APPROXIMATION OF TOPOLOGICAL SINGULARITIES THROUGH FREE DISCONTINUITY FUNCTIONALS: THE CRITICAL AND SUPER-CRITICAL REGIMES

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Abstract. We further investigate the properties of an approach to topological singularities through free discontinuity functionals of Mumford-Shah type proposed in [28]. We prove the variational equivalence between such energies, Ginzburg-Landau, and Core-Radius for anti-plane screw dislocations energies in dimension two, in the relevant energetic regimes $|\log\varepsilon|^a$, $a \geq 1$, where ε denotes the linear size of the process zone near the defects.

Further, we remove the *a priori* restrictive assumptions that the approximating order parameters have compact jump set. This is obtained by proving a new density result for $\mathbb{S}^1$-valued SBV^p functions, approximated through functions with essentially closed jump set, in the strong BV norm.

Keywords: Functions of Bounded Variation; Strict Convergence; Jacobian determinant; Topological Singularities; Γ-convergence; Ginzburg-Landau Model; Core-Radius Approach.

AMS subject classifications: 49J45 49Q20 26B30 74B15

Contents

Introduction

This paper concerns the analysis of topological singularities, which is a central topic in models arising in Physics and Materials Science. Vortices in superconductivity and superfluidity and (screw and edge) dislocations in single crystal plasticity are the main examples of such phenomenon [4, 41, 43, 49, 50, 51].

In the last decades several models have been introduced to describe the emergence of these objects. Among them, the most celebrated is the Ginzburg-Landau (GL) model, mainly studied in the context of superconductivity. In such a model, the order parameter is a function $u \in H^1(\Omega;\mathbb{R}^2)$ and the energy functional (in its simplest form) reads as

$$\mathcal{E}_\varepsilon^{\mathrm{GL}}(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \frac{1}{\varepsilon^2}\int_\Omega \left(1-|u|^2\right)^2 \mathrm{d}x, \tag{0.1}$$

where the parameter $\varepsilon > 0$ is referred to as *coherence length*. Here and below $\Omega \subset \mathbb{R}^2$ is a bounded open set with Lipschitz continuous boundary. A topological singularity is nothing but a point around which u has non-trivial winding number and hence the main object to look at is the Jacobian determinant (of u) $Ju := \det \nabla u$. Denoting by $\mathbb{S}^1$ the set of unitary vectors in the plane, we notice that close to a topological singularity, u cannot be $\mathbb{S}^1$-valued (a singularity can be seen

somehow as a zero of the order parameter); therefore, the parameter ε can be interpreted as the size of the region where u fails to take values in $\mathbb{S}^1$ and hence as the core-radius of the topological singularity.

The variational analysis of the (GL) functional has been first systematized in the monograph [6] (see also [59] and the references therein for the asymptotic analysis in terms of Γ-convergence), where the (GL) model is compared with (and shown somehow to be "equivalent" to) the so-called core-radius (CR) approach, in antiplane elasticity. Within this framework, the main variable is represented by the distribution of topological singularities $\mu = \sum \alpha_i \delta_{\xi_i}$ (with integer weights α_i) but the energy functional depends both on μ - which here plays the role of the Jacobian in (GL) - and on a map $u \in H^1(\Omega_\varepsilon(\mu); \mathbb{S}^1)$ "compatible with μ". Here, ε is the core-radius, $\Omega_\varepsilon(\mu) := \Omega \backslash \bigcup_i \overline{B}_\varepsilon(\xi_i)$ and the notion of compatibility is given by the fact that $\deg(u, \partial B_\varepsilon(\xi_i)) = \alpha_i$ (assuming that the balls $B_\varepsilon(\xi_i)$ are pairwise disjoint). The energy of the system thus writes

$$\mathcal{E}_\varepsilon^{\mathrm{CR}}(\mu, u) := \frac{1}{2} \int_{\Omega_\varepsilon(\mu)} |\nabla u|^2 \, \mathrm{d}x + |\mu|(\Omega). \tag{0.2}$$

Here, the quantity $|\mu|(\Omega)$ plays the same role of the potential term in $\mathcal{E}_\varepsilon^{\mathrm{GL}}$, namely, avoids that the cores cover the whole domain; in other words, it serves only to guarantee compactness and does not provide any energy contribution in the asymptotics as $\varepsilon \to 0$.

The (CR) approach is mostly used to model screw dislocations in semi-discrete theories. Loosely speaking, in pure (anti-plane) elasticity the bulk energy is determined by the Hooke's law, and reads as $\frac{1}{2} \int_\Omega |\nabla w|^2 \, \mathrm{d}x$, where the displacement w lies in $H^1(\Omega)$. In presence of a finite distribution $\mu = \sum_i \alpha_i \delta_{x_i}$ of (scalar) defects, the material has a purely plastic behavior in the cores $B_\varepsilon(\xi_i)$ and, oversimplifying, such a plastic contribution can be expressed by $|\mu|(\Omega)$. Moreover, along a closed circuit enclosing the singularity ξ_i, a displacement w compatible with μ should have a jump $[w]$ equal to α_i. Therefore, the displacement w is only in $SBV^2(\Omega_\varepsilon(\mu))$ with $[w] \in \mathbb{Z}$ and its elastic energy should be given by $\frac{1}{2} \int_{\Omega_\varepsilon(\mu)} |\nabla w|^2 \, \mathrm{d}x$, where ∇w is the absolutely continuous part of $\mathrm{D}w$. Setting

$$u = e^{2\pi \imath w}, \tag{0.3}$$

one obtains that the total energy associated to the pair (μ, u) is given by $\mathcal{E}_\varepsilon^{\mathrm{CR}}$.

In this paper, we adopt a different viewpoint, following the approach proposed in [28]. The main feature is that the order parameter is now an $\mathbb{S}^1$-valued map, as in the (CR) approach, defined on the whole Ω, as in the (GL) approach. Clearly, in presence of topological singularities, such a map cannot be in $H^1(\Omega; \mathbb{S}^1)$. But, instead of removing small disks around the singularities (as in (CR)) or to weaken the $\mathbb{S}^1$-constraint (as in (GL)), the map u is now allowed to jump. More precisely, u is a *special function of bounded variation* with square-integrable approximate gradient (i.e., $u \in SBV^2(\Omega; \mathbb{S}^1)$). The energy functional we consider is

$$\mathcal{F}_\varepsilon(u) := \int_\Omega \frac{1}{2} |\nabla u|^2 \, \mathrm{d}x + \frac{1}{\varepsilon} \mathcal{H}^1(S_u), \tag{0.4}$$

where $\varepsilon > 0$ is a small parameter determining the size of the jump set S_u of u. Here and throughout the paper $\mathcal{H}^1$ denotes the (one-dimensional) Hausdorff measure.

Formally, the functional $\mathcal{F}_\varepsilon$ has the structure of the Mumford-Shah functional [56], but the $\mathbb{S}^1$-constraint makes the analysis completely different. Indeed, having in mind the identity (0.3) and the (CR) approach for screw dislocations, jumps of the map u correspond to non-integer jumps of the displacement w and should pay energy. In other words, the (amplitude of the) jump $[w]$ of the displacement exhibits a transition between integers in a little portion of S_w. The transition is assumed to have length of the order of ε, and corresponds to the presence of singularities. In this respect, $\mathcal{H}^1(S_u)$ is the analogue of the potential term in (GL) and of the plastic term in (CR) and the parameter ε can be understood also in this case as the core-radius of the singularity.

We highlight that for SBV maps the definition of topological degree as well as that of Jacobian determinant are not so standard so that the notion of topological singularity is not so clear as in (GL) and in (CR). Nevertheless, in [28], using the *minimal lifting* in [45], a notion of Jacobian determinant is provided also for SBV functions; we recall such a definition in Section 1 (see [53]

where in a more restrictive setting this notion was first introduced, and see also [9] for a different definition of Jacobian determinant in dimension 2). In a nutshell, given a map $u \in SBV(\Omega;\mathbb{S}^1)$, the Jacobian Ju of u is defined as the boundary of the 1-current T_u defined by

$$\begin{aligned} T_u :=& \frac{1}{2}(-u^1\partial_{x_2}u^2 + u^2\partial_{x_2}u^1, u^1\partial_{x_1}u^2 - u^2\partial_{x_1}u^1) + \frac{1}{2}(u^+ \wedge u^-)\boldsymbol{\tau} \llcorner S_u \\ =:& T_u^{\mathrm{D}} + T_u^{\mathrm{S}}. \end{aligned} \tag{0.5}$$

In [28, Theorem 3.1] the Γ-convergence analysis of the functional

$$\mathcal{G}_\varepsilon(u) := \int_\Omega \frac{1}{2}|\nabla u|^2 \,\mathrm{d}x + \frac{1}{\varepsilon}\mathcal{H}^1(\overline{S}_u), \tag{0.6}$$

at the energy regime $|\log\varepsilon|$ has been developed: such an analysis revealed that the functional $\mathcal{G}_\varepsilon$ shares the same compactness and Γ-convergence properties of the functionals $\mathcal{E}_\varepsilon^{\mathrm{CR}}$ and $\mathcal{E}_\varepsilon^{\mathrm{GL}}$. Specifically, as $\varepsilon \to 0$, the Jacobian determinant tends to concentrate around a finite number of effective singularities and the Γ-limit of the functionals $\frac{\mathcal{G}_\varepsilon}{|\log\varepsilon|}$ is given (up to multiplicative constants) by the total variation of the limiting measure of the Jacobians. Since, in view of the possible presence of short dipoles, a uniform bound on the total variation of the dislocations' distributions is not available, the natural setting for such an asymptotic analysis is the (strong) flat convergence for Jacobian determinants, the flat topology being the strong topology in the dual of Lipschitz continuous functions with compact support in Ω.

We stress that the proof of compactness, which is a prominent issue in [28], is based on the *ball construction* technique introduced in [44, 58]: in order to start the corresponding construction, the original jump set needs to be covered by a finite family of pairwise disjoint balls in such a way that the sum of the diameters does not exceed a constant times the $\mathcal{H}^1$ measure of the jump set (the constant could be also taken arbitrarily close to 1, from above). This is easily ensured if the jump set is compact, whilst S_u could be even dense in Ω for a general $u \in SBV^2(\Omega;\mathbb{S}^1)$.

In the present paper we generalize the analysis done in [28] along two directions. On the one hand, we show that the penalization term can be "weakened" considering only the length of the jump set instead of its closure, i.e., working with the functional $\mathcal{F}_\varepsilon$ rather than with $\mathcal{G}_\varepsilon$. Therefore it is not needed to assume *a priori* that the jump set is compact. On the other hand, we show that the functional $\mathcal{F}_\varepsilon$ shares the same asymptotic behavior of the functionals $\mathcal{E}_\varepsilon^{\mathrm{CR}}$ and $\mathcal{E}_\varepsilon^{\mathrm{GL}}$ also in other energy regimes.

The first improvement is obtained by means of a density result in $SBV^p(\Omega;\mathbb{S}^1)$, $p > 1$, with respect to energies $\mathcal{F}_\varepsilon$ for fixed $\varepsilon > 0$, through functions in $SBV^p(\Omega;\mathbb{S}^1)$ with (essentially) closed jump set, converging in the strong BV norm and such that also the two unilateral traces of the approximants along the jump set converge; in particular, by using the characterization of Ju as the boundary of T_u in (0.5), the strong convergence of Jacobian determinants with respect to flat norm follows.

Our result hinges on tools developed in a slightly different setting, that is when only the symmetric part of the diffuse gradient is controlled in some L^p, for $p > 1$, rather than the whole diffuse gradient. Mechanically, this corresponds to consider fracture models for general linearized elasticity without the anti-plane assumption, described by the Griffith functional [40] instead of the Mumford-Shah one.

In fact, the main tool for density results developed in the context of Mumford-Shah functional is an approximated Poincaré-Wirtinger inequality for SBV^p functions with small ($\mathcal{H}^{d-1}$-measure of the) jump set, due to De Giorgi-Carriero-Leaci ([25]): given $u \in SBV^p$ there exists a truncation in $W^{1,p}$ such that u differs from w on an exceptional set ω whose volume is controlled by $(\mathcal{H}^{d-1}(S_u))^{1^*}$, $d \geq 2$ being the space dimension and $1^* := d/(d-1)$. In the same paper, this result has been used to prove that the jump set of Mumford-Shah minimizers is essentially closed, namely the $\mathcal{H}^{d-1}$ measure of the jump set equals that of its closure. After short time a generalization for $SBV^p(\Omega;\mathbb{S}^{k-1})$ maps has been proven in [16]; by combining such a generalization with an argument in [8] (cf. Lemma 5.2 therein), one can show that the Mumford-Shah functional in $SBV^p(\Omega;\mathbb{S}^1)$ can be approximated through $SBV^p(\Omega;\mathbb{S}^1)$ functions having essentially closed jump set and converging pointwise. Unfortunately, pointwise convergence of a sequence of functions

does not guarantee convergence of the corresponding Jacobians; since we need convergence in the flat norm such an approach is not satisfactory for our purposes. For functions with finite Griffith energy (with exponent p), that is in the space $GSBD^p$ ([23]), a fundamental tool is the approximated Poincaré-Korn inequality in [10], stating that for any $u \in GSBD^p$ with small jump set there exists an infinitesimal rigid motion a (i.e. an affine function with null symmetrized gradient) such that the $L^{\frac{dp}{d-1}}$-norm of $u - a$ is estimated by the L^p norm of $e(u)$, the symmetrized diffuse gradient of u, outside an exceptional set ω whose volume is controlled by $(\mathcal{H}^{d-1}(J_u))^{1^*}$. Moreover, a convolution of $u\chi_{\omega^c} + a\chi_\omega$ at the same scale of the domain provides a function with L^p-norm of the symmetrized diffuse gradient controlled by those of u.

This result, on which other contribution in this direction rest (see, e.g., the approximation in $GSBD^p$ through functions with essentially closed jump set [19, 11] and the analogue of [25] for the Griffith functional [20, 17, 12]), has been generalized by [18] and [13]: here, any $u \in GSBD^p$ is approximated, in terms of the Griffith energy, by functions $W^{1,p}$ on a slightly smaller domain, with essentially closed jump set, which differ from u on a set ω whose boundary is controlled by $\mathcal{H}^{d-1}(J_u)$; further, in [18] it is shown that in dimension two it is possible to guarantee that the approximants coincide with u in the boundary neighborhood where they are not in $W^{1,p}$. In [18] such an approximation is used to prove an integral representation result, while in [13] the main result is the approximation of any $u \in GSBD^p$, with respect to the Griffith energy, through functions with essentially closed jump set differing from u on sets of vanishing perimeter.

Moreover, Friedrich [32] proved a *piecewise Korn inequality* in dimension two, showing that up to subtracting piecewise rigid functions (finite sums of infinitesimal rigid motions multiplied by characteristic functions), any u in $GSBD^p$ can be approximated by functions in $SBV^q \cap L^\infty$, for $q < p$, in particular the diffuse gradient of the approximants is estimated on the whole domain by $e(u)$; this is a very powerful tool allowing to overcome the lack of a Coarea Formula in $GSBD$ and then to show, e.g., existence of quasi-static evolutions for Brittle Fracture models (see, for instance, [34]). In the same spirit, in [33] a similar result has been shown in the Mumford-Shah setting, namely dealing only with full diffuse gradients.

Eventually, we refer to [29] for the two dimensional analogue of [16] for maps in $SBV^{p(\cdot)}(\Omega; \mathbb{S}^{k-1})$, (with $\Omega \subset \mathbb{R}^2$) whose approximate gradient is integrable with respect to the variable exponent $p(\cdot)$ over Ω and whose jump set has finite $\mathcal{H}^1$-measure (see also [48] for the variable exponents analogue of [25]), obtained under the assumption that the function $p(\cdot)$ is regular enough and takes values in $(1, 2)$. This uses the analogue of the approximation of [18], proven by employing retractions $\mathcal{P}\colon \mathbb{R}^k \setminus \mathcal{X} \to \mathbb{S}^{k-1}$ with locally q-integrable gradient for $q \in [1, 2)$, where $\mathcal{X}$ is a smooth complex of codimension two (cf. e.g. [14]).

We then compare our main density result Corollary 2.2 with [13, Theorem 5.1]: we are in two dimensions and consider the full diffuse gradient instead of its symmetrized part, however we keep in the approximation the constraint of being $\mathbb{S}^1$-valued. We notice that also a version with symmetrized diffuse gradient is readily shown with essentially the same proof, see Theorem 2.6.

Since in our application the case $p = 2$ is the relevant one, we cannot follow a strategy based on retractions. Moreover, the proof of [18], [13], and [33] is not compatible with a non-convex target space such as $\mathbb{S}^1$.

Our approach is based on the existence of a *lifting* $\varphi \in SBV^p(\Omega)$ (i.e., such that $u = e^{2\pi\imath\varphi}$) with its BV-seminorm such that $\pi|\varphi|_{BV} \leq |u|_{BV}$ ([24], notice that therein the relation between u and φ is $u = e^{i\varphi}$ and the inequality is $|\varphi|_{BV} \leq 2|u|_{BV}$), for which we provide a suitable approximation (Theorem 2.1) and then compose the approximants with $e^{2\pi\imath\cdot}$.

We observe that, along this scheme, one could be tempted to directly employ the approximation provided by [13, Theorem 5.1] to φ, before composing with the complex exponential; however, despite the fact that the set on which the traces of the approximants - say φ_n - differ from those of φ has small measure, it could be even dense in S_φ, so that a subset of S_{u_n}, $u_n := e^{2\pi\imath\varphi_n}$, could be dense in the integer jump set S_φ^{int} of φ. As $\mathcal{H}^1(S_\varphi^{\text{int}})$ is in general comparable to $|u|_{BV}$, thus larger than $\mathcal{H}^1(S_u)$ by a factor $1/\varepsilon$, see (0.4), the procedure just described is not admissible for our purposes: in fact, since we need approximants with closed jump set we are forced to consider the closure of S_{u_n} or, in other words, if we cover S_{u_n} by a family of pairwise disjoint balls (in

order to start the ball construction) then the sum of their diameters is larger than S_u by a factor $1/\varepsilon$.

Therefore we need a more refined density result, which allows to approximate integer jumps with integer jumps as well. The strategy adopted in the proof of Theorem 2.1 is to work locally near points with integer jump at a scale for which the jump is almost flat and assumes a constant integer value (Step 1). Then locally most of the jump set of φ can be transferred (up to a small error) into a flat segment S on which the jump has the same value (Step 2.1, the modifications are defined in (2.29)); such S can be chosen in such a way that the remaining small jump set of φ, on any square with arbitrarily small sidelength with a side contained in S, is small compared to the sidelength: this follows by a two dimensional argument drawn from [18] (argument at the beginning of Step 2.1, in particular the need to work in dimension two is discussed below (2.27)).

In this way, the approximation for functions with small jump set of [18] (cf. Proposition 2.5) may be applied at every scale; therefore a Whitney-type argument combined with the fact that the approximants coincide with the original function on the boundary of any square ensures that the traces are the same on both sides of S, so the new jump is still integer (Step 2.1, from (2.41) till the end). We believe that this strategy may be replicated for different target manifolds, provided that the universal cover is convex (for instance $\mathbb{R}^k$) and a lifting with good BV bounds as in [24] exists (in this respect see e.g. [15]), but limiting to two dimensional domains. At the present stage, however, our techniques seem to be not applicable to the case of $\mathbb{S}^k$-valued maps, with $k > 1$. We anticipate here that the fact that our argument is two dimensional is not a real limitation for the application considered in the present paper, as further substantial issues arise in the three dimensional framework, see below.

The density result described above allows to develop the Γ-convergence analysis for the functional $\mathcal{F}_\varepsilon$ also in different energetic regimes (as it applies for fixed $\varepsilon > 0$), thus generalizing the setting of [28].

First, we develop the Γ-convergence analysis in the so-called critical regime, that is $|\log\varepsilon|^2$. Loosely speaking, since $|\log\varepsilon|$ is the energy cost of an isolated singularity, the fact that $\mathcal{F}_\varepsilon(u_\varepsilon) \sim |\log\varepsilon|^2$ implies that, for $\varepsilon > 0$, the number of singularities of the Jacobians Ju_ε is of order $|\log\varepsilon|$; therefore, the Jacobians Ju_ε, once rescaled by $|\log\varepsilon|$, should converge (in the flat norm) to a measure μ that is not anymore atomic but diffuse. Furthermore, we prove that such a measure μ lies also in $H^{-1}(\Omega)$. Indeed, by standard compactness results in $L^2(\Omega)$, also the fields $T^{\mathrm{D}}_{u_\varepsilon}$, once scaled by $|\log\varepsilon|$, should converge (weakly in $L^2(\Omega)$) to a field T^{D}, whose distributional divergence is shown to be given by $-\pi\mu$. As one may expect, the Γ-limit accounts both for the plastic contribution of μ as well as for the elastic energy of T^{D}. That is the reason why the $|\log\varepsilon|^2$ regime is called critical, since in such a case the elastic and plastic effects are of the same order. The Γ-convergence analysis for the functional $\mathcal{F}_\varepsilon$ is provided in Theorems 3.3 and 3.4 which are proved in Section 5. Second, the proofs of the compactness and of the lower bound are obtained combining the corresponding results for the core-radius approach together with the refined ball construction machinery introduced in [28] to analyze the $|\log\varepsilon|$ regime.

Finally, adopting the same strategy, in Theorem 3.4 we analyze also the super-critical regimes $|\log\varepsilon|^2 \ll N_\varepsilon \ll \frac{1}{\varepsilon}$. In such a case, the interaction elastic energy is larger and larger than the core energy, so that (unless scaling differently the two quantities Ju_ε and $T^{\mathrm{D}}_{u_\varepsilon}$) one keeps track only of T^{D} and the Jacobian determinants do not play any role when computing the effective energy.

We highlight that the Γ-convergence analysis for the functional $\mathcal{E}^{\mathrm{GL}}_\varepsilon$ in the regime $|\log\varepsilon|^2$ has been developed in [47, 60, 61], where the authors consider also the case with magnetic field. The analysis for $\mathcal{E}^{\mathrm{CR}}_\varepsilon$ is provided in Section 4 and is somehow a short self-contained resume of the results above, along the lines of [2].

However, a similar result in the context of edge dislocations within the (CR) approach is proven in [35] under the well-separation assumption for the singularities (see also [55] for such an analysis in the nonlinear elasticity framework); such an assumption has been removed in [39]. In view of the asymptotic equivalence result [1] between the Ginzburg-Landau model and the purely discrete models of XY spin systems and screw dislocations, we have that the analysis in the (GL) context extends also to such discrete models.

We remark that a similar analysis for space dimension $d \geq 3$ seems far from being addressed: besides the fact that the density result is proven only in two dimensions, for which the analogue of the *ball construction* ([44, 58]) cannot even start as discussed earlier, also the whole machinery requires in several occasion bounds which are available only in two dimensions.

The paper is organized as follows: After recalling some notations and preliminary results in Section 1, we prove in Section 2 a general density result for $SBV^p(\Omega)$ functions in Theorem 2.1 which implies, as a consequence, Corollary 2.2. This is the result we employ to obtain energy density in the Γ-convergence results of Section 3, actually allowing us to restrict such analysis to $\mathbb{S}^1$-valued functions with essentially closed jump set. The latter results are Theorems 3.3 and 3.4, stated in Section 3 after recalling the main features of our model. In order to prove them we recall in Section 4 the classical core radius approach, which is the starting point of our analysis. Finally, the proofs of Theorems 3.3 and 3.4 are given, respectively, in Sections 5 and 6.

Acknowledgements: The authors wish to thank the anonymous reviewers for their valuable comments.

The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

VC acknowledges the financial support of PRIN 2022J4FYNJ "Variational methods for stationary and evolution problems with singularities and interfaces", PNRR Italia Domani, funded by the European Union under NextGenerationEU, CUP B53D23009320006, and of Sapienza Università di Roma through a SEED PNR Project.

LDL acknowledges the financial support of PRIN 2022HKBF5C "Variational Analysis of complex systems in Materials Science, Physics and Biology", PNRR Italia Domani, funded by the European Union via the program NextGenerationEU, CUP B53D23009290006.

RS also acknowledges the partial financial support of the F-cur project number 2262-2022-SR-CONRICMIUR_PC-FCUR2022_002 of the University of Siena, and the of the PRIN project 2022PJ9EFL "Geometric Measure Theory: Structure of Singular Measures, Regularity Theory and Applications in the Calculus of Variations", PNRR Italia Domani, funded by the European Union via the program NextGenerationEU, CUP B53D23009400006.

Views and opinions expressed are however those of the authors only and do not necessarily reflect those of the European Union or The European Research Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

1. Preliminary results

In this section we collect some preliminary notions on the flat norm of measures and currents, as well as some properties of BV functions that will be used throughout the paper.

Flat norm of Radon measures. Let $n \geq 1$ be an integer and let $U \subset \mathbb{R}^n$ be a bounded and open set. We denote by $\mathcal{M}_b(U)$ the space of Radon measures on U with finite total variation. If $\mu \in \mathcal{M}_b(U)$, we denote by $|\mu|(U)$ the total variation of μ. We recall that a sequence $\mu_k \in \mathcal{M}_b(U)$ converges tightly to $\mu \in \mathcal{M}_b(U)$ if μ_k converges to μ weakly* as measure, and $|\mu_k|(U) \to |\mu|(U)$. We also introduce the concept of flat norm of a measure μ, denoted by $\|\mu\|_{\text{flat}}$, as

$$\|\mu\|_{\text{flat}} := \sup_{\substack{\varphi \in C_c^{0,1}(U) \\ \|\varphi\|_{C^{0,1}(U)} \leq 1}} \int_U \varphi \, \mathrm{d}\mu \,. \tag{1.1}$$

Here and below, the Lipschitz norm $\|\varphi\|_{C^{0,1}(U)}$ is defined by

$$\|\varphi\|_{C^{0,1}(U)} := \|\varphi\|_{L^\infty(U)} + \sup_{\substack{x,y \in U \\ x \neq y}} \frac{|\varphi(x) - \varphi(y)|}{|x-y|} \,.$$

By a density argument we easily see that the supremum in (1.1) can be equivalently computed among smooth and compactly supported (in U) functions φ with $\|\varphi\|_{C^{0,1}(U)} \leq 1$.

Flat norm of k-currents. Let $n \geq 2$ be an integer and let $U \subset \mathbb{R}^n$ be an open set. For every $k \in \mathbb{N}$ with $0 \leq k \leq n$, we denote by $\mathcal{D}^k(U)$ the topological vector space of smooth and compactly supported k-forms on U, and by $\mathcal{D}_k(U)$ its dual, i.e., the space of k-currents on U.

The mass $|T|$ of a current $T \in \mathcal{D}_k(U)$ is defined as

$$|T| = \sup\{\langle T, \omega\rangle \,:\, \omega \in \mathcal{D}^k(U), \|\omega\|_{L^\infty} \leq 1\}\,.$$

As done in (1.1) for measures, we define the *flat norm* of a current $T \in \mathcal{D}_k(U)$ in U by

$$\|T\|_{\mathrm{flat},U} := \sup_{\substack{\omega\in\mathcal{D}^k(U)\\ \|\omega\|_{F,U}\leq 1}} \langle T, \omega\rangle, \tag{1.2}$$

where

$$\|\omega\|_{F,U} := \|\omega\|_{L^\infty(U)} + \|\mathrm{d}\omega\|_{L^\infty(U)}\,.$$

In the special case that T is a 0-current and has finite mass, then it can be standardly identified with a measure, and the flat norm of T coincides with the flat norm of the measure T defined in (1.1). For $k \geq 1$, we recall the definition of boundary $\partial T \in \mathcal{D}_{k-1}(U)$ of a current $T \in \mathcal{D}_k(U)$, given by

$$\partial T(\omega) = T(d\omega) \qquad \forall \omega \in \mathcal{D}^{k-1}(U).$$

By convention the boundary of a 0-current is null.

