A GEOMETRICAL APPROACH TO THE SHARP HARDY INEQUALITY IN SOBOLEV–SLOBODECKIĬ SPACES

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Abstract. We give a partial negative answer to a question left open in a previous work by Brasco and the first and third-named authors concerning the sharp constant in the fractional Hardy inequality on convex sets. Our approach has a geometrical flavor and equivalently reformulates the sharp constant in the limit case $p = 1$ as the Cheeger constant for the fractional perimeter and the Lebesgue measure with a suitable weight. As a by-product, we obtain new lower bounds on the sharp constant in the 1-dimensional case, even for non-convex sets, some of which optimal in the case $p = 1$.

1. Introduction

1.1. Classical framework.

Given $N \in \mathbb{N}$ and $p \in (1,\infty)$, the celebrated *Hardy inequality* on a (non-empty) open convex set $\Omega \subsetneq \mathbb{R}^N$ states that

$$\int_{\mathbb{R}^N} |\nabla u|^p \,\mathrm{d}x \ge C \int_\Omega \frac{|u|^p}{d_\Omega^p}\,\mathrm{d}x \quad \text{for all } u \in C_0^\infty(\Omega), \tag{1.1}$$

see [32, 35] for a detailed introduction. Here and in the rest of the paper,

$$d_\Omega(x) = \inf_{y\in\partial\Omega} |x-y|, \quad \text{for all } x \in \Omega,$$

denotes the *distance* function from $\partial\Omega$. The sharp constant in (1.1), defined as

$$\mathfrak{h}_{1,p}(\Omega) = \inf_{u\in C_0^\infty(\Omega)} \left\{ \int_{\mathbb{R}^N} |\nabla u|^p \,\mathrm{d}x : \int_\Omega \frac{|u|^p}{d_\Omega^p}\,\mathrm{d}x = 1 \right\}, \tag{1.2}$$

can be explicitly computed as

$$\mathfrak{h}_{1,p}(\Omega) = \mathfrak{h}_{1,p}(\mathbb{H}^N_+) = \left(\frac{p-1}{p}\right)^p, \tag{1.3}$$

Date: July 21, 2024.

2020 *Mathematics Subject Classification.* Primary 49Q20. Secondary 35R11, 46E35.

Key words and phrases. Hardy inequality, fractional Sobolev spaces, fractional perimeter.

Acknowledgements. The authors thank Lorenzo Brasco for several precious comments on a preliminary version of the present work. The authors are members of the Istituto Nazionale di Alta Matematica (INdAM), Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA). The first-named author has received funding from INdAM under the INdAM–GNAMPA Project 2024 *Nuove frontiere nella capillarità non locale*, codice CUP_E53C23001670001. The second-named author has received funding from INdAM under the INdAM–GNAMPA Project 2024 *Ottimizzazione e disuguaglianze funzionali per problemi geometrico-spettrali locali e non-locali*, codice CUP_E53C23001670001, and the INdAM–GNAMPA 2023 Project *Problemi variazionali per funzionali e operatori non-locali*, codice CUP_E53C22001930001, and from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (grant agreement No. 945655). The third-named author has received funding from INdAM under the INdAM–GNAMPA Project 2024 *Proprietà geometriche e regolarità in problemi variazionali locali e non locali*, codice CUP_E53C23001670001.

see the original Hardy's works [21, 22] for $N = 1$ and [30, 31] for $N \geq 2$. As well-known, inequality (1.1) cannot hold for $p = 1$, that is,

$$\mathfrak{h}_{1,1}(\Omega) = \mathfrak{h}_{1,1}(\mathbb{H}^N_+) = 0 \tag{1.4}$$

in (1.2). Here and in the following, we set

$$\mathbb{H}^1_+ = (0,\infty) \quad \text{and} \quad \mathbb{H}^N_+ = (0,\infty)\times\mathbb{R}^{N-1} \quad \text{for } N \geq 2.$$

Noteworthy, the sharp constant (1.3) is never attained in (1.1), not even in the corresponding homogenous Sobolev space $\mathcal{W}^{1,p}_0(\Omega)$ obtained as the completion of $C^\infty_0(\mathbb{R}^N)$ with respect to the left-hand side of (1.1).

1.2. **Fractional framework.** The validity of the fractional analog of (1.1) was established in [5, Th. 1.1]. Precisely, given $s \in (0,1)$ and $p \in (1,\infty)$, it holds that

$$[u]^p_{W^{s,p}(\mathbb{R}^N)} \geq C \int_\Omega \frac{|u|^p}{d_\Omega^{sp}}\,\mathrm{d}x, \quad \text{for all } u \in C^\infty_0(\Omega). \tag{1.5}$$

Here and in the rest of the paper, given $s \in (0,1)$, $p \in [1,\infty)$, an open set $A \subset \mathbb{R}^N$, and a measurable function $u\colon A \to [-\infty,\infty]$, we let

$$[u]_{W^{s,p}(A)} = \left(\int_A\int_A \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{p}} \tag{1.6}$$

be the *Sobolev–Slobodeckiĭ* (s,p)*-fractional seminorm* of u on A. The sharp constant in (1.5), defined analogously as in (1.2) as

$$\mathfrak{h}_{s,p}(\Omega) = \inf_{u\in C^\infty_0(\Omega)} \left\{ [u]^p_{W^{s,p}(\mathbb{R}^N)} : \int_\Omega \frac{|u|^p}{d_\Omega^{sp}}\,\mathrm{d}x = 1 \right\}, \tag{1.7}$$

was investigated by Brasco and the first and third-named authors in the recent work [3]. In [3, Main Th.], it was proved that, for all $p \in (1,\infty)$ and $s \in (0,1)$,

$$\mathfrak{h}_{s,p}(\mathbb{H}^N_+) = C_{N,sp}\,\Lambda_{s,p} \tag{1.8}$$

and, in analogy with the classical result (1.3), if either $p = 2$ or $sp \geq 1$, then

$$\mathfrak{h}_{s,p}(\Omega) = \mathfrak{h}_{s,p}(\mathbb{H}^N_+). \tag{1.9}$$

Moreover, as in the classical case, the sharp constant in (1.9) is never attained in (1.5), not even in the homogenous fractional Sobolev–Slobodeckiĭ space $\mathcal{W}^{s,p}_0(\Omega)$ obtained as the completion of $C^\infty_0(\Omega)$ with respect to the seminorm (1.6) with $A = \mathbb{R}^N$. Here and below, we set

$$C_{N,q} = \begin{cases} (N-1)\,\omega_{N-1}\displaystyle\int_0^\infty \frac{t^{N-2}}{(1+t^2)^{\frac{N+q}{2}}}\,\mathrm{d}t & \text{for } N \geq 2,\\ 1 & \text{for } N = 1,\end{cases} \tag{1.10}$$

whenever $q \in [0,\infty)$, and

$$\Lambda_{s,p} = 2\int_0^1 \frac{\left|1-t^{\frac{sp-1}{p}}\right|^p}{(1-t)^{1+sp}}\,\mathrm{d}t + \frac{2}{sp}. \tag{1.11}$$

As mentioned in [3], concerning (1.9), the cases $p = 2$ for $\Omega = \mathbb{H}^N_+$, and $p = 2$ with $s \geq \frac{1}{2}$ for any open convex set $\Omega \subsetneq \mathbb{R}^N$, were previously established respectively in [4, Th. 1.1] and in [15, Th. 5] (see also [16]), with different techniques.

The strategy of [3] for proving (1.8) and (1.9) expands on the ideas of [4] and relies on an equivalent characterization of the constant (1.7) via the existence of positive local weak *supersolutions* of the corresponding non-local eigenvalue problem, see [3, Eq. (1.6)].

Unfortunately, this approach does not seem to work for determining the sharp constant $\mathfrak{h}_{s,p}(\Omega)$ for $\Omega \neq \mathbb{H}^N_+$ for the values of s and p not included in (1.9), see the discussion in [3, Sec. 1.3]. Nevertheless, in virtue of [3, Rem. 6.4], it still holds that

$$\frac{2\,C_{N,sp}}{sp} \le \mathfrak{h}_{s,p}(\Omega) \le \mathfrak{h}_{s,p}(\mathbb{H}^N_+) \tag{1.12}$$

for all $p \in (1,\infty)$ and $s \in (0,1)$.

1.3. **Main results.** Our main aim is to tackle the question left open in [3, Open Prob.] concerning the validity of (1.9) for $p \neq 2$ and $s < \frac{1}{p}$.

We first observe that a plain limiting argument yields (1.8) and (1.12), and thus (1.5), for $p = 1$—remarkably in contrast with (1.1), recall (1.4). We thus complete the picture of [3, 5] in the case $p = 1$, with the characterization of (1.7) for the half-space.

Theorem 1.1. *For $N \ge 1$ and $s \in (0,1)$, it holds that*

$$\mathfrak{h}_{s,1}(\mathbb{H}^N_+) = \Lambda_{s,1}\, C_{N,s} \quad \textit{with} \quad \Lambda_{s,1} = \frac{4}{s}, \tag{1.13}$$

and, whenever is $\Omega \subsetneq \mathbb{R}^N$ is a (non-empty) convex open set,

$$\frac{1}{2}\,\mathfrak{h}_{s,1}(\mathbb{H}^N_+) \le \mathfrak{h}_{s,1}(\Omega) \le \mathfrak{h}_{s,1}(\mathbb{H}^N_+). \tag{1.14}$$

Having (1.5) for $p = 1$ at disposal, our first main result yields a partial negative answer to [3, Open Prob.] for $p = 1$ and $\Omega \subsetneq \mathbb{R}^N$ a (non-empty) bounded convex open set.

Theorem 1.2. *Let $N \ge 1$ and let $\Omega \subsetneq \mathbb{R}^N$ be a (non-empty) bounded convex open set. If $N = 1$, then*

$$\mathfrak{h}_{s,1}(\Omega) = 2^{-s}\,\mathfrak{h}_{s,1}(\mathbb{H}^1_+) = \frac{2^{2-s}}{s} \quad \textit{for all } s \in (0,1). \tag{1.15}$$

If $N \ge 2$, then

$$\lim_{s\to 1^-} \frac{\mathfrak{h}_{s,1}(\Omega)}{\mathfrak{h}_{s,1}(\mathbb{H}^N_+)} = \frac{1}{2}. \tag{1.16}$$

Noteworthy, for $N \ge 2$, in the special case Ω is an open ball, the limit (1.16) can be slightly refined by exploiting the main result of [20], see Proposition 2.15 below.

Theorem 1.2 uncovers a remarkable difference between the classical inequality (1.1) and its fractional counterpart (1.5), revealing some unexpected features of the sharp constant (1.7) in the limiting case $p = 1$. In particular, the first inequality in (1.14) is asymptotically optimal as $s \to 1^-$ for (non-empty) bounded convex open sets. Moreover, for $N = 1$ and $p = 1$, the sharp constant (1.7) is attained. For additional discussions in this direction, we refer also to Remark 2.14 below.

As a consequence of Theorem 1.2, we can give the following partial negative answer to [3, Open Prob.] for $p > 1$, suggesting that the first inequality in (1.12) is asymptotically optimal as $s \to 1^-$ and $p \to 1^+$, with $sp < 1$, for (non-empty) bounded convex open sets.

Corollary 1.3. *If $N \geq 1$ and $\Omega \subsetneq \mathbb{R}^N$ is a (non-empty) bounded convex open set, then*

$$\lim_{s\to 1^-} \limsup_{p\to 1^+} \frac{\mathfrak{h}_{s,p}(\Omega)}{\mathfrak{h}_{s,p}(\mathbb{H}^N_+)} = \frac{1}{2}.$$

1.4. **Geometrical approach.** Besides refining and extending some of the results of [3] to the limiting case $p = 1$, the key idea of our approach is to reinterpret the sharp constant $\mathfrak{h}_{s,1}(\Omega)$ in (1.7) in a geometrical sense. More precisely, we prove the following result (in which we do not need to assume that Ω is convex).

Theorem 1.4. *Let $N \geq 1$, $s \in (0,1)$ and let $\Omega \subsetneq \mathbb{R}^N$ be a (non-empty) open set. Letting*

$$P_s(E) = [\mathbf{1}_E]_{W^{s,1}(\mathbb{R}^N)} \quad \text{and} \quad V_{s,\Omega}(E) = \int_E \frac{1}{d_\Omega^s}\,\mathrm{d}x$$

for any measurable set $E \subseteq \Omega$, it holds that

$$\mathfrak{h}_{s,1}(\Omega) = \mathfrak{g}_s(\Omega) \quad \text{in } [0,\infty], \tag{1.17}$$

where

$$\mathfrak{g}_s(\Omega) = \inf\left\{\frac{P_s(E)}{V_{s,\Omega}(E)} : E \subseteq \Omega,\ |E| > 0\right\} \in [0,\infty]. \tag{1.18}$$

Theorem 1.4 revisits the well-known connection between the *Cheeger constant* of a set with the *first* 1-*eigenvalue* of the underlying 1-*Laplacian* (see [17] and the references therein for a detailed account) in terms of the fractional Hardy inequality (1.5) for $p = 1$. In fact, the problem in (1.18) is the *Cheeger problem* in Ω with respect to the *fractional perimeter* P_s and the *weighted* volume $V_{s,\Omega}$, so that Theorem 1.4 can be seen as a particular case of [17, Th. 5.4] (see also [7, Th. 5.8] and [2, Th. 3.10] for similar results in the fractional/non-local setting). The idea behind the proof of Theorem 1.4 is essentially to exploit the *fractional coarea formula* (first observed in [37])

$$[u]_{W^{s,1}(\mathbb{R}^N)} = \int_{-\infty}^{\infty} P_s(\{u > t\})\,\mathrm{d}t \tag{1.19}$$

in order to pass from a function $u \in C_0^\infty(\Omega)$ in (1.7) to its superlevel sets $\{u > t\}$ in (1.18).

