A Finite-Differences Approximation of Fracture Energies for Non-linear Elastic Materials

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Abstract

We provide a variational approximation by finite-difference schemes of the weak formulation of Griffith's type functionals, i.e.,

$$\int_\Omega \psi(\nabla u)\, dx + \int_K g\left([u], \nu\right)\, d\mathcal{H}^1$$

defined for $u \in C^1(\Omega \setminus K; \mathbb{R}^N)$, where Ω is an open set in $\mathbb{R}^2$ and K is a closed hypersurface with normal ν. Here $[u]$ is the jump of u through K along ν and $\mathcal{H}^1$ is the 1-dimensional Hausdorff measure. These energies are related to variational models in fracture mechanics for non-linear elastic materials. Indeed, ψ is a quasi-convex function with p-growth, $p > 1$, and g satisfies standard lower semicontinuity conditions. The approximating functionals, in the sense of De Giorgi's Γ-convergence, are of the form

$$\sum_{\alpha \in \varepsilon\mathbb{Z}^2 \cap \Omega} \varepsilon^2 \psi_\varepsilon\left(D_\varepsilon u(\alpha)\right)$$

where ψ_ε is an interaction potential taking into account a separation of scales, u is a 'discrete' function defined on the lattice $\varepsilon\mathbb{Z}^2$ and $D_\varepsilon u$ is a suitable finite-difference discretization of the gradient.

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1 Introduction

Recently the theory of non-linear elasticity has turned out to be an useful tool to model phase transitions and fracture phenomena for hyperelastic and brittle media. The mechanical properties of such materials are characterized by a stored energy density ψ in terms of which the total stored energy for a smooth deformation u is given by

$$E(u) = \int_\Omega \psi\left(\nabla u\right)\, dx,$$

where $\Omega \subseteq \mathbb{R}^n$ is the reference configuration, $u : \Omega \to \mathbb{R}^N$ and ∇u is the deformation gradient.

The proper general mathematical assumption to require on ψ is that it be quasi-convex (see Definition 2.9). As observed by Ball [7], this condition is satisfied by stored energy functions complying with the constitutive assumptions of non-linear elasticity. Indeed, this condition is physically realistic for anisotropic homogeneous materials and, mathematically, it is sufficient and necessary to obtain the weak lower semicontinuity of the functional E, from which it is easy to infer the existence of minimum problems with given boundary data, and hence to state the existence of equilibria.

In the framework of Griffith's theory of fracture mechanics, the total stored energy consists of a volume and a surface term, the last one accounting for crack initiation. The natural class of deformations to be considered is the space $SBV\left(\Omega;\mathbb{R}^N\right)$ of special functions with bounded variation, introduced by De Giorgi and Ambrosio [19]. Thus, the total energy can be modelled by an elastic term independent of the crack, and a surface term which, in general, depends on the orientation of the crack surface, i.e.,

$$\mathcal{E}(u) = \int_\Omega \psi\left(\nabla u\right)\,dx + \int_{J_u} \varphi(\nu_u)\,d\mathcal{H}^{n-1}, \tag{1.1}$$

with $\varphi : \mathbb{S}^{n-1} \to [0,+\infty)$ a convex function such that $0 < \alpha \leq \varphi \leq \beta$. We recall that the formulation in (1.1) is the weak definition of the Griffith's energy still due to De Giorgi and Ambrosio (see [19],[5]).

According to a recent trend of research, which deduces a continuum theory starting from an atomistic description of the media (see [29],[12],[14],[15],[4],[28]), we provide a variational approximation, in dimension 2, of energies of the type (1.1). More precisely, we introduce a discrete model based on finite-differences discretization schemes, following the approach proposed by De Giorgi to treat the Mumford-Shah problem in Computer Vision (see [24]), and applied to Fracture Mechanics firstly by Braides, Dal Maso and Garroni [12] and then by Braides and Gelli [14],[15] (see also [29],[11],[28]).

The previous results study the scalar case; here we deal with the vectorial one. We prove the Γ-convergence to energies of type (1.1) of the family of approximating functionals defined as

$$\sum_{\alpha\in\mathcal{R}_\varepsilon} \frac{\varepsilon^2}{2}\psi_\varepsilon\left(\frac{1}{\varepsilon}D_\varepsilon u(\alpha)\right) \tag{1.2}$$

where $\mathcal{R}_\varepsilon$ is a portion of the lattice $\varepsilon\mathbb{Z}^2$, ψ_ε is a suitable non-convex interaction potential and, for $u : \varepsilon\mathbb{Z}^2 \to \mathbb{R}^N$, the matrix $D_\varepsilon u$ is an 'ε-discretization of the gradient'.

Since we are interested in non-isotropic bulk energy densities, a quite natural choice for $D_\varepsilon u$, in order to recover the global behaviour of the gradient matrix, is the finite-differences matrix below

$$D^+_\varepsilon u(\alpha) = \left(\langle u(\alpha+\varepsilon \mathrm{e}_i) - u(\alpha), e_k\rangle\right)_{\substack{i=1,2\\ k=1,\dots,N}}. \tag{1.3}$$

Indeed, the scalar models considered so far rely on discrete approximations, $d^i_\varepsilon u$, accounting for increments only along given integer directions, i.e., $D_\varepsilon u$ in (1.2) has to be replaced by

$$d^i_\varepsilon u(\alpha) = u(\alpha + \varepsilon \mathrm{e}_i) - u(\alpha);$$

while in the vectorial case, developed for linear elasticity, it is sufficient to choose the increment along a given direction of the component of u in the same direction, i.e.,

$$d^i_\varepsilon u(\alpha) = \langle u(\alpha + \varepsilon \mathrm{e}_i) - u(\alpha), \mathrm{e}_i \rangle.$$

These approaches allow to get a complete characterization of the limit by studying the asymptotic behaviour of 1-dimensional functionals. On the other hand, the only possible bulk energy densities obtained as limit are those determined by summing up all the contribution on fixed directions.

Note that definition (1.3) of $D^+_\varepsilon u(\alpha)$ takes into account the interactions of a given point α in the lattice $\varepsilon\mathbb{Z}^2$ with the nodes $\alpha + \varepsilon \mathrm{e}_1, \alpha + \varepsilon \mathrm{e}_2$, so that the unitary cell underlying such an interpolation is the simplex S_ε given by the convex hull of $\{(0,0), (\varepsilon, 0), (0, \varepsilon)\}$. Hence, to identify a discrete function $u : \varepsilon\mathbb{Z}^2 \to \mathbb{R}^N$ with a piecewise affine function defined on the whole $\mathbb{R}^2$ we are forced to introduce also the finite-differences matrix $D^-_\varepsilon u(\alpha)$ defined as

$$D^-_\varepsilon u(\alpha) = \big(\langle u(\alpha) - u(\alpha - \varepsilon \mathrm{e}_i), e_k \rangle \big)_{\substack{i=1,2 \\ k=1,\dots,N}}$$

that takes into account the interactions of α with $\alpha - \varepsilon \mathrm{e}_1$, $\alpha - \varepsilon \mathrm{e}_2$. This piecewise affine identification will be crucial to use lower semicontinuity results for functionals depending on the whole gradient variable and will determine the estimate on the volume part of the limit.

For what the crack term is concerned, the surface energy density is determined by the behaviour of $\psi_\varepsilon(X)$ for $|X| \geq T_\varepsilon$, where $T_\varepsilon \to +\infty$ and satisfies some technical conditions. This threshold condition is global, in the sense that high gradient elongations of each component of the deformation in every direction are penalized.

We give two different approximations related to different definitions of ψ_ε. To deal with the first one we can use an integral geometric reduction technique and we obtain as limit functionals of the form

$$\sum_{i=1,2} \int_{J_u} g_i \left(u^+ - u^- \right) |\langle \nu_u, \mathrm{e}_i \rangle| \, d\mathcal{H}^1$$

with $g_i : \mathbb{R}^N \setminus \{0\} \to [0, +\infty)$ subadditive, continuous functions, superlinear at 0 (for more details see Section 3).

The second model accounts for interactions between difference quotients in directions $\mathrm{e}_1, \mathrm{e}_2$. Such a model requires more sophisticated tools (see Lemma 2.14) and a new construction must be performed in order to have an estimate along direction $\mathrm{e}_2 - \mathrm{e}_1$, that is, a direction in which difference quotients are not involved.

This difficulty can be bypassed by considering the lattice generated by vectors $e_1, e_2 - e_1$ and by constructing a 1-dimensional profiled function, affine on the slanted unitary cell P_ε (see Proposition 3.8). The result of this model yields an anisotropic surface term that penalizes in different way crack sites, according to their orientation with respect to the basis $\{e_1, e_2\}$.

The plan of the paper is the following: Section 2 is devoted to definitions and preliminary results concerning (generalized) functions with bounded variation; Section 3 contains the main results that are Theorem 3.1 and Theorem 3.2; eventually in Section 4 we provide further results concerning the approximation of more general fracture energies penalizing crack sites in a different way.

2 Notations and Preliminaries

If $a, b \in \mathbb{R}$ we write $a \wedge b$ and $a \vee b$ for the minimum and maximum between a and b, respectively.

We denote by $\langle \cdot, \cdot \rangle$ the scalar product in $\mathbb{R}^n$ and with $|\cdot|$ the usual euclidean norm, without specifying the dimension n when there is no risk of confusion.
For every $x, y \in \mathbb{R}^n$, $[x, y]$ denotes the segment between x and y.

If Ω is a bounded open subset of $\mathbb{R}^n$, for every $\eta > 0$ set $\Omega_\eta := \{x \in \Omega : d(x, \partial\Omega) > \eta\}$.

Let $\mathcal{A}(\Omega)$ and $\mathcal{B}(\Omega)$ be the families of open and Borel subsets of Ω, respectively. If μ is a Borel measure and B is a Borel set, then the measure $\mu \llcorner B$ is defined as $\mu \llcorner B(A) = \mu(A \cap B)$. We denote by $\mathcal{L}^n$ the Lebesgue measure in $\mathbb{R}^n$ and by $\mathcal{H}^k$ the k-dimensional Hausdorff measure, $k \geq 0$.

The notation *a.e.* stands for almost everywhere with respect to the Lebesgue measure, unless otherwise specified.
We use standard notations for Lebesgue and Sobolev spaces.

Let us recall also the notion of convergence in measure. We say that a sequence of Borel functions u_j converges to u in measure if for every $\eta > 0$ we have $\lim_j \mathcal{L}^n(\{x \in \Omega : |u_j(x) - u(x)| > \eta\}) = 0$. The space $L^1(\Omega; \mathbb{R}^N)$, when endowed with this convergence, is metrizable, an example of metric being

$$d(u, v) := \int_\Omega \frac{|u(x) - v(x)|}{1 + |u(x) - v(x)|}\, dx$$

for $u, v \in L^1(\Omega; \mathbb{R}^N)$.

2.1 BV functions

Let Ω be a bounded open set of $\mathbb{R}^n$. If $u \in L^1(\Omega; \mathbb{R}^N)$, we denote by S_u the complement of the Lebesgue set of u, i.e., $x \notin S_u$ if and only if

$$\lim_{\rho \to 0} \rho^{-n} \int_{B_\rho(x)} |u(y) - z|\, dy = 0$$

for some $z \in \mathbb{R}^N$, where $B_\rho(x) := \{y \in \mathbb{R}^n : |y-x| < \rho\}$. If z exists then it is unique and we denote it by $\bar{u}(x)$. The set S_u is Lebesgue negligible and thus $\bar{u}$ is a Borel function equal to u a.e. in Ω.

Moreover, we say that $x \in \Omega$ is a *jump point* of u, and we denote by J_u the set of all such points for u, if there exist $a, b \in \mathbb{R}^N$ and $\nu \in \mathbb{S}^{n-1}$ such that $a \neq b$ and

$$\lim_{\rho \to 0} \rho^{-n} \int_{B_\rho^+(x,\nu)} |u(y) - a|\, dy = 0, \quad \lim_{\rho \to 0} \rho^{-n} \int_{B_\rho^-(x,\nu)} |u(y) - b|\, dy = 0, \tag{2.4}$$

where $B_\rho^\pm(x,\nu) := \{y \in B_\rho(x) : \pm \langle y - x, \nu \rangle > 0\}$.

The triplet (a, b, ν), uniquely determined by (2.4) up to a permutation of (a, b) and a change of sign of ν, will be denoted by $(u^+(x), u^-(x), \nu_u(x))$. Notice that J_u is a Borel subset of S_u.

We say that u is *approximately differentiable* at a Lebesgue point x if there exists $L \in \mathbb{R}^{N \times n}$ such that

$$\lim_{\rho \to 0} \rho^{-n-1} \int_{B_\rho(x)} |u(y) - \bar{u}(x) - L(y-x)|\, dy = 0. \tag{2.5}$$

If u is approximately differentiable at a Lebesgue point x, then L, uniquely determined by (2.5), will be denoted by $\nabla u(x)$ and will be called the *approximate gradient* of u at x.

Eventually, given a Borel set $J \subset \mathbb{R}^n$, we say that J is $\mathcal{H}^{n-1}$-*rectifiable* if

$$J = N \cup \bigcup_{i \geq 1} K_i$$

where $\mathcal{H}^{n-1}(N) = 0$ and each K_i is a compact subset of a C^1 $(n-1)$-dimensional manifold. Thus, for a $\mathcal{H}^{n-1}$-rectifiable set J it is possible to define $\mathcal{H}^{n-1}$ a.e. a unitary normal vector field ν.

We recall some definitions and properties concerning functions with bounded variation. The main reference is the book [6] (see also [21],[23]).

Definition 2.1 *Given $u \in L^1(\Omega; \mathbb{R}^N)$, we say that u is a* function with bounded variation *in Ω, and we write $u \in BV(\Omega; \mathbb{R}^N)$, if the distributional derivative Du of u is a $N \times n$ matrix-valued measure on Ω with finite total variation.*

If $u \in BV(\Omega; \mathbb{R}^N)$, then u is approximately differentiable a.e. in Ω, the set $S_u \setminus J_u$ is $\mathcal{H}^{n-1}$-negligible, i.e., $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$, and J_u turns out to be $\mathcal{H}^{n-1}$-rectifiable.

Let us consider the Lebesgue decomposition of Du with respect to $\mathcal{L}^n$, i.e., $Du = D^a u + D^s u$, where $D^a u$ is the absolutely continuous part and $D^s u$ is the singular one. The density of $D^a u$ with respect to $\mathcal{L}^n$ coincides a.e. with the approximate gradient ∇u of u. Define the *jump part* of Du, $D^j u$, to be the restriction of $D^s u$ to J_u, and the *Cantor part*, $D^c u$, to be the restriction of $D^s u$ to $\Omega \setminus S_u$, thus we have

$$Du = D^a u + D^j u + D^c u.$$

Moreover, the following representation formula holds true for the jump part of Du

$$D^j u = \left(u^+ - u^-\right) \otimes \nu_u \mathcal{H}^{n-1} \llcorner J_u,$$

where given $a \in \mathbb{R}^N$ and $b \in \mathbb{R}^n$ $a \otimes b$ denotes the $N \times n$ matrix whose entries are $a_i b_j$, $1 \leq i \leq N$ and $1 \leq j \leq n$.

Definition 2.2 *Given $u \in BV(\Omega;\mathbb{R}^N)$, we say that u is a* special function with bounded variation *in Ω, and we write $u \in SBV(\Omega;\mathbb{R}^N)$, if $D^c u = 0$.*

Functionals involved in free-discontinuity problems are often not coercive in $SBV(\Omega;\mathbb{R}^N)$, then it is useful to consider the following wider class (see [19],[6]).

