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From elastic shallow shells to beams with elastic hinges by Γ-convergence

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Abstract. In this paper, we study the Γ-limit of a properly rescaled family of energies, defined on a narrow strip, as the width of the strip tends to zero. The limit energy is one-dimensional and is able to capture (and penalize) concentrations of the midline curvature. At the best of our knowledge, it is the first paper in the Γ-convergence field for dimension reduction that predicts elastic hinges. In particular, starting from a purely elastic shell model with "smooth" solutions, we obtain a beam model where the derivatives of the displacement and/or of the rotation fields may have jump discontinuities. Mechanically speaking, elastic hinges can occur in the beam.

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Contents

1. Introduction

It is a common experience to bend by hands a sufficiently long portion of a carpenter tape measure. If the imposed rotation at the two ends is sufficiently large in magnitude, the deformation suddenly localizes approximately at the middle of the tape. Moreover, one can clearly feel the reduction of effort to maintain the equilibrium in the deformed state after the localization. If the constraints are released, the tape measure comes back to the initial, unbended configuration, suggesting the deformation is purely elastic. In Mechanics' jargon, we are actually bending a transversely curved shallow shell (or transversely curved ribbon), and the localized deformation is called *fold*.

Studies on the structure of elastic folds go back to the 1990s. The prototype is the sharp, straight fold of a piece of paper. In [21], Lobkovsky derived an asymptotic scaling for the boundary layer occurring in a thin elastic sheet around the sharp crease that would appear in the limit of vanishing thickness. He obtained, by a formal scaling argument, that the elastic energy (von Kármán type) per unit thickness

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h of the sheet scales as $h^{5/3}$. Note that a pure membrane (stretching) energy per unit thickness scales as h^0, while a pure bending energy per unit thickness scales as h^2. This intermediate regime indicates the activation of both membrane and bending energy in a non-trivial interplay. Later on, Venkataramani [31] proved via more rigorous variational techniques that minimizers of the von Kármán energy (per unit thickness) satisfying appropriate boundary conditions so to make the crease appear, scales as $h^{5/3}$. In the full nonlinear elasticity setting, Conti and Maggi [5] gave an optimal construction, inspired by paper origamis, and proved that the 3D, nonlinear elastic energy (per unit thickness) of a plate of thickness h scales as $h^{5/3}$, again under well-defined boundary conditions. We highlight *en passant* that the Γ-limit (with respect to some topology) for plates/shells is still not known if the nonlinear elastic energy (per unit thickness) scales as h^β with $\frac{5}{3} < \beta < 2$.

It must be pointed out that the occurrence of creases in a flat sheet relies on heavy (and *ad hoc*) boundary conditions. No singularities in fact appear when one tries to bend a rectangular sheet of paper by imposing rotations on two opposite sides. Moreover, it is questionable if in the real life such ridge singularities are fully reversible, i.e., if the deformation is elastic.

The experiment with the carpenter tape measure suggests that a "cleaner" way to obtain elastic folds is to introduce an initial transverse curvature in a rectangular plate (hence, a shell) of sufficiently large aspect ratio. In fact, just trivial Dirichlet boundary conditions at the short sides are needed to trigger the fold.

Despite the bending of cylindrical shells is actually a classical subject [4,22,33], not much is actually known if compared to the plate counterparts. The study of tape spring devices has been quite prolific in the last years in connection to deployable structures and mechanism-free actuation, especially for the aerospace field [32]. Numerical and experimental simulations can be found in [28,29], where the authors provided the rotation-reaction moment curve, showing a complicated and high-nonlinear behavior, characterized by a snap-back instability at a critical valued of the applied rotation. The behavior also depends on the sign of the mean curvature.

More recently, many rod models with thin-walled flexible cross section have been proposed in the literature: for instance in [18,19,26], the authors derived different enhanced rod models starting from the geometrically nonlinear Koiter shell energy. An interesting interpretation of the bending of tape springs as a multi-phase transition problem has also been proposed in [23]. The comparison is with the well-known regularized Ericksen bar. The two phases are represented by portions of cylindrical shells having misaligned zero-curvature axes: they are skew lines forming an angle of 90°. Dirichlet boundary conditions ("hard device" in Ericksen bar jargon) for bending imposes that both phases must be present. Hence, the whole problem is a matter of optimal transition between phases.

One may also wonder whether the scaling $h^{5/3}$ is still optimal for the bending of tape springs. In [6], a numerical study suggests that the energy (per unit thickness) is much lower and that the optimal scaling is close to h^2.

It is worth mentioning that a correlated problem to the bending of tape springs has been studied in [3]. The authors study the flattening and clamping, along one short side, of a cylindrical shallow shell with rectangular planform. Moreover, they propose mechanical non-linear rod models based on inextensible (on average) Koiter energy and on extensible von Kármán energy. We mention also the work of Percivale and Tomarelli [25] on variational principles for plastic hinges in beams.

In this paper, we study the Γ-limit of a sequence of variational problems for rectangular, transversely curved shallow shells, as the width of the planform goes to zero. Despite the energies are defined on sufficiently smooth functions, the limit energy penalizes concentrations of the longitudinal curvature. From a mathematical viewpoint, this behavior is entirely triggered by the presence of the transversal curvature in the natural state of the shell. The mechanical interpretation is the occurrence of localized elastic hinges, as actually happens, for instance, in the planar bending of tape spring devices.

The paper is structured as follows. In Sect. 2, we recall the main technical tools we will use later on. In Sect. 3 we introduce the sequence of variational problems we are interested in, together with some properties of the minimizers. In the same section, we also prove that the bending of a tape measure is not an isometry. In particular, Theorem 3 suggests that care must be paid in coupling the isometry requirement and Dirichlet boundary conditions (see also [15]). In Sect. 4 we study the compactness of rescaled sequences of displacements and identify the Γ-limit, the main result of the paper.

2. Preliminaries

The Euclidean (Frobenius) product in $\mathbb{R}^N$ is indicated with $\cdot$ and the corresponding induced norm by $|\cdot|$.

Let $\Omega \subset \mathbb{R}^N$ be open. If $r \in \mathbb{N}\cup\{\infty\}$, then $C^r(\Omega)$ denotes the space of real-valued, r-times continuously differentiable functions on Ω. $C^r_0(\Omega)$ denotes the completion with respect to the sup-norm of $C^r_c(\Omega)$, the space of functions belonging to $C^r(\Omega)$ that have compact support in Ω.

If not specified, we adopt Einstein' summation convention for indices, and C denotes a positive constant that may vary from line to line. We denote the integral average by $\fint_\Omega f\,\mathrm{d}x := \frac{1}{|\Omega|}\int\limits_\Omega f\,\mathrm{d}x$.

For real $p \geq 1$ and integer $M \geq 1$, we denote by

$$L^p(\Omega, \mathbb{R}^M) := \left\{ u : \Omega \to \mathbb{R}^M \, : \, \|u\|_{L^p(\Omega,\mathbb{R}^M)} < \infty \right\}$$

the Banach space of (equivalence classes of) Lebesgue-integrable functions on Ω with values in $\mathbb{R}^M$, where

$$\|u\|_{L^p(\Omega,\mathbb{R}^M)} := \left(\int\limits_\Omega |u|^p \,\mathrm{d}x \right)^{1/p} .$$

If $M = 1$, we will refrain to specify the codomain $\mathbb{R}$ in the notation of the functional spaces: for instance, we will simply write $L^p(\Omega)$ instead of $L^p(\Omega, \mathbb{R})$, and so forth.

The corresponding Sobolev' spaces of functions on Ω are the Banach spaces defined as follows:

$$W^{m,p}(\Omega) := \left\{ u : \Omega \to \mathbb{R} : u \in L^p(\Omega), \nabla^i u \in L^p(\Omega, \mathbb{R}^{N^i}) \; \forall i \leq M \right\}.$$

They are endowed with the norm

$$\|u\|^p_{W^{m,p}(\Omega)} := \|u\|^p_{L^p(\Omega)} + \sum_{i=1}^m \left\|\nabla^i u\right\|^p_{L^p(\Omega,\mathbb{R}^{N^i})} .$$

Let $1 \leq p_i \leq p_0 < \infty$, $i = 1, \ldots, N$. We introduce the following anisotropic Sobolev space:

$$W^{1,(p_0;p_1,\ldots,p_N)}(\Omega) := \{u \in L^{p_0}(\Omega) : \partial_i u \in L^{p_i}(\Omega), i = 1, \ldots, N\},$$

which is a Banach space with the norm

$$\|u\|_{W^{1,(p_0;p_1,\ldots,p_N)}(\Omega)} := \|u\|_{L^{p_0}(\Omega)} + \sum_{i=1}^N \|\partial_i u\|_{L^{p_i}(\Omega)} .$$

Given two functions $u, v : \mathbb{R}^n \to \mathbb{R}$ we define the convolution

$$u * v(x) := \int\limits_{\mathbb{R}^n} u(x-y)v(y)\,dy$$

whenever the integral exists.

We denote strong convergence (convergence in norm) with the symbol $\to$, while weak convergence will be denoted by $\rightharpoonup$.

We denote by $\mathcal{M}(\Omega)$ the space of finite Radon measures on Ω, by $\mathcal{L}^N$ the N-dimensional Lebesgue measure, and by $\mathcal{H}^1$ the one-dimensional Hausdorff measure in $\mathbb{R}^2$. Indicating by Du the distributional derivative of u, the spaces of functions of bounded variations and of bounded Hessian are defined as

$$BV(\Omega):=\{u \in L^1(\Omega) : Du \in \mathcal{M}(\Omega,\mathbb{R}^N)\} \tag{1}$$

and

$$\begin{aligned} BH(\Omega):=&\{u \in W^{1,1}(\Omega) : D^2u \in \mathcal{M}(\Omega,\mathbb{R}^{N\times N}_{\mathrm{sym}})\} \\ =&\{u \in W^{1,1}(\Omega) : \nabla u \in BV(\Omega,\mathbb{R}^N)\}. \end{aligned} \tag{2}$$

For $u \in BV(\Omega), BH(\Omega)$, we set

$$\begin{aligned} \|u\|_{BV(\Omega)}&:=\|u\|_{L^1(\Omega)}+|Du|(\Omega), \\ \|u\|_{BH(\Omega)}&:=\|u\|_{W^{1,1}(\Omega)}+|D^2u|(\Omega), \end{aligned}$$

where $|Du|(\Omega)$ is the (total) variation measure of u, defined as

$$|Du|(\Omega):=\sup\left\{\int_\Omega u \operatorname{div}\phi \,\mathrm{d}x : \phi \in C^1_c(\Omega,\mathbb{R}^N),\ |\phi|\le 1\right\},$$

and similarly

$$|D^2u|(\Omega):=\sup\left\{\int_\Omega \nabla u\cdot \operatorname{div}\phi \,\mathrm{d}x : \phi \in C^1_c(\Omega,\mathbb{R}^{N\times N}_{\mathrm{sym}}),\ |\phi|\le 1\right\}$$

is the (total) variation measure of ∇u. Note that if $u \in BV(\Omega)$, ∇u is the (approximate) pointwise differential of u and represents the density of Du with respect to the Lebesgue measure $\mathcal{L}^N$. It is a Sobolev function. In dimension one, following [25], for any $u \in BH(\mathbb{R})$ we use u'' for the second derivative in the sense of distributions, $\ddot{u}$ for its absolutely continuous part, $J_{\dot{u}}$ for the jump set of $\dot{u} = u'$, $[\dot{u}]:=\dot{u}^+ - \dot{u}^-$, $(u'')_s$, and $(u'')_c$ for the singular and Cantor part of u'', respectively.

