WEIGHTED TOTAL VARIATION MINIMIZATION PROBLEM WITH MIXED DIRICHLET-NEUMANN BOUNDARY CONDITIONS

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ABSTRACT. In this paper, we study the problem of minimizing the weighted total variation of a normalized BV function u plus a penalization on the weighted L^1 norm of the trace of u on the Neumann part Γ of the boundary, while assuming a Dirichlet condition $u = 0$ on the complement part $\Gamma^c \subset \partial\Omega$. We show that this problem is a relaxation of some shape optimization problem of type *Cheeger*, that is both problems have the same minimum. Then, we prove that the level sets of minimizers are optimal sets. Finally, we will also study the regularity as well as some properties of these optimal sets.

1. INTRODUCTION

Let Ω be an open bounded set in $\mathbb{R}^N$ and Γ is an open subset of $\partial\Omega$. Let ϕ and ψ be two nonnegative functions over $\overline{\Omega}$ and w be defined on Γ. Then, we are interested in studying the following minimization problem:

$$\inf\left\{\frac{\int_\Omega \psi|Du| - \int_\Gamma w|u|}{\int_\Omega \phi|u|}\ :\ u \neq 0 \in BV(\Omega),\ u = 0 \text{ on } \Gamma^c := \partial\Omega\backslash\Gamma\right\}. \tag{1.1}$$

The case when $\Gamma = \emptyset$ has been already considered in [6]. Moreover, the authors of [15] have also studied Problem (1.1) but in the case where $\Gamma \subset \partial\Omega$ and $w = 0$ on Γ. The interest in studying Problem (1.1) is motivated by a landslide model (see [10]) in which ϕ and ψ represent the body forces and the (inhomogeneous) yield limit distribution, respectively. When $\phi = \psi = 1$ (which is not a relevant assumption in landslides modeling) and $\Gamma = \emptyset$, the infimum in (1.1) can be restricted to characteristic functions $u = \chi_A$ and so, we get

$$\min\left\{\frac{Per(A)}{|A|}\ :\ A \subset \Omega\right\} \tag{1.2}$$

where $Per(A)$ denotes the perimeter of the set A in $\mathbb{R}^N$ in the sense of De Giorgi (see [2]). This problem is known as Cheeger's problem [9], its value $\lambda(\Omega)$ is called the Cheeger constant of Ω and its minimizers are called Cheeger sets of Ω (see aslo [16, 17]). Moreover, $\lambda(\Omega)$ is the first eigenvalue of the 1−Laplacian on Ω ([11, 12]). We note that the existence of an optimal set $A^\star$ in Problem (1.2) is very simple and it follows from the direct method in Calculus of Variations. In the case where the densities ϕ and ψ are not uniform, but $w = 0$ on $\Gamma \subset \partial\Omega$, Problem (1.1) will be the relaxation of the following problem:

$$\min\left\{\frac{\int_{\partial A\backslash\Gamma}\psi}{\int_A \phi}\ :\ A \subset \Omega\right\}. \tag{1.3}$$

This can be seen as a generalization of the Cheeger problem (1.2). However, up to our knowledge, the case when $w \neq 0$ has not been studied before in the literature. In this paper, we will show that Problem (1.1) is equivalent to the following generalization of (1.3):

$$\min\left\{\frac{\int_{\partial A\backslash\Gamma}\psi - \int_{\partial A\cap\Gamma} w}{\int_A \phi}\ :\ A \subset \Omega\right\}. \tag{1.4}$$

We note that the existence of an optimal set to Problem (1.4) is not guaranteed here for an arbitrary weight w on Γ. Indeed, the functional in (1.4) is not a priori lower semicontinuous with respect to the weak* convergence in BV. Thus, this additional difficulty imposed by the presence of a density $w \neq 0$ was the main motivation to write the present paper.

Inspired by [14], one can see that the variational formulation of the stationary anti-plane flow of an inhomogeneous Bingham (rigid visco-plastic) fluid can be stated as follows: there is a function $u \in H^1(\Omega)$ with $u = 0$ on Γ^c such that

$$\int_\Omega \nabla u \cdot \nabla(v-u) + \int_\Omega \psi|\nabla v| - \int_\Omega \psi|\nabla u| \geq \int_\Omega \phi(v-u) + \int_\Gamma w(v-u), \tag{1.5}$$

for all $v \in H^1(\Omega)$ such that $v = 0$ on Γ^c. The velocity field in the domain $D = \Omega \times \mathbb{R} \subset \mathbb{R}^3$ (here, we assume $N = 2$) is given by $\mathbf{u} = (0, 0, u)$ with $u = u(x_1, x_2)$. The viscosity distribution is equal to 1, ψ stands for the yield limit distribution, ϕ denotes the body forces in the x_3 direction and, w is an additional force acting on the Neumann part Γ. A particular case of the Bingham model lies in the presence of rigid zones located in the interior of the flow of the Bingham solid/fluid. As the yield limit ψ increases, these rigid zones become larger and may completely block the flow so that $u = 0$ is the solution of (1.5). In other words, the Bingham fluid is blocked if and only if

$$\int_\Omega \psi\,|\nabla v| - \int_\Gamma w\,v \geq \int_\Omega \phi\,v, \quad \text{for all } v \in H^1(\Omega), \ \ v = 0 \ \text{ on } \Gamma^c. \tag{1.6}$$

When considering oil transport in pipelines, in the process of oil drilling or in the case of metal forming, the blocking of the solid/fluid is a catastrophic event to be avoided. From (1.6), one can see the infimum in (1.1) as a safety coefficient. In other words, the Bingham fluid is not blocked if and only if $\inf$ (1.1) < 1. In a completely opposite context, when modeling landslides, the solid is blocked in its natural configuration and the beginning of a flow can be seen as a disaster. Here, the $1/\inf$ (1.1) appears as a safety coefficient.

Notice that Problem (1.1) can be seen as a study of the "eigenvalue problem" for the following degenerate inhomogeneous equation with mixed Dirichlet-Neumann boundary conditions (where the first eigenvalue is $\lambda^\star := \inf$ (1.1)):

$$\begin{cases} -\nabla \cdot [\psi \frac{\nabla u}{|\nabla u|}] = \lambda\,\phi, & \text{in } \ \Omega, \\ \psi[\frac{\nabla u}{|\nabla u|} \cdot \mathbf{n}] = w, & \text{on } \ \Gamma, \\ u = 0, & \text{on } \ \Gamma^c. \end{cases}$$

On the other hand, the properties of Cheeger sets (i.e. optimal sets in (1.2)) have been studied in several papers (see [8, 1] and the references therein). One of the very important results concerning the regularity of Cheeger sets, is that the internal boundary of Cheeger sets have constant curvature. In [15], the authors have also generalized some of these properties to optimal sets of the generalized Cheeger problem (1.3). More precisely, they show that the curvature of the boundary of any optimal set $A^\star$ at any point x in the interior of Ω is given by

$$\kappa(x) = \frac{\lambda^\star \phi(x) + \partial_{\mathbf{n}}\psi(x)}{\psi(x)},$$

where $\partial_{\mathbf{n}}\psi(x)$ is the inward normal derivative on $\partial A^\star$ at x (so, ψ should be at least of class C^1). Moreover, if $\partial A^\star$ crosses Γ at some point x where Γ is C^1 around x, then the tangent line to $\partial A^\star$ at x must be orthogonal to Γ.

This paper is organized as follows. In section 2, we will show that Problems (1.1) and (1.4) have the same minimal value and that each of these two problems has a solution. More precisely, we will show that from a minimizer of (1.1) one can construct an optimal set of (1.4) simply by considering its superlevel sets. Moreover, we will study in Section 3 the regularity properties of these optimal sets. Finally, we conclude the paper by some examples in Section 4.

2. Existence of solutions

Throughout this section, we assume that $\Omega \subset \mathbb{R}^N$ is an open bounded connected domain with Lipschitz boundary, $\phi(x) \geq \phi_0 > 0$ is a bounded function and, $\psi(x) \geq \psi_0 > 0$ is a continuous function on $\overline{\Omega}$ (where ψ_0, $\phi_0 \in \mathbb{R}^+$ are fixed). Let Γ be a closed subset of $\partial\Omega$ and w be a bounded function on Γ. Then, we consider the minimization problem:

$$\inf \left\{ \frac{\int_\Omega \psi|Du| - \int_\Gamma w|u|}{\int_\Omega \phi|u|} \ : \ u \in BV(\Omega),\ u = 0 \ \text{on} \ \Gamma^c \right\}. \tag{2.1}$$

We recall that proving existence of a minimizer for Problem (2.1) is a difficult task due to different facts. First, we do not have a priori compactness: if $(u_n)_n$ is a minimizing sequence then it is not clear if one can extract a subsequence converging weakly* in $BV(\Omega)$ and even so (i.e. assuming that $u_n \rightharpoonup^\star u$ in $BV(\Omega)$), since the trace map is not lower semicontinuous with respect to this topology then it is not true in general that $u_n \rightharpoonup u$ in $L^1(\partial\Omega)$ and so, we do not know whether the limit function u satisfies the Dirichlet condition $u = 0$ on Γ^c or not. In particular, it is possible that a solution to this problem (2.1) does not exist! So, the idea is to relax the boundary condition $u = 0$ on Γ^c by adding a penality term in the functional; this is a classical tool in the theory of Calculus of Variations and it has also been used to prove existence of a solution to the BV least gradient problem (see [19]).

Let $\tilde{\Omega}$ be an open bounded Lipschitz extension of Ω such that $\Gamma \subset \partial\tilde{\Omega}$ and $\Gamma^c \subset \tilde{\Omega}$. Then, we consider now the following relaxation of (2.1):

$$\inf \left\{ \frac{\int_{\tilde{\Omega}} \psi|Du| - \int_\Gamma w|u|}{\int_{\tilde{\Omega}} \phi|u|} \ : \ u \in BV(\tilde{\Omega}),\ u = 0 \ \text{on} \ \tilde{\Omega}\backslash\Omega \right\}. \tag{2.2}$$

Note that $\int_{\tilde{\Omega}} \psi|Du| = \int_\Omega \psi|Du| + \int_{\Gamma^c} \psi|u|$. But again, it is not easy to show existence of a solution to the relaxed version (2.2) since in general the map $u \mapsto -\int_\Gamma w|u|$ is not lower semicontinuous with respect to the weak* convergence in $BV(\tilde{\Omega})$. More precisely, we will show that the lower semicontinuity of the functional in (2.2) depends on the $L^\infty-$bounds of ψ and w as well as the regularity of the Neumann part Γ. To motivate this fact, we consider the following examples:

Example 2.0.1. *Assume that* $\Omega =]0,1[^2$, $\Gamma = (\{0\}\times[0,1])\cup([0,1]\times\{0\})$, $\psi = \psi_0 > 0$ *and* $w = w_0 \in \mathbb{R}$. *Set* $u_n(x_1,x_2) = n \cdot \chi_{E_n}$ *where* $E_n := \{(x_1,x_2) \in \Omega : x_1 + x_2 \leq \frac{1}{n}\}$. *Then, it is clear that* $u_n \rightharpoonup^\star 0$ *in* $BV(\Omega)$. *However,* $\int_{\tilde{\Omega}} \psi|Du_n| - \int_\Gamma w|u_n| = \sqrt{2}\,\psi_0 - 2\,w_0 < 0$ *as soon as* $\frac{w_0}{\psi_0} > \frac{\sqrt{2}}{2}$.

Example 2.0.2. *Assume that* $\Omega = B(0,1)$, Γ *is a smooth arc of* $\partial\Omega$, $\psi = \psi_0 > 0$ *and* $w = w_0 \in \mathbb{R}$. *Take* $u_n(x) = \min\{|x|,(n-1)(1-|x|)\}$. *Then, it is clear that* $u_n \rightharpoonup^\star u := |x|$ *in* $BV(\Omega)$. *But, we have* $\int_{\tilde{\Omega}} \psi|Du_n| - \int_\Gamma w|u_n| = \pi\,\psi_0[(1-\frac{1}{n})^2 + (n-1)(1-(1-\frac{1}{n})^2)] \to 3\pi\,\psi_0 < \int_{\tilde{\Omega}} \psi|Du| - \int_\Gamma w|u| = \psi_0(\pi + \mathcal{H}^1(\Gamma^c)) - w_0\,\mathcal{H}^1(\Gamma)$ *as soon as* $\frac{w_0}{\psi_0} < -1$.