Jacobian for $\mathbb{S}^1$-valued Sobolev maps in $\mathbb{R}^2$. Let $U \subset \mathbb{R}^2$ be a bounded and open set. Given a map $u \in W^{1,1}(U;\mathbb{S}^1)$ we recall that the distributional Jacobian $Ju = \mathrm{Det}(\nabla u)$ of u is defined by

$$\langle Ju, \varphi\rangle_U := \int_U \nabla\varphi \cdot \lambda_u \;\mathrm{d}x, \qquad \text{for every } \varphi \in C^\infty_{\mathrm{c}}(U), \tag{1.3}$$

where

$$\lambda_u := \frac{1}{2}\Big(- u^1\frac{\partial u^2}{\partial x_2} + u^2\frac{\partial u^1}{\partial x_2}, u^1\frac{\partial u^2}{\partial x_1} - u^2\frac{\partial u^1}{\partial x_1}\Big)\,; \tag{1.4}$$

notice that $\lambda_u \in L^1(U;\mathbb{R}^2)$.

Moreover, denoting by $j(u) \in L^1(U;\mathbb{R}^2)$ the *current* associated to u, i.e.,

$$j(u) := \frac{1}{2}(u^1\nabla u^2 - u^2\nabla u^1)\,, \tag{1.5}$$

one has $j^\perp(u) = \lambda_u$ and $j(u) = \pi\nabla w$, where w is a generic lifting of u, i.e., a map in $SBV^2(U)$ satisfying (0.3) and ∇w is the approximate gradient of w. Furthermore, it is easy to check that

$$Ju = -\mathrm{Div}\lambda_u = \mathrm{curl}\, j(u) = \pi\mathrm{curl}(\nabla w),$$

holds in the sense of distributions.

In the sequel we will use the fact that a function $u \in H^1(U;\mathbb{S}^1)$ satisfies $\mathrm{Det}(\nabla u) = 0$ in the sense of distributions. Moreover, if $u \in H^1(U \setminus \overline{B};\mathbb{S}^1)$, where $B \subset U$ is a ball, then, integrating by parts,

$$\int_{U\setminus\overline{B}} \lambda_u \cdot \nabla\varphi \;\mathrm{d}x = \int_{\partial B} \lambda_u \cdot \boldsymbol{\nu}\varphi \;\mathrm{d}\mathcal{H}^1 = \int_{\partial B} j(u)\cdot\boldsymbol{\tau}\varphi \;\mathrm{d}\mathcal{H}^1, \qquad \text{for every } \varphi \in C^\infty_{\mathrm{c}}(U),$$

where $\boldsymbol{\nu}$ is the inner normal vector to ∂B, $\boldsymbol{\tau} = -\boldsymbol{\nu}^\perp$ is the counter-clockwise tangent vector to ∂B. Notice that $j(u)\cdot\boldsymbol{\tau} = \frac{1}{2}(u^1\frac{\partial u^2}{\partial\boldsymbol{\tau}} - u^2\frac{\partial u^1}{\partial\boldsymbol{\tau}})$ on ∂B.

We recall that $\deg(u, \partial B) \in \mathbb{Z}$ is defined as

$$\deg(u,\partial B) := \frac{1}{\pi}\int_{\partial B} j(u)\cdot\boldsymbol{\tau}\;\mathrm{d}\mathcal{H}^1 = \frac{1}{\pi}\int_{\partial B}\lambda_u\cdot\boldsymbol{\nu}\;\mathrm{d}\mathcal{H}^1\,, \tag{1.6}$$

whenever $u \in H^{\frac{1}{2}}(\partial B;\mathbb{S}^1)$.

Jacobian for $\mathbb{S}^1$-valued SBV maps in $\mathbb{R}^2$. Let $U \subset \mathbb{R}^2$ be a bounded and open set. We denote by $|u|_{BV} := |\mathrm{D}u|(U)$ the BV-seminorm, and by $\|u\|_{BV} := \|u\|_{L^1} + |u|_{BV}$ the BV-norm of a function of bounded variation u on U. We say that a sequence $u_k \in BV(U;\mathbb{R}^2)$ converges strictly to $u \in BV(U;\mathbb{R}^2)$ if $u_k \to u$ in $L^1(U)$ and $|\mathrm{D}u_k|(U) \to |\mathrm{D}u|(U)$. In such a case we write $u_k \overset{\text{strict}}{\rightharpoonup} u$. For any $p \in [1,+\infty)$ the symbol $SBV^p(U;\mathbb{R}^2)$ denotes the space of functions $u \in BV(U;\mathbb{R}^2)$ such that the Cantor part $\mathrm{D}^{\mathrm{c}}u \equiv 0$, and $\nabla u \in L^p(U;\mathbb{R}^{2\times 2})$, where ∇u is the density of $\mathrm{D}^{\mathrm{a}}u$, i.e., $\mathrm{D}^{\mathrm{a}}u := \nabla u \mathcal{L}^2$. The space $SBV^p(U;\mathbb{S}^1)$ denotes the set of the functions $u \in SBV^p(U;\mathbb{R}^2)$ such that $|u| = 1$ a.e. in U.

The following result, proven in [28, Corollary 2.1], is specialized here to maps taking values in $\mathbb{R}^2$.

Proposition 1.1. *Let $u \in SBV(U;\mathbb{R}^2) \cap L^\infty(U;\mathbb{R}^2)$; then there exists a unique measure $\nu_u \in \mathcal{M}_b(U;\mathbb{R}^{2\times 2\times 2})$ such that, whenever $\{v_k\}_{k\in\mathbb{N}} \subset C^1(U;\mathbb{R}^2) \cap W^{1,1}(U;\mathbb{R}^2) \cap L^\infty(U;\mathbb{R}^2)$ satisfies $\|v_k\|_{L^\infty(U;\mathbb{R}^2)} \le C < +\infty$ for all $k \ge 1$ and $v_k \overset{\text{strict}}{\rightharpoonup} u$ in $BV(U;\mathbb{R}^2)$, then $v_k \otimes \nabla v_k \to \nu_u$, where $(\nu_u)^{i,h}_j$ is defined (for all $\varphi \in C_{\mathrm{c}}(U)$) by*

$$\int_U \varphi(x)\,\mathrm{d}(\nu_u)^{i,h}_j = \int_{U\setminus S_u} \phi(x) u^h(x) \partial_{x_j} u^i(x)\,\mathrm{d}x + \frac{1}{2}\int_{S_u} \phi(x)(u^{h,+}(x) + u^{h,-}(x))(u^{i,+}(x) - u^{i,-}(x))\boldsymbol{\nu}_j(x)\,\mathrm{d}\mathcal{H}^1(x),$$

for every $i,j,h \in \{1,2\}$. Finally, if $\{u_k\}_{k\in\mathbb{N}} \subset SBV(U;\mathbb{R}^2) \cap L^\infty(U;\mathbb{R}^2)$ with

$$\|u_k\|_{L^\infty(U;\mathbb{R}^2)} \le C, \tag{1.7}$$

for some constant $C > 0$, and $u_k \overset{\text{strict}}{\rightharpoonup} u$ in $BV(U;\mathbb{R}^2)$, then

$$\nu_{u_k} \overset{*}{\rightharpoonup} \nu_u \qquad \text{in } \mathcal{M}_b(U;\mathbb{R}^{2\times 2\times 2}). \tag{1.8}$$

In the following, for every map $u \in SBV(U;\mathbb{R}^2) \cap L^\infty(U;\mathbb{R}^2)$, we set

$$[u^h \mathrm{D}_j u^i] := (\nu_u)^{i,h}_j, \quad i,j,h \in \{1,2\},$$

For any map $u \in SBV(U;\mathbb{R}^2) \cap L^\infty(U;\mathbb{R}^2)$ we introduce the 1-current T_u defined by

$$\begin{aligned} T_u :=& \frac{1}{2}(-[u^1\mathrm{D}_2u^2] + [u^2\mathrm{D}_2u^1], [u^1\mathrm{D}_1u^2] - [u^2\mathrm{D}_1u^1]) \\ =& \frac{1}{2}(-u^1\partial_{x_2}u^2 + u^2\partial_{x_2}u^1, u^1\partial_{x_1}u^2 - u^2\partial_{x_1}u^1) + \frac{1}{2}(u^+ \wedge u^-)\boldsymbol{\tau} \llcorner S_u \\ =:& T^{\mathrm{D}}_u + T^{\mathrm{S}}_u. \end{aligned} \tag{1.9}$$

where we have noted $\alpha \wedge \beta = -\alpha \cdot \beta^\perp = \det(\alpha,\beta)$. Notice that $T^{\mathrm{D}}_u \in L^1(U;\mathbb{R}^2)$ and that, if $u \in W^{1,1}(U;\mathbb{S}^1)$, then $T_u = T^{\mathrm{D}}_u = \lambda_u$ with λ_u defined in (1.4). Finally, we highlight that if $u \in SBV(U;\mathbb{S}^1)$, for any lifting $w \in SBV(U)$ of u, i.e., satisfying (0.3), it holds that

$$T^{\mathrm{D}}_u := \pi\nabla^\perp w; \qquad T^{\mathrm{S}}_u := \frac{1}{2}\sin(2\pi(w^- - w^+))\boldsymbol{\tau} \llcorner S_u. \tag{1.10}$$

The distributional Jacobian $Ju \in \mathcal{D}_0(U)$ of u is defined as the boundary of T_u, namely

$$Ju := \partial T_u \qquad \text{in } \mathcal{D}_0(U). \tag{1.11}$$

Essentially by definition, it easily follows that

$$\|Ju\|_{\mathrm{flat},U} \le C\|u\|_{BV},$$

for all $u \in SBV(\Omega;\mathbb{S}^1)$, for a universal constant $C > 0$ (see [28, Theorems 2.2 and 2.5]).

Remark 1.2. We point out that in general Ju is not a Radon measure. This notion of Jacobian determinant was first introduced in [53] under some special hypotheses on u. Under these hypotheses it turns out that Ju is also a Radon measure.

2. Density results in $SBV^p(\Omega;\mathbb{S}^1)$

In this section we prove that any function $u \in SBV^p(\Omega;\mathbb{S}^1)$ can be approximated - in the strong BV norm - by (sequence of) functions in $SBV^p(\Omega;\mathbb{S}^1)$ with closed jump set. As a consequence (see Proposition 2.3), we deduce that the corresponding currents can be approximated in the flat norm.

In what follows for every function $\varphi \in SBV^p(\Omega)$ (with $p \geq 1$) we denote by $S_\varphi^{\rm frac}$ the *fractional jump set* of φ, i.e., $S_\varphi^{\rm frac} := \{x \in S_\varphi \colon [\varphi] \notin \mathbb{Z}\}$, and by $S_\varphi^{\rm int}$ the *integer jump set*, namely, $S_\varphi^{\rm int} := S_\varphi \setminus S_\varphi^{\rm frac}$. Observe that if $\varphi \in SBV^p(\Omega)$ is any lifting of $u \in SBV^p(\Omega;\mathbb{S}^1)$, then $S_\varphi^{\rm frac} = S_u$.

The main result of this section is the following.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with finite perimeter, $p \in (1,+\infty)$, and $\varepsilon > 0$. Then for every $\varphi \in SBV^p(\Omega)$ there exist:*

- *closed sets $\Gamma^{\rm int} = \Gamma_\varepsilon^{\rm int}$, $\Gamma^{\rm frac} = \Gamma_\varepsilon^{\rm frac}$, finite unions of disjoint C^1 curves;*
- *a set $\widetilde{\omega} = \widetilde{\omega}_\varepsilon$, finite union of cubes;*
- *a set of finite perimeter $\widehat{\omega} = \widehat{\omega}_\varepsilon$;*
- *a function $\theta = \theta_\varepsilon \in SBV^p(\Omega) \cap W^{1,p}(\Omega \setminus (\Gamma^{\rm int} \cup \Gamma^{\rm frac} \cup \overline{\widetilde{\omega}}))$;*

such that

$$\{\nabla\varphi \neq \nabla\theta\} \subset \widetilde{\omega} \cup \widehat{\omega}, \quad \mathcal{L}^2(\{\varphi \neq \theta\}) < \varepsilon, \quad \theta = 0 \text{ in } \widetilde{\omega}, \tag{2.1}$$

$[\theta](x) \in \mathbb{Z}$ for $\mathcal{H}^1$-a.e. $x \in \Gamma^{\rm int}$,

$$|\mathcal{H}^1(S_\varphi^{\rm int}) - \mathcal{H}^1(\Gamma^{\rm int})| + \mathcal{H}^1(S_\varphi^{\rm frac} \triangle \Gamma^{\rm frac}) + \mathcal{H}^1(\partial\widetilde{\omega}) + \mathcal{H}^1(\partial^*\widehat{\omega}) \leq \varepsilon, \tag{2.2}$$

and

$$\int_\Omega |\nabla\theta|^p \,\mathrm{d}x \leq (1+\varepsilon) \int_\Omega |\nabla\varphi|^p \,\mathrm{d}x. \tag{2.3}$$

Moreover, $\mathcal{H}^1(\Gamma^{\rm frac} \cap \{\theta^+ \neq \varphi^+\}) + \mathcal{H}^1(\Gamma^{\rm frac} \cap \{\theta^- \neq \varphi^-\}) \leq \varepsilon$, where $\theta^\pm$ and $\varphi^\pm$ denote the traces of θ and φ on the two sides of Γ.

Before providing the proof of Theorem 2.1 we state and prove our desired approximation results for maps in $SBV^p(\Omega;\mathbb{S}^1)$, in the following Corollary 2.2 and Proposition 2.3.

Corollary 2.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set of finite perimeter, $p \in (1,+\infty)$, and $\varepsilon > 0$. Then for every $u \in SBV^p(\Omega;\mathbb{S}^1)$ there exist:*

- *a closed set $\Gamma = \Gamma_\varepsilon$, finite union of disjoint C^1 curves;*
- *a set $\widetilde{\omega} = \widetilde{\omega}_\varepsilon$, finite union of cubes;*
- *a set of finite perimeter $\widehat{\omega} = \widehat{\omega}_\varepsilon$;*
- *a function $v = v_\varepsilon \in SBV^p(\Omega;\mathbb{S}^1) \cap W^{1,p}(\Omega \setminus (\Gamma \cup \overline{\widetilde{\omega}});\mathbb{S}^1)$;*

such that

$$\{\nabla u \neq \nabla v\} \subset \widetilde{\omega} \cup \widehat{\omega}, \quad \mathcal{L}^2(\{u \neq v\}) < \varepsilon, \quad \nabla v = 0 \ \mathcal{L}^2\text{-a.e. in } \widetilde{\omega}, \tag{2.4}$$

and

$$\mathcal{H}^1(S_u \triangle \Gamma) + \mathcal{H}^1(\partial\widetilde{\omega}) + \mathcal{H}^1(\partial^*\widehat{\omega}) \leq \varepsilon, \quad \int_\Omega |\nabla v|^p \,\mathrm{d}x \leq (1+\varepsilon) \int_\Omega |\nabla u|^p \,\mathrm{d}x. \tag{2.5}$$

Moreover, $\mathcal{H}^1(\Gamma \cap \{v^+ \neq u^+\}) + \mathcal{H}^1(\Gamma \cap \{v^- \neq u^-\}) \leq \varepsilon$, where $v^\pm$ and $u^\pm$ denote the traces of v and u on the two sides of Γ.

Proof. Let $u \in SBV^p(\Omega;\mathbb{S}^1)$ and let $\varepsilon > 0$. Then, by [24, Theorem 1.1], there exists $\varphi \in SBV^p(\Omega)$ such that $u = e^{2\pi\imath\varphi}$ with $\pi|\varphi|_{BV} \leq |u|_{BV}$. (Notice that in [24] the relation between u and φ is $u = e^{i\varphi}$ and the inequality is $|\varphi|_{BV} \leq 2|u|_{BV}$.) Let $\Gamma^{\rm int}$, $\Gamma^{\rm frac}$, $\widetilde{\omega}$, $\widehat{\omega}$ be the sets and let θ be the function provided by Theorem 2.1. We set $v := e^{2\pi\imath\theta}$. Then, $\{\nabla\varphi = \nabla\theta\} \equiv \{\nabla u = \nabla v\}$ $\mathcal{L}^2$-a.e. and $\{\varphi = \theta\} \subset \{u = v\}$, so that, by (2.1) we immediately deduce (2.4). Furthermore, since $S_\varphi^{\rm frac} \equiv S_u$, taking $\Gamma = \Gamma^{\rm frac}$, by Theorem 2.1 we deduce also the last part of the claim. □

Proposition 2.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with finite perimeter, let $p \in (1,+\infty)$, and let $u \in SBV^p(\Omega;\mathbb{S}^1)$. Then there exists $\{u_n\}_{n\in\mathbb{N}} \subset SBV^p(\Omega;\mathbb{S}^1)$ with $\mathcal{H}^1(\overline{S}_{u_n} \setminus S_{u_n}) = 0$ for all n such that*

$$\begin{aligned} &\|u_n - u\|_{BV(\Omega;\mathbb{R}^2)} \to 0,\\ &\|\nabla u_n\|_{L^p(\Omega;\mathbb{R}^{2\times 2})} \to \|\nabla u\|_{L^p(\Omega;\mathbb{R}^{2\times 2})},\\ &\mathcal{H}^1(S_{u_n}) \to \mathcal{H}^1(S_u). \end{aligned} \tag{2.6}$$

Furthermore

$$|T_{u_n} - T_u|(\Omega) \to 0, \tag{2.7}$$

where T_u and T_{u_n} are the measures provided by (1.9)*; in particular*

$$\|Ju - Ju_n\|_{\mathrm{flat},\Omega} \to 0,$$

with $\|\cdot\|_{\mathrm{flat},\Omega}$ is the norm defined in (1.2)*.*

Proof. Let $u \in SBV^p(\Omega;\mathbb{S}^1)$ be fixed. For every $n \in \mathbb{N}$ let u_n be the function provided by Corollary 2.2 for $\varepsilon = \varepsilon_n = \frac{1}{n}$. By (2.4) and (2.5) , we have that, for all $s \in [1,+\infty)$,

$$\mathcal{L}^2(\{u \neq u_n\}) \leq \frac{1}{n}, \qquad \|u - u_n\|_{L^s(\Omega;\mathbb{R}^2)} \leq \frac{1}{n^{\frac{1}{s}}}, \tag{2.8}$$

and

$$\begin{aligned} \|\nabla u - \nabla u_n\|_{L^1(\Omega;\mathbb{R}^{2\times 2})} &\leq \big(\|\nabla u\|_{L^s(\Omega;\mathbb{R}^{2\times 2})} + \|\nabla u_n\|_{L^s(\Omega;\mathbb{R}^{2\times 2})}\big)\big(|\widetilde{\omega}_n| + |\widehat{\omega}_n|\big)^{\frac{1}{s'}}\\ &\leq \frac{C}{n^{2/s'}}\|\nabla u\|_{L^s(\Omega;\mathbb{R}^{2\times 2})}\,. \end{aligned} \tag{2.9}$$

Furthermore, since

$$\mathrm{D}^{\mathrm{S}}u = (u^+ - u^-)\otimes \boldsymbol{\nu}_u \llcorner S_u \,\mathrm{d}\mathcal{H}^1, \qquad \mathrm{D}^{\mathrm{S}}u_n = (u_n^+ - u_n^-)\otimes \boldsymbol{\nu}_{u_n} \llcorner S_{u_n} \,\mathrm{d}\mathcal{H}^1\,,$$

by triangle inequality, using that $\boldsymbol{\nu}_u = \boldsymbol{\nu}_{u_n}$ on $S_u \cap S_{u_n}$, we get

$$\begin{aligned} |\mathrm{D}^{\mathrm{S}}u - \mathrm{D}^{\mathrm{S}}u_n|(\Omega) \leq& \big(|u^+ - u_n^+| + |u^- - u_n^-|\big)\mathcal{H}^1(S_u \cap S_{u_n})\\ &+ |u^+ - u^-|\mathcal{H}^1(S_u \setminus S_{u_n}) + |u_n^+ - u_n^-|\mathcal{H}^1(S_{u_n} \setminus S_u)\\ \leq& 4\mathcal{H}^1(\Gamma_n \cap \{u_n^+ \neq u^+\}) + 4\mathcal{H}^1(\Gamma_n \cap \{u_n^- \neq u^-\}) + 2\mathcal{H}^1(S_u \triangle S_{u_n})\\ \leq& \frac{C}{n}, \end{aligned} \tag{2.10}$$

where in the last inequality we have used (2.5) and the fact that $S_{u_n} \subset \Gamma_n \cup \partial\widetilde{\omega}_n$, to deduce that

$$\begin{aligned} \mathcal{H}^1(S_u \triangle S_{u_n}) \leq& \mathcal{H}^1(\Gamma_n \cap (S_u \setminus S_{u_n})) + \mathcal{H}^1(S_u \triangle \Gamma_n) + \mathcal{H}^1(\partial\widetilde{\omega}_n)\\ \leq& \mathcal{H}^1(\Gamma_n \cap \{u_n^+ \neq u^+\}) + \mathcal{H}^1(\Gamma_n \cap \{u_n^- \neq u^-\}) + \mathcal{H}^1(S_u \triangle \Gamma_n) + \mathcal{H}^1(\partial\widetilde{\omega}_n) \leq \frac{2}{n}. \end{aligned}$$

By (2.8), (2.9), (2.10), we immediately deduce the first claim in (2.6), whereas the other two easily follow from (2.5). Notice that (2.6) together with the fact that $u_n, u \in SBV^p(\Omega;\mathbb{S}^1)$ implies that

$$\int_{S_u \cap S_{u_n}} |[u] - [u_n]| \otimes \boldsymbol{\nu}_u \,\mathrm{d}\mathcal{H}^1 \to 0 \quad \text{and} \quad \mathcal{H}^1(S_u \triangle S_{u_n}) \to 0 \qquad \text{as } n \to +\infty. \tag{2.11}$$

Furthermore, using triangle inequality, (2.8) and (2.9), it is easy to check that

$$|T_u^{\mathrm{D}} - T_{u_n}^{\mathrm{D}}|(\Omega) \leq 4\|\nabla u\|_{L^1(\{u\neq u_n\};\mathbb{R}^{2\times 2})}\|u - u_n\|_{L^\infty(\Omega;\mathbb{R}^2)} + 4\|u_n\|_{L^\infty(\Omega;\mathbb{R}^2)}\|\nabla u - \nabla u_n\|_{L^1(\Omega;\mathbb{R}^{2\times 2})} \to 0$$

and that

$$\begin{aligned} |T_u^{\mathrm{S}} - T_{u_n}^{\mathrm{S}}|(\Omega) \leq& \frac{1}{2}\int_{S_u \cap S_{u_n}} |u_n^+ \wedge u_n^- - u^+ \wedge u^-| \,\mathrm{d}\mathcal{H}^1\\ &+ \frac{1}{2}\int_{S_u \setminus S_{u_n}} |u^+ \wedge u^-| \,\mathrm{d}\mathcal{H}^1 + \frac{1}{2}\int_{S_{u_n} \setminus S_u} |u_n^+ \wedge u_n^-| \,\mathrm{d}\mathcal{H}^1 \leq \mathcal{H}^1(\{[u] \neq [u_n]\}) \leq \frac{3}{n}, \end{aligned}$$

since $S_{u_n} \subset \Gamma_n \cup \partial\widetilde{\omega}_n \cup \partial^*\widehat{\omega}_n$ and $\{[u] \neq [u_n]\} = (S_u \triangle S_{u_n}) \cup (S_u \cap S_{u_n} \cap \{[u] \neq [u_n]\})$. □

In the proof of Theorem 2.1 we will employ a variant in SBV^p of the following fundamental technical tool, [18, Proposition 3.2]. In [18] the result is stated for balls, and it holds for cubes as well. Moreover, it holds true also for $GSBD^p$ functions, in place of SBD^p (see e.g. [17, Proof of Proposition 3.1]). As usual, $Q_\rho := (-\rho,\rho)^2$.

Proposition 2.4. *For every $p \in (1,\infty)$ there exist $c > 0$ and $\eta \in (0,1)$ such that if $u \in GSBD^p(Q_{2\varrho})$, $\varrho > 0$, satisfies*

$$\mathcal{H}^1(S_u \cap Q_{2\varrho}) < \eta(1-s)\varrho$$

for some $s \in (0,1)$, then there is a countable family $\mathscr{F} = \{B\}$ of closed balls of radius $r_B < 2(1-s)\varrho$ and center $x_B \in \overline{Q}_{2s\varrho}$ such that their union is compactly contained in $B_{2\varrho}$, and a field $w \in SBD^p(Q_{2\varrho})$ such that

(i) $\varrho^{-1}\sum_{B\in\mathscr{F}} \mathcal{L}^2(B) + \sum_{B\in\mathscr{F}} \mathcal{H}^1(\partial B) \le c/\eta\ \mathcal{H}^1(S_u \cap Q_{2\varrho})$;
(ii) $\mathcal{H}^1(S_u \cap \cup_{B\in\mathscr{F}}\partial B) = \mathcal{H}^1\big((S_u \cap Q_{2s\varrho}) \setminus \cup_{B\in\mathscr{F}} B\big) = 0$;
(iii) $w = u$ $\mathcal{L}^2$-a.e. on $Q_{2\varrho} \setminus \cup_{B\in\mathscr{F}} B$;
(iv) $w \in W^{1,p}(Q_{2s\varrho};\mathbb{R}^2)$ and $\mathcal{H}^1(S_w \setminus S_u) = 0$;
(v)

$$\int_{\cup_{B\in\mathscr{F}} B} |e(w)|^p \,\mathrm{d}x \le c \int_{\cup_{B\in\mathscr{F}} B} |e(u)|^p \,\mathrm{d}x. \tag{2.12}$$

We now present the modification of the result recalled just above, allowing to obtain the desired SBV^p version.

Proposition 2.5. *For every $p \in (1,\infty)$ there exist $c > 0$ and $\eta \in (0,1)$ such that if $\varphi \in SBV^p(Q_{2\varrho})$, $\varrho > 0$, satisfies*

$$\mathcal{H}^1(S_\varphi \cap Q_{2\varrho}) < \eta(1-s)\varrho$$

for some $s \in (0,1)$, then there is a countable family $\mathscr{F} = \{B\}$ of closed balls of radius $r_B < 2(1-s)\varrho$ and center $x_B \in \overline{Q}_{2s\varrho}$ such that their union is compactly contained in $Q_{2\varrho}$, and a field $w \in SBV^p(Q_{2\varrho})$ such that

(i) $\varrho^{-1}\sum_{B\in\mathscr{F}} \mathcal{L}^2(B) + \sum_{B\in\mathscr{F}} \mathcal{H}^1(\partial B) \le c/\eta\ \mathcal{H}^1(S_\varphi \cap Q_{2\varrho})$;
(ii) $\mathcal{H}^1(S_\varphi \cap \cup_{B\in\mathscr{F}}\partial B) = \mathcal{H}^1\big((S_\varphi \cap Q_{2s\varrho}) \setminus \cup_{\mathscr{F}} B\big) = 0$;
(iii) $w = \varphi$ $\mathcal{L}^2$-a.e. on $Q_{2\varrho} \setminus \cup_{B\in\mathscr{F}} B$;
(iv) $w \in W^{1,p}(Q_{2s\varrho})$ and $\mathcal{H}^1(S_w \setminus S_\varphi) = 0$;
(v)

$$\int_{\cup_{B\in\mathscr{F}} B} |\nabla w|^p \,\mathrm{d}x \le c \int_{\cup_{B\in\mathscr{F}} B} |\nabla \varphi|^p \,\mathrm{d}x. \tag{2.13}$$

Proof. We notice that it is enough to follow the proof of [18, Theorem 2.1], from which [18, Proposition 3.2] follows, and use the fact that, if $\varphi \in SBV^p$, one can control the components of the constant matrix $\nabla\phi(\varphi)$ in place of those of $e(\phi(\varphi))$ (see (2.12) in [18] and its consequences; therein the role of φ is played by u and $\phi(\varphi)$ corresponds to our modification w) by

$$\nabla\phi(\varphi)\cdot(x-y) = \phi(\varphi)(x) - \phi(\varphi)(y) = \int_{S_{x,y}} (\varphi_z^{\boldsymbol{\nu}})'(t)\,\mathrm{d}t,$$

where $\varphi_z^{\boldsymbol{\nu}}(t) := \varphi(z + t\boldsymbol{\nu})$, for $\boldsymbol{\nu} := \frac{x-y}{|x-y|}$, $z := (\mathrm{Id} - \boldsymbol{\nu}\otimes\boldsymbol{\nu})x$. Moreover, a constant in place of an infinitesimal rigid motion appears in the Poincaré's inequality for φ on $Q_{\overline{x},\overline{y}}$. □

We now turn to the proof of Theorem 2.1.