We recall that $(\mu_n) \subset \mathcal{M}(\Omega)$ converges weakly* to $\mu \in \mathcal{M}(\Omega)$ (and we write $\mu_n \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega)$) if for every $\phi \in C_0(\Omega)$ $\lim_{n\uparrow\infty}\int_\Omega \phi\,\mathrm{d}\mu_n = \int_\Omega \phi\,\mathrm{d}\mu$.

We say that $(u_n) \subset BV(\Omega)$ converges weakly* in $BV(\Omega)$ to $u \in BV(\Omega)$ (and we write $u_n \overset{*}{\rightharpoonup} u$ in $BV(\Omega)$) if $u_n \to u$ in $L^1(\Omega)$ and $Du_n \overset{*}{\rightharpoonup} Du$ in $\mathcal{M}(\Omega,\mathbb{R}^N)$. Similarly, $(u_n) \subset BH(\Omega)$ converges weakly* in $BH(\Omega)$ to $u \in BH(\Omega)$ (and we write $u_n \overset{*}{\rightharpoonup} u$ in $BH(\Omega)$) if $u_n \to u$ in $W^{1,1}(\Omega)$ and $D^2u_n \overset{*}{\rightharpoonup} D^2u$ in $\mathcal{M}(\Omega,\mathbb{R}^{N\times N}_{\mathrm{sym}})$.

We say that $(u_n) \subset BV(\Omega)$ converges strictly in $BV(\Omega)$ to $u \in BV(\Omega)$ (and we write $u_n \overset{s}{\rightharpoonup} u$ in $BV(\Omega)$) if $u_n \to u$ in $L^1(\Omega)$ and $|Du_n|(\Omega) \to |Du|(\Omega)$ as $n\uparrow\infty$. Similarly, $(u_n) \subset BH(\Omega)$ converges strictly in $BH(\Omega)$ to $u \in BH(\Omega)$ (and we write $u_n \overset{s}{\rightharpoonup} u$ in $BH(\Omega)$) if $u_n \to u$ in $W^{1,1}(\Omega)$ and $|D^2u_n|(\Omega) \to |D^2u|(\Omega)$ as $n\uparrow\infty$.

The restriction of a measure μ on Ω to a measurable set $E \subset \Omega$ is the measure $\mu \llcorner E$ defined as $\mu \llcorner E(F):=\mu(F\cap E)$ for all measurable sets $F \subset \Omega$.

Let A, B be two sets and let $\mu \in \mathcal{M}(A)$, $\nu \in \mathcal{M}(B)$. The product measure $\mu\times\nu \in \mathcal{M}(A\times B)$ is the measure satisfying $(\mu\times\nu)(E\times F) = \mu(E)\nu(F)$ for every Borel sets $E \in A$ and $F \in B$.

Let $\phi \in C^\infty_c(\Omega)$. For any sufficiently smooth function $u : \Omega \to \mathbb{R}$, we define the weak Hessian determinant as the following distribution of order one [7]:

$$Hu(\phi):=\int_\Omega \partial_1 u\,\partial_{12}u\,\partial_2\phi - \partial_1 u\,\partial_{22}u\,\partial_1\phi\,\mathrm{d}x. \tag{3}$$

It is well defined, for instance, for $u \in W^{2,\frac{4}{3}}(\Omega)$, and if $u \in W^{2,2}(\Omega)$ it is equivalent to the usual distributional Hessian determinant.

We will use also the following anisotropic Sobolev inequality and correlated embedding. To proceed, we say that a bounded domain $\Omega \subset \mathbb{R}^N$ satisfies the cube condition if there exist a finite family of subdomains $\Omega_k \subset \Omega$, $\cup_k \Omega_k = \Omega$ and a family of closed cubes Q_k having same edge length, one vertex at the origin, edges parallel to the coordinate axes, and such that $\Omega_k + Q_k \subset \Omega$ (Minkowski sum) for all k (see also [2, §8]).

Theorem 1. ([27, Theorems 1, 2]) *Let $\Omega \subset \mathbb{R}^N$ satisfy the cube condition. Then there exists a constant C such that*

$$\|u\|_{L^q(\Omega)} \leq C \sum_{i=1}^{N} \|\partial_i u\|_{L^{p_i}(\Omega)}$$

for every $u \in W^{1,(p_0;p_1,\ldots,p_N)}(\Omega)$, where

$$q := \begin{cases} N\left(\sum_{i=1}^N \frac{1}{p_i} - 1\right)^{-1} & \text{if } \sum_{i=1}^N \frac{1}{p_i} > 1, \\ [1,\infty) & \text{otherwise.} \end{cases}$$

Moreover, for every $1 \leq r < q$ the embedding

$$W^{1,(p_0;p_1,\ldots,p_N)}(\Omega) \hookrightarrow L^r(\Omega)$$

is compact.

It is easy to see that a rectangular domain satisfies the cube condition.

We conclude this section by recalling some useful properties of matrices. For every $A, B \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}$:

$$\begin{aligned} \det(A \pm B) &= \det A + \det B \pm A \cdot \operatorname{cof} B, \\ |A|^2 &\geq 2|\det A|, \\ A \cdot \operatorname{cof} A &= 2 \det A. \end{aligned} \tag{4}$$

where $(\operatorname{cof} A)_{\alpha\beta} := \mathcal{E}_{i\alpha}\mathcal{E}_{j\beta}A_{ij}$ and $\mathcal{E}$ is the Levi-Civita symbol.

3. The problem

Let ε be a sequence of positive numbers converging to zero. Let $\ell > 0$, $I := (-\ell/2, \ell/2)$, $W_\varepsilon := (-\varepsilon/2, \varepsilon/2)$ and let $\Omega_\varepsilon = I \times W_\varepsilon$. For the out-of-plane displacement $v \in W^{2,2}(\Omega_\varepsilon)$ we consider the following energy

$$I_\varepsilon(v) := \int_{\Omega_\varepsilon} \frac{1}{2}|\nabla^2 v - K_\varepsilon|^2 + c|\det \nabla^2 v| \, \mathrm{d}x, \tag{5}$$

with

$$K_\varepsilon := \frac{k}{\varepsilon} e_2 \otimes e_2, \tag{6}$$

k a nonzero real constant, and $c > 0$.

Remark. The function $v \mapsto I_\varepsilon(v)$ is strictly convex if $c < 1$.

The natural (stress-free) configuration of the shell is then described by a transversely curved, straight cylinder. With K_ε as in (6), the angle subtended to the curved cross section of the shell is the same for all terms in the sequence. Moreover, the cross-sections scale homothetically along the sequence. For a generic $\varepsilon > 0$, Fig. 1 shows the domain and the reference configuration of the shell.

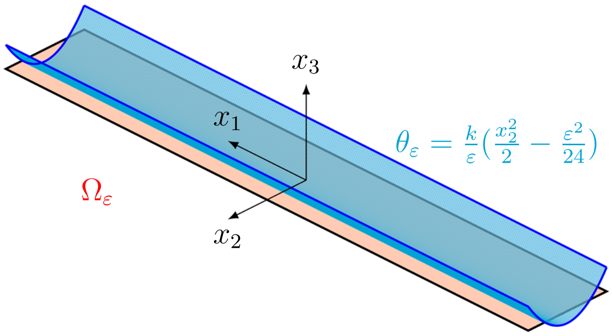

Fig. 1. Reference configuration and shell planform

3.1. Remarks on the choice of the energy

One could consider energies $\int_{\Omega_\varepsilon} \frac{1}{2}|\nabla^2 w - K_\varepsilon|^2 \mathrm{d}x$ to be minimized (in some function class) under the Monge-Ampére constraint $\det \nabla^2 w = 0$ a.e. in Ω_ε. This constrained, von Kármán-type energy has been rigorously derived via Γ-convergence for plates [16] and shells [20,30] and further studied in the case of planar ribbons [14]. We want to show that this energy is not suitable if one has in mind to study particular bending problems for shells.

Neglecting to specify the subscript $_\varepsilon$, let $\Omega = I \times W$. Let $\theta \in C^2(\bar{\Omega})$ describe the natural configuration of the shell. We look at problems where $w \in W^{2,2}(\Omega)$ must satisfy the following Dirichlet boundary conditions (in the sense of traces):

$$w\left(\mp\frac{\ell}{2}, \cdot\right) = \theta\left(\mp\frac{\ell}{2}, \cdot\right), \qquad \partial_1 w\left(\mp\frac{\ell}{2}, \cdot\right) = \pm\Phi \tag{7}$$

where $\Phi \neq 0$ is a given constant.

Given θ that further satisfies $\det \nabla^2\theta = 0$, we say $w \in W^{2,2}(\Omega)$ is a *linearized isometry* if and only if there exists a function $u \in W^{1,2}(\Omega, \mathbb{R}^2)$, unique up to an affine map with skew-symmetric gradient, such that

$$2Eu + \nabla w \otimes \nabla w - \nabla\theta \otimes \nabla\theta = 0 \qquad a.e.\ in\ \Omega,$$

where Eu denotes the symmetric part of ∇u.

Proposition 2. *Suppose Ω is a simply connected, bounded Lipschitz domain. Let $w \in W^{2,2}(\Omega)$, $\theta \in C^2(\bar{\Omega})$ with $\det \nabla^2\theta = 0$. The equation*

$$2Eu + \nabla w \otimes \nabla w - \nabla\theta \otimes \nabla\theta = 0$$

admits a solution $u \in W^{1,2}(\Omega, \mathbb{R}^2)$ if and only if $\det \nabla^2 w = 0$.

Proof. The proof is essentially the same of [16, Proposition 9]. We provide here the main ideas. Suppose firstly w is smooth, and let $E := -\frac{1}{2}\nabla w \otimes \nabla w + \frac{1}{2}\nabla\theta \otimes \nabla\theta$. We have

$$\operatorname{curl}\operatorname{curl} E = \det \nabla^2 w - \det \nabla^2\theta = \det \nabla^2 w,$$

which holds also in the sense of distributions if $w \in W^{2,2}(\Omega)$. The existence of $u \in W^{1,2}(\Omega, \mathbb{R}^2)$ such that $E = Eu \in L^2(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ is equivalent to the vanishing of the right-hand-side of the previous equation. $\square$

Theorem 3. *Let $\Omega = I \times W$. Let $\theta = \theta(x_2) \in C^2(\bar{\Omega})$ be such that $\nabla^2\theta$ is not identically zero. If $w \in W^{2,2}(\Omega)$ satisfies (7), then it is not a linearized isometry.*

Proof. Let us pose $\Gamma^{\pm}:=\{\pm\frac{\ell}{2}\}\times W$. Assume $w\in W^{2,2}(\Omega)$ satisfies (7). By Rellich theorem $w\in C^0(\bar\Omega)$.

If w is a linearized isometry, then $\det\nabla^2 w=0$. By [16, Proposition 10] $w\in C^1(\Omega)$. Moreover, at every point $x\in\Omega$ either (i) ∇w is constant (and so in a neighborhood of x), or (ii) there exists a line segment which intersects $\partial\Omega$ at both ends and on which ∇w is constant [24], [16, Theorem 9].