First of all, we start by showing that Problems (2.2) & (2.1) are completely equivalent.

Proposition 2.1. *Problems* (2.1) & (2.2) *have the same minimal value. If* u *is a solution for Problem* (2.1)*, then* u *solves Problem* (2.2)*. In addition, if* u *is a solution for Problem* (2.2) *with* $u = 0$ *on* Γ^c *then* u *solves Problem* (2.1).

Proof. It is obvious that inf (2.2) $\leq$ inf (2.1). On the other hand, fix $u \in BV(\tilde{\Omega})$ such that $u = 0$ on $\tilde{\Omega}\backslash\Omega$. For every $n \in \mathbb{N}^\star$, let η_n be a cutoff function such that $0 \leq \eta_n \leq 1$, $\eta_n(x) = 0$ on Γ and, $\eta_n(x) = 1$

for all $x \in \Omega$ with $\mathrm{dist}(x, \Gamma^c) > \frac{1}{n}$. Now, set $u_n = \eta_n u$. Then, we have $u_n = 0$ on Γ^c. In addition, it is clear that $u_n \to u$ in $L^1(\Omega)$ and so, $\int_\Omega \phi\,|u_n| \to \int_\Omega \phi\,|u|$. Moreover, one has

$$\int_\Omega \psi|Du_n| = \int_\Omega \psi|\eta_n Du + uD\eta_n| \le \int_\Omega \psi\eta_n|Du| + \int_\Omega \psi|D\eta_n||u| \to \int_\Omega \psi|Du| + \int_{\Gamma^c} \psi|u|\,d\mathcal{H}^{N-1}. \tag{2.3}$$

Yet, we also have

$$\int_\Gamma w\,|u_n| = \int_\Gamma w\,\eta_n|u| \to \int_\Gamma w\,|u|.$$

Hence,

$$\lim_n \left[\int_\Omega \psi|Du_n| - \int_\Gamma w|u_n|\right] \le \int_\Omega \psi|Du| + \int_{\Gamma^c} \psi|u| - \int_\Gamma w|u|.$$

Finally, we get that

$$\inf\,(2.1) \le \lim_n \left[\frac{\int_\Omega \psi|Du_n| - \int_\Gamma w|u_n|}{\int_\Omega \phi|u_n|}\right] \le \frac{\int_{\tilde\Omega} \psi|Du| - \int_\Gamma w|u|}{\int_{\tilde\Omega} \phi|u|}.$$

Since u is arbitrary, then we infer that $\inf\,(2.1) \le \inf\,(2.2)$. Consequently, the equality $\inf\,(2.1) = \inf\,(2.2)$ holds. The rest follows immediately from this equality. $\square$

Remark 2.1. *We clearly see that one can restrict Problem (2.2) to nonnegative functions (just by replacing u with $|u|$) and so,*

$$\min\,(2.2) = \min\left\{\frac{\int_{\tilde\Omega} \psi|Du| - \int_\Gamma wu}{\int_{\tilde\Omega} \phi u} \;:\; u \ne 0 \in BV(\tilde\Omega),\; u \ge 0,\; u = 0 \;\; on \;\; \tilde\Omega\backslash\Omega\right\}.$$

In the sequel, we will only consider nonnegative solutions to (2.2).

In order to prove existence of a solution to the relaxed problem (2.2), we need first to introduce the following constant:

$$\Lambda^\star := \sup\left\{\frac{\int_\Gamma w|u|}{\int_\Omega \psi|Du|} \;:\; u \ne 0 \in BV(\Omega),\;\; u = 0 \;\text{ on }\; \Gamma^c\right\}. \tag{2.4}$$

Moreover, the analysis will be performed under the following geometric assumption:

Definition 2.1. *Suppose that Γ is of class C^1. Then, we say that Ω satisfies a C^1-extension property near Γ^c if there exists an open bounded set $\tilde\Omega$ with C^1 boundary such that $\Omega \subset \tilde\Omega$ and $\partial\Omega \cap \partial\tilde\Omega = \Gamma$.*

Then, we have the following existence result:

Proposition 2.2. *Assume Γ is C^1, Ω satisfies a C^1-extension property near Γ^c, $||w||_\infty \le \psi_0$ and, $\Lambda^\star < 1$. Then, Problem (2.2) reaches a minimum.*

Proof. Let $(u_n)_n$ be a minimizing sequence in Problem (2.2). For every $n \in \mathbb{N}$, set $\tilde u_n := \frac{u_n}{\int_{\tilde\Omega} \phi|u_n|}$. So, it is clear that $(\tilde u_n)_n$ is also a minimizing sequence. In particular, there is a constant $C < \infty$ such that

$$\int_{\tilde\Omega} \psi|D\tilde u_n| - \int_\Gamma w|\tilde u_n| \le C, \;\text{ for all }\; n \in \mathbb{N}.$$

Hence,

$$(1 - \Lambda^\star)\int_{\tilde\Omega} \psi|D\tilde u_n| \le \int_{\tilde\Omega} \psi|D\tilde u_n| - \int_\Gamma w|\tilde u_n| \le C, \quad \text{for all }\; n \in \mathbb{N}.$$

Since $\Lambda^\star < 1$ and $\psi \ge \psi_0 > 0$, then we get that

$$\int_{\tilde\Omega} |D\tilde u_n| \le C, \;\text{ for all }\; n \in \mathbb{N}. \tag{2.5}$$

Yet, we have $||\phi\, \tilde{u}_n||_{L^1} = 1$ and $\phi \geq \phi_0 > 0$. Hence, up to a subsequence, $\tilde{u}_n$ converges weakly* in $BV(\tilde{\Omega})$ to some function $\tilde{u}$. In particular, $\tilde{u}_n \to \tilde{u}$ strongly in $L^1(\tilde{\Omega})$. This implies that $||\phi\, \tilde{u}_n||_{L^1} \to ||\phi\, \tilde{u}||_{L^1} = 1$ and $\tilde{u} = 0$ on $\tilde{\Omega}\backslash\Omega$.

On the other side, inspired by [21, Proposition 1.2], we also claim that the functional in (2.2) is lower semicontinuous with respect to the weak* convergence in $BV(\tilde{\Omega})$ and so, $\tilde{u}$ is a minimizer in (2.2). First, we clearly have

$$\int_{\tilde{\Omega}} \psi|D\tilde{u}_n| - \int_\Gamma w|\tilde{u}_n| - \int_{\tilde{\Omega}} \psi|D\tilde{u}| + \int_\Gamma w|\tilde{u}| \geq \int_{\tilde{\Omega}} \psi|D\tilde{u}_n| - \int_{\tilde{\Omega}} \psi|D\tilde{u}| - ||w||_\infty \int_\Gamma |\tilde{u}_n - \tilde{u}|.$$

Fix $\varepsilon > 0$. Then, we define $A_\varepsilon := \{x \in \tilde{\Omega} \,:\, d(x, \partial\tilde{\Omega}) \leq \varepsilon\}$. Let $\eta_\varepsilon \in C_0^\infty(\tilde{\Omega})$ be a cutoff function such that $0 \leq \eta_\varepsilon \leq 1$ and $\eta_\varepsilon = 1$ on $\tilde{\Omega}_\varepsilon := \tilde{\Omega}\backslash A_\varepsilon$. Set $v_{\varepsilon,n} := (1-\eta_\varepsilon)(\tilde{u}_n - \tilde{u})$. By the trace inequality for BV functions (see [3]), there are two constants c_1 and c_2 such that the following estimate holds:

$$\int_{\partial\Omega} |v_{\varepsilon,n}| \leq c_1 \int_{A_\varepsilon} |Dv_{\varepsilon,n}| + c_2 \int_{A_\varepsilon} |v_{\varepsilon,n}|.$$

Thus, we get that

$$\begin{aligned} \int_\Gamma |\tilde{u}_n - \tilde{u}| &\leq c_1 \int_{A_\varepsilon} (1-\eta_\varepsilon)|D(\tilde{u}_n - \tilde{u})| + c_1 \int_{A_\varepsilon} |\tilde{u}_n - \tilde{u}||D\eta_\varepsilon| + c_2 \int_{A_\varepsilon} (1-\eta_\varepsilon)|\tilde{u}_n - \tilde{u}| \\ &\leq \frac{c_1}{\psi_0} \int_{A_\varepsilon} \psi|D(\tilde{u}_n - \tilde{u})| + \frac{C}{\varepsilon} \int_{A_\varepsilon} |\tilde{u}_n - \tilde{u}|, \end{aligned} \tag{2.6}$$

where the constant C depends on c_2. Hence, one has

$$\begin{aligned} &\int_{\tilde{\Omega}} \psi|D\tilde{u}_n| - \int_\Gamma w|\tilde{u}_n| - \int_{\tilde{\Omega}} \psi|D\tilde{u}| + \int_\Gamma w|\tilde{u}| \\ &\geq \int_{\tilde{\Omega}} \psi|D\tilde{u}_n| - \int_{\tilde{\Omega}} \psi|D\tilde{u}| - \frac{c_1||w||_\infty}{\psi_0} \int_{A_\varepsilon} \psi|D(\tilde{u}_n - \tilde{u})| - \frac{C||w||_\infty}{\varepsilon} \int_{A_\varepsilon} |\tilde{u}_n - \tilde{u}| \\ &\geq \int_{\tilde{\Omega}} \psi|D\tilde{u}_n| - \int_{\tilde{\Omega}} \psi|D\tilde{u}| - \frac{c_1||w||_\infty}{\psi_0} \int_{A_\varepsilon} \psi|D\tilde{u}_n| - \frac{c_1||w||_\infty}{\psi_0} \int_{A_\varepsilon} \psi|D\tilde{u}| - \frac{C||w||_\infty}{\varepsilon} \int_{A_\varepsilon} |\tilde{u}_n - \tilde{u}| \\ &\geq \int_{\tilde{\Omega}_\varepsilon} \psi|D\tilde{u}_n| - \int_{\tilde{\Omega}_\varepsilon} \psi|D\tilde{u}| + \left(1 - \frac{c_1||w||_\infty}{\psi_0}\right) \int_{A_\varepsilon} \psi|D\tilde{u}_n| - \left(1 + \frac{c_1||w||_\infty}{\psi_0}\right) \int_{A_\varepsilon} \psi|D\tilde{u}| \\ &\quad - \frac{C||w||_\infty}{\varepsilon} \int_{A_\varepsilon} |\tilde{u}_n - \tilde{u}|. \end{aligned}$$

Since Γ is C^1 and Ω satisfies a C^1-extension property near Γ^c, then the boundary of $\tilde{\Omega}$ is of class C^1 and so, thanks to [3, Theorem 4], one can assume that in (2.6) the constants $c_1 = 1 + \delta$ and $c_2 = c_2(\Omega, \delta)$, where $\delta > 0$ can be chosen sufficiently small. Let us assume that $||w||_\infty < \psi_0$. Hence, choosing $\delta > 0$ small enough, we infer that

$$\begin{aligned} &\int_{\tilde{\Omega}} \psi|D\tilde{u}_n| - \int_\Gamma w|\tilde{u}_n| - \int_{\tilde{\Omega}} \psi|D\tilde{u}| + \int_\Gamma w|\tilde{u}| \\ &\geq \int_{\tilde{\Omega}_\varepsilon} \psi|D\tilde{u}_n| - \int_{\tilde{\Omega}_\varepsilon} \psi|D\tilde{u}| - (2+\delta) \int_{A_\varepsilon} \psi|D\tilde{u}| - \frac{C||w||_\infty}{\varepsilon} \int_{A_\varepsilon} |\tilde{u}_n - \tilde{u}|. \end{aligned}$$