Definition 2.3 *Given $u \in L^1(\Omega;\mathbb{R}^N)$, we say that u is a* generalized special function with bounded variation *in Ω, and we write $u \in GSBV(\Omega;\mathbb{R}^N)$, if $g(u) \in SBV(\Omega)$ for every $g \in C^1(\mathbb{R}^N)$ such that ∇g has compact support.*

In case $N = 1$, $u \in GSBV(\Omega)$ if and only if $u_T := (u \vee (-T) \wedge T) \in SBV(\Omega)$ for every $T > 0$; while for $N > 1$ the product space $(GSBV(\Omega))^N$ is strictly contained in $GSBV\left(\Omega;\mathbb{R}^N\right)$ even if $\Omega \subseteq \mathbb{R}$ (see Remark 4.27 of [6]). Functions u in $GSBV(\Omega;\mathbb{R}^N)$ are approximately differentiable a.e. in Ω and J_u turns out to be $\mathcal{H}^{n-1}$-rectifiable. Notice that $GSBV \cap L^\infty(\Omega;\mathbb{R}^N) = SBV \cap L^\infty(\Omega;\mathbb{R}^N)$. Moreover, define for any $p \geq 1$

$$GSBV^p(\Omega) := \{u \in GSBV(\Omega) : \nabla u \in L^p(\Omega;\mathbb{R}^n),\ \mathcal{H}^{n-1}(J_u) < +\infty\}. \tag{2.6}$$

Hence, if $u \in (GSBV^p(\Omega))^N \setminus SBV(\Omega;\mathbb{R}^N)$ then $|u^+ - u^-| \notin L^1\left(J_u;\mathcal{H}^{n-1}\right)$. Nevertheless, it is still possible to prove that for every $u \in (GSBV^p(\Omega))^N$

$$\mathcal{H}^{n-1}\left(\left\{x \in \Omega :\ \left|u^\pm(x)\right| = \infty\right\}\right) = 0 \tag{2.7}$$

(see Theorem 4.40 of [6]).

The main features of the space $GSBV\left(\Omega;\mathbb{R}^N\right)$ are the following closure and compactness theorems (see [6]). Let us first fix some notations. In the following we will denote by $\phi : [0,+\infty) \to [0,+\infty]$ and $\gamma : (0,+\infty) \to (0,+\infty]$ two lower semicontinuous increasing functions such that

$$\lim_{t\to+\infty} \frac{\phi(t)}{t} = +\infty, \quad \lim_{t\to 0^+} \frac{\gamma(t)}{t} = +\infty. \tag{2.8}$$

Theorem 2.4 *Let $(u_j) \subset GSBV\left(\Omega;\mathbb{R}^N\right)$ be satisfying*

$$\sup_j \left\{ \int_\Omega \phi\left(|\nabla u_j|\right) dx + \int_{J_{u_j}} \gamma\left(\left|u_j^+ - u_j^-\right|\right) d\mathcal{H}^{n-1} \right\} < +\infty. \tag{2.9}$$

If $u_j \to u$ in measure on Ω, then $u \in GSBV\left(\Omega;\mathbb{R}^N\right)$ and *(i)* $\nabla u_j \to \nabla u$ *weakly in* $L^1\left(\Omega;\mathbb{R}^{N\times n}\right)$*;*

(ii) $D^j u_j$ *converges weakly in the sense of measures to* $D^j u$*;*

(iii) if ϕ *is convex then*

$$\int_\Omega \phi\left(|\nabla u|\right)\,dx \le \liminf_j \int_\Omega \phi\left(|\nabla u_j|\right)\,dx;$$

(iv) if γ *is subadditive (see Definition 2.11 below) then*

$$\int_{J_u} \gamma\left(\left|u^+ - u^-\right|\right)\,d\mathcal{H}^{n-1} \le \liminf_j \int_{J_{u_j}} \gamma\left(\left|u_j^+ - u_j^-\right|\right)\,d\mathcal{H}^{n-1}.$$

Theorem 2.5 *Let* ϕ*,* γ *satisfy (2.8). Consider* $(u_j) \subset GSBV\left(\Omega;\mathbb{R}^N\right)$ *be satisfying (2.9) and assume, in addition, that* $\|u_j\|_{q,\Omega}$ *is uniformly bounded in* j *for some* $q \ge 1$*, then there exists a subsequence* (u_{j_k}) *and a function* $u \in GSBV\left(\Omega;\mathbb{R}^N\right)$*, such that* $u_{j_k} \to u$ *in measure on* Ω*.*

The original proofs of Theorem 2.4 and Theorem 2.5 make use of the one dimensional sections of BV functions which turned out to provide a useful tool also in the study of the variational approximation of free-discontinuity problems.

Before recalling the 'slicing' theorem (see [6]) let us fix some notations. Let $\xi \in \mathbb{S}^{n-1}$, let Π^ξ be the orthogonal space to ξ, i.e., $\Pi^\xi = \{y \in \mathbb{R}^n : \langle \xi, y\rangle = 0\}$. If $y \in \Pi^\xi$ and $E \subset \mathbb{R}^n$ define $E_y^\xi = \{t \in \mathbb{R} : y + t\xi \in E\}$, and given $u : E \to \mathbb{R}^N$ set $u^{\xi,y} : E_y^\xi \to \mathbb{R}^N$ by $u^{\xi,y}(t) := u(y + t\xi)$.

Theorem 2.6 *Let* $u \in (GSBV(\Omega))^N$*, then* $u^{\xi,y} \in \left(GSBV\left(\Omega_y^\xi\right)\right)^N$ *for all* $\xi \in \mathbb{S}^{n-1}$ *and* $\mathcal{H}^{n-1}$ *a.e.* $y \in \Pi^\xi$*. For such* y *we have*

(i) $\dot{u}^{\xi,y}(t) = \langle \nabla u\,(y + t\xi)\,, \xi\rangle$ *for* $\mathcal{L}^1$ *a.e.* $t \in \Omega_y^\xi$*;*

(ii) $J_{u^{\xi,y}} = (J_u)_y^\xi$*;*

(iii) $\left(u^{\xi,y}\right)^{\pm}(t) = u^{\pm}(y + t\xi)$ *or* $\left(u^{\xi,y}\right)^{\pm}(t) = u^{\mp}(y + t\xi)$ *according to the cases* $\langle \nu_u, \xi\rangle > 0$*,* $\langle \nu_u, \xi\rangle < 0$ *(the case* $\langle \nu_u, \xi\rangle = 0$ *being negligible).*

Moreover, for every open set $A \in \mathcal{A}(\Omega)$ *and for every function* $g \in L^1\left(J_u;\mathcal{H}^{n-1}\right)$ *there holds*

$$\int_{\Pi^\xi}\int_{J_{u^{\xi,y}}\cap A} g^{\xi,y}(t)\,d\mathcal{H}^0(t)\,d\mathcal{H}^{n-1}(y) = \int_{J_u\cap A} g(x)|\langle \nu_u(x),\xi\rangle|\,d\mathcal{H}^{n-1}(x). \quad (2.10)$$

2.1.1 Density of piecewise smooth functions in SBV

Let us now introduce a useful sub-space of SBV functions.

Definition 2.7 *Let $\mathcal{W}\left(\Omega;\mathbb{R}^N\right)$ be the space of all $u \in SBV\left(\Omega;\mathbb{R}^N\right)$ such that*

(i) J_u *is* essentially closed, *i.e.,* $\mathcal{H}^{n-1}\left(\overline{J_u}\setminus J_u\right)=0$;

(ii) $\overline{J_u}$ *is a* polyhedral set, *i.e.,* $\overline{J_u}$ *is the intersection of* Ω *with the union of a finite number of* $(n-1)$*-dimensional simplexes;*

(iii) $u \in W^{k,\infty}\left(\Omega\setminus\overline{J_u};\mathbb{R}^N\right)$ *for every* $k\in\mathbb{N}$.

The following theorem proved by Cortesani and Toader [17] provides a density result of the class $\mathcal{W}\left(\Omega;\mathbb{R}^N\right)$ in $SBV\cap L^\infty\left(\Omega;\mathbb{R}^N\right)$ with respect to anisotropic surface energies.

Theorem 2.8 *Let* $\Omega\subset\mathbb{R}^n$ *be a bounded open set with lipschitz boundary. Let* $u\in\left(GSBV^p(\Omega)\right)^N\cap L^\infty\left(\Omega;\mathbb{R}^N\right)$ *for some* $p\geq 1$*, then there exists a sequence* $(u_j)\subset\mathcal{W}\left(\Omega;\mathbb{R}^N\right)$ *such that*

(i) $u_j\to u$ *strongly in* $L^1\left(\Omega;\mathbb{R}^N\right)$;

(ii) $\nabla u_j\to\nabla u$ *strongly in* $L^p\left(\Omega;\mathbb{R}^{N\times n}\right)$;

(iii) $\limsup_j\|u_j\|_{\infty,\Omega}\leq\|u\|_{\infty,\Omega}$;

(iv) for every upper semicontinuous function $\varphi:\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{n-1}\to[0,+\infty)$ *such that* $\varphi(z_1,z_2,\nu)=\varphi(z_2,z_1,-\nu)$ *for every* $z_1,z_2\in\mathbb{R}^N$ *and* $\nu\in\mathbb{S}^{n-1}$*, there holds*

$$\limsup_j\int_{J_{u_j}}\varphi\left(u_j^-,u_j^+,\nu_{u_j}\right)\,d\mathcal{H}^{n-1}\leq\int_{J_u}\varphi\left(u^-,u^+,\nu_u\right)\,d\mathcal{H}^{n-1}. \tag{2.11}$$

2.1.2 Lower semicontinuity in GSBV

In this subsection we state some semicontinuity results for variational integrals along sequences satisfying the GSBV compactness assumption (2.9).

The first one, proved by Kristensen in [26] in a more general version (see also [6]), deals with the lower semicontinuity of volume energies. Let us first recall the notion of quasi-convexity.

Definition 2.9 *We say that a continuous integrand* $\psi:\mathbb{R}^{N\times n}\to[0,+\infty)$ *is* quasi-convex *if*

$$\psi(X)\,\mathcal{L}^n(\Omega)\leq\int_\Omega\psi\left(X+D\varphi(x)\right)\,dx \tag{2.12}$$

for every $X\in\mathbb{R}^{N\times n}$ *and* $\varphi\in C_c^1\left(\Omega;\mathbb{R}^N\right)$.

Theorem 2.10 *Let $\psi : \mathbb{R}^{N\times n} \to [0,+\infty)$ be a quasi-convex integrand such that*

$$|\psi(X)| \leq C\left(1+|X|^p\right)$$

for every $X \in \mathbb{R}^{N\times n}$ with $p>1$ and C a positive constant.
Let $(u_j) \subset GSBV\left(\Omega;\mathbb{R}^N\right)$ be satisfying (2.9) with $\phi(t)=t^p$; if $u_j \to u$ in measure then

$$\int_\Omega \psi(\nabla u)\, dx \leq \liminf_j \int_\Omega \psi(\nabla u_j)\, dx.$$

In order to state an analogous result for surface energies we need to introduce the notion of subadditivity.

Definition 2.11 *Let $\triangle := \left\{(z,z) : z \in \mathbb{R}^N\right\}$. We say that a function $\vartheta : \mathbb{R}^N \times \mathbb{R}^N \setminus \triangle \to [0,+\infty]$ is* subadditive *if for all distinct $z_i \in \mathbb{R}^N$, $i=1,2,3$, we have*

$$\vartheta(z_1,z_2) \leq \vartheta(z_1,z_3) + \vartheta(z_3,z_2).$$

We extend ϑ to the whole $\mathbb{R}^N \times \mathbb{R}^N$ setting $\vartheta \equiv 0$ on $\triangle$. Notice that, in case we consider traslation invariant functions, that is $\vartheta(z_1,z_2) = \vartheta(z_1-z_2,0) =: \theta(z_1-z_2)$ where $\theta : \mathbb{R}^N \setminus \{0\} \to [0,+\infty]$ and $\theta(0)=0$, the subadditivity condition on ϑ can be rewritten for θ as

$$\theta(z_1+z_2) \leq \theta(z_1) + \theta(z_2)$$

for every $z_i \in \mathbb{R}^N$, $i=1,2$.
The following result is an easy generalization to the vector-valued case of Theorem 4.3 of [10].

Theorem 2.12 *Let $\Omega \subset \mathbb{R}$ be a bounded open set, let $\vartheta : \mathbb{R}^N \times \mathbb{R}^N \to [0,+\infty]$ be a symmetric, subadditive and lower semicontinuous function.*

Let $(u_j) \subset SBV\left(\Omega;\mathbb{R}^N\right)$ be satisfying (2.9); if $u_j \to u$ in measure then $u \in SBV\left(\Omega;\mathbb{R}^N\right)$ and

$$\int_{J_u} \vartheta\left(u^+,u^-\right)\, d\mathcal{H}^0 \leq \liminf_j \int_{J_{u_j}} \vartheta\left(u_j^+,u_j^-\right)\, d\mathcal{H}^0.$$

Let us point out that the results stated in Theorem 2.12 heavily depend on the one-dimensional setting. Indeed, being $(u_j) \subset SBV\left(\Omega;\mathbb{R}^N\right)$ not equi-bounded a priori, by Theorem 2.4 we can only infer that $u \in (GSBV(\Omega))^N$. On the other hand, the superlinearity and the monotonicity assumptions on the function γ of (2.9) and the choice $n=1$ imply $u \in SBV\left(\Omega;\mathbb{R}^N\right)$.

2.2 Preliminary Lemmata

The following result will be used in the proof of the limsup inequality. Notice that the ideas and strategy used in the proof are strongly related to the regularity assumptions on the set J.

Lemma 2.13 *Let $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}$ and denote by $\xi^\perp := (-\xi_2, \xi_1)$. Let $\nu \in \mathbb{S}^1$ and let $J \subseteq \Pi^\nu$ be a closed set with $\mathcal{H}^1(J) < +\infty$. Define*

$$J_\varepsilon^{\xi,\pm} := \bigcup \left\{\alpha \pm S_\varepsilon^\xi : \left(\alpha \pm S_\varepsilon^\xi\right) \cap J \neq \emptyset\right\},$$

where $S^\xi := co\{(0,0); \xi; \xi^\perp\}$ and $S_\varepsilon^\xi := \varepsilon S^\xi$. Then

$$\limsup_{\varepsilon \to 0^+} \frac{\mathcal{L}^2\left(J_\varepsilon^{\xi,\pm}\right)}{\varepsilon} \leq \frac{1}{2} \int_J \varphi(\nu)\, d\mathcal{H}^1,$$

where $\varphi : \mathbb{S}^1 \to [0, +\infty)$ is defined as

$$\varphi(\nu) := \begin{cases} |\langle \nu, \mathrm{e}_1\rangle| \vee |\langle \nu, \mathrm{e}_2\rangle| & \text{if } \langle \nu, \mathrm{e}_1\rangle\langle \nu, \mathrm{e}_2\rangle \geq 0 \\ \\ |\langle \nu, \mathrm{e}_1\rangle| + |\langle \nu, \mathrm{e}_2\rangle| & \text{if } \langle \nu, \mathrm{e}_1\rangle\langle \nu, \mathrm{e}_2\rangle < 0. \end{cases} \tag{2.13}$$

Proof. Let $J^\eta := \{x \in \Pi^\nu : d(x, J) < \eta\}$, then there exists a sequence $(\eta_j) \subseteq (0,1)$ such that $\eta_j \to 0^+$ and $\mathcal{H}^1(J^{\eta_j}) = \mathcal{H}^1\left(\overline{J^{\eta_j}}\right) \to \mathcal{H}^1(J)$. It suffices then to prove the assertion for an open set $A \subseteq \Pi^\nu$ essentially closed, i.e., $\mathcal{H}^1(A) = \mathcal{H}^1\left(\overline{A}\right) < +\infty$.

