MULTISCALE YOUNG MEASURES IN ALMOST PERIODIC HOMOGENIZATION AND APPLICATIONS

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Abstract. We prove the existence of multiscale Young measures associated with almost periodic homogenization. We give applications of this tool in the homogenization of nonlinear partial differential equations with an almost periodic structure, such as scalar conservation laws, nonlinear transport equations, Hamilton-Jacobi equations and fully nonlinear elliptic equations. Motivated by the application to nonlinear transport equations, we also prove basic results on flows generated by Lipschitz almost periodic vector fields which are of interest in their own. In our analysis, an important role is played by the so called Bohr compactification $\mathbb{G}^N$ of $\mathbb{R}^N$; this is a natural parameter space for the Young measures. Our homogenization results provide also the asymptotic behaviour for the whole set of $\mathbb{G}^N$-translates of the solutions, which is in the spirit of recent studies on the homogenization of stationary ergodic processes.

1. Introduction

The purpose of this paper is to introduce the multiscale Young measure associated with almost periodic homogenization and to apply this tool to address some specific problems for nonlinear partial differential equations. Multiscale Young measures have been introduced in periodic problems by W. E [22] as a broader tool extending the previous concept of multiscale convergence introduced by Nguetseng [42] and further developed by Allaire [1]. It refines to multiple scale analysis the classical concept of Young measures introduced in [50], so fundamentally useful, especially after its striking applications in connection with problems concerning compactness of solution operators for nonlinear partial differential equations by Tartar [49], Murat [39], DiPerna [19, 20, 21], etc. .

The extension of the multiscale Young measures from the periodic setting to the almost periodic one requires the consideration of the so called Bohr compactification $\mathbb{G}^N$ of $\mathbb{R}^N$, which plays in the almost periodic case the same role played by the torus $\mathbb{T}^N := \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ in the periodic case as the domain of the fast scale variables. As opposed to the torus, the Bohr compactification is a nonseparable compact topological space and this lack of separability is the source of some difficulties in trying to adapt the arguments from the periodic context to the almost periodic one. Our analysis here is also motivated by the recent growing interest in the more general setting of homogenization of random stationary ergodic processes (see, e.g., [44], [35], [33], [17], [48], [38], [14]). This has led us to formulate our applications always considering together with each almost periodic nonlinear operator under homogenization process its translates by elements of $\mathbb{G}^N$, which are endowed with an additive group structure inherited from $\mathbb{R}^N$. In this way, we prove that the action of $\mathbb{R}^N$ on $\mathbb{G}^N$ by the addition operation is ergodic, so that the homogenization results presented may be viewed very explicitly as special instances of more general results for stationary ergodic processes. In our proofs, however, ergodicity does not play an explicit role: we need only the averaging properties of almost periodic functions and the invariance of the average under translations by elements of $\mathbb{G}^N$.

Our applications cover nonlinear transport equations, multidimensional conservation laws, Hamilton-Jacobi equations and fully nonlinear elliptic equations. We leave out from the applications given here some classical

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problems that are very common in general accounts on homogenization theory, such as linear equations, variational problems and monotone operators (see, e.g., [8], [11], [33], [2], [43], [41], [45], etc.), mostly because these may be well handled either by the classical methods or by the so-called multiscale convergence, even in the context of almost periodic homogenization.

The paper is organized as follows. In section 2 we recall concepts and basic facts of the theory of almost periodic functions. We end the section with the proof of the ergodicity of the action of $\mathbb{R}^N$ on $\mathbb{G}^N$ by addition. In section 3 we prove the main result concerning the construction of Young measures and also make some further remarks on this topic. In section 4 we analyse flows of Lipschitz almost periodic fields establishing some basic results of interest in their own, which will be needed in our study of the homogenization of nonlinear transport equations. In section 5 we consider the application to nonlinear transport equations extending to the almost periodic setting a previous result of W. E [22] in the periodic context. In section 6 we give another application concerning multidimensional scalar conservation laws with external oscillatory forces, extending to the multidimensional and almost periodic context a previous result by W. E and D. Serre [23]. In section 7 we give an application concerning the homogenization problem for the Hamilton-Jacobi equation with almost periodic Hamiltonian, following a previous analysis by Ishii [31]. Further, we establish links between the multiscale Young measures and the existence of correctors converting the weak convergence of the gradients into strong convergence in L^1_{loc}. Finally, in section 8, we consider the homogenization problem for fully nonlinear elliptic equations, establishing a result analogous to the one of the previous section. Also here we establish links between the multiscale Young measures and the existence of correctors.

2. Spaces of Almost Periodic Functions

In this section we recall the basic ingredients of the theory of almost periodic functions, initiated by H. Bohr. We also present extensions of this concept and prove a few results that will be used later in the paper. For the basic facts about almost periodic functions and generalizations of this concept we refer to the classical presentations of Bohr [10] and Besicovitch [9].

Definition 2.1. Let E be a Banach space. We denote by $\mathrm{BC}(\mathbb{R}^N;E)$ the space of bounded continuous functions on $\mathbb{R}^N$ with values in E endowed with the sup norm and by $\mathrm{BUC}(\mathbb{R}^N;E)$ the subspace of $\mathrm{BC}(\mathbb{R}^N;E)$ consisting of those functions which are uniformly continuous. Given $f\in\mathrm{BC}(\mathbb{R}^N;E)$ and $\varepsilon>0$, we say that $\tau\in\mathbb{R}^N$ is a ε-almost period for f if $\|f(x+\tau)-f(x)\|<\varepsilon$ for any $x\in\mathbb{R}^N$. A function $f\in\mathrm{BC}(\mathbb{R}^N;E)$ is said to be almost periodic if for any $\varepsilon>0$ the set of ε-almost periods of f is relatively dense, i.e., there is $l=l(\varepsilon)>0$ such that any cube with side length l contains at least one ε-almost period.

We denote by $\mathrm{AP}(\mathbb{R}^N;E)$ the space of almost periodic functions on $\mathbb{R}^N$ with values in E and set simply $\mathrm{AP}(\mathbb{R}^N)$ in case $E=\mathbb{R}$. From the above definition we easily deduce that $\mathrm{AP}(\mathbb{R}^N;E)\subseteq\mathrm{BUC}(\mathbb{R}^N;E)$. Below, we give an important characterization of $\mathrm{AP}(\mathbb{R}^N;E)$ due to Bochner which will be used frequently in this paper. We refer to, e.g., [9, 10, 18] for a proof in the case of scalar functions on $\mathbb{R}$ which immediately extends to functions on $\mathbb{R}^N$ with values in a Banach space E.

