ON FRACTIONAL HARDY-TYPE INEQUALITIES IN GENERAL OPEN SETS

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ABSTRACT. We show that, when $sp > N$, the sharp Hardy constant $\mathfrak{h}_{s,p}$ of the punctured space $\mathbb{R}^N \setminus \{0\}$ in the Sobolev-Slobodeckiĭ space provides an optimal lower bound for the Hardy constant $\mathfrak{h}_{s,p}(\Omega)$ of an open $\Omega \subsetneq \mathbb{R}^N$. The proof exploits the characterization of Hardy's inequality in the fractional setting in terms of positive local weak supersolutions of the relevant Euler-Lagrange equation and relies on the construction of suitable supersolutions by means of the distance function from the boundary of Ω. Moreover, we compute the limit of $\mathfrak{h}_{s,p}$ as $s \nearrow 1$, as well as the limit when $p \nearrow \infty$. Finally, we apply our results to establish a lower bound for the non-local eigenvalue $\lambda_{s,p}(\Omega)$ in terms of $\mathfrak{h}_{s,p}$ when $sp > N$, which, in turn, gives an improved Cheeger inequality whose constant does not vanish as $p \nearrow \infty$.

CONTENTS

1. INTRODUCTION

This paper deals with Hardy-type inequalities in fractional Sobolev spaces, with special interest to optimal lower bounds on their sharp contants.

We recall some well known facts on Hardy's inequalities in the classical (local) setting. For an open subset Ω of $\mathbb{R}^N$, let us define the distance function to the boundary as

$$d_\Omega(x) := \min_{y \in \partial\Omega} |x - y|, \quad \text{for all } x \in \Omega.$$

A classical result in the theory of Sobolev spaces states that, under suitable assumptions on the set Ω, there exists a positive constant C such that

$$C \int_\Omega \frac{|u|^p}{d_\Omega^p}\, dx \le \int_\Omega |\nabla u|^p\, dx, \qquad \text{for all } u \in C_0^\infty(\Omega). \tag{1.1}$$

In the huge existing literature concerning the Hardy inequality, some results establish necessary and sufficient conditions on the open set Ω ensuring the validity of (1.1) (see, for example, the references in [23]) while other papers are devoted to the interesting related question of determining

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the optimal constant C in (1.1), which can be defined in a variational way as

$$\mathfrak{h}_p(\Omega) = \inf_{u \in C_0^\infty(\Omega)} \left\{ \int_\Omega |\nabla u|^p : \int_\Omega \frac{|u|^p}{d_\Omega^p}\, dx = 1 \right\}.$$

This has been achieved in some particular cases, e.g. :

- if $\Omega = \mathbb{R}^N \setminus \{0\}$, $1 < p < \infty$, $p \neq N$, then $\mathfrak{h}_p(\Omega) = \left(\frac{|N-p|}{p}\right)^p$ (see [29] and the references therein);
- if Ω is convex, $1 < p < \infty$, then $\mathfrak{h}_p(\Omega) = \left(\frac{p-1}{p}\right)^p$ (for a proof, see [28, Theorem 11]).

In the particular case $p > N$, Lewis in [27] and Wannebo in [32] show that the Hardy inequality (1.1) holds on every open set $\Omega \subset \mathbb{R}^N$. Later, an alternative proof of this result has been given in [20] by means of a "pointwise Hardy inequality" and maximal function techniques. However, all these papers do not provide any explicit (lower) bound for $\mathfrak{h}_p(\Omega)$. The latter question is studied in [2, 18], where it is proved that, when $p > N$, the optimal Hardy constant of the punctured space $\mathbb{R}^N \setminus \{0\}$ provides an optimal lower bound for $\mathfrak{h}_p(\Omega)$, i.e. for every open set $\Omega \subset \mathbb{R}^N$ it holds:

$$\mathfrak{h}_p(\Omega) \geq \mathfrak{h}_p(\mathbb{R}^N \setminus \{0\}) = \left(\frac{p-N}{p}\right)^p. \tag{1.2}$$

Recently, much interest has been devoted to the study of fractional nonlocal operators, fractional Sobolev spaces and related functional inequalities. A natural question in this context is whether a fractional analogue of (1.1) holds true and whether one can determine the sharp constant, at least in some particular cases.

In order to state our main result, let us start by introducing our notation. For $1 \leq p < \infty$, $0 < s \leq 1$ and $\Omega \subseteq \mathbb{R}^N$, we define

$$W^{s,p}(\Omega) = \Big\{ \varphi \in L^p(\Omega) \,:\, [\varphi]_{W^{s,p}(\Omega)} < +\infty \Big\},$$

where

$$[\varphi]_{W^{s,p}(\Omega)} = \begin{cases} \left(\iint_{\Omega\times\Omega} \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{N+sp}}\, dx\, dy \right)^{\frac{1}{p}}, & \text{if } 0 < s < 1, \\ \left(\int_\Omega |\nabla \varphi|^p \, dx \right)^{\frac{1}{p}}, & \text{if } s = 1. \end{cases}$$

When $1 < p < \infty$, this is a reflexive space, when endowed with the norm

$$\|\varphi\|_{W^{s,p}(\Omega)} = \|\varphi\|_{L^p(\Omega)} + [\varphi]_{W^{s,p}(\Omega)}, \qquad \text{for every } \varphi \in W^{s,p}(\Omega).$$

We also indicate by $\widetilde{W}_0^{s,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{s,p}(\mathbb{R}^N)$.

The analogue of the Hardy inequality (1.1) in this context reads as follows:

$$C \int_\Omega \frac{|u|^p}{d_\Omega^{sp}}\, dx \leq [u]^p_{W^{s,p}(\mathbb{R}^N)}, \quad \text{for all } u \in C_0^\infty(\Omega). \tag{1.3}$$

For an open set $\Omega \subsetneq \mathbb{R}^N$, we introduce its sharp fractional (s,p)-Hardy constant defined as

$$\mathfrak{h}_{s,p}(\Omega) = \inf_{u \in C_0^\infty(\Omega)} \left\{ [u]^p_{W^{s,p}(\mathbb{R}^N)} : \int_\Omega \frac{|u|^p}{d_\Omega^{sp}}\, dx = 1 \right\}.$$

We explicitly note that $\mathfrak{h}_{1,p}(\Omega) = \mathfrak{h}_p(\Omega)$. Observe that, by definition of $\widetilde{W}_0^{s,p}(\Omega)$, we have

$$\mathfrak{h}_{s,p}(\Omega) = \inf_{u\in \widetilde{W}_0^{s,p}(\Omega)} \left\{ [u]^p_{W^{s,p}(\mathbb{R}^N)} \, : \, \int_\Omega \frac{|u|^p}{d_\Omega^{sp}}\, dx = 1 \right\}, \tag{1.4}$$

by a standard density argument.

A first result in the determination of the sharp constant in this fractional setting was established by Frank and Seiringer [16, Theorem 1.1], who proved that, if $N \geq 1$, $0 < s < 1$ and $1 \leq p < \infty$ are such that $sp \neq N$, then

$$\mathfrak{h}_{s,p} := \mathfrak{h}_{s,p}(\mathbb{R}^N \setminus \{0\}) = 2 \int_0^1 r^{sp-1} \left|1 - r^{\frac{N-sp}{p}}\right|^p \Phi_{N,s,p}(r)\, dr > 0,$$

where, for every $0 < r < 1$, the quantity $\Phi_{N,s,p}(r)$ is given by

$$\Phi_{N,s,p}(r) = |\mathbb{S}^{N-2}| \int_{-1}^1 \frac{(1-t^2)^{\frac{N-3}{2}}}{(1-2\,t\,r+r^2)^{\frac{N+sp}{2}}}\, dt, \qquad \text{for } N \geq 2,$$

and

$$\Phi_{1,s,p}(r) = \frac{1}{(1-r)^{1+sp}} + \frac{1}{(1+r)^{1+sp}}.$$

The case of convex sets has been considered recently in [4], where, in Theorems 6.3 and 6.6, it has been proved that, if Ω is convex, the optimal constant $\mathfrak{h}_{s,p}(\Omega)$ coincides with the one of the half-space $\mathbb{H}^N_+ := \mathbb{R}^{N-1} \times (0, +\infty)$ (whose explicit value is given in formulas (1.9)-(1.10) in [4]) in the following situations:

- for $1 < p < \infty$ and $1/p \leq s < 1$;
- for $p = 2$ and $0 < s < 1$.

We note that the paper [4] extends some previous results contained in [5, 15] for the case $p = 2$. More precisely in [5, Theorem 1.1], the explicit value of the optimal constant for the half-space has been computed for $p = 2$ and any $0 < s < 1$, while in [15, Theorem 5], it has been proved that if Ω is convex, then $\mathfrak{h}_{s,2}(\Omega) = \mathfrak{h}_{s,2}(\mathbb{H}^N_+)$ for any $1/2 \leq s < 1$.

When $sp > N$, the recent paper [31] shows that the fractional Hardy inequality (1.3) holds on every open set $\Omega \subset \mathbb{R}^N$, by adapting the technique in [20] to the non-local setting. However, such an approach does not give any lower estimate on the fractional Hardy constant $\mathfrak{h}_{s,p}(\Omega)$.

With the aim to provide an optimal lower bound on $\mathfrak{h}_{s,p}(\Omega)$ when $\Omega \subset \mathbb{R}^N$ is a general open set and $sp > N$, in this paper we give a different proof of the Hardy inequality (1.3) which comes out with a lower sharp estimate on $\mathfrak{h}_{s,p}(\Omega)$. In particular, our main result extends inequality (1.2) to the fractional case $sp > N$.