With Theorem 1.4 at disposal, Theorem 1.2 basically follows from the fact that

$$\mathfrak{h}_{s,1}(\Omega) = \mathfrak{g}_s(\Omega) \leq \frac{P_s(\Omega)}{V_{s,\Omega}(\Omega)}.$$

For $N \geq 2$, the idea is first to rewrite $V_{s,\Omega}(\Omega)$ in terms of the superlevel sets of d_Ω via the usual coarea formula and then to take advantage of the well-known limit (e.g., see [9, 27])

$$\lim_{s\to 1^-} (1-s)\,P_s(\Omega) = 2\,\omega_{N-1}\,P(\Omega), \tag{1.20}$$

valid for any measurable set $\Omega \subsetneq \mathbb{R}^N$ such that $\mathbf{1}_\Omega \in BV(\mathbb{R}^N)$. The case $N = 1$ in (1.15), instead, relies on a rearrangement argument (see Lemma 2.12 below) which, unfortunately, does not work in higher dimensions (for more details, see Remark 2.13).

1.5. **The non-convex case.** Theorem 1.4 can be also exploited to deal with the non-convex case $\Omega = \mathbb{R}^N \setminus \{0\}$. In fact, our geometrical approach allows us to naturally recover [19, Th. 1.1] for $p = 1$ (compare also with the argument in [19, Sec. 3.4]). In more precise terms, we can (re)prove the following result.

Corollary 1.5. *Given $N \geq 1$ and $s \in (0,1)$, it holds that*

$$[u]_{W^{s,1}(\mathbb{R}^N)} \geq \mathfrak{h}_{s,1}(\mathbb{R}^N \setminus \{0\}) \int_{\mathbb{R}^N} \frac{|u|}{|x|^s} \, \mathrm{d}x \tag{1.21}$$

for all $u \in \mathcal{W}_0^{s,1}(\mathbb{R}^N)$, with sharp constant given by

$$\mathfrak{h}_{s,1}(\mathbb{R}^N \setminus \{0\}) = \frac{4}{s} \frac{\pi^{\frac{N}{2}} \, \Gamma(1-s)}{\Gamma\left(\frac{N-s}{2}\right) \Gamma\left(1 - \frac{s}{2}\right)}. \tag{1.22}$$

Equality holds in (1.21) if and only if u is proportional to a symmetric decreasing function.

Noteworthy, for $N = 1$, the constant in (1.22) equals the one in (1.15) for all $s \in (0,1)$,

$$\mathfrak{h}_{s,1}\left(\mathbb{R} \setminus \{0\}\right) = \frac{2^{2-s}}{s}, \tag{1.23}$$

by well-known properties of the Gamma function (e.g., the *Legendre duplication formula*). The equality in (1.23) is actually a particular case of the following result (including (1.15) as a special instance), which comes as a natural by-product of our geometrical approach (as customary, $\lfloor x \rfloor \in \mathbb{Z}$ denotes the largest integer smaller than $x \in \mathbb{R}$).

Theorem 1.6. *Let $\Omega \subsetneq \mathbb{R}$ be the union of $n \in \mathbb{N}$ non-empty disjoint bounded open intervals of equal measure. For $s \in (0,1)$ and $p \in [1,\infty)$ such that $sp < 1$, it holds that*

$$\mathfrak{h}_{s,p}(\Omega) \geq \frac{\mathfrak{h}_{s,p}(\mathbb{R} \setminus \{0\})}{n^{sp}}, \tag{1.24}$$

with equality for $p = 1$ whenever $s \in (0,1)$ if Ω is equivalent to an interval. On the other hand, it holds that

$$\mathfrak{h}_{s,p}((-R,R) \setminus \mathbb{Z}) \leq \frac{R^{1-sp}}{\lfloor R \rfloor} \frac{4^{1-sp}}{sp} \tag{1.25}$$

for any $R \geq 1$, with $\mathfrak{h}_{s,p}(\mathbb{R} \setminus \mathbb{Z}) = 0$ in the limit case $R = \infty$.

We emphasize that the intervals in the first part of the statement of Theorem 1.6 do not have to be at a positive distance apart, but are only required to have the same measure. Even though our proof of Theorem 1.6 yields some hints for dealing with the more general case of intervals of different measures (see Remark 2.16 for a more detailed discussion), the explicit computations in that case become quite involved and beyond the scopes of this note, so we do not pursue this direction here.

Nevertheless, the case of intervals with different measures in Theorem 1.6 can be tackled in a different way, but up to assuming that the intervals are at a positive distance apart. This is our last main result, which can be stated as follows (see also Theorem 2.17 below).

Theorem 1.7. *Let $\Omega \subsetneq \mathbb{R}$ be the union of $n \in \mathbb{N}$ non-empty bounded open intervals, each with measure at most $\ell \in (0,\infty)$ and at distance at least $\delta \in (0,\infty)$ from the others. For*

$s \in (0,1)$ *and* $p \in [1,\infty)$ *such that* $sp \le 1$*, it holds that*

$$\mathfrak{h}_{s,p}(\Omega) \ge \frac{2^{2-sp}}{sp}\left(1-\left(\frac{\ell}{\ell+\delta}\right)^{sp}\right).$$

We point out that inequality (1.5) was addressed in the non-convex case in [8, 36]. The validity of (1.5) for any open set $\Omega \subsetneq \mathbb{R}^N$ was achieved in [36, Th. 1.10] with $s \in (0,1)$ and $p \in (1,\infty)$ such that $sp > N > 1$, but with an implicit constant. Recently, this result has been improved in [8, Th. 1.1], where it was shown that

$$\mathfrak{h}_{s,p}(\Omega) \ge \mathfrak{h}_{s,p}(\mathbb{R}^N \setminus \{0\})$$

for any open set $\Omega \subsetneq \mathbb{R}^N$ with $s \in (0,1)$ and $p \in (1,\infty)$ such that $sp > N \ge 1$. To the best of our knowledge, Theorems 1.6 and 1.7 are the first results establishing inequality (1.5) on a non-convex open set $\Omega \subsetneq \mathbb{R}$ for $s \in (0,1)$ and $p \in [1,\infty)$ such that $sp \le N = 1$.

Remark 1.8 (Regional fractional Hardy inequality)**.** For completeness, we mention that a vast literature has been devoted to the study of the *regional* version of inequality (1.5),

$$[u]^p_{W^{s,p}(\Omega)} \ge C\int_\Omega \frac{|u|^p}{d_\Omega^{sp}}\,\mathrm{d}x, \quad \text{for all } u \in C_0^\infty(\Omega). \tag{1.26}$$

Far from being complete, we refer to [10–14, 36] and to the references therein for sufficient and/or necessary conditions on s, p and Ω for (1.26) to hold. The problem of determining the sharp constant in (1.26) has been addressed in [4, 18, 28]. We underline that, in contrast to (1.5), the restriction $sp > 1$ is needed, since (1.26) cannot hold for $sp \le 1$ on bounded open Lipschitz sets $\Omega \subsetneq \mathbb{R}^N$, see [10, Sec. 2]. For further discussions, refer to [3, Rem. 1.1].

2. Proofs of the results

The rest of the paper is dedicated to the proofs of the results stated in the introduction.

2.1. Notation.

Let us start by collecting some general notation and well-known results we will use throughout our paper.

Given a non-empty open set $\Omega \subseteq \mathbb{R}^N$, we let $C_0^\infty(\Omega)$ be the set of all smooth functions on $\mathbb{R}^N$ with compact support contained in Ω. For $s \in (0,1)$ and $p \in [1,\infty)$, we let $\mathcal{W}_0^{s,p}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ with respect to the seminorm in (1.6) with $A = \mathbb{R}^N$. Thanks to [6, Th. 3.1], in the case $sp < N$ (which is the only one we actually need), we can identify

$$\mathcal{W}_0^{s,p}(\mathbb{R}^N) = \left\{u \in L^{\frac{Np}{N-sp}}(\mathbb{R}^N) : [u]_{W^{s,p}(\mathbb{R}^N)} < \infty\right\}.$$

For each $k \in \mathbb{N}$, we let

$$\omega_k = \frac{\pi^{\frac{k-1}{2}}}{\Gamma\left(\frac{k-1}{2}+1\right)} \tag{2.1}$$

be the k-dimensional Lebesgue measure of the unitary ball

$$\mathbb{B}^k = \{x \in \mathbb{R}^k : |x| < 1\}$$

in $\mathbb{R}^k$. Recall that $\mathscr{H}^{k-1}(\partial\mathbb{B}^k) = k\,\omega_k$, where $\mathscr{H}^k$ is the k-dimensional Hausdorff measure.

For $a, b > 0$, we let

$$\Gamma(a) = \int_0^\infty t^{a-1}\,e^{-t}\,\mathrm{d}t \quad \text{and} \quad \mathrm{B}(a,b) = \int_0^1 t^{a-1}\,(1-t)^{b-1}\,\mathrm{d}t \tag{2.2}$$

be the *Gamma* and *Beta functions*, respectively. We recall that

$$\mathrm{B}(a,b)=\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}\quad\text{for all } a,b>0. \tag{2.3}$$

In addition, for $x\in[0,1)$ and $a>0$ and $b\in\mathbb{R}$, we let

$$\mathrm{B}(x;a,b)=\int_0^x t^{a-1}\,(1-t)^{b-1}\,\mathrm{d}t \tag{2.4}$$

be the *incomplete Beta function*. We recall that, given $a>0$,

$$\mathrm{B}(x;a,-a)=\frac{x^a}{a\,(1-x)^a}\quad\text{for all } x\in[0,1). \tag{2.5}$$

Formula (2.5) plainly follows from the representation

$$\mathrm{B}(x;a,b)=\frac{x^a}{a}\,F(a,1-b;a+1;x),\quad\text{for all } x\in[0,1),$$

and the well-known properties of the *hypergeometric function* F.

The *fractional isoperimetric inequality* (e.g., see [20] and the references therein) states that, for all $s\in(0,1)$, if $E\subseteq\mathbb{R}^N$ is such that $|E|\in(0,\infty)$, then

$$\frac{P_s(E)}{|E|^{1-\frac{s}{N}}}\ge\frac{P_s(\mathbb{B}^N)}{|\mathbb{B}^N|^{1-\frac{s}{N}}}. \tag{2.6}$$

Given a measurable set $E\subset\mathbb{R}^N$ such that $|E|\in[0,\infty)$, we let

$$E^\star=\Big\{x\in\mathbb{R}^N : |x|<r_E\Big\} \tag{2.7}$$

where $r_E\in[0,\infty)$ is such that $|E^\star|=|E|$, and we set $\mathbf{1}_E^\star=\mathbf{1}_{E^\star}$. The *symmetric decreasing rearrangement* of a measurable function $u\colon\mathbb{R}^N\to\mathbb{R}$ *vanishing at infinity*, i.e., such that

$$\Big|\Big\{x\in\mathbb{R}^N : |u|>t\Big\}\Big|<\infty\quad\text{for all } t>0,$$

is defined as

$$u^\star(x)=\int_0^\infty \mathbf{1}^\star_{\{|u|>t\}}(x)\,\mathrm{d}t\quad\text{for all } x\in\mathbb{R}^N. \tag{2.8}$$

In particular, $u^\star$ is a lower semicountinuous, non-negative, radially symmetric, and non-increasing function such that $\{u^\star>t\}=\{|u|>t\}^\star$. Moreover, if $\Phi\colon[0,\infty)\to[0,\infty)$ is a non-decreasing function, then

$$(\Phi\circ|u|)^\star=\Phi\circ u^\star \tag{2.9}$$

for any measurable function $u\colon\mathbb{R}^N\to\mathbb{R}$. In particular, $(|u|^p)^\star=(u^\star)^p$ for all $p\in[1,\infty)$.

The *Hardy–Littlewood inequality* states that, if $u,v\colon\mathbb{R}^N\to\mathbb{R}$ are two measurable functions vanishing at infinity, then

$$\int_{\mathbb{R}^N}u(x)\,v(x)\,\mathrm{d}x\le\int_{\mathbb{R}^N}u^\star(x)\,v^\star(x)\,\mathrm{d}x. \tag{2.10}$$

For a detailed presentation concerning (2.8) and (2.10), we refer to [26, Ch. 3] for instance.

By [1, Th. 9.2], if $u\in\mathcal{W}^{s,p}_0(\mathbb{R}^N)$, then also $u^\star\in\mathcal{W}^{s,p}_0(\mathbb{R}^N)$, with

$$[u^\star]_{W^{s,p}(\mathbb{R}^N)}\le[u]_{W^{s,p}(\mathbb{R}^N)}. \tag{2.11}$$

2.2. **Properties of $C_{N,q}$ and $\Lambda_{s,p}$.** We study the constants defined in (1.10) and (1.11).

On the one hand, we have the following result concerning the constant in (1.10), which basically reformulates [3, Lem. B.1] and [18, Lem. 2.4] with minor refinements.

Lemma 2.1. *If $N \geq 1$ and $q \in [0, \infty)$, then the constant in (1.10) rewrites as*

$$C_{N,q} = \frac{1}{2} \int_{\mathbb{S}^{N-1}} |x_N|^q \, \mathrm{d}\mathscr{H}^{N-1}(x) = \pi^{\frac{N-1}{2}} \frac{\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{N+q}{2}\right)}. \tag{2.12}$$

In particular, for each $N \geq 1$, the function $q \mapsto C_{N,q}$ is continuous for $q \in [0, \infty)$.