Let $A = \cup_{r \geq 1} A_r$, where A_r are the connected components of A in Π^ν; since for every $M \in \mathbb{N}$

$$\mathcal{L}^2\left(A_\varepsilon^{\xi,\pm}\right) \leq \sum_{r=1}^{M} \mathcal{L}^2\left((A_r)_\varepsilon^{\xi,\pm}\right) + \mathcal{L}^2\left((\cup_{r \geq M} A_r)_\varepsilon^{\xi,\pm}\right), \tag{2.14}$$

we have that

$$\limsup_{\varepsilon \to 0^+} \frac{\mathcal{L}^2\left(A_\varepsilon^{\xi,\pm}\right)}{\varepsilon} \leq \sum_{r=1}^{M} \limsup_{\varepsilon \to 0^+} \frac{\mathcal{L}^2\left((A_r)_\varepsilon^{\xi,\pm}\right)}{\varepsilon} + 2\sqrt{2}|\xi|\mathcal{H}^1\left(\overline{\cup_{r \geq M} A_r}\right), \tag{2.15}$$

being the estimate on the second term in (2.14) due to a Minkowski's content argument (see Paragraph 2.13 of [6]). Since A is supposed to be essentially closed there follows

$$\mathcal{H}^1\left(\cup_{r \geq M} A_r\right) = \mathcal{H}^1\left(\overline{\cup_{r \geq M} A_r}\right).$$

Hence,

$$\sup_{r \geq M} \mathcal{H}^1\left(\overline{\cup_{r \geq M} A_r}\right) = 0,$$

and, by passing to the supremum on M in (2.15), we get

$$\limsup_{\varepsilon \to 0^+} \frac{\mathcal{L}^2\left(A_\varepsilon^{\xi,\pm}\right)}{\varepsilon} \leq \sum_{r \geq 1} \limsup_{\varepsilon \to 0^+} \frac{\mathcal{L}^2\left((A_r)_\varepsilon^{\xi,\pm}\right)}{\varepsilon}.$$

Thus, we may assume A to be an open interval in Π^ν and, without loss of generality, we may also assume $\xi = \mathrm{e}_1$. For $i = 1, 2$ let us define

$$\mathcal{I}_\varepsilon^{i,\pm} := \left\{\alpha \in \varepsilon\mathbb{Z}^2 : [\alpha, \alpha \pm \mathrm{e}_i] \cap A \neq \emptyset\right\};$$

then it can be easily proved that

$$\varepsilon \# \mathcal{I}_\varepsilon^{i,\pm} \leq \mathcal{H}^1(A)|\langle \nu, \mathrm{e}_i^\perp \rangle| + 2\varepsilon \tag{2.16}$$

(see Figure 1 (i), (ii)).

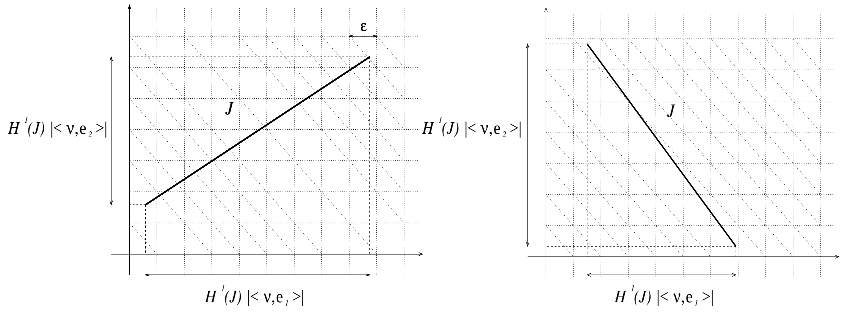

Figure 1: (i) case $\langle \nu, \mathrm{e}_1 \rangle \langle \nu, \mathrm{e}_2 \rangle \leq 0$ (ii) case $\langle \nu, \mathrm{e}_1 \rangle \langle \nu, \mathrm{e}_2 \rangle > 0$

Note that if $\langle \nu, \mathrm{e}_1 \rangle \langle \nu, \mathrm{e}_2 \rangle \leq 0$, then $\mathcal{I}_\varepsilon^{1,\pm} \cap \mathcal{I}_\varepsilon^{2,\pm} = \emptyset$, while if $\langle \nu, \mathrm{e}_1 \rangle \langle \nu, \mathrm{e}_2 \rangle > 0$, then either $\mathcal{I}_\varepsilon^{1,\pm} \subseteq \mathcal{I}_\varepsilon^{2,\pm}$ or $\mathcal{I}_\varepsilon^{2,\pm} \subseteq \mathcal{I}_\varepsilon^{1,\pm}$, according to the cases $|\langle \nu, \mathrm{e}_2 \rangle| \geq |\langle \nu, \mathrm{e}_1 \rangle|$, $|\langle \nu, \mathrm{e}_1 \rangle| \geq |\langle \nu, \mathrm{e}_2 \rangle|$. Hence, we will treat separately the two cases. Assume first that $\langle \nu, \mathrm{e}_1 \rangle \langle \nu, \mathrm{e}_2 \rangle \leq 0$, then by (2.16)

$$\frac{\mathcal{L}^2\left(A_\varepsilon^{\mathrm{e}_1,\pm}\right)}{\varepsilon} = \frac{\varepsilon}{2}\left(\#\mathcal{I}^{1,\pm} + \#\mathcal{I}_\varepsilon^{2,\pm}\right) \leq \frac{1}{2}\mathcal{H}^1(A)\varphi(\nu) + o(1)$$

and the thesis follows.
If, instead, $\langle \nu, \mathrm{e}_1 \rangle \langle \nu, \mathrm{e}_2 \rangle > 0$ and $|\langle \nu, \mathrm{e}_2 \rangle| \geq |\langle \nu, \mathrm{e}_1 \rangle|$, then $\mathcal{I}_\varepsilon^{1,\pm} \subseteq \mathcal{I}_\varepsilon^{2,\pm}$ and by (2.16)

$$\frac{\mathcal{L}^2\left(A_\varepsilon^{\mathrm{e}_1,\pm}\right)}{\varepsilon} = \frac{\varepsilon}{2}\#\mathcal{I}^{2,\pm} \leq \frac{1}{2}\mathcal{H}^1(A)\varphi(\nu) + o(1)$$

and the thesis follows. Analogously, we infer the thesis in the case $\langle \nu, \mathrm{e}_1 \rangle \langle \nu, \mathrm{e}_2 \rangle > 0$ and $|\langle \nu, \mathrm{e}_1 \rangle| \geq |\langle \nu, \mathrm{e}_2 \rangle|$. □

Eventually, we state the following result which will be useful in the sequel (see for instance [9]).

Lemma 2.14 *Let $\mu : \mathcal{A}(\Omega) \to [0, +\infty)$ be a superadditive function on disjoint open sets, let λ be a positive measure on Ω, let $\phi_h : \Omega \to [0, +\infty]$ be a countable family of Borel functions such that $\mu(A) \geq \int_A \phi_h d\lambda$ for every $A \in \mathcal{A}(\Omega)$.*

Set $\phi = \sup_{h\in\mathbb{N}} \phi_h$, then

$$\mu(A) \geq \int_A \phi \, d\lambda$$

for every $A \in \mathcal{A}(\Omega)$.

2.3 Γ-convergence

We recall the notion of Γ-convergence (see [20]). Let (X, d) be a metric space. A family $(F_\varepsilon)_{\varepsilon>0}$ of functionals $F_\varepsilon : X \to [0, +\infty]$ is said to *Γ-converge* to a functional $F : X \to [0, +\infty]$ at $u \in X$, and we write $F(u) = \Gamma\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon(u)$, if for every sequence (ε_j) of positive numbers decreasing to 0 the following two conditions hold:

(i) (*lower semicontinuity inequality*) for all sequences (u_j) converging to u in X we have $F(u) \leq \liminf_j F_{\varepsilon_j}(u_j)$;

(ii) (*existence of a recovery sequence*) there exists a sequence (u_j) converging to u in X such that $F(u) \geq \limsup_j F_{\varepsilon_j}(u_j)$.

We say that F_ε Γ-converges to F if $F(u) = \Gamma\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon(u)$ at all points $u \in X$ and that F is the *Γ-limit* of F_ε. If we define the *lower and upper Γ-limits* by

$$F'(u) = \Gamma\text{-}\liminf_{\varepsilon\to 0^+} F_\varepsilon(u) = \inf\{\liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon) : \ u_\varepsilon \to u\},$$

$$F''(u) = \Gamma\text{-}\limsup_{\varepsilon\to 0^+} F_\varepsilon(u) = \inf\{\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon) : \ u_\varepsilon \to u\},$$

respectively, then the conditions (i) and (ii) are equivalent to $F'(u) = F''(u) = F(u)$. Note that the functionals F' and F'' are lower semicontinuous.

In the sequel we will denote by $\Gamma(meas)$-lim inf, $\Gamma(meas)$-lim sup and $\Gamma(L^1)$-lim inf, $\Gamma(L^1)$-lim sup, the lower and upper Γ-limits on the space L^1 endowed with the metric of the L^1 strong convergence and the convergence in measure, respectively.

The variational characterization of this notion is explained by the following fundamental theorem.

Theorem 2.15 *Let $F = \Gamma\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon$, and let a compact set $K \subset X$ exist such that $\inf_X F_\varepsilon = \inf_K F_\varepsilon$ for all ε. Then*

$$\exists \min_X F = \lim_{\varepsilon\to 0^+} \inf_X F_\varepsilon. \tag{2.17}$$

Moreover, if (u_j) is a converging sequence such that $\lim_j F_{\varepsilon_j}(u_j) = \lim_j \inf_X F_{\varepsilon_j}$ then its limit is a minimum point for F.

We refer to [18] for an exposition of the main properties of Γ-convergence (see also [13]).

3 Main results

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set and denote $S_\varepsilon := S_\varepsilon^{e_1}$. For every subset $A \subseteq \Omega$ define the sets of indices

$$\mathcal{R}_\varepsilon^\pm(A) := \{\alpha \in \varepsilon\mathbb{Z}^2 : \ \alpha \pm S_\varepsilon \subseteq A\},$$
$$R_\varepsilon^\pm(A) := \{\alpha \in \varepsilon\mathbb{Z}^2 \setminus \mathcal{R}_\varepsilon^\pm(A) : \ (\alpha \pm S_\varepsilon) \cap A \neq \emptyset\},$$

which identify the simplexes properly contained in A and those intersecting its boundary ∂A, respectively. In case $A=\Omega$ we will drop the dependence on Ω in the definitions above.

Let us introduce a suitable discretization of the gradient in the coordinate directions. Let $\{\mathrm{e}_i\}_{i=1,2}$ and $\{e_k\}_{k=1,\dots,N}$ be the canonical bases of $\mathbb{R}^2$, $\mathbb{R}^N$, respectively. Let $u:\mathbb{R}^2\to\mathbb{R}^N$, for every $\varepsilon>0$ define $D_\varepsilon^\pm u$ to be the $N\times 2$ matrix

$$D_\varepsilon^\pm u(\alpha)=\left(\pm\langle u(\alpha\pm\varepsilon\mathrm{e}_i)-u(\alpha),e_k\rangle\right)_{\substack{i=1,2\\k=1,\dots,N}}$$

for every $\alpha\in\mathcal{R}_\varepsilon^\pm\cup R_\varepsilon^\pm$.

Moreover, set

$$\mathcal{A}_\varepsilon\left(\Omega;\mathbb{R}^N\right):=\{u:\Omega\to\mathbb{R}^N\,:u\equiv u(\alpha)\text{ on }(\alpha\pm S_\varepsilon)\cap\Omega\ \text{ for every }\alpha\in\varepsilon\mathbb{Z}^2\}.$$

Let $\psi:\mathbb{R}^{N\times 2}\to[0,+\infty)$ and $g_i:\mathbb{R}^N\setminus\{0\}\to[0,+\infty)$, $i=1,2$, satisfy

(h1) ψ is a quasi-convex function with superlinear growth, i.e., there exist $p>1$ and two positive constants $C_1,C_2>0$ such that

$$C_1\,|X|^p\le\psi\,(X)\le C_2\,(1+|X|^p)\tag{3.18}$$

for every $X\in\mathbb{R}^{N\times 2}$;

(h2) g_i is a symmetric, subadditive, lower semicontinuous function such that

$$\inf_{\mathbb{R}^N\setminus\{0\}}g_i>0;$$

(h3) g_i is an upper semicontinuous function bounded in a neighbourhood of $z=0$.

Notice that the subadditivity and the local boundedness assumptions on g_i imply the existence of a positive constant c_2 such that for every $z\in\mathbb{R}^N\setminus\{0\}$ and for $i=1,2$ there holds

$$c_1\le g_i(z)\le c_2(1+|z|),\tag{3.19}$$

where $c_1=\inf_{\mathbb{R}^N\setminus\{0\}}g_1\wedge\inf_{\mathbb{R}^N\setminus\{0\}}g_2$.

Let us also assume that

(h4) for every $u\in(GSBV^p(\Omega))^N$ there exists a sequence $(T_j)\subseteq[0,+\infty)$ with $T_j\to+\infty$ such that

$$\begin{aligned}\limsup_j\int_{J_u}g_i\left(\left(u_{T_j}\right)^+-\left(u_{T_j}\right)^-\right)|\langle\nu_u,\mathrm{e}_i\rangle|\,d\mathcal{H}^1\\=\int_{J_u}g_i\left(u^+-u^-\right)|\langle\nu_u,\mathrm{e}_i\rangle|\,d\mathcal{H}^1,\end{aligned}$$

where $u_T:=(\langle u,e_k\rangle_T)_{k=1,\dots,N}$.

Notice that this technical condition is fulfilled in case both the g_i are bounded on $\mathbb{R}^N \setminus \{0\}$. We will make further comments on this assumption in Remark 3.5 and Remark 3.11.

Let us introduce the approximating functionals. First, extend g_i to $\mathbb{R}^N$ setting $g_i(0) = 0$ thus preserving its lower semicontinuity property; then define $\psi_\varepsilon^g : \mathbb{R}^{N\times 2} \to [0,+\infty)$ as

$$\psi_\varepsilon^g(X) := \begin{cases} \psi(X) & \text{if } |X| \leq T_\varepsilon \\ \frac{1}{\varepsilon}\big(g_1(\varepsilon X \mathrm{e}_1) + g_2(\varepsilon X \mathrm{e}_2)\big) & \text{otherwise,} \end{cases}$$

where $(T_\varepsilon) \subset [0,+\infty)$ is such that $T_\varepsilon \to +\infty$ as $\varepsilon \to 0^+$ and $\sup_{\varepsilon>0}(\varepsilon T_\varepsilon^p) < +\infty$. Consider the family of functionals $\mathcal{F}_\varepsilon^g : L^1(\Omega;\mathbb{R}^N) \to [0,+\infty]$ given by

$$\mathcal{F}_\varepsilon^g(u) := \begin{cases} \sum\limits_{\alpha\in\mathcal{R}_\varepsilon^+} \frac{\varepsilon^2}{2}\psi_\varepsilon^g\big(\frac{1}{\varepsilon}D_\varepsilon^+ u(\alpha)\big) + \sum\limits_{\alpha\in\mathcal{R}_\varepsilon^-} \frac{\varepsilon^2}{2}\psi_\varepsilon^g\big(\frac{1}{\varepsilon}D_\varepsilon^- u(\alpha)\big) & \text{if } u \in \mathcal{A}_\varepsilon(\Omega;\mathbb{R}^N) \\ +\infty & \text{otherwise.} \end{cases} \tag{3.20}$$

Then the following result holds.

Theorem 3.1 *Let Ω be a bounded open set with Lipschitz boundary and assume* (h1)-(h4). *Then $(\mathcal{F}_\varepsilon^g)$ Γ-converges with respect to both the convergence in measure and strong $L^1(\Omega;\mathbb{R}^N)$ to the functional $\mathcal{F}^g : L^1(\Omega;\mathbb{R}^N) \to [0,+\infty]$ defined by*

$$\mathcal{F}^g(u) := \begin{cases} \int_\Omega \psi(\nabla u)dx + \int_{J_u} g(u^+ - u^-, \nu_u)\, d\mathcal{H}^1 & \text{if } u \in (GSBV(\Omega))^N \\ +\infty & \text{otherwise,} \end{cases}$$

where $g : \mathbb{R}^N \times \mathbb{S}^1 \to [0,+\infty)$ is defined by

$$g(z,\nu) := g_1(z)\,|\langle \nu, \mathrm{e}_1\rangle| + g_2(z)\,|\langle \nu, \mathrm{e}_2\rangle|.$$

Let us consider now the function $\psi_\varepsilon^\beta : \mathbb{R}^{N\times 2} \to [0,+\infty)$ defined by

$$\psi_\varepsilon^\beta(X) := \begin{cases} \psi(X) & \text{if } |X| \leq T_\varepsilon \\ \frac{1}{\varepsilon}\beta & \text{otherwise,} \end{cases}$$

where β is a positive constant. Note that even in case $g_1 = g_2 = \frac{\beta}{2}$ the functions ψ_ε^g, ψ_ε^β are different, since ψ_ε^g takes into account the values $X\mathrm{e}_1$, $X\mathrm{e}_2$ separately. If in definition (3.20) ψ_ε^g is substituted by ψ_ε^β, we prove the following result for the corresponding family of functionals $(\mathcal{F}_\varepsilon^\beta)$ (see Remark 3.6 for a comparison of the two results).