Proof of Theorem 2.1. Let ϱ, α_1, α_2 be small positive constants to be determined later. We divide the proof into three steps.

Step 1: Covering the jump set. Since the sets

$$\widehat{S}^0_\varphi := S^{\mathrm{frac}}_\varphi \qquad \text{and} \qquad \widehat{S}^z_\varphi := \{x \in S_\varphi \colon [\varphi](x) = z\} \qquad \text{for all } z \in \mathbb{Z}^* := \mathbb{Z}\setminus\{0\}$$

are countably $(\mathcal{H}^1,1)$ rectifiable with finite $\mathcal{H}^1$ measure, by [31, Theorem, 3.2.29] for every $z\in\mathbb{Z}$ there exists a countable family $\{M_k^z\}_{k\in\mathbb{N}}$ such that

$$\mathcal{H}^1\Big(\widehat{S}_\varphi^z\setminus\bigcup_{k=1}^{\infty}M_k^z\Big)=0 \tag{2.14}$$

and, by [3, Theorem 2.76] we may assume that for each $z\in\mathbb{Z}$ and $k\in\mathbb{N}$ the manifold M_k^z is a graph of a C^1 and Lipschitz function with Lipschitz constant less than α_1. Let us fix $z\in\mathbb{Z}$ such that $\mathcal{H}^1(\widehat{S}_\varphi^z)>0$ (in particular, for the application of the present theorem in this paper, this holds for $z=0$). Then, since x has $\mathcal{H}^1$-density 1 both for $\widehat{S}_\varphi^z$ and M_k^z for $\mathcal{H}^1$-a.e. $x\in\widehat{S}_\varphi^z\cap M_k^z$ (by Besicovitch-Marstrand-Mattila Theorem, see e.g. [3, Theorem 2.63], being both sets countably $(\mathcal{H}^1,1)$ rectifiable), for every $k\in\mathbb{N}$ and every such $x\in\widehat{S}_\varphi^z\cap M_k^z$ there exists $\overline{\varrho}(\alpha_2,x)\in(0,\varrho)$ such that

$$\begin{aligned}|\mathcal{H}^1(\overline{Q}_r(x)\cap\widehat{S}_\varphi^z)-2r|\le 2\alpha_2 r,\quad |\mathcal{H}^1(\overline{Q}_r(x)\cap M_k^z)-2r|\le 2\alpha_2 r,\\ |\mathcal{H}^1(\overline{Q}_r(x)\cap(\widehat{S}_\varphi^z\cap M_k^z))-2r|\le 2\alpha_2 r.\end{aligned} \tag{2.15}$$

for every $r\le\overline{\varrho}(\alpha_2,x)$, and moreover

$$\mathcal{H}^1(\overline{Q}_r(x)\cap(\widehat{S}_\varphi^z\triangle M_k^z))\le\alpha_2\mathcal{H}^1(\overline{Q}_r(x)\cap\widehat{S}_\varphi^z), \tag{2.16}$$

for every $r\le\overline{\varrho}(\alpha_2,x)$; here we recall that $Q_r(x)$ denotes the (open) cube $Q_r^{\nu(x)}(x)$, centered at x, with sidelength $2r$ and with a side normal to $\nu(x)$, the approximate normal to S_φ (or $\widehat{S}_\varphi^z$) at x. We notice that (2.15) holds also for S_φ in place of $\widehat{S}_\varphi^z$ or $\widehat{S}_\varphi^z\cap M_k^z$, that is, for $\mathcal{H}^1$-a.e. $x\in S_\varphi$, we may also assume

$$|\mathcal{H}^1(\overline{Q}_r(x)\cap S_\varphi)-2r|\le 2\alpha_2 r \tag{2.17}$$

for every $r\le\overline{\varrho}(\alpha_2,x)$. Now we introduce

$$M:=S_\varphi\cap\bigcup_{z\in\mathbb{Z},\,k\in\mathbb{N}}M_k^z.$$

We also denote by $\widetilde{M}\subset M$, the set of points x satisfying (2.15), (2.16), and (2.17). From what observed, $\mathcal{H}^1(M\setminus\widetilde{M})=0$; so, since the family $\{\overline{Q}_r(x)\colon x\in M,\,r\le\overline{\varrho}(\alpha_2,x)\}$ is a fine cover of $\widetilde{M}$, Vitali-Besicovitch's Covering Theorem (see [30, Theorem 1.10] for its version for cubes) ensures the existence of a disjoint subfamily $\{\overline{Q}_{r(\alpha_2,x)}(x)\colon x\in M'\}$, for a countable set $M'=\{x_j\}_{j\in\mathbb{N}}\subset\widetilde{M}\subset M$ and $r(\alpha_2,x)\le\overline{\varrho}(\alpha_2,x)$, such that

$$\mathcal{H}^1\Big(S_\varphi\setminus\bigcup_{j\in\mathbb{N}}\overline{Q}_j\Big)=0,$$

where we have denoted $Q_j:=Q_{r_j}(x_j)$ and $r_j:=r(\alpha_2,x_j)$ for every $j\in\mathbb{N}$. Then there exists $J=J(\alpha_2)\in\mathbb{N}$ such that

$$\mathcal{H}^1\Big(S_\varphi\setminus\bigcup_{j=1}^{J}\overline{Q}_j\Big)<\alpha_2. \tag{2.18}$$

For every $j\in\{1,\dots,J\}$, let $k_j\in\mathbb{N}$, $z_j\in\mathbb{Z}$ be the indexes such that $x_j\in\widehat{S}_\varphi^{z_j}\cap M_{k_j}^{z_j}$ and (2.15), (2.16), (2.17) hold (for all j, such indexes are unique). Then we set

$$\Gamma_j:=\overline{Q}_j\cap M_{k_j}^{z_j}\ .$$

We now see that for every $j \in \{1, \dots, J\}$, there hold

(a) Γ_j is the graph of a C^1 and Lipschitz function with Lipschitz constant less than α_1;

(b) $|\mathcal{H}^1(\overline{Q}_r(x_j) \cap \Gamma_j) - 2r| \le 2\alpha_2 r$ for all $0 < r \le r_j$;

(c) $\mathcal{H}^1(\overline{Q}_j \cap (\widehat{S}_\varphi^{z_j} \triangle \Gamma_j)) + \mathcal{H}^1(\overline{Q}_j \cap (S_\varphi \setminus \widehat{S}_\varphi^{z_j})) \le 8\alpha_2 r_j$;

(d) $\mathcal{H}^1\Big(S_\varphi \triangle \bigcup_{j=1}^J \Gamma_j\Big) \le \alpha_2(1 + 5\mathcal{H}^1(S_\varphi))$;

(e) $\mathcal{L}^2\Big(\bigcup_{j=1}^J \overline{Q}_j\Big) \le 3\varrho\, \mathcal{H}^1(S_\varphi)$.

Property (a) follows by definition, since M_k^z are graphs of Lipschitz maps with constant less than α_1. Property (b) follows immediately from (2.15). As for the proof of (c), by (2.16) and (2.15), choosing $0 < \alpha_2 \le 1$ we have that

$$\mathcal{H}^1(\overline{Q}_j \cap (\widehat{S}_\varphi^{z_j} \triangle \Gamma_j)) \le \alpha_2 \mathcal{H}^1(\overline{Q}_j \cap \widehat{S}_\varphi^{z_j}) \le 2\alpha_2(1+\alpha_2) r_j \le 4\alpha_2 r_j\,; \tag{2.19}$$

moreover, by (2.17) and (2.15) we have that

$$\begin{aligned}\mathcal{H}^1(\overline{Q}_j \cap (S_\varphi \setminus \widehat{S}_\varphi^{z_j})) &= \mathcal{H}^1(\overline{Q}_j \cap S_\varphi) - \mathcal{H}^1(\overline{Q}_j \cap \widehat{S}_\varphi^{z_j}) \\ &\le 2r_j(1+\alpha_2) - 2r_j(1-\alpha_2) = 4\alpha_2 r_j\,,\end{aligned}$$

which, combined with (2.19), yields property (c). Property (d) follows from the decomposition

$$S_\varphi \setminus \bigcup_{j=1}^J \Gamma_j = \left(S_\varphi \setminus \bigcup_{j=1}^J \overline{Q}_j\right) \cup \bigcup_{j=1}^J \big(\overline{Q}_j \cap (S_\varphi \setminus \Gamma_j)\big),$$

combined with (2.18) and the estimate

$$\begin{aligned}\mathcal{H}^1\big(\overline{Q}_j \cap (S_\varphi \triangle \Gamma_j)\big) &= \mathcal{H}^1\big(\overline{Q}_j \cap (\widehat{S}_\varphi^{z_j} \setminus \Gamma_j)\big) + \mathcal{H}^1\big(\overline{Q}_j \cap ((S_\varphi \setminus \widehat{S}_\varphi^{z_j}) \setminus \Gamma_j)\big) + \mathcal{H}^1\big(\overline{Q}_j \cap (\Gamma_j \setminus S_\varphi)\big) \\ &\le \mathcal{H}^1\big(\overline{Q}_j \cap (\widehat{S}_\varphi^{z_j} \triangle \Gamma_j)\big) + \mathcal{H}^1\big(\overline{Q}_j \cap (S_\varphi \setminus \widehat{S}_\varphi^{z_j})\big) \\ &\le 8\alpha_2 r_j \le 4\alpha_2\big(\mathcal{H}^1(\overline{Q}_j \cap S_\varphi) + 2\alpha_2 r_j)\big) \\ &\le 5\alpha_2 \mathcal{H}^1(\overline{Q}_j \cap S_\varphi)\end{aligned}$$

recalling that the cubes $\overline{Q}_j$ are pairwise disjoint. Here the first inequality follows from the fact that $\widehat{S}_\varphi^{z_j} \subset S_\varphi$, the second one from (c), the third one from (2.17), the fourth again from (2.17) choosing

$$\alpha_2 < \frac{1}{5}\,. \tag{2.20}$$

Eventually, recalling that $r_j \le \varrho$ for every j, by (2.17), since $\alpha_2 < \frac{1}{5} < \frac{1}{3}$, and using again that the cubes Q_j are pairwise disjoint, we have

$$\mathcal{L}^2\big(\bigcup_{j=1}^J \overline{Q}_j\big) = \sum_{j=1}^J 4r_j^2 \le 2\varrho \sum_{j=1}^J \big(\mathcal{H}^1(\overline{Q}_j \cap S_\varphi) + 2\alpha_2 r_j\big) \le 3\varrho \sum_{j=1}^J \mathcal{H}^1(\overline{Q}_j \cap S_\varphi) \le 3\varrho \mathcal{H}^1(S_\varphi),$$

from which (e) follows. Moreover, using (2.18) with (c), and arguing as done to prove (d), we obtain that

$$\widehat{\Gamma}^{\rm int} := \bigcup_{j\,:\,z_j \neq 0} \Gamma_j, \qquad \widehat{\Gamma}^{\rm frac} := \bigcup_{j\,:\,z_j = 0} \Gamma_j, \qquad \widehat{\Gamma} := \widehat{\Gamma}^{\rm int} \cup \widehat{\Gamma}^{\rm frac} = \bigcup_{j=1}^J \Gamma_j$$

are finite unions of disjoint C^1 curves and

$$\mathcal{H}^1(S_\varphi^{\rm int} \triangle \widehat{\Gamma}^{\rm int}) + \mathcal{H}^1(S_\varphi^{\rm frac} \triangle \widehat{\Gamma}^{\rm frac}) \le \alpha_2(1 + 5\mathcal{H}^1(S_\varphi)). \tag{2.21}$$

Step 2: Approximation in the cubes Q_j. We perform two different approximations depending on whether the cube $Q_j = Q_{r_j}(x_j)$ is such that $z_j = 0$ or not. To shorten the notation, we set $\nu_j := \boldsymbol{\nu}(x_j)$, where $\boldsymbol{\nu}(x_j)$ is the approximate normal to S_φ at x_j, and we denote by τ_j the unit vector such that $\tau_j \cdot \nu_j = 0$ and $\tau_j \wedge \nu_j = 1$, where we recall (cf. below (1.9)) the notation $\alpha \wedge \beta = -\alpha \cdot \beta^\perp = \det(\alpha, \beta)$.

Step 2.1: The case $x_j \notin \widehat{S}^0_\varphi$, i.e., $z_j \neq 0$. This implies that $[\varphi](x_j) = z_j \in \mathbb{Z}^*$. In this case we first show that there exists a "big" set of segments (parallel to τ_j) in the cube $\overline{Q}_j$ that do not intersect the jump set S_φ of φ and such that (small) stripes centered at each of these segments contain a "small" portion of $S_\varphi \setminus \Gamma_j$. To this end, for $x_j + \tau_j\mathbb{R}$ the straight line orthogonal to ν_j and passing through x_j , we define the (signed) distance from such a line as $\mathrm{dist}(x, x_j + \tau_j\mathbb{R}) := (x - x_j) \cdot \nu_j$. Moreover, for every $\gamma \in (-r_j/2, r_j/2)$ we define

$$T^\gamma_j := \overline{Q}_j \cap \{\mathrm{dist}(\cdot, x_j + \tau_j\mathbb{R}) = \gamma\}$$

and, for every $k \in \mathbb{N}$, we set

$$C^{\gamma,k}_j := \overline{Q}_j \cap \{\mathrm{dist}(\cdot, x_j + \tau_j\mathbb{R}) \in [\gamma - 2^{-k}r_j, \gamma + 2^{-k}r_j]\}.$$

Let

$$\tilde\eta := \frac{1}{160}\eta, \tag{2.22}$$

where η is the constant from Proposition 2.5. We set $S_j := \overline{Q}_j \cap (S_\varphi \setminus \Gamma_j)$. We claim that there exists a set $I^{\tilde\eta}_j \subset (-r_j/2, r_j/2)$ with

$$\mathcal{L}^1(I^{\tilde\eta}_j) \le \frac{160\,\alpha_2}{\tilde\eta} r_j \tag{2.23}$$

such that, for every $\gamma \in (-r_j/2, r_j/2) \setminus I^{\tilde\eta}_j$, it holds

$$\mathcal{H}^1(S_j \cap C^{\gamma,k}_j) < \tilde\eta 2^{-(k+1)} r_j \quad \text{for all } k \in \mathbb{N} \tag{2.24}$$

and

$$\mathcal{H}^1(T^\gamma_j \cap S_\varphi) = 0. \tag{2.25}$$

Indeed, for $\delta^k_j := 2^{-k}r_j$, we argue as in the proof of [18, Theorem 2.1], considering the family

$$\mathscr{I}^{\tilde\eta}_j := \left\{ [\gamma - \delta^k_j, \gamma + \delta^k_j] \quad : \quad \mathcal{H}^1(S_j \cap C^{\gamma,k}_j) \ge \frac{\tilde\eta}{2}\delta^k_j\,, \quad k \in \mathbb{N}\,, \quad \gamma \in (-r_j/2, r_j/2) \right\} \tag{2.26}$$

and $I^{\tilde\eta}_j := \bigcup_{I \in \mathscr{I}^{\tilde\eta}_j} I$. By Vitali's covering theorem, there exists a countable set $\{(\gamma^l, k^l)\}_{l\in\mathbb{N}}$ such that the intervals $[\gamma^l - \delta^{k^l}_j, \gamma^l + \delta^{k^l}_j]$ in $\mathscr{I}^{\tilde\eta}_j$ are pairwise disjoint and

$$I^{\tilde\eta}_j \subset \bigcup_{l\in\mathbb{N}} [\gamma^l - 5\delta^{k^l}_j, \gamma^l + 5\delta^{k^l}_j]\,.$$

By property (c) of Γ_j we have

$$8\alpha_2 r_j \ge \mathcal{H}^1(S_j \cap Q_j) \ge \sum_{l\in\mathbb{N}} \mathcal{H}^1(S_j \cap C^{\gamma^l,k^l}_j) \ge \sum_{l\in\mathbb{N}} \frac{\tilde\eta}{2}\delta^{k^l}_j \ge \frac{\tilde\eta}{20}\mathcal{L}^1(I^{\tilde\eta}_j). \tag{2.27}$$

Then (2.23) follows. We stress that in the above estimate it is crucial that the space dimension of the domain is d=2: in fact, taking for instance d=3, in order to have the analogue of Proposition 2.5 (see [13, Theorem 3.2]) one would consider the modified set $\mathscr{I}^{\tilde\eta,3}_j$ in place of $\mathscr{I}^{\tilde\eta}_j$ in (2.26), obtained by replacing the condition $\mathcal{H}^1(S_j \cap C^{\gamma,k}_j) \ge \frac{\tilde\eta}{2}\delta^k_j$ by $\mathcal{H}^2(S_j \cap C^{\gamma,k}_j) \ge \frac{\tilde\eta}{2}(\delta^k_j)^2$ (in order to guarantee that for all cubes $\widetilde{q}$ of sidelength δ^k_j in the strip $C^{\gamma,k}_j$ one has $\mathcal{H}^2(S_j \cap \widetilde{q}) \le \tilde\eta/2\,\mathcal{H}^2(\partial\widetilde{q})$), and then one would get

$$8\alpha_2 r_j \ge \mathcal{H}^2(S_j \cap Q_j) \ge \sum_{l\in\mathbb{N}} \mathcal{H}^2(S_j \cap C^{\gamma^l,k^l}_j) \ge \sum_{l\in\mathbb{N}} \frac{\tilde\eta}{2}(\delta^{k^l}_j)^2,$$

from which one cannot conclude the analogue of (2.27) since only $\sum_{l\in\mathbb{N}} \frac{\tilde\eta}{2}\delta_j^{k^l} \geq \frac{\tilde\eta}{20}\mathcal{L}^1(I_j^{\tilde\eta})$, and $(\delta_j^{k^l})^2 \leq \delta_j^{k^l}$.

From (2.23) and the definition of $I_j^{\tilde\eta}$, every $\gamma \in (-r_j/2, r_j/2)\setminus I_j^{\tilde\eta}$ does not belong to any interval of the family (2.26) and then satisfies (2.24). Since $\mathcal{H}^1(T_j^\gamma \cap S_\varphi) = 0$ for every $\gamma \in (-r_j/2, r_j/2)$ except at most countably many, we may enforce also (2.25). Then the claim is confirmed.

Let us choose $\gamma_j \in (-r_j/2, r_j/2) \setminus I_j^{\tilde\eta}$ satisfying (2.24), (2.25) with

$$\gamma_j \in \left(-\alpha_1 r_j - \frac{161\alpha_2}{\tilde\eta} r_j, -\alpha_1 r_j\right). \tag{2.28}$$

We notice that $\Gamma_j \subset \{\mathrm{dist}(\cdot, x_j + \tau_j\mathbb{R}) \in [-\alpha_1 r_j, \alpha_1 r_j]\}$, due to property (a) and since $x_j \in \Gamma_j$. Therefore $\Gamma_j \cap T_j^{\gamma_j} = \emptyset$ and Γ_j is "above" $T_j^{\gamma_j}$ with respect to the positive orientation of ν_j, i.e. for any two points $\overline{y}_1 \in \Gamma_j$, $\overline{y}_2 \in T_j^{\gamma_j}$ such that $(\overline{y}_1 - x_j)\cdot\tau_j = (\overline{y}_2 - x_j)\cdot\tau_j$, it holds that $(\overline{y}_1 - x_j)\cdot\nu_j > (\overline{y}_2 - x_j)\cdot\nu_j$. Such choice of γ_j is allowed by (2.23). We consider then the function

$$\widehat{\varphi}_j := \varphi + z_j\chi_{H_j} \quad \text{in } \overline{Q}_j, \tag{2.29}$$

where $H_j \subset Q_j$ is the closed region delimited by Γ_j, $T_j^{\gamma_j}$, and the two segments Σ_j^1, $\Sigma_j^2 \subset \partial Q_j$ joining the two couples of intersection points of Γ_j and $T_j^{\gamma_j}$ with the two boundary segments

$$B_j^\pm := \{x_j \pm r_j\tau_j + t\nu_j \colon |t| < r_j\} \subset \partial Q_j.$$

The motivation for introducing $\widehat{\varphi}_j$ is that this allows us to "transfer" the jump of φ, which is approximated by Γ_j, onto the jump set of $\widehat{\varphi}_j$ which is approximated by the segment $T_j^{\gamma_j}$. In fact, by property (c) we deduce that

$$\mathcal{H}^1(Q_j \cap (S_{\widehat{\varphi}_j} \setminus T_j^{\gamma_j})) \leq \mathcal{H}^1(Q_j \cap (S_\varphi \setminus \Gamma_j)) + \mathcal{H}^1(\Gamma_j \setminus \widehat{S}_\varphi^{z_j}) \leq 8\alpha_2 r_j; \tag{2.30}$$

by (2.25) it holds that

$$\mathcal{H}^1(T_j^{\gamma_j} \cap \{[\widehat{\varphi}_j] \neq z_j\}) = 0. \tag{2.31}$$

Further, for every $s \in (0, r_j)$, denoting $Q_{j,s} := x_j + Q_s$ and $\Sigma_{j,s}^1$, $\Sigma_{j,s}^2 \subset \partial Q_{j,s}$ the two segments joining the two couples of intersection points of Γ_i and $T_j^{\gamma_j}$ with the boundary segments

$$B_{j,s}^\pm := \{x_j \pm s\tau_j + t\nu_j \colon |t| < r_j\} \subset \partial Q_{j,s},$$

in view of (a) and (2.28)

$$\mathcal{H}^1(\Sigma_{j,s}^1 \cup \Sigma_{j,s}^2) \leq 2\left(2\alpha_1 s + \frac{161\alpha_2}{\tilde\eta} r_j\right). \tag{2.32}$$

Arguing as done before to ensure (2.24), (2.25) for γ outside a small set, provided

$$\sqrt{\alpha_2} < \frac{\tilde\eta}{320} \tag{2.33}$$

it is possible to find $\widehat{r}_j \in ((1-\sqrt{\alpha_2})r_j, r_j)$ such that, denoting

$$\begin{aligned} Q_{j,\widehat{r}_j}^+ &:= \{x \in Q_j \colon (x - x_j)\cdot\nu_j \in (\gamma_j, \gamma_j + \widehat{r}_j),\ (x - x_j)\cdot\tau_j \in (-\widehat{r}_j, \widehat{r}_j)\},\\ Q_{j,\widehat{r}_j}^- &:= \{x \in Q_j \colon (x - x_j)\cdot\nu_j \in (\gamma_j - \widehat{r}_j, \gamma_j),\ (x - x_j)\cdot\tau_j \in (-\widehat{r}_j, \widehat{r}_j)\}, \end{aligned} \tag{2.34}$$

it holds that

$$\begin{aligned} &\mathcal{H}^1(S_{\widehat{\varphi}_j} \cap \partial Q_{j,\widehat{r}_j}^\pm) = 0,\\ &\mathcal{H}^1\Big(S_{\widehat{\varphi}_j} \cap Q_{j,\widehat{r}_j}^\pm \cap \big(\partial Q_{j,\widehat{r}_j}^\pm + B_{2^{-k}r_j}(0)\big)\Big) < \tilde\eta 2^{-(k+1)} r_j \text{ for every } k \in \mathbb{N}. \end{aligned} \tag{2.35}$$

In fact, the same argument as above (in particular we use that the space dimension is 2, as discussed below (2.27)) shows that there exist sets $I_{\mathrm{hor}}^\pm$, $I_{\mathrm{ver}}^\pm \subset (-r_j, r_j)$ with $\mathcal{L}^1(I_{\mathrm{hor}}^\pm) \leq \frac{320\alpha_2}{\tilde\eta} r_j$, $\mathcal{L}^1(I_{\mathrm{ver}}^\pm) \leq \frac{320\alpha_2}{\tilde\eta} r_j$ such that, for

$$C_{\mathrm{hor},-}^{\widetilde\gamma,k} := \{x \in Q_j \colon (x - x_j)\cdot\nu_j \in [\widetilde\gamma, \widetilde\gamma + 2^{-k}r_j]\}, \quad C_{\mathrm{hor},+}^{\widetilde\gamma,k} := \{x \in Q_j \colon (x - x_j)\cdot\nu_j \in [\widetilde\gamma - 2^{-k}r_j, \widetilde\gamma]\},$$
$$C_{\mathrm{ver},-}^{\widetilde\gamma,k} := \{x \in Q_j \colon (x - x_j)\cdot\tau_j \in [\widetilde\gamma, \widetilde\gamma + 2^{-k}r_j]\}, \quad C_{\mathrm{ver},+}^{\widetilde\gamma,k} := \{x \in Q_j \colon (x - x_j)\cdot\tau_j \in [\widetilde\gamma - 2^{-k}r_j, \widetilde\gamma]\},$$

it holds that for every $\widetilde{\gamma} \notin I^{\pm}_{\rm hor}$, $I^{\pm}_{\rm ver}$

$$\mathcal{H}^1(S_j \cap C^{\widetilde{\gamma},k}_{{\rm hor},\pm}) \le \tilde{\eta} 2^{-(k+3)} r_j, \quad \mathcal{H}^1(S_j \cap C^{\widetilde{\gamma},k}_{{\rm ver},\pm}) \le \tilde{\eta} 2^{-(k+3)} r_j \quad \text{for every } k \in \mathbb{N}.$$

Therefore, since (2.33) implies that $\sqrt{\alpha_2} > \frac{320\alpha_2}{\tilde{\eta}}$, it is possible to find

$$\widehat{r}_j \in ((1-\sqrt{\alpha_2})r_j, r_j) \quad \text{such that } \gamma_j \pm \widehat{r}_j \notin I^{\pm}_{\rm hor},\ \pm\widehat{r}_j \notin I^{\pm}_{\rm ver},$$

so the second condition in (2.35) is satisfied. Then (2.35) follows, since the first condition holds true for every $\widehat{r}_j$ except at most countably many.

Let

$$\widehat{\delta}_j \in \Big(8\frac{\alpha_2}{\tilde{\eta}}, 16\frac{\alpha_2}{\tilde{\eta}}\Big) \cap \frac{\widehat{r}_j}{r_j\mathbb{N}}, \tag{2.36}$$

so that the rectangles $Q^{\pm}_{j,\widehat{r}_j}$ are partitioned into cubes of side length $\widehat{\delta}_j r_j$. Moreover, let $\widehat{k}_j \in \mathbb{N}$ be such that

$$\widehat{\delta}_j \in [2^{-(\widehat{k}_j+1)}, 2^{-\widehat{k}_j}). \tag{2.37}$$

By property (a), we have that

$$\mathcal{H}^1((Q_j \setminus Q_{j,\widehat{r}_j}) \cap \Gamma_j) \le 2\alpha_1\sqrt{\alpha_2} r_j. \tag{2.38}$$

We now subdivide $Q^{\pm}_{j,\widehat{r}_j}$ into cubes whose sidelength vanishes in a dyadic way towards the boundaries such that in any of them there is a small amount of jump of $\widehat{u}_j$ compared to the sidelength, in the sense of Proposition 2.5.

