\varepsilon_k$ solves

$$\begin{cases} -\operatorname{div}(a_{\varepsilon_k}\nabla z_{\varepsilon_k})=x_1 & \text{in } \Omega\\ z_{\varepsilon_k}=0 & \text{on } \partial\Omega.\end{cases}$$

By the G-convergence of $a_{\varepsilon_k}I$, we have

$$z_{\varepsilon_k}\rightharpoonup z \text{ in } H^1_0(\Omega),\qquad a_{\varepsilon_k}\nabla z_{\varepsilon_k}\rightharpoonup A\nabla z \text{ in } L^2(\Omega)^d,$$

with z the solution of

$$\begin{cases} -\operatorname{div}(A\nabla z)=x_1 & \text{in } \Omega\\ z=0 & \text{on } \partial\Omega.\end{cases}$$

Since we are assuming a_{ε_k} a solution of (1.2) and then of the relaxed problem (3.3), we deduce from (3.7) that the matrix with columns $\nabla u_{\varepsilon_k}, \nabla p_{\varepsilon_k}$ has rank one. By Morrey's theorem relative to the weak converges of the Jacobian ([19]), which in our case reduces to

$$\begin{aligned}&\partial_i u_{\varepsilon_k}\partial_j z_{\varepsilon_k} - \partial_i z_{\varepsilon_k}\partial_j u_{\varepsilon_k} = \partial_i(u_{\varepsilon_k}\partial_j z_{\varepsilon_k}) - \partial_j(u_{\varepsilon_k}\partial_i z_{\varepsilon_k})\\ &\rightharpoonup \partial_i(u\partial_j z) - \partial_j(u_0\partial_i z) = \partial_i u\partial_j z - \partial_i z\partial_j u_0 \quad \text{in } W^{-1,1}(\Omega),\end{aligned}$$

we have that ∇z is parallel a.e. to ∇u_0, and that z solves

$$\begin{cases} -\operatorname{div}\big(a_0\nabla z\big) = x_1 & \text{in } \Omega\\ z_0 = 0 & \text{on } \partial\Omega.\end{cases}$$

However, the solution of this problems is given by $z(x) = h(|x|)x_1$ with h the unique solution of

$$\begin{cases} -\big(a_0 h'\big)' - \dfrac{d+1}{r}a_0 h' - \dfrac{a_0'}{r}h = 1 & \text{in } (0,1)\\ h(1) = 0, \qquad \displaystyle\int_0^1 r^{d+1}|h'|^2 dr < \infty. & \end{cases}$$

Thus z is not a radial function and ∇z is not parallel to ∇u_0. □

5. Numerical simulations

In this section we present some numerical experiments for the resolution of problems of the kind of (1.2) in the 2-d case. We solve numerically the problems showed in Examples 2.5 and 2.6 for compliance optimization and the example given by problem (2.7) with ψ defined in (2.9) for energy optimization.

In the case of the compliance problem we put

$$J(a) = \int_\Omega \big(f(x)u + \psi(a)\big)dx,$$

then, having in mind that for u solution of the state equation (1.1), it is easy to compute that

$$\frac{dJ(a)}{da}\cdot\tilde a = \int_\Omega \tilde a\left(\psi'(a) - |\nabla u|^2\right)\,dx.$$

We will apply a gradient descent method with projection into the appropriate subspace functions a such that $\psi(a) < \infty$. For instance in Example 2.6 ψ is finite where $a \in [\alpha, \beta]$. The algorithm is the following:

- Initialization: choose an admisible function a_0.
- for $k \geq 0$, iterate until convergence as follow:
  - compute u_k solution of (1.1) for $a = a_k$.
  - compute $\bar a_k = -(\psi'(a) - |\nabla u|^2)$ descent direction associated to u_k.
  - update the function a_k:

    $$a_{k+1} = P_\psi(a_k + \epsilon_k \bar a_k)$$

    where P_ψ is a projection operator associated to the set $\{a : \psi(a) < \infty\}$, and where ϵ_k is small enough to ensure the decrease of the cost function.
- Stop if convergence: $\frac{|J(a_k) - J(a_{k-1})|}{|J(a_0)|} < tol$, for $tol > 0$ small.