Proof. The case $N = 1$ is trivial, so we assume $N \geq 2$.

The first equality in (2.12) has been already proved in [3, Lem. B.1]: one just needs to follow the very same argument letting $sp = q$. Note that the proof of [3, Lem. B.1] also works for $q = sp \in [0, 1]$. We leave the simple check to the reader.

For the second inequality in (2.12), it is enough to observe that simple changes of variables yield the formulas (recall the definition in (2.2))

$$\mathrm{B}(a, b) = \int_0^\infty \frac{t^{a-1}}{(1+t)^{a+b}} \, \mathrm{d}t = 2 \int_0^\infty \frac{t^{2a-1}}{(1+t^2)^{a+b}} \, \mathrm{d}t \quad \text{for } a, b > 0.$$

Thus, for $a = \frac{N-1}{2}$ and $b = \frac{q+1}{2}$, and owing to (2.3), we can rewrite

$$\int_0^\infty \frac{t^{N-2}}{(1+t^2)^{\frac{N+q}{2}}} \, \mathrm{d}t = \frac{1}{2} \mathrm{B}\left(\frac{N-1}{2}, \frac{q+1}{2}\right) = \frac{\Gamma\left(\frac{N-1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{N+q}{2}\right)}.$$

Hence, recalling (2.1), we easily get that

$$C_{N,q} = (N-1)\,\omega_{N-1} \frac{\Gamma\left(\frac{N-1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{N+q}{2}\right)} = \pi^{\frac{N-1}{2}} \frac{\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{N+q}{2}\right)}.$$

The last part of the statement follows either by the Dominated Convergence Theorem or by the known continuity properties of the Gamma function. □

On the other hand, concerning the constant in (1.11), we can prove the following result.

Lemma 2.2. *For $s \in (0, 1)$ and $p \in [1, \infty)$ such that $sp < 1$, it holds that*

$$\Lambda_{s,p} < \Lambda_{sp,1} = \frac{4}{sp}. \tag{2.13}$$

As a consequence, for each $s \in (0, 1)$ it holds that

$$\lim_{p \to 1^+} \Lambda_{s,p} = \Lambda_{s,1}. \tag{2.14}$$

Proof. Since $(1-t)^p < 1 - t^p$ for all $t \in (0, 1)$, we can estimate

$$\int_0^1 \frac{\left|1 - t^{\frac{sp-1}{p}}\right|^p}{(1-t)^{1+sp}} \, \mathrm{d}t = \int_0^1 \frac{t^{sp-1}\left(1 - t^{\frac{1-sp}{p}}\right)^p}{(1-t)^{1+sp}} \, \mathrm{d}t < \int_0^1 \frac{t^{sp-1}\left(1 - t^{1-sp}\right)}{(1-t)^{1+sp}} \, \mathrm{d}t.$$

Letting $q = sp \in (0,1)$, we can write

$$\begin{aligned}\int_0^1 \frac{t^{q-1}\left(1-t^{1-q}\right)}{(1-t)^{1+q}}\,\mathrm{d}t &= \lim_{\varepsilon\to 0^+} \mathrm{B}(1-\varepsilon; q, -q) - \int_0^{1-\varepsilon} \frac{1}{(1-t)^{1+q}}\,\mathrm{d}t \\ &= \frac{1}{q} + \lim_{\varepsilon\to 0^+} \mathrm{B}(1-\varepsilon; q, -q) - \frac{\varepsilon^{-q}}{q},\end{aligned}$$

where, for $\varepsilon \in (0,1)$, $\mathrm{B}(1-\varepsilon; q, -q)$ is as in (2.4). In view of (2.5), we have

$$\mathrm{B}(1-\varepsilon; q, -q) = \frac{(1-\varepsilon)^q}{q\,\varepsilon^q}$$

and thus we can easily compute

$$\lim_{\varepsilon\to 0^+} \mathrm{B}(1-\varepsilon; q, -q) - \frac{\varepsilon^{-q}}{q} = \lim_{\varepsilon\to 0^+} \frac{(1-\varepsilon)^q}{q\,\varepsilon^q} - \frac{\varepsilon^{-q}}{q} = 0.$$

We hence get that

$$\int_0^1 \frac{t^{q-1}\left(1-t^{1-q}\right)}{(1-t)^{1+q}}\,\mathrm{d}t = \frac{1}{q}$$

and the validity of (2.13) immediately follows by recalling the definition in (1.11). By Fatou's Lemma and by (2.13), we infer that

$$\Lambda_{s,1} \le \liminf_{p\to 1^+} \Lambda_{s,p} \le \limsup_{p\to 1^+} \Lambda_{s,p} \le \limsup_{p\to 1^+} \Lambda_{sp,1} = \lim_{p\to 1^+} \frac{4}{sp} = \frac{4}{s} = \Lambda_{s,1},$$

proving (2.14) and concluding the proof. □

2.3. **Proof of Theorem 1.4.** We now deal with our auxiliary result concerning the geometrical interpretation of the sharp constant (1.7) for $p = 1$.

Proof of Theorem 1.4. We start by showing that

$$\mathfrak{h}_{s,1}(\Omega) \ge \mathfrak{g}_s(\Omega) \quad \text{in } [0,\infty]. \tag{2.15}$$

Note that we can assume that $\mathfrak{h}_{s,1}(\Omega) < \infty$, otherwise (2.15) is trivial. We fix $u \in C_0^\infty(\Omega)$ and, without loss of generality, assume that $u \ge 0$. By Cavalieri's principle, we have that

$$\int_\Omega \frac{u}{d_\Omega^s}\,\mathrm{d}x = \int_0^{\|u\|_{L^\infty(\Omega)}} \int_{\{u>t\}} \frac{1}{d_\Omega^s}\,\mathrm{d}x\,\mathrm{d}t = \int_0^{\|u\|_{L^\infty(\Omega)}} V_{s,\Omega}(\{u>t\})\,\mathrm{d}x\,\mathrm{d}t.$$

Now, for each $t \in (0, \|u\|_{L^\infty(\Omega)})$, we have $\{u > t\} \Subset \Omega$ and thus, by the definition in (1.18),

$$P_s(\{u>t\}) \ge \mathfrak{g}_s(\Omega)\, V_{s,\Omega}(\{u>t\}).$$

Hence, thanks to the fractional coarea formula in (1.19), we infer that

$$\begin{aligned}[u]_{W^{s,1}(\mathbb{R}^N)} &= \int_0^{\|u\|_{L^\infty(\Omega)}} P_s(\{u>t\})\,\mathrm{d}t \ge \int_0^{\|u\|_{L^\infty(\Omega)}} \mathfrak{g}_s(\Omega)\, V_{s,\Omega}(\{u>t\})\,\mathrm{d}t \\ &= \mathfrak{g}_s(\Omega) \int_\Omega \frac{|u|}{d_\Omega^s}\,\mathrm{d}x,\end{aligned}$$

readily implying the validity of (2.15). We now prove converse inequality, that is,

$$\mathfrak{h}_{s,1}(\Omega) \le \mathfrak{g}_s(\Omega). \tag{2.16}$$

As before, it is not restrictive to assume that $\mathfrak{g}_s(\Omega) < \infty$, otherwise (2.16) is obvious. Thanks to a routine construction, we can find bounded open sets Ω_k, $k \in \mathbb{N}$, such that

$$\Omega_k \subset \Omega_{k+1}, \quad \Omega_k \Subset \Omega, \quad P(\Omega_k) < \infty \quad \text{and} \quad \Omega = \bigcup_{k=1}^{\infty} \Omega_k.$$

In particular, $P_s(\Omega_k) < \infty$ for all $k \in \mathbb{N}$. Now, given $E \subseteq \Omega$ such that $|E| > 0$ and $P_s(E) < \infty$, we clearly have that $E_k = E \cap \Omega_k$ satisfies

$$E_k \Subset \Omega, \quad P_s(E_k) < \infty \quad \text{and} \quad |E_k| > 0 \quad \text{for all } k \in \mathbb{N} \text{ sufficiently large.}$$

Moreover, by Fatou's Lemma and the Monotone Convergence Theorem, we get that

$$P_s(E) \le \liminf_{k\to\infty} P_s(E_k) \quad \text{and} \quad V_{s,\Omega}(E) = \lim_{k\to\infty} V_{s,\Omega}(E_k). \tag{2.17}$$

Now, for $\varepsilon > 0$ and $k \in \mathbb{N}$, we let

$$u_{\varepsilon,k} = \varrho_\varepsilon \star \mathbf{1}_{E_k} \in C_0^\infty(\mathbb{R}^N),$$

where $(\varrho_\varepsilon)_{\varepsilon>0} \subset C_0^\infty(\mathbb{R}^N)$ is a family of standard non-negative mollifiers. Given $k \in \mathbb{N}$ sufficiently large as above, we have that $\operatorname{supp} u_{\varepsilon,k} \Subset \Omega$ for all $\varepsilon > 0$ sufficiently small. In this case, owing to [25, Th. 6.62] and the Dominated Convergence Theorem, we have that

$$\mathfrak{h}_{s,1}(\Omega) \le \limsup_{\varepsilon\to 0^+} \frac{[u_{\varepsilon,k}]_{W^{s,1}(\mathbb{R}^N)}}{\displaystyle\int_\Omega \frac{u_{\varepsilon,k}}{d_\Omega^s}\,\mathrm{d}x} \le \limsup_{\varepsilon\to 0^+} \frac{P_s(E_k)}{\displaystyle\int_\Omega \frac{u_{\varepsilon,k}}{d_\Omega^s}\,\mathrm{d}x} = \frac{P_s(E_k)}{V_{s,\Omega}(E_k)}.$$

Thanks to (2.17), we can hence pass to the limit as $k \to \infty$ in the above to infer that

$$\mathfrak{h}_{s,1}(\Omega) \le \frac{P_s(E)}{V_{s,\Omega}(E)}$$

whenever $E \subseteq \Omega$ is such that $|E| > 0$, yielding (2.16) and concluding the proof. □

Actually, in Theorem 1.4, we can restrict the infimization in (1.18) only to smooth open sets compactly contained in Ω. Precisely, we have the following result.

Corollary 2.3. *Let $N \ge 1$, $s \in (0,1)$ and let $\Omega \subsetneq \mathbb{R}^N$ be a non-empty open set. Then,*

$$\mathfrak{g}_s(\Omega) = \inf\left\{\frac{P_s(E)}{V_{s,\Omega}(E)} : E \in \mathcal{O}(\Omega)\right\} \in [0,\infty], \tag{2.18}$$

where $\mathcal{O}(\Omega)$ is the family of all open sets with smooth boundary compactly contained in Ω.

Proof. Letting $\tilde{\mathfrak{g}}_s(\Omega) \in [0,\infty]$ be the right-hand side of (2.18), we have $\mathfrak{g}_s(\Omega) \le \tilde{\mathfrak{g}}_s(\Omega)$ by definition. On the other hand, if $u \in C_0^\infty(\Omega)$ is such that $u \ge 0$, then $\{u > t\} \in \mathcal{O}(\Omega)$ for a.e. $t \in (0, \|u\|_{L^\infty})$ (e.g., see [29, Th. 13.15]). Hence, arguing exactly as in the first part of the proof of Theorem 1.4, we see that $\mathfrak{h}_{s,1}(\Omega) \ge \tilde{\mathfrak{g}}_s(\Omega)$. Thus, by Theorem 1.4, we infer that $\tilde{\mathfrak{g}}_s(\Omega) \le \mathfrak{g}_s(\Omega)$, readily yielding the conclusion. □

A remarkable consequence of Corollary 2.3 is the following comparison result.

Corollary 2.4. *Let $N \ge 1$, $p \in (1,\infty)$ and $s \in (0,1)$ be such that $sp < 1$. For any non-empty open set $\Omega \subsetneq \mathbb{R}^N$, it holds that*

$$\mathfrak{h}_{s,p}(\Omega) \le \mathfrak{h}_{sp,1}(\Omega). \tag{2.19}$$

If $\Omega = \mathbb{H}_+^N$, then the inequality (2.19) is strict.

Proof. Given $E \Subset \Omega$ such that $|E| > 0$, we define $u_\varepsilon = \mathbf{1}_E * \varrho_\varepsilon \in C_0^\infty(\mathbb{R}^N)$ for all $\varepsilon > 0$, where $(\varrho_\varepsilon)_{\varepsilon>0} \subset C_0^\infty(\mathbb{R}^n)$ is a family of standard non-negative mollifiers. Clearly, $u_\varepsilon \in C_0^\infty(\mathbb{R}^N)$ and $0 \le u_\varepsilon \le 1$ for all $\varepsilon > 0$, with $\operatorname{supp} u_\varepsilon \Subset \Omega$ for all $\varepsilon > 0$ sufficiently small. Therefore, by [25, Th. 6.62] and the Dominated Convergence Theorem, we get that

$$\mathfrak{h}_{s,p}(\Omega) \le \lim_{\varepsilon\to0^+} \frac{[u_\varepsilon]^p_{W^{s,p}(\mathbb{R}^N)}}{\displaystyle\int_\Omega \frac{u_\varepsilon^p}{d_\Omega^{sp}}\,\mathrm{d}x} = \frac{[\mathbf{1}_E]^p_{W^{s,p}(\mathbb{R}^N)}}{\displaystyle\int_\Omega \frac{\mathbf{1}_E^p}{d_\Omega^{sp}}\,\mathrm{d}x} = \frac{[\mathbf{1}_E]_{W^{sp,1}(\mathbb{R}^N)}}{\displaystyle\int_\Omega \frac{\mathbf{1}_E}{d_\Omega^{sp}}\,\mathrm{d}x} = \frac{P_{sp}(E)}{V_{sp,\Omega}(E)}$$

whenever $E \Subset \Omega$ is such that $|E| > 0$, since $sp < 1$. Thanks to Corollary 2.3 and Theorem 1.4, we hence conclude that

$$\mathfrak{h}_{s,p}(\Omega) \le \mathfrak{g}_{sp,1}(\Omega) = \mathfrak{h}_{sp,1}(\Omega),$$

proving (2.19). For $\Omega = \mathbb{H}^N_+$, the inequality in (2.19) holds strict by (1.13) and (2.13). □

2.4. **Proof of Theorem 1.1.** In order to characterize the sharp constant in (1.7) for $\Omega = \mathbb{H}^N_+$ and $p = 1$, that is, to prove Theorem 1.1, we need some preliminary results.