We will prove that if w satisfies the boundary conditions (7), then both (i) and (ii) are false, so that $\det\nabla^2 w$ cannot vanish identically.

Step 1: Assume $w\in C^2(\bar\Omega)$. Near Γ^- ∇w is not constant because $\partial_2\theta$ is not constant. Furthermore, (ii) does not hold. In fact the fulfilling of the boundary conditions imposes that $\nabla w|_{\Gamma^-}=(\Phi,\partial_2\theta(x_2))$, from which ∇w would be constant on segments parallel to x_1 by continuity. Such segments must, however, intersect Γ^+, on which the boundary condition reads $\nabla w|_{\Gamma^+}=(-\Phi,\partial_2\theta)$. Hence the absurd.

Step 2: The conclusions of Step 1 remain valid even if $w\in W^{2,2}(\Omega)$. By approximation, there exists a sequence $(w_k)\subset W^{2,2}(\Omega)\cap C^\infty(\bar\Omega)$ such that $w_k\to w$ in $W^{2,2}(\Omega)$ as $k\uparrow\infty$ [10, Theorem 3, Section 4.2]. The trace operator on $\Gamma^\pm$, as defined for smooth functions up to the boundary, has then a unique continuous extension from $W^{1,2}(\Omega,\mathbb{R}^2)$ to $L^2(\Gamma^\pm,\mathbb{R}^2)$. □

To study the particular problems, we have in mind the constraint $\det\nabla^2 w=0$ needs to be relaxed. The term multiplied by c in the energy (5) can be thought as a relaxation of the constraint. Another way is to take into account the membrane energy. For the "standard" von Kármán energy, it is given by the squared $L^2(\Omega)$ norm of $Eu+\frac{1}{2}\nabla w\otimes\nabla w-\frac{1}{2}\nabla\theta\otimes\nabla\theta$. From what we have said, the penalization of the Hessian determinant of w in (5) can be interpreted as the deviation of w from a linearized isometry in the $L^1(\Omega)$ norm. Accordingly, c can be interpreted as the ratio between membrane and bending stiffness. We pursued this choice so to have the simplest model as possible (depending just on a scalar field) and to highlight/isolate the effect of the presence of initial curvature. The study of a "standard" Von Kármán energy will appear somewhere else.

3.2. Properties of the energy functional

We recall the following Theorem, specified for the bidimensional case.

Theorem 4. ([8]) *Let Ω be a bounded, open set in $\mathbb{R}^2$. For $v:\Omega\to\mathbb{R}^2$ let $M(\nabla v):=(\nabla v,\det\nabla v)$, and let $I(v):=\int_\Omega g(M(\nabla v))\,\mathrm{d}x$, where $g:\mathbb{R}^5\to\mathbb{R}$ is a convex, non-negative function. Then,*

$$I(v)\le\liminf_{n\uparrow\infty} I(v_n)$$

if $(v_n)\subset W^{1,2}(\Omega,\mathbb{R}^2)$ and $v_n\rightharpoonup v$ in $W^{1,1}(\Omega,\mathbb{R}^2)$ for $v\in W^{1,2}(\Omega,\mathbb{R}^2)$.

Proposition 5. *Let $\varepsilon>0$ be fixed. The functional I_ε has minimizer(s) on bounded, weakly closed subsets $\mathcal{A}_\varepsilon\subset W^{2,2}(\Omega_\varepsilon)$.*

Proof. We omit to specify the subscript $_\varepsilon$.

The energy is clearly well defined for functions in $W^{2,2}(\Omega)$. Let $(v_n)\subset\mathcal{A}$, $v\in\mathcal{A}$ such that $v_n\rightharpoonup v$ in $W^{2,2}(\Omega)$. Note that $I(v)$ is polyconvex. In fact $I(v)=\int_\Omega g(\nabla^2 v,\det\nabla^2 v)\,\mathrm{d}x$ where $(A,\mu)\mapsto g(A,\mu):=\frac{1}{2}|A-K|^2+c|\mu|$ is convex. To prove that $\liminf_{n\uparrow\infty}I(v_n)\ge I(v)$, we apply Theorem 4. □

Proposition 6. *Let $\varepsilon>0$ be fixed. Suppose there exists a global minimizer $v_\circ$ of I_ε (in some admissible class $\mathcal{A}$) such that: (i) the smallest singular value of $\nabla^2 v_\circ$ is uniformly bounded from below by some constant σ_0, (ii) there exists a positive constant $\lambda_\circ$ such that $\sup_x|\det\nabla^2 v_\circ(x)|\le\lambda_0$. If $c<\frac{\sigma_\circ^2}{\lambda_\circ}$, then $v_\circ$ is the unique global minimizer.*

Proof. We omit to specify the subscript $_\varepsilon$. Suppose $v_\circ$ is a minimizer for I. Let

$$dI(v)(\phi) := \int_\Omega ((\nabla^2 v - K) + c\,\mathrm{sgn}(\det \nabla^2 v)\,\mathrm{cof}\,\nabla^2 v) \cdot \nabla^2 \phi \, dx,$$

where sgn denotes the sign function, and the Bregman divergence

$$B(v, v_\circ) := \int_\Omega |\det \nabla^2 v| - |\det \nabla^2 v_\circ| - \mathrm{sgn}(\det \nabla^2 v_\circ)\,\mathrm{cof}\,\nabla^2 v_\circ \cdot (\nabla^2 v - \nabla^2 v_\circ)\,\mathrm{d}x.$$

Let v be in $\mathcal{A}$. We have

$$\begin{aligned} I(v) - I(v_\circ) &= I(v) - I(v_\circ) - dI(v_\circ)(v - v_\circ) \\ &= \frac{1}{2}\left\|\nabla^2 v - \nabla^2 v_\circ\right\|^2_{L^2(\Omega,\mathbb{R}^{2\times 2}_{\mathrm{sym}})} + cB(v, v_\circ). \end{aligned}$$

By applying the fundamental theorem of calculus we obtain

$$B(v, v_\circ) = \int_\Omega \int_0^1 \left(\mathrm{sgn}(\det \nabla^2 v_t)\,\mathrm{cof}\,\nabla^2 v_t - \mathrm{sgn}(\det \nabla^2 v_\circ)\,\mathrm{cof}\,\nabla^2 v_\circ\right) \cdot \frac{(\nabla^2 v_t - \nabla^2 v_\circ)}{t}\, dt\,\mathrm{d}x$$

where $v_t := tv - (1-t)v_\circ$ and where we have used the identity $\frac{(\nabla^2 v_t - \nabla^2 v_\circ)}{t} = \nabla^2 v - \nabla^2 v_\circ$.

In [17, Lemma 3.1], it has been proved that for any $A, B \in \mathbb{R}^{2\times 2}$

$$(\mathrm{sgn}(\det A)\,\mathrm{cof}\,A - \mathrm{sgn}(\det B)\,\mathrm{cof}\,B) \cdot (A - B) \geq -\frac{|\det A|}{\sigma^2}|A - B|^2$$

where σ is the smallest singular value of A. Hence,

$$\begin{aligned} B(v, v_\circ) &\geq -\int_\Omega \frac{|\det \nabla^2 v_\circ|}{\sigma_0^2} \int_0^1 t \left|\frac{\nabla^2 v_t - \nabla^2 v_\circ}{t}\right|^2 \mathrm{d}t\,\mathrm{d}x \\ &= -\frac{1}{2}\int_\Omega \frac{|\det \nabla^2 v_\circ|}{\sigma_\circ^2} |\nabla^2 v - \nabla^2 v_\circ|^2\,\mathrm{d}x \\ &\geq -\frac{\lambda_\circ}{2\sigma_\circ^2}\int_\Omega |\nabla^2 v - \nabla^2 v_\circ|^2\,\mathrm{d}x \end{aligned}$$

from which

$$I(v) \geq I(v_\circ) + \frac{1}{2}\left(1 - c\frac{\lambda_\circ}{\sigma_\circ^2}\right)\left\|\nabla^2 v - \nabla^2 v_\circ\right\|^2_{L^2(\Omega,\mathbb{R}^{2\times 2}_{\mathrm{sym}})}.$$

Hence, inasmuch $1 - c\frac{\lambda_\circ}{\sigma_\circ^2} > 0$, all $v \in \mathcal{A}$ have higher energy than $v_\circ$. □

3.3. The rescaled problem

As it is customary, we map (5) on a fixed domain. To this aim, let

$$W := W_1, \qquad \Omega := \Omega_1 = I \times W.$$

We introduce the scaled out-of-plane displacement $w : \Omega \to \mathbb{R}$ by setting

$$w(x_1, x_2) := v(x_1, \varepsilon x_2),$$

so that the scaled gradient and Hessian operators reads

$$\nabla_\varepsilon \cdot := \left(\partial_1 \cdot, \frac{1}{\varepsilon}\partial_2 \cdot\right)^T, \qquad \nabla^2_\varepsilon \cdot := \begin{pmatrix} \partial_{11}\cdot & \frac{1}{\varepsilon}\partial_{12}\cdot \\ \frac{1}{\varepsilon}\partial_{21}\cdot & \frac{1}{\varepsilon^2}\partial_{22}\cdot \end{pmatrix}.$$

Changing variable in (5), we obtain

$$J_\varepsilon(w) := \frac{I_\varepsilon(v)}{\varepsilon} = \int_\Omega \frac{1}{2}|\nabla^2_\varepsilon w - K_\varepsilon|^2 + c|\det \nabla^2_\varepsilon w| \, \mathrm{d}x. \tag{8}$$

Note that

$$\det \nabla^2_\varepsilon w = \frac{1}{\varepsilon^2} \det \nabla^2 w = \det \nabla^2 \frac{w}{\varepsilon}. \tag{9}$$

To avoid rigid motions, we consider functions in

$$W^{2,2}_{\langle 0\rangle}(\Omega) := \left\{ u \in W^{2,2}(\Omega) : \int_\Omega u(x)\, \mathrm{d}x = 0, \ \int_\Omega \nabla u(x)\, \mathrm{d}x = 0 \right\}.$$

With a slight abuse of notation, we still denote by $J_\varepsilon : BH(\Omega) \to [0,\infty]$ the augmented functional

$$J_\varepsilon(w) := \begin{cases} \int_\Omega \frac{1}{2}|\nabla^2_\varepsilon w - K_\varepsilon|^2 + c|\det \nabla^2_\varepsilon w| \, \mathrm{d}x & \text{if } w \in W^{2,2}_{\langle 0\rangle}(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{10}$$

4. Compactness and Γ-limit

Let us introduce the space

$$BH_{\langle 0\rangle}(S) := \left\{ u \in BH(S), \ \int_S u(x)\, \mathrm{d}x = 0 \ \int_S \nabla u(x)\, \mathrm{d}x = 0 \right\}.$$