Theorem 1.1. *Let $N \geq 1$, $0 < s < 1$ and $1 < p < \infty$ be such that $sp > N$. For every open set $\Omega \subsetneq \mathbb{R}^N$ we have*

$$\mathfrak{h}_{s,p}(\Omega) \geq \mathfrak{h}_{s,p}, \qquad \text{where } \mathfrak{h}_{s,p} := \mathfrak{h}_{s,p}(\mathbb{R}^N \setminus \{0\}). \tag{1.5}$$

The proof of this result is based on the so-called *supersolution method*, which was well known in the classical local case (see e.g. [1, 23, 13]) and was extensively studied in the fractional setting in [3, 4]. Such a method allows to give an equivalent "dual" definition of $\mathfrak{h}_{s,p}(\Omega)$, which relies on the existence of positive supersolutions to the nonlinear fractional equation:

$$(-\Delta_p)^s u = \lambda \frac{|u|^{p-2}u}{d_\Omega^{sp}} \quad \text{in } \Omega, \tag{1.6}$$

where $(-\Delta_p)^s$ denotes the fractional p-Laplacian, whose precise definition will be given later on in Section 2. We recall that, in [3, Theorem 1.1], it is proved that

$$\mathfrak{h}_{s,p}(\Omega) = \sup\{\lambda \geq 0 \,:\, \text{equation (1.6) admits a positive local weak supersolution}\}. \tag{1.7}$$

For the definition of local weak super/subsolution we refer to Definition 2.1 below.

Such result, as well explained in [3], is based on the equivalence between the strict positivity of $\mathfrak{h}_{s,p}(\Omega)$ and the existence of a positive (local weak) supersolution to (1.6) for some λ. For more details on the supersolution method see [3] and reference therein.

Thanks to the formula (1.7) above, in order to prove Theorem 1.1, it is enough to find a positive supersolution to (1.6) for $\lambda = \mathfrak{h}_{s,p}(\mathbb{R}^N \setminus \{0\})$. This is the content of Theorem 2.2 (and, more precisely, of Corollary 2.4 below), where we give an explicit supersolution to (1.6), in terms of powers of the distance function.

In the second part of the paper we study the asymptotics, when $s \nearrow 1$ of $\mathfrak{h}_{s,p}$, as well as its limit when $p \nearrow \infty$. The strategy adopted in [31] does not allow to perform a quantitative study of the optimal Hardy constant $\mathfrak{h}_{s,p}(\Omega)$ and of its behaviour as $s \nearrow 1$ and $p \nearrow \infty$. On the contrary, the application of the supersolution method permits us to prove, that, when $p > N$, it holds

$$\lim_{s\nearrow 1}(1-s)\mathfrak{h}_{s,p}(\mathbb{R}^N \setminus \{0\}) = K_{p,N}\left(\frac{p-N}{p}\right)^p = K_{p,N}\mathfrak{h}_p(\mathbb{R}^N \setminus \{0\}),$$

where $K_{p,N}$ is an explicit constant depending only on p and N (see Theorem 3.1). This will follow by combining a *limsup*-inequality (which is valid for any open subset of $\mathbb{R}^N$) and a *liminf*-inequality (proved for $\Omega = \mathbb{R}^N \setminus \{0\}$), which are established in Lemma 3.2-3.4, respectively. We emphasize that, while for proving the lim sup-inequality, it is sufficient to use the variational definition (1.4) of $\mathfrak{h}_{s,p}(\Omega)$, for establishing the liminf-inequality the dual formulation (1.7) is better situated (since there, the Hardy constant is written as a *supremum* rather than an *infimum*).

Moreover, by exploiting the lower bound (1.5), in Theorem 3.6 we show that, for every $0 < s < 1$, it holds

$$\lim_{p\to\infty}(\mathfrak{h}_{s,p}(\Omega))^{\frac{1}{p}} = 1, \tag{1.8}$$

generalising the result given in [9, Theorem 4.4] when $s = 1$. Again, this will follow by combining a *liminf* and a *limsup* inequality, both valid, now, for any open set Ω.

In the last section of this paper, we apply Theorem 1.1 to obtain the following Cheeger inequality

$$\lambda_{s,p}(\Omega) \geq \mathfrak{h}_{s,p}\left(\frac{h_1(\Omega)}{N}\right)^{sp}, \tag{1.9}$$

for $sp > N$ and $\Omega \subsetneq \mathbb{R}^N$ open, see Theorem 4.1. Here $h_1(\Omega)$ is the *Cheeger constant* of Ω defined by

$$h_1(\Omega) = \inf\left\{\frac{P(E)}{|E|} \,:\, E \Subset \Omega \text{ smooth}, |E| > 0\right\}, \tag{1.10}$$

and $\lambda_{s,p}(\Omega)$ is defined by the following sharp fractional Poincaré inequality

$$\lambda_{s,p}(\Omega) = \inf_{u\in C_0^\infty(\Omega)}\left\{[u]^p_{W^{s,p}(\mathbb{R}^N)} \,:\, \int_\Omega |u|^p\,dx = 1\right\}. \tag{1.11}$$

We explicitly note that, in the case $s = 1$, when $p > N$, combining (1.9) with (1.2), we get an improvement of the classical Cheeger inequality with a constant which does not vanish as $p \to \infty$ (for further details, see Remark 4.2). Finally, thanks to (1.8), we study the asymptotic behaviour of the family $(\lambda_{s,p}(\Omega))^{1/p}$ as $p \to \infty$ (see Corollary 4.3), getting a sharp estimate in the limit case $p = \infty$ (see (4.10)).

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2. An explicit supersolution and the proof of Theorem 1.1

For every $1 < p < \infty$, we indicate by $J_p : \mathbb{R} \to \mathbb{R}$ the monotone increasing function defined by

$$J_p(t) = |t|^{p-2}\, t, \qquad \text{for every } t \in \mathbb{R}.$$

For $x_0 \in \mathbb{R}^N$ and $R > 0$, we will denote by $B_R(x_0)$ the $N-$dimensional open ball centered at x_0, with radius R. We will use the standard notation ω_N for the $N-$dimensional Lebesgue measure of $B_1(0)$. For an open set $\Omega \subsetneq \mathbb{R}^N$, we denote by

$$d_\Omega(x) := \min_{y \in \partial\Omega} |x - y|, \qquad \text{for every } x \in \Omega,$$

the distance function from the boundary. We extend d_Ω by 0 outside Ω. Moreover, we denote by r_Ω the inradius of Ω, defined by

$$r_\Omega = \|d_\Omega\|_{L^\infty(\Omega)} = \sup\Big\{r > 0 \,:\, \text{there exists } x_0 \in \Omega \text{ such that } B_r(x_0) \subseteq \Omega\Big\}.$$

For a pair of open sets $E \subseteq \Omega \subseteq \mathbb{R}^N$, the symbol $E \Subset \Omega$ means that the closure $\overline{E}$ is a compact subset of Ω.

For $0 < \alpha < \infty$, we denote by $L^\alpha_{sp}(\mathbb{R}^N)$ the following weighted Lebesgue space

$$L^\alpha_{sp}(\mathbb{R}^N) = \left\{u \in L^\alpha_{\rm loc}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \frac{|u|^\alpha}{(1+|x|)^{N+sp}}\, dx < +\infty\right\}.$$

For $1 < p < \infty$, $0 < s < 1$, and $\Omega \subsetneq \mathbb{R}^N$ open, we will consider the equation

$$(-\Delta_p)^s u = \lambda \frac{|u|^{p-2}u}{d_\Omega^{sp}} \quad \text{in } \Omega, \tag{2.1}$$

where $\lambda \geq 0$. Here $(-\Delta_p)^s$ is the *fractional $p-$Laplacian of order s*, defined in its weak form by

$$\langle(-\Delta_p)^s u, \varphi\rangle := \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{J_p(u(x)-u(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\, dx\, dy, \qquad \text{for every } \varphi \in C^\infty_0(\mathbb{R}^N).$$

Definition 2.1. We say that $u \in W^{s,p}_{\rm loc}(\mathbb{R}^N) \cap L^{p-1}_{sp}(\mathbb{R}^N)$ is a

- *local weak supersolution* of (2.1) if

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{J_p(u(x)-u(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\, dx\, dy \geq \lambda \int_{\mathbb{R}^N} \frac{|u|^{p-2}u}{d_\Omega^{sp}}\varphi\, dx$$

  for every non-negative $\varphi \in C^\infty_0(\Omega)$;

- *local weak subsolution* of (2.1) if

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{J_p(u(x)-u(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy \le \lambda \int_{\mathbb{R}^N} \frac{|u|^{p-2}u}{d_\Omega^{sp}}\varphi\,dx$$

for every non-negative $\varphi \in C_0^\infty(\Omega)$.
- *local weak solution* of (2.1) if it is both a local weak supersolution and a local weak subsolution.

The aim of this section is to prove the following result:

Theorem 2.2. *Let $sp > N$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open set. Then, the function*

$$U_{\Omega,\beta} := d_\Omega^\beta, \qquad \beta \in \left(0, \frac{ps-N}{p-1}\right),$$

is a positive local weak supersolution of

$$(-\Delta_p)^s U_{\Omega,\beta} = \mathcal{C}(\beta)\frac{U_{\Omega,\beta}^{p-1}}{d_\Omega^{sp}}, \qquad \text{in } \Omega, \tag{2.2}$$

where the positive constant $\mathcal{C}(\beta)$ is given by

$$\mathcal{C}(\beta) := 4\pi\alpha_N \int_0^1 |1-\rho^\beta|^{p-2}(1-\rho^\beta)\left[\rho^{N-1}-\rho^{ps-\beta(p-1)-1}\right] G(\rho^2)\,d\rho, \tag{2.3}$$

with

$$\alpha_N := \frac{\pi^{\frac{N-3}{2}}}{\Gamma\left(\frac{N-1}{2}\right)}, \qquad G(t) := B\left(\frac{N-1}{2},\frac{1}{2}\right) F\left(\frac{N+ps}{2},\frac{ps+2}{2};\frac{N}{2};t\right).$$

Here Γ, B, and F denote the gamma, the beta, and the hypergeometric functions, respectively[1].