Let us assume, for simplicity of notation, that $x_j = 0$ and $\nu_j = e_2$. We introduce two sets $\mathcal{Q}^{\pm}_i$ of dyadic squares of sidelength $\widetilde{\delta}_k := 2^{-k}\widehat{\delta}_j\, r_j$, $k \in \mathbb{N}$, which refine towards $\partial Q^{\pm}_{j,\widehat{r}_j}$, as follows: let $\mathcal{Q}^{\pm}_{j,0}$ be the family of squares $q \in \{z + (0, \widehat{\delta}_j r_j]^2 \colon z \in \widehat{\delta}_j r_j \mathbb{Z}^2\}$, $q \subset Q^{\pm}_{j,\widehat{r}_j}$ such that ${\rm dist}(q, \partial Q^{\pm}_{j,\widehat{r}_j}) > \widehat{\delta}_j r_j$; recursively, for $k \ge 1$, let $\mathcal{Q}^{\pm}_{j,k}$ be the family of squares $q \in \{z + (0, \widetilde{\delta}_k]^2 \colon z \in \widetilde{\delta}_k \mathbb{Z}^2\}$, $q \subset Q^{\pm}_{j,\widehat{r}_j}$ such that ${\rm dist}(q, \partial Q^{\pm}_{j,\widehat{r}_j}) > \widetilde{\delta}_k$ and q does not intersect any cube in $\mathcal{Q}^{\pm}_{j,l}$, for $l < k$; we define

$$\mathcal{Q}^{\pm}_j := \bigcup_{k=0}^{\infty} \mathcal{Q}^{\pm}_{j,k}.$$

For each $q \in \mathcal{Q}^{\pm}_j$ let q' and q'' denote squares concentric with q with sidelength 10% and 20% longer, respectively, so that $l(q') = \frac{11}{12} l(q'')$ and $l(q'') = \frac{6}{5} l(q)$; here and below, $l(\tilde{q})$ denotes the sidelength of a cube $\tilde{q}$. By (2.22), (2.30), and (2.36), for any $q \in \mathcal{Q}^{\pm}_{j,0}$, we get that

$$\mathcal{H}^1(q'' \cap S_{\widehat{\varphi}_j}) \le 8\alpha_2 r_j \le \tilde{\eta}\widehat{\delta}_j r_j < \eta \frac{1}{40} l(q) = \eta \frac{1}{10}\frac{5}{6}\frac{l(q'')}{4} = \eta\Big(1 - \frac{11}{12}\Big)\frac{l(q'')}{4}, \tag{2.39}$$

so that all the squares q'' "coming from" squares $q \in \mathcal{Q}^{\pm}_{j,0}$ satisfy the hypotheses of Proposition 2.5 for $s = \frac{11}{12}$.

Moreover, let $k \in \mathbb{N}$. By (2.22), (2.35), (2.36), and (2.37), for any $q \in \mathcal{Q}^{\pm}_{j,k}$ we have

$$\mathcal{H}^1(q'' \cap S_{\widehat{\varphi}_j}) \le \tilde{\eta} 2^{-(k+\widehat{k}_j-1)} r_j < \frac{\eta}{40}\widetilde{\delta}_k = \frac{\eta}{10}\frac{5}{6}\frac{l(q'')}{4} = \eta\Big(1 - \frac{11}{12}\Big)\frac{l(q'')}{4}, \tag{2.40}$$

(in particular the first inequality follows from (2.35) and the fact that $q'' \subset \partial Q^{\pm}_{j,\widehat{r}_j} + B_{2^{-(k+\widehat{k}_j-2)}r_j}(0)$, which in turns follows from (2.37), the definition of $\mathcal{Q}^{\pm}_{j,k}$, and recalling $\widetilde{\delta}_k = 2^{-k}\widehat{\delta}_j\, r_j$) so that all the squares q'' "coming from" squares $q \in \mathcal{Q}^{\pm}_{j,k}$ satisfy the hypotheses of Proposition 2.5 for $s = \frac{11}{12}$. By (2.39) and (2.40) we thus deduce that all the squares q'' "coming from" squares $q \in \mathcal{Q}^{\pm}_j$ satisfy the hypotheses of Proposition 2.5 for $s = \frac{11}{12}$.

Therefore, by Proposition 2.5 applied to $\widehat{\varphi}_j \in SBV^p(q'')$, for each $q \in \mathcal{Q}_j^{\pm}$ there is a set of finite perimeter $\omega_q \subset q''$, $\omega_q = \cup_{\mathcal{F}_q} B$ and a function $w_q \in SBV^p(q'')$ such that

$$\begin{aligned} &w_q \in W^{1,p}(q'), \quad w_q = \widehat{\varphi}_j \text{ in } q'' \setminus \omega_q \\ &\mathcal{H}^1(\partial^* \omega_q) \le \frac{c}{\eta} \mathcal{H}^1(S_{\widehat{\varphi}_j} \cap q''), \\ &\int_{\omega_q} |\nabla w_q|^p \, \mathrm{d}x \le c \int_{\omega_q} |\nabla \widehat{\varphi}_j|^p \, \mathrm{d}x. \end{aligned} \tag{2.41}$$

We define

$$\omega_j^{\pm} := \bigcup_{q \in \mathcal{Q}_j^{\pm}} \omega_q.$$

Since the cubes q'' overlap at most 8 times, by the second and third property in (2.41) we deduce that

$$\begin{aligned} &\mathcal{H}^1(\partial^* \omega_j^{\pm}) \le 8 \frac{c}{\eta} \mathcal{H}^1(S_{\widehat{\varphi}_j} \cap Q_{j,\widehat{r}_j}^{\pm}), \\ &\int_{\omega_j^{\pm}} |\nabla w_q|^p \, \mathrm{d}x \le 8c \int_{\omega_j^{\pm}} |\nabla \widehat{\varphi}_j|^p \, \mathrm{d}x \end{aligned} \tag{2.42}$$

Following [13, proof of Theorem 4.1, p. 1198], we construct regularized functions $v_j^{\pm}$ on $Q_{j,\widehat{r}_j}^{\pm}$ which are convex combinations of the functions w_q. We notice that in our setting all the cubes are "good", according to the definition in [13], that is the jump inside has small $\mathcal{H}^1$-measure compared to the sidelength.

We set

$$\theta_j^{\pm} := \sum_{q \in \mathcal{Q}_j^{\pm}} w_q \phi_q, \tag{2.43}$$

where

$$\begin{aligned} &\phi_q := \frac{\psi_q}{\sum_{\widehat{q} \in \mathcal{Q}_j^{\pm}} \psi_{\widehat{q}}}, \quad \psi_q(x) := \psi\Big(\frac{x - c_q}{l(q)}\Big) \text{ for } q = c_q + \Big(-\frac{l(q)}{2}, \frac{l(q)}{2}\Big)^2, \\ &\psi \in C_{\mathrm{c}}^{\infty}\Big((-11/20, 11/20)^2; [0,1]\Big), \ \psi = 1 \text{ on } [-1/2, 1/2]^2. \end{aligned}$$

By construction, $\psi_q \in C_{\mathrm{c}}^{\infty}(q'; [0,1])$ and $\psi_q \equiv 1$ in q, for any $q \in \mathcal{Q}_j^{\pm}$. Since, by (2.41), $w_q \in W^{1,p}(q')$ for every $q \in \mathcal{Q}_j^{\pm}$, we deduce that

$$\theta_j^{\pm} \in W^{1,p}\Big(\bigcup_{q \in \mathcal{Q}_j^{\pm}} q'\Big). \tag{2.44}$$

Eventually, we define

$$\theta_j := \theta_j^+ \chi_{Q_{j,\widehat{r}_j}^+} + \theta_j^- \chi_{Q_{j,\widehat{r}_j}^-}, \quad \omega_j := \omega_j^+ \cup \omega_j^-. \tag{2.45}$$

By (2.30), (2.41), (2.42), (2.43), (2.45) it follows that

$$\begin{aligned} &\int_{\omega_j} |\nabla \theta_j|^p \, \mathrm{d}x \le 8c \int_{\omega_j} |\nabla \varphi|^p \, \mathrm{d}x, \quad \theta_j = \widehat{\varphi}_j \text{ in } Q_{j,\widehat{r}_j} \setminus \omega_j, \\ &\mathcal{H}^1(\partial^* \omega_j) \le 8 \frac{c}{\eta} \mathcal{H}^1\big(Q_{j,\widehat{r}_j} \cap (S_{\widehat{\varphi}_j} \setminus T_j^{\gamma_j})\big) \le 64 \frac{c}{\eta} \alpha_2 r_j. \end{aligned} \tag{2.46}$$

We observe that the first estimate above is obtained arguing as in [13, Step 3.3 in Theorem 5.1] with the full gradient in place of the symmetrized gradient.

Furthermore, by construction we have that

$$\theta_j = \widehat{\varphi}_j \text{ on } \partial Q_{j,\widehat{r}_j}, \quad [\theta_j] = [\widehat{\varphi}_j] \text{ on } T_j^{\gamma_j}. \tag{2.47}$$

We observe that the latter property above follows from the fact that we employed a Whitney-type approximation towards $T_j^{\gamma_j}$: for details we refer to [17, discussion between (27) and (28)]. In view of (2.32),

$$\mathcal{H}^1(\partial Q_{j,\widehat{r}_j} \cap \{\theta_j \neq \varphi\}) = \mathcal{H}^1(\Sigma^1_{j,\widehat{r}_j} \cup \Sigma^2_{j,\widehat{r}_j}) \leq 4\alpha_1 \widehat{r}_j + \frac{322\alpha_2}{\tilde{\eta}} r_j, \tag{2.48}$$

and, by property (b) and the definition of $T_j^{\gamma_j}$,

$$|\mathcal{H}^1(Q_{j,\widehat{r}_j} \cap \Gamma_j) - \mathcal{H}^1(Q_{j,\widehat{r}_j} \cap T_j^{\gamma_j})| \leq 2\alpha_2 \widehat{r}_j. \tag{2.49}$$

Step 2.2: The case $x_j \in \widehat{S}^0_\varphi$, i.e., $z_j = 0$. In this case, $x_j \in S^{\mathrm{frac}}_\varphi$, and as done before, we find radii $\widehat{r}_j \in ((1-\sqrt{\alpha_2})r_j, r_j)$ such that, denoting $Q_{j,s} := x_j + Q_s$, it holds that

$$\begin{aligned} &\mathcal{H}^1(S_{\widehat{\varphi}_j} \cap \partial Q_{j,\widehat{r}_j}) = 0, \\ &\mathcal{H}^1\Big(S_{\widehat{\varphi}_j} \cap Q_{j,\widehat{r}_j} \cap \big(Q_{j,\widehat{r}_j} + B_{2^{-k}r_j}(0)\big)\Big) < \tilde{\eta} 2^{-(k+1)} r_j \text{ for every } k \in \mathbb{N}. \end{aligned} \tag{2.50}$$

By this choice, we can slightly amend the construction in [13, Theorem 4.1]: denoting by $Q^\pm_{j,\widehat{r}_j}$ the two connected components of $Q_{j,\widehat{r}_j} \setminus \Gamma_j$, namely

$$Q^\pm_{j,\widehat{r}_j} := \{y \in Q_{j,\widehat{r}_j} : \pm(y \cdot \nu_j) > \Pi_{\Gamma_j}(y) \cdot \nu_j\} \text{ for } \Pi_{\Gamma_j}(y) \in \Gamma_j \text{ s.t. } \Pi_{\Gamma_j}(y) \cdot \tau_j = y \cdot \tau_j,$$

there exist two sets of finite perimeter $\omega_j^\pm \subset Q^\pm_{j,\widehat{r}_j}$ and functions $\theta_j^\pm \in W^{1,p}(Q^\pm_{j,\widehat{r}_j})$ such that, for suitable $c_j^\pm = c_j^\pm(p) > 0$,

$$\begin{aligned} &\theta_j^\pm = \varphi \text{ in } Q^\pm_{j,\widehat{r}_j} \setminus \omega_j^\pm, \quad \int_{\omega_j^\pm} |\nabla \theta_j^\pm|^p \, \mathrm{d}x \leq c_j^\pm \int_{\omega_j^\pm} |\nabla \varphi|^p \, \mathrm{d}x, \\ &\mathcal{H}^1(\partial^* \omega_j^\pm) \leq c_j^\pm \mathcal{H}^1(S_\varphi \cap Q^\pm_{j,\widehat{r}_j}), \quad \mathcal{H}^1(\partial Q_{j,\widehat{r}_j} \cap (\omega_j^+ \cup \omega_j^-)) = 0. \end{aligned} \tag{2.51}$$

We notice that the last condition is new with respect to [13, Theorem 4.1]: it comes from the Whitney-type construction as in the previous substep, in turn allowed by the choice of $\widehat{r}_j$ for which (2.50) holds, which is possible in 2d. Moreover, as in [13, proof of Theorem 5.1, Step 2.2], one proves that the constant

$$\tilde{c} := \max\{c_j^\pm : j \text{ s.t. } x_j \in S^{\mathrm{frac}}_\varphi\} \tag{2.52}$$

is bounded uniformly with respect to α_2 (in particular, even if the side lenghts of cubes decrease and the number of cubes increases; notice that increasing the number of cubes one may assume that the Lipschitz constant corresponding to Γ_j decreases). As above, we set

$$\theta_j := \theta_j^+ \chi_{Q^+_{j,\widehat{r}_j}} + \theta_j^- \chi_{Q^-_{j,\widehat{r}_j}}, \quad \omega_j := \omega_j^+ \cup \omega_j^-.$$

Step 3: Conclusion. Following the lines of [13, proof of Theorem 5.1, Step 3], let us consider $\delta \in (0, 0.4\sqrt{2}\alpha_2 \min_{j=1,\dots,J} r_j)$ and the families:

$$\begin{aligned} &\mathscr{Q}_1 := \Big\{ q_{z,\delta} = \delta z + [0,\delta]^2 : z \in \mathbb{Z}^2,\ q_{z,\delta} \cap \Big(\mathbb{R}^2 \setminus \bigcup_{j=1}^J Q_{j,\widehat{r}_j}\Big) \neq \emptyset \Big\}, \\ &\mathscr{Q}_2 := \Big\{ q_{z,\delta} = \delta z + [0,\delta]^2 : z \in \mathbb{Z}^2,\ q_{z,\delta} \notin \mathscr{Q}_1 \text{ and intersects some cubes in } \mathscr{Q}_1 \Big\}, \\ &\mathscr{Q} := \mathscr{Q}_1 \cup \mathscr{Q}_2. \end{aligned}$$

For each $q \in \mathscr{Q}$, let q' and q'' be the (closed) cubes concentric with q and having side length $l(q') = \frac{9}{8}\delta$ and $l(q'') = \frac{10}{8}\delta = \frac{10}{9} l(q')$, respectively. Let

$$\widehat{\theta}(x) := \begin{cases} \theta_j(x), & x \in Q_{j,\widehat{r}_j}, \\ \varphi(x), & x \in \Omega \setminus \bigcup_{j=1,\dots,J} Q_{i,\widehat{r}_j}, \end{cases} \tag{2.53}$$

and, recalling the definition of c, η from Proposition 2.5, set

$$\mathscr{Q}_g := \Big\{q \in \mathscr{Q} : \mathcal{H}^1(S_{\widehat{\theta}} \cap q'') \le \frac{1}{32}\eta\delta\Big\} = \Big\{q \in \mathscr{Q} : \mathcal{H}^1(S_{\widehat{\theta}} \cap q'') \le \eta\Big(1 - \frac{9}{10}\Big)\frac{l(q'')}{4}\Big\}, \quad \mathscr{Q}_b := \mathscr{Q} \setminus \mathscr{Q}_g.$$

For every $q \in \mathscr{Q}_g$, by Proposition 2.5 applied to $\widehat{v} \in SBV^p(q'')$ (in correspondence to $s = 0.9$) there exist $w_q \in SBV^p(q'')$ and $\omega_q \subset q''$, $\omega_q = \cup_{\mathcal{F}_q} B$ such that

$$\begin{aligned} w_q \in W^{1,p}(q'), \quad w_q = \widehat{\theta} \text{ in } q'' \setminus \omega_q \\ \mathcal{H}^1(\partial^* \omega_q) \le \frac{c}{\eta}\mathcal{H}^1(S_{\widehat{\theta}} \cap q''), \\ \int_{\omega_q} |\nabla w_q|^p \, \mathrm{d}x \le c \int_{\omega_q} |\nabla \widehat{\theta}|^p \, \mathrm{d}x. \end{aligned} \tag{2.54}$$

Up to reducing the threshold in the definition of $\mathscr{Q}_g$, it holds that if $q' \cap \Gamma_j \neq \emptyset$ for some $j = 1, \dots, J$, then $q \notin \mathscr{Q}_g$, so that if $q \in \mathscr{Q}_g$ is such that $q' \subset Q_{j,\widehat{r}_j}$ it holds that $q' \subset Q^{\pm}_{j,\widehat{r}_j}$ and then $w_q = \widehat{\theta}$ (and $\omega_q \cap q' = \emptyset$), since $\widehat{\theta} = \theta_j \in W^{1,p}(Q^{\pm}_{j,\widehat{r}_j})$.

We set, recalling (2.45) (and the analogue for j s.t. $x_j \in S^{\mathrm{frac}}_{\varphi}$)

$$G := \bigcup_{q \in \mathscr{Q}_g} q, \quad \widetilde{\omega} := \bigcup_{q \in \mathscr{Q}_b} q,$$

$$\Gamma^{\mathrm{int}} := \bigcup_{j\,:\,x_j \notin S^{\mathrm{frac}}_{\varphi}} (Q_{j,\widehat{r}_j} \cap T^{\gamma_j}_j), \quad \Gamma^{\mathrm{frac}} := \bigcup_{j\,:\,x_j \in S^{\mathrm{frac}}_{\varphi}} (Q_{j,\widehat{r}_j} \cap \Gamma_j), \qquad \widehat{\omega} := \bigcup_{q \in \mathscr{Q}_g} \omega_q \cup \bigcup_{j=1,\dots,J} \omega_j,$$

and

$$v := \begin{cases} \displaystyle\sum_{q \in \mathscr{Q}_g} w_q \phi_q, & \text{in } G, \\ 0, & \text{in } \widetilde{\omega}, \\ \widehat{v}, & \text{in } \Omega \setminus (G \cup \widetilde{\omega}), \end{cases} \tag{2.55}$$

where

$$\begin{aligned} \phi_q := \frac{\psi_q}{\sum_{\widehat{q} \in \mathcal{Q}_g} \psi_{\widehat{q}}}, \quad \psi_q(x) := \psi\Big(\frac{x - c_q}{l(q)}\Big) \text{ for } q = c_q + \Big(-\frac{l(q)}{2}, \frac{l(q)}{2}\Big)^2, \\ \psi \in C^\infty_{\mathrm{c}}\Big((-9/16, 9/16)^2; [0,1]\Big), \ \psi = 1 \text{ on } [-1/2, 1/2]^2. \end{aligned}$$

By triangle inequality, (d), (b), (2.17) using that $0 \le r_j - \widehat{r}_j \le \sqrt{\alpha_2} r_j$ and that the cubes Q_j are pairwise disjoint, we obtain

$$\begin{aligned} |\mathcal{H}^1(S^{\mathrm{int}}_{\varphi}) - \mathcal{H}^1(\Gamma^{\mathrm{int}})| &\le \mathcal{H}^1(S^{\mathrm{int}}_{\varphi} \setminus \widehat{\Gamma}^{\mathrm{int}}) + |\mathcal{H}^1(S^{\mathrm{int}}_{\varphi} \cap \widehat{\Gamma}^{\mathrm{int}}) - \mathcal{H}^1(\Gamma^{\mathrm{int}})| \\ &\le \mathcal{H}^1(S^{\mathrm{int}}_{\varphi} \triangle \widehat{\Gamma}^{\mathrm{int}}) + \sum_{j : x_j \notin S^{\mathrm{frac}}_{\varphi}} |\mathcal{H}^1(S^{\mathrm{int}}_{\varphi} \cap \Gamma_j) - \mathcal{H}^1(T^{\gamma_j}_j \cap Q_{j,\widehat{r}_j})| \\ &\le \mathcal{H}^1(S^{\mathrm{int}}_{\varphi} \triangle \widehat{\Gamma}^{\mathrm{int}}) + \sum_{j : x_j \notin S^{\mathrm{frac}}_{\varphi}} \mathcal{H}^1(\Gamma_j \setminus S^{\mathrm{int}}_{\varphi}) \\ &\qquad + \sum_{j : x_j \notin S^{\mathrm{frac}}_{\varphi}} |\mathcal{H}^1(\Gamma_j) - 2\widehat{r}_j| \\ &\le \mathcal{H}^1(S^{\mathrm{int}}_{\varphi} \triangle \widehat{\Gamma}^{\mathrm{int}}) + \alpha_2(1 + 5\mathcal{H}^1(S_\varphi)) + (\alpha_2 + \sqrt{\alpha_2})\mathcal{H}^1(S_\varphi). \end{aligned} \tag{2.56}$$

Furthermore, by construction,

$$\begin{aligned}
S_\varphi^{\rm frac}\triangle\Gamma^{\rm frac} &= \Big(S_\varphi^{\rm frac}\setminus\bigcup_{j:x_j\in S_\varphi^{\rm frac}}\overline{Q}_j\Big)\cup\bigcup_{j:x_j\in S_\varphi^{\rm frac}}\big(\overline{Q}_j\cap(S_\varphi^{\rm frac}\triangle(\Gamma_j\cap Q_{j,\widehat{r}_j}))\big)\\
&\subseteq \Big(S_\varphi^{\rm frac}\setminus\bigcup_{j:x_j\in S_\varphi^{\rm frac}}\overline{Q}_j\Big)\cup\bigcup_{j:x_j\in S_\varphi^{\rm frac}}\big(\overline{Q}_j\cap(S_\varphi^{\rm frac}\triangle\Gamma_j)\big)\\
&\quad\cup\bigcup_{j=1}^J\big(S_\varphi\cap(\overline{Q}_j\setminus Q_{j,\widehat{r}_j})\big)\\
&= \big(S_\varphi^{\rm frac}\triangle\widehat{\Gamma}^{\rm frac}\big)\cup\bigcup_{j=1}^J\big(S_\varphi\cap(\overline{Q}_j\setminus Q_{j,\widehat{r}_j})\big),
\end{aligned}$$

whence, using (2.38) and property (d), we deduce that

$$\begin{aligned}
\mathcal{H}^1(S_\varphi^{\rm frac}\triangle\Gamma^{\rm frac}) &\le \mathcal{H}^1(S_\varphi^{\rm frac}\triangle\widehat{\Gamma}^{\rm frac})+\sum_{j=1}^J\mathcal{H}^1(S_\varphi\cap(Q_j\setminus Q_{j,\widehat{r}_j}))\\
&\le \mathcal{H}^1(S_\varphi^{\rm frac}\triangle\widehat{\Gamma}^{\rm frac})+\sum_{j=1}^J\mathcal{H}^1(\Gamma_j\cap(Q_j\setminus Q_{j,\widehat{r}_j}))+\mathcal{H}^1\Big(S_\varphi\triangle\bigcup_{j=1}^J\Gamma_j\Big)\\
&\le \mathcal{H}^1(S_\varphi^{\rm frac}\triangle\widehat{\Gamma}^{\rm frac})+2\alpha_1\sqrt{\alpha_2}\mathcal{H}^1(S_\varphi)+\alpha_2(1+5\mathcal{H}^1(S_\varphi)).
\end{aligned}\tag{2.57}$$

By summing (2.56) and (2.57), using (2.21), we obtain

$$|\mathcal{H}^1(S_\varphi^{\rm int})-\mathcal{H}^1(\Gamma^{\rm int})|+\mathcal{H}^1(S_\varphi^{\rm frac}\triangle\Gamma^{\rm frac})\le\alpha_2(3+16\mathcal{H}^1(S_\varphi))+\sqrt{\alpha_2}(1+2\alpha_1)\mathcal{H}^1(S_\varphi).$$

By (2.31) and (2.47) it follows that $[v](x)\in\mathbb{Z}$ for $\mathcal{H}^1$-a.e. $x\in\Gamma^{\rm int}$.

By definition and (2.44), (2.51), (2.54) it is immediate that $v\in SBV^p(\Omega)\cap W^{1,p}(\Omega\setminus(\Gamma\cup\overline{\widetilde{\omega}}))$, that $\{\nabla\varphi\neq\nabla\theta\}\subset\widetilde{\omega}\cup\widehat{\omega}$ (since $\nabla(\widehat{\varphi}_j-\varphi)=0$ in Q_j, see (2.29)), that $\{\varphi\neq\theta\}\subset\{\nabla\varphi\neq\nabla\theta\}\cup\bigcup_{j:\,z_j\neq0}H_j$ (whose $\mathcal{L}^2$-measure vanishes with α_2 and ϱ from property (e)), and that $v=0$ in $\widetilde{\omega}$.

Summing up (2.54) over $q\in\mathscr{Q}_g$ we obtain (since the cubes q'' may overlap at most 8 times)

$$\mathcal{H}^1(\partial^*\bigcup_{q\in\mathscr{Q}_g}\omega_q)\le 8\frac{c\vee\tilde{c}}{\eta}\,\mathcal{H}^1\Big(S_{\widehat{v}}\setminus\bigcup_{j=1}^J Q_{j,\widehat{r}_j}\Big)\le C(\alpha_1,\alpha_2,p)\tag{2.58}$$

($\tilde{c}$ is the constant in (2.52)) with $C(\alpha_1,\alpha_2,p)$ vanishing with α_2 and $\alpha_1\varrho$ since

$$S_{\widehat{v}}\setminus\bigcup_{j=1,\dots,J}Q_{j,\widehat{r}_j}\subset\Big(S_\varphi\setminus\bigcup_{j=1,\dots,J}\overline{Q}_j\Big)\cup\bigcup_{j:\,x_j\notin S_\varphi^{\rm frac}}(\Sigma^1_{j,\widehat{r}_j}\cup\Sigma^2_{j,\widehat{r}_j})\cup\bigcup_{j:\,x_j\notin S_\varphi^{\rm frac}}((Q_j\setminus Q_{j,\widehat{r}_j})\cap(\Gamma_j\cup T_j^{\gamma_j})$$

and from the properties of Γ_j, (2.32), (2.38), (2.48), (2.49). Therefore, adding the estimates of the $\mathcal{H}^1$-measures of $\partial^*\omega_j$ in (2.46) over j such that $x_j\notin S_\varphi^{\rm frac}$ plus $\partial^*\omega_j^\pm$ in (2.51) over j such that $x_j\in S_\varphi^{\rm frac}$ together with (2.58), we conclude that $\mathcal{H}^1(\partial^*\widehat{\omega})$ vanishes with α_2 and $\alpha_1\varrho$. In view of the definition of $\widetilde{\omega}$ (in particular of $\mathscr{Q}_b$) we get

$$\mathcal{H}^1(\partial\widetilde{\omega})\le\frac{40}{9}\eta\,\mathcal{H}^1\Big(S_{\widehat{\theta}}\setminus\bigcup_{j=1,\dots,J}Q_{j,\widehat{r}_j}\Big),$$

where above a factor 8 accounts for the overlapping of squares q''; as well, $\mathcal{H}^1(\partial\widetilde{\omega})$ vanishes with α_2 and $\alpha_1\varrho$ by (2.58).