Theorem 3.2 *Let Ω be a bounded open set with lipschitz boundary and assume* (h1). *Then* $(\mathcal{F}^{\beta}_{\varepsilon})$ Γ*-converges with respect to both the convergence in measure and strong* $L^1(\Omega;\mathbb{R}^N)$ *to the functional* $\mathcal{F}^{\beta}: L^1(\Omega;\mathbb{R}^N) \to [0,+\infty]$ *defined by*

$$\mathcal{F}^{\beta}(u) := \begin{cases} \displaystyle\int_{\Omega} \psi(\nabla u)dx + \beta \int_{J_u} \varphi(\nu_u)\, d\mathcal{H}^1 & \text{if } u \in (GSBV(\Omega))^N \\ +\infty & \text{otherwise,} \end{cases} \tag{3.21}$$

where $\varphi : \mathbb{S}^1 \to [0,+\infty)$ *is given by (2.13).*

Remark 3.3 *The surface term in (3.21) can be explicitely written. Indeed, let*

$$J_u^+ := \{x \in J_u : \langle \nu_u(x), \mathrm{e}_1\rangle\langle \nu_u(x), \mathrm{e}_2\rangle \geq 0\},$$
$$J_u^- := \{x \in J_u : \langle \nu_u(x), \mathrm{e}_1\rangle\langle \nu_u(x), \mathrm{e}_2\rangle < 0\},$$

then

$$\int_{J_u} \varphi(\nu_u)\, d\mathcal{H}^1 = \int_{J_u^+} |\langle \nu_u, \mathrm{e}_1\rangle| \vee |\langle \nu_u, \mathrm{e}_2\rangle|\, d\mathcal{H}^1 + \int_{J_u^-} \left(|\langle \nu_u, \mathrm{e}_1\rangle| + |\langle \nu_u, \mathrm{e}_2\rangle|\right)\, d\mathcal{H}^1.$$

Remark 3.4 *Both in Theorem 3.1 and Theorem 3.2 the domain of the limit functional is contained in* $(GSBV^p(\Omega))^N$ *(see (2.6)). This identification derives substantially from the lower semicontinuity Theorem 2.10 as we will show in Step1 of the proof of Proposition 3.7. Actually, we can infer the same conclusion by using the 1-dimensional arguments of Step2 of the same proposition.*

Remark 3.5 *For instance, if we assume that*

$$h(|z|) \vee c_1 \leq g_i(z) \leq c_2\,(1 + h(|z|)) \tag{3.22}$$

with $h : [0,+\infty) \to [0,+\infty)$ *an increasing function and* c_1, c_2 *positive constants, then* (h4) *is satisfied.*

In case $h(t) = t$ *the control from above in* (3.22) *is automatically satisfied as noticed in* (3.19). *Moreover, the additional control from below implies that the domain of the limit functional* $\mathcal{F}^g$ *is contained in* $(GSBV^p(\Omega))^N \cap SBV(\Omega;\mathbb{R}^N)$.

Remark 3.6 *Let us point out the main differences between the approximating functionals involved in Theorem 3.1 and Theorem 3.2. We claim that one cannot deduce the second result from the first one since the lower bound inequality obtained in Proposition 3.7 below is not sharp for the family* $(\mathcal{F}^{\beta}_{\varepsilon})$.

Indeed, let us consider the functionals $\mathcal{F}^g_{\varepsilon}$ *with* $g_1(z) = g_2(z) = \frac{\beta}{2}$ *for* $z \neq 0$, *then* $\psi^g_{\varepsilon} \leq \psi^{\beta}_{\varepsilon}$, *so that* $\mathcal{F}^g_{\varepsilon}(u) \leq \mathcal{F}^{\beta}_{\varepsilon}(u)$ *for every* $u \in L^1(\Omega;\mathbb{R}^N)$. *Actually, there holds*

$$\mathcal{F}^{\beta}_{\varepsilon}(u) = \mathcal{F}^g_{\varepsilon}(u) + G_{\varepsilon}(u)$$

where

$$G_\varepsilon(u) := \frac{\beta}{2}\varepsilon \sum_{i=1,2} \big(\#\{\alpha \in \mathcal{I}^+_\varepsilon : \big|D^+_\varepsilon u(\alpha)\mathrm{e}_i\big| = 0\} + \#\{\alpha \in \mathcal{I}^-_\varepsilon : \big|D^-_\varepsilon u(\alpha)\mathrm{e}_i\big| = 0\}\big)\,.$$

Let $\Omega = B_1(0)$, $\nu \in \mathbb{S}^1$ *be such that* $\langle \nu, \mathrm{e}_1\rangle\langle \nu, \mathrm{e}_2\rangle < 0$ *and* $u : \Omega \to \mathbb{R}^N$ *be defined as*

$$u(x) := \begin{cases} z_1 & x \in B_1^+(0,\nu) \\ & \\ z_2 & x \in B_1^-(0,\nu), \end{cases}$$

for $z_1 \neq z_2$*. We will show in Proposition 3.10 and Proposition 3.12 that there exists a suitable piecewise constant interpolation of* u, (u_ε)*, which is a recovery sequence for both the functionals. Thus, according to the notations of Lemma 2.13, for such a sequence we have*

$$\mathcal{I}^{\pm,i}_\varepsilon = \{\alpha \in \mathcal{I}^\pm_\varepsilon : \big|D^\pm_\varepsilon u_\varepsilon(\alpha)\mathrm{e}_i\big| = 0\}.$$

Hence, by reasoning as in the proof of Lemma 2.13 it follows

$$\lim_{\varepsilon\to 0^+} G_\varepsilon(u_\varepsilon) = \frac{\beta}{2}\int_{J_u} (|\langle \nu, \mathrm{e}_1\rangle| + |\langle \nu, \mathrm{e}_2\rangle|)\; d\mathcal{H}^1 > 0,$$

that is, the extra-term $G_\varepsilon(u)$ *contributes significantly in the computation of the lower bound for the* Γ*-limit of the family* $\big(\mathcal{F}^\beta_\varepsilon\big)$*.*

3.1 Γ-liminf inequalities

In this section we prove the lower bound inequalities for Theorem 3.1 and Theorem 3.2. They will be deduced by a more general result, proved in Proposition 3.7 below, holding true in case the functions g_i, used in the definition of the functionals $\mathcal{F}^g_\varepsilon$, both satisfy the following milder growth condition:

(h5) g_i, $i = 1, 2$, is a symmetric, subadditive, lower semicontinuous function such that for every $z \in \mathbb{R}^N \setminus \{0\}$

$$g_i(z) \geq \gamma(|z|), \tag{3.23}$$

where $\gamma : (0, +\infty) \to (0, +\infty]$ is a lower semicontinuous increasing function satisfying

$$\lim_{t\to 0^+} \frac{\gamma(t)}{t} = +\infty. \tag{3.24}$$

Notice that we recover (h2) choosing γ in (h5) to be constant. Without loss of generality we may also assume γ to be subadditive up to replacing it with its lower semicontinuous and subadditive envelope, still satisfying assumptions (3.23), (3.24) above.

Let us then prove a lower bound inequality in case the functions g_i satisfy (h5) instead of (h2).

Proposition 3.7 *Let Ω be a bounded open set, assume* (h1) *and* (h5). *Then, for any $u \in L^1\left(\Omega;\mathbb{R}^N\right)$,*

$$\Gamma(meas)\text{-}\liminf_{\varepsilon\to 0} \mathcal{F}^g_\varepsilon(u) \geq \mathcal{F}^g(u).$$

Proof. Let $(u_j) \subset \mathcal{A}_{\varepsilon_j}\left(\Omega;\mathbb{R}^N\right)$ and $u \in L^1\left(\Omega;\mathbb{R}^N\right)$ be such that $u_j \to u$ in measure. Moreover, assume that

$$\liminf_j \mathcal{F}^g_{\varepsilon_j}(u_j) = \lim_j \mathcal{F}^g_{\varepsilon_j}(u_j) < +\infty. \tag{3.25}$$

Consider the sets of indices $\mathcal{I}^\pm_{\varepsilon_j} \subseteq \mathcal{R}^\pm_{\varepsilon_j}$ defined by

$$\mathcal{I}^\pm_{\varepsilon_j} := \{\alpha \in \mathcal{R}^\pm_{\varepsilon_j} :\ |D^\pm_\varepsilon u_j(\alpha)| > \varepsilon_j T_{\varepsilon_j}\}; \tag{3.26}$$

then, by taking into account the growth condition $g_i \geq \gamma$, $i = 1, 2$, the subadditivity and the monotonicity of γ, (3.25), we get

$$\sup_j \left(\varepsilon_j \gamma\left(\varepsilon_j T_{\varepsilon_j}\right) \#\mathcal{I}^\pm_{\varepsilon_j}\right) < +\infty. \tag{3.27}$$

In order to prove the Γ-liminf inequality we will show that the triangles determined by the sets of indices in (3.26) detect the jump set of u. Thus, we will divide $\mathcal{F}^g_{\varepsilon_j}$ into two terms contributing separately to the bulk and surface energies of the limit functional.

Step1. Bulk energy inequality. According to the scheme stated above we construct a sequence $(v_j) \subset SBV\left(\Omega;\mathbb{R}^N\right)$ such that $v_j \to u$ in measure, (v_j) satisfies locally all the assumptions of the GSBV Closure Theorem 2.4 and with fixed $\eta > 0$ we have

$$\begin{aligned}
&\sum_{\alpha\in\mathcal{R}^+_{\varepsilon_j}\setminus\mathcal{I}^+_{\varepsilon_j}} \frac{\varepsilon_j^2}{2}\psi\left(\frac{1}{\varepsilon_j}D^+_{\varepsilon_j}u_j(\alpha)\right) + \sum_{\alpha\in\mathcal{R}^-_{\varepsilon_j}\setminus\mathcal{I}^-_{\varepsilon_j}} \frac{\varepsilon_j^2}{2}\psi\left(\frac{1}{\varepsilon_j}D^-_{\varepsilon_j}u_j(\alpha)\right) \\
&\geq \int_{\Omega_\eta} \psi(\nabla v_j)\,dx + o(1)
\end{aligned} \tag{3.28}$$

for j sufficiently large. Indeed, let $v_j : \Omega \to \mathbb{R}^N$ be the function whose components are piecewise affine, uniquely determined by

$$\langle v_j(x), e_k\rangle := \begin{cases} \langle u_j(\alpha), e_k\rangle & x \in \left(\alpha \pm S_{\varepsilon_j}\right)\cap\Omega \\ & \alpha \in \mathcal{I}^\pm_{\varepsilon_j} \\ \langle u_j(\alpha), e_k\rangle + \frac{1}{\varepsilon_j}\langle D^\pm_{\varepsilon_j}u_j(\alpha)e_k, x-\alpha\rangle & x \in \left(\alpha \pm S_{\varepsilon_j}\right)\cap\Omega \\ & \alpha \in \varepsilon_j\mathbb{Z}^2 \setminus \mathcal{I}^\pm_{\varepsilon_j}, \end{cases} \tag{3.29}$$

for every $k = 1, \dots, N$. It is easy to check that $v_j \to u$ in measure and there holds

$$\nabla v_j = \begin{cases} 0 & x \in (\alpha \pm S_{\varepsilon_j}) \cap \Omega \\ & \alpha \in \mathcal{I}^{\pm}_{\varepsilon_j} \\ \frac{1}{\varepsilon_j} D^{\pm}_{\varepsilon_j} u_j(\alpha) & x \in (\alpha \pm S_{\varepsilon_j}) \cap \Omega \\ & \alpha \in \varepsilon_j \mathbb{Z}^2 \setminus \mathcal{I}^{\pm}_{\varepsilon_j}. \end{cases}$$

Hence

$$\begin{aligned} &\sum_{\alpha \in \mathcal{R}^+_{\varepsilon_j} \setminus \mathcal{I}^+_{\varepsilon_j}} \frac{\varepsilon_j^2}{2} \psi\left(\frac{1}{\varepsilon_j} D^+_{\varepsilon_j} u_j(\alpha)\right) + \sum_{\alpha \in \mathcal{R}^-_{\varepsilon_j} \setminus \mathcal{I}^-_{\varepsilon_j}} \frac{\varepsilon_j^2}{2} \psi\left(\frac{1}{\varepsilon_j} D^-_{\varepsilon_j} u_j(\alpha)\right) \\ &= \int_{\Omega^{\varepsilon_j}} \psi\left(\nabla v_j\right) \, dx - \frac{\varepsilon_j^2}{2} \psi(0) \left(\# \mathcal{I}^+_{\varepsilon_j} + \# \mathcal{I}^-_{\varepsilon_j}\right), \end{aligned}$$

where $\Omega^{\varepsilon_j} := \Omega^{\varepsilon_j,+} \cup \Omega^{\varepsilon_j,-}$ and $\Omega^{\varepsilon_j,\pm} := \cup_{\alpha \in \mathcal{R}^{\pm}_{\varepsilon_j}} \left(\alpha \pm S_{\varepsilon_j}\right)$. Since for j sufficiently large $\Omega_\eta \subseteq \Omega^{\varepsilon_j}$, by taking into account (3.27), the superlinearity of γ in 0 and the choice of T_{ε_j}, we get (3.28).

Conditions (3.18) and (3.28) yield

$$\int_{\Omega_\eta} |\nabla v_j|^p \, dx \le c \mathcal{F}^g_{\varepsilon_j}(u_j), \tag{3.30}$$

for some positive constant c. Moreover, notice that

$$J_{v_j} \subseteq \left(\cup_{\alpha \in \mathcal{I}^+_{\varepsilon_j}} \partial\left(\alpha + S_{\varepsilon_j}\right)\right) \cup \left(\cup_{\alpha \in \mathcal{I}^-_{\varepsilon_j}} \partial\left(\alpha - S_{\varepsilon_j}\right)\right),$$

and that $\left(v_j^+ - v_j^-\right)|_{\alpha \pm S_{\varepsilon_j}}$ is a convex combination of the finite-differences computed in the nodes of the square $\left(\alpha \pm Q_{\varepsilon_j}\right)$ for any $\alpha \in \mathcal{I}^{\pm}_{\varepsilon_j}$. Therefore by using the subadditivity and the monotonicity of γ it can be shown that

$$\int_{\Omega_\eta \cap J_{v_j}} \gamma\left(\left|v_j^+ - v_j^-\right|\right) \, d\mathcal{H}^1 \le c \mathcal{F}^g_{\varepsilon_j}(u_j), \tag{3.31}$$

for some positive constant c. Hence, the sequences $(\langle v_j, e_k \rangle)$, $k = 1, \dots, N$, satisfy all the assumptions of the GSBV Closure Theorem 2.4 on Ω_η, so that $u \in \left(GSBV\left(\Omega_\eta\right)\right)^N$ for every $\eta > 0$ and there holds

$$\begin{aligned} \int_{\Omega_\eta} |\nabla u|^p \, dx &\le \liminf_j \int_{\Omega_\eta} |\nabla v_j|^p \, dx, \\ \int_{\Omega_\eta \cap J_u} \gamma\left(\left|u^+ - u^-\right|\right) \, d\mathcal{H}^1 &\le \liminf_j \int_{\Omega_\eta \cap J_{v_j}} \gamma\left(\left|v_j^+ - v_j^-\right|\right) \, d\mathcal{H}^1. \end{aligned}$$

The last two inequalities together with conditions (3.25), (3.30) and (3.31) yield $u \in \left(GSBV\left(\Omega\right)\right)^N$.