Theorem 2.1 (Bochner's Characterization of AP). *A function $f\in\mathrm{BC}(\mathbb{R}^N;E)$ belongs to $\mathrm{AP}(\mathbb{R}^N,E)$ if and only the family of translates $\{f(\cdot+t)\}_{t\in\mathbb{R}^N}$ is relatively compact in $\mathrm{BC}(\mathbb{R}^N;E)$.*

The following is the fundamental fact in the theory of almost periodic functions.

Theorem 2.2 (Bohr). *A function in $\mathrm{BC}(\mathbb{R}^N)$ is in $\mathrm{AP}(\mathbb{R}^N)$ if and only if it may be uniformly approximated by finite linear combinations of functions in the set $\{\cos(\lambda\cdot x),\ \sin(\lambda\cdot x)\,:\,\lambda\in\mathbb{R}^N\}$.*

The following result was first obtained by Gelfand, Raikov and Chilov [27] as an application of their theory of commutative Banach algebras. We give here a multidimensional version of the statement presented in [18].

Theorem 2.3 (cf. [18], Theorem XI.2.2)**.** *The space $\mathbb{R}^N$, with the usual addition operation, may be embedded as a dense subgroup of a compact Abelian topological group $\mathbb{G}^N$ in such a way as to make* $\mathrm{AP}(\mathbb{R}^N)$ *the family of all restrictions $f|\mathbb{R}^N$ to $\mathbb{R}^N$ of functions f in $C(\mathbb{G}^N)$. The operation $f \mapsto f|\mathbb{R}^N$ is an isometric $*$-isomorphism of $C(\mathbb{G}^N)$ onto* $\mathrm{AP}(\mathbb{R}^N)$*. Moreover, the addition operation $+ : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ extends uniquely to the continuous group operation of $\mathbb{G}^N$, $+ : \mathbb{G}^N \times \mathbb{G}^N \to \mathbb{G}^N$. The group $\mathbb{G}^N$ is called the Bohr compactification of $\mathbb{R}^N$.*

Given any $f \in \mathrm{AP}(\mathbb{R}^N)$ we denote in the following by $\underline{f}$ its canonical extension to $\mathbb{G}^N$. As a consequence of the above result and the Riez Representation Theorem we have:

Theorem 2.4 (cf. [18], Theorem XI.2.5)**.** *The space* $\mathrm{AP}(\mathbb{R}^N)^*$*, dual to* $\mathrm{AP}(\mathbb{R}^N)$*, is isometrically isomorphic to the space $\mathcal{M}(\mathbb{G}^N)$ of all Radon measures on $\mathbb{G}^N$, the Bohr compactification of $\mathbb{R}^N$. The isomorphism $x^* \mapsto \mu_{x^*} \in \mathcal{M}(\mathbb{G}^N)$ is given by the formula*

$$x^* f = \int_{\mathbb{G}^N} \underline{f}\, d\mu_{x^*}(y), \qquad f \in \mathrm{AP}(\mathbb{R}^N).$$

In Abelian groups an important role is payed by the so called characters.

Definition 2.2. If G is an Abelian group and e its identity element, then a *character* of G is a complex valued function χ defined on G which is such that $\chi(e) = 1$ and $\chi(st) = \chi(s)\chi(t)$ for all s, t in G.

We recall the following basic result on compact Abelian groups of Peter and Weyl.

Theorem 2.5 (Peter-Weyl, cf. [18], Theorem XI.1.5)**.** *Let G be a compact Abelian group, with Σ its Borel field and μ its Haar measure. Then the set of continuous characters is fundamental both in $C(G)$ and in $L^2(G, \Sigma, \mu)$.*

The characters of $\mathbb{G}$ are determined in the following lemma.

Lemma 2.1 (cf. [18], Lemma XI.2.3)**.** *The continuous characters of the compact Abelian group $\mathbb{G}$ are the functions $\underline{e^{i\lambda\cdot}} : \mathbb{G} \to \mathbb{C}$, $\lambda \in \mathbb{R}$.*

The following fact is a consequence of the averaging properties of almost periodic functions and of Theorem 2.4 (*cf.* [28], Proposition 5.7).

Proposition 2.1. *For any $f \in \mathrm{AP}(\mathbb{R}^N)$ we have*

$$\int_{\mathbb{G}^N} \underline{f}\, dz = \fint_{\mathbb{R}^N} f\, dx, \tag{2.1}$$

where dz is the Haar measure in $\mathbb{G}^N$, normalized to be a probability measure, dx is the usual Lebesgue measure in $\mathbb{R}^N$, and by $\fint_{\mathbb{R}^N}$ we denote the asymptotic mean value, given by

$$\fint_{\mathbb{R}^N} f\, dx := \lim_{L\to\infty} \fint_{[-L,L]^N} f\, dx.$$

Moreover, we have

$$\fint_{\mathbb{R}^N} f\, dx = \lim_i \fint_{Q_i} f\, dx$$

where (Q_i) is any sequence of cubes in $\mathbb{R}^N$ whose sides lengths tend to $+\infty$ as $i \to +\infty$.

It is also easy to check that the trigonometric polynomials satisfy

$$\fint_{\mathbb{R}^N} e^{i\lambda\cdot x} e^{-i\lambda'\cdot x}\, dx = 0 \qquad \text{whenever } \lambda \neq \lambda'. \tag{2.2}$$

Next, we establish a compactness criterion for families of almost periodic functions that easily follows from Theorem 2.1. It is equivalent to a well known compactness criterion of Lyusternik (see, e.g., [40]).