Passing to the limit when $n \to \infty$ and using the lower semicontinuity of the weighted total variation (see [6, Corollary 1])

$$\liminf_n \int_{\tilde{\Omega}_\varepsilon} \psi|D\tilde{u}_n| \geq \int_{\tilde{\Omega}_\varepsilon} \psi|D\tilde{u}|$$

as well as the L^1 convergence, we get

$$\liminf_n \left[\int_{\tilde{\Omega}} \psi|D\tilde{u}_n| - \int_\Gamma w|\tilde{u}_n|\right] - \int_{\tilde{\Omega}} \psi|D\tilde{u}| + \int_\Gamma w|\tilde{u}| \geq -(2+\delta) \int_{A_\varepsilon} \psi|D\tilde{u}|.$$

Let $\varepsilon \to 0^+$, this yields that

$$\liminf_n \left[\int_{\tilde{\Omega}} \psi |D\tilde{u}_n| - \int_\Gamma w|\tilde{u}_n| \right] - \int_{\tilde{\Omega}} \psi |D\tilde{u}| + \int_\Gamma w|\tilde{u}| \geq 0.$$

Finally, assume that $||w||_\infty = \psi_0$. So, we will prove lower semicontinuity of the functional in (2.2) by approximation. More precisely, fix $\zeta > 0$ small enough. Then, we have

$$\int_{\tilde{\Omega}} \psi |D\tilde{u}_n| - \int_\Gamma w|\tilde{u}_n| = \int_{\tilde{\Omega}} \psi |D\tilde{u}_n| - (1-\zeta) \int_\Gamma w|\tilde{u}_n| - \zeta \int_\Gamma w|\tilde{u}_n|.$$

Recalling (2.5) and the fact that $\Lambda^\star < 1$, we infer that

$$\begin{aligned} \liminf_n \left[\int_{\tilde{\Omega}} \psi |D\tilde{u}_n| - \int_\Gamma w|\tilde{u}_n| \right] &\geq \liminf_n \left[\int_{\tilde{\Omega}} \psi |D\tilde{u}_n| - (1-\zeta) \int_\Gamma w|\tilde{u}_n| \right] - C\zeta \\ &\geq \int_{\tilde{\Omega}} \psi |D\tilde{u}| - (1-\zeta) \int_\Gamma w|\tilde{u}| - C\zeta. \end{aligned}$$

Since $\zeta > 0$ is arbitrarily small, then this concludes the proof of our claim. □

Remark 2.2. *We note that if $w = 0$ then the C^1 regularity of Γ is not needed and, the existence of a solution to Problem (2.2) is trivial in this case. In Example 2.0.1, take $w_0 = \psi_0 = 1$ then the condition $||w||_\infty \leq \psi_0$ is well satisfied but, the functional $u \mapsto \int_{\tilde{\Omega}} \psi |Du| - \int_\Gamma w|u|$ is not lower semicontinuous due to the lack of C^1−regularity of the arc Γ. However, in Example 2.0.2, the arc Γ is smooth but the functional is always not lower semicontinuous provided that $w_0 < -\psi_0$. This shows the necessity of the assumptions we made in Proposition 2.2.*

Remark 2.3. *Although Problem (2.2) has a solution u but it is still not clear whether this solution solves Problem (2.1), or equivalently if this solution u satisfies the Dirichlet condition ($u = 0$ on Γ^c). In fact, we will see that this is not necessarily the case and, a solution to (2.1) may not exist.*

On the other hand, one can also study the summability of a solution u in Problem (2.2). Inspired by the proof of [12, Proposition 7] (see also [6, Theorem 4]), one can show that any solution of (2.2) must be bounded. For this aim, we start by the following:

Proposition 2.3. *Let H be a Lipschitz nondecreasing function on $\mathbb{R}_+$ with $H(0) = 0$. For any nonnegative solution u of Problem (2.2), the function $H(u)$ is also a solution for (2.2).*

Proof. We note that this proof follows the lines of the proof of [6, Proposition 1]. First, let us assume that H is smooth. Then, we consider the Cauchy problem:

$$\begin{cases} \partial_t y(t,v) = -H(y(t,v)), & t \geq 0 \\ y(0,v) = v. \end{cases} \tag{2.7}$$

Let $y(t,v)$ be the solution of (2.7). Thanks to our assumptions on H, $y(t,v)$ is smooth. For every $t \geq 0$, we define $u_t = y(t,u)$ (so, we have $u_0 = u$). Now, we consider the map

$$h(t) = \int_{\tilde{\Omega}} \psi |Du_t| - \int_\Gamma w\, u_t - \lambda^\star \int_{\tilde{\Omega}} \phi\, u_t,$$

where we recall that $\lambda^\star = \min\,(2.2)$. Since u_0 is a minimizer for Problem (2.2) and $u_t = 0$ on $\tilde{\Omega} \backslash \Omega$ for every $t \geq 0$ (this follows from the fact that $y(t,0) = 0$ and the uniqueness of the solution in (2.7)), then h has a minimum at $t = 0$. In particular, we have

$$\lim_{t \to 0^+} \frac{h(t) - h(0)}{t} \geq 0.$$

Yet,

$$\frac{h(t)-h(0)}{t} = \int_{\tilde\Omega} \psi \, \frac{|Du_t|-|Du_0|}{t} - \int_\Gamma w \, \frac{u_t-u_0}{t} - \lambda^\star \int_{\tilde\Omega} \phi \, \frac{u_t-u_0}{t}.$$

For every $x \in \tilde\Omega \cup \Gamma$, we have

$$\frac{u_t(x)-u_0(x)}{t} = \frac{y(t,u(x))-y(0,u(x))}{t} \longrightarrow -H(u(x)).$$

Taking the derivative with respect to v in (2.7), we get that

$$\begin{cases} \partial_t[\partial_v y(t,v)] = -H'(y(t,v))\, \partial_v y(t,v), & t \geq 0 \\ \partial_v y(0,v) = 1. \end{cases}$$

Hence,

$$\partial_v y(t,v) = e^{-\int_0^t H'(y(s,v))\,\mathrm{d}s} \geq 0.$$

By the chain rule for BV functions (see [2]), we have

$$|Du_t| = \partial_v y(t,u)|\tilde D u| + [y(t,u^+)-y(t,u^-)] \cdot \mathcal{H}^{N-1} \llcorner J_u,$$

where u^+ and u^- are respectively the approximate upper and lower limits, J_u is the jump set of u, and the nonnegative measure $|\tilde D u|$ is the sum of the absolutely continuous part and the Cantor part of $|Du|$. Consequently, we have

$$\begin{aligned} \frac{|Du_t|-|Du_0|}{t} &= \frac{\partial_v y(t,u)-\partial_v y(0,u)}{t}\,|\tilde D u| + \frac{[y(t,u^+)-y(0,u^+)]-[y(t,u^-)-y(0,u^-)]}{t} \cdot \mathcal{H}^{N-1} \llcorner J_u \\ &\longrightarrow -H'(u)|\tilde D u| - [H(u^+)-H(u^-)] \cdot \mathcal{H}^{N-1} \llcorner J_u. \end{aligned}$$

Therefore,

$$\int_{\tilde\Omega} \psi\, H'(u)|\tilde D u| + \int_{J_u} \psi\,[H(u^+)-H(u^-)]\mathcal{H}^{N-1} - \int_\Gamma w\, H(u) - \lambda^\star \int_{\tilde\Omega} \phi\, H(u) \leq 0.$$

Since $H' \geq 0$ and $|D(H(u))| = H'(u)|\tilde D u| + [H(u^+)-H(u^-)] \cdot \mathcal{H}^{N-1} \llcorner J_u$, this yields that $H(u)$ also minimizes Problem (2.2). Finally, it remains to extend the result to the case when H is not smooth; but this can be done by approximation. In fact, one can approximate H with a sequence of smooth Lipschitz increasing functions H_n with $H_n(0) = 0$ such that $H_n(u)$ converges weakly* to $H(u)$ in $BV(\tilde\Omega)$. Hence, $H_n(u)$ is a solution to (2.2), for every n. Yet, recalling the proof of Proposition 2.2, we know that the functional in (2.2) is lower semicontinuous w.r.t. the weak* convergence in $BV(\tilde\Omega)$. This yields that $H(u)$ is also a solution. □

Under the assumptions of Proposition 2.2, we get as a consequence of Proposition 2.3 the following summability result.

Proposition 2.4. *Let u be a solution for Problem (2.2), then u belongs to $L^\infty(\Omega)$.*

Proof. Fix $M > 0$ large enough. Thanks to Proposition 2.3, we see that $u_M := \min\{u, M\}$ is a solution for Problem (2.2). Therefore, we have

$$\int_{\tilde\Omega} \psi |Du_M| - \int_\Gamma w u_M = \lambda^\star \int_{\tilde\Omega} \phi\, u_M \leq \lambda^\star ||\phi||_\infty ||u_M||_1. \tag{2.8}$$

Since $\Lambda^\star < 1$ and $\psi \geq \psi_0 > 0$, then by (2.8) we get

$$\int_{\tilde\Omega} |Du_M| \leq \frac{\lambda^\star ||\phi||_\infty}{\psi_0(1-\Lambda^\star)} ||u_M||_1. \tag{2.9}$$

Yet, one has

$$||u_M||_{\frac{N}{N-1}} \leq C \int_{\tilde{\Omega}} |Du_M|.$$

Hence,

$$||u_M||_{\frac{N}{N-1}} \leq C||u_M||_1. \tag{2.10}$$

But, it is clear that u_M^p is also a solution for Problem (2.2), for all $p \geq 1$. Then, thanks to (2.10), we also have

$$||u_M^p||_{\frac{N}{N-1}} \leq C||u_M^p||_1.$$

This yields that

$$||u_M||_{\frac{Np}{N-1}} \leq C^{\frac{1}{p}}||u_M||_p.$$

Fix $n \in \mathbb{N}$. Then, by induction, we get that

$$||u_M||_{(\frac{N}{N-1})^n} \leq C^{(\frac{N-1}{N})^{n-1}}||u_M||_{(\frac{N}{N-1})^{n-1}} \leq C^{[(\frac{N-1}{N})^{n-1}+(\frac{N-1}{N})^{n-2}+\ldots+1]}\,||u_M||_1.$$

Consequently,

$$||u_M||_{(\frac{N}{N-1})^n} \leq C^{N[1-(\frac{N-1}{N})^n]}||u||_1, \quad \text{for all} \quad n \in \mathbb{N}.$$

Passing to the limit when $n \to \infty$, this yields that

$$||u_M||_\infty \leq C^N||u||_1. \tag{2.11}$$

Finally, letting $M \to \infty$ in (2.11), this concludes the proof that $u \in L^\infty(\Omega)$. □

In addition, one can show that Problem (2.2) is also equivalent to a shape optimization problem of type *Cheeger* and that any superlevel set of a solution u is an optimal set (see [7, 15, 16] for similar level-sets approach for variational problems involving total variation minimization). More precisely, we introduce the following problem:

$$\min\left\{\frac{Per_\psi(A) - \int_{\partial^\star A\cap\Gamma} w}{\int_A \phi} \,:\, A \subset \Omega\right\} \tag{2.12}$$

where $Per_\psi(A) := \int_{\tilde{\Omega}} \psi|D\chi_A| = \int_{\Omega\cup\Gamma^c} \psi|D\chi_A| = \int_{\partial^\star A\setminus\Gamma} \psi \,\mathrm{d}\mathcal{H}^{N-1}$ is the weighted perimeter of A that is taken relative to $\tilde{\Omega}$ (or equivalently, relative to $\Omega\cup\Gamma^c$ since A is assumed to be a subset of Ω) and $\partial^\star A$ denotes the reduced boundary of A. Under the assumptions of Proposition 2.2, we have the following:

Proposition 2.5. *The values of Problems* (2.2) *and* (2.12) *coincide (i.e.,* $\min$ (2.2) $= \min$ (2.12)*). In addition, a function u solves* (2.2) *if and only if the superlevel sets $A_t := \{u > t\}$ solve* (2.12)*, for almost all $t \geq 0$. In particular, Problem* (2.12) *admits an optimal set $A^\star$.*