Eventually, arguing as in [13, Step 3.1 in Theorem 5.1] for the cubes Q_j such that $x_j\in S_\varphi^{\rm frac}$ one proves that $\mathcal{H}^1(\Gamma^{\rm frac}\cap\{\theta^\pm\neq\varphi^\pm\})$ vanishes with α_2, while (again following [13, Step 3.3 in Theorem 5.1] with the full gradient in place of the symmetrized gradient) one deduces from the

last estimate in (2.54) that

$$\int_{\bigcup_{q\in\mathscr{Q}_g}\omega_q}|\nabla\theta|^p\,\mathrm{d}x\le 8c\int_{\bigcup_{q\in\mathscr{Q}_g}\omega_q}|\nabla\widehat{\theta}|^p\,\mathrm{d}x,$$

which together with the estimates on the gradients in (2.46) and (2.51) gives that

$$\int_{\widehat{\omega}}|\nabla\theta|^p\,\mathrm{d}x\le C'(\alpha_2,p)\int_{\widehat{\omega}}|\nabla\varphi|^p\,\mathrm{d}x,$$

for $C'(\alpha_2,p)$ a positive constant vanishing with α_2 and $\alpha_1\varrho$. Being $\{\nabla\varphi\neq\nabla\theta\}\subset\widetilde{\omega}\cup\widehat{\omega}$, $\theta=0$ in $\widetilde{\omega}$, and since the measure of $\widehat{\omega}$ vanishes with α_2 and $\alpha_1\varrho$, we obtain

$$\int_{\Omega}|\nabla\theta|^p\,\mathrm{d}x\le(1+C''(\alpha_2,\varrho,p))\int_{\Omega}|\nabla\varphi|^p\,\mathrm{d}x,$$

where $C''(\alpha_2,p)>0$ vanishes with α_2 and $\alpha_1\varrho$.

We conclude since α_1, α_2, ϱ may be fixed arbitrarily small. □

By arguing as in the proof of Theorem 2.1, using Proposition 2.4 in place of Proposition 2.5, one can show that also the following result holds true.

Theorem 2.6. *Let $\Omega\subset\mathbb{R}^2$ be a bounded open set of finite perimeter, $p\in(1,+\infty)$, $u\in GSBD^p(\Omega)$, and $\varepsilon>0$. Then there exist:*

- *closed sets $\Gamma^{\rm int}$, $\Gamma^{\rm frac}$, finite unions of disjoint C^1 curves;*
- *a set $\tilde{\omega}$, finite union of cubes;*
- *a set of finite perimeter $\widehat{\omega}$;*
- *a function $v\in GSBD^p(\Omega)\cap W^{1,p}(\Omega\setminus(\Gamma\cup\overline{\widetilde{\omega}});\mathbb{R}^2)$, where $\Gamma:=\Gamma^{\rm int}\cup\Gamma^{\rm frac}$;*

such that $\{\nabla u\neq\nabla v\}\subset\tilde{\omega}\cup\widehat{\omega}$, $\mathcal{L}^2(\{u\neq v\})<\varepsilon$, $v=0$ in $\tilde{\omega}$, $[v](x)\in\mathbb{Z}^2$ for $\mathcal{H}^1$-a.e. $x\in\Gamma^{\rm int}$, and

$$|\mathcal{H}^1(S_u^{\rm int})-\mathcal{H}^1(\Gamma^{\rm int})|+\mathcal{H}^1(S_u^{\rm frac}\triangle\Gamma^{\rm frac})+\mathcal{H}^1(\partial\tilde{\omega})+\mathcal{H}^1(\partial^*\widehat{\omega})\le\varepsilon,\qquad\int_{\Omega\setminus\tilde{\omega}}|e(v)|^p\,\mathrm{d}x\le(1+\varepsilon)\int_{\Omega}|e(u)|^p\,\mathrm{d}x,$$

where $S_u^{\rm frac}:=\{x\in S_u\colon[u]\notin\mathbb{Z}^2\}$. Moreover, $\mathcal{H}^1(\Gamma\cap\{v^\pm\neq u^\pm\})\le\varepsilon$, where $v^\pm$ and $u^\pm$ denote the traces of v and u on the two sides of Γ, and, if $u\in SBD^p(\Omega)$, then also $v\in SBD^p(\Omega)$.

3. Description of the problem

Let Ω be a bounded and open subset of $\mathbb{R}^2$ with Lipschitz continuous boundary and let $\Omega'\subset\subset\Omega$ be an open set. We introduce

$$\mathcal{AD}(\Omega,\Omega'):=\{u\in SBV^2(\Omega;\mathbb{S}^1)\,:\,\overline{S}_u\subset\overline{\Omega'}\},\tag{3.1}$$

where S_u denotes the jump set of u. For every $\varepsilon>0$, let $\mathcal{G}_\varepsilon:SBV^2(\Omega;\mathbb{S}^1)\to[0,\infty]$ be the functional defined by

$$\mathcal{G}_\varepsilon(u):=\begin{cases}\displaystyle\int_\Omega\frac{1}{2}|\nabla u|^2\,\mathrm{d}x+\frac{1}{\varepsilon}\mathcal{H}^1(\overline{S}_u) & \text{if } u\in\mathcal{AD}(\Omega,\Omega')\\ +\infty & \text{elsewhere in } SBV^2(\Omega;\mathbb{S}^1)\,.\end{cases}\tag{3.2}$$

In what follows, we will adopt also localized versions of the functional $\mathcal{G}_\varepsilon$; more precisely, for any $u\in\mathcal{AD}(\Omega,\Omega')$ and for any open set A with $\Omega'\subset\subset A\subset\subset\Omega$, we will denote by $\mathcal{G}_\varepsilon(u;A)$ the functional in (3.2) with Ω replaced by A.

Notice that, since $u\in H^1(\Omega\setminus\overline{\Omega}';\mathbb{S}^1)$, it follows that

$$\operatorname{supp}Ju\subseteq\overline{\Omega}'\qquad\text{for every } u\in\mathcal{AD}(\Omega,\Omega').\tag{3.3}$$

Indeed, let $\varphi \in C_c^\infty(\Omega \setminus \overline{\Omega}')$, and write

$$\begin{aligned}\langle Ju, \varphi\rangle_\Omega &= \frac{1}{2}\int_\Omega \frac{\partial \varphi}{\partial x_2}\,\mathrm{d}([u^1 \mathrm{D}_1 u^2] - [u^2 \mathrm{D}_1 u^1]) - \frac{1}{2}\int_\Omega \frac{\partial \varphi}{\partial x_1}\,\mathrm{d}([u^1 \mathrm{D}_2 u^2] - [u^2 \mathrm{D}_2 u^1])\\ &= \frac{1}{2}\int_{\Omega\setminus\overline{\Omega}'} \frac{\partial \varphi}{\partial x_2}\Big(u^1 \frac{\partial u^2}{\partial x_1} - u^2 \frac{\partial u^1}{\partial x_1}\Big)\,\mathrm{d}x - \frac{1}{2}\int_{\Omega\setminus\overline{\Omega}'} \frac{\partial \varphi}{\partial x_1}\Big(u^1 \frac{\partial u^2}{\partial x_2} - u^2 \frac{\partial u^1}{\partial x_2}\Big)\,\mathrm{d}x\\ &= \langle Ju, \varphi\rangle_{\Omega\setminus\overline{\Omega}'} = 0,\end{aligned}$$

where the last equality follows since $u \in H^1(\Omega\setminus\overline{\Omega}'; \mathbb{S}^1)$ has null distributional Jacobian determinant in $\Omega \setminus \overline{\Omega}'$.

3.1. **Γ-convergence in the subcritical regime.** We introduce the class of atomic measures, namely

$$X(\Omega) := \Big\{\mu \in \mathcal{M}(\Omega) \,:\, \mu = \sum_{n=1}^N z^n \delta_{x^n},\ x^n \in \Omega,\ z^n \in \mathbb{Z}\setminus\{0\},\ N \in \mathbb{N}\Big\}.$$

In [28, Theorem 3.1], the authors show that the rescaled functional $|\log\varepsilon|^{-1}\mathcal{G}_\varepsilon$ Γ-converges to the functional $\mathcal{F} : X(\Omega) \to \mathbb{R}^+$ defined as $\mathcal{F}(\mu) = \pi|\mu|(\Omega)$. Using a density argument, and in particular Proposition 2.3, this result can be easily extended to the following setting, where the energy functional does not take into account of the closure of the jump set: We introduce

$$\mathcal{F}_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega \frac{1}{2}|\nabla u|^2\,\mathrm{d}x + \frac{1}{\varepsilon}\mathcal{H}^1(S_u) & \text{if } u \in \mathcal{AD}(\Omega, \Omega')\\ +\infty & \text{elsewhere in } SBV^2(\Omega; \mathbb{S}^1)\,. \end{cases} \tag{3.4}$$

Then the following Γ-convergence result holds:

Theorem 3.1. *Let Ω and Ω' be as above; then it holds*

(i) *(Compactness) Let $\{u_\varepsilon\}_\varepsilon \subset SBV^2(\Omega; \mathbb{S}^1)$ be such that*

$$\sup_{\varepsilon>0} \frac{\mathcal{F}_\varepsilon(u_\varepsilon)}{|\log\varepsilon|} \le C, \tag{3.5}$$

for some $C > 0$. Then there exists $\mu \in X(\Omega)$ with $\operatorname{supp}\mu \subseteq \overline{\Omega}'$ such that, up to a subsequence, $\|Ju_\varepsilon - \pi\mu\|_{\mathrm{flat},\Omega} \to 0$ (as $\varepsilon \to 0$).

(ii) *(Γ-liminf inequality) For every $\mu \in X(\Omega)$ with $\operatorname{supp}\mu \subseteq \overline{\Omega}'$ and for every $\{u_\varepsilon\}_\varepsilon \subset SBV^2(\Omega; \mathbb{S}^1)$ such that $\|Ju_\varepsilon - \pi\mu\|_{\mathrm{flat},\Omega} \to 0$ (as $\varepsilon \to 0$), it holds*

$$\pi|\mu|(\Omega) \le \liminf_{\varepsilon\to 0} \frac{\mathcal{F}_\varepsilon(u_\varepsilon)}{|\log\varepsilon|}\,. \tag{3.6}$$

(iii) *(Γ-limsup inequality) For every $\mu \in X(\Omega)$ with $\operatorname{supp}\mu \subseteq \overline{\Omega}'$, there exists $\{u_\varepsilon\}_\varepsilon \subset SBV^2(\Omega; \mathbb{S}^1)$ with $\|Ju_\varepsilon - \pi\mu\|_{\mathrm{flat},\Omega} \to 0$ (as $\varepsilon \to 0$), such that*

$$\pi|\mu|(\Omega) \ge \limsup_{\varepsilon\to 0} \frac{\mathcal{F}_\varepsilon(u_\varepsilon)}{|\log\varepsilon|}\,. \tag{3.7}$$

Actually, by arguing as above and going through the proof of [28, Theorem 3.1], one can prove the following more general result.

Theorem 3.2. *Let Ω and Ω' be as above; and let $\{E_\varepsilon\}_\varepsilon \subset (0, +\infty)$ with $c|\log\varepsilon| \le E_\varepsilon \ll |\log\varepsilon|^2$ for some constant $c > 0$ (independent of ε). Then the following Γ-convergence result holds true.*

(i) *(Compactness) Let $\{u_\varepsilon\}_\varepsilon \subset SBV^2(\Omega; \mathbb{S}^1)$ be such that*

$$\sup_{\varepsilon>0} \frac{\mathcal{F}_\varepsilon(u_\varepsilon)}{E_\varepsilon} \le C,$$

for some $C > 0$. Then there exists $\mu \in X(\Omega)$ with $\operatorname{supp}\mu \subseteq \overline{\Omega}'$ such that, up to a subsequence, $\|\frac{|\log\varepsilon|}{E_\varepsilon} Ju_\varepsilon - \pi\mu\|_{\mathrm{flat},\Omega} \to 0$ (as $\varepsilon \to 0$).

(ii) *(Γ-liminf inequality) For every $\mu \in X(\Omega)$ with $\operatorname{supp}\mu \subseteq \overline{\Omega}'$ and for every $\{u_\varepsilon\}_\varepsilon \subset SBV^2(\Omega;\mathbb{S}^1)$ such that $\|\frac{|\log\varepsilon|}{E_\varepsilon}Ju_\varepsilon - \pi\mu\|_{\mathrm{flat},\Omega} \to 0$ (as $\varepsilon \to 0$), it holds*

$$\pi|\mu|(\Omega) \le \liminf_{\varepsilon\to 0} \frac{\mathcal{F}_\varepsilon(u_\varepsilon)}{E_\varepsilon}\,. \tag{3.8}$$

(iii) *(Γ-limsup inequality) For every $\mu \in X(\Omega)$ with $\operatorname{supp}\mu \subseteq \overline{\Omega}'$, there exists $\{u_\varepsilon\}_\varepsilon \subset SBV^2(\Omega;\mathbb{S}^1)$ with $\|\frac{|\log\varepsilon|}{E_\varepsilon}Ju_\varepsilon - \pi\mu\|_{\mathrm{flat},\Omega} \to 0$ (as $\varepsilon \to 0$), such that*

$$\pi|\mu|(\Omega) \ge \limsup_{\varepsilon\to 0} \frac{\mathcal{F}_\varepsilon(u_\varepsilon)}{E_\varepsilon}\,. \tag{3.9}$$

Proof of Theorem 3.1. Although the argument is standard, we briefly discuss how to prove points (i) and (ii), (iii) being identical to the case of [28]. Assume (3.5); by Proposition 2.3 (applied to the domain Ω'), for all $\varepsilon > 0$ we choose $\widehat{u}_\varepsilon$ such that

$$\begin{aligned} &\int_\Omega \frac{1}{2}|\nabla\widehat{u}_\varepsilon|^2 dx \le \int_\Omega \frac{1}{2}|\nabla u_\varepsilon|^2 dx + \varepsilon,\\ &\mathcal{H}^1(\overline{S}_{\widehat{u}_\varepsilon}) = \mathcal{H}^1(S_{\widehat{u}_\varepsilon}) \le \mathcal{H}^1(S_{u_\varepsilon}) + \varepsilon,\\ &\|J\widehat{u}_\varepsilon - Ju_\varepsilon\|_{\mathrm{flat},\Omega} \le \varepsilon, \end{aligned} \tag{3.10}$$

so that it follows

$$\sup_{\varepsilon>0} \frac{\mathcal{G}_\varepsilon(\widehat{u}_\varepsilon)}{|\log\varepsilon|} \le C + 1.$$

The compactness result in [28, Theorem 3.1 (i)] and the third condition in (3.10) imply (i). In a similar way also (ii) is a consequence of [28, Theorem 3.1 (ii)] and of the same density result. □

We do not discuss the proof of Theorem 3.2 since it follows from the same result with $\mathcal{G}_\varepsilon$ in place of $\mathcal{F}_\varepsilon$, which in turn has the same proof of [28, Theorem 3.1].

3.2. **Γ-convergence in the critical and supercritical regimes.** Our main results are the following.

Theorem 3.3. *The following Γ-convergence result holds true.*

(i) *(Compactness) Let $\{u_\varepsilon\}_\varepsilon \subset SBV^2(\Omega;\mathbb{S}^1)$ be such that*

$$\sup_{\varepsilon>0} \frac{\mathcal{F}_\varepsilon(u_\varepsilon)}{|\log\varepsilon|^2} \le C, \tag{3.11}$$

for some $C > 0$. Then there exist a measure $\mu \in \mathcal{M}(\Omega) \cap H^{-1}(\Omega)$ with $\operatorname{supp}\mu \subseteq \overline{\Omega}'$ and a map $T^{\mathrm{D}} \in L^2(\Omega;\mathbb{R}^2)$ with $-\operatorname{Div} T^{\mathrm{D}} = \pi\mu$ such that, up to a subsequence,

$$\Big\|\frac{Ju_\varepsilon}{\pi|\log\varepsilon|} - \mu\Big\|_{\mathrm{flat},\Omega} \to 0 \tag{FJ}$$

$$\frac{T^{\mathrm{D}}_{u_\varepsilon}}{|\log\varepsilon|} \rightharpoonup T^{\mathrm{D}} \ \text{ in } L^2(\Omega;\mathbb{R}^2)\,. \tag{ACJ}$$

(ii) *(Γ-liminf inequality) For every $(\mu, T^{\mathrm{D}}) \in \big(\mathcal{M}(\Omega) \cap H^{-1}(\Omega)\big) \times L^2(\Omega;\mathbb{R}^2)$ as in (i) and for every $\{u_\varepsilon\}_\varepsilon \subset SBV^2(\Omega;\mathbb{S}^1)$ satisfying* (FJ) *and* (ACJ)*, it holds*

$$\pi|\mu|(\Omega) + 2\int_\Omega |T^{\mathrm{D}}|^2\,\mathrm{d}x \le \liminf_{\varepsilon\to 0} \frac{\mathcal{F}_\varepsilon(u_\varepsilon)}{|\log\varepsilon|^2}\,. \tag{3.12}$$

(iii) *(Γ-limsup inequality) For every $(\mu, T^{\mathrm{D}}) \in (\mathcal{M}(\Omega) \cap H^{-1}(\Omega)) \times L^2(\Omega;\mathbb{R}^2)$ as in (i) there exists $\{u_\varepsilon\}_\varepsilon \subset SBV^2(\Omega;\mathbb{S}^1)$ satisfying* (FJ) *and* (ACJ)*, such that*

$$\pi|\mu|(\Omega) + 2\int_\Omega |T^{\mathrm{D}}|^2\,\mathrm{d}x \ge \limsup_{\varepsilon\to 0} \frac{\mathcal{F}_\varepsilon(u_\varepsilon)}{|\log\varepsilon|^2}\,. \tag{3.13}$$

Theorem 3.4. *Let $\{N_\varepsilon\}_{\varepsilon>0}$ be such that $|\log\varepsilon| \ll N_\varepsilon \ll \varepsilon^{-1}$. The following Γ-convergence result holds true.*

(i) *(Compactness) Let* $\{u_\varepsilon\}_\varepsilon \subset SBV^2(\Omega;\mathbb{S}^1)$ *be such that*

$$\sup_{\varepsilon>0} \frac{\mathcal{F}_\varepsilon(u_\varepsilon)}{N_\varepsilon^2} \leq C, \tag{3.14}$$

for some $C>0$ *. Then there exist a field* $T^{\mathrm{D}} \in L^2(\Omega;\mathbb{R}^2)$ *such that, up to a subsequence,* $\frac{T^{\mathrm{D}}_{u_\varepsilon}}{N_\varepsilon} \rightharpoonup T^{\mathrm{D}}$ *in* $L^2(\Omega;\mathbb{R}^2)$ *.*

(ii) *(*Γ*-liminf inequality) For every* $T^{\mathrm{D}} \in L^2(\Omega;\mathbb{R}^2)$ *and for every* $\{u_\varepsilon\}_\varepsilon \subset SBV^2(\Omega;\mathbb{S}^1)$ *with* $\frac{T^{\mathrm{D}}_{u_\varepsilon}}{N_\varepsilon} \rightharpoonup T^{\mathrm{D}}$ *in* $L^2(\Omega;\mathbb{R}^2)$ *, it holds*

$$2\int_\Omega |T^{\mathrm{D}}|^2 \,\mathrm{d}x \leq \liminf_{\varepsilon\to 0} \frac{\mathcal{F}_\varepsilon(u_\varepsilon)}{N_\varepsilon^2}\,. \tag{3.15}$$

(iii) *(*Γ*-limsup inequality) For every* $T^{\mathrm{D}} \in L^2(\Omega;\mathbb{R}^2)$ *there exists* $\{u_\varepsilon\}_\varepsilon \subset SBV^2(\Omega;\mathbb{S}^1)$ *with* $\frac{T^{\mathrm{D}}_{u_\varepsilon}}{N_\varepsilon} \rightharpoonup T^{\mathrm{D}}$ *in* $L^2(\Omega;\mathbb{R}^2)$ *such that*

$$2\int_\Omega |T^{\mathrm{D}}|^2 \,\mathrm{d}x \geq \limsup_{\varepsilon\to 0} \frac{\mathcal{F}_\varepsilon(u_\varepsilon)}{N_\varepsilon^2}\,. \tag{3.16}$$

By using the density result in Proposition 2.3, one can show that Theorems 3.3 and 3.4 hold true also when replacing $\mathcal{F}_\varepsilon$ with $\mathcal{G}_\varepsilon$.

In order to prove Theorems 3.3 and 3.4, we will make use of the corresponding core radius approach results that for the sake of completeness we state and prove in Section 4 below.

4. Core radius approach

We first introduce some notation. Let $V \subset \mathbb{R}^2$ be a bounded and open set with Lipschitz continuous boundary. For every finite family $\mathscr{B} := \{B^n\}_{n=1,\dots,N}$ (with $N \in \mathbb{N}$) of open balls with mutually disjoint closures we set

$$V(\mathscr{B}) := V \setminus \bigcup_{n=1}^N \overline{B}^n,$$

and we denote by $\mathcal{R}ad(\mathscr{B})$ the sum of the radii of the balls B^n, namely

$$\mathcal{R}ad(\mathscr{B}) := \sum_{n=1}^N r(B^n)\,,$$

where $r(B)$ denotes the radius of the ball B. Moreover, for every $\mu \in X(V)$ with $\mu \neq 0$ of the form

$$\mu := \sum_{n=1}^N z^n \delta_{x(B^n)} \qquad \text{with } z^n \in \mathbb{Z}\setminus\{0\}\,, \tag{4.1}$$

we set

$$\mathscr{A}(\mathscr{B},\mu,V) := \{u \in H^1(V(\mathscr{B});\mathbb{S}^1) \,:\, \deg(u,\partial B^n) = z^n \text{ for every } n=1,\dots,N\}\,. \tag{4.2}$$

Here and below, $x(B)$ denotes the center of the ball B.

Now we provide the notion of merging procedure used in the ball construction introduced by [58] and [44] (see also [27]). Although we do not re-prove the ball construction, stated in Proposition 4.2, we need it to start the proof of Theorem 3.3.

Definition 4.1 (Merging procedure)**.** Given a finite family $\mathscr{B} = \{B_{r^i}(x^i)\}_{i=1,\dots,I}$ ($I \in \mathbb{N}$) of balls in $\mathbb{R}^2$, we define a new family $\widehat{\mathscr{B}}$ as follows. If the closures of two balls in $\mathscr{B}$ are not disjoint, then we replace the two balls with a unique ball which contains both of them and has radius less than or equal to the sum of the radii of the original balls. After this, we repeat this replacement recursively, until as all the balls in the family have mutually disjoint closures. The final family is $\widehat{\mathscr{B}}$. The procedure of passing from $\mathscr{B}$ to $\widehat{\mathscr{B}}$ is called merging procedure applied to $\mathscr{B}$. Notice that a merging procedure does not increase the sum of all the radii of the balls in the family.

The following result is proven in [27, Proposition 2.2].

Proposition 4.2. *Let $V \subset \mathbb{R}^2$ be a bounded and open set with Lipschitz continuous boundary, let $\mathscr{B}$ be a finite family of open balls with mutually disjoint closures in $\mathbb{R}^2$, let $\mu \in X(V)$ be of the form (4.1), and let $u \in \mathscr{A}(\mathscr{B}, \mu, V)$. Then, there exists a one-parameter family of open balls $\mathscr{B}(t)$ with $t \geq 0$ such that, setting $U(t) := \bigcup_{B \in \mathscr{B}(t)} B$, the following properties hold true:*

(1) $\mathscr{B}(0) = \mathscr{B}$;
(2) $U(t_1) \subset U(t_2)$ *for any* $0 \leq t_1 < t_2$;
(3) *the balls in* $\mathscr{B}(t)$ *are have mutually disjoint closures for every* $t > 0$;
(4) *for any* $0 \leq t_1 < t_2$ *and for any open set* $A \subseteq V$,

$$\frac{1}{2} \int_{(U(t_2) \setminus \overline{U}(t_1)) \cap A} |\nabla u|^2 \, \mathrm{d}x \geq \pi \sum_{\substack{B \in \mathscr{B}(t_2) \\ B \subseteq A}} |\mu(B)| \log \frac{1 + t_2}{1 + t_1} ;$$

(5) *for every* $t > 0$: $\displaystyle\sum_{B \in \mathscr{B}(t)} r(B) \leq (1 + t) \sum_{B \in \mathscr{B}} r(B)$, *where* $r(B)$ *denotes the radius of* B.

For every $\mathscr{B}$ and μ as in Proposition 4.2, for every $t > 0$, we set $\mathscr{C}(t) := \{B \in \mathscr{B}(t) : \overline{B} \subset V\}$ and we define

$$\widetilde{\mu} := \sum_{B \in \mathscr{C}(1)} \mu(B) \delta_{x(B)} . \tag{4.3}$$

We can now state the crucial result which will be the starting point of the proof of Theorem 3.3.

Theorem 4.3. *Let V be a bounded open set with Lipschitz continuous boundary. For every $\varepsilon > 0$ let $\mathscr{B}_\varepsilon := \{B^n_\varepsilon\}_{n=1,\dots,N_\varepsilon}$ (with $N_\varepsilon \in \mathbb{N}$) be a (finite) family of open balls having mutually disjoint closures with $\mathcal{R}ad(\mathscr{B}_\varepsilon) \to 0$ as $\varepsilon \to 0$, $\mu_\varepsilon := \sum_{n=1}^{N_\varepsilon} z^n_\varepsilon \delta_{x(B^n_\varepsilon)}$ with $z^n_\varepsilon \in \mathbb{Z} \setminus \{0\}$ for every $n = 1, \dots, N_\varepsilon$. Let moreover $\{u_\varepsilon\}_\varepsilon$ be such that $u_\varepsilon \in \mathscr{A}(\mathscr{B}_\varepsilon, \mu_\varepsilon, V)$. Assume that*

$$\sup_{\varepsilon > 0} \frac{1}{2|\log \mathcal{R}ad(\mathscr{B}_\varepsilon)|^2} \int_{V(\mathscr{B}_\varepsilon)} |\nabla u_\varepsilon|^2 \, \mathrm{d}x \leq C , \tag{4.4}$$

for some constant $C > 0$ independent of ε. Then, the following facts hold true.

(i) *Let $\widetilde{\mu}_\varepsilon$ be the measures defined in (4.3) with $\mathscr{C}(1) = \mathscr{C}_\varepsilon(1) = \{B \in \mathscr{B}_\varepsilon(1) : \overline{B} \subset V\}$; then $|\widetilde{\mu}_\varepsilon|(V) \leq C|\log \mathcal{R}ad(\mathscr{B}_\varepsilon)|^2$ for all $\varepsilon > 0$ with a constant $C > 0$ independent of ε, and there exist a measure $\mu \in \mathcal{M}(V)$ and a function $\lambda \in L^2(V; \mathbb{R}^2)$ such that, up to a subsequence, as $\varepsilon \to 0$*

$$\frac{\widetilde{\mu}_\varepsilon \llcorner V}{|\log \mathcal{R}ad(\mathscr{B}_\varepsilon)|} \overset{\text{flat}}{\to} \mu, \tag{4.5}$$

$$\frac{\lambda_{u_\varepsilon} \chi_{V(\mathscr{B}_\varepsilon)}}{|\log \mathcal{R}ad(\mathscr{B}_\varepsilon)|} \rightharpoonup \lambda \qquad \textit{weakly in } L^2(V; \mathbb{R}^2) ; \tag{4.6}$$

(ii) $\pi|\mu|(V) + 2 \int_V |\lambda|^2 \, \mathrm{d}x \leq \liminf_{\varepsilon \to 0} \frac{1}{2|\log \mathcal{R}ad(\mathscr{B}_\varepsilon)|^2} \int_{V(\mathscr{B}_\varepsilon)} |\nabla u_\varepsilon|^2 \, \mathrm{d}x$.

Notice that, as $u_\varepsilon \in \mathscr{A}(\mathscr{B}_\varepsilon, \mu_\varepsilon, V)$, we have $\lambda_{u_\varepsilon} \in L^2(V(\mathscr{B}_\varepsilon); \mathbb{R}^2)$, where λ_u is defined in (1.4). In formula (4.6), symbol $\lambda_{u_\varepsilon} \chi_{V(\mathscr{B}_\varepsilon)}$ denotes the extension of λ_{u_ε} to the constant $(0, 0)$ in $V \setminus \overline{V(\mathscr{B}_\varepsilon)}$.