Remark 2.3. Observe that, for the choice $\beta = (sp-N)/p$, we have that

$$\mathcal{C}\left(\frac{sp-N}{p}\right) = \mathfrak{h}_{s,p}.$$

Indeed, with simple algebraic manipulations, this choice gives

$$\begin{aligned}\left|1-\rho^{\frac{sp-N}{p}}\right|^{p-2}\left(1-\rho^{\frac{sp-N}{p}}\right)\left[\rho^{N-1}-\rho^{ps-\frac{sp-N}{p}(p-1)-1}\right] &= \left(1-\rho^{\frac{sp-N}{p}}\right)^p\rho^{N-1}\\ &= \rho^{sp-1}\,\rho^{N-sp}\left(1-\rho^{\frac{sp-N}{p}}\right)^p\\ &= \rho^{sp-1}\left|1-\rho^{\frac{N-sp}{p}}\right|^p.\end{aligned}$$

Moreover, as observed in [16], according to [19, equation (3.665)] we have

$$\Phi_{N,s,p}(t) = |\mathbb{S}|^{N-2}\,B\left(\frac{N-1}{2},\frac{1}{2}\right)F\left(\frac{N+ps}{2},\frac{ps+2}{2};\frac{N}{2};t\right) = |\mathbb{S}^{N-2}|G(t^2),$$

and

$$|\mathbb{S}^{N-2}| = 2\,\frac{\pi^{\frac{N-1}{2}}}{\Gamma\left(\frac{N-1}{2}\right)} = 2\,\pi\,\alpha_N.$$

[1] We refer for example to [25, Chapter 1] and [25, Chapter 9] for the definitions and properties of these functions.

This finally gives

$$\mathfrak{h}_{s,p} = 2\int_0^1 \rho^{sp-1}\left|1-\rho^{\frac{N-sp}{p}}\right|^p \Phi_{N,s,p}(\rho)\,d\rho = \mathcal{C}\left(\frac{sp-N}{p}\right).$$

According to the previous remark, when $\beta = (sp-N)/p$ from Theorem 2.2 we immediately get the following

Corollary 2.4. *Let $sp > N$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open set. Then, the function*

$$U_\Omega := d_\Omega^{\frac{sp-N}{p}},$$

is a positive local weak supersolution of

$$(-\Delta_p)^s U_\Omega = \mathfrak{h}_{s,p}\frac{U_\Omega^{p-1}}{d_\Omega^{sp}} \quad \text{in } \Omega.$$

In order to show Theorem 2.2, we recall that, by [11, Theorem 1.1] (see also [16, Lemma 3.1]), given $z \in \mathbb{R}^N$, $sp > N$ and $0 < \beta < (sp-N)/(p-1)$, the function

$$V(x) := |x-z|^\beta = d^\beta_{\mathbb{R}^N\setminus\{z\}}(x),$$

belongs to $W^{s,p}_{\rm loc}(\mathbb{R}^N) \cap L^{p-1}_{sp}(\mathbb{R}^N)$ and is a local weak solution to

$$(-\Delta_p)^s V = \mathcal{C}(\beta)\frac{V^{p-1}}{d^{sp}_{\mathbb{R}^N\setminus\{z\}}} \quad \text{in } \mathbb{R}^N\setminus\{z\}. \tag{2.4}$$

where $\mathcal{C}(\beta)$ is given by (2.3). Using this fact, we can prove the following preliminary result.

Proposition 2.5. *Let $sp > N$ and $0 < \beta < (sp-N)/(p-1)$. Let $x_0,\, x_1 \in \mathbb{R}^N$. Then, the function*

$$U_1 := d^\beta_{\mathbb{R}^N\setminus\{x_0,x_1\}}$$

is a local weak supersolution of

$$(-\Delta_p)^s U_1 = \mathcal{C}(\beta)\frac{U_1^{p-1}}{d^{sp}_{\mathbb{R}^N\setminus\{x_0,x_1\}}}, \qquad \text{in } \mathbb{R}^N\setminus\{x_0,x_1\}, \tag{2.5}$$

where $\mathcal{C}(\beta)$ is given by (2.3).

Proof. We start with the obvious (yet crucial) observation that, since $\beta > 0$, we have

$$\begin{aligned} U_1 = d^\beta_{\mathbb{R}^N\setminus\{x_0,x_1\}} &= \left(\min\{d_{\mathbb{R}^N\setminus\{x_0\}}, d_{\mathbb{R}^N\setminus\{x_1\}}\}\right)^\beta \\ &= \min\left\{d^\beta_{\mathbb{R}^N\setminus\{x_0\}}, d^\beta_{\mathbb{R}^N\setminus\{x_1\}}\right\} = \min\{V_0, V_1\}, \end{aligned} \tag{2.6}$$

where

$$V_i(x) = |x-x_i|^\beta, \qquad \text{for } i = 0,\, 1.$$

Then the function U_1 belongs to $W^{s,p}_{\rm loc}(\mathbb{R}^N)\cap L^{p-1}_{sp}(\mathbb{R}^N)$. In order to prove that U_1 is a local weak supersolution of (2.5), we can suitably adapt the strategy applied in the proof of [24, Theorem 1.1], where it is shown that the minimum of two locally weakly (s,p)-superharmonic functions is itself a locally weakly (s,p)-superharmonic function. In our case we need to handle the additional nonlinear term on the right-hand side, which is however local.

Let $\varphi \in C_0^\infty(\mathbb{R}^N \setminus \{x_0, x_1\})$ be a nonnegative test function. Then it is admissible for the weak formulation of the equations satisfied by V_0 and V_1 (i.e. equation (2.4) with z replaced by x_0 and x_1, respectively). For every $0 < \varepsilon < 1/4$ we define

$$\theta_\varepsilon := \min\left\{1, \frac{(V_0 - V_1)_+}{\varepsilon}\right\}.$$

Then we consider $\eta_1 = (1-\theta_\varepsilon)\varphi$ as a test function in the equation satisfied by V_0, and $\eta_2 = \theta_\varepsilon\varphi$ as a test function in the equation satisfied by V_1. By summing up the corresponding integrals for V_0 and V_1, we obtain

$$\mathcal{C}(\beta)\int_{\mathbb{R}^N} \frac{|V_0|^{p-2}V_0}{d^{sp}_{\mathbb{R}^N\setminus\{x_0\}}}(1-\theta_\varepsilon)\varphi dx + \mathcal{C}(\beta)\int_{\mathbb{R}^N} \frac{|V_1|^{p-2}V_1}{d^{sp}_{\mathbb{R}^N\setminus\{x_1\}}}\theta_\varepsilon\varphi dx = \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\Phi_\varepsilon(x,y)}{|x-y|^{N+sp}}\,dx\,dy, \tag{2.7}$$

where

$$\begin{aligned}\Phi_\varepsilon(x,y) &= J_p(V_0(x)-V_0(y))((1-\theta_\varepsilon(x))\varphi(x) - (1-\theta_\varepsilon(y))\varphi(y)) \\ &\quad + J_p(V_1(x)-V_1(y))(\theta_\varepsilon(x)\varphi(x) - \theta_\varepsilon(y)\varphi(y)).\end{aligned}$$

As shown in [24, Theorem 1.1], we have that

$$\limsup_{\varepsilon\to 0}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\Phi_\varepsilon(x,y)}{|x-y|^{N+sp}}\,dx\,dy \le \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{J_p(U_1(x)-U_1(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy. \tag{2.8}$$

We further observe that, by (2.6), we have

$$\frac{U_1^{p-1}(x)}{d^{sp}_{\mathbb{R}^N\setminus\{x_0,x_1\}}(x)} = \frac{\min\{V_0^{p-1}(x), V_1^{p-1}(x)\}}{\min\left\{d^{sp}_{\mathbb{R}^N\setminus\{x_0\}}(x), d^{sp}_{\mathbb{R}^N\setminus\{x_1\}}(x)\right\}} = \begin{cases} \dfrac{V_0^{p-1}(x)}{d^{sp}_{\mathbb{R}^N\setminus\{x_0\}}(x)} & \text{if } x \in S_1, \\[2ex] \dfrac{V_1^{p-1}(x)}{d^{sp}_{\mathbb{R}^N\setminus\{x_1\}}(x)} & \text{if } x \in S_2, \end{cases} \tag{2.9}$$

where

$$S_1 := \{x \in \mathbb{R}^N : |x-x_0| \le |x-x_1|\} \quad\text{and}\quad S_2 := \{x\in\mathbb{R}^N : |x-x_1| < |x-x_0|\}.$$

Moreover, by definition of θ_ε, S_1 and S_2, we have that

$$\theta_\varepsilon = 0 \text{ on } S_1 \quad\text{and}\quad \theta_\varepsilon \to 1 \text{ pointwise on } S_2.$$

Then, taking into account that $\varphi \in C_0^\infty(\mathbb{R}^N\setminus\{x_0,x_1\})$, we can apply the Lebesgue Dominated Convergence Theorem and we get

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} \frac{V_0^{p-1}}{d^{sp}_{\mathbb{R}^N\setminus\{x_0\}}}(1-\theta_\varepsilon)\varphi dx + \int_{\mathbb{R}^N}\frac{V_1^{p-1}}{d^{sp}_{\mathbb{R}^N\setminus\{x_1\}}}\theta_\varepsilon\varphi dx \\ &= \int_{S_1}\frac{V_0^{p-1}}{d^{sp}_{\mathbb{R}^N\setminus\{x_0\}}}\varphi dx + \lim_{\varepsilon\to 0}\left(\int_{S_2}\frac{V_0^{p-1}}{d^{sp}_{\mathbb{R}^N\setminus\{x_0\}}}(1-\theta_\varepsilon)\varphi dx + \int_{S_2}\frac{V_1^{p-1}}{d^{sp}_{\mathbb{R}^N\setminus\{x_1\}}}\theta_\varepsilon\varphi dx\right) \\ &= \int_{S_1}\frac{V_0^{p-1}}{d^{sp}_{\mathbb{R}^N\setminus\{x_0\}}}\varphi dx + \int_{S_2}\frac{V_1^{p-1}}{d^{sp}_{\mathbb{R}^N\setminus\{x_1\}}}\varphi dx = \int_{\mathbb{R}^N}\frac{U_1^{p-1}}{d^{sp}_{\mathbb{R}^N\setminus\{x_1,x_0\}}}\varphi dx,\end{aligned} \tag{2.10}$$

where the last identity follows thanks to (2.9). Hence, combining (2.8), (2.7) and (2.10), we get the desired conclusion. □

By repeatedly applying Proposition 2.5, we have the following corollary.

Corollary 2.6. *Let $sp > N$ and $0 < \beta < (sp-N)/(p-1)$. Let $n \in \mathbb{N}$ and $x_0, \dots, x_n \in \mathbb{R}^N$. Then, the function*

$$U_n = d^{\beta}_{\mathbb{R}^N \setminus \{x_0,\dots,x_n\}},$$

is a local weak supersolution of

$$(-\Delta_p)^s U_n = \mathcal{C}(\beta) \frac{U_n^{p-1}}{d^{sp}_{\mathbb{R}^N\setminus\{x_0,\dots,x_n\}}}, \qquad \text{in } \mathbb{R}^N \setminus \{x_0,\dots,x_n\}, \tag{2.11}$$

where $\mathcal{C}(\beta)$ is given by (2.3).