In the case of the energy problem, we use a similar algorithm based on formulation (3.4).

On the other hand, we are interested also in showing the numerical evidence of the non-existence of an optimal coefficient for the general case. We propose to solve the relaxed formulation of (1.2), given by (3.3), in $\Omega = B(0,1)$ unit disc of $\mathbb{R}^2$, and with the data given in (4.1). By convex minimization and following the system of optimality, we propose the following algorithm to compute (t_{opt}, A_{opt}).

- Initialization: choose an admissible (t_0, A_0) such that $0 \leq t_0 \leq 1$ and $\nu_t I \leq A_0 \leq \mu_t I$.
- For $k \geq 0$, iterate until convergence as follows:
  - compute the solutions u_k and p_k of (3.6) for $A_{opt} = A_k$. Then, we define N^+, N^- by (3.9);
  - compute $\hat{t}$ given by (3.8);
  - compute $\hat{A}$ defined by (3.7), considering $\hat{u} = u_k$, $\hat{p} = p_k$, $t_{opt} = \hat{t}$ and such that the spectrum of $\hat{A}$ belongs to $[\nu_{t_k}, \mu_{t_k}]$;
  - for $\varepsilon_k \in (0,1]$, update the function (t_k, A_k) as:
$$t_{k+1} = t_k + \varepsilon_k(\hat{t} - t_k), \qquad A_{k+1} = A_k + \varepsilon_k(\hat{A} - A_k).$$
- Stop if convergence: $\frac{|J(t_{k+1},A_{k+1}) - J(t_k,A_k)|}{|J(t_0,A_0)|} < tol$, for $tol > 0$ small and J corresponding with the cost function in (3.3) .

We now show some numerical experiments based on the algorithms described above. The computation has been carried out using the free software FreeFem++ v4.5 ([16], available in http://www.freefem.org). The picture of figures are made in Paraview 5.10.1 (available at https://www.kitware.com/open-source/# paraview), which is free too. We use P_1-Lagrange finite element approximations for u and p, solutions of the state and adjoint state equations respectively, and P_0-Lagrange finite element approximations for control variables, a for scalar problems or (t, A) for the matrix problems. For all simulations where the parameters α and β appear, we consider a normalized value $\alpha = 1$, and $\beta = 2$.

Example 5.1. We consider two cases in the framework of compliance optimization, i.e., $j(x,u) = f(x)u$. In Example 2.5 for $\psi(s) = s^2/2$ we provided an explicit solution when Ω is the unit ball and $f \equiv 1$. Here we solve numerically this problem in the non-radial case, considering the square $\Omega = [0,1]^2$. In Figure 2 we show the computed optimal solutions, the optimal density a on the left, and the optimal function u on the right.

The second case corresponds to ψ given by (2.6). We solve numerically this problem in the cube $\Omega = [0,1]^2$, and considering the Lagrange multiplier $\gamma = 0.01141$ in order to work with a volumen constraint of 50% of each phase α and β. In Figure 3 we show the computed optimal solutions, the optimal density a on the left, and optimal function u on the right.

Example 5.2. We consider a case in the framework of energy optimization, i.e., $j(x,u) = -f(x)u$. We assume $f \equiv 1$ and ψ defined as (2.9) with $\gamma = 0.0142$ in order to assure a volumen constraint of 50% of each phase α and β and we deal with the relaxed formulation (3.4). In Figure 4 we show the computed optimal solutions, the optimal density a on the left, and optimal function u on the right.