Lemma 2.2 (Compactness criterion). *A family $\{u_\alpha\}_{\alpha\in\Lambda} \subseteq \mathrm{AP}(\mathbb{R}^N)$ is relatively compact if and only if the following holds:*

(i) *$\{u_\alpha\}_{\alpha\in\Lambda}$ is uniformly bounded and equi-continuous.*
(ii) *Given $\varepsilon > 0$, there are $t_1, \dots, t_r \in \mathbb{R}^N$ such that for any $t \in \mathbb{R}^N$, there is a $t_j \in \{t_1, \dots, t_r\}$ with $\|u_\alpha(\cdot + t) - u_\alpha(\cdot + t_j)\|_\infty < \varepsilon$ for all $\alpha \in \Lambda$.*

Proof. First, we prove that properties (i) and (ii) imply compactness in $\mathrm{AP}(\mathbb{R}^N)$. Indeed, given any sequence $\{u_{\alpha_j}\}_{j\in\mathbb{N}}$, by property (i), there is a subsequence $\{u_{\alpha_{j(k)}}\}_{k\in\mathbb{N}}$ converging locally uniformly to a function $u \in \mathrm{BUC}(\mathbb{R}^N)$. Passing to the limit in condition (ii) we find that for any $\varepsilon > 0$, there exist $t_1, \dots, t_r \in \mathbb{R}^N$ such that for any $t \in \mathbb{R}^N$, there is a $t_j \in \{t_1, \dots, t_r\}$ with

$$\sup_{\alpha\in\Lambda} \|u_\alpha(\cdot + t) - u_\alpha(\cdot + t_j)\| \le \varepsilon, \qquad \|u(\cdot + t) - u(\cdot + t_j)\|_\infty \le \varepsilon. \tag{2.3}$$

Since ε is arbitrary, this proves that $u \in \mathrm{AP}(\mathbb{R}^N)$. It remains to show that $u_{\alpha_{j(k)}} - u$ converge to 0 uniformly. Indeed, by contradiction we can assume, possibly passing to a subsequence, that $|u_{\alpha_{j(k)}}(x_k) - u(x_k)| \ge 3\varepsilon$ for some $\varepsilon > 0$ and some $x_k \in \mathbb{R}^N$. By applying (2.3) with $t = x_k$ we can find t_j such that both $|u_{\alpha_{j(k)}}(x_k) - u_{\alpha_{j(k)}}(t_j)| \le \varepsilon$ and $|u(x_k) - u(t_j)| \le \varepsilon$ hold for infinitely many k. Then, the triangle inequality immediately gives that $|u_{\alpha_{j(k)}}(t_j) - u(t_j)| \ge \varepsilon$ for infinitely many k, a contradiction. In the proof of the converse implication we use the fact that property (ii) is true for finite families $\{u_1, \dots, u_p\}$ of almost periodic functions: indeed, introducing the map $U : \mathbb{R}^n \to \mathbb{R}^p$ whose components are u_i, Bochner's criterion and a diagonal argument give that the collection of translates $\{U(t+\cdot)\}_{t\in\mathbb{R}^N}$ is relatively compact in $\mathrm{BUC}(\mathbb{R}^N; \mathbb{R}^p)$. The total boundedness of this collection immediately gives property (ii) for the family $\{u_1, \dots, u_p\}$. Assume now that $\{u_\alpha\} \subseteq \mathrm{AP}(\mathbb{R}^N)$ is relatively compact. Then, given $\varepsilon > 0$, there are functions $u_1, \dots, u_p \in \mathrm{AP}(\mathbb{R}^N)$, such that, for any $\alpha \in \Lambda$, there is $j \in \{1, \dots, p\}$ such that $\|u_\alpha - u_j\|_\infty < \varepsilon/3$. Then, since property (ii) is true for finite families of functions, there are points $t_1, \dots, t_r$, such that, for any $t \in \mathbb{R}^N$, for some $k \in \{1, \dots, r\}$, we have $\|u_j(\cdot + t) - u_j(\cdot + t_k)\|_\infty < \varepsilon/3$ for any j. Hence, the points $t_1, \dots, t_r$ verify (ii). Assertion (i) is obvious. □

The previous compactness criterion yields the extension of Theorem 2.3 to E-valued almost periodic maps, where E is any Banach space.

Theorem 2.6. *The space $\mathrm{AP}(\mathbb{R}^N; E)$ is canonically isomorphic and isometric to $C(\mathbb{G}^N; E)$. The isomorphism associates to $f \in \mathrm{AP}(\mathbb{R}^N; E)$ a map $\tilde f \in C(\mathbb{G}^N; E)$ such that*

$$\underline{\langle L, f\rangle} = \langle L, \tilde f\rangle \in C(\mathbb{G}^N) \qquad \forall L \in E^*. \tag{2.4}$$

Furthermore, for any sequence of cubes $Q_i \subseteq \mathbb{R}^N$ whose sides lengths tend to ∞ the mean values $\fint_{Q_i} f\,dx$ weakly converge in E to the vector $\fint_{\mathbb{R}^N} f\,dx$, characterized by

$$\langle L, \fint_{\mathbb{R}^N} f\rangle = \fint_{\mathbb{R}^N} \langle L, f\rangle\,dx \qquad \forall L \in E^*.$$

Proof. Fix $f \in \mathrm{AP}(\mathbb{R}^N; E)$. For $L \in E^*$ we consider the function $L_f(x) = \langle L, f(x)\rangle$. It is immediate to check, using the definition of $\mathrm{AP}(\mathbb{R}^N; E)$, that $L_f \in \mathrm{AP}(\mathbb{R}^N)$. Furthermore $L \mapsto L_f$ is linear and the family

$$\mathcal{F} := \{L_f :\ L \in E^*,\ \|L\| \le 1\}$$

is compact in $\mathrm{AP}(\mathbb{R}^N)$. To see this, notice that all functions in $\mathcal{F}$ are equi-bounded and have the same modulus of continuity, so that the family is relatively compact with respect to local uniform convergence in $\mathbb{R}^N$. On the other hand, the compactness of the closed unit ball of E^* yields that any limit point is still in $\mathcal{F}$. These two facts imply that condition (i) in Lemma 2.2 is fulfilled. Condition (ii) follows immediately by the fact that for any $\varepsilon > 0$ we can find $t_1, \dots, t_r \in \mathbb{R}^N$ such that for any $t \in \mathbb{R}^N$ we have $\sup_{x\in\mathbb{R}^N} \|f(\cdot + t) - f(\cdot + t_j)\|_E < \varepsilon$ for some $j \in \{1, \dots, r\}$. Now for any $\omega \in \mathbb{G}^N$ we consider the map

$L \mapsto \underline{L_f}(\omega)$. This is a linear map on E^* and the compactness of $\mathcal{F}$ yields that this map is continuous with respect to the topology $\sigma(E^*, E)$: indeed, if $L^i \to L$ in the w^*-topology, then the maps $L^i_f \to L_f$ pointwise and compactness yields that they converge also in $\mathrm{AP}(\mathbb{R}^N)$. As a consequence $\underline{L^i_f}$ converges uniformly in $\mathbb{G}^N$ to $\underline{L_f}$. Hence, for any $\omega \in \mathbb{G}^N$ we can find an element of E, that we denote by $\tilde{f}(\omega)$, such that $\underline{L_f}(\omega) = \langle L, \tilde{f}(\omega)\rangle$ for any $L \in E^*$.