Proof. By considering characteristic functions $u := \chi_A$ where $A \subset \Omega$ in (2.2), it is obvious that we get $\min$ (2.2) $\leq \min$ (2.12). Now, let us show the reverse inequality. Fix $u \in BV(\tilde{\Omega})$ with $u \geq 0$ and $u = 0$ on $\tilde{\Omega}\setminus\Omega$. Using the coarea formula, we have

$$\begin{aligned}
&\int_{\tilde{\Omega}} \psi|Du| - \int_\Gamma wu \\
&= \int_0^{+\infty}\int_{\partial^\star A_t\setminus\Gamma} \psi \,\mathrm{d}\mathcal{H}^{N-1} - \int_0^{+\infty}\int_{\partial^\star A_t\cap\Gamma} w \,\mathrm{d}\mathcal{H}^{N-1}\,\mathrm{d}t \\
&= \int_0^{+\infty} \frac{Per_\psi(A_t) - \int_{\partial^\star A_t\cap\Gamma} w \,\mathrm{d}\mathcal{H}^{N-1}}{\int_{A_t}\phi}\left(\int_{A_t}\phi\right)\mathrm{d}t \\
&\geq \min\left\{\frac{Per_\psi(A) - \int_{\partial^\star A\cap\Gamma} w}{\int_A \phi} \,:\, A \subset \Omega\right\}\int_0^{+\infty}\left(\int_{A_t}\phi\right)\mathrm{d}t
\end{aligned} \tag{2.13}$$

$$= \min\left\{\frac{Per_\psi(A) - \int_{\partial^\star A\cap\Gamma} w}{\int_A \phi} \ : \ A \subset \Omega\right\} \int_{\tilde\Omega} \phi u$$

Hence,

$$\min(2.2) \geq \min\left\{\frac{Per_\psi(A) - \int_{\partial^\star A\cap\Gamma} w}{\int_A \phi} \ : \ A \subset \Omega\right\}.$$

This yields that $\min(2.2) = \min(2.12)$. Moreover, if u is a minimizer in Problem (2.2) then the inequality in (2.13) becomes equality. Yet, this means that for almost every $t \geq 0$, we have

$$\frac{Per_\psi(A_t) - \int_{\partial^\star A_t\cap\Gamma} w}{\int_{A_t} \phi} = \min\left\{\frac{Per_\psi(A) - \int_{\partial^\star A\cap\Gamma} w}{\int_A \phi} \ : \ A \subset \Omega\right\}.$$

Consequently, the superlevel sets $A_t = \{u > t\}$ solve (2.12), for a.e. $t \geq 0$. The last statement follows directly from Proposition 2.2. □

Remark 2.4. *In fact, one can show in Proposition 2.5 that for every $t \geq 0$, the superlevel set $A_t = \{u > t\}$ is optimal for Problem (2.12). Indeed, let $(t_n)_n$ be a decreasing sequence such that $t_n \to t$ and A_{t_n} is optimal in (2.12), for all n. Recalling the estimate (2.9), we have*

$$Per_\psi(A_{t_n}) \leq \frac{\lambda^\star ||\phi||_\infty}{\psi_0(1-\Lambda^\star)}|A_{t_n}| \leq \frac{\lambda^\star ||\phi||_\infty}{\psi_0(1-\Lambda^\star)}|\Omega|.$$

Hence, $\chi_{A_{t_n}}$ is bounded in $BV(\Omega)$ and so, up to a subsequence, $\chi_{A_{t_n}} \rightharpoonup^\star \chi_{A_t}$ in $BV(\Omega)$. In particular, one has $Per(A_t) < \infty$. Finally, the lower semicontinuity of the functional in (2.2) yields that χ_{A_t} is also a solution for Problem (2.2).

Remark 2.5. *Similarly to the proof of Proposition 2.5 about the equivalence between Problems (2.2) & (2.12), one can show using the coarea formula again that*

$$\Lambda^\star = \sup\left\{\frac{\int_\Gamma w|u|}{\int_\Omega \psi|Du|} \ : \ u \neq 0 \in BV(\Omega), \ \ u = 0 \ \text{ on } \ \Gamma^c\right\} = \sup\left\{\frac{\int_{\partial^\star A\cap\Gamma} w}{Per_\psi(A)} \ : \ A \subset \Omega\right\}.$$

In particular, we have $\Lambda^\star < 1$ if and only if

$$\int_{\partial^\star A\cap\Gamma} w < Per_\psi(A), \qquad \text{for all} \ \ A \subset \Omega. \tag{2.14}$$

This condition is always satisfied as soon as $w \leq 0$. Otherwise, it holds obviously if for all $A \subset \Omega$, we have

$$\mathcal{H}^{N-1}(\partial^\star A \cap \Gamma) < \frac{\psi_0}{||w^+||_\infty} Per(A).$$

For instance, if $||w^+||_\infty \leq \psi_0$ and Γ is a line segment, then the inequality above is clearly satisfied. Now, assume that Γ is not a line segment, the distance between the endpoints of Γ is D and the length of Γ is L. Then, we see that when the ratio L/D increases, the factor $\frac{\psi_0}{||w^+||_\infty}$ should be large enough in order to guarantee the existence of a solution to Problem (2.12).

We conclude this section by showing that any solution u has a flat part $\{u = ||u||_\infty\}$. This result has already been proven in [6, Theorem 5] but the proof here is completely different and we also consider it much simpler. More precisely, we have the follwing:

Proposition 2.6. *Let u be a solution of Problem (2.2). Then, we have $|\{u = ||u||_\infty\}| > 0$.*

Proof. Let $A_t := \{u \geq t\} \neq \emptyset$ be a superlevel set of u. Thanks to Proposition 2.5, we know that A_t is an optimal set in Problem (2.12). Hence, one has

$$\int_{\tilde\Omega} \psi|D\chi_{A_t}| - \int_{\partial^\star A_t\cap\Gamma} w = \lambda^\star \int_{A_t} \phi. \tag{2.15}$$

From Remark 2.5 and since $\Lambda^\star < 1$, we get

$$\lambda^\star ||\phi||_\infty\, |A_t| \geq \lambda^\star \int_{A_t} \phi = \int_{\tilde{\Omega}} \psi |D\chi_{A_t}| - \int_{\partial^\star A_t \cap \Gamma} w \geq (1-\Lambda^\star) \int_{\tilde{\Omega}} \psi |D\chi_{A_t}| \geq c(1-\Lambda^\star)\, \psi_0\, |A_t|^{\frac{N-1}{N}},$$

where $c > 0$ is a universal constant. Therefore, we infer the following estimate:

$$|A_t| \geq \left(\frac{c(1-\Lambda^\star)\psi_0}{\lambda^\star ||\phi||_\infty}\right)^N.$$

In particular, this yields that

$$|\{u = ||u||_\infty\}| \geq \left(\frac{c(1-\Lambda^\star)\psi_0}{\lambda^\star ||\phi||_\infty}\right)^N > 0. \quad \square$$

3. Regularity properties of optimal sets

In this section, we study the regularity of an optimal set $A^\star$ in Problem (2.12). In [15, Theorem 5], the authors have already studied the regularity of $\partial A^\star$ but in the particular case when $w = 0$ on Γ and $N = 2$. However, there is a gap in their proof since in order to prove regularity on $\partial A^\star$ they assume that $\partial A^\star$ is in $W^{1,1}$; but it is not clear why an arc of $\partial A^\star$ cannot be for instance the graph of a Cantor function. Fortunately, this is not the case as the results below show.

Proposition 3.1. *Assume that ψ is locally Lipschitz in Ω. Then, there exists a relatively closed set $\Sigma \subset \partial A^\star \cap \Omega$ such that $\mathcal{H}^{N-2}(\Sigma) = 0$ and for every $x \in (\partial A^\star \backslash \Sigma) \cap \Omega$, $\partial A^\star$ is of class $C^{1,\frac{1}{2}}$ around x.*

Proof. First of all, it is clear that if $A^\star$ minimizes (2.12) then $A^\star$ solves also the following problem:

$$\min\left\{ Per_\psi(A) - \int_{\partial^\star A \cap \Gamma} w \,-\, \lambda^\star \int_A \phi \,:\, A \subset \Omega \right\} \tag{3.1}$$

where $\lambda^\star = \min\,(2.2)$. Fix $x_0 \in \partial A^\star \cap \Omega$ and $0 < r_0 < d(x_0, \partial\Omega)$. Let $E \subset \mathbb{R}^N$ be a set with finite perimeter such that $A^\star \Delta E \subset B(x_0, r_0)$. In particular, we have $E \subset \Omega$. Thanks to the minimality of $A^\star$ in (3.1), we get that

$$Per_\psi(A^\star) - \int_{\partial^\star A^\star \cap \Gamma} w \,-\, \lambda^\star \int_{A^\star} \phi \leq Per_\psi(E) - \int_{\partial^\star E \cap \Gamma} w \,-\, \lambda^\star \int_E \phi.$$

But, we clearly have $\partial^\star A^\star \cap \Gamma = \partial^\star E \cap \Gamma$, since $A^\star \Delta E \subset B(x_0, r_0)$ and $r_0 < d(x_0, \partial\Omega)$. Hence, we infer that

$$Per_\psi(A^\star) \,-\, \lambda^\star \int_{A^\star} \phi \leq Per_\psi(E) \,-\, \lambda^\star \int_E \phi.$$

Consequently, we get that

$$Per_\psi(A^\star) \leq Per_\psi(E) \,+\, \lambda^\star ||\phi||_\infty\, |A^\star \Delta E|.$$

In other words, $A^\star$ is a $(\Lambda, r_0)-$ minimizer of $Per_\psi(E)$ in Ω with $\Lambda = \lambda^\star ||\phi||_\infty$ (see [22]). Then, thanks to [22, Theorem 1.10], we infer that $A^\star$ has boundary of class $C^{1,\frac{1}{2}}$, out of a closed singular set $\Sigma \subset \partial A^\star$ of dimension $d < N - 2$. $\square$

Remark 3.1. *In fact, we can reduce the dimension of the singular set Σ in Proposition 3.1 to $N - 8$ but perhaps with less regularity on $\partial A^\star$. More precisely, thanks to* [18, Theorem 3.2]*, one can show that $\partial A^\star$ is of class $C^{1,\frac{1}{4}}$ inside Ω, except at a singular set of dimension $N - 8$. For this aim, we just need to show that $A^\star$ is an almost minimal set in $B(x_0, r_0)$, for every point $x_0 \in \partial A^\star$ and $r_0 > 0$ small enough such that $\overline{B(x_0, r_0)} \subset \Omega$. Indeed, let $x \in \partial A^\star \cap B(x_0, r_0)$ and $r > 0$ be small enough so* *that $B_r := B(x,r) \subset B(x_0,r_0)$. Recalling the proof of Proposition 3.1, for any subset $A \subset \Omega$ such that $A\Delta A^\star \subset B_r$, one has*

$$\int_{\overline{B_r}} \psi\, |D\chi_{A^\star}| - \lambda^\star \int_{A^\star\cap B_r} \phi \le \int_{\overline{B_r}} \psi\, |D\chi_A| - \lambda^\star \int_{A\cap B_r} \phi.$$

In particular,

$$\int_{\overline{B_r}} |D\chi_{A^\star}| \le \frac{1}{\psi_0} \int_{\overline{B_r}} \psi\, |D\chi_{A^\star}| \le \frac{1}{\psi_0}\left[\int_{\partial B_r} \psi - \lambda^\star \int_{B_r} \phi + \lambda^\star \int_{A^\star\cap B_r} \phi\right] \le C\, r^{N-1}.$$