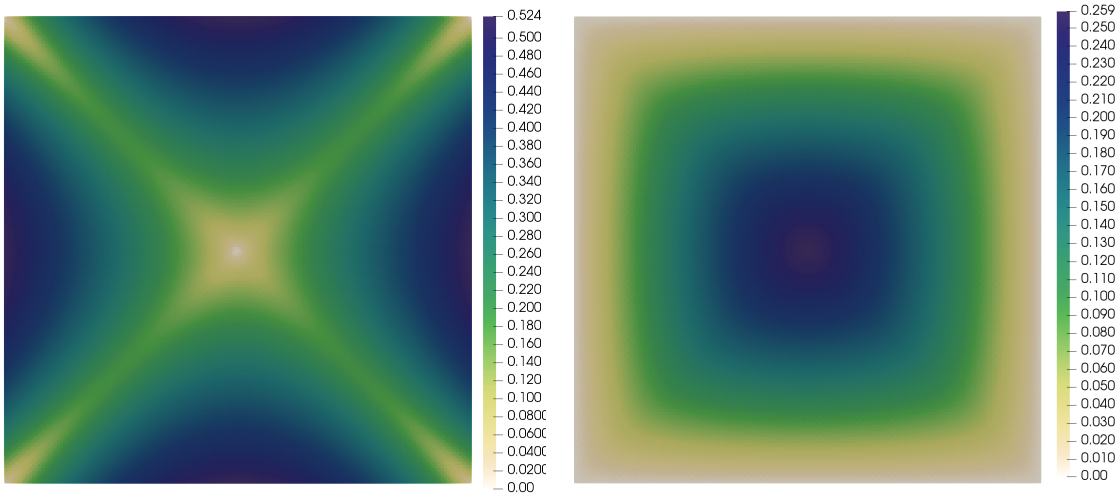

FIGURE 2. Example 1.1 Optimal a (left), and optimal u (right).

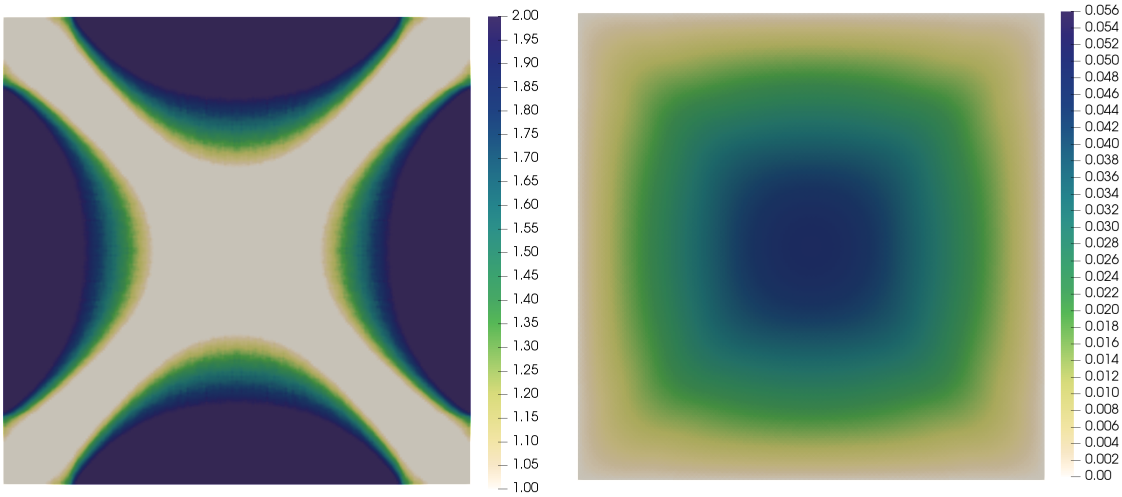

FIGURE 3. Example 1.2 Optimal a (left), and optimal u (right).

Example 5.3. In the last case we show a numerical evidence of non existence of an optimal density a_{opt} for problem of kind of (1.2). We follow the counterexample showed in Section 4. We consider $\Omega = B(0,1)$ the unit ball in $\mathbb{R}^2$ and the rest of data as in (4.1) with $\tau = 0.23539$. We have solved the relaxed formulation of problem (1.2) searching an optimal density t_{opt} and an optimal matrix A_{opt}. Firstly, we have considered the problem with $\varepsilon = 0$. In this case, the computed optimal matrix is $A_{opt} = \mu_{t_{opt}} I$, a scalar matrix. We show in Figure 5 on the left the optimal value of $\mu_{t_{opt}}$ corresponding to $\varepsilon = 0$. As hoped, it agrees with the function a_0 defined by (4.2). On the other hand, we consider the problem with $\varepsilon = 0.5$. In this case the computed optimal matrix A_{opt} is not scalar. We show in Figure 5 on the right the ratio λ_1/λ_2 with λ_i, $i = 1, 2$ the eigenvalues of A_{opt}. Observe that this ratio is not identically one, and therefore A_{opt} is a non-isotropic matrix.