Eventually, by applying the lower semicontinuity Theorem 2.10 in (3.28) and then by passing to the limit on $\eta \to 0^+$, we get

$$\liminf_j \left(\sum_{\alpha \in \mathcal{R}^+_{\varepsilon_j} \setminus \mathcal{I}^+_{\varepsilon_j}} \frac{\varepsilon_j^2}{2} \psi\left(\frac{1}{\varepsilon_j} D^+_{\varepsilon_j} u_j(\alpha)\right) + \sum_{\alpha \in \mathcal{R}^-_{\varepsilon_j} \setminus \mathcal{I}^-_{\varepsilon_j}} \frac{\varepsilon_j^2}{2} \psi\left(\frac{1}{\varepsilon_j} D^-_{\varepsilon_j} u_j(\alpha)\right) \right)$$
$$\geq \int_\Omega \psi\left(\nabla u\right)\, dx \tag{3.32}$$

Step2. Surface energy inequality. With fixed $i = 1, 2$ and $\eta > 0$ we will prove the following inequalities

$$\liminf_j \sum_{\alpha \in \mathcal{I}^\pm_{\varepsilon_j}} \frac{\varepsilon_j}{2} g_i\left(D^\pm_{\varepsilon_j} u_j(\alpha) \mathrm{e}_i\right)$$
$$\geq \frac{1}{2} \int_{J_u} g_i\left(u^+ - u^-\right) \left|\langle \nu_u, \mathrm{e}_i\rangle\right| d\mathcal{H}^1, \tag{3.33}$$

which summed up together yield the desired inequality on the surface term, i.e.,

$$\liminf_j \sum_{i=1,2} \left(\sum_{\alpha \in \mathcal{I}^+_{\varepsilon_j}} \frac{\varepsilon_j}{2} g_i\left(D^+_{\varepsilon_j} u_j(\alpha)\mathrm{e}_i\right) + \sum_{\alpha \in \mathcal{I}^-_{\varepsilon_j}} \frac{\varepsilon_j}{2} g_i\left(D^-_{\varepsilon_j} u_j(\alpha)\mathrm{e}_i\right) \right)$$
$$\geq \int_{J_u} g\left(u^+ - u^-, \nu_u\right)\, d\mathcal{H}^1. \tag{3.34}$$

For the sake of simplicity we estimate only the term depending on '+', the other case being analogous. To this aim we construct a sequence $(v^i_j) \subset SBV\left(\Omega; \mathbb{R}^N\right)$ with one-dimensional profile along e_i, which is locally pre-compact in SBV in this given direction but not in general globally in GSBV. More precisely, let $Q_\varepsilon := \varepsilon[0,1)^2$ and for every $k = 1, \ldots, N$, define

$$\langle v^i_j(x), e_k\rangle := \begin{cases} \langle u_j(\alpha), e_k\rangle & x \in \left(\alpha + Q_{\varepsilon_j}\right) \cap \Omega \\ & \alpha \in \mathcal{I}^+_{\varepsilon_j} \\ & \\ \langle u_j(\alpha), e_k\rangle + \frac{1}{\varepsilon_j}\langle D^+_{\varepsilon_j} u_j(\alpha)\mathrm{e}_i, e_k\rangle\langle x - \alpha, \mathrm{e}_i\rangle & x \in \left(\alpha + Q_{\varepsilon_j}\right) \cap \Omega \\ & \alpha \in \varepsilon_j \mathbb{Z}^2 \setminus \mathcal{I}^+_{\varepsilon_j}, \end{cases} \tag{3.35}$$

then $(v_j^i) \subset SBV\left(\Omega;\mathbb{R}^N\right)$ and $v_j^i \to u$ in measure. Notice that, by construction

$$\frac{\partial}{\partial \mathrm{e}_i}\langle v_j^i, e_k\rangle = \begin{cases} 0 & x \in (\alpha + Q_{\varepsilon_j}) \cap \Omega \\ & \alpha \in \mathcal{I}^+_{\varepsilon_j} \\ \frac{1}{\varepsilon_j}\langle D^+_{\varepsilon_j} u_j(\alpha)\mathrm{e}_i, e_k\rangle & x \in (\alpha + Q_{\varepsilon_j}) \cap \Omega \\ & \alpha \in \varepsilon_j \mathbb{Z}^2 \setminus \mathcal{I}^+_{\varepsilon_j}, \end{cases}$$

hence, by (3.18) we get

$$\int_{\Omega_\eta} \left|\frac{\partial v_j^i}{\partial \mathrm{e}_i}\right|^p dx \le c\mathcal{F}^g_{\varepsilon_j}(u_j), \tag{3.36}$$

for some positive constant c. Moreover, since $\nu_{v_j^i} \in \{\mathrm{e}_1, \mathrm{e}_2\}$ $\mathcal{H}^1$ a.e. in $J_{v_j^i}$ there holds

$$\begin{aligned} &\sum_{\alpha \in \mathcal{I}^+_{\varepsilon_j}} \frac{\varepsilon_j}{2} g_i\left(D^+_{\varepsilon_j} u_j(\alpha)\mathrm{e}_i\right) \\ &\quad\ge \frac{1}{2}\int_{\Omega_\eta \cap J_{v_j^i}} g_i\left((v_j^i)^+ - (v_j^i)^-\right) |\langle \nu_{v_j^i}, \mathrm{e}_i\rangle|\, d\mathcal{H}^1 \\ &\quad= \frac{1}{2}\int_{\Pi^{\mathrm{e}_i}} d\mathcal{H}^1 \int_{\Omega_\eta \cap (J_{v_j^i})^{\mathrm{e}_i}_y} g_i\left(\left((v_j^i)^{\mathrm{e}_i,y}\right)^+ - \left((v_j^i)^{\mathrm{e}_i,y}\right)^-\right) d\mathcal{H}^0, \end{aligned} \tag{3.37}$$

where the last equality follows by the slicing formula (2.10). By passing to the limit on $j \to +\infty$ in (3.37) and by applying Fatou's lemma we have

$$\begin{aligned} &\liminf_j \sum_{\alpha \in \mathcal{I}^+_{\varepsilon_j}} \frac{\varepsilon_j}{2} g_i\left(D^+_{\varepsilon_j} u_j(\alpha)\mathrm{e}_i\right) \\ &\quad\ge \frac{1}{2}\int_{\Pi^{\mathrm{e}_i}} d\mathcal{H}^1 \liminf_j \int_{\Omega_\eta \cap (J_{v_j^i})^{\mathrm{e}_i}_y} g_i\left(\left((v_j^i)^{\mathrm{e}_i,y}\right)^+ - \left((v_j^i)^{\mathrm{e}_i,y}\right)^-\right) d\mathcal{H}^0, \end{aligned}$$

hence by (3.25) we deduce that for $\mathcal{H}^1$ a.e. $y \in \Pi^{\mathrm{e}_i}$ there holds

$$\liminf_j \int_{\Omega_\eta \cap (J_{v_j^i})^{\mathrm{e}_i}_y} g_i\left(\left((v_j^i)^{\mathrm{e}_i,y}\right)^+ - \left((v_j^i)^{\mathrm{e}_i,y}\right)^-\right) d\mathcal{H}^0 < +\infty. \tag{3.38}$$

Thus, for $\mathcal{H}^1$ a.e. $y \in \Pi^{\mathrm{e}_i}$, up to extracting subsequences depending on such a fixed y, we may assume $\left(v_j^i\right)^{\mathrm{e}_i,y} \to u^{\mathrm{e}_i,y}$ in measure on $(\Omega_\eta)^{\mathrm{e}_i}_y$, the inferior limit in (3.38) to be a limit and by taking into account (3.36) also that

$$\sup_j \int_{(\Omega_\eta)^{\mathrm{e}_i}_y} \left|(\dot{v}_j^i)^{\mathrm{e}_i,y}\right|^p dt < +\infty.$$

Hence, the slices $\left(\left(v_j^i\right)^{\mathrm{e}_i,y}\right)$ satisfy on $\left(\Omega_\eta\right)_y^{\mathrm{e}_i}$ all the assumptions of the GSBV Closure Theorem 2.4, then Theorem 2.12 yields

$$
\begin{aligned}
&\liminf_j \sum_{\alpha\in\mathcal{I}^+_{\varepsilon_j}} \frac{\varepsilon_j}{2} g_i\left(D^+_{\varepsilon_j} u_j(\alpha)\mathrm{e}_i\right) \\
&\geq \frac{1}{2}\int_{\Pi^{\mathrm{e}_i}} d\mathcal{H}^1 \liminf_j \int_{\Omega_\eta\cap(J_{v_j^i})_y^{\mathrm{e}_i}} g_i\left(\left(\left(v_j^i\right)^{\mathrm{e}_i,y}\right)^+ - \left(\left(v_j^i\right)^{\mathrm{e}_i,y}\right)^-\right) d\mathcal{H}^0 \\
&\geq \frac{1}{2}\int_{\Pi^{\mathrm{e}_i}} d\mathcal{H}^1 \int_{\Omega_\eta\cap(J_u)_y^{\mathrm{e}_i}} g_i\left((u^{\mathrm{e}_i,y})^+ - (u^{\mathrm{e}_i,y})^-\right) d\mathcal{H}^0 \\
&= \frac{1}{2}\int_{\Omega_\eta\cap J_u} g_i\left(u^+ - u^-\right) |\langle \nu_u, \mathrm{e}_i\rangle|\, d\mathcal{H}^1,
\end{aligned}
$$

from which we deduce (3.33) passing to the limit on $\eta\to 0^+$.

To conclude it suffices to use the subadditivity of the inferior limit and (3.32), (3.34). □

We now prove the lower semicontinuity inequality for the family of functionals $\left(\mathcal{F}^\beta_\varepsilon\right)$. To this aim we need to 'localize' the functionals $\mathcal{F}^\beta_\varepsilon$. For every $A\in\mathcal{A}(\Omega)$ and $u\in\mathcal{A}_\varepsilon\left(\Omega;\mathbb{R}^N\right)$ let

$$
\mathcal{F}^\beta_\varepsilon(u,A) := \sum_{\alpha\in\mathcal{R}^+_\varepsilon(A)} \frac{\varepsilon^2}{2}\psi^\beta_\varepsilon\left(\frac{1}{\varepsilon}D^+_\varepsilon u(\alpha)\right) + \sum_{\alpha\in\mathcal{R}^-_\varepsilon(A)} \frac{\varepsilon^2}{2}\psi^\beta_\varepsilon\left(\frac{1}{\varepsilon}D^-_\varepsilon u(\alpha)\right),
$$

where recall that $\mathcal{R}^\pm_\varepsilon(A) = \{\alpha\in\varepsilon\mathbb{Z}^2 : \alpha\pm S_\varepsilon\subseteq A\}$.

We obtain separate estimates on the bulk and on the surface terms which we 'glue' together by means of Lemma 2.14. Besides using the same tecniques applied in the proof of Proposition 3.7 above, we will perform an additional construction with profile along the diagonal direction $\mathrm{e}_2-\mathrm{e}_1$.

Proposition 3.8 *Let Ω be a bounded open set, assume* (h1). *Then, for any $u\in L^1\left(\Omega;\mathbb{R}^N\right)$,*

$$
\Gamma(meas)\text{-}\liminf_{\varepsilon\to 0}\mathcal{F}^\beta_\varepsilon(u) \geq \mathcal{F}^\beta(u).
$$

Proof. Let $(u_j)\subset\mathcal{A}_{\varepsilon_j}\left(\Omega;\mathbb{R}^N\right)$ and $u\in L^1\left(\Omega;\mathbb{R}^N\right)$ be such that $u_j\to u$ in measure. By using analogous arguments of those of Proposition 3.7 it is possible to show that for every $A\in\mathcal{A}(\Omega)$ the following inequalities hold true:

$$
\begin{aligned}
&\liminf_j \left(\sum_{\alpha\in\mathcal{R}^+_{\varepsilon_j}(A)\setminus\mathcal{I}^+_{\varepsilon_j}} \frac{\varepsilon_j^2}{2}\psi\left(\frac{1}{\varepsilon_j}D^+_{\varepsilon_j}u_j(\alpha)\right) + \sum_{\alpha\in\mathcal{R}^-_{\varepsilon_j}(A)\setminus\mathcal{I}^-_{\varepsilon_j}} \frac{\varepsilon_j^2}{2}\psi\left(\frac{1}{\varepsilon_j}D^-_{\varepsilon_j}u_j(\alpha)\right)\right) \\
&\geq \int_A \psi\left(\nabla u\right)\, dx, \qquad (3.39)
\end{aligned}
$$

$$\liminf_j \frac{\beta}{2}\varepsilon_j \left(\#\left(\mathcal{R}^+_{\varepsilon_j}(A)\cap\mathcal{I}^+_{\varepsilon_j}\right)+\#\left(\mathcal{R}^-_{\varepsilon_j}(A)\cap\mathcal{I}^-_{\varepsilon_j}\right)\right)$$
$$\geq \beta\int_{A\cap J_u}|\langle\nu_u,\mathrm{e}_i\rangle|\,d\mathcal{H}^1, \tag{3.40}$$

for $i=1,2$. Thus to conclude it suffices to show that there holds

$$\liminf_j \frac{\beta}{2}\varepsilon_j \left(\#\left(\mathcal{R}^+_{\varepsilon_j}(A)\cap\mathcal{I}^+_{\varepsilon_j}\right)+\#\left(\mathcal{R}^-_{\varepsilon_j}(A)\cap\mathcal{I}^-_{\varepsilon_j}\right)\right)$$
$$\geq \beta\int_{A\cap J_u}|\langle\nu_u,\mathrm{e}_2-\mathrm{e}_1\rangle|\,d\mathcal{H}^1. \tag{3.41}$$

Indeed, by using Lemma 2.14 with $\left(\delta^i_h\right)_h=\mathbb{Q}\cap[0,1]$, $i=1,2$, and

$$\mu(A) := \liminf_j \mathcal{F}^\beta_{\varepsilon_j}(u_j,A),$$
$$\lambda := \mathcal{L}^2 \llcorner (\Omega\setminus J_u)+\mathcal{H}^1\llcorner J_u,$$
$$\phi_h(x) := \begin{cases}\psi(\nabla u) & \text{on } \Omega\setminus J_u\\[2ex] \beta\big(\,\delta^1_h|\langle\nu_u,\mathrm{e}_1\rangle|+\delta^2_h|\langle\nu_u,\mathrm{e}_2\rangle|+\left(1-\delta^1_h-\delta^2_h\right)|\langle\nu_u,\mathrm{e}_2-\mathrm{e}_1\rangle|\big) & \text{on } J_u,\end{cases}$$

the statement follows by noticing that if $x\in J^+_u$ then

$$|\langle\nu_u(x),\mathrm{e}_2-\mathrm{e}_1\rangle|\leq|\langle\nu_u,\mathrm{e}_1\rangle|\vee|\langle\nu_u,\mathrm{e}_2\rangle|,$$

and if $x\in J^-_u$

$$|\langle\nu_u(x),\mathrm{e}_2-\mathrm{e}_1\rangle|=|\langle\nu_u(x),\mathrm{e}_1\rangle|+|\langle\nu_u(x),\mathrm{e}_2\rangle|\geq|\langle\nu_u,\mathrm{e}_1\rangle|\vee|\langle\nu_u,\mathrm{e}_2\rangle|.$$

To prove inequality (3.41) we will construct a sequence $(w_j)\subset SBV\left(\Omega;\mathbb{R}^N\right)$ with one-dimensional profile in $\mathrm{e}_2-\mathrm{e}_1$ which is locally pre-compact in SBV in this given direction but not in general globally in GSBV. Suppose

$$\liminf_j \mathcal{F}^\beta_{\varepsilon_j}(u_j)=\lim_j \mathcal{F}^\beta_{\varepsilon_j}(u_j)<+\infty, \tag{3.42}$$

consider the sets of indices $\mathcal{I}^\pm_{\varepsilon_j}:=\{\alpha\in\mathcal{R}^\pm_{\varepsilon_j}:\ |D^\pm_\varepsilon u_j(\alpha)|>\varepsilon_j T_{\varepsilon_j}\}$, then by (3.42) we get

$$\sup_j \varepsilon_j\left(\#\mathcal{I}^+_{\varepsilon_j}+\#\mathcal{I}^-_{\varepsilon_j}\right)<+\infty. \tag{3.43}$$