This proves (2.4) and it remains to show that $\tilde{f}$ is a continuous map. This is again a compactness argument based on the compactness of the family $\underline{\mathcal{F}} := \{\underline{f} : f \in \mathcal{F}\}$: if $\omega_i \to \omega$ then, by the compactness of $\underline{\mathcal{F}}$ and the Hahn-Banach theorem, $\underline{L_f}(\omega_i) \to \underline{L_f}(\omega)$ uniformly with respect to L in the unit ball of E^*. As a consequence $\tilde{f}(\omega_i) \to \tilde{f}(\omega)$ in E. □

Since functions $f \in \mathrm{AP}(\mathbb{R}^N)$ correspond to restrictions, $f = \underline{f}|\mathbb{R}^N$, of functions $\underline{f} \in C(\mathbb{G}^N)$, a natural question is whether it is possible to define a class of functions $f : \mathbb{R}^N \to \mathbb{R}$ which correspond to "restrictions", $\underline{f}|\mathbb{R}^N$ of functions $\underline{f} \in L^1(\mathbb{G}^N)$. This motivates the following definition.

Definition 2.3. Given $p \in [1, \infty)$ and a Banach space E, the space $\mathrm{BAP}^p(\mathbb{R}^N; E)$, of Besicovitch's generalized almost periodic functions on $\mathbb{R}^N$, with values in E, consists of those functions $f \in L^p_{loc}(\mathbb{R}^N; E)$ [1] for which there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathrm{AP}(\mathbb{R}^N; E)$ satisfying

$$\lim_{n\to\infty} \limsup_{L\to\infty} ⨍_{[-L,L]^N} \|f_n(x) - f(x)\|^p_E \, dx = 0. \tag{2.5}$$

We denote $\mathrm{BAP}^p(\mathbb{R}^N; \mathbb{R})$ simply by $\mathrm{BAP}^p(\mathbb{R}^N)$ and $\mathrm{BAP}^1(\mathbb{R}^N; E)$ by $\mathrm{BAP}(\mathbb{R}^N; E)$.

The space of generalized almost periodic functions $\mathrm{BAP}^p(\mathbb{R})$ was introduced by Besicovitch, who also gave them a structural characterization. We refer to [9] for more details about functions in $\mathrm{BAP}^p(\mathbb{R})$. Intuitively, the space $\mathrm{BAP}^p(\mathbb{R}^N; E)$ corresponds to $L^p(\mathbb{G}; E)$ in a way similar to the one in which the space $\mathrm{AP}(\mathbb{R}^N; E)$ corresponds to $C(\mathbb{G}^N; E)$. Indeed, notice first that the definition of $\mathrm{BAP}^p(\mathbb{R}^N; E)$ immediately gives that the asymptotic mean value $⨍_{\mathbb{R}^N} \|f\|^p \, dx$ of a function in $\mathrm{BAP}^p(\mathbb{R}^N; E)$ is well defined; moreover, any approximating sequence $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathrm{AP}(\mathbb{R}^N; E)$ satisfying (2.5) can be viewed as a Cauchy sequence in $L^p(\mathbb{G}^N; E)$ and, hence, there exists $\underline{f} \in L^p(\mathbb{G}^N; E)$ such that $\underline{f_n} \to \underline{f}$ in $L^p(\mathbb{G}^N; E)$. Since $\underline{f}$ is easily seen to be independent of the approximating sequence, in this way we may associate with each $f \in \mathrm{BAP}^p(\mathbb{R}^N; E)$ a well determined function $\underline{f} \in L^p(\mathbb{G}^N; E)$ which we may view as an "extension" of f to $\mathbb{G}^N$. Notice that $f \mapsto \underline{f}$ is a linear map and that the approximation procedure together with Proposition 2.1 show that

$$⨍_{\mathbb{R}^N} \|f\|^p \, dx = \int_{\mathbb{G}^N} \|\underline{f}\|^p \, dz \qquad \forall f \in \mathrm{BAP}^p(\mathbb{R}^N; E). \tag{2.6}$$

As a consequence, the kernel of the map $f \mapsto \underline{f}$ is made by the functions f such that the asymptotic mean value of $\|f\|^p$ is 0.

We endow $\mathrm{BAP}^2(\mathbb{R}^N)$ with the scalar product

$$\langle f, g\rangle = ⨍_{\mathbb{R}^N} f(x)g(x) \, dx = \int_{\mathbb{G}^N} \underline{f}\,\underline{g} \, dz,$$

and set $\|f\|_2 := \langle f, f\rangle^{1/2}$ (the second equality follows by (2.6) with $p = 2$, implying that the scalar product is preserved under $f \mapsto \underline{f}$). Notice that this norm vanishes on the kernel of $f \mapsto \underline{f}$, and we denote by $\mathrm{BAP}^2(\mathbb{R}^N)/\sim$ the relative quotient space, with the induced scalar product. Concerning the surjectivity of the map, we are able to give a positive answer only in the case $p = 2$.

Proposition 2.2. *$f \mapsto \underline{f}$ is surjective as a map between $\mathrm{BAP}^2(\mathbb{R}^N)$ and $L^2(\mathbb{G}^N)$.*

[1]When E is not separable we denote by $L^1(\Omega; E)$ the space of strongly measurable and summable maps from $\Omega \subseteq \mathbb{R}^N$ to E, or equivalently the L^1 closure of $C_0(\Omega; E)$

This proposition is an almost direct consequence of the following theorem of Besicovitch (see [9], p.110).

Theorem 2.7 (Besicovitch-Riesz-Fischer). *To any pair of sequences $(a_n) \subseteq \mathbb{R}$ and $(\lambda_n) \subseteq \mathbb{R}$ with $\sum |a_n|^2 < +\infty$ there corresponds a function in $\mathrm{BAP}^2(\mathbb{R})$ having these coefficients as Fourier coefficients: this means that, setting*

$$a_f(\lambda) := \fint_{\mathbb{R}} f(x) e^{i\lambda x}\, dx,$$

we have $a_f(\lambda) = a_n$ if $\lambda = \lambda_n$ and $a_f(\lambda) = 0$ otherwise.

The proof of Theorem 2.7 works with minor modifications for functions in $\mathbb{R}^N$.