Proof. We start by proving (i). Our proof closely resembles that of [2, Theorem 3.2] where the compactness result is proven in the energy regime $|\log \mathcal{R}ad(\mathscr{B}_\varepsilon)|$.

For every $0 < p < 1$ and for every $\varepsilon > 0$ we set

$$t^p_\varepsilon := \frac{1}{\mathcal{R}ad^{1-p}(\mathscr{B}_\varepsilon)} - 1 , \qquad \nu^p_\varepsilon := \nu[t^p_\varepsilon] , \tag{4.7}$$

where we have set, for $t \geq 0$,

$$\nu[t] := \sum_{B \in \mathscr{C}_\varepsilon(t)} \mu_\varepsilon(B) \delta_{x(B)} .$$

Fix $0 < p < 1$. Then, by applying Proposition 4.2(4) (with $t_1 = 0$ and $t_2 = 1$ and $t_2 = t_\varepsilon^p$) and by the energy bound (4.4), we have that

$$|\widetilde{\mu}_\varepsilon|(V) \le C|\log \mathcal{R}ad(\mathscr{B}_\varepsilon)|^2\,, \qquad |\nu_\varepsilon^p|(V) \le C(1-p)^{-1}|\log \mathcal{R}ad(\mathscr{B}_\varepsilon)|\,, \tag{4.8}$$

whence we deduce the first statement in claim (i) and the existence of a measure $\mu^p \in \mathcal{M}_b(V)$ such that (up to a not-relabeled subsequence)

$$\frac{\nu_\varepsilon^p}{|\log \mathcal{R}ad(\mathscr{B}_\varepsilon)|} \overset{*}{\rightharpoonup} \mu^p \qquad \text{as } \varepsilon \to 0\,. \tag{4.9}$$

Now we prove that

$$\frac{1}{|\log \mathcal{R}ad(\mathscr{B}_\varepsilon)|}\big(\widetilde{\mu}_\varepsilon - \nu_\varepsilon^p\big) \overset{\text{flat}}{\to} 0 \qquad \text{for every } 0 < p < 1\,, \tag{4.10}$$

from which we deduce also that $\mu^p \equiv \mu$ for any $0 < p < 1$. To this purpose, we first observe that $\widetilde{\mu}_\varepsilon(B) = \nu_\varepsilon^p(B)$ for any $B \in \mathscr{C}_\varepsilon(t_\varepsilon^p) = \{B \in \mathscr{B}_\varepsilon(t_\varepsilon^p) : \overline{B} \subset V\}$; therefore, using (4.8) and Proposition 4.2(5) together with the very definition of t_ε^p, for every sequence $\{\varphi_\varepsilon\}_\varepsilon \subset C_{\rm c}^{0,1}(V)$ with $\|\varphi_\varepsilon\|_{C^{0,1}} \le 1$, we have

$$\begin{aligned}
&\frac{1}{|\log \mathcal{R}ad(\mathscr{B}_\varepsilon)|}\big|\big\langle \widetilde{\mu}_\varepsilon - \nu_\varepsilon^p, \varphi_\varepsilon\big\rangle\big| \\
&\le \frac{1}{|\log \mathcal{R}ad(\mathscr{B}_\varepsilon)|}\left|\sum_{B\in\mathscr{C}_\varepsilon(t_\varepsilon^p)} \int_B \Big(\varphi_\varepsilon - \fint_B \varphi_\varepsilon\,\mathrm{d}x\Big)\,\mathrm{d}(\widetilde{\mu}_\varepsilon - \nu_\varepsilon^p)\right| \\
&\quad + \frac{1}{|\log \mathcal{R}ad(\mathscr{B}_\varepsilon)|}\left|\sum_{B\in\mathscr{B}_\varepsilon(t_\varepsilon^p)\setminus\mathscr{C}_\varepsilon(t_\varepsilon^p)} \int_{B\cap V} \varphi_\varepsilon\,\mathrm{d}(\widetilde{\mu}_\varepsilon - \nu_\varepsilon^p)\right| \\
&\le \frac{1}{|\log \mathcal{R}ad(\mathscr{B}_\varepsilon)|}\sum_{B\in\mathscr{B}_\varepsilon(t_\varepsilon^p)} \big(\sup_{B\cap V}\varphi_\varepsilon - \inf_{B\cap V}\varphi_\varepsilon\big)\big(|\widetilde{\mu}_\varepsilon|(B) + |\nu_\varepsilon^p|(B)\big) \\
&\le \frac{1}{|\log \mathcal{R}ad(\mathscr{B}_\varepsilon)|}\sum_{B\in\mathscr{B}_\varepsilon(t_\varepsilon^p)} \mathrm{diam}(B)\big(|\widetilde{\mu}_\varepsilon|(V) + |\nu_\varepsilon^p|(V)\big) \\
&\le C\mathcal{R}ad^p(\mathscr{B}_\varepsilon)|\log \mathcal{R}ad(\mathscr{B}_\varepsilon)|\,,
\end{aligned} \tag{4.11}$$

whence (4.10) follows. We highlight that the second inequality in (4.11) follows from the fact that for every $B \in \mathscr{B}_\varepsilon(t_\varepsilon^p) \setminus \mathscr{C}_\varepsilon(t_\varepsilon^p)$ we have that $\inf_{B\cap V}\varphi_\varepsilon \le 0$ so that

$$\int_{B\cap V} \varphi_\varepsilon\,\mathrm{d}(\widetilde{\mu}_\varepsilon - \nu_\varepsilon^p) \le \sup_{B\cap V}\varphi_\varepsilon\big(|\widetilde{\mu}_\varepsilon|(V) + |\nu_\varepsilon^p|(V)\big) \le \big(\sup_{B\cap V}\varphi_\varepsilon - \inf_{B\cap V}\varphi_\varepsilon\big)\big(|\widetilde{\mu}_\varepsilon|(V) + |\nu_\varepsilon^p|(V)\big).$$

Moreover, by the very definition of λ_{u_ε} in (1.4) and by the energy bound (4.4), we immediately have that

$$\frac{1}{2}\int_{V(\mathscr{B}_\varepsilon)} |2\lambda_{u_\varepsilon}|^2\,\mathrm{d}x = \frac{1}{2}\int_{V(\mathscr{B}_\varepsilon)} |\nabla u_\varepsilon|^2\,\mathrm{d}x \le C|\log \mathcal{R}ad(\mathscr{B}_\varepsilon)|^2\,, \tag{4.12}$$

thus, up to extracting a further subsequence, there exists $\lambda \in L^2(V;\mathbb{R}^2)$ such that (4.6) holds. Notice that, for $p \in (0,1)$ fixed, since $|U(t_\varepsilon^p)| \to 0$ as $\varepsilon \to 0$, we also deduce

$$\frac{\lambda_{u_\varepsilon}}{|\log \mathcal{R}ad(\mathscr{B}_\varepsilon)|}\chi_{V(\mathscr{B}_\varepsilon(t_\varepsilon^p))} \rightharpoonup \lambda \qquad \text{weakly in } L^2(V;\mathbb{R}^2). \tag{4.13}$$

Now we prove (ii). To this end, let $p \in (0,1)$ be fixed; by (4.9) and by Proposition 4.2(4), we get

$$\begin{aligned}
\liminf_{\varepsilon\to 0}\frac{1}{2|\log \mathcal{R}ad(\mathscr{B}_\varepsilon)|^2}\int_{(U(t_\varepsilon^p)\setminus U(0))\cap V} |\nabla u_\varepsilon|^2\,\mathrm{d}x &\ge \pi(1-p)\liminf_{\varepsilon\to 0}\frac{|\nu_\varepsilon^p|(V)}{|\log \mathcal{R}ad(\mathscr{B}_\varepsilon)|} \\
&\ge \pi(1-p)|\mu|(V)\,.
\end{aligned} \tag{4.14}$$

Furthermore, by (4.13), we have that

$$\liminf_{\varepsilon\to 0}\frac{1}{2|\log \mathcal{R}ad(\mathscr{B}_\varepsilon)|^2}\int_{V(\mathscr{B}_\varepsilon(t_\varepsilon^p))}|\nabla u_\varepsilon|^2\,\mathrm{d}x = \liminf_{\varepsilon\to 0}\frac{1}{2}\int_V\Big|\frac{2\lambda_{u_\varepsilon}}{|\log \mathcal{R}ad(\mathscr{B}_\varepsilon)|}\Big|^2\chi_{V(\mathscr{B}_\varepsilon(t_\varepsilon^p))}\,\mathrm{d}x$$
$$\geq 2\int_V|\lambda|^2\,\mathrm{d}x\,,$$

which, together with (4.14) (letting $p\to 0$), yields (ii). □

5. Proof of Theorem 3.3

This section is devoted to the proof of Theorem 3.3. Using the density argument as in the proof of Theorem 3.1, we can prove Theorem 3.3 with $\mathcal{F}_\varepsilon$ replaced by $\mathcal{G}_\varepsilon$ in (3.2).

The proofs of the compactness and of the lower bound are addressed in Subsection 5.1 and closely resemble those in the $|\log\varepsilon|$ regime treated in [28], whereas the construction of the recovery sequence is provided in Subsection 5.2.

5.1. **Proof of compactness and lower bound.** By the energy bound (3.11), together with Corollary 2.2, we have that

$$\mathcal{H}^1(\overline{S}_{u_\varepsilon})\leq C\varepsilon|\log\varepsilon|^2\,, \tag{5.1}$$

for every $\varepsilon>0$. By the very definition of Hausdorff measure, since $\overline{S}_{u_\varepsilon}$ is compact, there exists a finite family $\mathscr{B}_\varepsilon$ of open balls (in $\mathbb{R}^2$) such that $\overline{S}_{u_\varepsilon}\subset\bigcup_{B\in\mathscr{B}_\varepsilon}B$ and $\mathcal{R}ad(\mathscr{B}_\varepsilon)\leq C\varepsilon|\log\varepsilon|^2$. Notice that we can always assume (just by enlarging an arbitrarily chosen ball in $\mathscr{B}_\varepsilon$) that $\mathcal{R}ad(\mathscr{B}_\varepsilon)\geq\varepsilon$, so that, from now on

$$\varepsilon\leq\mathcal{R}ad(\mathscr{B}_\varepsilon)\leq C\varepsilon|\log\varepsilon|^2\,, \tag{5.2}$$

for some $C>0$. Moreover, by construction,

$$u_\varepsilon\in H^1(\Omega(\mathscr{B}_\varepsilon);\mathbb{S}^1)\,, \tag{5.3}$$

where we recall that $\Omega(\mathscr{B}_\varepsilon):=\Omega\setminus\bigcup_{B\in\mathscr{B}_\varepsilon}\overline{B}$. By (5.2) and recalling that $\overline{S}_{u_\varepsilon}\subset\overline{\Omega'}$, we can assume that, for ε small enough,

$$\bigcup_{B\in\mathscr{B}_\varepsilon}\overline{B}\subset\Omega\,. \tag{5.4}$$

Up to applying a merging procedure (as described in Definition 4.1) to the balls in $\mathscr{B}_\varepsilon$, we can assume without loss of generality that these balls having mutually disjoint closures, and still satisfy (5.2) and (5.4). For $\varepsilon>0$ small enough we set

$$\mu_\varepsilon:=\sum_{B\in\mathscr{B}_\varepsilon}\deg(u_\varepsilon,\partial B)\delta_{x(B)}\,. \tag{5.5}$$

By (3.11), (5.2), and (5.3), for ε small enough it holds

$$\frac{1}{2}\int_{\Omega(\mathscr{B}_\varepsilon)}|\nabla u_\varepsilon|^2\,\mathrm{d}x\leq\mathcal{G}_\varepsilon(u_\varepsilon)\leq C|\log\varepsilon|^2\leq C|\log\mathcal{R}ad(\mathscr{B}_\varepsilon)|^2\,. \tag{5.6}$$

Therefore we can apply Theorem 4.3 to the family $\{(\mathscr{B}_\varepsilon;\mu_\varepsilon)\}_\varepsilon$. Notice that, in view of the very definition of $\mathcal{G}_\varepsilon$, we have that also the family $\mathscr{B}_\varepsilon(1)$ satisfies (5.4) (for ε small enough), so that $\mathscr{C}_\varepsilon(1)\equiv\mathscr{B}_\varepsilon(1)$. Setting

$$\widetilde{\mu}_\varepsilon:=\sum_{B\in\mathscr{B}_\varepsilon(1)}\mu_\varepsilon(B)\delta_{x(B)}\,,$$

by Theorem 4.3(i) (more precisely, by (4.5)), using (5.2), we have that, up to a subsequence,

$$\frac{\widetilde{\mu}_\varepsilon}{|\log\varepsilon|}\overset{\text{flat}}{\to}\mu\,, \tag{5.7}$$

for some $\mu \in \mathcal{M}(\Omega)$. By construction, $\operatorname{supp}\mu \subseteq \overline{\Omega'}$. Moreover, by arguing verbatim as in the proof of [28, formula (3.17)] one can prove that

$$\frac{1}{|\log\varepsilon|}\|Ju_\varepsilon - \pi\widetilde{\mu}_\varepsilon\|_{\mathrm{flat},\Omega} \to 0\,,$$

which, combined with (5.7), yields (FJ). Furthermore, by (1.10) and by (3.11), we have that

$$\frac{1}{|\log\varepsilon|^2}\int_\Omega |2T^{\mathrm{D}}_{u_\varepsilon}|^2\,\mathrm{d}x = \frac{1}{|\log\varepsilon|^2}\int_\Omega |\nabla u_\varepsilon|^2\,\mathrm{d}x \le C\,,$$

so that, up to a further subsequence,

$$\frac{T^{\mathrm{D}}_{u_\varepsilon}}{|\log\varepsilon|} \rightharpoonup T^{\mathrm{D}} \qquad \text{in } L^2(\Omega;\mathbb{R}^2)\,, \tag{5.8}$$

for some field $T^{\mathrm{D}} \in L^2(\Omega;\mathbb{R}^2)$. This proves (ACJ).

It remains to show that $-\mathrm{Div}\,T^{\mathrm{D}} = \pi\mu$, which will imply also that $\mu \in H^{-1}(\Omega)$. To this end, let $\varphi \in C^\infty_{\mathrm{c}}(\Omega)$; then, by (1.10),

$$\langle Ju_\varepsilon, \varphi\rangle = \langle T_{u_\varepsilon}, \nabla\varphi\rangle = \langle T^{\mathrm{D}}_{u_\varepsilon}, \nabla\varphi\rangle + \langle T^{S}_{u_\varepsilon}, \nabla\varphi\rangle\,, \tag{5.9}$$

for every $\varepsilon > 0$. By (ACJ), we have that

$$\frac{1}{|\log\varepsilon|}\langle T^{\mathrm{D}}_{u_\varepsilon}, \nabla\varphi\rangle \to \langle T^{\mathrm{D}}, \nabla\varphi\rangle \qquad \text{as } \varepsilon \to 0\,; \tag{5.10}$$

moreover, by (1.10), (3.11), and (5.1), we have that

$$\big|\langle T^{\mathrm{S}}_{u_\varepsilon}, \nabla\varphi\rangle\big| \le C\varepsilon\|\nabla\varphi\|_{L^\infty}|\log\varepsilon|^2\,. \tag{5.11}$$

By (FJ), (5.9), (5.10), and (5.11),

$$\langle \pi\mu, \varphi\rangle = \lim_{\varepsilon\to 0}\frac{1}{|\log\varepsilon|}\langle Ju_\varepsilon, \varphi\rangle = \lim_{\varepsilon\to 0}\frac{1}{|\log\varepsilon|}\langle T^{\mathrm{D}}_{u_\varepsilon}, \nabla\varphi\rangle = \langle T^{\mathrm{D}}, \nabla\varphi\rangle = \langle -\mathrm{Div}T^{\mathrm{D}}, \phi\rangle\,,$$

thus concluding the proof of (i).

Now we prove the lower bound (ii). We can assume without loss of generality that (3.11) holds true. By the first inequality in (5.6) and by Theorem 4.3(ii) we have immediately that

$$\liminf_{\varepsilon\to 0}\frac{\mathcal{G}_\varepsilon(u_\varepsilon)}{|\log\varepsilon|^2} \ge \pi|\mu|(\Omega) + 2\int_\Omega |T^{\mathrm{D}}|^2\,\mathrm{d}x\,,$$

where we have used also that the function T^{D} coincides with the field λ in Theorem 4.3. The claim thus follows.

5.2. **Proof of the upper bound.** In order to construct the recovery sequence, we first introduce some notation. Let $r > 0$ be fixed. For every finite sum of Dirac deltas $\mu := \sum_{n=1}^N \delta_{x^n}$ with $|x^{n_1} - x^{n_2}| \ge 2r$ for $n_1 \ne n_2$ and for every $0 < \rho < r$ we set

$$\widehat{\mu}^\rho := \frac{1}{2\pi\rho}\sum_{n=1}^N \mathcal{H}^1 \llcorner \partial B_\rho(x^n)\,, \qquad \widetilde{f}^\rho := \frac{1}{\pi\rho^2}\sum_{n=1}^N \chi_{B_\rho(x^n)}\,, \qquad \text{and} \qquad \widetilde{\mu}^\rho := \widetilde{f}^\rho\,\mathrm{d}x\,. \tag{5.12}$$

For every $r > 0$ and for every $z \in \mathbb{R}^2$, we recall that $Q_r(z)$ denotes the (open) square centered at z with sides parallel to the cartesian axes and side-length equal to $2r$, i.e., $Q_r(z) := z + Q_r(0)$, with $Q_r(0) := (-r, r)^2$.

Lemma 5.1. *Let $\mu := \sum_{l=1}^L m^l\chi_{\omega^l}\mathrm{d}x$, where $L \in \mathbb{N}$, $m^l \in \mathbb{R}$ and $\{\omega^l\}_{l=1,\dots,L}$ is a partition of Ω into sets with Lipschitz continuous boundary. Let $N_\varepsilon \to +\infty$ as $\varepsilon \to 0$. For every $\varepsilon > 0$ and for every $l = 1,\dots,L$ with $m^l \ne 0$, set*

$$r^l_\varepsilon := \frac{1}{2\sqrt{N_\varepsilon|m^l|}}. \tag{5.13}$$

For every $l=1,\dots,L$ with $m^l\neq 0$, let $\mathcal{Z}^l_\varepsilon:=\{z\in 2r^l_\varepsilon\mathbb{Z}^2\,:\,\overline{Q}_{r^l_\varepsilon}(z)\subset\omega^l\}$ and $N^l_\varepsilon:=\sharp\mathcal{Z}^l_\varepsilon$. Then, for every $l=1,\dots,L$ with $m^l\neq 0$,

$$\frac{N^l_\varepsilon}{N_\varepsilon}\to|m^l||\omega^l|\qquad \text{as } \varepsilon\to 0\,. \tag{5.14}$$

Moreover, setting $\mu^l_\varepsilon:=\sum_{n=1}^{N^l_\varepsilon}\delta_{x^{l,n}_\varepsilon}$ (where $\{x^{l,n}_\varepsilon\}_{n=1}^{N^l_\varepsilon}$ is the set of points in $\mathcal{Z}^l_\varepsilon$) for every $l=1,\dots,L$ with $m^l\neq 0$, $\mu^l_\varepsilon\equiv 0$ whenever $m^l=0$, and $\mu_\varepsilon:=\sum_{l=1}^L\mu^l_\varepsilon$, we have that the sequence $\{\mu_\varepsilon\}_\varepsilon\subset\mathcal{M}(\Omega)$ satisfies

(a) *for every $\varepsilon>0$ and for every $l=1,\dots,L$ with $m^l\neq 0$ and $n=1,\dots,N^l_\varepsilon$, the balls $B_{r^l_\varepsilon}(x^{l,n}_\varepsilon)$ are pairwise disjoint and contained in ω^l;*

(b) *$\frac{\mu_\varepsilon}{N_\varepsilon}\overset{*}{\rightharpoonup}\mu$ in $\mathcal{M}(\Omega)$ as $\varepsilon\to 0$;*

(c) *$\left\|\frac{\widetilde{\mu}_\varepsilon}{N_\varepsilon}-\mu\right\|_{H^{-1}(\Omega)}\le CN_\varepsilon^{-\frac14}$ (for ε small enough),*

where $\widetilde{\mu}_\varepsilon:=\sum_{l=1}^L\widetilde{\mu}^{l,r^l_\varepsilon}_\varepsilon$, with $\widetilde{\mu}^{l,r^l_\varepsilon}_\varepsilon$ defined as in (5.12) starting from μ^l_ε.

Proof. For every $l=1,\dots,L$ with $m^l\neq 0$ we set $\overline{\omega}^l_\varepsilon:=\bigcup_{z\in\mathcal{Z}^l_\varepsilon}\overline{Q}_{r^l_\varepsilon}(z)$ and we denote by $\omega^l_\varepsilon:=\mathrm{int}(\overline{\omega}^l_\varepsilon)$ the set of interior points of $\overline{\omega}^l_\varepsilon$. We set $R_\varepsilon:=\max\{r^l_\varepsilon\,:\,l=1,\dots,L,\ m^l\neq 0\}$, and we notice that $R_\varepsilon\to 0$ as $\varepsilon\to 0$.

Property (5.14) is straightforward. Indeed, let $l\in\{1,\dots,L\}$ with $m^l\neq 0$; setting $(\partial\omega^l)_{2r^l_\varepsilon}:=\{x\in\omega^l:\mathrm{dist}(x,\partial\omega^l)<2r^l_\varepsilon\}$, it is sufficient to observe that $\left|(\omega^l\setminus\overline{\omega}^l_\varepsilon)\setminus(\partial\omega^l)_{2r^l_\varepsilon}\right|=0$, and hence, by the Lipschitz continuity of $\partial\omega^l$,

$$|\omega^l\setminus\overline{\omega}^l_\varepsilon|=\mathrm{O}(r^l_\varepsilon)=\mathrm{O}(R_\varepsilon)\,, \tag{5.15}$$

where $\limsup_{\varepsilon\to 0}\mathrm{O}(R_\varepsilon)R_\varepsilon^{-1}<C<+\infty$. On the other hand, $|\overline{\omega}^l_\varepsilon||m^l|=4(r^l_\varepsilon)^2N^l_\varepsilon|m^l|=\frac{N^l_\varepsilon}{N_\varepsilon}$, which, together with (5.15), yields $|\omega^l|=|\omega^l\setminus\overline{\omega}^l_\varepsilon|+|\overline{\omega}^l_\varepsilon|=\frac{\mathrm{O}(R_\varepsilon)}{|m^l|}+\frac{N^l_\varepsilon}{|m^l|N_\varepsilon}$, and hence (5.14).

Property (a) follows by construction. Now we prove property (b). Let $\phi\in C_{\mathrm{c}}(\Omega)$, and let m_ϕ be the modulus of continuity of ϕ. Then, setting

$$M:=\max\{|m^l|:l=1,\dots,L\}, \tag{5.16}$$

we have that, as $\varepsilon\to 0$,

$$\begin{aligned}\left|\left\langle\frac{\mu_\varepsilon}{N_\varepsilon}-\mu,\phi\right\rangle_\Omega\right|&\le\sum_{\substack{l=1\\ m^l\neq 0}}^L m^l\sum_{n=1}^{N^l_\varepsilon}\int_{Q_{r^l_\varepsilon}(x^{l,n}_\varepsilon)}|\phi(x^{l,n}_\varepsilon)-\phi(x)|\,\mathrm{d}x+\sum_{\substack{l=1\\ m^l\neq 0}}^L m^l\int_{\omega^l\setminus\overline{\omega}^l_\varepsilon}|\phi(x)|\,\mathrm{d}x\\&\le 4M\sum_{\substack{l=1\\ m^l\neq 0}}^L N^l_\varepsilon(r^l_\varepsilon)^2\max_{t\in[0,2\sqrt{2}r^l_\varepsilon]}m_\phi(t)+M\|\phi\|_{L^\infty}\sum_{\substack{l=1\\ m^l\neq 0}}^L|\omega^l\setminus\overline{\omega}^l_\varepsilon|\to 0,\end{aligned}$$

where we have used (5.13), (5.14), and (5.15); this proves (b).

We are left with the proof of (c). We set

$$\omega:=\bigcup_{\substack{l=1\\ m^l\neq 0}}^L\omega^l,\qquad \omega_\varepsilon:=\bigcup_{\substack{l=1\\ m^l\neq 0}}^L\omega^l_\varepsilon,\qquad \eta_\varepsilon:=(\frac{\widetilde{\mu}^{r_\varepsilon}_\varepsilon}{N_\varepsilon}-\mu)\chi_{\omega_\varepsilon},$$

so that

$$\frac{\widetilde{\mu}^{r_\varepsilon}_\varepsilon}{N_\varepsilon}-\mu=\eta_\varepsilon-\mu\chi_{\omega\setminus\omega_\varepsilon}\,. \tag{5.17}$$

Let $\bar{v}\in H^1(Q_1(0))$ be a solution to

$$\begin{cases}\Delta v=\frac{4}{\pi}\chi_{B_1}-1 & \text{in } Q_1(0)\\ \partial_\nu v=0 & \text{on } \partial Q_1(0)\,.\end{cases}$$

We then define for every $\varepsilon > 0$ and for every $l = 1, \dots, L$, and $n = 1, \dots, N_\varepsilon^l$,

$$v_\varepsilon^{l,n}(x) := \frac{1}{2N_\varepsilon}\bar{v}\Big(\frac{x - x_\varepsilon^{l,n}}{r_\varepsilon^l}\Big);$$

notice that $\nabla v_\varepsilon^{l,n}(x) = \sqrt{\frac{m^l}{N_\varepsilon}}\nabla\bar{v}(\frac{x-x_\varepsilon^{l,n}}{r_\varepsilon^l})$, so

$$\|\nabla v_\varepsilon^{l,n}\|^2_{L^2(Q_{r_\varepsilon^l}(x_\varepsilon^{l,n});\mathbb{R}^2)} = \frac{1}{4N_\varepsilon^2}\|\nabla\bar{v}\|^2_{L^2(Q_1(0);\mathbb{R}^2)} \tag{5.18}$$

and

$$-\Delta v_\varepsilon^{l,n} = \eta_\varepsilon \qquad \text{in} \quad Q_{r_\varepsilon^l}(x_\varepsilon^{l,n}), \qquad\qquad \partial_{\boldsymbol{\nu}} v_\varepsilon^{l,n} = 0 \quad \text{on} \quad \partial Q_{r_\varepsilon^l}(x_\varepsilon^{l,n}).$$

Integrating by parts, using Hölder inequality, (5.18), and Young inequality, it follows that

$$\begin{aligned}
\|\eta_\varepsilon\|_{H^{-1}(\Omega)} &= \sup_{\substack{\phi\in H_0^1(\Omega)\\ \|\phi\|_{H^1(\Omega)}\le 1}} \int_\Omega \phi \,\mathrm{d}\eta_\varepsilon = \sup_{\substack{\phi\in H_0^1(\Omega)\\ \|\phi\|_{H^1(\Omega)}\le 1}} \sum_{\substack{l=1\\ m^l\neq 0}}^{L}\sum_{n=1}^{N_\varepsilon^l}\int_{Q_{r_\varepsilon^l}(x_\varepsilon^{l,n})} -\phi\Delta v_\varepsilon^{l,n}\,\mathrm{d}x\\
&= \sup_{\substack{\phi\in H_0^1(\Omega)\\ \|\phi\|_{H^1(\Omega)}\le 1}} \sum_{\substack{l=1\\ m^l\neq 0}}^{L}\left(\sum_{n=1}^{N_\varepsilon^l}\int_{Q_{r_\varepsilon^l}(x_\varepsilon^{l,n})} \nabla\phi\cdot\nabla v_\varepsilon^{l,n}\,\mathrm{d}x\right)\\
&\le \sup_{\substack{\phi\in H_0^1(\Omega)\\ \|\phi\|_{H^1(\Omega)}\le 1}} \sum_{\substack{l=1\\ m^l\neq 0}}^{L}\sum_{n=1}^{N_\varepsilon^l}\|\nabla\phi\|_{L^2(Q_{r_\varepsilon^l}(x_\varepsilon^{l,n});\mathbb{R}^2)}\|\nabla v_\varepsilon^{l,n}\|_{L^2(Q_{r_\varepsilon^l}(x_\varepsilon^{l,n});\mathbb{R}^2)}\\
&= \sup_{\substack{\phi\in H_0^1(\Omega)\\ \|\phi\|_{H^1(\Omega)}\le 1}} \sum_{\substack{l=1\\ m^l\neq 0}}^{L}\frac{1}{2N_\varepsilon}\sum_{n=1}^{N_\varepsilon^l}\|\nabla\phi\|_{L^2(Q_{r_\varepsilon^l}(x_\varepsilon^{l,n});\mathbb{R}^2)}\|\nabla\bar{v}\|_{L^2(Q_1(0);\mathbb{R}^2)}\\
&\le \sup_{\substack{\phi\in H_0^1(\Omega)\\ \|\phi\|_{H^1(\Omega)}\le 1}} \sum_{\substack{l=1\\ m^l\neq 0}}^{L}\frac{1}{4N_\varepsilon}\sum_{n=1}^{N_\varepsilon^l}\Big(\frac{1}{N_\varepsilon^{\frac12}}\|\nabla\bar{v}\|^2_{L^2(Q_1(0);\mathbb{R}^2)} + N_\varepsilon^{\frac12}\|\nabla\phi\|^2_{L^2(Q_{r_\varepsilon^l}(x_\varepsilon^{l,n});\mathbb{R}^2)}\Big)\\
&\le \frac12\sum_{\substack{l=1\\ m^l\neq 0}}^{L}\frac{N_\varepsilon^l}{N_\varepsilon^{\frac32}}\|\nabla\bar{v}\|^2_{L^2(Q_1(0);\mathbb{R}^2)} + \frac{1}{4N_\varepsilon^{\frac12}} \le \frac{C}{N_\varepsilon^{\frac12}},
\end{aligned} \tag{5.19}$$

where the last inequality follows from (5.14) and (5.13). Finally, by Hölder inequality, (5.13) and (5.15), we obtain

$$\|\mu\chi_{\omega\setminus\omega_\varepsilon}\|_{H^{-1}(\Omega)} = \sup_{\substack{\phi\in H_0^1(\Omega)\\ \|\nabla\phi\|_{L^2(\Omega;\mathbb{R}^2)}\le 1}} \int_{\omega\setminus\omega_\varepsilon}\phi\,\mathrm{d}\mu \le \sup_{\substack{\phi\in H_0^1(\Omega)\\ \|\nabla\phi\|_{L^2(\Omega;\mathbb{R}^2)}\le 1}} M\|\phi\|_{L^2(\Omega)}|\omega\setminus\omega_\varepsilon|^{\frac12} \le \frac{C}{N_\varepsilon^{\frac14}}, \tag{5.20}$$

with M defined in (5.16); this, combined with (5.17) and (5.19), yields (c).