We begin with the following stability result for $\mathfrak{h}_{s,p}(\Omega)$ as $p \to 1^+$.

Lemma 2.5. *Let $s \in (0,1)$ and let $\Omega \subsetneq \mathbb{R}^N$ be a (non-empty) convex open set. If $u \in C_0^\infty(\Omega)$, then*

$$\lim_{p\to1^+} \int_\Omega \frac{|u|^p}{d_\Omega^{sp}}\,\mathrm{d}x = \int_\Omega \frac{|u|}{d_\Omega^{s}}\,\mathrm{d}x \tag{2.20}$$

and

$$\lim_{p\to1^+} [u]^p_{W^{s,p}(\mathbb{R}^N)} = [u]_{W^{s,1}(\mathbb{R}^N)}. \tag{2.21}$$

As a consequence, it holds that

$$\limsup_{p\to1^+} \mathfrak{h}_{s,p}(\Omega) \le \mathfrak{h}_{s,1}(\Omega). \tag{2.22}$$

Proof. By Hölder's inequality, we can estimate

$$\int_\Omega \frac{|u|}{d_\Omega^s}\,\mathrm{d}x = \int_{\operatorname{supp} u} \frac{|u|}{d_\Omega^s}\,\mathrm{d}x \le \left(\int_\Omega \frac{|u|^p}{d_\Omega^{sp}}\,\mathrm{d}x\right)^{\frac{1}{p}} |\operatorname{supp} u|^{\frac{p-1}{p}}, \tag{2.23}$$

and, similarly,

$$\int_\Omega \frac{|u|^p}{d_\Omega^{sp}}\,\mathrm{d}x \le \left\| \frac{u}{d_\Omega^s} \right\|^{p-1}_{L^\infty(\Omega)} \int_\Omega \frac{|u|}{d_\Omega^s}\,\mathrm{d}x. \tag{2.24}$$

Hence the validity of (2.20) immediately follows by combining (2.23) and (2.24). Concerning (2.21), owing to Fatou's Lemma, we just need to observe that

$$[u]^p_{W^{s,p}(\mathbb{R}^N)} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-1}}{|x-y|^{s(p-1)}}\,\frac{|u(x)-u(y)|}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y \le [u]^{p-1}_{C^{0,s}(\mathbb{R}^N)}\,[u]_{W^{s,1}(\mathbb{R}^N)}.$$

Lastly, the validity of (2.22) readily follows by combining (1.7) with (2.20) and (2.21). □

In the case $\Omega = \mathbb{H}^N_+$, Lemma 2.5 can be complemented with the following result, yielding useful estimates on the energies of product test functions for $p = 1$.

Lemma 2.6. *If* $\varphi \in C_0^\infty(\mathbb{R}^{N-1})$ *and* $\psi \in C_0^\infty(\mathbb{H}^1_+)$, *then* $u \in C_0^\infty(\mathbb{H}^N_+)$, *defined as* $u(x) = \varphi(x')\,\psi(x_N)$ *for all* $x = (x', x_N) \in \mathbb{H}^N_+$, *satisfies*

$$\int_{\mathbb{H}^N_+} \frac{|u(x)|}{x_N^s}\,\mathrm{d}x = \|\varphi\|_{L^1(\mathbb{R}^{N-1})} \int_{\mathbb{H}^1_+} \frac{\psi(t)}{t^s}\,\mathrm{d}t \tag{2.25}$$

and

$$[u]_{W^{s,1}(\mathbb{R}^N)} \le C_{N,s}\,\|\varphi\|_{L^1(\mathbb{R}^{N-1})}\,[\psi]_{W^{s,1}(\mathbb{R})} + \|\psi\|_{L^1(\mathbb{H}^1_+)}\,[\varphi]_{W^{s,1}(\mathbb{R}^{N-1})} \int_0^\infty \frac{\mathrm{d}t}{(1+t^2)^{\frac{N+s}{2}}}. \tag{2.26}$$

Proof. To prove (2.25), we simply observe that

$$\int_{\mathbb{H}^N_+} \frac{|u(x)|}{x_N^s}\,\mathrm{d}x = \int_{\mathbb{R}^{N-1}} \varphi(x')\,\mathrm{d}x' \int_0^\infty \frac{\psi(x_N)}{x_N^s}\,\mathrm{d}x_N = \|\varphi\|_{L^1(\mathbb{R}^{N-1})} \int_{\mathbb{H}^1_+} \frac{\psi(t)}{t^s}\,\mathrm{d}t.$$

To see (2.26), we first estimate

$$\begin{aligned}[u]_{W^{s,1}(\mathbb{R}^N)} \le &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\varphi(x')|\,\frac{|\psi(x_N)-\psi(y_N)|}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y \\ &+ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\psi(y_N)|\,\frac{|\varphi(x')-\varphi(y')|}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{2.27}$$

The first term in the right-hand side of (2.27) rewrites as

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\varphi(x')|\,\frac{|\psi(x_N)-\psi(y_N)|}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|\psi(x_N)-\psi(y_N)|}{|x_N-y_N|^{N+s}} \int_{\mathbb{R}^{N-1}} |\varphi(x')| \int_{\mathbb{R}^{N-1}} \frac{\mathrm{d}y'}{\left(1+\frac{|x'-y'|^2}{|x_N-y_N|^2}\right)^{\frac{N+s}{2}}}\,\mathrm{d}x'\,\mathrm{d}x_N\,\mathrm{d}y_N \\ &\quad= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|\psi(x_N)-\psi(y_N)|}{|x_N-y_N|^{1+s}} \int_{\mathbb{R}^{N-1}} |\varphi(x')| \int_{\mathbb{R}^{N-1}} \frac{\mathrm{d}z'}{(1+|z'|)^{\frac{N+s}{2}}}\,\mathrm{d}x'\,\mathrm{d}x_N\,\mathrm{d}y_N \\ &\quad= C_{N,s}\,\|\varphi\|_{L^1(\mathbb{R}^{N-1})}\,[\psi]_{W^{s,1}(\mathbb{R})},\end{aligned} \tag{2.28}$$

while the second term in (2.27) similarly corresponds to

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\psi(y_N)|\,\frac{|\varphi(x')-\varphi(y')|}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad= \int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}^{N-1}} \frac{|\varphi(x')-\varphi(y')|}{|x'-y'|^{N+s}} \int_{\mathbb{R}} |\psi(y_N)| \int_{\mathbb{R}} \frac{\mathrm{d}x_N}{\left(1+\frac{|x_N-y_N|^2}{|x'-y'|^2}\right)^{\frac{N+s}{2}}}\,\mathrm{d}y_N\,\mathrm{d}x'\,\mathrm{d}y' \\ &\quad= \int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}^{N-1}} \frac{|\varphi(x')-\varphi(y')|}{|x'-y'|^{N-1+s}} \int_{\mathbb{R}} |\psi(y_N)| \int_{\mathbb{R}} \frac{\mathrm{d}z_N}{(1+|z_N|^2)^{\frac{N+s}{2}}}\,\mathrm{d}y_N\,\mathrm{d}x'\,\mathrm{d}y' \\ &\quad= 2\,\|\psi\|_{L^1(\mathbb{H}^1_+)}\,[\varphi]_{W^{s,1}(\mathbb{R}^{N-1})} \int_0^\infty \frac{\mathrm{d}t}{(1+t^2)^{\frac{N+s}{2}}}.\end{aligned} \tag{2.29}$$

The conclusion hence readily follows by combining (2.27), (2.28) and (2.29). □

As a consequence of the above results, we get the following partial step towards (1.13).

Corollary 2.7. *Given* $N \ge 1$ *and* $s \in (0,1)$, *it holds that*

$$C_{N,s}\,\Lambda_{s,1} \le \mathfrak{h}_{s,1}(\mathbb{H}^N_+) \le C_{N,s}\,\mathfrak{h}_{s,1}(\mathbb{H}^1_+). \tag{2.30}$$

Proof. Owing to (2.22), (1.8), Lemma 2.1 and Lemma 2.2, we plainly get that

$$\mathfrak{h}_{s,1}(\mathbb{H}^N_+) \geq \limsup_{p\to 1^+} \mathfrak{h}_{s,p}(\mathbb{H}^N_+) = \lim_{p\to 1^+} C_{N,sp}\,\Lambda_{s,p} = C_{N,s}\,\Lambda_{s,1},$$

proving the first inequality in (2.30).

To prove the second inequality in (2.30), instead, we argue as follows. Let $\varphi \in C^\infty_0(\mathbb{R}^{N-1})$ and $\psi \in C^\infty_0(\mathbb{H}^1_+)$ be fixed. For each $k \in \mathbb{N}$, we define $\varphi_k(x') = \varphi\left(\frac{x'}{k}\right)$ for all $x' \in \mathbb{R}^{N-1}$ and $u_k(x) = \varphi_k(x')\,\psi(x_N)$ for all $x = (x', x_N) \in \mathbb{R}^{N-1} \times (0,\infty) = \mathbb{H}^N_+$. Owing to Lemma 2.6 and observing that

$$[\varphi_k]_{W^{s,1}(\mathbb{R}^{N-1})} = k^{N-1-s}\,[\varphi]_{W^{s,1}(\mathbb{R}^{N-1})} \quad \text{and} \quad \|\varphi_k\|_{L^1(\mathbb{R}^{N-1})} = k^{N-1}\,\|\varphi\|_{L^1(\mathbb{R}^{N-1})},$$

we plainly get that

$$\begin{aligned}
\mathfrak{h}_{s,1}(\mathbb{H}^N_+) &\leq \frac{[u_k]_{W^{s,1}(\mathbb{R}^N)}}{\displaystyle\int_{\mathbb{H}^N_+} \frac{|u_k(x)|}{x_N^s}\,\mathrm{d}x} \leq C_{N,s}\,\frac{[\psi]_{W^{s,1}(\mathbb{R})}}{\displaystyle\int_{\mathbb{H}^1_+} \frac{\psi(t)}{t^s}\,\mathrm{d}t} + \frac{[\varphi_k]_{W^{s,1}(\mathbb{R}^{N-1})}}{\|\varphi_k\|_{L^1(\mathbb{R}^{N-1})}}\,\frac{\displaystyle\|\psi\|_{L^1(\mathbb{H}^1_+)}\int_0^\infty \frac{\mathrm{d}t}{(1+t^2)^{\frac{N+s}{2}}}}{\displaystyle\int_{\mathbb{H}^1_+} \frac{\psi(t)}{t^s}\,\mathrm{d}t}\\
&= C_{N,s}\,\frac{[\psi]_{W^{s,1}(\mathbb{R})}}{\displaystyle\int_{\mathbb{H}^1_+} \frac{\psi(t)}{t^s}\,\mathrm{d}t} + \frac{1}{k^s}\,\frac{[\varphi]_{W^{s,1}(\mathbb{R}^{N-1})}}{\|\varphi\|_{L^1(\mathbb{R}^{N-1})}}\,\frac{\displaystyle\|\psi\|_{L^1(\mathbb{H}^1_+)}\int_0^\infty \frac{\mathrm{d}t}{(1+t^2)^{\frac{N+s}{2}}}}{\displaystyle\int_{\mathbb{H}^1_+} \frac{\psi(t)}{t^s}\,\mathrm{d}t}
\end{aligned}$$

for all $k \in \mathbb{N}$. The second inequality in (2.30) hence follows first by passing to the limit as $k \to \infty$ and then by taking the infimum with respect to $\psi \in C^\infty_0(\mathbb{H}^1_+)$. □

At this point, the proof of Theorem 1.1 reduces to a simple direct computation.

Proof of Theorem 1.1. For $N = 1$, we easily see that

$$P_s((0,1)) = \frac{4}{s(1-s)} \quad \text{and} \quad V_{s,\mathbb{H}^1_+}((0,1)) = \frac{1}{1-s} \quad \text{for all } s \in (0,1). \tag{2.31}$$

Thus, thanks to (1.17) in Theorem 1.4, we infer that

$$\mathfrak{h}_{s,1}(\mathbb{H}^1_+) = \mathfrak{g}_s(\mathbb{H}^1_+) \leq \frac{P_s((0,1))}{V_{s,\mathbb{H}^1_+}((0,1))} = \frac{4}{s}$$

for all $s \in (0,1)$. The validity of (1.13) hence follows owing to (2.13) and (2.30). Consequently, we get (1.14) by simply passing to the limit as $p \to 1^+$ in (1.12). □

As an immediate consequence of Theorem 1.1 and Lemmas 2.1 and 2.2, we get the following stability result, whose simple proof is omitted.