Proof of Proposition 2.2. Notice first that Theorem 2.5, Lemma 2.1, the invariance of the scalar product and (2.2) give that $\{\underline{e^{i\lambda\cdot x}}\}_{\lambda\in\mathbb{R}^N}$ is a complete orthonormal system in $L^2(\mathbb{G}^N)$. Given $F \in L^2(\mathbb{G}^N)$ we define

$$a_F(\lambda) = \int_{\mathbb{G}} F(z) \underline{e^{i\lambda\cdot z}}\, dz$$

and use Bessel's inequality to obtain a countable number of λ's for which $a_F(\lambda) \neq 0$, with $\sum a_F^2(\lambda) < +\infty$. By the N-dimensional version of Theorem 2.7, there exists a function $f \in \mathrm{BAP}^2(\mathbb{R}^N)$ such that $a_f(\lambda) = a_F(\lambda)$ for any $\lambda \in \mathbb{R}^N$. We claim that $\underline{f} = F$, a.e. in $\mathbb{G}^N$. Indeed, using again the fact that $f \mapsto \underline{f}$ preserves the scalar product, we have

$$\int_{\mathbb{G}^N} (F(z) - \underline{f}) \underline{e^{i\lambda\cdot z}}\, dz = 0 \qquad \text{for all } \lambda \in \mathbb{R}^N.$$

Hence $F - \underline{f}$ is orthogonal to $C(\mathbb{G}^N)$ and so $F - \underline{f}$ must vanish a.e. in $\mathbb{G}^N$. □

The following Corollary is a direct consequence of Proposition 2.2 and its proof.

Corollary 2.1. *The correspondence $f \mapsto \underline{f}$ is an isometric isomorphism between the Hilbert spaces $\mathrm{BAP}^2(\mathbb{R}^N)/\sim$ and $L^2(\mathbb{G}^N)$. Moreover $\{\sin\lambda\cdot x, \cos\lambda\cdot x\}_{\lambda\in\mathbb{R}^N}$ is a complete orthogonal basis of $L^2(\mathbb{G}^N)$.*

The following proposition will also be useful in some applications.

Proposition 2.3. *Let μ be a Radon measure in $\mathbb{G}^N$ and assume that $\langle \mu, \underline{\nabla_i \varphi}\rangle = 0$ for any $i = 1, \dots, N$ and any $\varphi \in \mathrm{AP}(\mathbb{R}^N)$ with $\nabla\varphi \in \mathrm{AP}(\mathbb{R}^N)$. Then μ is a constant multiple of the Haar measure.*

Proof. We clearly have $\langle \mu, \underline{\cos\lambda\cdot z}\rangle = \langle \mu, \underline{\sin\lambda\cdot z}\rangle = 0$ whenever $\lambda \neq 0$. Hence, considering functions φ that are finite sums of $\cos\lambda\cdot z$ and $\sin\lambda\cdot z$ we immediately see that $|\langle \mu, \underline{\varphi}\rangle| \leq \mu(\mathbb{G}^N)\|\underline{\varphi}\|_2$, so that the density of this class of functions in $L^2(\mathbb{G}^N)$ (ensured by Proposition 2.2) immediately gives that μ is representable by some $f \in L^2(\mathbb{G}^N)$. Since f is orthogonal to all functions $\underline{\cos\lambda\cdot z}$, $\underline{\sin\lambda\cdot z}$ with $\lambda \neq 0$, it must be a constant. □

We now briefly recall the concept of almost periodic distribution and some of its basic properties, for the sake of later reference (cf. [47]).

Definition 2.4. We denote by $\mathcal{B}(\mathbb{R}^N)$ the space of C^∞ functions in $\mathbb{R}^N$ which are uniformly bounded together with all their derivatives of any order, with the topology induced by the seminorms $\|\partial^\alpha\phi\|_\infty$, $\alpha = (\alpha_1, \dots, \alpha_N)$, $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_N}^{\alpha_N}$ and by $\mathcal{D}(\mathbb{R}^N)$ the subspace of elements of $\mathcal{B}(\mathbb{R}^N)$ with compact support. We denote by $\mathcal{B}'(\mathbb{R}^N)$ the dual of $\mathcal{B}(\mathbb{R}^N)$ and by $\mathcal{D}'(\mathbb{R}^N)$ the dual of $\mathcal{D}(\mathbb{R}^N)$.

Definition 2.5. We denote by $\mathcal{B}\mathrm{AP}(\mathbb{R}^N)$ the subspace of $\mathcal{B}(\mathbb{R}^N)$ consisting of those functions which, together with all their derivatives of any order, are in $\mathrm{AP}(\mathbb{R}^N)$. We say that $T \in \mathcal{B}'(\mathbb{R}^N)$ is an *almost periodic distribution* if $T * \varphi \in \mathcal{B}\mathrm{AP}(\mathbb{R}^N)$ for any $\varphi \in \mathcal{D}(\mathbb{R}^N)$. We denote by $\mathcal{B}'\mathrm{AP}(\mathbb{R}^N)$ the space of almost periodic distributions.

Proposition 2.4 (Schwartz [47]). *Each of the two following properties is equivalent to $T \in \mathcal{B}'\mathrm{AP}(\mathbb{R}^N)$:*

(i) $T \in \mathcal{B}'(\mathbb{R}^N)$ *is a finite sum of derivatives of functions in* $\mathrm{AP}(\mathbb{R}^N)$*;*
(iii) $T \in \mathcal{B}'(\mathbb{R}^N)$ *and the set of translates* $\{\tau_z T\}_{z\in\mathbb{R}^N}$ *is relatively compact in* $\mathcal{B}'(\mathbb{R}^N)$*.*

It is also immediate to check that $\phi T \in \mathcal{B}'\mathrm{AP}(\mathbb{R}^N)$ whenever $\phi \in \mathcal{B}\mathrm{AP}(\mathbb{R}^N)$ and $T \in \mathcal{B}'\mathrm{AP}(\mathbb{R}^N)$. The following lemma shows also the possibility of embedding canonically $\mathrm{BAP}(\mathbb{R}^N)$ in $\mathcal{B}'\mathrm{AP}(\mathbb{R}^N)$.