□

Proof of Theorem 3.3(iii). We divide the proof into two cases.

Case 1: $\mu = \sum_{l=1}^L m^l\chi_{\omega^l}\mathrm{d}x$, where $L\in\mathbb{N}$, $m^l\in\mathbb{R}$, and $\{\omega^l\}_{l=1,\dots,L}$ is a partition of Ω into sets with Lipschitz continuous boundary.

We divide the proof into two steps. In the first one we construct the recovery sequence $\{(\mu_\varepsilon,\overline{\beta}_\varepsilon)\}_\varepsilon$ for the core-radius problem; in the second step, we exploit the structure of $\{(\mu_\varepsilon,\overline{\beta}_\varepsilon)\}_\varepsilon$ to build up the recovery sequence $\{u_\varepsilon\}_\varepsilon$ for the functional $\mathcal{G}_\varepsilon$.

First step: Construction of the discrete measure μ_ε and of the core-radius field $\overline{\beta}_\varepsilon$. For every $\varepsilon > 0$, we set

$$N_\varepsilon := \lfloor |\log \varepsilon| \rfloor \tag{5.21}$$

and let

$$\mu_\varepsilon := \sum_{l=1}^{L} \mu_\varepsilon^l = \sum_{\substack{l=1\\ m^l\neq 0}}^{L} \sum_{n=1}^{N_\varepsilon^l} \delta_{x_\varepsilon^{l,n}} \tag{5.22}$$

be the measure provided by Lemma 5.1. Set

$$S := \{(0, x_2) \,:\, x_2 < 0\} \tag{5.23}$$

and let $\vartheta \in C^\infty(\mathbb{R}^2 \setminus \overline{S})$ be the function defined by

$$\vartheta(x) := \begin{cases} \arctan \frac{x_2}{x_1} & \text{if } x_1 > 0 \\ \frac{\pi}{2} & \text{if } x_1 = 0 \text{ and } x_2 > 0 \\ \pi + \arctan \frac{x_2}{x_1} & \text{if } x_1 < 0 \\ \frac{3}{2}\pi & \text{if } x_1 = 0 \text{ and } x_2 < 0\,. \end{cases} \tag{5.24}$$

For every $l = 1, \dots, L$ with $m^l \neq 0$ and for every $n = 1, \dots, N_\varepsilon^l$, let $\widehat{K}_\varepsilon^{l,n} \in L^2_{\rm loc}(\mathbb{R}^2 \setminus \{x_\varepsilon^{l,n}\}; \mathbb{R}^2)$ and $\widetilde{K}_\varepsilon^{l,n} \in L^2_{\rm loc}(\mathbb{R}^2; \mathbb{R}^2)$ be the functions defined by

$$\widehat{K}_\varepsilon^{l,n}(x) := \frac{1}{2\pi} \nabla\vartheta(x - x_\varepsilon^{l,n}) = \frac{1}{2\pi|x - x_\varepsilon^{l,n}|^2}(-(x_2 - x_{\varepsilon,2}^{l,n}), x_1 - x_{\varepsilon,1}^{l,n})\,,$$

and

$$\widetilde{K}_\varepsilon^{l,n}(x) := \frac{|x - x_\varepsilon^{l,n}|^2}{2\pi(r_\varepsilon^l)^2} \nabla\vartheta(x - x_\varepsilon^{l,n}) = \frac{1}{2\pi(r_\varepsilon^l)^2}(-(x_2 - x_{\varepsilon,2}^{l,n}), x_1 - x_{\varepsilon,1}^{l,n})\,,$$

respectively. Recalling that $A_{r,R}(x) := B_R(x) \setminus \overline{B}_r(x)$ (for every $0 < r < R$ and for every $x \in \mathbb{R}^2$), we define

$$\widehat{K}_\varepsilon := \sum_{\substack{l=1\\ m^l\neq 0}}^{L} \sum_{n=1}^{N_\varepsilon^l} \widehat{K}_\varepsilon^{l,n} \chi_{A_{\varepsilon, r_\varepsilon^l}(x_\varepsilon^{l,n})}, \qquad \widetilde{K}_\varepsilon := \sum_{\substack{l=1\\ m^l\neq 0}}^{L} \sum_{n=1}^{N_\varepsilon^l} \widetilde{K}_\varepsilon^{l,n} \chi_{B_{r_\varepsilon^l}(x_\varepsilon^{l,n})}\,. \tag{5.25}$$

Now, as in (5.12), for all $l = 1, \dots, L$ with $m^l \neq 0$ and for every $0 < \rho < r_\varepsilon^l$ we set

$$\widetilde{\mu}_\varepsilon^{l,\rho} := \frac{1}{\pi\rho^2} \sum_{n=1}^{N_\varepsilon^l} \chi_{B_\rho(x_\varepsilon^{l,n})}\, \mathrm{d}x\,, \qquad \text{and} \qquad \widetilde{\mu}_\varepsilon := \sum_{l=1}^{L} \widetilde{\mu}_\varepsilon^{l,r_\varepsilon^l},$$

and analogously

$$\zeta_\varepsilon^{l,\rho} := \frac{1}{2\pi\rho} \sum_{n=1}^{N_\varepsilon^l} \mathcal{H}^1 \llcorner \partial B_\rho(x_\varepsilon^{l,n})\,, \qquad \text{and} \qquad \zeta_\varepsilon := \sum_{l=1}^{L} \zeta_\varepsilon^{l,r_\varepsilon^l}.$$

Eventually, we introduce

$$\zeta_\varepsilon^\varepsilon := \sum_{l=1}^{L} \zeta_\varepsilon^{l,\varepsilon}$$

Then, using the notation just introduced, we have

$$\operatorname{curl} \widehat{K}_\varepsilon = \zeta_\varepsilon^\varepsilon - \zeta_\varepsilon \qquad \text{and} \qquad \operatorname{curl} \widetilde{K}_\varepsilon = \widetilde{\mu}_\varepsilon - \widehat{\mu}_\varepsilon\,. \tag{5.26}$$

Let $v_\varepsilon \in H^1(\Omega)$ be the solution to

$$\begin{cases} -\Delta v = \widetilde{\mu}_\varepsilon - N_\varepsilon \mu & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{5.27}$$

Then, by Poincaré inequality and by Lemma (5.1)(c), we get

$$\|\nabla v_\varepsilon\|^2_{L^2(\Omega;\mathbb{R}^2)} \leq \|\widetilde{\mu}_\varepsilon - N_\varepsilon\mu\|_{H^{-1}(\Omega)} \|v_\varepsilon\|_{H^1_0(\Omega)} \leq C(\Omega) N_\varepsilon^{\frac{3}{4}} \|\nabla v_\varepsilon\|_{L^2(\Omega;\mathbb{R}^2)}\,,$$

whence, recalling (5.21) we get

$$\frac{\nabla v_\varepsilon}{\sqrt{N_\varepsilon|\log\varepsilon|}} \to 0 \qquad \text{in } L^2(\Omega;\mathbb{R}^2)\,. \tag{5.28}$$

Let

$$\beta := -\frac{1}{\pi}(T^{\mathrm{D}})^\perp \in L^2(\Omega;\mathbb{R}^2)\,, \tag{5.29}$$

and set

$$\overline{\beta}_\varepsilon := N_\varepsilon\beta + \widehat{K}_\varepsilon - \widetilde{K}_\varepsilon + \nabla^\perp v_\varepsilon\,. \tag{5.30}$$

By (5.26) and (5.27),

$$\operatorname{curl}\overline{\beta}_\varepsilon \llcorner \Omega = N_\varepsilon\mu + \zeta^\varepsilon_\varepsilon - \zeta_\varepsilon - \widetilde{\mu}_\varepsilon + \zeta_\varepsilon + \widetilde{\mu}_\varepsilon - N_\varepsilon\mu = \zeta^\varepsilon_\varepsilon\,, \tag{5.31}$$

so that $\operatorname{curl}\overline{\beta}_\varepsilon = 0$ in $\Omega_\varepsilon(\mu_\varepsilon) := \Omega \setminus \bigcup^L_{\substack{l=1\\ m^l\neq 0}} \bigcup_{n=1}^{N^l_\varepsilon} \overline{B}_\varepsilon(x^{l,n}_\varepsilon)$. Furthermore, by (5.31), for any $l = 1,\dots,L$ with $m^l \neq 0$ and for any $n = 1,\dots,N^l_\varepsilon$, we have

$$\int_{\partial B_\rho(x^{l,n}_\varepsilon)} \overline{\beta}_\varepsilon \cdot \boldsymbol{\tau}\,\mathrm{d}\mathcal{H}^1 = 1 \qquad \text{for a.e. } \rho \in (\varepsilon, r^l_\varepsilon)\,.$$

Again by (5.31), setting

$$S^{l,n}_\varepsilon := x^{l,n}_\varepsilon + S, \tag{5.32}$$

(with S defined in (5.23)) for every $l = 1,\dots,L$ with $m^l \neq 0$ and $n = 1,\dots,N^l_\varepsilon$, there exists a function $\overline{\vartheta}_\varepsilon \in H^1\Big(\Omega_\varepsilon(\mu_\varepsilon) \setminus \bigcup^L_{\substack{l=1\\ m^l\neq 0}} \bigcup_{n=1}^{N^l_\varepsilon} S^{l,n}_\varepsilon\Big)$ such that

$$\overline{\beta}_\varepsilon = \nabla\overline{\vartheta}_\varepsilon \qquad \text{a.e. on } \Omega_\varepsilon(\mu_\varepsilon) \setminus \bigcup_{\substack{l=1\\ m^l\neq 0}}^{L} \bigcup_{n=1}^{N^l_\varepsilon} S^{l,n}_\varepsilon. \tag{5.33}$$

In what follows, with a little abuse of notation, we denote by $\overline{\vartheta}_\varepsilon$ and $\overline{\beta}_\varepsilon$ the zero-extensions of $\overline{\vartheta}_\varepsilon$ and $\overline{\beta}_\varepsilon$ to $\bigcup^L_{\substack{l=1\\ m^l\neq 0}} \bigcup_{n=1}^{N^l_\varepsilon} B_\varepsilon(x^{l,n}_\varepsilon)$, respectively. We now prove that

$$\frac{\overline{\beta}_\varepsilon}{\sqrt{N_\varepsilon|\log\varepsilon|}} \rightharpoonup \beta \qquad \text{in } L^2(\Omega;\mathbb{R}^2)\,. \tag{5.34}$$

On the one hand, by (5.13) and (5.14),

$$\begin{aligned}\frac{1}{\sqrt{N_\varepsilon|\log\varepsilon|}}\int_\Omega |\widehat{K}_\varepsilon|\,\mathrm{d}x &= \frac{1}{\sqrt{N_\varepsilon|\log\varepsilon|}} \sum_{\substack{l=1\\ m^l\neq 0}}^{L} \sum_{n=1}^{N^l_\varepsilon} \int_{A_{\varepsilon,r^l_\varepsilon}(x^{l,n}_\varepsilon)} |\widehat{K}^{l,n}_\varepsilon|\,\mathrm{d}x \\ &= \frac{2\pi}{\sqrt{N_\varepsilon|\log\varepsilon|}} \sum_{\substack{l=1\\ m^l\neq 0}}^{L} (r^l_\varepsilon - \varepsilon)N^l_\varepsilon \to 0\,.\end{aligned}$$

On the other hand, recalling (5.14), we also have

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{1}{N_\varepsilon|\log\varepsilon|}\int_\Omega |\widehat{K}_\varepsilon|^2\,\mathrm{d}x &= \lim_{\varepsilon\to 0}\frac{1}{N_\varepsilon|\log\varepsilon|} \sum_{\substack{l=1\\ m^l\neq 0}}^{L} \sum_{n=1}^{N^l_\varepsilon} \int_{A_{\varepsilon,r^l_\varepsilon}(x^{l,n}_\varepsilon)} |\widehat{K}^{l,n}_\varepsilon|^2\,\mathrm{d}x \\ &= \lim_{\varepsilon\to 0}\frac{\sum^L_{\substack{l=1\\ m^l\neq 0}} N^l_\varepsilon \log\frac{r^l_\varepsilon}{\varepsilon}}{N_\varepsilon|\log\varepsilon|}\frac{1}{2\pi} = \frac{1}{2\pi}|\mu|(\Omega)\,,\end{aligned} \tag{5.35}$$

so that

$$\frac{\widehat{K}_\varepsilon}{\sqrt{N_\varepsilon|\log\varepsilon|}} \rightharpoonup 0 \qquad \text{in } L^2(\Omega;\mathbb{R}^2)\,. \tag{5.36}$$

Moreover, by construction,

$$\begin{aligned}\frac{1}{N_\varepsilon|\log\varepsilon|}\int_\Omega |\widetilde{K}_\varepsilon|^2\,\mathrm{d}x \\ = \frac{1}{N_\varepsilon|\log\varepsilon|}\sum_{\substack{l=1\\ m^l\neq 0}}^{L}\frac{1}{4\pi^2(r^l_\varepsilon)^2}\sum_{n=1}^{N^l_\varepsilon}\int_{B_{r^l_\varepsilon}(x^{l,n}_\varepsilon)}|x-x^{l,n}_\varepsilon|^2\,\mathrm{d}x \to 0 \quad \text{in } L^2(\Omega;\mathbb{R}^2)\,;\end{aligned} \tag{5.37}$$

therefore, by the very definition of $\overline{\beta}_\varepsilon$ in (5.30), using (5.36), (5.37), and (5.28), we deduce (5.34). Moreover, by (5.35), (5.37) and (5.28), we easily get

$$\lim_{\varepsilon\to 0}\frac{1}{N_\varepsilon|\log\varepsilon|}\int_\Omega|\overline{\beta}_\varepsilon|^2\,\mathrm{d}x = \frac{1}{2\pi}|\mu|(\Omega)+\int_\Omega|\beta|^2\,\mathrm{d}x\,. \tag{5.38}$$

Second step: Construction of the recovery sequence $\{u_\varepsilon\}_\varepsilon$. Let $\bar l\in\{1,\dots,L\}$ be such that $m^{\bar l}\neq 0$ and $\bar n\in\{1,\dots,N^{\bar l}_\varepsilon\}$ be fixed. Recalling (5.32) and that the mutual distance between the points $x^{\bar l,\bar n}_\varepsilon$ is at least 2ε, the set $A_{\varepsilon,2\varepsilon}(x^{\bar l,\bar n}_\varepsilon)\setminus\bigcup_{\substack{l=1\\ m^l\neq 0}}^{L}\bigcup_{n=1}^{N^l_\varepsilon}S^{l,n}_\varepsilon$ is either connected or is given by the union of the two sets

$$A^{\pm}_{\varepsilon,2\varepsilon}(x^{\bar l,\bar n}_\varepsilon) := A_{\varepsilon,2\varepsilon}(x^{\bar l,\bar n}_\varepsilon)\cap\{x_1 \gtrless x^{\bar l,\bar n}_{\varepsilon,1}\}.$$

We set $a^{\bar l,\bar n,+}_\varepsilon := \fint_{A^+_{\varepsilon,2\varepsilon}(x^{\bar l,\bar n}_\varepsilon)}\overline{\vartheta}_\varepsilon\,\mathrm{d}x$ and $a^{\bar l,\bar n,-}_\varepsilon := \fint_{A^-_{\varepsilon,2\varepsilon}(x^{\bar l,\bar n}_\varepsilon)}\overline{\vartheta}_\varepsilon\,\mathrm{d}x$, where $\overline{\vartheta}_\varepsilon$ is the function in (5.33). By construction $\overline{\vartheta}_\varepsilon\in H^1(A^+_{\varepsilon,2\varepsilon}(x^{\bar l,\bar n}_\varepsilon))$ and $\overline{\vartheta}_\varepsilon\in H^1(A^-_{\varepsilon,2\varepsilon}(x^{\bar l,\bar n}_\varepsilon))$, so that, since the sets $A^{\pm}_{\varepsilon,2\varepsilon}(x^{\bar l,\bar n}_\varepsilon)$ have Lipschitz continuous boundary, we can apply the Poincaré-Wirtinger inequality in $H^1(A^+_{\varepsilon,2\varepsilon}(x^{\bar l,\bar n}_\varepsilon))$ and $H^1(A^-_{\varepsilon,2\varepsilon}(x^{\bar l,\bar n}_\varepsilon))$, thus getting

$$\|\overline{\vartheta}_\varepsilon - a^{\bar l,\bar n,+}_\varepsilon\|^2_{L^2(A^+_{\varepsilon,2\varepsilon}(x^{\bar l,\bar n}_\varepsilon))} + \|\overline{\vartheta}_\varepsilon - a^{\bar l,\bar n,-}_\varepsilon\|^2_{L^2(A^-_{\varepsilon,2\varepsilon}(x^{\bar l,\bar n}_\varepsilon))} \le C\varepsilon^2\|\overline{\beta}_\varepsilon\|^2_{L^2(A_{\varepsilon,2\varepsilon}(x^{\bar l,\bar n}_\varepsilon);\mathbb{R}^2)}\,, \tag{5.39}$$

for some universal constant $C>0$.

Let $\sigma_\varepsilon\in C^\infty(B_{2\varepsilon}(0);[0,1])$ be such that $\sigma_\varepsilon\equiv 0$ in $B_\varepsilon(0)$, $\sigma_\varepsilon\equiv 1$ in $A_{\frac{3}{2}\varepsilon,2\varepsilon}(0)$ and that

$$|\nabla\sigma_\varepsilon(x)|\le\frac{C}{\varepsilon}\quad\text{for every } x\in B_{2\varepsilon}(0)\,, \tag{5.40}$$

for some constant $C>0$ independent of ε (and of x). For every $\varepsilon>0$ we set

$$\vartheta_\varepsilon(x) := \begin{cases}\sigma_\varepsilon(x-x^{\bar l,\bar n}_\varepsilon)\overline{\vartheta}_\varepsilon(x)+(1-\sigma_\varepsilon(x-x^{\bar l,\bar n}_\varepsilon))a^{\bar l,\bar n,-}_\varepsilon & \text{if } x\in B^-_{2\varepsilon}(x^{\bar l,\bar n}_\varepsilon) \text{ for some } \bar l=1,\dots,L,\ \bar n=1,\dots,N^{\bar l}_\varepsilon\\ \sigma_\varepsilon(x-x^{\bar l,\bar n}_\varepsilon)\overline{\vartheta}_\varepsilon(x)+(1-\sigma_\varepsilon(x-x^{\bar l,\bar n}_\varepsilon))a^{\bar l,\bar n,+}_\varepsilon & \text{if } x\in B^+_{2\varepsilon}(x^{\bar l,\bar n}_\varepsilon) \text{ for some } \bar l=1,\dots,L,\ \bar n=1,\dots,N^{\bar l}_\varepsilon\\ \overline{\vartheta}_\varepsilon(x) & \text{if } x\in\Omega\setminus\bigcup_{\substack{l=1\\ m^l\neq 0}}^{L}\bigcup_{n=1}^{N^l_\varepsilon}\overline{B}_{2\varepsilon}(x^{l,n}_\varepsilon)\,,\end{cases}$$

(where $\overline{B}^{\pm}_{2\varepsilon}(x^{l,n}_\varepsilon) := \overline{B}_{2\varepsilon}(x^{l,n}_\varepsilon)\cap\{x_1\gtrless x^{l,n}_{\varepsilon,1}\}$ and $\bar l$ is such that $m^{\bar l}\neq 0$) and we define $u_\varepsilon:\Omega\to\mathbb{S}^1$ as

$$u_\varepsilon(\cdot) := e^{2\pi\imath\vartheta_\varepsilon(\cdot)}\,. \tag{5.41}$$

By construction,

$$\overline{S}_{u_\varepsilon}\subset\bigcup_{\substack{l=1\\ m^l\neq 0}}^{L}\bigcup_{n=1}^{N^l_\varepsilon}\overline{B}_{\frac{3}{2}\varepsilon}(x^{l,n}_\varepsilon)\qquad\text{and}\qquad\mathcal{H}^1(\overline{S}_{u_\varepsilon})\le\sum_{\substack{l=1\\ m^l\neq 0}}^{L}N^l_\varepsilon 4\varepsilon\,, \tag{5.42}$$

which, in view of (5.14), implies

$$\lim_{\varepsilon\to 0}\frac{1}{N_\varepsilon|\log\varepsilon|}\frac{1}{\varepsilon}\mathcal{H}^1(\overline{S}_{u_\varepsilon})=0\,. \tag{5.43}$$

We claim that

$$\frac{\nabla\vartheta_\varepsilon}{\sqrt{N_\varepsilon|\log\varepsilon|}} \rightharpoonup \beta \qquad \text{in } L^2(\Omega;\mathbb{R}^2)\,, \tag{5.44}$$

which, together with (5.43), in view of (1.10) and (5.29), will imply (ACJ). To show (5.44), we start by observing that, by the very definition of ϑ_ε in $\bigcup_{\substack{l=1\\ m^l\neq 0}}^{L}\bigcup_{n=1}^{N_\varepsilon^l} A_{\varepsilon,2\varepsilon}(x_\varepsilon^{l,n})$ and by (5.39) (applied to every $l=1,\dots,L$ with $m^l\neq 0$ and $n=1,\dots,N_\varepsilon^l$), we get

$$\begin{aligned}
\frac{1}{N_\varepsilon|\log\varepsilon|}\|\nabla\vartheta_\varepsilon\|^2_{L^2(\bigcup_{\substack{l=1\\ m^l\neq 0}}^{L}\bigcup_{n=1}^{N_\varepsilon^l} A_{\varepsilon,2\varepsilon}(x_\varepsilon^{l,n});\mathbb{R}^2)} &\le \frac{2}{N_\varepsilon|\log\varepsilon|}\sum_{\substack{l=1\\ m^l\neq 0}}^{L}\sum_{n=1}^{N_\varepsilon^l}\|\overline{\beta}_\varepsilon\|^2_{L^2(A_{\varepsilon,2\varepsilon}(x_\varepsilon^{l,n});\mathbb{R}^2)}\\
&\quad+\frac{2}{N_\varepsilon|\log\varepsilon|}\frac{C}{\varepsilon^2}\sum_{\substack{l=1\\ m^l\neq 0}}^{L}\sum_{n=1}^{N_\varepsilon^l}\|\overline{\vartheta}_\varepsilon-a_\varepsilon^{l,n,+}\|^2_{L^2(A^+_{\varepsilon,2\varepsilon}(x_\varepsilon^{l,n}))}\\
&\quad+\frac{2}{N_\varepsilon|\log\varepsilon|}\frac{C}{\varepsilon^2}\sum_{\substack{l=1\\ m^l\neq 0}}^{L}\sum_{n=1}^{N_\varepsilon^l}\|\overline{\vartheta}_\varepsilon-a_\varepsilon^{l,n,-}\|^2_{L^2(A^-_{\varepsilon,2\varepsilon}(x_\varepsilon^{l,n}))}\\
&\le \frac{C}{N_\varepsilon|\log\varepsilon|}\sum_{\substack{l=1\\ m^l\neq 0}}^{L}\sum_{n=1}^{N_\varepsilon^l}\|\overline{\beta}_\varepsilon\|^2_{L^2(A_{\varepsilon,2\varepsilon}(x_\varepsilon^{l,n});\mathbb{R}^2)}\\
&\le \frac{C}{|\log\varepsilon|}+C\|\beta\|^2_{L^2(\bigcup_{\substack{l=1\\ m^l\neq 0}}^{L}\bigcup_{n=1}^{N_\varepsilon^l} A_{\varepsilon,2\varepsilon}(x_\varepsilon^{l,n});\mathbb{R}^2)}+\mathrm{o}(1)\,,
\end{aligned}\tag{5.45}$$

as $\varepsilon\to 0$, where in the last inequality we used (5.37), (5.35) (with r_ε replaced by 2ε), and (5.28), to deduce that

$$\begin{aligned}
\frac{1}{N_\varepsilon|\log\varepsilon|}\sum_{\substack{l=1\\ m^l\neq 0}}^{L}\sum_{n=1}^{N_\varepsilon^l}\|\overline{\beta}_\varepsilon\|^2_{L^2(A_{\varepsilon,2\varepsilon}(x_\varepsilon^{l,n});\mathbb{R}^2)} &\le \frac{2}{N_\varepsilon|\log\varepsilon|}\sum_{\substack{l=1\\ m^l\neq 0}}^{L}\sum_{n=1}^{N_\varepsilon^l}\|\widehat{K}_\varepsilon^{l,n}\|^2_{L^2(A_{\varepsilon,2\varepsilon}(x_\varepsilon^{l,n});\mathbb{R}^2)}\\
&\quad+\frac{N_\varepsilon}{|\log\varepsilon|}\sum_{\substack{l=1\\ m^l\neq 0}}^{L}\sum_{n=1}^{N_\varepsilon^l}\|\beta\|^2_{L^2(A_{\varepsilon,2\varepsilon}(x_\varepsilon^{l,n});\mathbb{R}^2)}+\mathrm{o}(1)\\
&\le \frac{C}{|\log\varepsilon|}+C\|\beta\|^2_{L^2\left(\bigcup_{\substack{l=1\\ m^l\neq 0}}^{L}\bigcup_{n=1}^{N_\varepsilon^l} A_{\varepsilon,2\varepsilon}(x_\varepsilon^{l,n});\mathbb{R}^2\right)}+\mathrm{o}(1)\,.
\end{aligned}$$

Therefore, by (5.45) and (5.34), using the very definition of ϑ_ε, we deduce (5.44). Furthermore, using (5.38) and again (5.45), recalling (5.29), we get

$$\begin{aligned}
\lim_{\varepsilon\to 0}\frac{1}{N_\varepsilon|\log\varepsilon|}\frac{1}{2}\int_\Omega|\nabla u_\varepsilon|^2\,\mathrm{d}x &= \lim_{\varepsilon\to 0}\frac{1}{N_\varepsilon|\log\varepsilon|}2\pi^2\int_\Omega|\beta_\varepsilon|^2\,\mathrm{d}x\\
&=\pi|\mu|(\Omega)+2\pi^2\int_\Omega|\beta|^2\,\mathrm{d}x=\pi|\mu|(\Omega)+2\int_\Omega|T^{\mathrm{D}}|^2\,\mathrm{d}x\,,
\end{aligned}\tag{5.46}$$

which, combined with (5.43), implies that the sequence $\{u_\varepsilon\}_\varepsilon$ satisfies (3.13).