Yet, we have

$$\begin{aligned}&\psi(x) \int_{\overline{B_r}} |D\chi_{A^\star}| + \int_{\overline{B_r}} [\psi - \psi(x)]\, |D\chi_{A^\star}| - \lambda^\star \int_{A^\star\cap B_r} \phi \\ &\le \psi(x) \int_{\overline{B_r}} |D\chi_A| + \int_{\overline{B_r}} [\psi - \psi(x)]\, |D\chi_A| - \lambda^\star \int_{A\cap B_r} \phi.\end{aligned}$$

Since ψ is Lipschitz in $\overline{B(x_0,r_0)}$, this implies that $|\psi - \psi(x)| \le Cr$ on B_r and so, we get the following estimate:

$$\int_{\overline{B_r}} |D\chi_{A^\star}| \le \int_{\overline{B_r}} |D\chi_A| + C\, r^N.$$

Proposition 3.2. *Assume that $\phi \in C(\Omega)$ and $\psi \in C^1(\Omega)$. Then, the boundary of $A^\star$, out of the singular set Σ, is of class $C^{1,\alpha}$, for all $\alpha < 1$. Moreover, $\partial A^\star \backslash \Sigma$ is $C^{2,\alpha}$ inside Ω as soon as $\phi \in C^{0,\alpha}(\Omega)$ and $\psi \in C^{1,\alpha}(\Omega)$. Moreover, the mean curvature $H_{A^\star}$ of $\partial A^\star$ at any point $x \notin \Sigma$ is given by the following formula (where $\partial_{\mathbf{n}}\psi$ denotes the interior normal derivative of ψ on $\partial A^\star$):*

$$(N-1)H_{A^\star}(x) = \frac{\lambda^\star\, \phi(x) + \partial_{\mathbf{n}}\psi(x)}{\psi(x)}.$$

Proof. First, we recall from Proposition 3.1 that there is a closed set $\Sigma \subset \partial A^\star$ such that $\partial A^\star \backslash \Sigma$ is $C^{1,\frac{1}{2}}$ inside Ω. Fix a point $x_0 \in (\partial A^\star \backslash \Sigma) \cap \Omega$. Without loss of generality, we assume that x_0 is the origin. We may also assume that near x_0, $\partial A^\star$ is the graph of a function $v^\star : B_\varepsilon \mapsto \mathbb{R}$, for some $\varepsilon > 0$ small enough. So, we already know that $v^\star \in C^{1,\frac{1}{2}}(B_\varepsilon)$. It is clear that $v^\star$ minimizes the following problem:

$$\min\left\{\int_{B_\varepsilon} \psi(x, v(x))\, \sqrt{1 + |\nabla v(x)|^2}\, \mathrm{d}x + \lambda^\star \int_{B_\varepsilon} \int_0^{v(x)} \phi(x,t)\, \mathrm{d}t\, \mathrm{d}x : v \in BV(B_\varepsilon),\ v|\partial B_\varepsilon = v^\star|\partial B_\varepsilon\right\}.$$

From the optimality conditions on $v^\star$, we have

$$\nabla \cdot \left[\psi(x, v^\star(x)) \frac{\nabla v^\star(x)}{\sqrt{1 + |\nabla v^\star(x)|^2}}\right] = \partial_{x_N} \psi(x, v^\star(x))\, \sqrt{1 + |\nabla v^\star(x)|^2} + \lambda^\star\, \phi(x, v^\star(x)). \tag{3.2}$$

Due to the regularity of $v^\star$, (3.2) can be written as

$$\nabla \cdot \left[\frac{\nabla v^\star(x)}{\sqrt{1 + |\nabla v^\star(x)|^2}}\right]$$

$$= \frac{-\frac{[\nabla_x \psi(x, v^\star(x)) + \partial_{x_N} \psi(x, v^\star(x)) \nabla v^\star(x)] \cdot \nabla v^\star(x)}{\sqrt{1 + |\nabla v^\star(x)|^2}} + \partial_{x_N} \psi(x, v^\star(x))\, \sqrt{1 + |\nabla v^\star(x)|^2} + \lambda^\star\, \phi(x, v^\star(x))}{\psi(x, v^\star(x))} \tag{3.3}$$

or equivalently,

$$\sum_{i.j} a_{ij} v^\star_{ij} = f$$

where

$$(3.4) \qquad a_{ij} = \frac{(1+|\nabla v^\star|^2)\delta_{ij} - v_i^\star v_j^\star}{(1+|\nabla v^\star|^2)^{3/2}}$$

and f is the right hand side in (3.3), which is clearly bounded and, it is also Hölder continuous with exponent α as soon as $\phi \in C^{0,\alpha}$ and $\psi \in C^{1,\alpha}$. It is easy to check that there are two positive constants $0 < \lambda < \Lambda < \infty$ such that $\lambda|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2$. Moreover, $a_{ij} \in C^{0,1/2}(B_\varepsilon)$, for all i, j. Thanks to the Calderón-Zygmund estimates, we infer that $v^\star$ is in $W^{2,p}(B_{\frac{\varepsilon}{2}})$ for any $p < \infty$, in particular $v^\star \in C^{1,\alpha}(B_{\frac{\varepsilon}{2}})$ for any $\alpha < 1$. Then, by Schauder estimates (see also [5]), this implies that $v^\star$ is $C^{2,\alpha}$ in $B_{\frac{\varepsilon}{2}}$ provided that $\phi \in C^{0,\alpha}$ and $\psi \in C^{1,\alpha}$. In addition, the mean curvature $H_{A^\star}$ of $\partial A^\star$ at a point $(x, v^\star(x))$ is given by

$$(N-1)H_{A^\star} = \nabla \cdot \left[\frac{\nabla v^\star(x)}{\sqrt{1+|\nabla v^\star(x)|^2}}\right] = \frac{\lambda^\star\, \phi(x, v^\star(x)) + \partial_{\mathbf{n}}\psi(x, v^\star(x))}{\psi(x, v^\star(x))}. \qquad \square$$

In order to extend our regularity result on $\partial A^\star$ up to the boundary $\partial\Omega$, we need to introduce the following definition that generalizes the notion that the mean curvature of Γ^c is bounded from below in the case when $\psi = 1$ (see also [4, Definition 1] and [20]). Let us assume that ψ is extendable to a locally Lipschitz function in $\tilde{\Omega}$.

Definition 3.1. *We say that Γ^c is a $\psi-$almost minimal set if for every $x_0 \in \Gamma^c$ there are constants $r_0 > 0$ small enough and $C < \infty$ such that the following holds*

$$Per_\psi(E \cap \Omega) \leq Per_\psi(E) + C|E \backslash \Omega|, \quad \textit{for every set } E \subset \mathbb{R}^N \textit{ such that } E\backslash\Omega \subset B(x_0, r_0).$$

Remark 3.2. *Assume that $N = 2$, $\psi = 1$ and, Γ^c is convex (as an arc of $\partial\Omega$). Fix a point $x_0 \in \Gamma^c$. Then, it is not difficult to check that*

$$Per(E \cap \Omega) \leq Per(E),$$

for every set $E \subset \mathbb{R}^2$ such that $E\backslash\Omega \subset B(x_0, r_0)$, where $r_0 > 0$ is small enough. In particular, this implies that Γ^c is an almost minimal set in the sense of Definition 3.1.

Proposition 3.3. *Assume that Γ^c is a $\psi-$almost minimal set. Then, there is a relatively closed singular set $\Sigma_b \subset \partial A^\star \cap \Gamma^c$ with dimension $d < N-2$ such that $\partial A^\star \cap \Gamma^c$ is of class $C^{1,\frac{1}{2}}$, outside Σ_b.*

Proof. Fix a point $x_0 \in \partial A^\star \cap \Gamma^c$ and $r_0 > 0$ small enough. We claim that $A^\star$ is a $(\Lambda, r_0)-$ minimizer of $Per_\psi(E)$ in $B(x_0, r_0)$. Let $E \subset \mathbb{R}^N$ be a set with finite perimeter such that $A^\star \Delta E \subset B(x_0, r_0)$. We note that here E is not necessarily contained in Ω. However, $E \cap \Omega$ is always admissible in (3.1). Hence, by minimality of $A^\star$ in (3.1), we infer that

$$Per_\psi(A^\star) - \int_{\partial^\star A^\star \cap \Gamma} w \,-\, \lambda^\star \int_{A^\star} \phi \leq Per_\psi(E \cap \Omega) - \int_{\partial^\star(E\cap\Omega)\cap\Gamma} w \,-\, \lambda^\star \int_{E\cap\Omega} \phi.$$

Let us choose $r_0 > 0$ small enough so that $B(x_0, r_0) \cap \Gamma = \emptyset$. Then, one has $\partial^\star A^\star \cap \Gamma = \partial^\star(E \cap \Omega) \cap \Gamma$, since $A^\star \Delta E \subset B(x_0, r_0)$. Hence,

$$Per_\psi(A^\star) \,-\, \lambda^\star \int_{A^\star} \phi \leq Per_\psi(E \cap \Omega) \,-\, \lambda^\star \int_{E\cap\Omega} \phi$$

and so,

$$Per_\psi(A^\star) \leq Per_\psi(E \cap \Omega) \,+\, \lambda^\star ||\phi||_\infty \,|A^\star \backslash E|.$$

On the other hand, Γ^c is an almost minimizer of $Per_\psi(E)$, $x_0 \in \Gamma^c$ and $E\backslash\Omega \subset B(x_0, r_0)$. Hence, one has

$$Per_\psi(E \cap \Omega) \leq Per_\psi(E) + C|E\backslash\Omega|.$$

Therefore, we get that

$$Per_\psi(A^\star) \le Per_\psi(E) + C|E\backslash\Omega| \,+\, \lambda^\star ||\phi||_\infty \,|A^\star \Delta E|.$$

Yet, $E\backslash\Omega \subset A^\star\Delta E$. Hence, $A^\star$ is a $(\Lambda, r_0)-$ minimizer of $Per_\psi(E)$ in $B(x_0, r_0)$. Thanks again to [22, Theorem 1.10], this yields that $\partial A^\star \cap \Gamma^c$ has boundary of class $C^{1,\frac{1}{2}}$, out of a closed singular set $\Sigma_b \subset \partial A^\star \cap \Gamma^c$ of dimension $d < N-2$. ☐

Proposition 3.4. *Assume that Γ^c is $C^{1,1}$, $\phi \in C(\overline{\Omega})$ and $\psi \in C^1(\overline{\Omega})$. Then, $(\partial A^\star \cap \Gamma^c)\backslash\Sigma_b$ is of class $C^{1,1}$.*

Proof. Thanks to Proposition 3.3, there is a relatively closed singular set $\Sigma_b \subset \partial A^\star \cap \Gamma^c$ such that $(\partial A^\star \cap \Gamma^c)\backslash\Sigma_b$ is of class $C^{1,\frac{1}{2}}$. Fix $x_0 \in (\partial A^\star \cap \Gamma^c)\backslash\Sigma_b$. After rotation and translation of axes, we may assume that $x_0 = 0$ and that the tangent space to Γ^c at x_0 is the hyperplane $x_N = 0$. Let us assume that near x_0, Γ^c is the graph of $h : B_r \mapsto \mathbb{R}$ (where $r > 0$ is small enough) and $\partial A^\star$ is the graph of $v^\star : B_r \mapsto \mathbb{R}$. So, we have that $v^\star \in C^{1,\frac{1}{2}}(B_r)$, $h \in C^{1,1}(B_r)$, $h(0) = v^\star(0) = 0$ and $\nabla h(0) = \nabla v^\star(0) = 0$. Again, we see that $v^\star$ minimizes the following problem:

$$\min\left\{\int_{B_r}\psi(x,v(x))\sqrt{1+|\nabla v(x)|^2}\mathrm{d}x + \lambda^\star\int_{B_r}\int_0^{v(x)}\phi(x,t)\mathrm{d}t\mathrm{d}x : v \in BV(B_r),\, v \ge h,\, v|\partial B_r = v^\star|\partial B_r\right\}.$$