Lemma 7. (Compactness) *Let $(w_\varepsilon) \subset BH(\Omega)$ be a sequence such that $\sup_\varepsilon J_\varepsilon(w_\varepsilon) < \infty$. Then, up to a subsequence, there exist $w \in BH_{\langle 0\rangle}(\Omega)$, $\vartheta \in BH_{\langle 0\rangle}(I)$, $r \in BH_{\langle 0\rangle}(I)$, $\gamma \in L^2(\Omega)$ such that*

$$w(x_1, x_2) = r(x_1) + x_2\vartheta(x_1) + k\left(\frac{x_2^2}{2} - \frac{1}{24}\right), \tag{11}$$

$$\frac{w_\varepsilon}{\varepsilon} \overset{*}{\rightharpoonup} w \qquad \text{in } BH(\Omega), \tag{12}$$

$$\nabla^2 \frac{w_\varepsilon}{\varepsilon}\mathcal{L}^2 \llcorner \Omega \overset{*}{\rightharpoonup} \begin{pmatrix} D_{11}w & \dot\vartheta\mathcal{L}^2 \llcorner \Omega \\ \dot\vartheta\mathcal{L}^2 \llcorner \Omega & k\mathcal{L}^2 \llcorner \Omega \end{pmatrix} \qquad \text{in } \mathcal{M}(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}}), \tag{13}$$

$$\nabla^2_\varepsilon w_\varepsilon - K_\varepsilon \rightharpoonup \begin{pmatrix} 0 & \dot\vartheta \\ \dot\vartheta & \gamma \end{pmatrix} \qquad \text{in } L^2(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}}). \tag{14}$$

Proof. Let $(w_\varepsilon) \subset BH(\Omega)$ be a sequence such that $\sup_\varepsilon J_\varepsilon(w_\varepsilon) < \infty$. Since the energy is uniformly bounded from above, we deduce that

$$\sup_\varepsilon \left\|\nabla^2_\varepsilon w_\varepsilon - K_\varepsilon\right\|_{L^2(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})} < \infty, \qquad \sup_\varepsilon \left\|\det \nabla^2_\varepsilon w_\varepsilon\right\|_{L^1(\Omega)} < \infty \tag{15}$$

from which, up to a subsequence,

$$\nabla^2_\varepsilon w_\varepsilon - K_\varepsilon \rightharpoonup A \ \text{ in } L^2(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}}), \tag{16}$$

with $A \in L^2(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$.

By (4), it follows that for every $\varepsilon > 0$

$$\int_\Omega \left|\det \nabla_\varepsilon^2 w_\varepsilon - k\frac{\partial_{11} w_\varepsilon}{\varepsilon}\right| \,\mathrm{d}x = \int_\Omega \left|\det(\nabla_\varepsilon^2 w_\varepsilon - K_\varepsilon)\right| \,\mathrm{d}x \leq \frac{1}{2}\left\|\nabla_\varepsilon^2 w_\varepsilon - K_\varepsilon\right\|^2_{L^2(\Omega,\mathbb{R}^{2\times 2}_{\mathrm{sym}})} < C < \infty \quad (17)$$

and hence, by (15),

$$|k|\left\|\partial_{11}\frac{w_\varepsilon}{\varepsilon}\right\|_{L^1(\Omega)} \leq C + \left\|\det \nabla_\varepsilon^2 w_\varepsilon\right\|_{L^1(\Omega)} \leq C < \infty. \quad (18)$$

Moreover, still from (15), by using Hölder inequality, we have that for any $\varepsilon > 0$

$$\left\|\partial_{12}\frac{w_\varepsilon}{\varepsilon}\right\|_{L^1(\Omega)} \leq C\left\|\partial_{12}\frac{w_\varepsilon}{\varepsilon}\right\|_{L^2(\Omega)} < C, \quad (19)$$

and

$$\left\|\partial_{22}\frac{w_\varepsilon}{\varepsilon}\right\|_{L^1(\Omega)} \leq C\left\|\partial_{22}\frac{w_\varepsilon}{\varepsilon}\right\|_{L^2(\Omega)} < k + \varepsilon C, \quad (20)$$

so that together with (18) we conclude that $(\nabla^2\frac{w_\varepsilon}{\varepsilon})$ is uniformly bounded in $L^1(\Omega,\mathbb{R}^{2\times 2}_{\mathrm{sym}})$. Identifying $L^1(\Omega,\mathbb{R}^{2\times 2}_{\mathrm{sym}})$ with a subspace of $(C_0(\Omega,\mathbb{R}^{2\times 2}_{\mathrm{sym}}))' = \mathcal{M}(\Omega,\mathbb{R}^{2\times 2}_{\mathrm{sym}})$, there exists $Z \in \mathcal{M}(\Omega,\mathbb{R}^{2\times 2}_{\mathrm{sym}})$ such that $\nabla^2\frac{w_\varepsilon}{\varepsilon}\mathcal{L}^2 \llcorner \Omega \overset{*}{\rightharpoonup} Z$ in the sense of measures. By Poincaré-Wirtinger inequality, $(w_\varepsilon/\varepsilon)$ is uniformly bounded in $W^{2,1}(\Omega)$. By the compact embedding $W^{2,1}(\Omega) \hookrightarrow W^{1,1}(\Omega)$, $\frac{w_\varepsilon}{\varepsilon} \to w$ strongly in $W^{1,1}(\Omega)$. Hence, $\frac{w_\varepsilon}{\varepsilon} \overset{*}{\rightharpoonup} w$ in $BH(\Omega)$ for some $w \in BH_{\langle 0\rangle}(\Omega)$, so that $Z = D^2 w$ (in the sense of measures).

From (19), (20), and (15) we have also that $\partial_{12}\frac{w_\varepsilon}{\varepsilon} \rightharpoonup A_{12}$ in $L^2(\Omega)$ and $\partial_{22}\frac{w_\varepsilon}{\varepsilon} \to k$ in $L^2(\Omega)$.

Hence, for $\phi \in C_c^\infty(\Omega)$, we have

$$\int_\Omega \phi d(D_{22}w) = \lim_{\varepsilon\downarrow 0}\int_\Omega \phi d\left(\partial_{22}\frac{w_\varepsilon}{\varepsilon}\mathcal{L}^2\right) = \lim_{\varepsilon\downarrow 0}\int_\Omega \phi\partial_{22}\frac{w_\varepsilon}{\varepsilon}d\mathcal{L}^2 = \int_\Omega \phi k d\mathcal{L}^2,$$

so that $D_{22}w = k\mathcal{L}^2 \llcorner \Omega$.

The element w has a continuous representative up to the boundary (see [9, Theorem 3.3 and Remark 3.2]), hence, by integration, we deduce

$$w(x_1,x_2) = k\left(\frac{x_2^2}{2} - \frac{1}{24}\right) + \vartheta(x_1)x_2 + r(x_1)$$

with $\vartheta, r \in BH_{\langle 0\rangle}(I)$.

Similarly as before, we deduce that for $\phi \in C_c^\infty(\Omega)$

$$\int_\Omega \phi d(\dot\vartheta\mathcal{L}^2) = \int_\Omega \phi d(D_{12}w) = \lim_{\varepsilon\downarrow 0}\int_\Omega \phi d\left(\partial_{12}\frac{w_\varepsilon}{\varepsilon}\mathcal{L}^2\right) = \lim_{\varepsilon\downarrow 0}\int_\Omega \phi\partial_{12}\frac{w_\varepsilon}{\varepsilon}d\mathcal{L}^2 = \int_\Omega \phi A_{12} d\mathcal{L}^2$$

so that $\partial_{12}w = \dot\vartheta(x_1)$ in the sense of distributions. Moreover, $A_{12} = \vartheta$ by the uniqueness of the limit. It is further clear that, by (18), $A_{11} = 0$. We conclude the characterization of A by posing $\gamma{:=}A_{22}$. □

Remark. The 11-entry in the limit matrix in (14) vanishes because of the presence of curvature in the natural state. In fact, since $k \neq 0$, we have at our disposal the uniform bound (18), which is not available in naturally flat ribbons as in [13].

In the next lemma we prove that the jump set of D^2w is made of segments orthogonal to e_1. Moreover, only $\partial_1 w$ has a jump part.

Lemma 8. *Let w be as in Lemma 7. The following representation holds:*

$$D^2 w = \nabla^2 w\mathcal{L}^2 \llcorner \Omega + [\nabla w\cdot e_1]e_1 \otimes e_1\mathcal{H}^1 \llcorner J_{\nabla w} + C(\nabla w). \quad (21)$$

Proof. Since $w \in BH(\Omega)$, by definition $\nabla w \in BV(\Omega, \mathbb{R}^2)$, so that we have the following representation (see [12])

$$D^2 w = D(\nabla w) = \nabla^2 w \mathcal{L}^2 \llcorner \Omega + [\nabla w] \otimes \nu_{\nabla w} \mathcal{H}^1 \llcorner J_{\nabla w} + C(\nabla w)$$

where $[\cdot]$ is the jump of $\cdot$, $J_{\nabla w}$ is the jump set of ∇w, $\nu_{\nabla w}$ is the unit normal across $J_{\nabla w}$ and $C(\nabla w)$ is the Cantor part of $D^2 w$, singular to the measure $\mathcal{L}^2 \llcorner \Omega + \mathcal{H}^1 \llcorner J_{\nabla w}$.

Note that

$$[\nabla w] = \left[\begin{pmatrix} \dot{r} + x_2 \dot{\vartheta} \\ \vartheta + k x_2 \end{pmatrix} \right] = \begin{pmatrix} [\dot{r} + x_2 \dot{\vartheta}] \\ 0 \end{pmatrix} = \begin{pmatrix} [\dot{r}] + x_2 [\dot{\vartheta}] \\ 0 \end{pmatrix}$$

because $\vartheta + k x_2$ is a continuous function. Since $[\nabla w] \otimes \nu_{\nabla w}$ must be a symmetric, rank one, tensor, we readily deduce that $\nu_{\nabla w}(x) = e_1$ for $\mathcal{H}^1-$a.e. $x \in J_{\nabla w}$. Moreover, since $\dot{r}, \dot{\vartheta}$ are functions of x_1 only, $J_{\nabla w}$ are segments running through the width of Ω. □

Proposition 9. *Under the same assumptions of Lemma 7, we have*

$$(\det \nabla_\varepsilon^2 w_\varepsilon) \mathcal{L}^2 \llcorner \Omega \overset{*}{\rightharpoonup} (\det \nabla^2 w) \mathcal{L}^2 \llcorner \Omega + k (D_{11} w)_s \qquad \textit{in} \;\; \mathcal{M}(\Omega). \tag{22}$$

Proof. By (9) and Lemma 7, we deduce that, up to a subsequence, $(\det \nabla^2 (w_\varepsilon / \varepsilon)) \mathcal{L}^2 \llcorner \Omega \overset{*}{\rightharpoonup} \eta$ for some measure $\eta \in \mathcal{M}(\Omega)$. We shall now characterize η.