Lemma 2.3. *Given any* $f \in \mathrm{BAP}(\mathbb{R}^N)$*, the linear form* $T_f : \mathcal{B}\mathrm{AP}(\mathbb{R}^N) \to \mathbb{R}$ *given by*

$$\langle T_f, \phi\rangle = \fint_{\mathbb{R}^N} f\phi\, dx \tag{2.7}$$

defines T_f *as an element of* $\mathcal{B}'\mathrm{AP}(\mathbb{R}^N)$*. The map* $f \mapsto T_f$ *is a continuous embedding of* BAP *into* $\mathcal{B}'\mathrm{AP}$*.*

Proof. It is easy to check that $fg \in \mathrm{BAP}(\mathbb{R}^N)$ whenever $f \in \mathrm{BAP}(\mathbb{R}^N)$ and $g \in \mathrm{AP}(\mathbb{R}^N)$, and

$$\left| \fint_{\mathbb{R}^N} fg\, dx \right| \le \fint_{\mathbb{R}^N} |f|\, dx\, \|\phi\|_\infty.$$

Hence, $T_f \in \mathcal{B}'(\mathbb{R}^N)$. Since $\tau_z T_f = T_{\tau_z f}$, and the above inequality with $\tau_z f$ replacing f in the left-hand side also holds, for any $z \in \mathbb{R}^N$, the assertion follows from Proposition 2.4 (ii) together with well known facts on the topology of $\mathcal{B}'(\mathbb{R}^N)$. □

We conclude this section with a brief discussion about the group of translations $\{\tau_x\}_{x\in\mathbb{R}^N}$ of $\mathbb{G}^N$, defined by $\tau_x w = w + x$, where the extended addition is given by Theorem 2.3. The family is clearly a group since $\tau_x\tau_{x'}w = \tau_{x+x'}w$. Also, by the invariance of the Haar measure, the group is measure preserving. Moreover, we have the following.

Theorem 2.8. *The measure preserving group* $\{\tau_x\}_{x\in\mathbb{R}^N}$ *is ergodic, that is, for any Borel set* $A \subseteq \mathbb{G}^N$ *invariant under the action of the group we have that either* $m_{\mathbb{G}^N}(A) = 0$ *or* $m_{\mathbb{G}^N}(A) = 1$*, where* $m_{\mathbb{G}^N}(A)$ *denotes the normalized Haar measure of* A*. Moreover* $m_{\mathbb{G}^N}(\mathbb{R}^N) = 0$*.*

Proof. Let $A \subseteq \mathbb{G}^N$ be an invariant Borel set. We have

$$m_{\mathbb{G}^N}(A) = m_{\mathbb{G}^N}(A \cap \tau_{-x}A) = \int_{\mathbb{G}^N} \chi_A(z)\chi_A(z+x)\, dz. \tag{2.8}$$

Now, translations are strongly continuous on $L^2(\mathbb{G}^N)$. Indeed, this is a standard consequence of the density of $C(\mathbb{G}^N)$ in $L^2(\mathbb{G}^N)$, which follows from Theorem 2.5, and of the invariance of the Haar measure. Therefore, the right-hand side is a continuous function of x, and so the identity still holds with $x \in \mathbb{G}^N$. Hence we get, using Fubini theorem and the invariance of the Haar measure,

$$m_{\mathbb{G}^N}(A) = \int_{\mathbb{G}^N}\int_{\mathbb{G}^N} \chi_A(z)\chi_A(z+w)\, dz\, dw = \int_{\mathbb{G}^N} \chi_A(z)\, dz \int_{\mathbb{G}^N} \chi_A(w)\, dw = m^2_{\mathbb{G}^N}(A),$$

from which it follows that $m_{\mathbb{G}^N}(A) \in \{0,1\}$, as asserted.

It remains to show that $m_{\mathbb{G}^N}(\mathbb{R}^N) = 0$. First we observe that $\mathbb{R}^N$ is a Borel subset of $\mathbb{G}^N$, since it is the union of a countable family of compact sets, e.g., the images of the cubes $[-k,k]^N$, $k \in \mathbb{N}$. Since $\mathbb{R}^N$ is invariant under the action of $\{\tau_x\}_{x\in\mathbb{R}^N}$ we have $m_{\mathbb{G}^N}(\mathbb{R}^N) \in \{0,1\}$. But, for any $\omega \in \mathbb{G}^N \setminus \mathbb{R}^N$, $\omega + \mathbb{R}^N$ is also an invariant Borel set and $\mathbb{R}^N \cap \{\omega + \mathbb{R}^N\} = \emptyset$. By the invariance of the Haar measure $m_{\mathbb{G}^N}(\omega + \mathbb{R}^N) = m_{\mathbb{G}^N}(\mathbb{R}^N)$. Hence, we conclude that $m_{\mathbb{G}^N}(\mathbb{R}^N) = 0$. □

3. Almost Periodic Multiscale Young Measures

In this section we state and prove our general result on the existence of multiscale Young measure on almost periodic test functions. Before stating our theorem, let us briefly recall the concepts of net and subnet in a general topological space (*cf.* [34], Ch. 2). We say that a binary relation $\ge$ directs a set D if D is non-void and

(a) if $m, n, p \in D$, $m \ge n$ and $n \ge p$, then $m \ge p$;

(b) if $m \in D$, then $m \geq m$;
(c) if $m, n \in D$, then there is $p \in D$ such that $p \geq m$ and $p \geq n$.

A directed set is a pair $(D, \geq)$ such that $\geq$ directs D. A *net* is a triple $(S, D, \geq)$ such that S is a function with domain D and $\geq$ directs D. A net $\{S, D, \geq\}$ is eventually in a set A if there is an element m of D such that $S_n \in A$ for all $n \in D$ satisfying $n \geq m$. It is frequently in A if for each $m \in D$ there is $n \in D$ such that $n \geq m$ and $S_n \in A$. A net $(S, D, \geq)$ in a topological space $(X, \mathcal{T})$ converges to $s \in X$ if it is eventually in each $\mathcal{T}$-neighborhood of s.
Let $(D, \geq_D)$ and $(E, \geq_E)$ be two directed sets. A set $\{T_m,\ m \in E\}$ is a *subnet* of a net $\{S_n,\ n \in D\}$ if there is a function $N : E \to D$ such that

(a) $T = S \circ N$, or equivalently, $T_i = S_{N_i}$, for each $i \in E$;
(b) for each $m \in D$ there is $n \in E$ with the property that, if $p \geq_E n$, then $N_p \geq_D m$.