Taking into account the presence of the obstacle $v \ge h$ on B_r, we get instead of (3.3) the following inequality:

$$-\sum_{i.j} a_{ij} v^\star_{ij} + f \ge 0, \tag{3.5}$$

where a_{ij} and f are defined exactly as in the proof of Proposition 3.2 (see (3.3) & (3.4)). Moreover, the equality in (3.5) holds inside the open set $O := \{x \in B_r : v^\star(x) > h(x)\}$. Since $v^\star \ge h$ on B_r and $h \in C^{1,1}(B_r)$, then we have

$$-Cr^2 \le h(x) \le v^\star(x) \quad \text{on } B_r.$$

In order to show that $v^\star$ is $C^{1,1}$ at the origin, we just need to show the following estimate:

$$-Cr^2 \le v^\star(x) \le Cr^2, \quad \text{for all } x \in B_{\frac{r}{2}}. \tag{3.6}$$

The proof of (3.6) will follow the one in [8, Theorem 2] with some simplification (coming from the fact that one can always assume that the tangent space to Γ^c at x_0 is the hyperplane $x_N = 0$). Set $w^\star = v^\star + Cr^2 \ge 0$. Then, $w^\star$ satisfies the following inequality:

$$-\sum_{i.j} a_{ij} w^\star_{ij} + f \ge 0.$$

Let w_0 be the solution of

$$-\sum_{i.j} a_{ij} w_{0\,ij} + f = 0$$

with $w_0 = w^\star \ge 0$ on ∂B_r. Then, by the comparison principle (see [5]), we get that $w_0 \le w^\star$ on B_r. Let $x^\star \in B_r$ be such that

$$w^\star(x^\star) - w_0(x^\star) = \max_{B_r}(w^\star - w_0) \ge 0.$$

Then, we have two possibilities: either $x^\star \in O$ or $x^\star \in \partial O$. Assume the latter holds. Hence, we get that

$$w^\star(x) - w_0(x) \le w^\star(x^\star) - w_0(x^\star) = v^\star(x^\star) + Cr^2 - w_0(x^\star) = h(x^\star) + Cr^2 - w_0(x^\star). \tag{3.7}$$

Now, assume that the maximum point $x^\star \in O$ and set $r^\star = \mathrm{dist}(x^\star, \partial O) > 0$. Then, we should have

$$-\sum_{i.j} a_{ij}(w^\star - w_0)_{ij} = 0 \quad \text{in} \quad B(x^\star, r^\star),$$

But, by the maximum principle [13, Theorem 9.6], $w^\star - w_0$ cannot achieve a (nonnegative) maximum in $B(x^\star, r^\star)$ unless it is a constant. Therefore, $w^\star(x^\star) - w_0(x^\star) = w^\star(y^\star) - w_0(y^\star)$ where $y^\star \in \partial O \cap \partial B(x^\star, r^\star)$. Hence, we may always assume that $x^\star \in \partial O$ and so, (3.7) holds.

Now, consider the quadratic function $V(x) = \frac{\gamma}{2}(|x|^2 - r^2)$, where $\gamma > 0$ is to be chosen later. So, we have $V = 0$ on ∂B_r. Moreover, one can choose the constant γ large enough so that V solves

$$-\sum_{i.j} a_{ij} V_{ij} + f \leq 0.$$

Indeed,

$$-\sum_{i.j} a_{ij} V_{ij} + f = -\gamma \sum_{i,j} a_{ij}\delta_{ij} + f = -\gamma \sum_{i} a_{ii} + f \leq -N\lambda\gamma + ||f||_\infty,$$

where in the last inequality we used that $a_{ii} \geq \lambda > 0$. Thanks again to the comparison principle and the fact that $V \leq w_0$ on ∂B_r, we get

$$V \leq w_0 \ \text{ on } \ B_r.$$

Recalling (3.7) and thanks to the fact that Γ^c is $C^{1,1}$ and $V(x) \geq -\frac{\gamma}{2}r^2$ for all $x \in B_r$, we get

$$w^\star(x) - w_0(x) \leq h(x^\star) + Cr^2 - V(x^\star) \leq Cr^2, \quad \text{for all } \ x \in B_r. \tag{3.8}$$

But, we have

$$-\sum_{i.j} a_{ij}(w_0 - V)_{ij} + f = \gamma \sum_{i} a_{ii}.$$

Recalling (3.4), we see that $a_{ii} \in C(\overline{B_r})$, for all i. Thanks to [13, Corollary 9.18], we infer that $w_0 - V \in W^{2,p}_{loc}(B_r) \cap C(\overline{B_r})$, for all $p < \infty$. By [13, Theorem 9.20], we have for any $p > 0$ that

$$\sup_{B_{\frac{r}{2}}}(w_0 - V) \leq C\bigg(\bigg(\frac{1}{|B_r|}\int_{B_r}(w_0 - V)^p\bigg)^{\frac{1}{p}} + \frac{r}{\lambda}||f||_{L^N(B_r)}\bigg) \tag{3.9}$$

where the constant C does not depend on r. Yet, [13, Theorem 9.22] yields that there are constants p and C depending only on N, λ and Λ such that

$$\bigg(\frac{1}{|B_r|}\int_{B_r}(w_0 - V)^p\bigg)^{\frac{1}{p}} \leq C\bigg(\inf_{B_r}(w_0 - V) + \frac{r}{\lambda}\,||f||_{L^N(B_r)}\bigg). \tag{3.10}$$

Combining (3.9) & (3.10), we get the following Harnack inequality:

$$\sup_{B_{\frac{r}{2}}}(w_0 - V) \leq C\bigg(\inf_{B_r}(w_0 - V) + \frac{r}{\lambda}||f||_{L^N(B_r)}\bigg) \leq C(w_0(0) - V(0) + r^{N+1}),$$

since $f \in L^\infty(B_r)$. But, $w_0(0) \leq w^\star(0) = v^\star(0) + Cr^2 = Cr^2$ and $V(0) = -\frac{\gamma}{2}r^2$. Hence, we infer that

$$\sup_{B_{\frac{r}{2}}}(w_0 - V) \leq Cr^2.$$

Recalling (3.8), we infer that

$$w^\star(x) \leq w_0(x) + Cr^2 \leq V(x) + Cr^2 \leq Cr^2, \quad \text{for all } \ x \in B_{\frac{r}{2}}.$$

But, this concludes our claim (3.6). $\square$

Now, our aim is to study the shape of $\partial A^\star$ near Γ. For this, we need to restrict ourselves to dimension $N = 2$. The following proposition is a generalization of [15, Theorem 5] to the case when $w \neq 0$.

Proposition 3.5. *Assume that $\partial A^\star$ touches the interior of Γ at some point x and suppose that Γ is C^1 around x. Then, $\partial A^\star\backslash\{x\}\cap B(x,\delta)$ is composed of two arcs C_1 and C_2 such that $C_2 \subset \Gamma$. Let $\theta(x)$ be the angle between the tangent vectors to C_1 and C_2 at x. Then, we have the following estimate:*

$$\tan^{-1}\left(\frac{\psi(x)^2 - w(x)^2}{2\,\psi(x)w(x)}\right) \le \theta(x) \le \frac{\pi}{2} \qquad \text{if} \quad w(x) \ge 0$$

and

$$\frac{\pi}{2} \le \theta(x) \le \tan^{-1}\left(\frac{\psi(x)^2 - w(x)^2}{2\,\psi(x)w(x)}\right) \qquad \text{if} \quad w(x) \le 0.$$

Proof. First, we recall from Proposition 3.2 that $\partial A^\star$ is $C^{2,\alpha}$ inside Ω, for all $\alpha < 1$. Moreover, the curvature of $\partial A^\star$ is uniformly bounded in Ω,

$$\kappa = \frac{1}{\psi(x)}\Big[\lambda^\star\,\phi(x) + \partial_{\mathbf{n}}\psi(x)\Big].$$

Fix $x \in \partial A^\star$ in the interior of Γ such that $\partial A^\star \cap B(x,\delta) \cap \Omega \neq \emptyset$, for every $\delta > 0$ small enough. Assume that Γ is C^1 around x. Let C_1 and C_2 be two different arcs of $\partial A^\star \cap B(x,\delta)$ such that x is an endpoint of both C_1 and C_2.

Assume that C_1 and C_2 are contained in Ω (we note that the case when $C_1 \cap \Gamma \neq \emptyset$ or $C_2 \cap \Gamma \neq \emptyset$ can be treated similarly) with $0 \le \theta(x) < \pi$. After rotation and translation of axes, one can assume that x is the origin and $(s,\alpha(s))$ is a parametrization of C_1 ($s \in (0,\delta)$) and C_2 ($s \in (-\delta,0)$) such that $\alpha(0) = 0$, $\alpha'(0^-) < 0$ and $\alpha'(0^+) > 0$. For $\varepsilon > 0$ small enough, let $s_\varepsilon < 0$ be such that $\alpha(s_\varepsilon) = \alpha(\varepsilon)$. Let us denote by $\mathcal{C}_\varepsilon := \{(s,\alpha(s)) : s \in (s_\varepsilon,\varepsilon)\}$ and by $\hat{\mathcal{C}}_\varepsilon \subset \Omega$ the line segment joining the points $(s_\varepsilon,\alpha(s_\varepsilon))$ and $(\varepsilon,\alpha(\varepsilon))$. Let A_ε be such that $\partial A_\varepsilon = (\partial A^\star\backslash\mathcal{C}_\varepsilon)\cup\hat{\mathcal{C}}_\varepsilon$. Thanks to the minimality of $A^\star$ in (3.1), we have

$$Per_\psi(A_\varepsilon) - \int_{\partial A_\varepsilon\cap\Gamma} w - \lambda^\star\int_{A_\varepsilon}\phi - \left[Per_\psi(A^\star) - \int_{\partial A^\star\cap\Gamma} w - \lambda^\star\int_{A^\star}\phi\right] \ge 0.$$

Hence,

$$\int_{s_\varepsilon}^{\varepsilon}\psi(s,\alpha(\varepsilon))\,\mathrm{d}s - \int_{s_\varepsilon}^{\varepsilon}\psi(s,\alpha(s))\sqrt{1+\alpha'(s)^2}\,\mathrm{d}s - \lambda^\star\int_{A_\varepsilon}\phi + \lambda^\star\int_{A^\star}\phi \ge 0. \tag{3.11}$$

Yet, we have

$$\left|\int_{A_\varepsilon}\phi - \int_{A^\star}\phi\right| = \left|\int_{s_\varepsilon}^{\varepsilon}\int_{\alpha(s)}^{\alpha(\varepsilon)}\phi(s,t)\,\mathrm{d}t\,\mathrm{d}s\right| \le ||\phi||_\infty\, o(\varepsilon),$$

where the last inequality comes from the fact that the map $\varepsilon \mapsto s(\varepsilon) := s_\varepsilon$ is Lipschitz with $s(0) = 0$. Dividing (3.11) by ε and letting $\varepsilon \to 0^+$, we get

$$\alpha'(0^+)\left[1 - \sqrt{1+\alpha'(0^-)^2}\right] - \alpha'(0^-)\left[1 - \sqrt{1+\alpha'(0^+)^2}\right] \ge 0.$$

But, this is clearly a contradiction.