Now, in order to conclude the proof of (iii) of Theorem 3.3 in the case $\mu=\sum_{l=1}^{L}m^l\chi_{\omega^l}\,\mathrm{d}x$, it remains to prove that also (FJ) is satisfied. To this end, we first observe that, by Hölder inequality and the very definition of u_ε in (5.41), for every $l=1,\dots,L$ with $m^l\neq 0$ and for every

$n = 1, \dots, N_\varepsilon^l$

$$\int_{A_{\frac{3}{2}\varepsilon, 2\varepsilon}(x_\varepsilon^{l,n})} |\nabla u_\varepsilon| \, \mathrm{d}x \le C\varepsilon \|\nabla \vartheta_\varepsilon\|_{L^2(A_{\frac{3}{2}\varepsilon, 2\varepsilon}(x_\varepsilon^{l,n}); \mathbb{R}^2)}$$

which, by Fubini Theorem and by the Mean Value Theorem, implies that (for every $l = 1, \dots, L$ with $m^l \neq 0$ and for every $n = 1, \dots, N_\varepsilon^l$) there exists $\frac{3}{2}\varepsilon < \rho_\varepsilon^{l,n} < 2\varepsilon$ such that

$$\int_{\partial B_{\rho_\varepsilon^{l,n}}(x_\varepsilon^{l,n})} |\nabla u_\varepsilon| \, \mathrm{d}\mathcal{H}^1 \le C \|\nabla \vartheta_\varepsilon\|_{L^2(A_{\frac{3}{2}\varepsilon, 2\varepsilon}(x_\varepsilon^{l,n}); \mathbb{R}^2)} \, .$$

Therefore, recalling (5.42), by (5.45), for ε small enough we get

$$\begin{aligned} \sum_{\substack{l=1 \\ m^l \neq 0}}^{L} \sum_{n=1}^{N_\varepsilon^l} \int_{\partial B_{\rho_\varepsilon^{l,n}}(x_\varepsilon^{l,n})} \mathrm{d}|T_{u_\varepsilon}| &= \sum_{\substack{l=1 \\ m^l \neq 0}}^{L} \sum_{n=1}^{N_\varepsilon^l} \int_{\partial B_{\rho_\varepsilon^{l,n}}(x_\varepsilon^{l,n})} |T_{u_\varepsilon}^{\mathrm{D}}| \, \mathrm{d}\mathcal{H}^1 \\ &\le C\sqrt{N_\varepsilon} \|\nabla \vartheta_\varepsilon\|_{L^2\left(\bigcup_{\substack{l=1 \\ m^l \neq 0}}^{L} \bigcup_{n=1}^{N_\varepsilon^l} A_{\varepsilon, 2\varepsilon}(x_\varepsilon^{l,n}); \mathbb{R}^2\right)} \\ &\le C N_\varepsilon |\log \varepsilon|^{\frac{1}{2}} \, . \end{aligned} \tag{5.47}$$

Analogously, again by (5.45) and by (5.42), using also Hölder inequality, we have

$$\begin{aligned} |T_{u_\varepsilon}|\Big(\bigcup_{\substack{l=1 \\ m^l \neq 0}}^{L} \bigcup_{n=1}^{N_\varepsilon^l} B_{\rho_\varepsilon^{l,n}}(x_\varepsilon^{l,n}) \Big) &\le 2|\mathrm{D}u_\varepsilon|\Big(\bigcup_{\substack{l=1 \\ m^l \neq 0}}^{L} \bigcup_{n=1}^{N_\varepsilon^l} B_{\rho_\varepsilon^{l,n}}(x_\varepsilon^{l,n}) \Big) \\ &\le 2\varepsilon \|\nabla \vartheta_\varepsilon\|_{L^2(\bigcup_{\substack{l=1 \\ m^l \neq 0}}^{L} \bigcup_{n=1}^{N_\varepsilon^l} B_{\rho_\varepsilon^{l,n}}(x_\varepsilon^{l,n}); \mathbb{R}^2)} \sqrt{\sum_{\substack{l=1 \\ m^l \neq 0}}^{L} N_\varepsilon^l \pi} + C \sum_{\substack{l=1 \\ m^l \neq 0}}^{L} N_\varepsilon^l \varepsilon \\ &\le C\varepsilon N_\varepsilon |\log \varepsilon|^{\frac{1}{2}} \, . \end{aligned}$$

Recalling the definition of ϑ in (5.24), we define

$$v_\varepsilon(\cdot) := \exp\Big(\imath \sum_{\substack{l=1 \\ m^l \neq 0}}^{L} \sum_{n=1}^{N_\varepsilon^l} \vartheta(\cdot - x_\varepsilon^{l,n}) \Big);$$

by construction, $v_\varepsilon \in W^{1,p}(\Omega; \mathbb{S}^1)$ for any $1 \le p < 2$, and

$$Jv_\varepsilon = \pi \mu_\varepsilon \qquad \text{in } \mathcal{M}(\Omega). \tag{5.48}$$

Moreover for every $x \in \Omega$

$$|T_{v_\varepsilon}(x)| = |T_{v_\varepsilon}^{\mathrm{D}}(x)| = 2|\lambda_{v_\varepsilon}(x)| \le 2\pi \sum_{\substack{l=1 \\ m^l \neq 0}}^{L} \sum_{n=1}^{N_\varepsilon^l} |\nabla \vartheta(x - x_\varepsilon^n)| = 2\pi \sum_{\substack{l=1 \\ m^l \neq 0}}^{L} \sum_{n=1}^{N_\varepsilon^l} \frac{1}{|x - x_\varepsilon^{l,n}|} \, . \tag{5.49}$$

Therefore, for every $\bar{l}=1,\dots,L$ with $m^{\bar{l}}\neq 0$ and for every $\bar{n}=1,\dots,N_\varepsilon^{\bar{l}}$, and for ε small enough we have

$$\begin{aligned}|T_{v_\varepsilon}|(\partial B_{\rho_\varepsilon^{\bar{l},\bar{n}}}(x_\varepsilon^{\bar{l},\bar{n}}))\leq&\sum_{\substack{l=1\\ m^l\neq 0}}^{L}\sum_{n=1}^{N_\varepsilon^l}\int_{\partial B_{\rho_\varepsilon^{\bar{l},\bar{n}}}(x_\varepsilon^{\bar{l},\bar{n}})}\frac{2\pi}{|x-x_\varepsilon^{l,n}|}\,\mathrm{d}\mathcal{H}^1\\ \leq&\,4\pi^2+\sum_{\substack{n=1\\ n\neq\bar{n}}}^{N_\varepsilon^{\bar{l}}}\int_{\partial B_{\rho_\varepsilon^{\bar{l},\bar{n}}}(x_\varepsilon^{\bar{l},\bar{n}})}\frac{2\pi}{|x-x_\varepsilon^{\bar{l},n}|}+\sum_{\substack{l=1\\ l\neq\bar{l}\\ m^l\neq 0}}^{L}\sum_{n=1}^{N_\varepsilon^l}\frac{2\pi}{|x-x_\varepsilon^{\bar{l},n}|}\,\mathrm{d}\mathcal{H}^1\\ \leq&\,4\pi^2+4\pi\varepsilon\frac{N_\varepsilon^{\bar{l}}}{r_\varepsilon^{\bar{l}}}+\sum_{\substack{l=1\\ l\neq\bar{l}\\ m^l\neq 0}}^{L}N_\varepsilon^l\frac{4\pi\varepsilon}{r_\varepsilon^l}\leq 4\pi^2+4\pi\varepsilon\sum_{l=1}^{L}|m^l||\omega^l|N_\varepsilon^{\frac{3}{2}}\leq C\,,\end{aligned}$$

where in the last but one inequality we have used (5.14) together with the fact that

$$\inf_{\substack{n=1,\dots,N_\varepsilon^{\bar{l}}\\ n\neq\bar{n}}}|x_\varepsilon^{\bar{l},n}-x_\varepsilon^{\bar{l},\bar{n}}|\geq r_\varepsilon^{\bar{l}},\qquad\qquad \inf_{\substack{l=1,\dots,L\\ l\neq\bar{l}}}\inf_{n=1,\dots,N_\varepsilon^l}|x_\varepsilon^{l,n}-x_\varepsilon^{\bar{l},\bar{n}}|\geq r_\varepsilon^{\bar{l}},$$

so that (for ε small enough) $\mathrm{dist}(\partial B_{\rho_\varepsilon^n}(x_\varepsilon^{\bar{l},\bar{n}}),x_\varepsilon^{l,n})\geq r_\varepsilon^{\bar{l}}\geq 2\varepsilon$. It follows that

$$\sum_{\substack{l=1\\ m^l\neq 0}}^{L}\sum_{n=1}^{N_\varepsilon^l}\int_{\partial B_{\rho_\varepsilon^{l,n}}(x_\varepsilon^{l,n})}|T_{v_\varepsilon}|\,\mathrm{d}\mathcal{H}^1\leq CN_\varepsilon=C|\log\varepsilon|\,. \tag{5.50}$$

Analogously, by (5.49), for every $\bar{l}=1,\dots,L$ with $m^{\bar{l}}\neq 0$, for every $\bar{n}=1,\dots,N_\varepsilon^{\bar{l}}$ and for ε small enough we have

$$\begin{aligned}|T_{v_\varepsilon}|(B_{\rho_\varepsilon^{\bar{l},\bar{n}}}(x_\varepsilon^{\bar{l},\bar{n}}))\leq&\,2\pi\int_{B_{\rho_\varepsilon^{\bar{l},\bar{n}}}(x_\varepsilon^{\bar{l},\bar{n}})}\frac{1}{|x-x_\varepsilon^{\bar{l},\bar{n}}|}\,\mathrm{d}x+2\pi\sum_{\substack{n=1\\ n\neq\bar{n}}}^{N_\varepsilon^{\bar{l}}}\int_{B_{\rho_\varepsilon^{\bar{l},\bar{n}}}(x_\varepsilon^{\bar{l},\bar{n}})}\frac{1}{|x-x_\varepsilon^{\bar{l},n}|}\,\mathrm{d}x\\ &+2\pi\sum_{\substack{l=1\\ l\neq\bar{l}\\ m^l\neq 0}}^{L}\sum_{n=1}^{N_\varepsilon^l}\int_{B_{\rho_\varepsilon^{\bar{l},\bar{n}}}(x_\varepsilon^{\bar{l},\bar{n}})}\frac{1}{|x-x_\varepsilon^{l,n}|}\,\mathrm{d}x\\ \leq&\,C\varepsilon+CN_\varepsilon^{\bar{l}}\frac{\varepsilon}{r_\varepsilon^{\bar{l}}}+C\sum_{\substack{l=1\\ l\neq\bar{l}\\ m^l\neq 0}}^{L}N_\varepsilon^l\frac{\varepsilon}{r_\varepsilon^l}\leq CN_\varepsilon^{\frac{3}{2}}\varepsilon\,,\end{aligned}$$

whence we deduce that

$$\frac{1}{|\log\varepsilon|}|T_{v_\varepsilon}|\Big(\bigcup_{\substack{l=1\\ m^l\neq 0}}^{L}\bigcup_{n=1}^{N_\varepsilon^l}B_{\rho_\varepsilon^{l,n}}(x_\varepsilon^{l,n})\Big)\to 0\qquad\text{as }\varepsilon\to 0\,. \tag{5.51}$$

Let finally $\varphi\in C_{\mathrm{c}}^1(\Omega)$ be such that $\|\varphi\|_{C_{\mathrm{c}}^{0,1}(\Omega)}\leq 1$. By the very definition of distributional Jacobian, integrating by parts and using that

$$Ju_\varepsilon\llcorner\Big(\Omega\setminus\bigcup_{\substack{l=1\\ m^l\neq 0}}^{L}\bigcup_{n=1}^{N_\varepsilon^l}B_{\rho_\varepsilon^{l,n}}(x_\varepsilon^{l,n})\Big)=Jv_\varepsilon\llcorner\Big(\Omega\setminus\bigcup_{\substack{l=1\\ m^l\neq 0}}^{L}\bigcup_{n=1}^{N_\varepsilon^l}B_{\rho_\varepsilon^{l,n}}(x_\varepsilon^{l,n})\Big)=0,$$

we obtain, using (5.48),

$$\begin{aligned}\langle Ju_\varepsilon,\varphi\rangle_\Omega &= \sum_{\substack{l=1\\ m^l\neq 0}}^{L}\sum_{n=1}^{N_\varepsilon^l}\int_{B_{\rho_\varepsilon^{l,n}}(x_\varepsilon^{l,n})}\nabla\varphi\cdot\,\mathrm{d}T_{u_\varepsilon}+\int_{\Omega\setminus\bigcup_{\substack{l=1\\ m^l\neq 0}}^{L}\bigcup_{n=1}^{N_\varepsilon^l}B_{\rho_\varepsilon^{l,n}}(x_\varepsilon^{l,n})}\nabla\varphi\cdot\,\mathrm{d}T_{u_\varepsilon}\\ &= \sum_{\substack{l=1\\ m^l\neq 0}}^{L}\sum_{n=1}^{N_\varepsilon^l}\int_{B_{\rho_\varepsilon^{l,n}}(x_\varepsilon^{l,n})}\nabla\varphi\cdot\,\mathrm{d}T_{u_\varepsilon}-\sum_{\substack{l=1\\ m^l\neq 0}}^{L}\sum_{n=1}^{N_\varepsilon^l}\int_{\partial B_{\rho_\varepsilon^{l,n}}(x_\varepsilon^{l,n})}\varphi T_{u_\varepsilon}\cdot\boldsymbol{\nu}\,\mathrm{d}\mathcal{H}^1\,,\end{aligned}$$

and

$$\begin{aligned}\langle Jv_\varepsilon,\varphi\rangle_\Omega &= \sum_{\substack{l=1\\ m^l\neq 0}}^{L}\sum_{n=1}^{N_\varepsilon^l}\int_{B_{\rho_\varepsilon^{l,n}}(x_\varepsilon^{l,n})}\nabla\varphi\cdot\,\mathrm{d}T_{v_\varepsilon}+\int_{\Omega\setminus\bigcup_{\substack{l=1\\ m^l\neq 0}}^{L}\bigcup_{n=1}^{N_\varepsilon^l}B_{\rho_\varepsilon^{l,n}}(x_\varepsilon^{l,n})}\nabla\varphi\cdot\,\mathrm{d}T_{v_\varepsilon}\\ &= \sum_{\substack{l=1\\ m^l\neq 0}}^{L}\sum_{n=1}^{N_\varepsilon^l}\int_{B_{\rho_\varepsilon^{l,n}}(x_\varepsilon^{l,n})}\nabla\varphi\cdot\,\mathrm{d}T_{v_\varepsilon}-\sum_{\substack{l=1\\ m^l\neq 0}}^{L}\sum_{n=1}^{N_\varepsilon^l}\int_{\partial B_{\rho_\varepsilon^{l,n}}(x_\varepsilon^{l,n})}\varphi T_{v_\varepsilon}\cdot\boldsymbol{\nu}\,\mathrm{d}\mathcal{H}^1\,.\end{aligned}$$

Therefore, by (5.47)-(5.51), and using that

$$Ju_\varepsilon(B_{\rho_\varepsilon^{l,n}}(x_\varepsilon^{l,n}))=Jv_\varepsilon(B_{\rho_\varepsilon^{l,n}}(x_\varepsilon^{l,n})),\qquad \text{for every } l=1,\dots,L \text{ with } m^l\neq 0,\ n=1,\dots,N_\varepsilon^l,$$

we obtain

$$\begin{aligned}\big|\langle Ju_\varepsilon-\pi\mu_\varepsilon,\varphi\rangle_\Omega\big| &\leq \sum_{\substack{l=1\\ m^l\neq 0}}^{L}\sum_{n=1}^{N_\varepsilon^l}\Big|\int_{\partial B_{\rho_\varepsilon^{l,n}}(x_\varepsilon^{l,n})}\varphi(T_{u_\varepsilon}-T_{v_\varepsilon})\cdot\boldsymbol{\nu}\,\mathrm{d}\mathcal{H}^1\Big|\\ &\quad+|T_{u_\varepsilon}|\Big(\bigcup_{\substack{l=1\\ m^l\neq 0}}^{L}\bigcup_{n=1}^{N_\varepsilon^l}B_{\rho_\varepsilon^{l,n}}(x_\varepsilon^{l,n})\Big)+|T_{v_\varepsilon}|\Big(\bigcup_{\substack{l=1\\ m^l\neq 0}}^{L}\bigcup_{n=1}^{N_\varepsilon^l}B_{\rho_\varepsilon^{l,n}}(x_\varepsilon^{l,n})\Big)\\ &\leq \sum_{\substack{l=1\\ m^l\neq 0}}^{L}\sum_{n=1}^{N_\varepsilon^l}\mathrm{osc}_{B_{\rho_\varepsilon^{l,n}}(x_\varepsilon^{l,n})}(\varphi)\int_{\partial B_{\rho_\varepsilon^{l,n}}(x_\varepsilon^{l,n})}\big(|T_{u_\varepsilon}|+|T_{v_\varepsilon}|\big)\,\mathrm{d}\mathcal{H}^1\\ &\quad+|\log\varepsilon|\mathrm{o}(1)\,.\end{aligned}$$

Since $\mathrm{osc}_{B_{\rho_\varepsilon^{l,n}}(x_\varepsilon^{l,n})}(\varphi)\leq C\varepsilon$, it follows that

$$\frac{1}{|\log\varepsilon|}\,\|Ju_\varepsilon-\pi\mu_\varepsilon\|_{\mathrm{flat},\Omega}\to 0\qquad\text{as } \varepsilon\to 0\,,$$

whence (FJ) follows by Lemma 5.1(b).

Case 2: General case. We argue by density, namely we show that for every $(\mu,T^{\mathrm{D}})\in(\mathcal{M}(\Omega)\cap H^{-1}(\Omega))\times L^2(\Omega;\mathbb{R}^2)$ with $\mathrm{supp}\,\mu\subset\subset\Omega$ and $-\mathrm{Div}\,T^{\mathrm{D}}=\pi\mu$ there exists a sequence $\{(\mu_k,T_k^{\mathrm{D}})\}_{k\in\mathbb{N}}\subset(\mathcal{M}(\Omega)\cap H^{-1}(\Omega))\times L^2(\Omega;\mathbb{R}^2)$ with $\mathrm{supp}\,\mu_k\subset\subset\Omega$ and $-\mathrm{Div}T_k^{\mathrm{D}}=\pi\mu_k$ for every $k\in\mathbb{N}$ such that μ_k is locally constant for every k (and takes the form as in Case 1, i.e., $\mu=\sum_{l=1}^{L}m^l\chi_{\omega^l}\mathrm{d}x$, where $L\in\mathbb{N}$, $m^l\in\mathbb{R}$, and $\{\omega^l\}_{l=1,\dots,L}$ is a partition of Ω into sets with Lipschitz continuous boundary), and

$$\mu_k\overset{*}{\rightharpoonup}\mu\,,\qquad|\mu_k|(\Omega)\to|\mu|(\Omega)\,,\qquad T_k^{\mathrm{D}}\to T^{\mathrm{D}}\quad\text{in } L^2(\Omega;\mathbb{R}^2)\qquad\text{as } k\to+\infty\,.\tag{5.52}$$

First, let $\{\rho_h\}_{h>0}$ be a sequence of standard mollifiers. We define

$$f_h:=\mu*\rho_h\,,\qquad\mu_h:=f_h\,\mathrm{d}x\,,\qquad T_h^{\mathrm{D}}:=(T^{\mathrm{D}}*\rho_h)\llcorner\Omega$$

By construction, $-\mathrm{Div}\, T_h^{\mathrm{D}} = 2\pi\mu_h$ for every $h > 0$ and, for h small enough, $|\mu_h|(\partial\Omega) = 0$. Moreover,

$$\mu_h \overset{*}{\rightharpoonup} \mu\,, \qquad |\mu_h|(\Omega) \to |\mu|(\Omega)\,, \qquad T_h^{\mathrm{D}} \to T^{\mathrm{D}} \text{ in } L^2(\Omega;\mathbb{R}^2)\,.$$

Furthermore, since $\{f_h\}_{h>0} \subset C^\infty(\Omega)$, for every $h > 0$, there exists a sequence $\{f_h^j\}_{j\in\mathbb{N}}$ with f_h^j as in Case 1 (i.e., $f_h^j = \sum_{l=1}^{L_h^j} (m_h^j)^l \chi_{(\omega_h^j)^l}$, where $L_h^j \in \mathbb{N}$, $(m_h^j)^l \in \mathbb{R}$, and $\{(\omega_h^j)^l\}_{l=1,\dots,L_h^j}$ is a partition of Ω into sets with Lipschitz continuous boundary), such that

$$\|f_h^j - f_h\|_{L^\infty(\Omega)} \to 0 \qquad \text{and} \qquad \int_\Omega (f_h^j - f_h)\,\mathrm{d}x = 0\,.$$

For every $h > 0$ and $j \in \mathbb{N}$, let w_h^j be the solution to

$$\begin{cases} -\Delta w = f_h^j - f_h & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega\,. \end{cases}$$

By standard elliptic estimates, we have

$$\|\nabla w_h^j\|_{L^2(\Omega;\mathbb{R}^2)} \le C\|f_h^j - f_h\|_{L^2(\Omega)}\,.$$

Finally, for every $h > 0$ and for any $j \in \mathbb{N}$, we set $(T^{\mathrm{D}})_h^j := T_h^{\mathrm{D}} + 2\pi\nabla w_h^j$, so that $-\mathrm{Div}\,(T^{\mathrm{D}})_h^j = 2\pi\mu_h^j$, and, for every $h > 0$,

$$(T^{\mathrm{D}})_h^j \to T_h^{\mathrm{D}} \qquad \text{in } L^2(\Omega;\mathbb{R}^2) \qquad (\text{as } j \to +\infty)\,.$$

Using a standard diagonal argument one can find a sequence $\{(\mu_k, T_k^{\mathrm{D}})\}_{k\in\mathbb{N}}$ satisfying (5.52). Finally, by arguing as in the second step of Case 1, we can construct the recovery sequence for the functional $\mathcal{G}_\varepsilon$. □

6. Proof of Theorem 3.4

This section is devoted to the proof of Theorem 3.4.

Proof of Theorem 3.4. The compactness statement follows immediately by (3.14) and (1.10).

Analogously, the lower bound (3.15) is a consequence of (1.10) and of the lower semicontinuity of the L^2 norm with respect to the weak convergence.

Therefore, it remains to prove only the upper bound. The proof is fully analogous to that of Theorem 3.3(iii) in Subsection 5.2. We briefly sketch it. Let $\beta \in L^2(\Omega;\mathbb{R}^2)$ be such that $\beta^\perp = T^{\mathrm{D}}$ and we set $\mu := \frac{1}{2\pi}\mathrm{curl}\,\beta = -\frac{1}{2\pi}\mathrm{Div} T^{\mathrm{D}}$. Then $\mu \in \mathcal{M}(\Omega) \cap H^{-1}(\Omega)$. Moreover, by construction, $\mathrm{supp}\,\mu \subset\subset \Omega$.

We show how to prove the claim only in the case $\mu := \sum_{l=1}^L m^l \chi_{\omega^l}\mathrm{d}x$, where $L \in \mathbb{N}$, $m^l \in \mathbb{R}$ and $\{\omega^l\}_{l=1,\dots,L}$ is a partition of Ω into sets with Lipschitz continuous boundary. Indeed, the general case follows by the former by arguing verbatim as in the proof of Theorem 3.3(iii). Let

$$\mu_\varepsilon := \sum_{l=1}^{L}\sum_{n=1}^{N_\varepsilon^l} \delta_{x_\varepsilon^{l,n}} \tag{6.1}$$

be the measure provided by Lemma 5.1 and let $\overline{\beta}_\varepsilon \in L^2(\Omega;\mathbb{R}^2)$ be the field defined in (5.30); in particular, we recall the fields $\widehat{K}_\varepsilon$ and $\widetilde{K}_\varepsilon$ defined in (5.25). By arguing verbatim as in Case 1 (first step) of the proof of Theorem 3.3(iii), and using that here $N_\varepsilon \gg |\log\varepsilon|$, we have that

$$\begin{aligned} \lim_{\varepsilon\to 0}\frac{1}{N_\varepsilon^2}\int_\Omega |\widehat{K}_\varepsilon|^2\,\mathrm{d}x &= \lim_{\varepsilon\to 0}\frac{1}{N_\varepsilon^2}\sum_{l=1}^{L}\sum_{n=1}^{N_\varepsilon^l}\int_{A_{\varepsilon,r_\varepsilon}(x_\varepsilon^n)} |\widehat{K}_\varepsilon|^2\,\mathrm{d}x \\ &= \lim_{\varepsilon\to 0}\sum_{l=1}^{L}\sum_{n=1}^{N_\varepsilon^l}\frac{1}{2\pi}\frac{\log r_\varepsilon^l + |\log\varepsilon|}{N_\varepsilon^2} = 0\,; \end{aligned}$$

analogously, by arguing as in (5.37) and (5.28), we get

$$\lim_{\varepsilon\to 0}\frac{1}{N_\varepsilon^2}\int_\Omega |\widetilde{K}_\varepsilon|^2\,\mathrm{d}x = \lim_{\varepsilon\to 0}\frac{1}{N_\varepsilon^2}\int_\Omega |\nabla v_\varepsilon|^2\,\mathrm{d}x = 0\,.$$

Therefore,

$$\frac{\overline{\beta}_\varepsilon}{N_\varepsilon} \to 0 \qquad \text{strongly in } L^2(\Omega;\mathbb{R}^2)\,. \tag{6.2}$$

Finally, defining the sequence $\{u_\varepsilon\}_\varepsilon$ as in (5.41), by (5.42) and (5.45), we have that

$$\lim_{\varepsilon\to 0}\frac{1}{N_\varepsilon^2}\mathcal{G}_\varepsilon(u_\varepsilon) = \frac{1}{2}\int_\Omega |\beta|^2\,\mathrm{d}x = \frac{1}{2}\int_\Omega |T^{\mathrm{D}}|^2\,\mathrm{d}x\,,$$

which concludes the proof of the claim in this case. □

Conflict of interest. The authors declare that they have no conflict of interest and guarantee the compliance with the Ethics Guidelines of the journal.

Data Availibility Statement. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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