We are now ready to prove Theorem 2.2. The idea is to argue by approximation: we pick a countable dense subset $\cup_{i\in\mathbb{N}}\{x_i\}$ of $\partial\Omega$, we apply Corollary 2.6 on the finite set $\cup_{i=0}^{n}\{x_i\}$, and finally we pass to the limit as $n \to \infty$.

Proof of Theorem 2.2. Let $D = \cup_{i\in\mathbb{N}}\{x_i\}$ be a dense subset of $\partial\Omega$ and for every $n \in \mathbb{N}$ we define

$$E_n = \mathbb{R}^N \setminus \{x_0,\dots,x_n\} \qquad \text{and} \qquad U_n = d^{\beta}_{E_n}.$$

By Corollary 2.6, we know that, for every $n \in \mathbb{N}$, the function $U_n \in L^{p-1}_{sp}(\mathbb{R}^N)$ is a local weak supersolution of (2.11) in $E_n \supseteq \Omega$. Moreover, $\{d_{E_n}\}_{n\in\mathbb{N}}$ and $\{U_n\}_{n\in\mathbb{N}}$ are decreasing sequences (being $\beta > 0$) and, since D is dense in $\partial\Omega$, we have that for every $x \in \Omega$

$$d_\Omega(x) = \min_{z\in\partial\Omega} |x-z| = \inf_{i\in\mathbb{N}} |x-x_i| = \inf_{n\in\mathbb{N}} d_{E_n}(x) = \lim_{n\to\infty} d_{E_n}(x), \tag{2.12}$$

and

$$U_{\Omega,\beta}(x) = d^{\beta}_{\Omega}(x) = \min_{z\in\partial\Omega} |x-z|^{\beta} = \inf_{i\in\mathbb{N}} |x-x_i|^{\beta} = \inf_{n\in\mathbb{N}} U_n(x) = \lim_{n\to\infty} U_n(x). \tag{2.13}$$

In particular, $U_{\Omega,\beta} \in L^{p-1}_{sp}(\mathbb{R}^N)$. In order to show that the function $U_{\Omega,\beta}$ is a local weak supersolution of (2.2), let $\varphi \in C^\infty_0(\Omega)$ be a non-negative function and we observe that it is an admissible test function for the weak formulation of the equation satisfied by U_n, for any $n \in \mathbb{N}$. Indeed, we have $\Omega \subseteq E_n$ by costruction. Hence, for any $n \in \mathbb{N}$, we have

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{J_p(U_n(x)-U_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy \ge \mathcal{C}(\beta)\int_\Omega \frac{U_n^{p-1}\varphi}{d^{sp}_{E_n}}\,dx. \tag{2.14}$$

Since the integrand in the right-hand side of (2.14) is non-negative, Fatou's Lemma in conjunction with (2.12) and (2.13) gives

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{U_n^{p-1}\varphi}{d^{sp}_{E_n}}\,dx \ge \int_{\mathbb{R}^N} \frac{U^{p-1}_{\Omega,\beta}\varphi}{d^{sp}_{\Omega}}\,dx. \tag{2.15}$$

Let us consider now the left-hand side of (2.14). We recall that φ is compactly supported in Ω, thus, denoting by S_φ its support, we have that

$$\delta := \operatorname{dist}(\partial\Omega, S_\varphi) > 0.$$

Moreover, we set

$$K_\delta(\varphi) = \{y \in \Omega \,:\, \operatorname{dist}(y, S_\varphi) < \delta/2\}.$$

We can write

$$
\begin{aligned}
&\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{J_p(U_n(x)-U_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy\\
&\quad= \iint_{S_\varphi\times S_\varphi} \frac{J_p(U_n(x)-U_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy\\
&\qquad+2\iint_{S_\varphi\times(\mathbb{R}^N\setminus S_\varphi)} \frac{J_p(U_n(x)-U_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy\\
&\quad= \iint_{S_\varphi\times S_\varphi} \frac{J_p(U_n(x)-U_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy\\
&\qquad+2\iint_{S_\varphi\times(K_\delta(\varphi)\setminus S_\varphi)} \frac{J_p(U_n(x)-U_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy\\
&\qquad+2\iint_{S_\varphi\times(\mathbb{R}^N\setminus K_\delta(\varphi))} \frac{J_p(U_n(x)-U_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy =: \mathcal{I}_n^1+\mathcal{I}_n^2+\mathcal{I}_n^3.
\end{aligned}
$$

Taking into account the pointwise convergence (2.13), we want to apply the Dominated Convergence Theorem in order to pass to the limit in each $\mathcal{I}_n^i$. To this aim, we observe that for $x, y \in \Omega$ we have

$$|U_n(x)-U_n(y)| \le \beta(d_{E_n}(x)^{\beta-1}+d_{E_n}(y)^{\beta-1})|d_{E_n}(x)-d_{E_n}(y)|.$$

This simply follows from the Fundamental Theorem of Calculus, applied to the function $t \mapsto t^\beta$. Moreover, by using that $d_\Omega \le d_{E_n}$, that $\beta-1<0$, and the 1−Lipschitz character of the distance function, we get

$$|U_n(x)-U_n(y)| \le \beta(d_\Omega(x)^{\beta-1}+d_\Omega(y)^{\beta-1})|x-y|, \qquad \text{for } x,y\in\Omega,$$

and, in particular, we have that

$$|U_n(x)-U_n(y)| \le C(\beta,\delta)|x-y|, \quad \text{for any } x,\ y\ \in K_\delta(\varphi). \tag{2.16}$$

Using (2.16) and the Lipschitz continuity of φ, we deduce

$$\frac{|J_p(U_n(x)-U_n(y))||\varphi(x)-\varphi(y)|}{|x-y|^{N+sp}} \le C(\beta,\delta)^{(p-1)}||\nabla\varphi||_{L^\infty(\Omega)}\,\frac{1}{|x-y|^{N+p(s-1)}} \in L^1(K_\delta(\varphi)\times K_\delta(\varphi)).$$

This allows to pass to the limit in both $\mathcal{I}_n^1$ and $\mathcal{I}_n^2$, showing that

$$
\begin{aligned}
\lim_{n\to+\infty}(\mathcal{I}_n^1+\mathcal{I}_n^2) &= \iint_{S_\varphi\times S_\varphi} \frac{J_p(U_{\Omega,\beta}(x)-U_{\Omega,\beta}(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy\\
&\quad+2\iint_{S_\varphi\times(K_\delta(\varphi)\setminus S_\varphi)} \frac{J_p(U_{\Omega,\beta}(x)-U_{\Omega,\beta}(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy.
\end{aligned}
\tag{2.17}
$$

It remains to show that

$$\lim_{n\to+\infty}\mathcal{I}_n^3 = 2\iint_{S_\varphi\times(\mathbb{R}^N\setminus K_\delta(\varphi))} \frac{J_p(U_{\Omega,\beta}(x)-U_{\Omega,\beta}(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy. \tag{2.18}$$

We start by observing that, since $\{U_n\}_{n\in\mathbb{N}}$ is a decreasing sequence, we have

$$\begin{aligned}&\frac{2|J_p(U_n(x)-U_n(y)||\varphi(x)-\varphi(y)|}{|x-y|^{N+sp}}\\&\leq 4\|\varphi\|_{L^\infty(\Omega)}\frac{|U_n(x)-U_n(y)|^{p-1}}{|x-y|^{N+sp}}\\&\leq 2^p\|\varphi\|_{L^\infty(\Omega)}\frac{|U_0(x)|^{p-1}+|U_0(y)|^{p-1}}{|x-y|^{N+sp}}.\end{aligned}$$

Equality (2.18) will follow, by applying again the Dominated Convergence Theorem, if we show that

$$g(x,y):=\frac{|U_0(x)|^{p-1}+|U_0(y)|^{p-1}}{|x-y|^{N+sp}}\in L^1(S_\varphi\times(\mathbb{R}^N\setminus K_\delta(\varphi))). \tag{2.19}$$

In order to do that, we note that

$$|x-y|>\frac{\delta}{2}\qquad\text{for every } x\in S_\varphi \text{ and } y\in\mathbb{R}^N\setminus K_\delta(\varphi). \tag{2.20}$$

Hence we have that

$$\begin{aligned}\iint_{S_\varphi\times(\mathbb{R}^N\setminus K_\delta(\varphi))} g(x,y)\,dx\,dy &= \int_{S_\varphi}|U_0(x)|^{p-1}\,dx\int_{\mathbb{R}^N\setminus K_\delta(\varphi)}\frac{1}{|x-y|^{N+sp}}\,dy\\&\quad+\iint_{S_\varphi\times(\mathbb{R}^N\setminus K_\delta(\varphi))}\frac{|U_0(y)|^{p-1}}{|x-y|^{N+sp}}\,dx\,dy\\&\leq \frac{N\omega_N}{sp}\left(\frac{2}{\delta}\right)^{sp}\int_{S_\varphi}|U_0(x)|^{p-1}\,dx+\iint_{S_\varphi\times(\mathbb{R}^N\setminus K_\delta(\varphi))}\frac{|U_0(y)|^{p-1}}{|x-y|^{N+sp}}\,dx\,dy.\end{aligned} \tag{2.21}$$

Since S_φ is a compact set, the first integral on the right-hand side in (2.21) is finite. For the second integral, we observe that for every $x\in S_\varphi$ and for every $y\in\mathbb{R}^N\setminus K_\delta(\varphi)$, by applying again (2.20), it holds

$$\frac{|y|+1}{|x-y|}\leq\frac{|y-x|+|x|+1}{|x-y|}\leq 1+\frac{|x|+1}{|x-y|}\leq 1+\frac{2(M+1)}{\delta}=C,$$

where $M=\max_{x\in S_\varphi}|x|$. This implies that

$$\frac{1}{|x-y|^{N+sp}}\leq\frac{C^{N+sp}}{(1+|y|)^{N+sp}}\qquad\text{for every } x\in S_\varphi \text{ and } y\in\mathbb{R}^N\setminus K_\delta(\varphi). \tag{2.22}$$

Since U_0 belongs to $L^{p-1}_{sp}(\mathbb{R}^N)$, the above estimate implies that also the second term on the right-hand side of (2.21) is finite and thus (2.19) holds true.