Let $C \subset \Omega$ be an arc of $\partial A^\star \cap B(x,\delta)\cap\Omega$ such that x is an endpoint of C. Let $(s,\beta(s))$, $s \in (-\delta,\delta)$, be a parametrization of Γ such that $\beta(0) = \beta'(0) = 0$. Assume that the angle between the tangent vector to C at x and $< 1, 0 >$ is less than $\frac{\pi}{2}$. Let $(s,\alpha(s))$, $s \in (0,\delta)$, be a parametrization of C with $\alpha(0) = 0$. For $\varepsilon > 0$ small enough, we define the set A_ε as follows:

$$\partial A_\varepsilon = [\partial A^\star\backslash C_\varepsilon] \cup [(\varepsilon,\alpha(\varepsilon)),(\varepsilon,\beta(\varepsilon))] \cup \Gamma_\varepsilon$$

where $C_\varepsilon := \{(s,\alpha(s)) : s \in (0,\varepsilon)\}$ and Γ_ε denotes the arc of Γ between $(0,0)$ and $(\varepsilon,\beta(\varepsilon))$. Here, we assume that $\Gamma_\varepsilon \cap \partial A^\star = \emptyset$. So, one can see easily that

$$Per_\psi(A_\varepsilon) - \int_{\partial A_\varepsilon \cap \Gamma} w - \lambda^\star \int_{A_\varepsilon} \phi - \left[Per_\psi(A^\star) - \int_{\partial A^\star \cap \Gamma} w - \lambda^\star \int_{A^\star} \phi\right]$$

$$= \int_{\beta(\varepsilon)}^{\alpha(\varepsilon)} \psi(\varepsilon,t)\,\mathrm{d}t - \int_0^\varepsilon w(s,\beta(s))\sqrt{1+\beta'(s)^2}\,\mathrm{d}s - \int_0^\varepsilon \psi(s,\alpha(s))\sqrt{1+\alpha'(s)^2}\,\mathrm{d}s - \lambda^\star \int_{A_\varepsilon} \phi + \lambda^\star \int_{A^\star} \phi.$$

But, one has

$$\lim_{\varepsilon\to 0^+} \frac{\int_{\beta(\varepsilon)}^{\alpha(\varepsilon)} \psi(\varepsilon,t)\,\mathrm{d}t - \int_0^\varepsilon \psi(s,\alpha(s))\sqrt{1+\alpha'(s)^2}\,\mathrm{d}s}{\varepsilon} = \left[\alpha'(0^+) - \sqrt{1+\alpha'(0^+)^2}\right]\psi(0,0)$$

and

$$\left|\int_{A_\varepsilon} \phi - \int_{A^\star} \phi\right| = \left|\int_0^\varepsilon \int_{\beta(s)}^{\alpha(s)} \phi(s,t)\,\mathrm{d}t\,\mathrm{d}s\right| \le ||\phi||_\infty\, o(\varepsilon).$$

Thanks to the optimality of $A^\star$ in (3.1), we get that

$$\left[\alpha'(0^+) - \sqrt{1+\alpha'(0^+)^2}\right]\psi(0,0) - w(0,0) \ge 0. \tag{3.12}$$

Hence, $w(0,0) \le 0$. Using the above estimates, it is easy to check that if $w(0,0) \le 0$ then Γ_ε cannot intersect $\partial A^\star$. Moreover, we have

$$\alpha'(0^+) \ge \frac{w(0,0)^2 - \psi(0,0)^2}{2\,w(0,0)\,\psi(0,0)}.$$

Now, let us assume that the angle between the tangent vector to C at x and $< 1,0 >$ is greater than $\frac{\pi}{2}$. Let $(s,\alpha(s))$, $s \in (-\delta,0)$, be a parametrization of C with $\alpha(0)=0$. For $\varepsilon > 0$ small enough, we define the set A_ε as follows:

$$\partial A_\varepsilon = [\partial A^\star \backslash C_\varepsilon] \cup [(-\varepsilon,\alpha(-\varepsilon)),(-\varepsilon,\beta(-\varepsilon))]$$

where $C_\varepsilon := \{(s,\alpha(s)) : s \in (-\varepsilon,0)\} \cup \{(s,\beta(s)) : s \in (-\varepsilon,0)\}$. Here, we assume that $C_\varepsilon \subset \partial A^\star$. Again from the optimality of $A^\star$ in (3.1), we must have the following inequality:

$$Per_\psi(A_\varepsilon) - \int_{\partial A_\varepsilon \cap \Gamma} w - \lambda^\star \int_{A_\varepsilon} \phi - \left[Per_\psi(A^\star) - \int_{\partial A^\star \cap \Gamma} w - \lambda^\star \int_{A^\star} \phi\right] \ge 0.$$

Hence,

$$\int_{\beta(-\varepsilon)}^{\alpha(-\varepsilon)} \psi(-\varepsilon,t)\,\mathrm{d}t + \int_{-\varepsilon}^0 w(s,\beta(s))\sqrt{1+\beta'(s)^2}\,\mathrm{d}s - \int_{-\varepsilon}^0 \psi(s,\alpha(s))\sqrt{1+\alpha'(s)^2}\,\mathrm{d}s - \lambda^\star \int_{A_\varepsilon} \phi + \lambda^\star \int_{A^\star} \phi \ge 0.$$

Dividing by $\varepsilon > 0$ and letting $\varepsilon \to 0^+$, we get that

$$\left[-\alpha'(0^-) - \sqrt{1+\alpha'(0^-)^2}\right]\psi(0,0) + w(0,0) \ge 0. \tag{3.13}$$

In particular, this yields that $w(0,0) \ge 0$. If $w(0,0) \ge 0$ then one can also see that C_ε must be contained in $\partial A^\star$. In addition, we have the following estimate:

$$\alpha'(0^-) \le \frac{w(0,0)^2 - \psi(0,0)^2}{2\,w(0,0)\,\psi(0,0)}.$$

Yet, this concludes the proof. $\square$

We finish this section by some remarks on optimal sets.

Remark 3.3. *We note that an optimal set $A^\star$ is a priori not connected. For instance, this may happen when ψ has two minima or when Ω is not convex (like two disks connected by a tube). However, one can always show that there is an open connected set $A^\star$ that minimizes Problem (2.12). Indeed, if $A^\star$ is an optimal set then it is not difficult to check that the interior of $A^\star$ is optimal too. So, let us assume that $A^\star$ is open. Now, let $\{A_i^\star\}_{i\in\mathbb{N}}$ be the family of disjoint open connected components of $A^\star$ (i.e. $A^\star = \cup_{i\in\mathbb{N}} A_i^\star$ and $A_i^\star \cap A_j^\star = \emptyset$, for all $i \neq j$). In fact, the optimality of $A^\star$ also implies that $\overline{A_i^\star} \cap \overline{A_j^\star} = \emptyset$, for all i, j. Yet, we have*

$$\lambda^\star = \frac{Per_\psi(A^\star) - \int_{\partial^\star A^\star \cap \Gamma} w}{\int_{A^\star} \phi} \leq \frac{Per_\psi(A_i^\star) - \int_{\partial^\star A_i^\star \cap \Gamma} w}{\int_{A_i^\star} \phi}, \quad \text{for all } i.$$

Hence,

$$\lambda^\star \int_{A_i^\star} \phi \leq Per_\psi(A_i^\star) - \int_{\partial^\star A_i^\star \cap \Gamma} w. \tag{3.14}$$

Since the closures of these sets $A_i^\star$ are mutually disjoint, then taking the sum over i in (3.14), we get the following inequality:

$$\lambda^\star \int_{A^\star} \phi = \lambda^\star \sum_i \int_{A_i^\star} \phi \leq \sum_i Per_\psi(A_i^\star) - \sum_i \int_{\partial^\star A_i^\star \cap \Gamma} w = Per_\psi(A^\star) - \int_{\partial^\star A^\star \cap \Gamma} w. \tag{3.15}$$

But so, the inequality in (3.15) must be an equality. In particular, it implies that for all i, the inequality in (3.14) is an equality:

$$\lambda^\star \int_{A_i^\star} \phi = Per_\psi(A_i^\star) - \int_{\partial^\star A_i^\star \cap \Gamma} w.$$

In other words, this means that $A_i^\star$ is an optimal set for Problem (2.12), for all i.

In addition, one can show in 2D that any connected optimal set $A^\star$ is convex as soon as ψ is a constant function.

Remark 3.4. *Assume that $\psi = 1$. For every point $x \in \partial A^\star \cap \Omega$, there is an $\varepsilon > 0$ such that $A^\star \cap B(x,\varepsilon)$ is convex. Moreover, if $x \in \partial A^\star \cap \partial\Omega$ and $\Omega \cap B(x,\varepsilon)$ is convex, then $A^\star \cap B(x,\varepsilon)$ is convex. To see this, assume that there are two points $x^\star$, $y^\star \in \partial A^\star \cap \Omega$ such that $]x^\star, y^\star[\subset \Omega \backslash A^\star$. Let E be the small region delimited by $[x^\star, y^\star]$ and $\partial A^\star$. Now, we define $\tilde{A} = A^\star \cup E$. Then, it is easy to see that $Per(\tilde{A}) < Per(A^\star)$. Yet, we also have $\int_{\tilde{A}} \phi > \int_{A^\star} \phi$ and $\partial\tilde{A} \cap \Gamma = \partial A^\star \cap \Gamma$. Thanks to (2.14), we infer that*

$$0 < \frac{Per(\tilde{A}) - \int_{\partial\tilde{A}\cap\Gamma} w}{\int_{\tilde{A}} \phi} < \frac{Per(A^\star) - \int_{\partial A^\star \cap \Gamma} w}{\int_{A^\star} \phi}.$$

But, this contradicts the optimality of $A^\star$ in (2.12). We note that this argument does not work in higher dimension since it is not true when $N > 2$ that the perimeter of the convex hull of a set is less than the perimeter of the set itself.

Remark 3.5. *Assume that $\phi = 1$. Then, one can show that any connected optimal set $A^\star$ in (2.12) has to intersect the boundary $\partial\Omega$. Indeed, assume that $A^\star$ is contained in the interior of Ω. Take $t > 1$ such that $tA^\star \subset \Omega$. Then, it is clear that*

$$Per(tA^\star) = tPer(A^\star) \qquad \text{and} \qquad |tA^\star| = t^2|A^\star|.$$

Hence, we have

$$\frac{Per(tA^\star)}{|tA^\star|} < \frac{Per(A^\star)}{|A^\star|},$$

which is a contradiction. Hence, $A^\star$ touches $\partial\Omega$. Moreover, if $\partial A^\star \cap \Gamma = \emptyset$ then $A^\star$ cannot be translated inside Ω, since if A is a translation of $A^\star$ inside Ω then we have

$$Per(A) = Per(A^\star) \qquad \text{and} \qquad |A| = |A^\star|.$$

In addition, assume Ω is convex. Then, $A^\star$ must intersect Γ with $\mathcal{H}^1(\partial A^\star \cap \{x \in \Gamma : w(x) > -1\}) > 0$. Indeed, if $\mathcal{H}^1(\partial A^\star \cap \{x \in \Gamma : w(x) > -1\}) = 0$ then one can move $A^\star$ in $\overline{\Omega}$ until we obtain a new set A such that $\mathcal{H}^1(\partial A \cap \{x \in \Gamma : w(x) > -1\}) > 0$. Yet, we clearly have $|A| = |A^\star|$. Moreover, we have

$$Per(A) - \int_{\partial A \cap \Gamma} w = Per(A) - \int_{\partial A \,\cap\, \{x \in \Gamma \,:\, w(x) = -1\}} w - \int_{\partial A \,\cap\, \{x \in \Gamma \,:\, w(x) > -1\}} w$$

$$< Per(A, \mathbb{R}^2) = Per(A^\star, \mathbb{R}^2) = Per(A^\star) - \int_{\partial A^\star \,\cap\, \{x \in \Gamma \,:\, w(x) = -1\}} w.$$

Consequently, we get

$$Per(A) - \int_{\partial A \cap \Gamma} w < Per(A^\star) - \int_{\partial A^\star \cap \Gamma} w.$$

4. Examples

We conclude the paper by some examples in 2D where we can find explicitly the optimal set $A^\star$ in Problem (2.12).