A point s is a *cluster point* of a net S if S is frequently in every neighborhood of s. Consequently, a point s is a cluster point of a net S if and only if some subnet of S converges to s. We will use the following well known property of compact topological spaces (see [34], e.g., Ch. 5): *A topological space X is compact if and only if each net has a subnet which converges to some point of X.*

Let $(F, \mathcal{F})$ be a measurable space and let K be a compact metric space. We say that a a family $\{\nu_x\}_{x\in F}$ of probability measures in K is weakly $\mathcal{F}$-measurable if $x \mapsto \langle \nu_x, F\rangle$ is $\mathcal{F}$-measurable for any $F \in C(K)$. It is easy to show (see e.g. Section 2.5 in [4]) that the map

$$x \mapsto \int_K f(x, y)\, d\nu_x(y)$$

is also $\mathcal{F}$-measurable for any nonnegative function f, measurable with respect to the product σ-algebra $\mathcal{F} \otimes \mathbb{B}(K)$, where $\mathbb{B}(K)$ is the Borel σ-algebra of K.

We are now ready to state the main result of this section.

Theorem 3.1. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set and let $\{u_\varepsilon(x)\}_{\varepsilon>0}$ be a family of functions in $L^\infty(\Omega; K)$, for some compact and separable metric space K. Then, given any sequence $\{u_{\varepsilon_i}\}_{i\in\mathbb{N}}$, with $\varepsilon_i \to 0$ as $i \to \infty$, there exist a subnet $\{u_{\varepsilon_{i(d)}}\}_{d\in D}$, indexed by a certain directed set D, and a family of probability measures on K, $\{\nu_{z,x}\}_{z\in\mathbb{G}^N, x\in\mathbb{R}^N}$, weakly measurable with respect to the product of the Borel σ-algebras in $\mathbb{G}^N$ and $\mathbb{R}^N$, such that for any $\Phi \in \mathrm{AP}(\mathbb{R}^N; C_0(\Omega \times K))$ we have*

$$\lim_D \int_\Omega \Phi(\frac{x}{\varepsilon_{i(d)}}, x, u_{\varepsilon_{i(d)}}(x))\, dx = \int_\Omega \int_{\mathbb{G}^N} \langle \nu_{z,x}, \underline{\Phi}(z, x, \cdot)\rangle\, dz\, dx, \tag{3.1}$$

where $\underline{\Phi} \in C(\mathbb{G}^N; C_0(\Omega \times K))$ denotes the unique extension of Φ. Moreover, equality (3.1) *can be extended to functions Φ in the following function spaces:*

(a) $\mathrm{BAP}(\mathbb{R}^N; C_0(\Omega \times K))$*;*
(b) $\mathrm{BAP}^p(\mathbb{R}^N; C(\bar\Omega \times K))$ *with $p > 1$;*
(c) $L^1(\Omega; \mathrm{AP}(\mathbb{R}^N; C(K)))$. [2]

Proof. Step 1. (Construction of $\nu_{z,x}$) We consider the following linear forms over $\mathrm{AP}(\mathbb{R}^N; C_0(\Omega \times K))$:

$$\langle \mu_i, \Phi\rangle := \int_\Omega \Phi(\frac{x}{\varepsilon_i}, x, u_{\varepsilon_i}(x))\, dx.$$

We trivially have $|\langle \mu_i, \Phi\rangle| \leq |\Omega|\, \|\Phi\|_\infty$, where $|\Omega|$ denotes the Lebesgue measure of Ω. Then, recalling that by Theorem 2.4 any function in $\mathrm{AP}(\mathbb{R}^N; C_0(\Omega \times K))$ can be seen as the restriction to $\mathbb{R}^N$ of a function in $C(\mathbb{G}^N; C_0(\Omega \times K)) \equiv C_0(\mathbb{G}^N \times \Omega \times K)$, by Riesz Representation Theorem we can view $\{\mu_i\}$ as a bounded

[2] As in Section 2, this space has to be understood as the L^1 closure of $C(\Omega; \mathrm{AP}(\mathbb{R}^N; C(K)))$

sequence of Radon measures on $\mathbb{G}^N \times \Omega \times K$. By Banach-Alaoglu theorem there exists $\mu \in \mathcal{M}(\mathbb{G}^N \times \Omega \times K)$ which is a cluster point of $\{\mu_i\}$, hence there exists a subnet $\{\mu_{i(d)}\}_{d\in D}$ such that

$$\langle \mu, \Phi\rangle = \lim_D \langle \mu_{i(d)}, \Phi\rangle, \qquad \text{for all } \Phi \in C_0(\mathbb{G}^N \times \Omega \times K). \tag{3.2}$$

Now, given $F \in C(K)$, consider the Radon measure μ_F on $\mathbb{G}^N \times \Omega$ given by

$$\langle \mu_F, \phi\rangle := \langle \mu, F\phi\rangle, \qquad \text{for all } \phi \in C_0(\mathbb{G}^N \times \Omega).$$

For any $f \in \mathrm{AP}(\mathbb{R}^N)$ and any $\varphi \in C_0(\Omega)$, with $\operatorname{supp}\varphi \subseteq \{x_0 + [0, L]^N\}$ for some $x_0 \in \mathbb{R}^N$ and $L > 0$, we have

$$\begin{aligned}
|\langle \mu_F, f\varphi\rangle| &\le \|F\|_\infty \limsup_D \int_\Omega |\varphi(x)||f(\frac{x}{\varepsilon_{i(d)}})|\, dx \\
&\le \|F\|_\infty \|\varphi\|_\infty \limsup_D \int_{x_0+[0,L]^N} |f(\frac{x}{\varepsilon_{i(d)}})|\, dx \\
&\le L^N \|F\|_\infty \|\varphi\|_\infty \limsup_D ⨍_{\frac{x_0}{\varepsilon_{i(d)}}+[0,\frac{L}{\varepsilon_{i(d)}}]^N} |f(x)|\, dx \\
&= L^N \|F\|_\infty \|\varphi\|_\infty ⨍_{\mathbb{R}^N} |f(x)|\, dx.
\end{aligned}$$

Now, if $x_0 + [0, L]^N \subseteq \Omega$, we can take a sequence $\{\varphi_n\} \subseteq C_0(\Omega)$, with $\operatorname{supp}\varphi_n \subseteq \{x_0 + (0, L)^N\}$, $\|\varphi_n\|_\infty \le 1$ and converging everywhere to $\chi_{\{x_0+(0,L)^N\}}$. Taking the limit as $n \to \infty$ in the inequality

$$|\langle \mu_F, f\varphi_n\rangle| \le L^N \|F\|_\infty \int_{\mathbb{G}^N} |\underline{f}|\, dz,$$

using Lebesgue's dominated convergence theorem in the left-hand side, we obtain

$$|\langle \mu_F, f\chi_{\{x_0+(0,L)^N\}}\rangle| \le \|F\|_\infty \int_{\mathbb{G}^N} |\underline{f}|\, dz \int_\Omega \chi_{\{x_0+(0,L)^N\}}\, dx.$$