Corollary 2.8. *Given $N \geq 1$ and $s \in (0,1)$, it holds that*

$$\lim_{p\to 1^+} \mathfrak{h}_{s,p}(\mathbb{H}^N_+) = \mathfrak{h}_{s,1}(\mathbb{H}^N_+).$$

2.5. **Proof of Theorem 1.2.** We begin with the case $N = 1$, proving the following preliminary rearrangement result.

Lemma 2.9. *Let $s \in (0,1)$ and let $I \subsetneq \mathbb{R}$ be a non-empty bounded open interval. The function $\delta_{s,I} \colon \mathbb{R} \to [0,\infty]$, defined as*

$$\delta_{s,I}(x) = \frac{\mathbf{1}_I(x)}{d_I^s(x)} \quad \text{for all } x \in \mathbb{R},$$

satisfies

$$|\{\delta_{s,I} > t\}| = |I|\, \mathbf{1}_{(0,2^s|I|^{-s}]}(t) + 2\, t^{-\frac{1}{s}}\, \mathbf{1}_{(2^s|I|^{-s},\infty)}(t) \quad \text{for all } t > 0, \tag{2.32}$$

and thus its symmetric decreasing rearrangement $\delta^\star_{s,I} \colon \mathbb{R} \to [0,\infty]$ is given by

$$\delta^\star_{s,I}(x) = \frac{\mathbf{1}_{I^\star}(x)}{|x|^s} \quad \text{for all } x \in \mathbb{R}. \tag{2.33}$$

Proof. Let $r \in (0,\infty)$ and $c \in \mathbb{R}$ be such that $I = (c-r, c+r)$, so that $d_I(x) = r - |x-c|$ for all $x \in I$. Since $\delta_{s,I}(x) > t$ if and only if $|x-c| > r - t^{-\frac{1}{s}}$ for all $x \in I$, we have

$$\{\delta_{s,I} > t\} = \begin{cases} I & 0 < t \le r^{-s}, \\ \left(c-r, c-r+t^{-\frac{1}{s}}\right) \cup \left(c+r-t^{-\frac{1}{s}}, c+r\right) & t > r^{-s}, \end{cases}$$

from which we readily deduce (2.32). We hence get that

$$\{\delta_{s,I} > t\}^\star = \begin{cases} (-r, r) & 0 < t \le r^{-s}, \\ \left(-t^{-\frac{1}{s}}, t^{-\frac{1}{s}}\right) & t > r^{-s}, \end{cases}$$

and thus, recalling the definition in (2.8), we can compute

$$\begin{aligned} \delta^\star_{s,I}(x) &= \int_0^\infty \mathbf{1}_{\{\delta_{s,I}>t\}^\star}(x)\, \mathrm{d}t = \int_0^{r^{-s}} \mathbf{1}_{(-r,r)}(x)\, \mathrm{d}t + \int_{r^{-s}}^\infty \mathbf{1}_{\left(-t^{-1/s}, t^{-1/s}\right)}(x)\, \mathrm{d}t \\ &= r^{-s}\, \mathbf{1}_{I^\star}(x) + \mathbf{1}_{I^\star}(x) \int_{r^{-s}}^\infty \mathbf{1}_{(0,|x|^{-s})}(t)\, \mathrm{d}t = |x|^{-s}\, \mathbf{1}_{I^\star}(x) \end{aligned}$$

for all $x \in \mathbb{R}$, proving (2.33) and concluding the proof. □

Remark 2.10. With the same notation of Lemma 2.9, if $I, J \subsetneq \mathbb{R}$ are two non-empty bounded open intervals such that $|I| = |J|$, then $\delta^\star_{s,I} = \delta^\star_{s,J}$ for all $s \in (0,1)$.

Remark 2.11. Lemma 2.9 also holds for $N \ge 2$, but the formula corresponding to (2.33) has a completely different appearance. More precisely, letting $\Omega = \mathbb{B}^N$ for simplicity and, correspondingly, defining $\delta_{s,\mathbb{B}^N} \colon \mathbb{R}^N \to [0,\infty]$ as

$$\delta_{s,\mathbb{B}^N}(x) = \frac{\mathbf{1}_{\mathbb{B}^N}(x)}{d^s_{\mathbb{B}^N}(x)} = \mathbf{1}_{\mathbb{B}^N}(x)\,(1-|x|)^{-s} \quad \text{for all } x \in \mathbb{R}^N,$$

analogous computations yields that

$$\delta^\star_{s,\mathbb{B}^N}(x) = \mathbf{1}_{\mathbb{B}^N}(x) \left(1 - \left(1 - |x|^N\right)^{\frac{1}{N}}\right)^{-s} \quad \text{for all } x \in \mathbb{R}^N.$$

We omit the detailed derivation. See Remark 2.13 below for further discussions.

We can now exploit Lemma 2.9 to deal with the case $N = 1$ in Theorem 1.2, as follows.

Lemma 2.12. *It holds that* $\mathfrak{h}_{s,1}(\mathbb{B}^1) = \frac{2^{2-s}}{s}$ *for all* $s \in (0,1)$.

Proof. As in (2.31), we easily get that

$$P_s(\mathbb{B}^1) = 2^{1-s}\, P_s((0,1)) = 2^{1-s}\, \frac{4}{s(1-s)} \quad \text{and} \quad V_{s,\mathbb{B}^1}(\mathbb{B}^1) = \frac{2}{(1-s)},$$

so that, owing to (1.17) in Theorem 1.4, we can estimate

$$\mathfrak{h}_{s,1}(\mathbb{B}^1) = \mathfrak{g}_s(\mathbb{B}^1) \le \frac{P_s(\mathbb{B}^1)}{V_{s,\mathbb{B}^1}(\mathbb{B}^1)} = \frac{2^{2-s}}{s}$$

for all $s \in (0,1)$. To prove the converse inequality, let $E \subseteq \mathbb{B}^1$ be such that $|E| > 0$ and define $E^\star = \left(-\frac{|E|}{2}, \frac{|E|}{2}\right) \subseteq \mathbb{B}^1$ as in (2.7). By (2.6), we have that

$$P_s(E) \ge P_s(E^\star) = P_s((0,1))\, |E|^{1-s} = \frac{4}{s(1-s)}\, |E|^{1-s} \tag{2.34}$$

for all $s \in (0,1)$. On the other hand, by (2.10) and Lemma 2.9, we have that

$$V_{s,\mathbb{B}^1}(E) = \int_{\mathbb{R}} \mathbf{1}_E(x)\, \delta_s(x)\, \mathrm{d}x \le \int_{\mathbb{R}} \mathbf{1}_{E^\star}(x)\, \delta_s^\star(x)\, \mathrm{d}x = \int_{-\frac{|E|}{2}}^{\frac{|E|}{2}} \frac{\mathrm{d}x}{|x|^s} = \frac{2^s}{(1-s)}\, |E|^{1-s}. \tag{2.35}$$

By combining (2.34) and (2.35), we thus get that

$$\frac{P_s(E)}{V_{s,\mathbb{B}^1}(E)} \ge \frac{4\,|E|^{1-s}}{s(1-s)}\, \frac{(1-s)}{2^s\, |E|^{1-s}} = \frac{2^{2-s}}{s}$$

for all $s \in (0,1)$ whenever $E \subseteq \mathbb{B}^1$ is such that $|E| > 0$, readily yielding the conclusion. ☐

Remark 2.13. The strategy of the proof of Lemma 2.12 is ineffectual in the higher dimensional case $N \ge 2$. Indeed, by applying Remark 2.11, we analogously get that

$$\frac{P_s(E)}{V_{s,\mathbb{B}^N}(E)} \ge \frac{P_s(\mathbb{B}^N)}{|\mathbb{B}^N|^{1-\frac{s}{N}}}\, \frac{|E|^{1-\frac{s}{N}}}{\displaystyle\int_{E^\star} \left(1 - \left(1 - |x|^N\right)^{\frac{1}{N}}\right)^{-s} \mathrm{d}x} \tag{2.36}$$

whenever $E \subseteq \mathbb{B}^N$ is such that $|E| > 0$, where $E^\star$ is as in (2.7). However, we have that

$$\inf_{R \in (0,1]} \frac{|B_R|^{1-\frac{s}{N}}}{\displaystyle\int_{B_R} \left(1 - \left(1 - |x|^N\right)^{\frac{1}{N}}\right)^{-s} \mathrm{d}x} = 0,$$

so that (2.36) leads to the trivial lower bound $\mathfrak{h}_{s,1}(\mathbb{B}^N) \ge 0$.

We now pass to the case $N \ge 2$ in Theorem 1.2.

Proof of Theorem 1.2. Thanks to Lemma 2.12, we can assume $N \ge 2$. Let

$$r_\Omega = \sup_{x \in \Omega} d_\Omega(x) \in (0, \infty)$$

be the *inradius* of Ω. By [23, Th. 1.2] (see also [24]), we have that

$$P(\Omega_t) \ge \left(1 - \frac{t}{r_\Omega}\right)^{N-1} P(\Omega) \quad \text{for all } t \in (0, r_\Omega), \tag{2.37}$$

where

$$\Omega_t = \{x \in \Omega : d_\Omega(x) > t\}.$$

By scale invariance of (1.7), we can assume that $r_\Omega = 1$ without loss of generality. Since d_Ω is 1-Lipschitz, $|\nabla d_\Omega| \le 1$ a.e. in Ω and thus, by the usual coarea formula, we have that

$$V_{s,\Omega}(\Omega) = \int_\Omega \frac{1}{d_\Omega^s}\,\mathrm{d}x \ge \int_\Omega \frac{|\nabla d_\Omega|}{d_\Omega^s}\,\mathrm{d}x = \int_0^1 \int_{\{d_\Omega = t\}} \frac{1}{t^s}\,\mathrm{d}\mathscr{H}^{N-1}\,\mathrm{d}t = \int_0^1 \frac{P(\Omega_t)}{t^s}\,\mathrm{d}t,$$

since

$$\partial\Omega_t = \{x \in \Omega : d_\Omega(x) = t\} \quad \text{and} \quad P(\Omega_t) = \mathscr{H}^{N-1}(\partial\Omega_t) \quad \text{for all } t \in (0,1).$$

Owing to (2.37), we hence get that

$$V_{s,\Omega}(\Omega) \ge \int_0^1 \frac{P(\Omega_t)}{t^s}\,\mathrm{d}t \ge P(\Omega) \int_0^1 t^{-s}\,(1-t)^{N-1}\,\mathrm{d}t = P(\Omega)\;\mathrm{B}(N, 1-s).$$

By Theorem 1.4 and (2.3), we can thus estimate

$$\mathfrak{h}_{s,1}(\Omega) \le \frac{P_s(\Omega)}{V_{s,\Omega}(\Omega)} \le \frac{(1-s)\,P_s(\Omega)}{P(\Omega)}\,\frac{\Gamma(N+1-s)}{\Gamma(N)\,\Gamma(2-s)} \tag{2.38}$$

and so, thanks to (1.13) and (2.12), we get that

$$\frac{\mathfrak{h}_{s,1}(\Omega)}{\mathfrak{h}_{s,1}(\mathbb{H}^N_+)} \le \frac{(1-s)\,P_s(\Omega)}{P(\Omega)}\,\frac{\Gamma(N+1-s)}{\Gamma(N)\,\Gamma(2-s)}\,\frac{s\,\Gamma\left(\frac{N+s}{2}\right)}{4\,\pi^{\frac{N-1}{2}}\,\Gamma\left(\frac{s+1}{2}\right)}$$

for all $s \in (0,1)$. Hence, by (1.20) and (2.1), we infer that

$$\begin{aligned}\limsup_{s\to1^-} \frac{\mathfrak{h}_{s,1}(\Omega)}{\mathfrak{h}_{s,1}(\mathbb{H}^N_+)} &\le \lim_{s\to1^-} \frac{(1-s)\,P_s(\Omega)}{P(\Omega)}\,\frac{\Gamma(N+1-s)}{\Gamma(N)\,\Gamma(2-s)}\,\frac{s\,\Gamma\left(\frac{N+s}{2}\right)}{4\,\pi^{\frac{N-1}{2}}\,\Gamma\left(\frac{s+1}{2}\right)}\\ &= 2\,\omega_{N-1}\,\frac{\Gamma(N)}{\Gamma(N)\,\Gamma(1)}\,\frac{\Gamma\left(\frac{N+1}{2}\right)}{4\,\pi^{\frac{N-1}{2}}\,\Gamma(1)} = \frac{1}{2}\end{aligned}$$

and the conclusion immediately follows by the first inequality in (1.14). ☐

Remark 2.14. Given $N \ge 2$ and a (non-empty) bounded convex open set $\Omega \subsetneq \mathbb{R}^N$, from (2.38) and (1.20) we get that

$$\limsup_{s\to1^-} \mathfrak{h}_{s,1}(\Omega) \le \lim_{s\to1^-} \frac{(1-s)\,P_s(\Omega)}{P(\Omega)}\,\frac{\Gamma(N+1-s)}{\Gamma(N)\,\Gamma(2-s)} = 2\,\omega_{N-1},$$

while, thanks to Theorem 1.1 and Lemma 2.1,

$$\liminf_{s\to1^-} \mathfrak{h}_{s,1}(\Omega) \ge \frac{1}{2} \lim_{s\to1^-} \mathfrak{h}_{s,1}(\mathbb{H}^N_+) = 2\,C_{N,1} = 2\,\omega_{N-1}.$$

This implies that (1.16) can be equivalently reformulated as

$$\lim_{s\to1^-} \mathfrak{h}_{s,1}(\Omega) = 2\,\omega_{N-1} < 4\,\omega_{N-1} = \lim_{s\to1^-} \mathfrak{h}_{s,1}(\mathbb{H}^N_+).$$

Noteworthy, thanks to [33, Th. 3] (see also [34]), the first inequality in (2.38), Lemma 2.1 and Theorem 1.1, we also have that

$$\limsup_{s\to0^+} s\,\mathfrak{h}_{s,1}(\Omega) \le \lim_{s\to0^+} \frac{s\,P_s(\Omega)}{V_{s,\Omega}(\Omega)} = \frac{2\,N\,\omega_N\,|\Omega|}{|\Omega|} = 2\,N\,\omega_N = 4\,C_{N,0} = \lim_{s\to0^+} s\,\mathfrak{h}_{s,1}(\mathbb{H}^N_+).$$

Due to (1.15), in the case $N = 1$, the above improves to

$$\lim_{s\to 0^+} s\,\mathfrak{h}_{s,1}(\Omega) = \lim_{s\to 0^+} s\,\mathfrak{h}_{s,1}(\mathbb{H}^1_+),$$

but we do not know if this is also the case for $N \geq 2$.