Finally, combining (2.17) and (2.18), we deduce that

$$\begin{aligned}&\lim_{n\to\infty}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{J_p(U_n(x)-U_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy\\&=\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{J_p(U_{\Omega,\beta}(x)-U_{\Omega,\beta}(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy.\end{aligned}$$

Putting together the limit above with (2.14) and (2.15), we conclude that $U_{\Omega,\beta}$ is a positive local weak supersolution of (2.2). □

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. By combining formula (1.7) with Theorem 2.2 when $\beta = (sp-N)/p$, we obtain that

$$\mathfrak{h}_{s,p}(\Omega) \geq \mathcal{C}\left(\frac{sp-N}{p}\right) = \mathfrak{h}_{s,p}.$$

The last equality follows from Remark 2.3. □

We conclude this section with the following result. It gives a sufficient condition for the constant $\mathfrak{h}_{s,p}(\Omega)$ not to be attained.

Proposition 2.7. *Let $1 < p < \infty$, $0 < s < 1$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open set. Suppose that there exists a positive local weak supersolution u of (2.1) with $\lambda = \mathfrak{h}_{s,p}(\Omega)$, such that*

$$u \geq \frac{1}{C}\, d_\Omega^{\frac{sp-N}{p}},$$

for some positive constant C. Then, the infimum $\mathfrak{h}_{s,p}(\Omega)$ is not attained.

In particular, when $sp > N$, the constant $\mathfrak{h}_{s,p}(\Omega)$ is not attained for every open set $\Omega \subsetneq \mathbb{R}^N$ such that

$$\mathfrak{h}_{s,p}(\Omega) = \mathfrak{h}_{s,p}.$$

Proof. The proof follows the one of [4, Proposition 3.5] with some minor changes, and uses some integrability properties of the distance function. We argue by contradiction and suppose that $v \in \widetilde{W}_0^{s,p}(\Omega)$ is a minimizer for $\mathfrak{h}_{s,p}(\Omega)$. Thus, in such a case, we have $\mathfrak{h}_{s,p}(\Omega) > 0$. Following [4, Proposition 3.5], we can assume that v is positive. Let us consider now a sequence of functions $v_n \in C_0^\infty(\Omega)$ (which we can assume to be nonnegative) approximating v in $W^{s,p}(\mathbb{R}^N)$ and almost everywhere.

By using as a test function in the weak formulation of the inequality satisfied by u, the function

$$\varphi := \frac{v_n^p}{u^{p-1}},$$

and proceeding as in [4], the equality cases of the fractional Picone inequality permits to infer that

$$u = Cv, \qquad \text{a. e. in } \Omega,$$

for some positive constant C. Thus, that there exists another, possibly different, positive constant C such that

$$v \geq \frac{1}{C} d_\Omega^{\frac{sp-N}{p}}, \qquad \text{in } \Omega.$$

This contradicts the minimality of v, since we would have

$$[v]^p_{W^{s,p}(\mathbb{R}^N)} = \mathfrak{h}_{s,p}(\Omega) \int_\Omega \frac{v^p}{d_\Omega^{sp}}\, dx \geq \frac{\mathfrak{h}_{s,p}(\Omega)}{C^p} \int_\Omega \frac{1}{d_\Omega^N}\, dx = +\infty,$$

where the last equality follows from [3, Lemma 3.4].

Finally, if $sp > N$, let us suppose that $\mathfrak{h}_{s,p}(\Omega) = \mathfrak{h}_{s,p}$. Then, thanks to Corollary 2.4, the function $U = d_\Omega^{(sp-N)/p}$ is a positive local weak supersolution of (2.1) with $\lambda = \mathfrak{h}_{s,p} = \mathfrak{h}_{s,p}(\Omega)$. From the first part of the proof we get the desired conclusion. □

3. Asymptotics for the sharp constant

3.1. The case $s \nearrow 1$. This subsection is devoted to study the limit

$$\lim_{s\nearrow 1}(1-s)\,\mathfrak{h}_{s,p}(\Omega),$$

for $p > N$. In the case when Ω is a convex set or the punctured space $\mathbb{R}^N \setminus \{0\}$ we can show that

$$\lim_{s\nearrow 1}(1-s)\,\mathfrak{h}_{s,p}(\Omega) = K_{p,N}\,\mathfrak{h}_p(\Omega),$$

where

$$\mathfrak{h}_p(\Omega) = \inf_{u\in C_0^\infty(\Omega)}\left\{\|\nabla u\|^p_{L^p(\Omega)}\,:\,\int_\Omega \frac{|u|^p}{d_\Omega^p}\,dx = 1\right\} \quad \text{and} \quad K_{p,N} = \frac{1}{p}\int_{\mathbb{S}^{N-1}} |\langle \omega, \mathbf{e}_1\rangle|^p\, d\mathcal{H}^{N-1}(\omega).$$

We start by recalling the celebrated *Bourgain-Brezis-Mironescu formula*

$$\lim_{s\nearrow 1}(1-s)\,[\varphi]^p_{W^{s,p}(\mathbb{R}^N)} = K_{p,N}\,[\varphi]^p_{W^{1,p}(\mathbb{R}^N)}, \qquad \text{for every } \varphi \in C_0^\infty(\mathbb{R}^N), \tag{3.1}$$

which will be useful in the sequel. For a proof of (3.1), see for example [14, Corollary 3.20].

Now we are in a position to state the main result of this section.

Theorem 3.1. *Let $p > N$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open set such that $\mathfrak{h}_p(\Omega) = \mathfrak{h}_p$. Then*

$$\lim_{s\nearrow 1}(1-s)\,\mathfrak{h}_{s,p}(\Omega) = K_{p,N}\mathfrak{h}_p = K_{p,N}\left(\frac{p-N}{p}\right)^p.$$

In order to show Theorem 3.1, we start proving the lim sup inequality, which holds true for any open set and every p.

Lemma 3.2. *Let $1 < p < \infty$, for every open set $\Omega \subsetneq \mathbb{R}^N$, it holds*

$$\limsup_{s\nearrow 1}(1-s)\,\mathfrak{h}_{s,p}(\Omega) \le K_{p,N}\,\mathfrak{h}_p(\Omega). \tag{3.2}$$

Proof. For every $\varepsilon > 0$, there exists $u_\varepsilon \in C_0^\infty(\Omega)$ such that

$$\mathfrak{h}_p(\Omega) + \varepsilon \ge \int_\Omega |\nabla u_\varepsilon|^p dx \qquad \text{and} \qquad \left\|\frac{u_\varepsilon}{d_\Omega}\right\|_{L^p(\Omega)} = 1.$$

Then, by using Fatou's lemma and (3.1), we get

$$\begin{aligned}\limsup_{s\nearrow 1}(1-s)\,\mathfrak{h}_{s,p}(\Omega) &\le \limsup_{s\nearrow 1}\frac{(1-s)\,[u_\varepsilon]^p_{W^{s,p}(\mathbb{R}^N)}}{\left\|\frac{u_\varepsilon}{d_\Omega^s}\right\|_{L^p(\Omega)}}\\ &\le K_{p,N}\,\|\nabla u_\varepsilon\|^p_{L^p(\mathbb{R}^N)} \le K_{p,N}\,\Big(\mathfrak{h}_p(\Omega)+\varepsilon\Big).\end{aligned}$$

By arbitrariness of $\varepsilon > 0$, the latter gives (3.2). □

In order to prove the lim inf inequality, we are going to show that for a Sobolev function u, as $s \nearrow 1$, we have

$$(1-s)\,(-\Delta_p)^s u \to -\Delta_p u,$$

in weak sense, up to a normalization constant. In other words, for every test function φ we consider the limit, as s goes to 1, of

$$(1-s)\,\langle(-\Delta_p)^s u, \varphi\rangle := (1-s)\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{J_p(u(x)-u(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy,$$

and show that this coincides with

$$K_{p,N}\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,dx.$$

This extends [10, Theorem 2.8] and [12, Lemma 5.1], by considerably relaxing the assumptions on the involved functions. The proof will exploit the convexity of the function $t\mapsto |t|^p$ and (3.1).

Lemma 3.3. *Let $\Omega\subseteq\mathbb{R}^N$ be an open set and let $\varphi\in C_0^\infty(\Omega)$. Let $0<s<1$, $1<p<\infty$, and assume that $u\in W^{1,p}_{\rm loc}(\Omega)\cap L^{p-1}_{sp}(\mathbb{R}^N)$. Then,*

$$\lim_{s\nearrow 1}(1-s)\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{J_p(u(x)-u(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}dx\,dy = K_{p,N}\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,dx.$$

Proof. Let us denote by S_φ the support of φ and let $\Omega'\Subset\Omega$ be an open set with Lipschitz boundary such that $S_\varphi\subseteq\Omega'$. By convexity of the map $t\to J_p(t)$, for every $t\in(0,1)$ we have:

$$\frac{1}{p}[u+t\varphi]^p_{W^{s,p}(\Omega')}-\frac{1}{p}[u]^p_{W^{s,p}(\Omega')}\geq t\iint_{\Omega'\times\Omega'}\frac{J_p(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy.$$

Multiplying the above inequality by $(1-s)$, letting $s\nearrow 1$, and applying [30, Corollary 1], we deduce

$$\begin{aligned}&\frac{K_{p,N}}{p}\left(\|\nabla u+t\nabla\varphi\|^p_{L^p(\Omega')}-\|\nabla u\|^p_{L^p(\Omega')}\right)\\&\qquad\geq t\,\limsup_{s\nearrow 1}(1-s)\iint_{\Omega'\times\Omega'}\frac{J_p(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy.\end{aligned}$$

Dividing by $t\in(0,1)$ and letting $t\searrow 0$, we have

$$\begin{aligned}&K_{p,N}\int_\Omega|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,dx\\&\qquad\geq\limsup_{s\nearrow 1}(1-s)\iint_{\Omega'\times\Omega'}\frac{J_p(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy.\end{aligned}\tag{3.3}$$