We note that for every $\phi \in C_c^\infty(\Omega)$ and $y \in W^{2,2}(\Omega)$, we have

$$\begin{aligned} \int_\Omega \phi \det \nabla^2 y \, \mathrm{d}x &= \int_\Omega (\partial_{11} y \partial_{22} y - \partial_{12} y \partial_{12} y) \; \phi \, \mathrm{d}x \\ &= \int_\Omega \partial_1 y \, \partial_2 (\phi \partial_{12} y) - \partial_1 y \, \partial_1 (\phi \partial_{22} y) \, \mathrm{d}x \\ &= \int_\Omega \partial_1 y \, \partial_{12} y \, \partial_2 \phi - \partial_1 y \, \partial_{22} y \, \partial_1 \phi \, \mathrm{d}x, \\ &= Hy(\phi), \end{aligned}$$

as defined in (3). In our setting, we have

$$H \frac{w_\varepsilon}{\varepsilon}(\phi) = \int_\Omega \phi \det \nabla^2 \frac{w_\varepsilon}{\varepsilon} \, \mathrm{d}x \quad = \int_\Omega \partial_1 \frac{w_\varepsilon}{\varepsilon} \, \partial_{12} \frac{w_\varepsilon}{\varepsilon} \, \partial_2 \phi - \partial_1 \frac{w_\varepsilon}{\varepsilon} \, \partial_{22} \frac{w_\varepsilon}{\varepsilon} \, \partial_1 \phi \, \mathrm{d}x.$$

For w as in Lemma 7, we have that $\partial_{12} w$ and $\partial_{22} w$ belong to $L^2(\Omega)$, and thus the weak Hessian of w is still well defined:

$$Hw(\phi) = \int_\Omega \partial_1 w \, \partial_{12} w \, \partial_2 \phi - \partial_1 w \, \partial_{22} w \, \partial_1 \phi \, \mathrm{d}x.$$

We claim that

$$\lim_{\varepsilon \downarrow 0} \left| H \frac{w_\varepsilon}{\varepsilon}(\phi) - Hw(\phi) \right| = 0.$$

We have:

$$
\begin{aligned}
\left| H\frac{w_\varepsilon}{\varepsilon}(\phi) - Hw(\phi)\right| &= \Bigg| \int_\Omega \left(\partial_1 \frac{w_\varepsilon}{\varepsilon}\right)\,(\partial_{12}\frac{w_\varepsilon}{\varepsilon})\,\partial_2\phi\,\mathrm{d}x - \int_\Omega (\partial_1 w)\,(\partial_{12}w)\,\partial_2\phi\,\mathrm{d}x \\
&\quad - \int_\Omega \left(\partial_1 \frac{w_\varepsilon}{\varepsilon}\right)\left(\partial_{22}\frac{w_\varepsilon}{\varepsilon}\right)\,\partial_1\phi\,\mathrm{d}x + \int_\Omega (\partial_1 w)\,(\partial_{22}w)\,\partial_1\phi\,\mathrm{d}x \Bigg| \\
&= \Bigg| \int_\Omega \partial_1 \frac{w_\varepsilon}{\varepsilon}\,\partial_{12}\frac{w_\varepsilon}{\varepsilon}\,\partial_2\phi\,\mathrm{d}x - \int_\Omega \partial_1 w\,\partial_{12}w\,\partial_2\phi\,\mathrm{d}x \pm \int_\Omega \partial_1 w\,\partial_{12}\frac{w_\varepsilon}{\varepsilon}\,\partial_2\phi\,\mathrm{d}x \\
&\quad - \int_\Omega \partial_1 \frac{w_\varepsilon}{\varepsilon}\,\partial_{22}\frac{w_\varepsilon}{\varepsilon}\,\partial_1\phi\,\mathrm{d}x + \int_\Omega \partial_1 w\,\partial_{22}w\,\partial_1\phi\,\mathrm{d}x \pm \int_\Omega \partial_1 w\,\partial_{22}\frac{w_\varepsilon}{\varepsilon}\,\partial_1\phi\,\mathrm{d}x \Bigg| \\
&\le \left| \int_\Omega \partial_1 w\left(\partial_{12}\frac{w_\varepsilon}{\varepsilon} - \partial_{12}w\right)\partial_2\phi\,\mathrm{d}x \right| \\
&\quad + \left| \int_\Omega \partial_1 w\left(\partial_{22}\frac{w_\varepsilon}{\varepsilon} - \partial_{22}w\right)\partial_1\phi\,\mathrm{d}x \right| \\
&\quad + \left| \int_\Omega \left(\partial_1\frac{w_\varepsilon}{\varepsilon} - \partial_1 w\right)\left(\partial_{12}\frac{w_\varepsilon}{\varepsilon}\partial_2\phi - \partial_{22}\frac{w_\varepsilon}{\varepsilon}\partial_1\phi\right)\,\mathrm{d}x \right|. \\
&=: S_1 + S_2 + S_3.
\end{aligned}
$$

We show the three summands S_i converge to zero.

For the first term, notice that $\partial_1 w \in BV(\Omega) \hookrightarrow L^2(\Omega)$, so that $\partial_1 w \partial_2\phi \in L^2(\Omega)$. Hence, we conclude because of the weak convergence of $\partial_{12}\frac{w_\varepsilon}{\varepsilon}$ in $L^2(\Omega)$ to $\partial_{12}w$.

For the second term, by Hölder inequality we have

$$
S_2 \le \|\partial_1 w\|_{L^2(\Omega)} \left\|\partial_{22}\frac{w_\varepsilon}{\varepsilon} - \partial_{22}w\right\|_{L^2(\Omega)} \le C\left\|\partial_{22}\frac{w_\varepsilon}{\varepsilon} - \partial_{22}w\right\|_{L^2(\Omega)}
$$

and we conclude by the strong convergence of $\partial_{22}\frac{w_\varepsilon}{\varepsilon}$ in $L^2(\Omega)$ to $\partial_{22}w(=k)$.

For the third summand, we need a finer argument. Note that from Lemma 7 we actually have $\sup_\varepsilon \left\|\partial_1\nabla\frac{w_\varepsilon}{\varepsilon}\right\|_{L^1(\Omega,\mathbb{R}^2)} < \infty$ and $\sup_\varepsilon \left\|\partial_2\nabla\frac{w_\varepsilon}{\varepsilon}\right\|_{L^2(\Omega,\mathbb{R}^2)} < \infty$. Moreover, by the embedding $W^{1,1}(\Omega,\mathbb{R}^2) \hookrightarrow L^2(\Omega,\mathbb{R}^2)$, we have also $\sup_\varepsilon \left\|\nabla\frac{w_\varepsilon}{\varepsilon}\right\|_{L^2(\Omega,\mathbb{R}^2)} < \infty$. Hence, $(\nabla\frac{w_\varepsilon}{\varepsilon})$ is uniformly bounded in $W^{1,(2;1,2)}(\Omega,\mathbb{R}^2)$. By Theorem 1, we conclude that

$$
\nabla\frac{w_\varepsilon}{\varepsilon} \to \nabla w \qquad \text{in } L^2(\Omega,\mathbb{R}^2). \tag{23}
$$

We can thus apply again Hölder inequality to the third summand, so to have

$$
S_3 \le C\left\|\partial_1\frac{w_\varepsilon}{\varepsilon} - \partial_1 w\right\|_{L^2(\Omega)} \to 0
$$

thanks to (23).

To conclude, it suffices to note that

$$
\begin{aligned}
Hw(\phi) &= \int_\Omega \partial_1 w\, \partial_{12} w\, \partial_2 \phi - \partial_1 w\, \partial_{22} w\, \partial_1 \phi \,\mathrm{d}x \\
&= \int_\Omega (\dot r + x_2 \dot\vartheta)\, \dot\vartheta\, \partial_2 \phi - (\dot r + x_2\dot\vartheta) k\, \partial_1 \phi \,\mathrm{d}x \\
&= \int_\Omega -\dot\vartheta^2\, \phi \,\mathrm{d}x + k \int_\Omega \phi\; d(r'' + x_2 \vartheta'') \\
&= \int_\Omega \phi\; d(k(r'' + x_2\vartheta'') - \dot\vartheta^2 \mathcal{L}^2) \\
&= \int_\Omega \phi\, \underbrace{(k(\ddot r + x_2 \ddot\vartheta) - \dot\vartheta^2)}_{\det \nabla^2 w} \,\mathrm{d}x + k \int_\Omega \phi d\, \underbrace{(r'' + x_2\vartheta'')_s}_{(D_{11} w)_s}.
\end{aligned}
$$

□

Remark. The convergence of the absolute continuous part of the limit measure in (22) could have been obtained by applying the following theorem to the sequence $(\nabla \frac{w_\varepsilon}{\varepsilon})$.

Theorem 10. ([11, Theorem 2]) *Let (y_n) be bounded in $W^{1,N-1}(\Omega,\mathbb{R}^N)$, $(\operatorname{cof}\nabla y_n) \subset L^{\frac{N}{N-1}}(\Omega,\mathbb{R}^{N\times N})$. Suppose $y_n \to y$ in $L^1(\Omega,\mathbb{R}^N)$ to $y \in BV(\Omega,\mathbb{R}^N)$ and that $\det\nabla y_n \overset{*}{\rightharpoonup} \eta$ in $\mathcal{M}(\Omega)$. Then, for $\mathcal{L}^N$ a.e. $x \in \Omega$ $\det\nabla y(x) = \frac{d\eta}{d\mathcal{L}^N}(x)$.*

Note that, in dimension two, $(\operatorname{cof}\nabla y_n)$ need not be bounded in $L^2(\Omega,\mathbb{R}^{2\times 2})$. Moreover, the condition $(\operatorname{cof}\nabla y_n) \subset L^2(\Omega,\mathbb{R}^{2\times 2})$ is equivalent to $(\nabla y_n) \subset L^2(\Omega,\mathbb{R}^{2\times 2})$. In our setting, we have at our disposal more ingredients, so that we can characterize also the singular part of the limit measure.

The following results will be particularly useful for the construction of the recovery sequence.

Lemma 11. *Let $u \in BH(I)$. Then, there exists $(u_n) \subset C^\infty(I) \cap BH(I)$ such that $u_n \overset{s}{\rightharpoonup} u$ in $BH(I)$. Moreover,*

- (i) *if $(s_n) \subset \mathbb{R}$ is a sequence such that $s_n \downarrow 0$, as $n \uparrow \infty$, it is possible to choose (u_n) so that it also satisfies $\lim_{n\uparrow\infty} \|s_n u_n''\|_{L^2(I)} = 0$;*
- (ii) *if u is affine in $(-\ell/2, -\ell/2 + b_1)$ and $(\ell/2 - b_1, \ell/2)$ for some $b_1 > 0$, it is also possible to choose (u_n) such that $u_n = u$ in $(-\ell/2, -\ell/2 + b_2) \cup (\ell/2 - b_2, \ell/2)$ for some $0 < b_2 < b_1$;*
- (iii) *if $v \in BH(I)$ there exists $(v_n) \subset C^\infty(I)$ such that $v_n \overset{s}{\rightharpoonup} v$ in $BH(I)$ and such that $u_n + \alpha v_n \overset{s}{\rightharpoonup} u + \alpha v$ in $BH(I)$ for every $\alpha \in \mathbb{R}$;*
- (iv) *if $(g_n) \subset L^1(I)$ and $g \in L^1(I)$ are such that $g_n \to g$ in $L^1(I)$, then*

$$
\lim_{n\uparrow\infty} \int_I |u_n'' + g_n| \,\mathrm{d}x_1 = |u'' + g\mathcal{L}|(I).
$$

Proof. The result essentially follows by a standard mollification procedure. We briefly sketch the proof. Recall that $I = (-\ell/2, \ell/2)$ and set $\widetilde{I} := (-\ell, \ell)$ and

$$
\widetilde{u}(x) := \begin{cases} u(-\ell/2) + \dot u(-\ell/2)(x + \ell/2) & x \in (-\ell, -\ell/2), \\ u(x) & x \in I, \\ u(\ell/2) + \dot u(\ell/2)(x - \ell/2) & x \in (\ell/2, \ell). \end{cases}
$$

Let $\eta : \mathbb{R} \to [0, +\infty)$ be a smooth even function with support in $(-1, 1)$ and such that $\int_{\mathbb{R}} \eta(x) \,\mathrm{d}x = 1$. Let $(\varepsilon_n) \subset \mathbb{R}$ a sequence for which $\varepsilon_n \downarrow 0$ as $n \uparrow \infty$. Let $u_n := \eta_{\varepsilon_n} * u$ where $\eta_{\varepsilon_n}(x) := 1/\varepsilon_n \eta(x/\varepsilon_n)$. Then,

$u_n \to u$ in $W^{1,1}(I)$ and as a consequence $\liminf_{n\uparrow\infty} |u_n''|(I) \geq |u''|(I)$. Since for any $\phi \in C_c^1(I)$ such that $|\phi| \leq 1$ we have that

$$\int_I u_n' \phi' \, \mathrm{d}x = \int_I \widetilde{u}'(\eta_{\varepsilon_n} * \phi)' \, \mathrm{d}x \leq |\widetilde{u}''|(\widetilde{I}) = |u''|(I), \tag{24}$$

it follows that $|u_n''|(I) \leq |u''|(I)$ and, as a consequence, $u_n \xrightharpoonup{s} u$ in $BH(I)$.