Let

$$P_{\varepsilon_j}:=\varepsilon_j\left\{x\in\mathbb{R}^2: x=\lambda(-\mathrm{e}_1)+\mu(\mathrm{e}_2-\mathrm{e}_1)\ \lambda,\mu\in[0,1]\right\}$$

and define for every $k = 1, \ldots, N$

$$\langle w_j(x), e_k\rangle := \begin{cases} \langle u_j(\alpha), e_k\rangle & \begin{array}{l} x \in \left(\alpha + P_{\varepsilon_j}\right) \cap \Omega \\ (\alpha - \varepsilon_j \mathrm{e}_1) \in \mathcal{I}^+_{\varepsilon_j} \end{array} \\ \langle u_j(\alpha), e_k\rangle + \frac{1}{\sqrt{2}\varepsilon_j}\langle D^+_{\varepsilon_j} u_j(\alpha - \varepsilon_j \mathrm{e}_1)(\mathrm{e}_2 - \mathrm{e}_1), e_k\rangle\langle x - \alpha, \mathrm{e}_2 - \mathrm{e}_1\rangle & \begin{array}{l} x \in \left(\alpha + P_{\varepsilon_j}\right) \cap \Omega \\ (\alpha - \varepsilon_j \mathrm{e}_1) \in \varepsilon_j \mathbb{Z}^2 \setminus \mathcal{I}^+_{\varepsilon_j}. \end{array} \end{cases}$$

Notice the analogy with the construction in (3.35): in this case the unitary cubic cell Q_{ε_j} is replaced by the unitary slanted one P_{ε_j}.
We have that $(w_j) \subset SBV\left(\Omega; \mathbb{R}^N\right)$, $w_j \to u$ in measure and also

$$\frac{\partial}{\partial(\mathrm{e}_2 - \mathrm{e}_1)}\langle w_j, e_k\rangle = \begin{cases} 0 & \begin{array}{l} x \in \left(\alpha + P_{\varepsilon_j}\right) \cap \Omega \\ (\alpha - \varepsilon_j \mathrm{e}_1) \in \mathcal{I}^+_{\varepsilon_j} \end{array} \\ \frac{1}{\sqrt{2}\varepsilon_j}\langle D^+_{\varepsilon_j} u_j(\alpha - \varepsilon_j \mathrm{e}_1)(\mathrm{e}_2 - \mathrm{e}_1), e_k\rangle & \begin{array}{l} x \in \left(\alpha + P_{\varepsilon_j}\right) \cap \Omega \\ (\alpha - \varepsilon_j \mathrm{e}_1) \in \varepsilon_j \mathbb{Z}^2 \setminus \mathcal{I}^+_{\varepsilon_j}. \end{array} \end{cases}$$

Moreover, for every $\eta > 0$ and $A \in \mathcal{A}(\Omega)$ by (3.18)

$$\int_{A_\eta} \left|\frac{\partial w_j}{\partial(\mathrm{e}_2 - \mathrm{e}_1)}\right|^p dx \leq c\mathcal{F}^\beta_{\varepsilon_j}(u_j, A), \tag{3.44}$$

for some positive constant c, and since $\nu_{w_j} \in \{\mathrm{e}_2, \mathrm{e}_1 + \mathrm{e}_2\}$ $\mathcal{H}^1$ a.e. in J_{w_j} we have

$$\int_{A_\eta \cap J_{w_j}} \left|\langle \nu_{w_j}, \mathrm{e}_2 - \mathrm{e}_1\rangle\right| d\mathcal{H}^1 \leq \varepsilon_j \# \left(\mathcal{R}^+_{\varepsilon_j}(A) \cap \mathcal{I}^+_{\varepsilon_j}\right). \tag{3.45}$$

Notice that (3.42), (3.43) together with (3.44), (3.45) for $A = \Omega$ assure that for $\mathcal{H}^1$ a.e. $y \in \Pi^{\mathrm{e}_2 - \mathrm{e}_1}$, up to subsequences depending on such a fixed y, the slices $\left((w_j)^{\mathrm{e}_2 - \mathrm{e}_1, y}\right)$ satisfy on $(\Omega_\eta)^{\mathrm{e}_2 - \mathrm{e}_1}_y$ all the assumptions of the GSBV Closure Theorem 2.4. Thus, by taking into account Fatou's lemma and Theorem 2.12 we get passing to the inferior limit on $j \to +\infty$ in (3.45)

$$\frac{\beta}{2} \liminf_j \varepsilon_j \# \left(\mathcal{R}^+_{\varepsilon_j}(A) \cap \mathcal{I}^+_{\varepsilon_j}\right) \geq \frac{\beta}{2} \int_{A_\eta \cap J_u} \left|\langle \nu_u, \mathrm{e}_2 - \mathrm{e}_1\rangle\right| d\mathcal{H}^1. \tag{3.46}$$

Eventually, (3.41) follows by applying an analogous argument to show that

$$\frac{\beta}{2} \liminf_j \varepsilon_j \# \left(\mathcal{R}^-_{\varepsilon_j}(A) \cap \mathcal{I}^-_{\varepsilon_j}\right) \geq \frac{\beta}{2} \int_{A_\eta \cap J_u} \left|\langle \nu_u, \mathrm{e}_2 - \mathrm{e}_1\rangle\right| d\mathcal{H}^1, \tag{3.47}$$

summing up inequalities (3.46), (3.47) and then passing to the limit on $\eta \to 0^+$. $\square$

Remark 3.9 *We claim that, by proceeding as in Step 1 of the proof of Proposition 3.7, one can prove that the families of functions* $(u_\varepsilon) \subseteq \mathcal{A}_\varepsilon\left(\Omega;\mathbb{R}^N\right)$ *such that*

$$\sup_{\varepsilon>0}\left\{\mathcal{F}^g_\varepsilon(u_\varepsilon)+\|u_\varepsilon\|_{1,\Omega}\right\}<+\infty$$

are pre-compact in $L^1\left(\Omega;\mathbb{R}^N\right)$. *Indeed, to get the result it suffices to apply Theorem 2.5 to the family* (v_ε) *constructed in (3.29) and to use Proposition A.1 of [6].*

3.2 Γ-limsup inequalities

In this section we prove the upper bound inequalities for Theorem 3.1 and Theorem 3.2. In both the following propositions we will reduce ourselves to prove the inequalities for a class of more regular functions, in this approach will be crucial Lemma 2.13.

Proposition 3.10 *Let* Ω *be a bounded open set with lipschitz boundary, assume* (h1)-(h4). *Then, for any* $u \in L^1\left(\Omega;\mathbb{R}^N\right)$,

$$\Gamma\left(L^1\right)\text{-}\limsup_{\varepsilon\to 0^+}\mathcal{F}^g_\varepsilon(u)\leq\mathcal{F}^g(u).$$

Proof. It suffices to prove the inequality above for $u \in \left(GSBV^p(\Omega)\right)^N$ such that $\mathcal{F}^g(u)<+\infty$.

Step1. Let Ω' be an open set such that $\Omega' \supset\supset \Omega$ and suppose $u \in \mathcal{W}\left(\Omega';\mathbb{R}^N\right)$, then there exists a finite number of polyhedral sets S^r such that

1. $\mathcal{H}^1\left(\overline{J_u}\setminus J_u\right)=0$;
2. $\overline{J_u}=\Omega'\cap\cup_{r=1}^M S^r$;
3. for every $1\leq r\leq M$ the set S^r is contained in a 1-dimensional hyperplane Π^r with normal ν_r and $\Pi^r\neq\Pi^s$ for $r\neq s$.

Let us first fix some notations. With fixed $m\in\mathbb{N}$, let

$$J_u^m:=\left\{x\in J_u:\left|u^+(x)-u^-(x)\right|\geq\frac{1}{m}\right\},$$

$\{J_u^m\}_{m\in\mathbb{N}}$ is an increasing family of sets such that $J_u=\cup_{m\in\mathbb{N}}J_u^m$ and so

$$\lim_{m\to+\infty}\mathcal{H}^1(J_u^m)=\mathcal{H}^1(J_u).$$

Moreover, let $J:=\overline{J_u}$ and define the set of indices

$$\mathcal{J}_\varepsilon^\pm:=\left\{\alpha\in\varepsilon\mathbb{Z}^2:[\alpha;\alpha\pm \mathrm{e}_i]\cap J\neq\emptyset\ \text{ for }\ i=1,2\right\},$$

and for $m \in \mathbb{N} \setminus \{0\}$ set

$$\mathcal{J}^{\pm}_{m,\varepsilon} := \mathcal{R}^{\pm}_{\varepsilon}\left(J^m_u\right) \cup R^{\pm}_{\varepsilon}\left(J^m_u\right).$$

Up to infinitesimal translations we may assume that $J \cap \varepsilon\mathbb{Z}^2 = \emptyset$ for every $\varepsilon > 0$, then let

$$u_\varepsilon(x) := u(\alpha)$$

if $x \in (\alpha \pm S_\varepsilon) \cap \Omega \neq \emptyset$. Notice that $u_\varepsilon \in \mathcal{A}_\varepsilon\left(\Omega;\mathbb{R}^N\right)$ and $u_\varepsilon \to u$ strongly in $L^1\left(\Omega;\mathbb{R}^N\right)$. Eventually, denote as usual $\mathcal{I}^{\pm}_\varepsilon := \{\alpha \in \mathcal{R}^{\pm}_\varepsilon : |D^{\pm}_\varepsilon u_\varepsilon(\alpha)| > \varepsilon T_\varepsilon\}$. By taking into account Theorem 2.6 we have for $i = 1, 2$

$$D^{\pm}_\varepsilon u_\varepsilon(\alpha)\mathrm{e}_i = \int_0^\varepsilon \nabla u\left(\alpha \pm t\mathrm{e}_i\right)\mathrm{e}_i\, dt + \sum_{y \in J_u \cap [\alpha;\alpha \pm \varepsilon \mathrm{e}_i)} [u](y), \tag{3.48}$$

where $[u](y) := (u^+(y) - u^-(y))\, sgn\left(\langle \nu_u(y), \mathrm{e}_i\rangle\right)$. Thus, if $\alpha \in \mathcal{R}^{\pm}_\varepsilon \setminus \mathcal{J}^{\pm}_\varepsilon$, we have

$$D^{\pm}_\varepsilon u_\varepsilon(\alpha)\mathrm{e}_i = \int_0^\varepsilon \nabla u\left(\alpha \pm t\mathrm{e}_i\right)\mathrm{e}_i\, dt, \tag{3.49}$$

for $i = 1, 2$, from which it follows $|D^{\pm}_\varepsilon u_\varepsilon(\alpha)| \leq \varepsilon\|\nabla u\|_{\infty,\Omega'}$. Define the vector fields $W_\varepsilon : \Omega \to \mathbb{R}^{N\times 2}$ by

$$W_\varepsilon(x) := \left(\frac{1}{\varepsilon}\int_0^\varepsilon \nabla u\left(\alpha \pm t\mathrm{e}_i\right)\mathrm{e}_i\, dt\right)_{i=1,2},$$

for $x \in (\alpha \pm S_\varepsilon) \cap \Omega \neq \emptyset$, then $W_\varepsilon \to \nabla u$ strongly in $L^p\left(\Omega;\mathbb{R}^{N\times 2}\right)$ and by (3.49) for ε sufficiently small it follows

$$\begin{aligned}
&\limsup_{\varepsilon\to 0^+}\left(\sum_{\alpha\in\mathcal{R}^+_\varepsilon\setminus\mathcal{J}^+_\varepsilon}\frac{\varepsilon^2}{2}\psi^g_\varepsilon\left(\frac{1}{\varepsilon}D^+_\varepsilon u_\varepsilon(\alpha)\right) + \sum_{\alpha\in\mathcal{R}^-_\varepsilon\setminus\mathcal{J}^-_\varepsilon}\frac{\varepsilon^2}{2}\psi^g_\varepsilon\left(\frac{1}{\varepsilon}D^-_\varepsilon u_\varepsilon(\alpha)\right)\right)\\
&= \limsup_{\varepsilon\to 0^+}\left(\sum_{\alpha\in\mathcal{R}^+_\varepsilon\setminus\mathcal{J}^+_\varepsilon}\frac{\varepsilon^2}{2}\psi\left(W_\varepsilon(\alpha)\right) + \sum_{\alpha\in\mathcal{R}^-_\varepsilon\setminus\mathcal{J}^-_\varepsilon}\frac{\varepsilon^2}{2}\psi\left(W_\varepsilon(\alpha)\right)\right)\\
&\leq \lim_{\varepsilon\to 0^+}\int_\Omega \psi\left(W_\varepsilon\right)\, dx = \int_\Omega \psi\left(\nabla u\right)\, dx,
\end{aligned} \tag{3.50}$$

the last equality holding by the growth condition (3.18).

Consider the decomposition $\mathcal{J}^{\pm}_\varepsilon = (\mathcal{J}^{\pm}_\varepsilon \cap \mathcal{I}^{\pm}_\varepsilon) \cup (\mathcal{J}^{\pm}_\varepsilon \setminus \mathcal{I}^{\pm}_\varepsilon)$, then it follows

$$\begin{aligned}
\sum_{\alpha\in\mathcal{J}^{\pm}_\varepsilon}\frac{\varepsilon^2}{2}\psi^g_\varepsilon\left(\frac{1}{\varepsilon}D^{\pm}_\varepsilon u_\varepsilon(\alpha)\right) &= \sum_{i=1,2}\sum_{\alpha\in\mathcal{J}^{\pm}_\varepsilon\cap\mathcal{I}^{\pm}_\varepsilon}\frac{\varepsilon}{2}g_i\left(D^{\pm}_\varepsilon u_\varepsilon(\alpha)\mathrm{e}_i\right)\\
&+ \sum_{\alpha\in\mathcal{J}^{\pm}_\varepsilon\setminus\mathcal{I}^{\pm}_\varepsilon}\frac{\varepsilon^2}{2}\psi\left(\frac{1}{\varepsilon}D^{\pm}_\varepsilon u_\varepsilon(\alpha)\right) =: I^{1,\pm}_\varepsilon + I^{2,\pm}_\varepsilon.
\end{aligned} \tag{3.51}$$

Let us estimate separately the two terms in (3.51) above.