We define

$$T_s:=2\int_{\Omega'}\int_{\mathbb{R}^N\setminus\Omega'}\frac{J_p(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy=2\int_{\Omega'}\varphi(x)\left(\int_{\mathbb{R}^N\setminus\Omega'}\frac{J_p(u(x)-u(y))}{|x-y|^{N+sp}}dy\right)dx.$$

We claim that T_s is uniformly bounded for $s_0<s<1$, with a bound degenerating as s_0 goes to 0. Indeed, we note that

$$\delta:=\mathrm{dist}(\partial\Omega',S_\varphi)>0.$$

Then, we have

$$\begin{aligned}|T_s|&\leq 2\int_{S_\varphi}|\varphi(x)|\left(\int_{\mathbb{R}^N\setminus\Omega'}\frac{|J_p(u(x)-u(y))|}{|x-y|^{N+sp}}dy\right)dx\\&\leq 2\|\varphi\|_{L^\infty(\Omega)}\iint_{S_\varphi\times\{y\in\mathbb{R}^N\,:\,d(y,S_\varphi)>\delta/2\}}\frac{|u(x)-u(y))|^{p-1}}{|x-y|^{N+sp}}\,dx\,dy\\&\leq C_p\|\varphi\|_{L^\infty(\Omega)}\iint_{S_\varphi\times\{y\in\mathbb{R}^N\,:\,d(y,S_\varphi)>\delta/2\}}\frac{|u(x)|^{p-1}+|u(y)|^{p-1}}{|x-y|^{N+sp}}\,dx\,dy\\&\leq C_p\|\varphi\|_{L^\infty(\Omega)}\left(\frac{N\omega_N}{sp}\left(\frac{2}{\delta}\right)^{sp}\int_{S_\varphi}|u(x)|^{p-1}\,dx+\iint_{S_\varphi\times\{y\in\mathbb{R}^N\,:\,d(y,S_\varphi)>\delta/2\}}\frac{|u(y)|^{p-1}}{|x-y|^{N+sp}}\,dx\,dy\right).\end{aligned}$$

By applying (2.22), for every $x \in S_\varphi$ we have that

$$\frac{|u(y)|^{p-1}}{|x-y|^{N+sp}} \le C^{N+sp} \frac{|u(y)|^{p-1}}{(1+|y|)^{N+sp}}, \qquad \text{for every } y \in \mathbb{R}^N \text{ such that } d(y, S_\varphi) > \delta/2.$$

Since u belongs to $L^{p-1}_{sp}(\mathbb{R}^N)$, the above estimate easily implies that $\{T_s\}_{s_0<s<1}$ is bounded. Thus, we get

$$\lim_{s \nearrow 1} (1-s) T_s = 0,$$

and (3.3) gives

$$K_{p,N} \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx \ge \limsup_{s \nearrow 1} (1-s) \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{J_p(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}} \, dx \, dy.$$

Finally, by replacing φ with $-\varphi$, we get

$$K_{p,N} \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx \le \liminf_{s \nearrow 1} (1-s) \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{J_p(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}} \, dx \, dy.$$

Joining the last two equations, we eventually conclude the proof. □

We can now give the proof of the lim inf inequality, which holds true for the specific case of the punctured space $\Omega = \mathbb{R}^N \setminus \{0\}$.

Lemma 3.4. *Let $p > N$. Then*

$$\liminf_{s \nearrow 1} (1-s) \mathfrak{h}_{s,p} \ge K_{p,N} \mathfrak{h}_p.$$

Proof. Observe that

$$\beta := \frac{p-N}{p} \in \left(0, \frac{sp-N}{p-1}\right),$$

for s sufficiently close to 1. Hence, for such values of s, by applying again [11, Theorem 1.1], the function $u(x) = |x|^\beta$ is a positive local weak solution to

$$(-\Delta_p)^s u = \mathcal{C}_{p,s} \frac{u^{p-1}}{|x|^{sp}}, \quad \text{in } \mathbb{R}^N \setminus \{0\},$$

where $\mathcal{C}_{p,s}$ is the constant given by (2.3) when $\beta = (p-N)/p$.

By using again (1.7), we obtain that

$$\mathfrak{h}_{s,p} \ge \mathcal{C}_{p,s}, \tag{3.4}$$

for s sufficiently close to 1. Moreover, by a direct computation, we have that u satisfies

$$-\Delta_p u = \mathfrak{h}_p \frac{u^{p-1}}{|x|^p}, \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

Now, let $s_j \nearrow 1$ be such that

$$\liminf_{s \nearrow 1} (1-s) \mathfrak{h}_{s,p} = \lim_{j \to \infty} (1-s_j) \mathfrak{h}_{p,s_j}.$$

Now we apply Lemma 3.3, with a fixed non-negative function $\varphi \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ such that $\varphi \neq 0$ and u defined above. From the equation satisfied by u, we then obtain

$$
\begin{aligned}
K_{p,N}\mathfrak{h}_p \int_{\mathbb{R}^N} \frac{u^{p-1}}{|x|^p}\varphi\,dx &= K_{p,N}\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,dx \\
&= \lim_{j\to\infty}(1-s_j)\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{J_p(u(x)-u(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+s_jp}}\,dx\,dy \\
&= \lim_{j\to\infty}\left((1-s_j)\mathcal{C}_{p,s_j}\int_{\mathbb{R}^N}\frac{u^{p-1}}{|x|^{s_jp}}\varphi\,dx\right).
\end{aligned}
\tag{3.5}
$$

If we now apply (3.4) with s_j in the place of s, (3.5) implies that

$$
K_{p,N}\mathfrak{h}_p \int_{\mathbb{R}^N}\frac{u^{p-1}}{|x|^p}\varphi\,dx \le \lim_{j\to\infty}(1-s_j)\mathfrak{h}_{p,s_j}\int_{\mathbb{R}^N}\frac{u^{p-1}}{|x|^{s_jp}}\varphi\,dx = \liminf_{s\nearrow 1}(1-s)\mathfrak{h}_{s,p}\int_{\mathbb{R}^N}\frac{u^{p-1}}{|x|^p}\varphi\,dx.
$$

Hence, the desired conclusion follows, by canceling the common factor. □

Proof of Theorem 3.1. By applying Lemma 3.2, Theorem 1.1 and Lemma 3.4, we have that for every open set $\Omega \subsetneq \mathbb{R}^N$ it holds

$$
K_{p,N}\mathfrak{h}_p(\Omega) \ge \limsup_{s\nearrow 1}(1-s)\,\mathfrak{h}_{s,p}(\Omega) \ge \liminf_{s\nearrow 1}(1-s)\,\mathfrak{h}_{s,p} \ge K_{p,N}\mathfrak{h}_p
$$

When $\mathfrak{h}_p(\Omega) = \mathfrak{h}_p$, this implies that

$$
\lim_{s\nearrow 1}(1-s)\,\mathfrak{h}_{s,p}(\Omega) = \mathfrak{h}_p.
$$

□

Remark 3.5. Actually, with the same proof, one can prove the analogue of Theorem 3.1 for convex sets. In other words, if $1 < p < \infty$ and $\Omega \subseteq \mathbb{R}^N$ is a convex set, then we can obtain

$$
\lim_{s\nearrow 1}(1-s)\,\mathfrak{h}_{s,p}(\Omega) = K_{p,N}\mathfrak{h}_p(\Omega) = K_{p,N}\left(\frac{p-1}{p}\right)^p. \tag{3.6}
$$

This is possible since, for fixed $p > 1$, we have that $sp \ge 1$ when s is sufficiently close to 1 and, in this range, for any convex set Ω it holds

$$
\mathfrak{h}_{s,p}(\Omega) = \mathfrak{h}_{s,p}(\mathbb{H}^N_+), \qquad \text{where } \mathbb{H}^+_N := \mathbb{R}^{N-1}\times(0,+\infty),
$$

see [4, Theorems 6.3]. Moreover, in the specific case of the half-space $\mathbb{H}^N_+$, for s sufficiently close to 1, by [4, Theorem 5.2], we have that $u(x) = |x|^{(p-1)/p}$ is a positive local weak solution to the equation

$$
(-\Delta_p)^s V = C_{p,s}\frac{V^{p-1}}{d_\Omega^{sp}}, \qquad \text{in } \mathbb{H}^+_N,
$$

with a suitable positive costant $C_{p,s}$. By using again formula (1.7), we obtain that

$$
\mathfrak{h}_{s,p}(\mathbb{H}^+_N) \ge C_{p,s}.
$$

Moreover, by direct verification we see that such a function u is also a (actually classical) positive solution of the equation

$$
-\Delta_p V = \left(\frac{p-1}{p}\right)^p \frac{V^{p-1}}{d_\Omega^p}, \qquad \text{in } \mathbb{H}^+_N.
$$

Hence, by using such a function u and arguing exactly as in (3.5), we can conclude that

$$\liminf_{s\nearrow 1}(1-s)\mathfrak{h}_{s,p}(\Omega) = \liminf_{s\nearrow 1}(1-s)\mathfrak{h}_{s,p}(\mathbb{H}^+_N) \geq K_{p,N}\,\mathfrak{h}_p(\mathbb{H}^+_N) = K_{p,N}\left(\frac{p-1}{p}\right)^p.$$

The $\limsup$ inequality is provided by Lemma 3.2. Hence the limit (3.6) follows.

3.2. **The case $p\nearrow\infty$.** In this subsection we show the following theorem.