As already mentioned in the introduction, the limit in (1.16) can be slightly improved if Ω is an open ball by exploiting the main result of [20], as follows.

Proposition 2.15. *If $N \geq 2$, then*

$$\mathfrak{h}_{s,1}(\mathbb{B}^N) \leq \frac{\pi^{\frac{1}{2}}}{\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(\frac{1}{2}+\frac{s}{2}\right)} \frac{\Gamma\left(\frac{N+s}{2}\right)}{\Gamma\left(\frac{N-s}{2}\right)} \frac{\Gamma(N-s)}{\Gamma(N)}\,\mathfrak{h}_{s,1}(\mathbb{H}^N_+) \quad \textit{for all } s \in (0,1). \tag{2.39}$$

Proof of Proposition 2.15. Since $d_{\mathbb{B}^N}(x) = 1 - |x|$ for all $x \in \mathbb{B}^N$, we can compute

$$V_{s,\mathbb{B}^N}(\mathbb{B}^N) = \int_{\mathbb{B}^N} (1-|x|)^{-s}\,\mathrm{d}x = N\omega_N\,\mathrm{B}(N,1-s) = N\omega_N \frac{\Gamma(N)\,\Gamma(1-s)}{\Gamma(N+1-s)}$$

for all $s \in (0,1)$. On the other hand, thanks to [20, Prop. 1.1], we can write

$$P_s(\mathbb{B}^N) = \omega_N^{1-\frac{s}{N}} \frac{N\,\pi^{\frac{N+s}{2}}\,\Gamma(1-s)}{\frac{s}{2}\,\Gamma\left(\frac{N}{2}+1\right)^{\frac{s}{N}}\,\Gamma\left(1-\frac{s}{2}\right)\,\Gamma\left(\frac{N-s}{2}+1\right)} \tag{2.40}$$

for all $s \in (0,1)$. Therefore, recalling (2.1), we get that

$$\begin{aligned}
\mathfrak{h}_{s,1}(\mathbb{B}^N) \leq \frac{P_s(\mathbb{B}^N)}{V_{s,\mathbb{B}^N}(\mathbb{B}^N)} &= \omega_N^{1-\frac{s}{N}} \frac{N\,\pi^{\frac{N+s}{2}}\,\Gamma(1-s)}{\frac{s}{2}\,\Gamma\left(\frac{N}{2}+1\right)^{\frac{s}{N}}\,\Gamma\left(1-\frac{s}{2}\right)\,\Gamma\left(\frac{N-s}{2}+1\right)} \frac{\Gamma(N+1-s)}{N\omega_N\,\Gamma(N)\,\Gamma(1-s)} \\
&= \omega_N^{-\frac{s}{N}} \frac{\pi^{\frac{N+s}{2}}}{\frac{s}{2}\,\Gamma\left(\frac{N}{2}+1\right)^{\frac{s}{N}}\,\Gamma\left(1-\frac{s}{2}\right)\,\Gamma\left(\frac{N-s}{2}+1\right)} \frac{\Gamma(N+1-s)}{\Gamma(N)} \\
&= \frac{2\,\pi^{\frac{N}{2}}\,\Gamma(N+1-s)}{s\,\Gamma(N)\,\Gamma\left(1-\frac{s}{2}\right)\,\Gamma\left(\frac{N-s}{2}+1\right)}
\end{aligned}$$

for all $s \in (0,1)$. We can thus estimate

$$\begin{aligned}
\frac{\mathfrak{h}_{s,1}(\mathbb{B}^N)}{\mathfrak{h}_{s,1}(\mathbb{H}^N_+)} &\leq \frac{2\,\pi^{\frac{N}{2}}\,\Gamma(N+1-s)}{s\,\Gamma(N)\,\Gamma\left(1-\frac{s}{2}\right)\,\Gamma\left(\frac{N-s}{2}+1\right)} \frac{s\,\Gamma\left(\frac{N+s}{2}\right)}{4\,\pi^{\frac{N-1}{2}}\,\Gamma\left(\frac{s+1}{2}\right)} \\
&= \frac{\pi^{\frac{1}{2}}}{\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(\frac{1}{2}+\frac{s}{2}\right)} \frac{\Gamma\left(\frac{N+s}{2}\right)}{\Gamma\left(\frac{N-s}{2}\right)} \frac{\Gamma(N-s)}{\Gamma(N)}
\end{aligned}$$

for all $s \in (0,1)$, proving (2.39) and concluding the proof. □

2.6. **Proof of Corollary 1.3.** We can now study the sharp constant in (1.7) for $s \in (0,1)$ and $p \in (1,\infty)$ such that $sp < 1$.

Proof of Corollary 1.3. By (1.12), Theorem 1.1, (2.22) and Corollary 2.8, we get that

$$\frac{1}{2} \leq \limsup_{p\to 1^+} \frac{\mathfrak{h}_{s,p}(\Omega)}{\mathfrak{h}_{s,p}(\mathbb{H}^N_+)} \leq \frac{\mathfrak{h}_{s,1}(\Omega)}{\mathfrak{h}_{s,1}(\mathbb{H}^N_+)}$$

for all $s \in (0,1)$. The conclusion hence plainly follows from Theorem 1.2. □

2.7. **Proof of Corollary 1.5.** We now pass to the study of the non-convex case $\Omega = \mathbb{R}^N \setminus \{0\}$, (re)proving [19, Th. 1.1] in the case $p = 1$.

Proof of Corollary 1.5. Let us set $\mathbb{U}^N = \mathbb{R}^N \setminus \{0\}$ for brevity. On the one hand, since

$$V_{s,\mathbb{U}^N}(\mathbb{B}^N) = \int_{\mathbb{B}^N} \frac{dx}{|x|^s} = N\,\omega_N \int_0^1 r^{N-1-s}\,dr = \frac{N\,\omega_N}{N-s},$$

we can estimate

$$\mathfrak{g}_s(\mathbb{U}^N) \le \frac{P_s(\mathbb{B}^N)}{V_{s,\mathbb{U}^N}(\mathbb{B}^N)} = P_s(\mathbb{B}^N)\,\frac{N-s}{N\,\omega_N} \tag{2.41}$$

for all $s \in (0,1)$. On the other hand, given a measurable set $E \subseteq \mathbb{R}^N$ such that $|E| \in (0,\infty)$, for all $s \in (0,1)$, by (2.6), we can estimate

$$P_s(E) \ge \frac{P_s(\mathbb{B}^N)}{|\mathbb{B}^N|^{1-\frac{s}{N}}}\,|E|^{1-\frac{s}{N}} = \frac{P_s(\mathbb{B}^N)}{\omega_N^{1-\frac{s}{N}}}\,|E|^{1-\frac{s}{N}},$$

while, by (2.10) and owing to the fact that $x \mapsto |x|^{-s}$ is radially symmetric decreasing,

$$V_{s,\mathbb{U}^N}(E) = \int_E \frac{1}{|x|^s}\,dx \le \int_{E^\star} \frac{1}{|x|^s}\,dx = \frac{N\,\omega_N}{N-s}\left(\frac{|E|}{\omega_N}\right)^{1-\frac{s}{N}}$$

where E^* is as in (2.7). Thus, we get that

$$\frac{P_s(E)}{V_{s,\mathbb{U}^N}(E)} \ge \frac{P_s(\mathbb{B}^N)}{\omega_N^{1-\frac{s}{N}}}\,|E|^{1-\frac{s}{N}}\,\frac{N-s}{N\,\omega_N}\left(\frac{\omega_N}{|E|}\right)^{1-\frac{s}{N}} = P_s(\mathbb{B}^N)\,\frac{N-s}{N\,\omega_N} \tag{2.42}$$

for all $s \in (0,1)$, whenever $E \subseteq \mathbb{R}^N$ is such that $|E| \in (0,\infty)$. By combining (2.41) and (2.42), by [20, Prop. 1.1], that is, by (2.40), and recalling (2.1), we conclude that

$$\mathfrak{g}_s(\mathbb{U}^N) = P_s(\mathbb{B}^N)\,\frac{N-s}{N\,\omega_N} = \frac{4}{s}\,\frac{\pi^{\frac{N}{2}}\,\Gamma(1-s)}{\Gamma\left(\frac{N-s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)}$$

for all $s \in (0,1)$, which yields (1.22) in virtue of Theorem 1.4. The characterization of the equality cases in (1.21) follows as in [19], so we omit the details. □

2.8. **Proofs of Theorems 1.6 and 1.7.** We now conclude our note with the study of the non-convex case in dimension $N = 1$, starting with the proof of Theorem 1.6.

Proof of Theorem 1.6. Before dealing with (1.24), we need some preliminary work, generalizing the rearrangement result achieved in Lemma 2.9 to the present situation.

Let $\Omega = \bigcup_{i=1}^n I_i$, with $I_i = (c_i - r, c_i + r)$ with $r \in (0,\infty)$ and $c_i \in \mathbb{R}$ for each $i = 1,\dots,n$. In particular, note that it may happen that $c_i + r = c_j - r$ for some $i \ne j$, as we are not assuming that the distance between any two I_i's is strictly positive. Due to the scale invariance of (1.7), we can assume $r = 1$ without loss of generality.

Adopting the same notation of Lemma 2.9, we define $\delta_{s,\Omega}\colon \mathbb{R} \to [0,\infty]$ as

$$\delta_{s,\Omega}(x) = \frac{\mathbf{1}_\Omega(x)}{d_\Omega(x)^s} \quad \text{for all } x \in \mathbb{R}.$$

As the I_i's are disjoint, we can write

$$\delta_{s,\Omega}(x) = \sum_{i=1}^{n} \delta_{s,I_i}(x), \quad \text{for all } x \in \mathbb{R}, \tag{2.43}$$

where $\delta_{s,I_i} \colon \mathbb{R} \to [0,\infty]$, $i = 1,\dots,n$, is defined as

$$\delta_{s,I_i}(x) = \frac{\mathbf{1}_{I_i}(x)}{d_{I_i}(x)^s}, \quad \text{for all } x \in \mathbb{R},$$

exactly with the same notation of Lemma 2.9. The decomposition in (2.43) implies that

$$\{\delta_{s,\Omega} > t\} = \bigcup_{i=1}^{n} \{\delta_{s,I_i} > t\} \quad \text{for all } t > 0,$$

with disjoint union. By Lemma 2.9 and since $|I_i| = 2$ for all $i = 1,\dots,n$, we hence get

$$\begin{aligned}
|\{\delta_{s,\Omega} > t\}| &= \sum_{i=1}^{n} |\{\delta_{s,I_i} > t\}| \\
&= \sum_{i=1}^{n} |I_i| \, \mathbf{1}_{(0,2^s|I_i|^{-s}]}(t) + 2\, t^{-\frac{1}{s}} \, \mathbf{1}_{(2^s|I_i|^{-s},\infty)}(t) \\
&= \sum_{i=1}^{n} 2\, \mathbf{1}_{(0,1]}(t) + 2\, t^{-\frac{1}{s}} \, \mathbf{1}_{(1,\infty)}(t) \\
&= 2\, n\, \mathbf{1}_{(0,1]}(t) + 2\, n\, t^{-\frac{1}{s}} \, \mathbf{1}_{(1,\infty)}(t)
\end{aligned}$$

for all $t > 0$. We thus obtain that

$$\{\delta_{s,\Omega} > t\}^\star = \begin{cases} (-n, n) & 0 < t \le 1, \\ \left(-n\, t^{-\frac{1}{s}}, n\, t^{-\frac{1}{s}}\right) & t > 1, \end{cases}$$

and thus

$$\begin{aligned}
\delta^\star_{s,\Omega}(x) &= \int_0^\infty \mathbf{1}_{\{\delta_{s,\Omega} > t\}^\star}(x)\,\mathrm{d}t = \int_0^1 \mathbf{1}_{(-n,n)}(x)\,\mathrm{d}t + \int_1^\infty \mathbf{1}_{(-n\,t^{-1/s}, n\,t^{-1/s})}(x)\,\mathrm{d}t \\
&= \mathbf{1}_{(-n,n)}(x) + \mathbf{1}_{(-n,n)}(x) \int_1^\infty \mathbf{1}_{(0,n^s|x|^{-s})}(t)\,\mathrm{d}t = n^s\, \mathbf{1}_{(-n,n)}(x)\, |x|^{-s}
\end{aligned}$$

for all $x \in \mathbb{R}$, which, observing that $\Omega^\star = (-n, n)$, can be equivalently reformulated as

$$\delta^\star_{s,\Omega}(x) = n^s \, \frac{\mathbf{1}_{\Omega^\star}(x)}{|x|^s} \quad \text{for all } x \in \mathbb{R}. \tag{2.44}$$

We can now deal with (1.24) by distinguishing the cases $p = 1$ and $p \in \left(1, \frac{1}{s}\right)$.