Example 4.0.1. *Assume that $\Omega := [-1, 1] \times [0, 1]$, $\Gamma = [-1, 1] \times \{0\}$, $\psi = \phi = 1$ and $w = w_0 \in [0, 1]$. Thanks to Proposition 3.5, we know that any arc of $\partial A^\star$ inside Ω is an arc of circle with radius $R^\star = 1/\lambda^\star$ and, $\partial A^\star$ is also of class C^1 on Γ^c. Using Proposition 3.5, one can see that if $\partial A^\star$ touches Γ at some point x, then the tangent line to $\partial A^\star \cap \Omega$ at x must be orthogonal to Γ. Moreover, by Remark 3.5, one has $\mathcal{H}^1(\partial A^\star \cap \Gamma) > 0$. Thanks to Remark 3.4, $A^\star$ is also convex.*

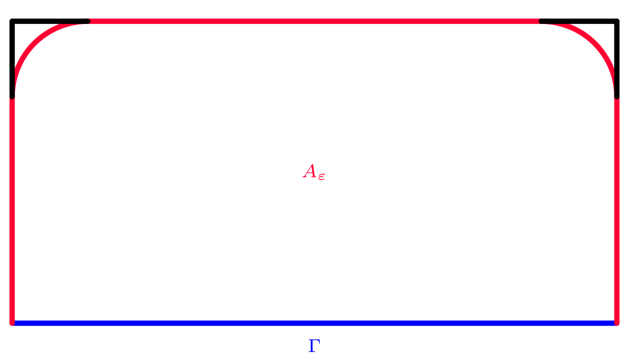
For every $\varepsilon \in]0, 1[$, let A_ε be the "rounded" rectangle Ω where the corners $(1, 1)$ and $(-1, 1)$ are cutted off and replaced by arcs of circles with radius ε and centers $(1 - \varepsilon, 1 - \varepsilon)$ and $(-1 + \varepsilon, 1 - \varepsilon)$. We have

$$Per(A_\varepsilon) = 4 + (\pi - 4)\varepsilon \qquad \textit{and} \qquad |A_\varepsilon| = 2 - 2(1 - \frac{\pi}{4})\varepsilon^2.$$

Hence,

$$\mathcal{J}(\varepsilon) := \frac{Per(A_\varepsilon) - \int_{\partial A_\varepsilon \cap \Gamma} w}{|A_\varepsilon|} = \frac{4 + (\pi - 4)\varepsilon - 2w_0}{2 - 2(1 - \frac{\pi}{4})\varepsilon^2}.$$

Yet, this function $\mathcal{J}(\varepsilon)$ reaches a minimum at $\varepsilon^\star = (4 - 2w_0 - 2\sqrt{\pi - 4 + (w_0 - 2)^2})/(4 - \pi)$. Then, we infer that the optimal set $A^\star = A_{\varepsilon^\star}$ and $\lambda^\star = 1/\varepsilon^\star$.



Example 4.0.2. *Now, assume that $\Gamma = ([-1, 1] \times \{0\}) \cup (\{1\} \times [0, 1])$ and $w_0 = 0$. For every $\varepsilon \in]0, 1[$, let us denote by A_ε the "rounded" rectangle Ω where the corner point $(-1, 1)$ is cutted off and replaced*

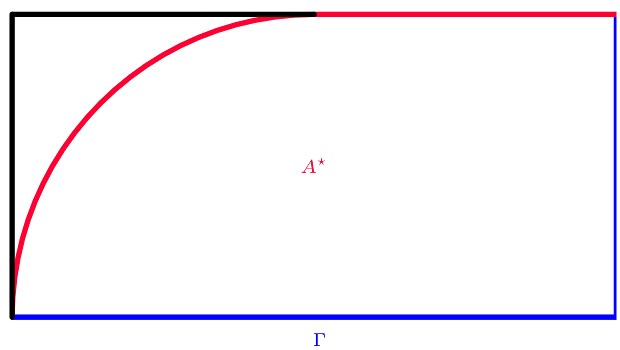
by an arc of circle with center $(-1+\varepsilon, 1-\varepsilon)$ *and radius* ε. *Again, it is clear that*

$$Per(A_\varepsilon) = 3 + (\frac{\pi}{2} - 2)\varepsilon \qquad \text{and} \qquad |A_\varepsilon| = 2 - (1 - \frac{\pi}{4})\varepsilon^2.$$

Then, we get that

$$\frac{Per(A_\varepsilon)}{|A_\varepsilon|} = \frac{3 + (\frac{\pi}{2} - 2)\varepsilon}{2 - (1 - \frac{\pi}{4})\varepsilon^2}$$

attains a minimum at $\varepsilon = 1$. *Consequently, this implies that the optimal set* $A^\star$ *in* (2.12) *is nothing else than* A_1.



Example 4.0.3. *In this example, we will see that the situation becomes much complicated when the penalization* w *on* Γ *is negative. Again, assume that* $\Omega := [-1,1] \times [0,1]$, $\Gamma = [-1,1] \times \{0\}$, $\psi = \phi = 1$ *and* $w = w_0$, *where* $-1 < w_0 < 0$. *Let* $A^\star$ *be a convex optimal set in* (2.12). *We recall that any part of* $\partial A^\star$ *in the interior of* Ω *is an arc of circle with radius* $R^\star = 1/\lambda^\star$ *and, that* $\partial A^\star$ *is* C^1 *on* Γ *with* $\mathcal{H}^1(\partial A^\star \cap \Gamma) > 0$. *Moreover, we know that if* $\partial A^\star$ *touches* Γ *at a point* x *then the angle* $\theta \in]\frac{\pi}{2}, \pi[$ *between the tangent line to* $\partial A^\star \cap \Omega$ *at* x *and* Γ *should satisfy:*

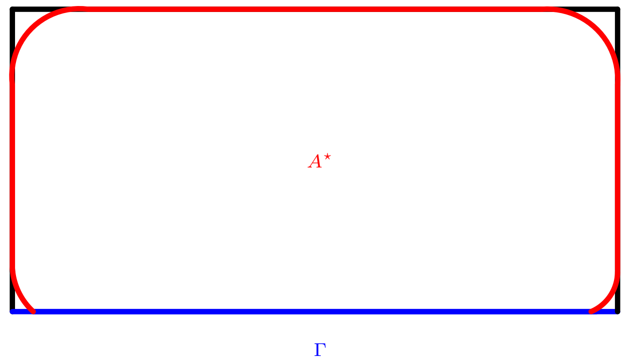
$$\theta \leq \tan^{-1}\left(\frac{1 - w_0^2}{2\, w_0}\right). \tag{4.1}$$

For all $\varepsilon \in]0,1[$ *and* $\delta \in]0, \varepsilon[$, *we define* $A_{\varepsilon,\delta}$ *as the "rounded" rectangle* Ω *where the corners* $(1,1)$ *and* $(-1,1)$ *are cutted off and replaced by arcs of circles with radius* ε *and centers* $(1-\varepsilon, 1-\varepsilon)$ *and* $(-1+\varepsilon, 1-\varepsilon)$, *while the corners* $(-1,0)$ *and* $(1,0)$ *are cutted off and replaced by arcs of the circles* $(x_1 - \varepsilon)^2 + (x_2 - \delta)^2 = \varepsilon^2$ *and* $(x_1 - 1 + \varepsilon)^2 + (x_2 - \delta)^2 = \varepsilon^2$. *Then, it is not difficult to check that*

$$\mathcal{J}(\varepsilon,\delta) = \frac{Per(A_{\varepsilon,\delta}) - \int_{\partial A_{\varepsilon,\delta} \cap \Gamma} w}{|A_{\varepsilon,\delta}|} = \frac{4 + (\pi - 4)\varepsilon + 2[\varepsilon \cos^{-1}(\frac{\sqrt{\varepsilon^2-\delta^2}}{\varepsilon}) - \delta] - 2[1 - \varepsilon + \sqrt{\varepsilon^2 - \delta^2}]w_0}{2 - \frac{(4-\pi)}{2}\varepsilon^2 - 2\delta[\varepsilon - \sqrt{\varepsilon^2-\delta^2}] + \varepsilon\delta\sqrt{\varepsilon^2-\delta^2} - \varepsilon^2\cos^{-1}(\frac{\sqrt{\varepsilon^2-\delta^2}}{\varepsilon})}.$$

If $(\varepsilon^\star, \delta^\star)$ *is a minimizer of* $\mathcal{J}(\varepsilon,\delta)$, *then the optimal set* $A^\star$ *will be* $A_{\varepsilon^\star,\delta^\star}$. *Notice that, thanks to* (4.1), *we must have the following estimate:*

$$\delta^\star \leq \frac{-2w_0}{\sqrt{(1-w_0^2)^2 + 4w_0^2}}\,\varepsilon^\star.$$



References

[1] F. Alter and V. Caselles, Uniqueness of the Cheeger set of a convex body, *Nonlinear Analysis: Theory, Methods & Applications*, 70 (1), 2009, 32-44.

[2] L. Ambrosio, N. Fusco and D. Pallara, Functions of bounded variation and free discontinuity problems, *Mathematical Monographs*, Oxford University Press, New York, 2000.

[3] G. Anzellotti and M. Giaquinta, Funzioni BV e tracce, *Rend. Sem. Mat. Univ. Padova*, Vol. 60, 1978, pp. 1-22.

[4] E. Barozzi, U. Massari, Regularity of minimal boundaries with obstacles, *Rendiconti del Seminario Matematico della Università di Padova*, 66 (1982), 129–135.

[5] L. A. Caffarelli and Xavier Cabré, Fully nonlinear elliptic equations, *Amer. Math. Soc. Colloq. Publ.*, 43, 1995.

[6] G. Carlier and M. Comte, On a weighted total variation minimization problem, *Journal of Functional Analysis*, 250 (1), 214–226, 2007.

[7] V. Caselles and A. Chambolle, Anisotropic curvature-driven flow of convex sets, *Nonlinear Anal., Theory Methods Appl.* 65 (2006), No. 8 (A), 1547-1577.

[8] V. Caselles, M. Novaga, and A. Chambolle, "Some remarks on uniqueness and regularity of Cheeger sets", *Rendiconti del Seminario Matematico della Università di Padova*, 123 (2010): 191-202.

[9] J. Cheeger, A lower Bound for the Smallest Eigenvalue of the Laplacian, Problems in Analysis, A Symposium in Honor of Salomon Bochner, Ed. R.C. Gunning, Princeton Univ. Press (1970), 195–199.

[10] N. Cristescu, O. Cazacu and C. Cristescu, A model for landslides, *Canadian Geotechnical Journal* 39, (2002), 924–937.

[11] F. Demengel, Théorèmes d'existence pour des équations avec l'opérateur $1-\Delta$, première valeur propre de $-\Delta_1$, *C. R. Math. Acad. Sci.* Paris 334 (12) (2002), 1071–1076.

[12] F. Demengel, Some existence's results for noncoercive "$1-$Laplacian" operator, *Asymptot. Anal.* 43 (2005), 287–322.

[13] D. Gilbarg and N.S. Trudinger, *Elliptic partial Differential Equations of Second Order*, Springer Verlag, 1998.

[14] P. Hild, I. R. Ionescu, T. Lachand-Robert & I. Rosca, The blocking of an inhomogeneous Bingham fluid, Applications to landslides, *M2AN* 36-6 (2002), 1013-1026.

[15] I. R. Ionescu and T. Lachand-Robert, Generalized Cheeger sets related to landslides, *Calc. Var. Partial Differential Equations*, 23 (2005) 227–249.

[16] B. Kawohl, V. Fridman, Isoperimetric estimates for the first eigenvalue of the $p-$Laplace operator and the Cheeger constant, *Comment. Math. Univ. Carolin.* 44 (4) (2003), 659–667.

[17] B. Kawohl, T. Lachand-Robert, Characterization of Cheeger sets for convex subsets of the plane, *Pacific J. Math.* 225 (2006), 103–118.

[18] U. Massari, Frontiere orientate di curvatura media assegnata in L^p, Rend. Sem. Mat. Univ. Padova 53, 37-52, 1975.

[19] J. M. Mazon, J. D. Rossi and S. S. de Leon, Functions of least gradient and 1-harmonic functions, *Indiana Univ. J. Math*, 63, 1067-1084.

[20] M. Miranda, Frontiere minimali con ostacoli, *Ann. Univ. Ferrara*, 16 (1971), pp. 29–37.

[21] L. Modica, Gradient theory of phase transitions with boundary contact energy, *Ann. Inst. Henri Poincaré*, 4 (5), 1987, 487-512.

[22] Philippis G.D., Maggi F. Regularity of Free Boundaries in Anisotropic Capillarity Problems and the Validity of Young's Law, *Arch Rational Mech Anal* 216, 473–568 (2015).

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