Theorem 3.6. *Let $0<s\leq 1$ and let $\Omega\subsetneq\mathbb{R}^N$ be an open set. Then*

$$\lim_{p\to\infty}(\mathfrak{h}_{s,p}(\Omega))^{\frac{1}{p}} = 1. \tag{3.7}$$

Proof. The case $s=1$ is contained in [9, Theorem 4.4]. In order to show the limit in (3.7) for $\mathfrak{h}_{s,p}(\Omega)$ when $0<s<1$, first we show that the $\limsup$ is smaller than or equal to 1. To this aim, it is sufficient to use a suitable test function. For every $x_0\in\Omega$, take $r<d_\Omega(x_0)$ and define

$$\varphi_\varepsilon(x) = \big(\varepsilon + (r-|x-x_0|)_+\big)^s - \varepsilon^s, \qquad \text{for } x\in\mathbb{R}^N. \tag{3.8}$$

Since $\varphi_\varepsilon\in C^{0,1}(\mathbb{R}^N)$ and vanishes on $\mathbb{R}^N\setminus B_r(x_0)$, we have that $\varphi_\varepsilon\in\widetilde{W}^{s,p}_0(B_r(x_0))\subseteq\widetilde{W}^{s,p}_0(\Omega)$. Thus, by recalling (1.4), we have that

$$(\mathfrak{h}_{s,p}(\Omega))^{\frac{1}{p}} \leq \frac{[\varphi_\varepsilon]_{W^{s,p}(\mathbb{R}^N)}}{\left\|\dfrac{\varphi_\varepsilon}{d_\Omega^s}\right\|_{L^p(\Omega)}}.$$

By sending p to ∞ and by using [8, Lemma 2.4], we obtain that

$$\limsup_{p\to\infty}\Big(\mathfrak{h}_{s,p}(\Omega)\Big)^{\frac{1}{p}} \leq \limsup_{p\to\infty}\frac{[\varphi_\varepsilon]_{W^{s,p}(\mathbb{R}^N)}}{\left\|\dfrac{\varphi_\varepsilon}{d_\Omega^s}\right\|_{L^p(\Omega)}} = \frac{[\varphi_\varepsilon]_{C^{0,s}(\mathbb{R}^N)}}{\left\|\dfrac{\varphi_\varepsilon}{d_\Omega^s}\right\|_{L^\infty(B_r(x_0))}}. \tag{3.9}$$

We now observe that for every $x,y\in\mathbb{R}^N$

$$\begin{aligned}|\varphi_\varepsilon(x)-\varphi_\varepsilon(y)| &= \Big|\big(\varepsilon+(r-|x-x_0|)_+\big)^s - \big(\varepsilon+(r-|y-x_0|)_+\big)^s\Big|\\ &\leq |(r-|x-x_0|)_+ - (r-|y-x_0|)_+|^s\\ &\leq \big||x-x_0|-|y-x_0|\big|^s \leq |x-y|^s,\end{aligned}$$

which shows that

$$[\varphi_\varepsilon]_{C^{0,s}(\mathbb{R}^N)} \leq 1. \tag{3.10}$$

Thus, from (3.9) we get

$$\begin{aligned}\limsup_{p\to\infty}\Big(\mathfrak{h}_{s,p}(\Omega)\Big)^{\frac{1}{p}} \leq \frac{1}{\left\|\dfrac{\varphi_\varepsilon}{d_\Omega^s}\right\|_{L^\infty(B_r(x_0))}} &= \inf_{x\in B_r(x_0)}\frac{d_\Omega(x)^s}{\big(\varepsilon+(r-|x-x_0|)_+\big)^s-\varepsilon^s}\\ &\leq \frac{d_\Omega(x_0)^s}{(r+\varepsilon)^s-\varepsilon^s}.\end{aligned}$$

By first taking the limit as ε goes to 0 and then as r goes to $d_\Omega(x_0)$, we finally obtain

$$\limsup_{p\to\infty}\Big(\mathfrak{h}_{s,p}(\Omega)\Big)^{\frac{1}{p}} \leq 1.$$

If we show that

$$\liminf_{p\to\infty} \left(\mathfrak{h}_{s,p}\right)^{\frac{1}{p}} \geq 1, \tag{3.11}$$

in view of Theorem 1.1 and of the previous lim sup, we obtain the desired conclusion (3.7).

We recall that

$$\mathfrak{h}_{s,p} = 2\int_0^1 r^{sp-1} \left|1 - r^{\frac{N-sp}{p}}\right|^p \Phi_{N,s,p}(r)\,dr > 0,$$

where, for every $0 < r < 1$, the quantity $\Phi_{N,s,p}(r)$ is given by

$$\Phi_{N,s,p}(r) = \begin{cases} |\mathbb{S}^{N-2}| \displaystyle\int_{-1}^1 \frac{(1-t^2)^{\frac{N-3}{2}}}{(1-2\,t\,r+r^2)^{\frac{N+sp}{2}}}\,dt, & \text{if } N \geq 2,\\ \dfrac{1}{(1-r)^{1+sp}} + \dfrac{1}{(1+r)^{1+sp}}, & \text{if } N = 1. \end{cases}$$

By a simple computation, one can see that

$$\mathfrak{h}_{s,p} = 2\int_0^1 r^{N-1} \left(1 - r^{\frac{sp-N}{p}}\right)^p \Phi_{N,s,p}(r)\,dr. \tag{3.12}$$

In the case $N \geq 2$, we observe that, for any $r \in (0,1)$ and for any $t \geq 1/2$, one has

$$1 - 2tr + r^2 = 1 + r(r-2t) \leq 1 + r(r-1) \leq 1.$$

Hence,

$$\Phi_{N,s,p}(r) \geq |\mathbb{S}^{N-2}| \int_{1/2}^1 (1-t^2)^{\frac{N-3}{2}}\,dt.$$

In the case $N = 1$ we have that

$$\Phi_{N,s,p}(r) \geq \Phi_{N,s,p}(0) = 2.$$

Thus, (3.12) implies that

$$\left(\mathfrak{h}_{s,p}\right)^{\frac{1}{p}} \geq C_N^{\frac{1}{p}} \left(\int_0^1 r^{N-1} \left(1 - r^{\frac{sp-N}{p}}\right)^p dr\right)^{\frac{1}{p}},$$

where we have set

$$C_N := \begin{cases} 2|\mathbb{S}^{N-2}| \int_{1/2}^1 (1-t^2)^{\frac{N-3}{2}}\,dt, & \text{if } N \geq 2\\ 2 & \text{if } N = 1. \end{cases}$$

Since, $C_N^{\frac{1}{p}} \to 1$ as $p \nearrow \infty$, in order to prove (3.11), it is sufficient to show that, for every $N \geq 1$, it holds

$$\liminf_{p\to\infty} \left(\int_0^1 r^{N-1} \left(1 - r^{\frac{sp-N}{p}}\right)^p dr\right)^{\frac{1}{p}} \geq 1. \tag{3.13}$$

In order to do that, we observe that, for any $r \in (0,1)$ and for any $p \geq p_0$, with p_0 fixed such that $p_0\, s > N$, it holds $r^{s-N/p} \leq r^{s-N/p_0}$, and thus

$$\left(\int_0^1 r^{N-1} \left(1 - r^{\frac{sp-N}{p}}\right)^p dr\right)^{\frac{1}{p}} \geq \left(\int_0^1 r^{N-1} \left(1 - r^{s-\frac{N}{p_0}}\right)^p dr\right)^{\frac{1}{p}} = \left\|1 - r^{s-\frac{N}{p_0}}\right\|_{L^p_{\mu((0,1))}},$$

where we have denoted by $\|\cdot\|_{L^p_{\mu((0,1))}}$ the L^p-norm with respect to the measure $d\mu = r^{N-1}\,dr$. Finally, taking the limit as $p \nearrow \infty$, we deduce that

$$\liminf_{p\to\infty} \left(\int_0^1 r^{N-1} \left(1 - r^{\frac{sp-N}{p}}\right)^p dr\right)^{\frac{1}{p}} \geq \sup_{r\in(0,1)} |1 - r^{s-\frac{N}{p_0}}| = 1.$$

This shows (3.13), thus concluding the proof of (3.11). □

4. A Cheeger type inequality

In the main theorem of this section, we provide a lower bound for $\lambda_{s,p}(\Omega)$ given by (1.11), in terms of the classical and fractional *Cheeger constants* $h_1(\Omega)$ and $h_s(\Omega)$. We recall that

$$h_s(\Omega) = \inf\left\{\frac{P_s(E)}{|E|} \ :\ E \Subset \Omega \text{ smooth, } |E| > 0\right\}$$

where

$$P_s(E) = [1_E]_{W^{s,1}(\mathbb{R}^N)} = \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|1_E(x) - 1_E(y)|}{|x-y|^{N+s}}\,dx\,dy$$

is the nonlocal s-perimeter of E. We explicitly note that our result covers also the case $s = 1$.

Theorem 4.1. *Let $0 < s \leq 1$ and $sp > N$. Let $\Omega \subsetneq \mathbb{R}^N$ be an open set and define*

$$\lambda_{s,p}(\Omega) = \inf_{u\in C_0^\infty(\Omega)} \left\{[u]^p_{W^{s,p}(\mathbb{R}^N)} : \int_\Omega |u|^p\,dx = 1\right\}.$$

If $r_\Omega < +\infty$ then it holds

$$\lambda_{s,p}(\Omega) \geq \mathfrak{h}_{s,p}\left(\frac{h_1(\Omega)}{N}\right)^{sp}, \tag{4.1}$$

where $h_1(\Omega)$ is defined by (1.10). In particular, for $0 < s < 1$ we also have

$$\lambda_{s,p}(\Omega) \geq \frac{\mathfrak{h}_{s,p}}{N^{sp}}\left(\frac{(1-s)\,s}{2N\omega_N} h_s(\Omega)\right)^p. \tag{4.2}$$

Proof. Let us suppose that $r_\Omega < +\infty$. Since $sp > N$, thanks to Theorem 1.1, we have that Ω satisfies the Hardy inequality with $\mathfrak{h}_{s,p}(\Omega) \geq \mathfrak{h}_{s,p} > 0$. Then

$$\int_\Omega |u|^p\,dx \leq r_\Omega^{sp} \int_\Omega \frac{|u|^p}{d_\Omega^{sp}}\,dx \leq \frac{1}{\mathfrak{h}_{s,p}(\Omega)}\, r_\Omega^{sp}\,[u]^p_{W^{s,p}(\mathbb{R}^N)} \qquad \text{for every } u \in C_0^\infty(\Omega).$$

By taking the infimum on $C_0^\infty(\Omega)$, we easily get

$$\lambda_{s,p}(\Omega) \geq \frac{\mathfrak{h}_{s,p}(\Omega)}{r_\Omega^{sp}} \geq \frac{\mathfrak{h}_{s,p}}{r_\Omega^{sp}}. \tag{4.3}$$

The above estimate, combined with the well known inequality

$$h_1(\Omega) \leq \frac{N}{r_\Omega}, \tag{4.4}$$

gives (4.1). In order to show (4.2), it is sufficient to note that, thanks to [6, Corollary 4.4], for every open bounded set $E \Subset \Omega$ with smooth boundary, it holds

$$\left(\frac{P(E)}{|E|}\right)^s \geq \frac{P_s(E)}{|E|}\,\frac{(1-s)s}{2N\omega_N}.$$