To prove *i)* it suffices to notice that

$$u_n''(x) = \frac{1}{\varepsilon_n^2} \int_{-1}^{1} \eta''(z) \widetilde{u}(x - \varepsilon_n z) \, dz \quad \Rightarrow \quad \|u_n''\|_{L^2(I)} \leq \frac{C}{\varepsilon_n^2} \|\widetilde{u}\|_{L^2(\widetilde{I})}$$

and choose $(\varepsilon_n) \subset \mathbb{R}$ so that $s_n/\varepsilon_n^2 \downarrow 0$ as n goes to infinity.

Statement *ii)* follows by taking ε_n smaller than $b_1/3$, for instance, and by noticing that, since η is an even function whose integral is 1, the mollification of an affine function is the affine function itself.

For statement *(iii)* we define (v_n) in the same way we have defined (u_n). Clearly, $u_n + \alpha v_n \to u + \alpha v$ in $W^{1,1}(I)$ and as a consequence $\liminf_{n\uparrow\infty} |u_n'' + \alpha v_n''|(I) \geq |u'' + \alpha v''|(I)$. We conclude by using (24) with $u_n + \alpha v_n$ in place of u_n.

Finally, for statement *iv)*, fix $\delta > 0$ and let $\phi \in C_c^1(I)$ such that $|\phi| \leq 1$ and

$$\int_I |u_n'' + g_n| \, \mathrm{d}x - \delta \leq \int_I (u_n'' + g_n)\phi \, \mathrm{d}x.$$

Then

$$\begin{aligned}
\int_I (u_n'' + g_n)\phi \, \mathrm{d}x &= \int_I -\widetilde{u}'(\eta_{\varepsilon_n} * \phi)' + g(\eta_{\varepsilon_n} * \phi) - g(\eta_{\varepsilon_n} * \phi) + g_n \phi \, \mathrm{d}x \\
&= \int_I \eta_{\varepsilon_n} * \phi \, d(\widetilde{u}'' + g\mathcal{L}) + \int_I -g(\eta_{\varepsilon_n} * \phi) + g_n \phi \, \mathrm{d}x \\
&\leq |u'' + g\mathcal{L}|(I) + \int_I -g(\eta_{\varepsilon_n} * \phi) + g_n \phi \, \mathrm{d}x.
\end{aligned}$$

By letting $n \uparrow \infty$ and then δ to zero, we deduce that $\limsup_{n\uparrow\infty} \int_I |u_n'' + g_n| \, \mathrm{d}x \leq |u + g\mathcal{L}|(I)$. Also, for $\psi \in C_c^1(I)$ such that $|\psi| \leq 1$ we have

$$\liminf_{n\uparrow\infty} \int_I |u_n'' + g_n| \, \mathrm{d}x \geq \lim_{n\uparrow\infty} \int_I (u_n'' + g_n)\psi \, \mathrm{d}x = \int_I \psi \, d(\widetilde{u}'' + g\mathcal{L})$$

which implies $\liminf_{n\uparrow\infty} \int_I |u_n'' + g_n| \, \mathrm{d}x \geq |u'' + g\mathcal{L}|(I)$. □

Lemma 12. *Let λ, μ be two (not necessarily positive) Radon measures defined on I. Let $(\mu + x_2\lambda) \times \mathcal{L}$ the measure defined on $\Omega = I \times W$ by*

$$((\mu + x_2\lambda) \times \mathcal{L})(B) := \int_I \int_W \chi_B(x_1, x_2) \, d(\mu + x_2\lambda)(x_1) dx_2,$$

for every Borel set $B \subset \Omega$. Then,

$$|(\mu + x_2\lambda) \times \mathcal{L}|(\Omega) = \int_W |\mu + x_2\lambda|(I) \, dx_2.$$

Proof. By applying the definition of total variation we have

$$
\begin{aligned}
|(\mu + x_2\lambda) \times \mathcal{L}|(\Omega) &= \sup\left\{ \int_\Omega \phi \, d(\mu + x_2\lambda) \times \mathcal{L} : \phi \in C_c(\Omega),\ |\phi| \le 1 \right\} \\
&\ge \sup\left\{ \int_W \phi_2 \int_I \phi_1 \, d(\mu + x_2\lambda) dx_2 : \phi_1 \in C_c(I), \phi_2 \in C_c(W), \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. |\phi_1| \le 1,\ |\phi_2| \le 1 \right\} \\
&= \sup\left\{ \int_W \phi_2 |\mu + x_2\lambda|(I) dx_2 : \phi_2 \in C_c(W),\ |\phi_2| \le 1 \right\} \\
&= \int_W |\mu + x_2\lambda|(I)\, dx_2.
\end{aligned}
$$

To deduce the opposite inequality it suffices to note that $(\mu + x_2\lambda) \times \mathcal{L} \le |\mu + x_2\lambda| \times \mathcal{L}$ and hence $|(\mu + x_2\lambda) \times \mathcal{L}| \le |\mu + x_2\lambda| \times \mathcal{L}$. $\square$

To state our first Γ-convergence result, we need to make a couple of definitions. Let $\hat{J} : BH(I) \times BH(I) \to [0, \infty)$ be defined by

$$
\hat{J}(r, \vartheta) := \int_I \dot{\vartheta}^2 \, dx_1 + c|k(r'' + x_2\vartheta'') \times \mathcal{L} - \dot{\vartheta}^2 \mathcal{L}^2|(\Omega), \tag{25}
$$

and let

$$
\mathcal{A}(\Omega) := \left\{ w \in BH_{\langle 0 \rangle}(\Omega) :\ \exists r, \vartheta \in BH_{\langle 0 \rangle}(I),\ w = r + x_2\vartheta + k\left(\frac{x_2^2}{2} - \frac{1}{24} \right) \right\},
$$

and

$$
J(w) := \begin{cases} \hat{J}(r, \vartheta) & \text{if } w \in \mathcal{A}(\Omega) \ \&\ w = r + x_2\vartheta + k\left(\frac{x_2^2}{2} - \frac{1}{24} \right), \\ +\infty & \text{otherwise in } BH(\Omega). \end{cases}
$$

Theorem 13. *As $\varepsilon \downarrow 0$, the sequence of functionals J_ε Γ-converges to J in the following sense:*

(a) *(Liminf inequality) for every sequence $(w_\varepsilon) \subset BH(\Omega)$ and every $w \in BH(\Omega)$ such that*

$$
\frac{w_\varepsilon}{\varepsilon} \overset{*}{\rightharpoonup} w \quad \text{in } BH(\Omega),
$$

we have

$$
\liminf_{\varepsilon \downarrow 0} J_\varepsilon(w_\varepsilon) \ge J(w);
$$

(b) *(Recovery sequence) for every $w \in BH(\Omega)$ there exists a sequence $(w_\varepsilon^R) \subset BH(\Omega)$, called recovery sequence, such that*

$$
\frac{w_\varepsilon^R}{\varepsilon} \overset{*}{\rightharpoonup} w \quad \text{in } BH(\Omega)
$$

and

$$
\limsup_{\varepsilon \downarrow 0} J_\varepsilon(w_\varepsilon^R) \le J(w).
$$

Proof. (a) (Liminf inequality) Without loss of generality, let us assume $\liminf_{\varepsilon\downarrow 0} J_\varepsilon(w_\varepsilon) < \infty$, otherwise there is nothing to prove. Hence, up to a subsequence, $\sup_\varepsilon J_\varepsilon(w_\varepsilon) < \infty$. From Lemma 7, we deduce that $w \in \mathcal{A}(\Omega)$ and thus w can be written as $w = r + x_2\vartheta + k(\frac{x_2^2}{2} - \frac{1}{24})$ where $r, \vartheta \in BH_{\langle 0\rangle}(I)$. Still by Lemma 7, we have that

$$\nabla_\varepsilon^2 w_\varepsilon - K_\varepsilon \rightharpoonup \begin{pmatrix} 0 & \dot{\vartheta} \\ \dot{\vartheta} & \gamma \end{pmatrix} \qquad \textit{in } L^2(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}}),$$

for some $\gamma \in L^2(\Omega)$. Let

$$\mu_\varepsilon := (\det \nabla_\varepsilon^2 w_\varepsilon)\mathcal{L}^2 \llcorner \Omega \quad \text{and} \quad \mu := (\det \nabla^2 w)\mathcal{L}^2 \llcorner \Omega + k(D_{11}w)_s.$$

By Proposition 9 we have that $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega)$. We therefore have

$$\begin{aligned}
\liminf_{\varepsilon\downarrow 0} J_\varepsilon(w_\varepsilon) &= \liminf_{\varepsilon\downarrow 0} \left(\int_\Omega \frac{1}{2}|\nabla_\varepsilon^2 w - K_\varepsilon|^2 \mathrm{d}x + c|\mu_\varepsilon|(\Omega) \right) \\
&\geq \int_I \dot{\vartheta}^2\, dx_1 + \frac{1}{2}\int_\Omega |\gamma|^2 \mathrm{d}x + c \liminf_{\varepsilon\downarrow 0} |\mu_\varepsilon|(\Omega) \\
&\geq \int_I \dot{\vartheta}^2\, dx_1 + c \liminf_{\varepsilon\downarrow 0} |\mu_\varepsilon|(\Omega).
\end{aligned}$$

Up to a subsequence, we have that $|\mu_\varepsilon| \overset{*}{\rightharpoonup} \lambda$ for some $\lambda \in \mathcal{M}(\Omega)$. Then, $\liminf_{\varepsilon\downarrow 0} |\mu_\varepsilon|(\Omega) \geq \lambda(\Omega)$ and since $\lambda \geq |\mu|$ we have that $\liminf_{\varepsilon\downarrow 0} |\mu_\varepsilon|(\Omega) \geq |\mu|(\Omega)$. Hence,

$$\begin{aligned}
\liminf_{\varepsilon\downarrow 0} J_\varepsilon(w_\varepsilon) &\geq \int_I \dot{\vartheta}^2\, dx_1 + c|\mu|(\Omega) \\
&= \int_I \dot{\vartheta}^2\, dx_1 + c\int_\Omega |\det \nabla^2 w|\, \mathrm{d}x + c|k(D_{11}w)_s|(\Omega) \\
&= \int_I \dot{\vartheta}^2\, dx_1 + c\int_\Omega |k(\ddot{r} + x_2\ddot{\vartheta}) - \dot{\vartheta}|\, \mathrm{d}x + c|k(r'' + x_2\vartheta'')_s|(\Omega),
\end{aligned}$$

and since $(r'' + x_2\vartheta'')_s = ((r'' + x_2\vartheta'') - \dot{\vartheta}^2)_s$, we deduce that

$$\liminf_{\varepsilon\downarrow 0} J_\varepsilon(w_\varepsilon) \geq \int_I \dot{\vartheta}^2\, dx_1 + c|k(r'' + x_2\vartheta'') \times \mathcal{L} - \dot{\vartheta}^2\mathcal{L}^2|(\Omega) = \hat{J}(r, \vartheta).$$