Case $p = 1$. Let $E \subseteq \Omega$ be a measurable set with $|E| > 0$. On the one hand, setting $E^\star = \left(-\frac{|E|}{2}, \frac{|E|}{2}\right)$, by (2.6), we have that

$$P_s(E) \ge P_s(E^\star) = P_s((0,1))\, |E|^{1-s} = \frac{4}{s\,(1-s)}\, |E|^{1-s} \tag{2.45}$$

for all $s \in (0,1)$. On the other hand, by (2.10), we get that

$$V_{s,\Omega}(E) = \int_{\mathbb{R}} \mathbf{1}_E(x)\, \delta_{s,\Omega}(x)\,\mathrm{d}x \le \int_{\mathbb{R}} \mathbf{1}_{E^\star}(x)\, \delta^\star_{s,\Omega}(x)\,\mathrm{d}x$$

and thus, by (2.44) and since $E^\star \subseteq \Omega^\star = (-n, n)$, we can estimate

$$V_{s,\Omega}(E) \le \int_{-\frac{|E|}{2}}^{\frac{|E|}{2}} \frac{n^s}{|x|^s}\,\mathrm{d}x = n^s\,\frac{2\,|E|^{1-s}}{2^{1-s}\,(1-s)} = 2^s\,n^s\,\frac{|E|^{1-s}}{(1-s)} \tag{2.46}$$

for all $s \in (0,1)$. Therefore, by combining (2.45) and (2.46), we get that

$$\frac{P_s(E)}{V_{s,\Omega}(E)} \ge \frac{4}{s\,(1-s)}\,|E|^{1-s}\,2^{-s}\,n^{-s}\,\frac{(1-s)}{|E|^{1-s}} = \frac{2^{2-s}}{s\,n^s}$$

for all $s \in (0,1)$, whenever $E \subseteq \Omega$ is such that $|E| > 0$, so that, by Theorem 1.4,

$$\mathfrak{h}_{s,1}(\Omega) = \mathfrak{g}_s(\Omega) \ge \frac{2^{2-s}}{s\,n^s}$$

for all $s \in (0,1)$, proving (1.24) in the case $p = 1$ thanks to (1.23). Moreover, if Ω is equivalent to an interval, then we may assume that $\Omega = \Omega^\star = (-n, n)$, so that

$$\mathfrak{h}_{s,1}(\Omega) = \mathfrak{g}_s(\Omega) \le \frac{P_s(\Omega^\star)}{V_{s,\Omega^\star}(\Omega^\star)} = \frac{2^{2-s}}{s\,n^s}$$

by the very same computations performed above.

Case $p \in \left(1, \frac{1}{s}\right)$. Let $u \in C_0^\infty(\Omega)$. By (2.10) and (2.9), we can estimate

$$\int_\Omega \frac{|u|^p}{d_\Omega^{sp}}\,\mathrm{d}x = \int_{\mathbb{R}} |u|^p\,\delta_{sp,\Omega}\,\mathrm{d}x \le \int_{\mathbb{R}} (|u|^p)^\star\,\delta^\star_{sp,\Omega}\,\mathrm{d}x = \int_{\mathbb{R}} (u^\star)^p\,\delta^\star_{sp,\Omega}\,\mathrm{d}x.$$

Since $sp < 1$, we can exploit (2.44) (with sp in place of s) to infer that

$$\int_\Omega \frac{|u|^p}{d_\Omega^{sp}}\,\mathrm{d}x \le \int_{\mathbb{R}} (u^\star)^p\,\delta^\star_{sp,\Omega}\,\mathrm{d}x = n^{sp}\int_{\mathbb{R}} \frac{(u^\star)^p}{|x|^{sp}}\,\mathrm{d}x, \tag{2.47}$$

in view of the fact that $\operatorname{supp} u^\star \subseteq \Omega^\star$. By combining (2.11) and (2.47), we thus get that

$$\frac{[u]^p_{W^{s,p}(\mathbb{R})}}{\displaystyle\int_\Omega \frac{|u|^p}{d_\Omega^{sp}}\,\mathrm{d}x} \ge \frac{[u^\star]^p_{W^{s,p}(\mathbb{R})}}{\displaystyle n^{sp}\int_{\mathbb{R}} \frac{(u^\star)^p}{|x|^{sp}}\,\mathrm{d}x} \ge \frac{\mathfrak{h}_{s,p}\left(\mathbb{R}^N \setminus \{0\}\right)}{n^{sp}}$$

owing to Corollary 1.5. This concludes the proof of (1.24).

We now pass to the proof of (1.25). By Corollary 2.4 and Theorem 1.4, for any $R \ge 1$, we can easily estimate

$$\mathfrak{h}_{s,p}((-R,R)\setminus\mathbb{Z}) \le \mathfrak{h}_{sp,1}((-R,R)\setminus\mathbb{Z}) = \mathfrak{g}_{sp}((-R,R)\setminus\mathbb{Z}). \tag{2.48}$$

Now, since $|\mathbb{Z}| = 0$, we have that

$$P_{sp}((-R,R)\setminus\mathbb{Z}) = P_{sp}((-R,R)) = (2R)^{1-sp}\,P_{sp}((0,1)) = (2R)^{1-sp}\,\frac{4}{sp\,(1-sp)}, \tag{2.49}$$

thanks to (2.31) (with $sp < 1$ in place of s), and also

$$\begin{aligned} V_{sp,(-R,R)\setminus\mathbb{Z}}((-R,R)\setminus\mathbb{Z}) &\ge V_{sp,(-R,R)\setminus\mathbb{Z}}((-\lfloor R\rfloor,\lfloor R\rfloor)) = V_{sp,(-\lfloor R\rfloor,\lfloor R\rfloor)\setminus\mathbb{Z}}((-\lfloor R\rfloor,\lfloor R\rfloor)) \\ &= 2\,\lfloor R\rfloor\,V_{sp,(0,1)}((0,1)) = 4\,\lfloor R\rfloor\int_0^{\frac{1}{2}} r^{-sp}\,\mathrm{d}r = \lfloor R\rfloor\,\frac{2^{1+sp}}{1-sp}. \end{aligned} \tag{2.50}$$

Hence, by combining (2.48), (2.49) and (2.50), we infer that

$$\mathfrak{h}_{s,p}((-R,R)\setminus\mathbb{Z}) \le \mathfrak{g}_{sp}((-R,R)\setminus\mathbb{Z}) \le \frac{P_{sp}((-R,R)\setminus\mathbb{Z})}{V_{sp,(-R,R)\setminus\mathbb{Z}}((-R,R)\setminus\mathbb{Z})} = \frac{R^{1-sp}}{\lfloor R\rfloor}\,\frac{4^{1-sp}}{sp}$$

proving (1.25) for any $R \ge 1$. For the limit case $R=\infty$, we can analogously bound

$$\mathfrak{h}_{s,p}(\mathbb{R}\setminus\mathbb{Z}) \le \mathfrak{h}_{sp,1}(\mathbb{R}\setminus\mathbb{Z}) = \mathfrak{g}_{sp}(\mathbb{R}\setminus\mathbb{Z}) \le \mathfrak{g}_{sp}((-m,m)\setminus\mathbb{Z}) \le \frac{1}{m^{sp}}\,\frac{4^{1-sp}}{sp}$$

for all $m\in\mathbb{N}$ because $V_{s,\mathbb{R}\setminus\mathbb{Z}}(E) = V_{s,(-m,m)\setminus\mathbb{Z}}(E)$ for any $E\subset(-m,m)\setminus\mathbb{Z}$, readily yielding $\mathfrak{h}_{s,p}(\mathbb{R}\setminus\mathbb{Z})=0$ and concluding the proof. □

Remark 2.16. Let $\Omega=\bigcup_{i=1}^n I_i$, with $I_i=(c_i-r_i,c_i+r_i)$, $0<r_1\le r_2\le\dots\le r_n<\infty$ and $c_i\in\mathbb{R}$, for each $i=1,\dots,n$. By arguing as in the proof of Theorem 1.6, we get that

$$\{\delta_{s,\Omega}>t\}^\star = (-r(t),r(t)) \quad \text{for all } t>0,$$

where, with a slight abuse of notation,

$$r(t) = \sum_{k=0}^{n} \mathbf{1}_{\left(r_{n+1-k}^{s},\,r_{n-k}^{-s}\right]}(t)\left(k\,t^{-\frac{1}{s}} + \sum_{i=0}^{n-k} r_i\right), \tag{2.51}$$

where we have set $r_0=0$ and $r_{n+1}=\infty$ with the convention that $\frac{1}{\infty}=0$ and $\frac{1}{0}=\infty$. The radius function in (2.51) can be exploited to compute the symmetric decreasing rearrangement $\delta^\star_{s,\Omega}$ of $\delta_{s,\Omega}$ depending to the sizes of the r_i's, but the resulting formula is not as neat as the one in (2.44) (in which the r_i's are all equal).

We now turn to the proof of Theorem 1.7—actually, of the following more precise result. Here and below, given an non-empty set $A\subsetneq\mathbb{R}$ and $\delta\in(0,\infty)$, we let $A^\delta=\{x\in\mathbb{R}:\operatorname{dist}(x,A)<\delta\}$ be the *open δ-neighborhood* of A.

Theorem 2.17. *Let $\Omega\subsetneq\mathbb{R}$, $n\in\mathbb{N}$ and $\ell,\delta\in(0,\infty)$ be as in Theorem 1.7. For $s\in(0,1)$ and $p\in[1,\infty)$ such that $sp\le 1$, it holds that*

$$[u]^p_{W^{s,p}(\Omega^\delta)} \ge \frac{2}{sp}\left(1-\left(\frac{\ell}{\ell+\delta}\right)^{sp}\right)\int_\Omega \frac{|u|^p}{d_\Omega^{sp}}\,\mathrm{d}x$$

for all $u\in C_0^\infty(\Omega)$.

To prove Theorem 2.17, we need the following preliminary result, that is, Theorem 2.17 in the case of a single interval.

Proposition 2.18. *Let $I\subsetneq\mathbb{R}$ be a non-empty bounded open interval with $|I|=\ell\in(0,\infty)$. For $s\in(0,1)$ and $p\in[1,\infty)$ such that $sp\le1$, it holds that*

$$[u]^p_{W^{s,p}(I^\delta)} \ge \frac{2^{2-sp}}{sp}\left(1-\left(\frac{\ell}{\ell+\delta}\right)^{sp}\right)\int_I \frac{|u|^p}{d_I^{sp}}\,\mathrm{d}x$$

for any $\delta\in(0,\infty)$ and $u\in C_0^\infty(I)$.

Proof. For convenience, let $a,b\in\mathbb{R}$, $a<b$, be such that $I=(a,b)$. We let $\varphi\in\operatorname{Lip}(\mathbb{R})$ be a Lipschitz function such that $\operatorname{supp}\varphi\subset I$ and $\varphi\ge0$. For any $\varepsilon>0$, we define

$$\triangle_\varepsilon = \{(x,y)\in\mathbb{R}^2 : |x-y|\le\varepsilon\}.$$

With this notation in force, we can write

$$\begin{aligned}\iint_{(I^\delta\times I^\delta)\setminus\triangle_\varepsilon} &\frac{J_p(\mathbf{1}_I(x)-\mathbf{1}_I(y))}{|x-y|^{1+sp}}\,(\varphi(x)-\varphi(y))\,\mathrm{d}x\,\mathrm{d}y\\ &=2\iint_{(I^\delta\times I^\delta)\setminus\triangle_\varepsilon}\frac{J_p(\mathbf{1}_I(x)-\mathbf{1}_I(y))}{|x-y|^{1+sp}}\,\varphi(x)\,\mathrm{d}x\,\mathrm{d}y\\ &=2\int_I\varphi(x)\int_{\{y\in I^\delta\setminus I\,:\,|y-x|>\varepsilon\}}\frac{\mathrm{d}y}{|y-x|^{1+sp}}\,\mathrm{d}x,\end{aligned}$$

where we have set

$$J_p(t)=\begin{cases}|t|^{p-2}\,t & \text{for } p\in(1,\infty),\\ \operatorname{sgn}(t) & \text{for } p=1,\end{cases}$$

for all $t\neq0$, and $J_p(0)=0$. We now observe that

$$\frac{1}{(x-a+\delta)^{sp}}\le\frac{C_\delta^{sp}}{(x-a)^{sp}}\quad\text{and}\quad\frac{1}{(b+\delta-x)^{sp}}\le\frac{C_\delta^{sp}}{(b-x)^{sp}}\quad\text{for all } x\in I,$$

where

$$C_\delta=\frac{b-a}{b-a+\delta}\in(0,1).$$

We can hence estimate

$$\begin{aligned}\lim_{\varepsilon\to0^+}\int_{\{y\in I^\delta\setminus I:|y-x|>\varepsilon\}}&\frac{\mathrm{d}y}{|y-x|^{1+sp}}\\ &=\frac{1}{sp}\left(\frac{1}{(x-a)^{sp}}-\frac{1}{(x-a+\delta)^{sp}}+\frac{1}{(b-x)^{sp}}-\frac{1}{(b+\delta-x)^{sp}}\right)\\ &\ge\frac{(1-C_\delta^{sp})}{sp}\left(\frac{1}{(x-a)^{sp}}+\frac{1}{(b-x)^{sp}}\right)\end{aligned}$$

for all $x\in I$. Since $sp\le1$, for each $x\in I$, we further have that

$$\frac{1}{(x-a)^{sp}}+\frac{1}{(b-x)^{sp}}=\frac{(x-a)^{sp}+(b-x)^{sp}}{(x-a)^{sp}\,(b-x)^{sp}}\ge\frac{2^{1-sp}\,(b-a)^{sp}}{(x-a)^{sp}\,(b-x)^{sp}}\ge\frac{2^{1-sp}}{d_I(x)^{sp}}$$

and thus, by Fatou's Lemma,

$$\begin{aligned}\iint_{I^\delta\times I^\delta}&\frac{J_p(\mathbf{1}_I(x)-\mathbf{1}_I(y))}{|x-y|^{1+sp}}\,(\varphi(x)-\varphi(y))\,\mathrm{d}x\,\mathrm{d}y\\ &=\lim_{\varepsilon\to0^+}2\int_I\varphi(x)\int_{\{y\in I^\delta\setminus I\,:\,|y-x|>\varepsilon\}}\frac{\mathrm{d}y}{|y-x|^{1+sp}}\,\mathrm{d}x\ge\frac{2^{2-sp}\,(1-C_\delta^{sp})}{sp}\int_I\frac{\varphi}{d_I^{sp}}\,\mathrm{d}x.\end{aligned}$$