Hence, by arbitrariness of E we get

$$(h_1(\Omega))^s \geq \frac{(1-s)\,s}{2N\omega_N} \inf\left\{\frac{P_s(E)}{|E|} \ : \ E \Subset \Omega \text{ smooth}, |E|>0\right\} = \frac{(1-s)\,s}{2N\omega_N} h_s(\Omega).$$

Joining (4.1) with the above inequality, we obtain (4.2). □

Remark 4.2. We note that our estimate (4.1) appears to be new already in the case $s=1$, where it improves (for $p>N$) the celebrated Cheeger inequality

$$\lambda_p(\Omega) \geq \left(\frac{h_1(\Omega)}{p}\right)^p, \tag{4.5}$$

valid for every $1<p<\infty$ and for every open set Ω (for a proof, see [26, 22]). Indeed, when $s=1$, by joining (4.5) and (4.1), we now get

$$\lambda_p(\Omega) \geq \max\left\{\left(\frac{p-N}{N}\right)^p, 1\right\} \left(\frac{h_1(\Omega)}{p}\right)^p,$$

which holds for every open set $\Omega \subseteq \mathbb{R}^N$. The main interest of this results is that this is stable as p goes ∞, i.e. we have

$$\lim_{p\nearrow\infty}\left(\max\left\{\left(\frac{p-N}{N}\right)^p, 1\right\} \left(\frac{h_1(\Omega)}{p}\right)^p\right)^{\frac{1}{p}} = \frac{h_1(\Omega)}{N},$$

while the right-hand side of (4.5) raised to the power $1/p$ converges to 0.

By combining the estimate (4.3) with the asymptotic behaviour of $(\mathfrak{h}_{s,p}(\Omega))^{1/p}$ as $p\to\infty$, we get the next result, which clarifies the interest of Remark 4.2.

Proposition 4.3. *Let $0<s\leq 1$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open set. Then*[2]

$$\lim_{p\to\infty}(\lambda_{s,p}(\Omega))^{\frac{1}{p}} = \frac{1}{r_\Omega^s} = \lambda_{s,\infty}(\Omega), \tag{4.6}$$

where $\lambda_{s,\infty}(\Omega)$ is defined through the following minimization problem

$$\lambda_{s,\infty}(\Omega) = \inf_{u\in C^{0,s}(\overline{\Omega})}\left\{[u]_{C^{0,s}(\overline{\Omega})} : \|u\|_{L^\infty(\Omega)} = 1, \ u=0 \text{ on } \partial\Omega\right\}.$$

Moreover, when $r_\Omega < +\infty$, a minimizer of the last problem is given by

$$U = \left(\frac{d_\Omega}{r_\Omega}\right)^s.$$

Proof. First of all, we prove that

$$\limsup_{p\to\infty}\Big(\lambda_{s,p}(\Omega)\Big)^{\frac{1}{p}} \leq \frac{1}{r_\Omega^s}. \tag{4.7}$$

We take $r<r_\Omega$, thus there exists $x_0\in\Omega$ such that $B_r(x_0)\subseteq\Omega$. For every $\varepsilon>0$, we take the same function φ_ε defined in (3.8). This implies that

$$\Big(\lambda_{s,p}(\Omega)\Big)^{\frac{1}{p}} \leq \frac{[\varphi_\varepsilon]_{W^{s,p}(\mathbb{R}^N)}}{\|\varphi_\varepsilon\|_{L^p(\Omega)}}.$$

[2] It is intended that $1/r_\Omega^s = 0$, in the case $r_\Omega = +\infty$.

By sending p to ∞ (and by using [8, Lemma 2.4] when $0 < s < 1$), we obtain that

$$\limsup_{p\to\infty} \Big(\lambda_{s,p}(\Omega)\Big)^{\frac{1}{p}} \le \limsup_{p\to\infty} \frac{[\varphi_\varepsilon]_{W^{s,p}(\mathbb{R}^N)}}{\|\varphi_\varepsilon\|_{L^p(\Omega)}} = \begin{cases} \dfrac{[\varphi_\varepsilon]_{C^{0,s}(\mathbb{R}^N)}}{\|\varphi_\varepsilon\|_{L^\infty(\Omega)}}, & \text{if } 0 < s < 1, \\[2ex] \dfrac{\|\nabla\varphi_\varepsilon\|_{L^\infty(\mathbb{R}^N)}}{\|\varphi_\varepsilon\|_{L^\infty(\Omega)}}, & \text{if } s = 1. \end{cases}$$

By observing that

$$\|\nabla\varphi_\varepsilon\|_{L^\infty(\mathbb{R}^N)} = [\varphi_\varepsilon]_{C^{0,1}(\mathbb{R}^N)},$$

we thus obtain

$$\limsup_{p\to\infty} \Big(\lambda_{s,p}(\Omega)\Big)^{\frac{1}{p}} \le \frac{[\varphi_\varepsilon]_{C^{0,s}(\mathbb{R}^N)}}{\|\varphi_\varepsilon\|_{L^\infty(\Omega)}}, \qquad \text{for } 0 < s \le 1. \tag{4.8}$$

We now observe that

$$\|\varphi_\varepsilon\|_{L^\infty(\Omega)} = (r+\varepsilon)^s - \varepsilon^s.$$

By recalling also (3.10), from (4.8) we thus obtain for every $\varepsilon > 0$

$$\limsup_{p\to\infty} \Big(\lambda_{s,p}(\Omega)\Big)^{\frac{1}{p}} \le \frac{1}{(r+\varepsilon)^s - \varepsilon^s}, \qquad \text{for } 0 < s \le 1.$$

By taking the limit as ε goes to 0 and using the arbitrariness of $r < r_\Omega$, we get (4.7). In particular, if $r_\Omega = +\infty$, (4.7) implies that

$$\lim_{p\to\infty} (\lambda_{s,p}(\Omega))^{\frac{1}{p}} = 0 = \frac{1}{r_\Omega^s}.$$

In the case when $r_\Omega < \infty$, for every $0 < s \le 1$, we can use (4.3) and (3.7), to obtain that

$$\liminf_{p\to\infty} (\lambda_{s,p}(\Omega))^{\frac{1}{p}} \ge \liminf_{p\to\infty} \frac{(\mathfrak{h}_{s,p}(\Omega))^{\frac{1}{p}}}{r_\Omega^s} = \frac{1}{r_\Omega^s}.$$

We now prove that

$$\frac{1}{r_\Omega^s} = \lambda_{s,\infty}(\Omega). \tag{4.9}$$

Let $\varphi \in C^{0,s}(\overline{\Omega})$ be admissible for the problem defining $\lambda_{s,\infty}(\Omega)$. For every $x \in \Omega$, we take $y_x \in \partial\Omega$ such that $d_\Omega(x) = |x - y_x|$. Thus, we have

$$|\varphi(x)| = |\varphi(x) - \varphi(y_x)| \le |x - y_x|^s\, [\varphi]_{C^{0,s}(\overline{\Omega})} = d_\Omega(x)^s\, [\varphi]_{C^{0,s}(\overline{\Omega})} \le r_\Omega^s\, [\varphi]_{C^{0,s}(\overline{\Omega})}.$$

This shows that

$$\frac{1}{r_\Omega^s} \le [\varphi]_{C^{0,s}(\overline{\Omega})},$$

thanks to the normalization on φ. It is intended that the left-hand side is zero, in the case $r_\Omega = +\infty$. The previous inequality in turn implies that

$$\lambda_{s,\infty}(\Omega) \ge \frac{1}{r_\Omega^s}.$$

Finally, if $r_\Omega < +\infty$ we take the function

$$\varphi = \frac{d_\Omega^s}{r_\Omega^s},$$

which is admissible for $\lambda_{s,\infty}(\Omega)$. This gives

$$\lambda_{s,\infty}(\Omega) = \frac{1}{r_\Omega^s}\,[d_\Omega^s]_{C^{0,s}(\overline{\Omega})} \leq \frac{1}{r_\Omega^s},$$

thanks to the fact that d_Ω^s is $s-$Hölder continuous, with Hölder constant less than or equal to 1. This shows (4.9) and that $r_\Omega^{-s} d_\Omega^s$ is a minimizer for the problem defining $\lambda_{s,\infty}(\Omega)$.

In the case $r_\Omega = +\infty$, it is sufficient to take $M > 0$ and use the test function

$$\varphi_M = \frac{\min\{d_\Omega^s, M^s\}}{M^s}.$$

This would give

$$\lambda_{s,\infty}(\Omega) \leq \lim_{M\to\infty} \frac{1}{M^s}\left[\min\{d_\Omega^s, M^s\}\right]_{C^{0,s}(\overline{\Omega})} = \lim_{M\to\infty} \frac{1}{M^s}\,[d_\Omega^s]_{C^{0,s}(\overline{\Omega})} \leq \lim_{M\to\infty} \frac{1}{M^s} = 0,$$

thus proving (4.9) in the case $r_\Omega = +\infty$, as well. □

Remark 4.4. We recall that, when $s = 1$, the limit

$$\lim_{p\to\infty} (\lambda_p(\Omega))^{1/p} = \frac{1}{r_\Omega}$$

has been shown in [17, Theorem 3.1] and [21, Lemma 1.2], when $\Omega \subseteq \mathbb{R}^N$ is a bounded open set. Later this result has been extended to every open set in [9, Corollary 6.1].

Remark 4.5. Thanks to the previous result, we can observe that the lower bound (4.1) becomes sharp in the limit, as p goes to ∞. Indeed, for every $0 < s \leq 1$ and for every open set $\Omega \subseteq \mathbb{R}^N$, by combining (4.6) and (4.4), we get the following inequality

$$\lambda_{s,\infty}(\Omega) = \lim_{p\to\infty} (\lambda_{s,p}(\Omega))^{\frac{1}{p}} \geq \left(\frac{h_1(\Omega)}{N}\right)^s. \tag{4.10}$$

Such an estimate *is sharp*, since it becomes an identity when $\Omega = B_R(x_0)$, thanks to the fact that

$$\lambda_{s,\infty}(B_R(x_0)) = \frac{1}{R^s} = \left(\frac{h_1(B_R(x_0))}{N}\right)^s.$$

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