(b) (Recovery sequence) We may suppose $J(w)$ finite. Hence, there exist $r, \vartheta \in BH_{\langle 0\rangle}(I)$ such that $w = r + x_2\vartheta + k(\frac{x_2^2}{2} - \frac{1}{24})$. By applying Lemma 11, we find a sequence $(\widetilde{r}_\varepsilon) \subset C^\infty(I) \cap BH(I)$ such that $\widetilde{r}_\varepsilon \overset{s}{\rightharpoonup} r$ in $BH(I)$ and $\varepsilon\widetilde{r}_\varepsilon'' \to 0$ in $L^2(I)$. Set

$$r_\varepsilon := \widetilde{r}_\varepsilon - \fint_I \widetilde{r}_\varepsilon\, dx_1 - x_1 \fint_I \widetilde{r}_\varepsilon'\, dx_1$$

and note that $r_\varepsilon \in W^{2,2}_{\langle 0\rangle}(I)$ and that, since $r \in BH_{\langle 0\rangle}(I)$, $r_\varepsilon \overset{s}{\rightharpoonup} r$ in $BH(I)$ and $\varepsilon r_\varepsilon'' \to 0$ in $L^2(I)$. Similarly we find $\vartheta_\varepsilon \in W^{2,2}_{\langle 0\rangle}(I)$ such that $\vartheta_\varepsilon \overset{s}{\rightharpoonup} \vartheta$ in $BH(I)$ and $\varepsilon\vartheta_\varepsilon'' \to 0$ in $L^2(I)$. We define the recovery sequence as

$$w_\varepsilon^R(x_1, x_2) := \varepsilon\left(r_\varepsilon(x_1) + x_2\vartheta_\varepsilon(x_1) + k\left(\frac{x_2^2}{2} - \frac{1}{24}\right)\right).$$

Then, $w_\varepsilon^R/\varepsilon \to w$ in $W^{1,1}(\Omega)$ and

$$\nabla^2 \frac{w_\varepsilon^R}{\varepsilon} = \begin{pmatrix} r_\varepsilon'' + x_2\vartheta_\varepsilon'' & \dot\vartheta_\varepsilon \\ \dot\vartheta_\varepsilon & k \end{pmatrix} \overset{*}{\rightharpoonup} \begin{pmatrix} (r'' + x_2\vartheta'') \times \mathcal{L} & \dot\vartheta\mathcal{L}^2 \\ \dot\vartheta\mathcal{L}^2 & k\mathcal{L}^2 \end{pmatrix}.$$

By using *(iii)* of Lemma 11 and Lemma 12, we conclude that $w_\varepsilon^R/\varepsilon \overset{s}{\rightharpoonup} w$ in $BH(\Omega)$.

Since

$$J(w_\varepsilon^R) = \frac{1}{2}\int_\Omega |\varepsilon(r_\varepsilon'' + x_2\vartheta_\varepsilon'')|^2\,\mathrm{d}x + \int_I |\dot\vartheta_\varepsilon|^2\,dx_1 + c\int_\Omega |k(r_\varepsilon'' + x_2\vartheta_\varepsilon'') - \dot\vartheta_\varepsilon^2|\,\mathrm{d}x$$

using that $\dot\vartheta_\varepsilon \to \vartheta$ in $L^2(I)$, as follows by the compact embedding $BV(I) \hookrightarrow L^2(I)$, and *(iii)* and *(iv)* of Lemma 11, we conclude that

$$\lim_{\varepsilon\downarrow 0} J(w_\varepsilon^R) = \int_I |\dot\vartheta|^2\,dx_1 + c\int_{-1/2}^{1/2} |k(r'' + x_2\vartheta'') \times \mathcal{L} - \dot\vartheta^2\mathcal{L}^2|(I)\,dx_2.$$

The recovery condition statement then follows by applying Lemma 12. □

The Γ-limit can be rewritten using the simple following result.

Lemma 14. *For every $a, b \in \mathbb{R}$ we have*

$$\int_{-1/2}^{1/2} |a + bx|\,dx = \begin{cases} |a| & \text{if } |a| \geq \frac{|b|}{2}, \\ \frac{4a^2+b^2}{4|b|} & \text{otherwise.} \end{cases}$$

Proof. Let $t{:=}a + bx$ and let us change variable within the integral:

$$\begin{aligned} \int_{-1/2}^{1/2} |a + bx|\mathrm{d}x &= \frac{1}{b}\int_{a-\frac{b}{2}}^{a+\frac{b}{2}} |t|\mathrm{d}t \\ &= \frac{1}{2b}\left(\left(a + \frac{b}{2}\right)\left|a + \frac{b}{2}\right| - \left(a - \frac{b}{2}\right)\left|a - \frac{b}{2}\right|\right) \\ &= \begin{cases} |a| & \text{if } |a| \geq \frac{|b|}{2}, \\ \frac{4a^2+b^2}{4|b|} & \text{otherwise.} \end{cases} \end{aligned}$$

□

Accordingly, let $\digamma : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be defined by

$$\digamma(a, b){:=}\begin{cases} |a| & \text{if } |a| \geq \frac{|b|}{2}, \\ \frac{4|a|^2+|b|^2}{4|b|} & \text{otherwise.} \end{cases}$$

We then have

$$
\begin{aligned}
\hat{J}(r,\vartheta) &= \int_I \dot{\vartheta}^2\, dx_1 + c|k(r''+x_2\vartheta'')\times\mathcal{L} - \dot{\vartheta}^2\mathcal{L}^2|(\Omega)\\
&= \int_I \dot{\vartheta}^2\, dx_1 + c\int_I \int_{-1/2}^{1/2} |k(\ddot{r}+x_2\ddot{\vartheta}) - \dot{\vartheta}^2|\, dx_2\, dx_1\\
&\quad + c|k| \sum_{J_{\dot r}\cup J_{\dot\vartheta}} \int_{-1/2}^{1/2} |[\dot r] + x_2[\dot\vartheta]|\, dx_2 + c|k| \int_{-1/2}^{1/2} |(r'')_c + x_2(\vartheta'')_c|(I)\, dx_2 \qquad (26)\\
&= \int_I |\dot\vartheta|^2 + cF(k\ddot r - \dot\vartheta^2, k\ddot\vartheta)\, dx_1 + c|k| \sum_{J_{\dot r}\cup J_{\dot\vartheta}} F([\dot r],[\dot\vartheta])\\
&\quad + c|k|\int_{-1/2}^{1/2} |(r'')_c + x_2(\vartheta'')_c|(I)\, dx_2.
\end{aligned}
$$

4.1. The case with Dirichlet boundary conditions

We here study the constrained problem discussed in Sect. 3.1. We set

$$
J_\varepsilon^C(w) := \begin{cases} \int_\Omega \frac{1}{2}|\nabla_\varepsilon^2 w - K_\varepsilon|^2 + c|\det \nabla_\varepsilon^2 w|\, \mathrm{d}x & \text{if } w\in\mathcal{A}_\varepsilon^C(\Omega),\\ +\infty & \text{otherwise in } BH(\Omega).\end{cases} \qquad (27)
$$

where

$$
\mathcal{A}_\varepsilon^C(\Omega) := \left\{ u\in W^{2,2}(\Omega) : u\left(\mp\frac{\ell}{2}, x_2\right) = \frac{k}{\varepsilon}\left(\frac{(\varepsilon x_2)^2}{2} - \frac{\varepsilon^2}{24}\right),\ \partial_1 u\left(\mp\frac{\ell}{2},x_2\right) = \pm\varepsilon\Phi\right\}
$$

(equalities are in the sense of traces), where $\Phi\in\mathbb{R}$.

The difference with respect to the previous case consists in dealing with boundary traces.

Let

$$
\mathcal{A}^C(\Omega) := \left\{ w\in BH(\Omega) : \exists r,\vartheta\in BH(I),\ w = r + x_2\vartheta + k\left(\frac{x_2^2}{2}-\frac{1}{24}\right),\ r\left(\mp\frac{\ell}{2}\right) = \vartheta\left(\mp\frac{\ell}{2}\right) = 0\right\}
$$

and

$$
J^C(w) := \begin{cases} \hat{J}(r,\vartheta) + J^{BC}(r,\vartheta) & \text{if } w\in\mathcal{A}^C(\Omega)\ \&\ w = r + x_2\vartheta + k\left(\frac{x_2^2}{2} - \frac{1}{24}\right),\\ +\infty & \text{otherwise in } BH(\Omega),\end{cases} \qquad (28)
$$

where $\hat{J}$ has been defined in (25) and the functional J^{BC}, that takes into account the jumps at $\pm\ell/2$, is defined by

$$
J^{BC}(r,\vartheta) := c|k| \int_{-1/2}^{1/2} \left|(\dot r\left(-\frac{\ell}{2}\right) - \Phi + x_2\dot\vartheta\left(-\frac{\ell}{2}\right)\right| + \left|(\dot r\left(\frac{\ell}{2}\right) + \Phi + x_2\dot\vartheta\left(\frac{\ell}{2}\right)\right|\, dx_2.
$$

where $\dot{r}(\pm\ell/2)$ and $\dot{\vartheta}(\pm\ell/2)$ represent the traces of $\dot{r}$ and of $\dot{\vartheta}$. By means of the function $\digamma$ it is possible to rewrite the functional J^{BC} as

$$J^{BC}(r,\vartheta) = c|k|\left(\digamma\left(\dot{r}\left(-\frac{\ell}{2}\right) - \Phi, \dot{\vartheta}\left(-\frac{\ell}{2}\right)\right) + \digamma\left(\dot{r}\left(\frac{\ell}{2}\right) + \Phi, \dot{\vartheta}\left(\frac{\ell}{2}\right)\right)\right).$$

The following diagonalization result will be useful.

Lemma 15. ([1, Corollary 1.18]) *Let (X,τ) be a metrizable space and $\{x_{\nu,\mu} : \nu \in \mathbb{N}, \mu \in \mathbb{N}\}$ a double indexed sequence in X such that:*

$$x_{\nu,\mu} \xrightarrow[\nu\uparrow\infty]{\tau} x_\mu$$

and

$$x_\mu \xrightarrow[\mu\uparrow\infty]{\tau} x.$$

Then, there exists a mapping $\nu \mapsto \mu(\nu)$ increasing to $+\infty$ such that

$$x_{\nu,\mu(\nu)} \xrightarrow[\nu\uparrow\infty]{\tau} x.$$

We now prove the Γ-convergence theorem for the constrained case.