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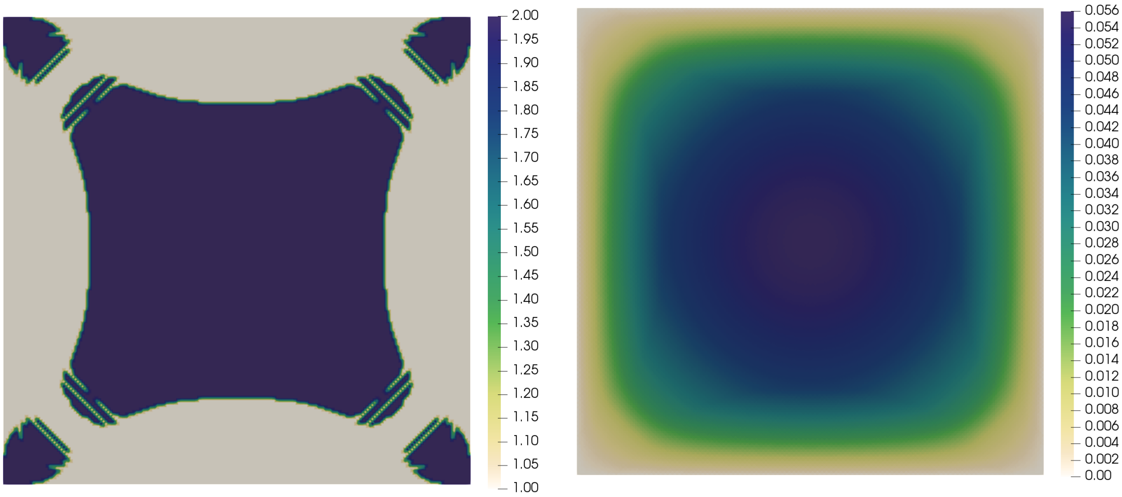

FIGURE 4. Example 2. Optimal a (left), and optimal u (right).

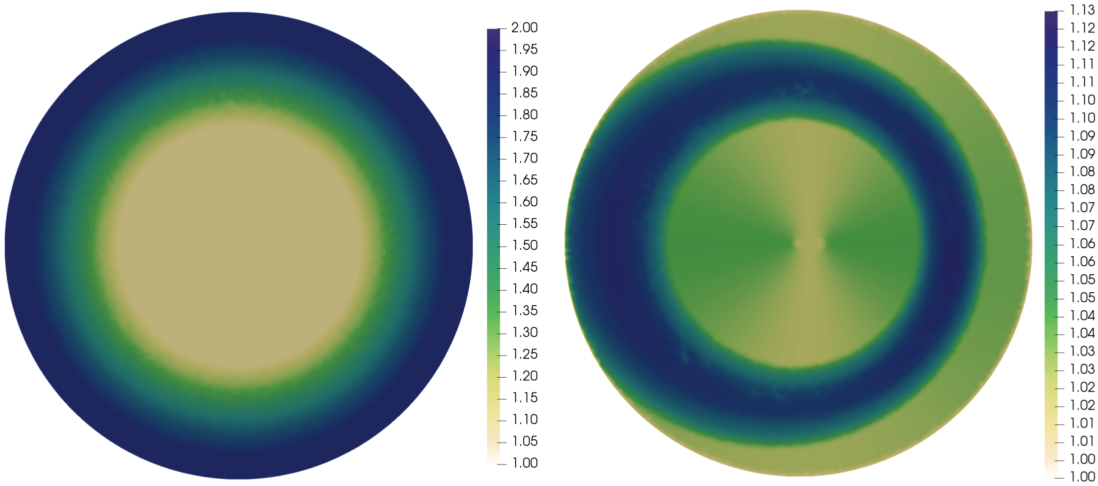

FIGURE 5. Example 3, $\lambda_1 = \lambda_2$ for $\varepsilon = 0$ (left), and λ_2/λ_1 for $\varepsilon = 0.5$.

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