We now choose the Lipschitz function

$$\varphi=\frac{|u|^p}{(\varepsilon+\mathbf{1}_I)^{p-1}},$$

where $u\in C_0^\infty(I)$. Noticing that, owing to [3, Lem. 2.4], it holds that

$$J_p(\mathbf{1}_I(x)-\mathbf{1}_I(y))\left(\frac{|u(x)|^p}{(\varepsilon+\mathbf{1}_I(x))^{p-1}}-\frac{|u(y)|^p}{(\varepsilon+\mathbf{1}_I(y))^{p-1}}\right)\le|u(x)-u(y)|^p$$

for all $x, y \in I^\delta$ and $\varepsilon > 0$, we get that

$$\begin{aligned}[u]^p_{W^{s,p}(I^\delta)} &\ge \liminf_{\varepsilon\to 0^+} \iint_{I^\delta\times I^\delta} \frac{J_p(\mathbf{1}_I(x)-\mathbf{1}_I(y))}{|x-y|^{1+sp}} \left(\frac{|u(x)|^p}{(\varepsilon+\mathbf{1}_I(x))^{p-1}} - \frac{|u(y)|^p}{(\varepsilon+\mathbf{1}_I(y))^{p-1}}\right) \mathrm{d}x\,\mathrm{d}y \\ &\ge \liminf_{\varepsilon\to 0^+} \frac{2^{2-sp}\,(1-C_\delta^{sp})}{sp} \int_I \frac{|u|^p}{d_I^{sp}(\varepsilon+\mathbf{1}_I)^{p-1}}\,\mathrm{d}x \ge \frac{2^{2-sp}\,(1-C_\delta^{sp})}{sp} \int_I \frac{|u|^p}{d_I^{sp}}\,\mathrm{d}x,\end{aligned}$$

and the proof is complete. □

We can now prove Theorem 2.17, and thus Theorem 1.7, by exploiting Proposition 2.18.

Proof of Theorem 2.17. Let I_i, $i = 1, \dots, n$, be the n intervals composing Ω, that is, $\Omega = \bigcup_{i=1}^n I_i$ with disjoint union. By assumption, the δ-enlarged intervals I_i^δ, $i = 1, \dots, n$, are pairwise disjoint. Thus, thanks to Proposition 2.18, we can estimate

$$\begin{aligned}[u]^p_{W^{s,p}(\Omega^\delta)} &\ge \sum_{i=1}^n [u]^p_{W^{s,p}(I_i^\delta)} \ge \sum_{i=1}^n \frac{2^{2-sp}}{sp}\left(1-\left(\frac{|I_i|}{|I_i|+\delta}\right)^{sp}\right)\int_{I_i}\frac{|u|^p}{d_{I_i}^{sp}}\,\mathrm{d}x \\ &\ge \frac{2^{2-sp}}{sp}\left(1-\left(\frac{\ell}{\ell+\delta}\right)^{sp}\right)\sum_{i=1}^n\int_{I_i}\frac{|u|^p}{d_{I_i}^{sp}}\,\mathrm{d}x = \frac{2^{2-sp}}{sp}\left(1-\left(\frac{\ell}{\ell+\delta}\right)^{sp}\right)\int_{\Omega}\frac{|u|^p}{d_{\Omega}^{sp}}\,\mathrm{d}x\end{aligned}$$

whenever $u \in C_0^\infty(\Omega)$ and the proof is complete. □

Remark 2.19. Concerning the constant appearing in Theorem 2.17 and Proposition 2.18, we observe that

$$\lim_{\delta\to 0^+} \frac{2^{2-sp}}{sp}\left(1-\left(\frac{\ell}{\ell+\delta}\right)^{sp}\right) = 0,$$

in accordance with the fact that the *regional* fractional Hardy inequality (1.26) cannot hold for $sp \le 1$ on bounded intervals, recall Remark 1.8.

References

[1] F. J. Almgren Jr. and E. H. Lieb, *Symmetric decreasing rearrangement is sometimes continuous*, J. Amer. Math. Soc. **2** (1989), no. 4, 683–773, DOI 10.2307/1990893. MR1002633

[2] K. Bessas and G. Stefani, *Non-local BV functions and a denoising model with L^1 fidelity*, Adv. Calc. Var. (2024), DOI 10.1515/acv-2023-0082.

[3] F. Bianchi, L. Brasco, and A. C. Zagati, *On Hardy's inequality in Sobolev-Slobodeckiĭ spaces*, Math. Ann. (2023), DOI 10.1007/s00208-023-02770-z.

[4] K. Bogdan and B. Dyda, *The best constant in a fractional Hardy inequality*, Math. Nachr. **284** (2011), no. 5-6, 629–638, DOI 10.1002/mana.200810109. MR2663757

[5] L. Brasco and E. Cinti, *On fractional Hardy inequalities in convex sets*, Discrete Contin. Dyn. Syst. **38** (2018), no. 8, 4019–4040, DOI 10.3934/dcds.2018175. MR3814363

[6] L. Brasco, D. Gómez-Castro, and J. L. Vázquez, *Characterisation of homogeneous fractional Sobolev spaces*, Calc. Var. Partial Differential Equations **60** (2021), no. 2, Paper No. 60, 40, DOI 10.1007/s00526-021-01934-6. MR4225499

[7] L. Brasco, E. Lindgren, and E. Parini, *The fractional Cheeger problem*, Interfaces Free Bound. **16** (2014), no. 3, 419–458, DOI 10.4171/IFB/325. MR3264796

[8] E. Cinti and F. Prinari, *On fractional Hardy-type inequalities in general open sets* (2024), arXiv preprint, available at arXiv:2407.06568.

[9] J. Dávila, *On an open question about functions of bounded variation*, Calc. Var. Partial Differential Equations **15** (2002), no. 4, 519–527, DOI 10.1007/s005260100135. MR1942130

[10] B. Dyda, *A fractional order Hardy inequality*, Illinois J. Math. **48** (2004), no. 2, 575–588. MR2085428

[11] B. Dyda and M. Kijaczko, *On density of compactly supported smooth functions in fractional Sobolev spaces*, Ann. Mat. Pura Appl. (4) **201** (2022), no. 4, 1855–1867, DOI 10.1007/s10231-021-01181-8. MR4454384

[12] B. Dyda and A. V. Vähäkangas, *A framework for fractional Hardy inequalities*, Ann. Acad. Sci. Fenn. Math. **39** (2014), no. 2, 675–689, DOI 10.5186/aasfm.2014.3943. MR3237044

[13] ———, *Characterizations for fractional Hardy inequality*, Adv. Calc. Var. **8** (2015), no. 2, 173–182, DOI 10.1515/acv-2013-0019. MR3331699

[14] D. E. Edmunds, R. Hurri-Syrjänen, and A. V. Vähäkangas, *Fractional Hardy-type inequalities in domains with uniformly fat complement*, Proc. Amer. Math. Soc. **142** (2014), no. 3, 897–907, DOI 10.1090/S0002-9939-2013-11818-6. MR3148524

[15] S. Filippas, L. Moschini, and A. Tertikas, *Sharp trace Hardy-Sobolev-Maz'ya inequalities and the fractional Laplacian*, Arch. Ration. Mech. Anal. **208** (2013), no. 1, 109–161, DOI 10.1007/s00205-012-0594-4. MR3021545

[16] ———, *Correction to: Sharp trace Hardy-Sobolev-Maz'ya inequalities and the fractional Laplacian [MR3021545]*, Arch. Ration. Mech. Anal. **229** (2018), no. 3, 1281–1286, DOI 10.1007/s00205-018-1264-y. MR3814603

[17] V. Franceschi, A. Pinamonti, G. Saracco, and G. Stefani, *The Cheeger problem in abstract measure spaces*, J. London Math. Soc. **109** (2024), no. 1, Paper No. e12840, 55 pp., DOI 10.1112/jlms.12840.

[18] R. L. Frank and R. Seiringer, *Sharp fractional Hardy inequalities in half-spaces*, Around the research of Vladimir Maz'ya. I, Int. Math. Ser. (N. Y.), vol. 11, Springer, New York, 2010, pp. 161–167, DOI 10.1007/978-1-4419-1341-8_6. MR2723817

[19] ———, *Non-linear ground state representations and sharp Hardy inequalities*, J. Funct. Anal. **255** (2008), no. 12, 3407–3430, DOI 10.1016/j.jfa.2008.05.015. MR2469027

[20] N. Garofalo, *On the best constant in the nonlocal isoperimetric inequality of Almgren and Lieb*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **31** (2020), no. 2, 465–470, DOI 10.4171/rlm/900. MR4120279

[21] G. H. Hardy, *Note on a theorem of Hilbert*, Math. Z. **6** (1920), no. 3-4, 314–317, DOI 10.1007/BF01199965. MR1544414

[22] ———, *An inequality between integrals*, Messenger of Math. **54** (1925), 150–156.

[23] S. Larson, *A bound for the perimeter of inner parallel bodies*, J. Funct. Anal. **271** (2016), no. 3, 610–619, DOI 10.1016/j.jfa.2016.02.022. MR3506959

[24] ———, *Corrigendum to "A bound for the perimeter of inner parallel bodies" [J. Funct. Anal. 271 (3) (2016) 610–619]*, J. Funct. Anal. **279** (2020), no. 5, 108574, 2, DOI 10.1016/j.jfa.2020.108574. MR4097280

[25] G. Leoni, *A first course in fractional Sobolev spaces*, Graduate Studies in Mathematics, vol. 229, American Mathematical Society, Providence, RI, 2023. MR4567945

[26] E. H. Lieb and M. Loss, *Analysis*, 2nd ed., Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI, 2001. MR1817225

[27] L. Lombardini, *Fractional perimeters from a fractal perspective*, Adv. Nonlinear Stud. **19** (2019), no. 1, 165–196, DOI 10.1515/ans-2018-2016. MR3912427

[28] M. Loss and C. Sloane, *Hardy inequalities for fractional integrals on general domains*, J. Funct. Anal. **259** (2010), no. 6, 1369–1379, DOI 10.1016/j.jfa.2010.05.001. MR2659764

[29] F. Maggi, *Sets of finite perimeter and geometric variational problems*, Cambridge Studies in Advanced Mathematics, vol. 135, Cambridge University Press, Cambridge, 2012. MR2976521

[30] M. Marcus, V. J. Mizel, and Y. Pinchover, *On the best constant for Hardy's inequality in* $\mathbf{R}^n$, Trans. Amer. Math. Soc. **350** (1998), no. 8, 3237–3255, DOI 10.1090/S0002-9947-98-02122-9. MR1458330

[31] T. Matskewich and P. E. Sobolevskii, *The best possible constant in generalized Hardy's inequality for convex domain in* $\mathbf{R}^n$, Nonlinear Anal. **28** (1997), no. 9, 1601–1610, DOI 10.1016/S0362-546X(96)00004-1. MR1431208

[32] V. Maz'ya, *Sobolev spaces with applications to elliptic partial differential equations*, augmented edition, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 342, Springer, Heidelberg, 2011. MR2777530

[33] V. Maz'ya and T. Shaposhnikova, *On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces*, J. Funct. Anal. **195** (2002), no. 2, 230–238, DOI 10.1006/jfan.2002.3955. MR1940355

[34] ———, *Erratum to: "On the Bourgain, Brezis and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces" [J. Funct. Anal.* **195** *(2002), no. 2, 230–238; MR1940355 (2003j:46051)]*, J. Funct. Anal. **201** (2003), no. 1, 298–300, DOI 10.1016/S0022-1236(03)00002-8. MR1986163

[35] B. Opic and A. Kufner, *Hardy-type inequalities*, Pitman Research Notes in Mathematics Series, vol. 219, Longman Scientific & Technical, Harlow, 1990. MR1069756

[36] F. Sk, *Characterization of fractional Sobolev-Poincaré and (localized) Hardy inequalities*, J. Geom. Anal. **33** (2023), no. 7, Paper No. 223, 20, DOI 10.1007/s12220-023-01293-y. MR4581748

[37] A. Visintin, *Generalized coarea formula and fractal sets*, Japan J. Indust. Appl. Math. **8** (1991), no. 2, 175–201, DOI 10.1007/BF03167679. MR1111612

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