Theorem 16. *As $\varepsilon \downarrow 0$, the sequence of functionals J^C_ε Γ-converges to J^C in the following sense:*

(a) *(Liminf inequality) for every sequence $(w_\varepsilon) \subset BH(\Omega)$ and every $w \in BH(\Omega)$ such that*

$$\frac{w_\varepsilon}{\varepsilon} \overset{*}{\rightharpoonup} w \quad \text{in } BH(\Omega),$$

we have

$$\liminf_{\varepsilon\downarrow 0} J^C_\varepsilon(w_\varepsilon) \geq J^C(w);$$

(b) *(Recovery sequence) for every $w \in BH(\Omega)$ there exists a sequence $(w^R_\varepsilon) \subset BH(\Omega)$, called recovery sequence, such that*

$$\frac{w^R_\varepsilon}{\varepsilon} \overset{*}{\rightharpoonup} w \quad \text{in } BH(\Omega)$$

and

$$\limsup_{\varepsilon\downarrow 0} J^C_\varepsilon(w^R_\varepsilon) \leq J^C(w).$$

Proof. (a) (Liminf inequality) Consider the extended domain $\widetilde{\Omega}{:=}\widetilde{I}\times W$, where $\widetilde{I}{:=}(-\ell,\ell)$ and let

$$J^C_\varepsilon(w;B){:=}\int\limits_B \frac{1}{2}|\nabla^2_\varepsilon w - K_\varepsilon|^2 + c|\det\nabla^2_\varepsilon w|\,\mathrm{d}x,$$

for every Borel set B. For every $w \in \mathcal{A}^C_\varepsilon(\Omega)$ we denote by $\widetilde{w} \in BH(\widetilde{\Omega})$ the function defined as

$$\widetilde{w}(x_1,x_2){:=}\begin{cases} \frac{k}{\varepsilon}\left(\frac{(\varepsilon x_2)^2}{2} - \frac{\varepsilon^2}{24}\right) + \varepsilon\Phi\left(x_1 + \frac{\ell}{2}\right) & \text{if } -\ell < x_1 \leq -\frac{\ell}{2}, \\ w & \text{if } x_1 \in I, \\ \frac{k}{\varepsilon}\left(\frac{(\varepsilon x_2)^2}{2} - \frac{\varepsilon^2}{24}\right) - \varepsilon\Phi\left(x_1 - \frac{\ell}{2}\right) & \text{if } \frac{\ell}{2} \leq x_1 < \ell. \end{cases}$$

We note that $J^C_\varepsilon(w;\Omega) = J^C_\varepsilon(\widetilde{w};\widetilde{\Omega})$ for every $\varepsilon > 0$.

Let $(w_\varepsilon) \subset BH(\Omega)$ and $w \in BH(\Omega)$ such that $\frac{w_\varepsilon}{\varepsilon} \overset{*}{\rightharpoonup} w$ in $BH(\Omega)$. Then, $\frac{\widetilde{w}_\varepsilon}{\varepsilon} \overset{*}{\rightharpoonup} \widehat{w}$ in $BH(\widetilde{\Omega})$ for some $\widehat{w} \in BH(\widetilde{\Omega})$. From the definition of $\widetilde{w}$, it follows that

$$\widehat{w}(x_1, x_2) = \begin{cases} k\left(\frac{x_2^2}{2} - \frac{1}{24}\right) + \Phi\left(x_1 + \frac{\ell}{2}\right) & \text{if } -\ell < x_1 \le -\frac{\ell}{2}, \\ w & \text{if } x_1 \in I, \\ k\left(\frac{x_2^2}{2} - \frac{1}{24}\right) - \Phi\left(x_1 - \frac{\ell}{2}\right) & \text{if } \frac{\ell}{2} \le x_1 < \ell. \end{cases}$$

Without loss of generality, we may assume that, up to a subsequence, $\sup_\varepsilon J_\varepsilon^C(w_\varepsilon) < \infty$, and, as a consequence $J_\varepsilon^C(w_\varepsilon) = J_\varepsilon^C(w_\varepsilon; \Omega) = J_\varepsilon^C(\widetilde{w}_\varepsilon; \widetilde{\Omega})$. By using the boundary conditions, the conclusions of Lemmata 7, 8, 9 still hold. Thus, there exist $r, \vartheta \in BH(I)$ and $\widehat{r}, \widehat{\vartheta} \in BH(\widetilde{I})$ such that

$$w = r + x_2\vartheta + k\left(\frac{x_2^2}{2} - \frac{1}{24}\right) \quad \text{and} \quad \widehat{w} = \widehat{r} + x_2\widehat{\vartheta} + k\left(\frac{x_2^2}{2} - \frac{1}{24}\right).$$

It follows that $r = \widehat{r}$ and $\vartheta = \widehat{\vartheta}$ in I and $\widehat{\vartheta}' = 0$ in $(-\ell, -\ell/2) \cup (\ell/2, \ell)$. As in the proof of Theorem 13, we deduce that

$$\begin{aligned} \liminf_{\varepsilon \downarrow 0} J_\varepsilon^C(w_\varepsilon) = \liminf_{\varepsilon \downarrow 0} J_\varepsilon^C(\widetilde{w}_\varepsilon) &\ge \int_{\widetilde{I}} (\widehat{\vartheta}')^2 \, dx_1 + c|k(\widehat{r}'' + x_2\widehat{\vartheta}'') - (\widehat{\vartheta}')^2|(\widetilde{\Omega}), \\ &= \int_I \dot{\vartheta}^2 \, dx_1 + c|k(r'' + x_2\vartheta'') - \dot{\vartheta}^2|(\Omega) \\ &\quad + c \sum_{x_1 \in \{\pm \ell/2\}} |k(r'' + x_2\vartheta'') - \dot{\vartheta}^2| \left(\{x_1\} \times \left(-\frac{1}{2}, \frac{1}{2}\right)\right) \\ &= \hat{J}(r, \vartheta) + c|k| \sum_{x_1 \in \{\pm \ell/2\}} \int_{-1/2}^{1/2} |\dot{r}(x_1) \mp \Phi + x_2\dot{\vartheta}(x_1)| \, dx_2. \end{aligned}$$

From the continuity of the traces, we immediately deduce that $r(\pm\frac{\ell}{2}) = \vartheta(\pm\frac{\ell}{2}) = 0$.

(b) (Recovery sequence) Fix $w \in \mathcal{A}^C(\Omega)$ such that $J^C(w)$ is finite. There exist $r, \vartheta \in BH(I)$, $r(\pm\frac{\ell}{2}) = \vartheta(\pm\frac{\ell}{2}) = 0$, such that $w = r + x_2\vartheta + k(\frac{x_2^2}{2} - \frac{1}{24})$. Fix T such that $0 < T < \ell/4$ and let $I_T := (-\frac{\ell}{2} + T, \frac{\ell}{2} - T)$. Define

$$r_T'(x_1) := \begin{cases} \Phi & -\frac{\ell}{2} < x_1 \le -\frac{\ell}{2} + T \\ r' - \fint_{I_T} r'(t) \, dt & x_1 \in I_T \\ -\Phi & \frac{\ell}{2} - T \le x_1 \le \frac{\ell}{2} \end{cases},$$

$$\vartheta_T'(x_1) := \begin{cases} 0 & -\frac{\ell}{2} < x_1 \le -\frac{\ell}{2} + T \\ \vartheta' - \fint_{I_T} \vartheta'(t) \, \mathrm{d}t & x_1 \in I_T \\ 0 & \frac{\ell}{2} - T \le x_1 \le \frac{\ell}{2} \end{cases},$$

and

$$r_T(x_1) := \int_{-\ell/2}^{x_1} r_T'(s) \, \mathrm{d}s, \qquad \vartheta_T(x_1) := \int_{-\ell/2}^{x_1} \vartheta_T'(s) \, \mathrm{d}s.$$

Note that $r_T(\mp\frac{\ell}{2}) = \vartheta_T(\mp\frac{\ell}{2}) = 0$ and $r_T' \in \{\pm\Phi\}$, $\vartheta_T' = 0$ in $I \setminus I_T$.

For every T, by applying Lemma 11 we find $(r_{T,\varepsilon}), (\vartheta_{T,\varepsilon}) \subset BH(I) \cap C^\infty(I)$ such that $r_{T,\varepsilon} = r_T$ and $\vartheta_{T,\varepsilon} = \vartheta_T$ near $\pm\ell/2$. Let

$$w^R_{T,\varepsilon}(x_1, x_2) := \varepsilon \left(r_{T,\varepsilon}(x_1) + x_2 \vartheta_{T,\varepsilon}(x_1) + k \left(\frac{x_2^2}{2} - \frac{1}{24} \right) \right).$$

It is clear that $(w^R_{T,\varepsilon}) \subset \mathcal{A}^C_\varepsilon(\Omega)$ and that $\frac{w^R_{T,\varepsilon}}{\varepsilon} \overset{s}{\rightharpoonup} w_T := r_T + x_2\vartheta_T + k\left(\frac{x_2^2}{2} - \frac{1}{24}\right)$ in $BH(\Omega)$ as $\varepsilon \downarrow 0$. It follows that $\frac{w^R_{T,\varepsilon}}{\varepsilon} \overset{*}{\rightharpoonup} w_T$ in $BH(\Omega)$ as $\varepsilon \downarrow 0$, and by proceeding as in the proof of Theorem 13, we deduce

$$\begin{aligned}
\lim_{\varepsilon \downarrow 0} J^C_\varepsilon(w^R_{T,\varepsilon}) &= \int_I |\dot\vartheta_T|^2 dx_1 + c \left| k(r_T'' + x_2 \vartheta_T'') - \dot\vartheta_T^2 \right| (\Omega) \\
&= \int_{I_T} |\dot\vartheta|^2 dx_1 + c \left| k(r'' + x_2\vartheta'') - \dot\vartheta^2 \right| \left(I_T \times \left(-\frac{1}{2}, \frac{1}{2} \right) \right) \\
&\quad + c|k| \sum_{x_1 \in \{\mp\ell/2 \pm T\}} \int_{-1/2}^{1/2} |\dot r(x_1) \mp \Phi + x_2 \dot\vartheta(x_1)| \, dx_2,
\end{aligned}$$

and hence

$$\lim_{T\downarrow 0} \lim_{\varepsilon\downarrow 0} J^C_\varepsilon(w^R_{T,\varepsilon}) = \hat J(r, \vartheta) + J^{BC}(r, \vartheta) = J^C(w).$$

We also have

$$\lim_{T\downarrow 0} \lim_{\varepsilon\downarrow 0} \|w^R_{T,\varepsilon} - w\|_{W^{1,1}(\Omega)} = 0.$$

By applying Lemma 15, we can find a map $\varepsilon \mapsto T(\varepsilon)$ decreasing to 0 such that

$$\lim_{\varepsilon\downarrow 0} J^C_\varepsilon(w^R_{T(\varepsilon),\varepsilon}) = J^C(w) \quad \text{and} \quad \lim_{\varepsilon\downarrow 0} \|w^R_{T(\varepsilon),\varepsilon} - w\|_{W^{1,1}(\Omega)} = 0.$$

We conclude by noticing that $|\nabla^2 w^R_{T(\varepsilon),\varepsilon}/\varepsilon|(\Omega) \le C|\nabla^2 w|(\Omega)$ and hence $w^R_{T(\varepsilon),\varepsilon}/\varepsilon \overset{*}{\rightharpoonup} w$ in $BH(\Omega)$. $\square$

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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