Let $\overline{B} := \overline{B}_{(0,\|\nabla u\|_{\infty,\Omega}+2M\|u\|_{\infty,\Omega})}$ and let $K = \sup_{\overline{B}}(g_1+g_2)$, then for every $m \in \mathbb{N}$ it follows

$$\sum_{i=1,2}\sum_{\alpha\in\left(\mathcal{J}_\varepsilon^\pm\setminus\mathcal{J}_{m,\varepsilon}^\pm\right)\cap\mathcal{I}_\varepsilon^\pm}\frac{\varepsilon}{2}g_i\left(D_\varepsilon^\pm u_\varepsilon(\alpha)\mathrm{e}_i\right)\le K\frac{\varepsilon}{2}\#\left(\mathcal{J}_\varepsilon^\pm\setminus\mathcal{J}_{m,\varepsilon}^\pm\right)$$

hence, by Lemma 2.13 there holds

$$\limsup_{\varepsilon\to0^+}\sum_{i=1,2}\sum_{\alpha\in\left(\mathcal{J}_\varepsilon^\pm\setminus\mathcal{J}_{m,\varepsilon}^\pm\right)\cap\mathcal{I}_\varepsilon^\pm}\frac{\varepsilon}{2}g_i\left(D_\varepsilon^\pm u_\varepsilon(\alpha)\mathrm{e}_i\right)$$
$$\le K\limsup_{\varepsilon\to0^+}\frac{\mathcal{L}^2\left(\left(\overline{J_u\setminus J_u^m}\right)_\varepsilon^{\mathrm{e}_1,\pm}\right)}{\varepsilon}\le\frac{K}{2}\int_{\overline{J_u\setminus J_u^m}}\varphi\left(\nu_u\right)\,d\mathcal{H}^1,\qquad(3.52)$$

the last term being infinitesimal as $m\to+\infty$.
Let $\omega_m:[0,+\infty)\to[0,+\infty)$ be the maximum of the moduli of continuity of g_i, $i=1,2$, on the compact set $\overline{B}\setminus B_{(0,\frac{1}{m})}$, then for ε sufficiently small we get by (3.48)

$$\sum_{\alpha\in\mathcal{J}_\varepsilon^\pm\cap\mathcal{J}_{m,\varepsilon}^\pm\cap\mathcal{I}_\varepsilon^\pm}\frac{\varepsilon}{2}g_i\left(D_\varepsilon^\pm u_\varepsilon(\alpha)\mathrm{e}_i\right)$$
$$\le\sum_{\alpha\in\mathcal{J}_\varepsilon^\pm\cap\mathcal{J}_{m,\varepsilon}^\pm}\frac{\varepsilon}{2}g_i\left(\sum_{y\in J_u\cap[\alpha;\alpha\pm\varepsilon\mathrm{e}_i)}[u](y)\right)+\frac{\varepsilon}{2}\#\mathcal{J}_{m,\varepsilon}^\pm\,\omega_m\left(\varepsilon\|\nabla u\|_{\infty,\Omega'}\right)$$
$$\le\sum_{\alpha\in\mathcal{J}_\varepsilon^\pm\cap\mathcal{J}_{m,\varepsilon}^\pm}\frac{\varepsilon}{2}\sum_{y\in J_u\cap[\alpha;\alpha\pm\varepsilon\mathrm{e}_i)}g_i\left(u^+(y)-u^-(y)\right)+\frac{\varepsilon}{2}\#\mathcal{J}_{m,\varepsilon}^\pm\,\omega_m\left(\varepsilon\|\nabla u\|_{\infty,\Omega'}\right),$$

the last inequality holding by the subadditivity and the symmetry of g_i, $i=1,2$.
It is easy to check that the regularity assumption $u\in\mathcal{W}\left(\Omega';\mathbb{R}^N\right)$ and the continuity of g_i on $\mathbb{R}^N\setminus\{0\}$ yield for every $1\le r\le M$

$$\limsup_{\varepsilon\to0^+}\sum_{\alpha\in\mathcal{J}_\varepsilon^\pm\cap\mathcal{J}_{m,\varepsilon}^\pm}\frac{\varepsilon}{2}\sum_{y\in J_u\cap[\alpha;\alpha\pm\varepsilon\mathrm{e}_i)}g_i\left(u^+(y)-u^-(y)\right)$$
$$\le\frac{1}{2}\int_J g_i\left(u^+-u^-\right)\left|\langle\nu_u,\mathrm{e}_i\rangle\right|d\mathcal{H}^1,$$

hence by Lemma 2.13, (3.52) and since $\mathcal{H}^1\left(\overline{J_u}\setminus J_u\right)=0$ we infer

$$\limsup_{\varepsilon\to0^+}\sum_{i=1,2}\sum_{\alpha\in\mathcal{J}_\varepsilon^\pm\cap\mathcal{I}_\varepsilon^\pm}\frac{\varepsilon}{2}g_i\left(D_\varepsilon^\pm u_\varepsilon(\alpha)\mathrm{e}_i\right)\le\frac{1}{2}\int_{J_u}g\left(u^+-u^-,\nu_u\right)\,d\mathcal{H}^1.\qquad(3.53)$$

By collecting (3.52), (3.53) and by passing to the limit on $m \to +\infty$ we get

$$\limsup_{\varepsilon\to 0^+} I_\varepsilon^{1,\pm} \leq \frac{1}{2}\int_{J_u} g\left(u^+ - u^-, \nu_u\right)\, d\mathcal{H}^1. \tag{3.54}$$

To estimate $I_\varepsilon^{2,\pm}$ notice that since $\alpha \in \mathcal{R}_\varepsilon^\pm \setminus \mathcal{I}_\varepsilon^\pm$ by (3.18) there holds

$$I_\varepsilon^{2,\pm} = \sum_{\alpha\in\mathcal{J}_\varepsilon^\pm\setminus\mathcal{I}_\varepsilon^\pm} \frac{\varepsilon^2}{2}\psi\left(\frac{1}{\varepsilon}D_\varepsilon^\pm u_\varepsilon(\alpha)\right) \leq C_2\frac{\varepsilon^2}{2}\left(1+T_\varepsilon^p\right)\#\left(\mathcal{J}_\varepsilon^\pm\setminus\mathcal{I}_\varepsilon^\pm\right), \tag{3.55}$$

and the term on the right hand side above is infinitesimal as $\varepsilon \to 0^+$. Indeed, with fixed $m \in \mathbb{N}$ there holds

$$\#\left(\mathcal{J}_\varepsilon^\pm\setminus\mathcal{I}_\varepsilon^\pm\right) \leq \#\left(\mathcal{J}_{m,\varepsilon}^\pm\setminus\mathcal{I}_\varepsilon^\pm\right) + \#\left(\mathcal{J}_\varepsilon^\pm\setminus\mathcal{J}_{m,\varepsilon}^\pm\right),$$

and since for ε sufficiently small $\#\left(\mathcal{J}_{m,\varepsilon}^\pm\setminus\mathcal{I}_\varepsilon^\pm\right) \leq c$, where c is a positive constant independent of $m \in \mathbb{N}$, by Lemma 2.13 we deduce

$$\limsup_{\varepsilon\to 0^+} \varepsilon\#\left(\mathcal{J}_\varepsilon^\pm\setminus\mathcal{I}_\varepsilon^\pm\right) \leq \limsup_{\varepsilon\to 0^+} \varepsilon\#\left(\mathcal{J}_\varepsilon^\pm\setminus\mathcal{J}_{m,\varepsilon}^\pm\right) \leq \int_{\overline{J_u\setminus J_u^m}} \varphi\left(\nu_u\right)\, d\mathcal{H}^1.$$

Moreover, assumption $\sup_{\varepsilon>0}\left(\varepsilon T_\varepsilon^p\right) < +\infty$ yields

$$\limsup_{\varepsilon\to 0^+} I_\varepsilon^{2,\pm} \leq c\int_{\overline{J_u\setminus J_u^m}} \varphi\left(\nu_u\right)\, d\mathcal{H}^1 \tag{3.56}$$

for some positive constant c independent of $m \in \mathbb{N}$, so that passing to the limit on $m \to +\infty$ we infer that $I_\varepsilon^{2,\pm}$ is infinitesimal as $\varepsilon \to 0^+$.

Eventually, by collecting (3.50), (3.54) and (3.56) we get the conclusion, i.e.,

$$\limsup_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon^g(u_\varepsilon) \leq \int_\Omega \psi\left(\nabla u\right)\, dx + \int_{J_u} g\left(u^+ - u^-, \nu_u\right)\, d\mathcal{H}^1.$$

Step2. Assume $u \in \left(GSBV^p(\Omega)\right)^N \cap L^\infty\left(\Omega;\mathbb{R}^N\right)$. By taking into account the SBV Extension Theorem (see Lemma 4.11 [9]), with fixed an open and bounded set $\Omega' \supset\supset \Omega$, there exists a function $\tilde{u} \in SBV \cap L^\infty\left(\Omega';\mathbb{R}^N\right)$ such that $\tilde{u}|_\Omega \equiv u$, $\nabla\tilde{u} \in L^p\left(\Omega';\mathbb{R}^{N\times 2}\right)$, $\mathcal{H}^1\left(J_{\tilde{u}}\right) < +\infty$ and $\mathcal{H}^1\left(\partial\Omega\cap J_{\tilde{u}}\right) = 0$.
Let $(u_j) \subset \mathcal{W}\left(\Omega';\mathbb{R}^N\right)$ be the sequence provided by Theorem 2.8, since the continuity and local boundedness of g, we may apply statement (iv) of the same Theorem 2.8 to obtain

$$\lim_{j\to+\infty}\int_{\Omega\cap J_{u_j}} g\left(u_j^+ - u_j^-, \nu_{u_j}\right)\, d\mathcal{H}^1 = \int_{J_u} g\left(u^+ - u^-, \nu_u\right)\, d\mathcal{H}^1.$$

Hence, by *Step1* and the lower semicontinuity of the upper Γ-limit the statement follows.

Step3. Let $u \in (GSBV^p(\Omega))^N$, and let $u_T = (\langle u, e_k\rangle_T)_{k=1,\dots,N}$ for any $T>0$. Then, $u_T \in (GSBV^p(\Omega))^N \cap L^\infty(\Omega;\mathbb{R}^N)$ and $J_{u_T} \subseteq J_u$. Moreover, by taking into account (2.7) we have

$$\lim_{T\to+\infty} g_i\left((u_T)^+(x) - (u_T)^-(x)\right) = g_i\left(u^+(x) - u^-(x)\right) \tag{3.57}$$

for $\mathcal{H}^1$ a.e. $x \in J_u$, $i = 1, 2$. Eventually, by assumption (h4) and (3.57) we may apply the Dominated Convergence Theorem to conclude. □

Remark 3.11 *Let us point out that the assumption* (h4) *is technical and needed only to recover the limsup estimate on* $(GSBV^p(\Omega))^N \setminus SBV(\Omega;\mathbb{R}^N)$. *Indeed, assume* $u \in SBV(\Omega;\mathbb{R}^N)$ *to be such that* $\mathcal{F}^g(u) < +\infty$, *by following the notations of Step3 in Proposition 3.10 above and by taking into account (3.19) we get*

$$c_1 \le g\left((u_T)^+ - (u_T)^-, \nu_{u_T}\right) \le 2c_2\left(1 + \left|u^+ - u^-\right|\right).$$

Moreover, since $u \in SBV(\Omega;\mathbb{R}^N)$ *implies* $|u^+ - u^-| \in L^1(J_u;\mathcal{H}^1)$, *the Dominated Convergence Theorem yields*

$$\lim_{T\to+\infty} \int_{J_u} g\left((u_T)^+ - (u_T)^-, \nu_{u_T}\right) d\mathcal{H}^1 = \int_{J_u} g\left(u^+ - u^-, \nu_u\right) d\mathcal{H}^1,$$

hence in this case (h4) *is automatically satisfied.*

Proposition 3.12 *Let* Ω *be a bounded open set with lipschitz boundary, assume* (h1). *Then, for any* $u \in L^1(\Omega;\mathbb{R}^N)$,

$$\Gamma\left(L^1\right)\text{-}\limsup_{\varepsilon\to0^+} \mathcal{F}^\beta_\varepsilon(u) \le \mathcal{F}^\beta(u).$$

Proof. Let $u \in (GSBV^p(\Omega))^N$ be such that $\mathcal{F}^\beta(u) < +\infty$. Let us first prove the inequality for regular functions. Let Ω' be an open set such that $\Omega' \supset\supset \Omega$ and suppose $u \in \mathcal{W}(\Omega';\mathbb{R}^N)$. Arguing like in *Step1* of the proof of Proposition 3.10 above and using the same notations, we have

$$\begin{aligned}
&\limsup_{\varepsilon\to0^+}\left(\sum_{\alpha\in\mathcal{R}^+_\varepsilon\setminus\mathcal{J}^+_\varepsilon} \frac{\varepsilon^2}{2}\psi^\beta_\varepsilon\left(\frac{1}{\varepsilon}D^+_\varepsilon u_\varepsilon(\alpha)\right) + \sum_{\alpha\in\mathcal{R}^-_\varepsilon\setminus\mathcal{J}^-_\varepsilon} \frac{\varepsilon^2}{2}\psi^\beta_\varepsilon\left(\frac{1}{\varepsilon}D^-_\varepsilon u_\varepsilon(\alpha)\right)\right)\\
&= \limsup_{\varepsilon\to0^+}\left(\sum_{\alpha\in\mathcal{R}^+_\varepsilon\setminus\mathcal{J}^+_\varepsilon} \frac{\varepsilon^2}{2}\psi\left(W_\varepsilon(\alpha)\right) + \sum_{\alpha\in\mathcal{R}^-_\varepsilon\setminus\mathcal{J}^-_\varepsilon} \frac{\varepsilon^2}{2}\psi\left(W_\varepsilon(\alpha)\right)\right)\\
&\le \lim_{\varepsilon\to0^+}\int_\Omega \psi\left(W_\varepsilon\right) dx = \int_\Omega \psi\left(\nabla u\right) dx,
\end{aligned} \tag{3.58}$$

Moreover, by the very definition of ψ_ε^β and $\mathcal{I}_\varepsilon^\pm$ we have

$$\begin{aligned}
&\sum_{\alpha\in\mathcal{J}_\varepsilon^+\cap\mathcal{I}_\varepsilon^+}\frac{\varepsilon^2}{2}\psi_\varepsilon^\beta\left(\frac{1}{\varepsilon}D_\varepsilon^+u_\varepsilon(\alpha)\right)+\sum_{\alpha\in\mathcal{J}_\varepsilon^-\cap\mathcal{I}_\varepsilon^-}\frac{\varepsilon^2}{2}\psi_\varepsilon^\beta\left(\frac{1}{\varepsilon}D_\varepsilon^-u_\varepsilon(\alpha)\right)\\
&\quad=\beta\frac{\varepsilon}{2}\left(\#\left(\mathcal{J}_\varepsilon^+\cap\mathcal{I}_\varepsilon^+\right)+\#\left(\mathcal{J}_\varepsilon^-\cap\mathcal{I}_\varepsilon^-\right)\right)\\
&\quad\le\frac{\beta}{\varepsilon}\left(\mathcal{L}^2\left(\left(\overline{\Omega\cap J_u}\right)_\varepsilon^{+,\mathrm{e}_1}\right)+\mathcal{L}^2\left(\left(\overline{\Omega\cap J_u}\right)_\varepsilon^{-,\mathrm{e}_1}\right)\right),
\end{aligned}$$

and then by Lemma 2.13

$$\begin{aligned}
&\limsup_{\varepsilon\to0^+}\left(\sum_{\alpha\in\mathcal{J}_\varepsilon^+\cap\mathcal{I}_\varepsilon^+}\frac{\varepsilon^2}{2}\psi_\varepsilon^\beta\left(\frac{1}{\varepsilon}D_\varepsilon^+u_\varepsilon(\alpha)\right)+\sum_{\alpha\in\mathcal{J}_\varepsilon^-\cap\mathcal{I}_\varepsilon^-}\frac{\varepsilon^2}{2}\psi_\varepsilon^\beta\left(\frac{1}{\varepsilon}D_\varepsilon^-u_\varepsilon(\alpha)\right)\right)\\
&\qquad\le\beta\int_{\overline{\Omega\cap J_u}}\varphi(\nu_u)\,d\mathcal{H}^1. \qquad (3.59)
\end{aligned}$$

Moreover, by taking into account (3.18), we get

$$\begin{aligned}
&\sum_{\alpha\in\mathcal{J}_\varepsilon^\pm\setminus\mathcal{I}_\varepsilon^\pm}\frac{\varepsilon^2}{2}\psi_\varepsilon^\beta\left(\frac{1}{\varepsilon}D_\varepsilon^\pm u_\varepsilon(\alpha)\right)\\
&\quad=\sum_{\alpha\in\mathcal{J}_\varepsilon^\pm\setminus\mathcal{I}_\varepsilon^\pm}\frac{\varepsilon^2}{2}\psi\left(\frac{1}{\varepsilon}D_\varepsilon^\pm u_\varepsilon(\alpha)\right)\le C_2\frac{\varepsilon^2}{2}\left(1+T_\varepsilon^p\right)\#\left(\mathcal{J}_\varepsilon^\pm\setminus\mathcal{I}_\varepsilon^\pm\right), \qquad (3.60)
\end{aligned}$$

the term on the right hand side above being infinitesimal as proved in (3.56) of Proposition 3.10. Eventually, by collecting (3.58), (3.59) and (3.60) we get

$$\limsup_{\varepsilon\to0^+}\mathcal{F}_\varepsilon^\beta(u_\varepsilon)\le\int_\Omega\psi\left(\nabla u\right)\,dx+\beta\int_{J_u}\varphi(\nu_u)\,d\mathcal{H}^1.$$

To infer the result for every $u\in L^1\left(\Omega;\mathbb{R}^N\right)$ it suffices to argue like in *Step2* and *Step3* of the proof of Proposition 3.10. □

4 Further extensions

In this section we show how the techniques so far developed can be applied to prove other approximation results.

4.1 Non-traslation invariant surface energy densities

The same arguments used to prove Theorem 3.1 (see Proposition 3.7 and Proposition 3.10 above) can be applied to the family of functionals defined in (4.62) below to obtain as limit more general surface energy densities.

Indeed, let $\vartheta_i:\mathbb{R}^N\times\mathbb{R}^N\setminus\triangle\to[0,+\infty)$, $i=1,2$, be such that (h2') ϑ_i is a symmetric, subadditive, continuous function with

$$\inf_{\mathbb{R}^N \times \mathbb{R}^N \setminus \triangle} \vartheta_i > 0;$$

(h3') ϑ_i is locally bounded in a neighbourhood of $\triangle$;

(h4') for every $u \in (GSBV^p(\Omega))^N$ there exists a sequence $(T_j) \subseteq [0, +\infty)$ with $T_j \to +\infty$ such that

$$\begin{aligned}\limsup_j \int_{J_u} \vartheta_i \left(\left(u_{T_j}\right)^+, \left(u_{T_j}\right)^- \right) |\langle \nu_u, \mathrm{e}_i \rangle| \, d\mathcal{H}^1 \\ = \int_{J_u} \vartheta_i \left(u^+, u^-\right) |\langle \nu_u, \mathrm{e}_i \rangle| \, d\mathcal{H}^1.\end{aligned}$$

Extend, as usual, ϑ_i to $\mathbb{R}^N \times \mathbb{R}^N$ setting $\vartheta_i \equiv 0$ on $\triangle$.

Define the 'values' matrices of $u : \mathbb{R}^2 \to \mathbb{R}^N$ to be the $N \times 2$ matrices

$$V_\varepsilon^{\mathrm{e}_i, \pm} u(\alpha) := \left(\begin{array}{cc} \langle u\left(\alpha \pm \varepsilon \mathrm{e}_i\right), e_k \rangle & \langle u(\alpha), e_k \rangle \end{array} \right)_{k=1,\dots,N} \tag{4.61}$$

for $i = 1, 2$, and let $\mathcal{I}_\varepsilon^\pm := \{\alpha \in \mathcal{R}_\varepsilon^\pm : |D_\varepsilon^\pm u(\alpha)| > \varepsilon T_\varepsilon\}$.

Consider the functionals $\mathcal{V}_\varepsilon : L^1\left(\Omega; \mathbb{R}^N\right) \to [0, +\infty]$ defined as

$$\mathcal{V}_\varepsilon(u) := \begin{cases} \sum\limits_{\alpha \in \mathcal{R}_\varepsilon^+ \setminus \mathcal{I}_\varepsilon^+} \frac{\varepsilon^2}{2} \psi\left(\frac{1}{\varepsilon} D_\varepsilon^+ u(\alpha)\right) + \sum\limits_{i=1,2} \sum\limits_{\alpha \in \mathcal{I}_\varepsilon^+} \frac{\varepsilon}{2} \vartheta_i \left(V_\varepsilon^{\mathrm{e}_i,+} u(\alpha)\right) \\ + \sum\limits_{\alpha \in \mathcal{R}_\varepsilon^- \setminus \mathcal{I}_\varepsilon^-} \frac{\varepsilon^2}{2} \psi\left(\frac{1}{\varepsilon} D_\varepsilon^- u(\alpha)\right) + \sum\limits_{i=1,2} \sum\limits_{\alpha \in \mathcal{I}_\varepsilon^-} \frac{\varepsilon}{2} \vartheta_i \left(V_\varepsilon^{\mathrm{e}_i,-} u(\alpha)\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } u \in \mathcal{A}_\varepsilon\left(\Omega; \mathbb{R}^N\right) \\ +\infty \qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise,} \end{cases}$$

then the following Γ-convergence result holds true.

Theorem 4.1 *Let Ω be a bounded open set with lipschitz boundary, assume* (h1), (h2'), (h3') *and* (h4'). *Then $(\mathcal{V}_\varepsilon)$ Γ-converges with respect to both the convergence in measure and strong $L^1\left(\Omega; \mathbb{R}^N\right)$ to the functional $\mathcal{V} : L^1\left(\Omega; \mathbb{R}^N\right) \to [0, +\infty]$ defined by*

$$\mathcal{V}(u) := \begin{cases} \displaystyle\int_\Omega \psi(\nabla u) \, dx + \int_{J_u} \vartheta\left(u^+, u^-, \nu_u\right) \, d\mathcal{H}^1 & \text{if } u \in (GSBV(\Omega))^N \\ +\infty & \text{otherwise,} \end{cases}$$

where $\vartheta : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^1 \to [0, +\infty)$ is defined by

$$\vartheta\left(z_1, z_2, \nu\right) := \vartheta_1\left(z_1, z_2\right) |\langle \nu, \mathrm{e}_1 \rangle| + \vartheta_2\left(z_1, z_2\right) |\langle \nu, \mathrm{e}_2 \rangle|.$$

4.2 A model for long-range interactions

In this subsection we deal with potentials taking into account multi-range interactions. For the sake of simplicity we assume $g = g_1 = g_2$. For $u \in \mathcal{A}_\varepsilon\left(\Omega;\mathbb{R}^N\right)$ and $\xi \in \mathbb{Z}^2 \setminus \{0\}$ define the discretization of the gradient with respect to the basis $\{\xi, \xi^\perp\}$ as

$$D_\varepsilon^{\pm,\xi} u(\alpha) := \pm \left(\ \langle u\left(\alpha \pm \varepsilon\xi\right) - u(\alpha), e_k\rangle \quad \langle u\left(\alpha \pm \varepsilon\xi^\perp\right) - u(\alpha), e_k\rangle \ \right)_{k=1,\dots,N}, \tag{4.62}$$

then set

$$\mathcal{F}_\varepsilon^{g,\xi}(u) := \sum_{\alpha \in \mathcal{R}_\varepsilon^{+,\xi}} \frac{\varepsilon^2}{2}\psi_\varepsilon^g\left(\frac{1}{\varepsilon}D_\varepsilon^{+,\xi}u(\alpha)\right) + \sum_{\alpha \in \mathcal{R}_\varepsilon^{-,\xi}} \frac{\varepsilon^2}{2}\psi_\varepsilon^g\left(\frac{1}{\varepsilon}D_\varepsilon^{-,\xi}u(\alpha)\right) \tag{4.63}$$

where

$$\mathcal{R}_\varepsilon^{\pm,\xi} := \left\{\alpha \in \varepsilon\mathbb{Z}^2 : \alpha \pm S_\varepsilon^\xi \subseteq \Omega\right\},$$

and recall that $S_\varepsilon^\xi = \varepsilon\ co\{(0,0), \xi, \xi^\perp\}$.
Consider the functional $F_\varepsilon^g : L^1\left(\Omega;\mathbb{R}^N\right) \to [0,+\infty]$ defined as

$$F_\varepsilon^g(u) = \begin{cases} \displaystyle\sum_{\xi\in\mathbb{Z}^2} \rho(\xi)\mathcal{F}_\varepsilon^{g,\xi}(u) & \text{if } u \in \mathcal{A}_\varepsilon\left(\Omega;\mathbb{R}^N\right) \\ +\infty & \text{otherwise} \end{cases} \tag{4.64}$$

where $\rho : \mathbb{Z}^2 \to [0,+\infty)$ is such that $0 < \sum_{\xi\in\mathbb{Z}^2} |\xi|^p \rho(\xi) < +\infty$. By taking into account analogous arguments of those used in Proposition 3.7 and Proposition 3.10 and by using a superposition principle we claim the following result.

Theorem 4.2 *Let Ω be a bounded open set with lipschitz boundary and assume* (h1)-(h4). *Then (F_ε^g) Γ-converges with respect to both the convergence in measure and strong $L^1\left(\Omega;\mathbb{R}^N\right)$ to the functional $F^g : L^1\left(\Omega;\mathbb{R}^N\right) \to [0,+\infty]$ defined by*

$$F^g(u) := \begin{cases} \displaystyle\int_\Omega \Psi(\nabla u)\,dx + \int_{J_u} G\left(u^+ - u^-, \nu_u\right)\,d\mathcal{H}^1 & \textit{if } u \in (GSBV(\Omega))^N \\ +\infty & \textit{otherwise,} \end{cases}$$

where, denoting by $(\xi\ \xi^\perp)$ the 2×2 matrix whose columns are ξ,$\xi^\perp$, $\Psi : \mathbb{R}^{N\times 2} \to [0,+\infty)$ is given by

$$\Psi(X) := \sum_{\xi\in\mathbb{Z}^2} \rho(\xi)\psi\left(X(\xi\ \xi^\perp)\right),$$

and $G : \mathbb{R}^N \times \mathbb{S}^1 \to [0,+\infty)$ is given by

$$G(z,\nu) := \sum_{\xi\in\mathbb{Z}^2} \rho(\xi)\left(|\langle\nu,\xi\rangle| + |\langle\nu,\xi^\perp\rangle|\right) g(z).$$

It is easily seen that an analogous result can be proved by replacing ψ_ε^g with ψ_ε^β.

4.3 A remark on bulk energy densities with linear growth

In this subsection we show that with the finite-differences schemes we can also perform an approximation for energies with bulk densities satisfying a linear growth condition. The main tool will be a relaxation result of Ambrosio and Dal Maso stated below (see Theorem 5.47 of [6]).

Let $\psi : \mathbb{R}^{N\times 2} \to [0,+\infty)$ be a quasi-convex function with linear growth, i.e., there exist two positive constants $C_1, C_2 > 0$ such that

$$C_1 |X| \le \psi(X) \le C_2 (1 + |X|) \tag{4.65}$$

for every $X \in \mathbb{R}^{N\times 2}$. Let $\psi_\infty : \mathbb{R}^{N\times 2} \to [0,+\infty)$ be the recession function of ψ, defined by

$$\psi_\infty(X) := \limsup_{t\to+\infty} \frac{\psi(tX)}{t}.$$

Theorem 4.3 *Let $F : L^1(\Omega;\mathbb{R}^N) \to [0,+\infty]$ be defined by*

$$F(u) := \begin{cases} \displaystyle\int_\Omega \psi(\nabla u)\,dx & \text{if } u \in W^{1,1}(\Omega;\mathbb{R}^N) \\ +\infty & \text{otherwise.} \end{cases}$$

Set $\mathcal{F}(u) := \Gamma(L^1)\text{-}\lim F(u)$, then

$$\mathcal{F}(u) := \begin{cases} \displaystyle\int_\Omega \psi(\nabla u)\,dx + \int_{J_u} \psi_\infty\left(\left(u^+ - u^-\right)\otimes \nu_u\right)\, d\mathcal{H}^1 & \\ \displaystyle + \int_\Omega \psi_\infty\left(\frac{dD^c u}{d|D^c u|}\right) d|D^c u| & \text{if } u \in BV(\Omega;\mathbb{R}^N) \\ +\infty & \text{otherwise.} \end{cases} \tag{4.66}$$

Define $\mathcal{F}_\varepsilon : L^1(\Omega;\mathbb{R}^N) \to [0,+\infty]$ as

$$\mathcal{F}_\varepsilon(u) := \begin{cases} \displaystyle\sum_{\alpha\in\mathcal{R}_\varepsilon^+} \tfrac{\varepsilon^2}{2}\psi\left(\tfrac{1}{\varepsilon}D_\varepsilon^+ u(\alpha)\right) + \sum_{\alpha\in\mathcal{R}_\varepsilon^-} \tfrac{\varepsilon^2}{2}\psi\left(\tfrac{1}{\varepsilon}D_\varepsilon^- u(\alpha)\right) & \text{if } u \in \mathcal{A}_\varepsilon(\Omega;\mathbb{R}^N) \\ +\infty & \text{otherwise.} \end{cases}$$

Then the following result holds.

Theorem 4.4 *Let Ω be a bounded open set with lipschitz boundary. Then $(\mathcal{F}_\varepsilon)$ Γ-converges with respect to the strong convergence $L^1(\Omega;\mathbb{R}^N)$ to the functional $\mathcal{F} : L^1(\Omega;\mathbb{R}^N) \to [0,+\infty]$ defined in (4.66).*

Proof. Let $(u_j) \subset \mathcal{A}_{\varepsilon_j}\left(\Omega; \mathbb{R}^N\right)$ and $u \in L^1\left(\Omega; \mathbb{R}^N\right)$ be such that $u_j \to u$ in $L^1\left(\Omega; \mathbb{R}^N\right)$. Moreover, assume that

$$\liminf_j \mathcal{F}^g_{\varepsilon_j}(u_j) = \lim_j \mathcal{F}^g_{\varepsilon_j}(u_j) < +\infty. \tag{4.67}$$

Let $v_j : \Omega \to \mathbb{R}^N$ be the piecewise affine interpolation of u_j given by

$$\langle v_j(x), e_k \rangle := \langle u_j(\alpha), e_k \rangle + \frac{1}{\varepsilon_j} \langle D^{\pm}_{\varepsilon_j} u_j(\alpha) e_k, x - \alpha \rangle \tag{4.68}$$

for $x \in \left(\alpha \pm S_{\varepsilon_j}\right) \cap \Omega$ and $\alpha \in \varepsilon_j \mathbb{Z}^2$, for every $k = 1, \dots, N$. It is easy to check that $v_j \to u$ in $L^1\left(\Omega; \mathbb{R}^N\right)$, and by (4.65), (4.67) it follows $\sup_j |Dv_j|(\Omega) < +\infty$, so that by the BV compactness theorem $u \in BV\left(\Omega; \mathbb{R}^N\right)$. Moreover, for every $\eta > 0$ we have

$$\mathcal{F}_{\varepsilon_j}(u_j) \geq \int_{\Omega_\eta} \psi(\nabla u_j)\, dx;$$

hence by Theorem 4.3 and by letting $\eta \to 0^+$ we get

$$\liminf_j \mathcal{F}_{\varepsilon_j}(u_j) \geq \mathcal{F}(u).$$

Let us now prove the limsup inequality. First notice that by condition (4.65) $\mathcal{F}|_{W^{1,1}(\Omega;\mathbb{R}^N)}$ is continuous with respect to the strong $W^{1,1}\left(\Omega; \mathbb{R}^N\right)$ convergence. Let $\Omega' \supset\supset \Omega$ and let $u \in C^1\left(\Omega'; \mathbb{R}^N\right)$, define $u_\varepsilon(x) = u(\alpha)$ for $x \in (\alpha \pm S_\varepsilon) \cap \Omega$ and $\alpha \in \varepsilon \mathbb{Z}^2$. Then $u_\varepsilon \to u$ in $L^1\left(\Omega; \mathbb{R}^N\right)$ and the functions v_ε defined in (4.68) converges to u in $W^{1,1}\left(\Omega; \mathbb{R}^N\right)$. Moreover, we have

$$\mathcal{F}_\varepsilon(u_\varepsilon) \leq \int_\Omega \psi(\nabla v_\varepsilon)\, dx,$$

and then

$$\limsup_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon) \leq \int_\Omega \psi(\nabla u)\, dx.$$

The regularity assumption on Ω assures that for every $u \in W^{1,1}\left(\Omega; \mathbb{R}^N\right)$ there exists a sequence $(u_j) \subset C^1\left(\Omega'; \mathbb{R}^N\right)$ such that $u_j \to u$ in $W^{1,1}\left(\Omega; \mathbb{R}^N\right)$. Thus, the lower semicontinuity of the Γ-upper limit yields the limsup inequality in this case.

The conclusion then follows by applying Theorem 4.3. □

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