Now, given any $\varphi \in C_0(\Omega)$, we may define a sequence of functions g_n with compact support in Ω of the form $\sum_{j=1}^{J_n} a_j^n \chi_{\{x_j^n+(0,L_j^n)\}}$, such that $g_n(x) \to \varphi(x)$ for all $x \in \Omega$ and $\|g_n\|_\infty \le \|\varphi\|_\infty$. Since g_n clearly satisfies

$$|\langle \mu_F, fg_n\rangle| \le \|F\|_\infty \int_{\mathbb{G}^N} |\underline{f}|\, dz \int_\Omega |g_n|\, dx,$$

taking the limit as $n \to \infty$ and using again Lebesgue's dominated convergence theorem, now in both sides of the inequality, we arrive at

$$|\langle \mu_F, f\varphi\rangle| \le \|F\|_\infty \int_{\mathbb{G}^N} |\underline{f}|\, dz \int_\Omega |\varphi|\, dx. \tag{3.3}$$

Equation (3.3) implies that

$$|\mu_F| \le \|F\|_\infty\, dz\, dx.$$

If $\psi_F(z, x)$ is a Borel map representing the Radon-Nikodym derivative of μ_F with respect to $dz\, dx$, we then have

$$|\psi_F(z, x)| \le \|F\|_\infty, \qquad \text{for a.e. } (z, x) \in \mathbb{G}^N \times \Omega.$$

Let $\mathbf{S}$ be a countable dense set in $C(K)$ that is also a vector space on the field of rational numbers $\mathbb{Q}$: such a set exists since K is separable. By the uniqueness of the Radon-Nikodym derivative and the monotonicity of the depedence of μ_F on F we can find a Borel negligible set $\mathcal{N} \subseteq \mathbb{G}^N \times \Omega$ such that $F \mapsto \psi_F(z, x)$ is $\mathbb{Q}$-linear and monotone on $\mathbf{S}$ and moreover

$$|\psi_F(z, x)| \le \|F\|_\infty \tag{3.4}$$

for any $(z,x)\in\mathbb{G}^N\times\Omega\setminus\mathcal{N}$. For any such (z,x) the $\mathbb{Q}$-linear form $\langle\nu_{z,x},F\rangle=\psi_F(z,x)$ may be uniquely extended to a monotone and continuous linear functional on $C(K)$. Hence, by Riesz Representation Theorem, $\nu_{z,x}\in\mathcal{M}(K)$ for all $(z,x)\in\mathbb{G}^N\times\Omega\setminus\mathcal{N}$. Moreover, proceeding as above, we see that μ_1 satisfies

$$\langle\mu_1,f\chi_Q\rangle=|Q|\int_{\mathbb{G}^N}\underline{f}\,dz,$$

for any N-dimensional cube $Q\subseteq\Omega$ and any $f\in\mathrm{AP}(\mathbb{R}^N)$. Hence, as above, we conclude that

$$\langle\mu_1,f\varphi\rangle=\int_{\mathbb{G}^N}\underline{f}\,dz\int_\Omega\varphi\,dx,$$

for all $f\in\mathrm{AP}(\mathbb{R}^N)$ and $\varphi\in C_0(\Omega)$. Therefore, we have $\psi_1(z,x)\equiv1$ for a.e. $(z,x)\in\mathbb{G}^N\times\Omega$ and we conclude $\langle\nu_{z,x},1\rangle=1$ for almost all $(z,x)\in\mathbb{G}^N\times\Omega\setminus\mathcal{N}$, which means that $\nu_{z,x}$ are probability measures on K for a.e. (z,x). Defining $\nu_{z,x}$ in an arbitrary (but Borel) way on $\mathcal{N}$, the weak measurability of $\nu_{z,x}$ with respect to $(z,x)\in\mathbb{G}^N\times\Omega$ follows directly from the fact that $\langle\nu_{z,x},F\rangle=\psi_F(z,x)$ for any $F\in C(K)$ and $(z,x)\notin\mathcal{N}$.

Recalling the definition of μ_F, this proves that

$$\langle\mu,\Phi\rangle=\int_\Omega\int_{\mathbb{G}^N}\langle\nu_{z,x},\Phi(z,x,\lambda)\rangle\,dz\,dx$$

for all test functions Φ of the form $F(\lambda)\varphi(z,x)$ with $F\in C(K)$ and $\varphi\in C_0(\mathbb{G}^N\times\Omega)$. Since μ is uniquely determined, as a measure in $\mathbb{G}^N\times\Omega\times K$, by the duality with this class of functions, we obtain that the identity above holds for any $\Phi\in C_0(\mathbb{G}^N\times\Omega\times K)$. Taking (3.2) into account, this proves (3.1).

Step 2. (More general test functions) Now we prove that (3.1) can be extended to $\Phi\in\mathrm{BAP}(\mathbb{R}^N;C_0(\Omega\times K))$. For Φ_1, $\Phi_2\in\mathrm{BAP}(\mathbb{R}^N;C_0(\Omega\times K))$ we have

$$\begin{aligned}
&\limsup_D\left|\int_\Omega\left(\Phi_1(\frac{x}{\varepsilon_{i(d)}},x,u^{\varepsilon_{i(d)}})-\Phi_2(\frac{x}{\varepsilon_{i(d)}},x,u^{\varepsilon_{i(d)}})\right)\,dx\right|\\
\le\ &\limsup_D\int_\Omega\|\Phi_1(\frac{x}{\varepsilon_{i(d)}},\cdot,\cdot)-\Phi_2(\frac{x}{\varepsilon_{i(d)}},\cdot,\cdot)\|_{C_0(\Omega\times K)}\,dx\\
\le\ &\limsup_D\int_{\{[-L,L]^N\}}\|\Phi_1(\frac{x}{\varepsilon_{i(d)}},\cdot,\cdot)-\Phi_2(\frac{x}{\varepsilon_{i(d)}},\cdot,\cdot)\|_{C_0(\Omega\times K)}\,dx\\
\le\ &(2L)^N\|\Phi_1-\Phi_2\|_{\mathrm{BAP}(\mathbb{R}^N;C_0(\Omega\times K))},
\end{aligned}$$

where $L>0$ is such that $\Omega\subseteq[-L,L]^N$. On the other hand, we have

$$\begin{aligned}
\left|\int_\Omega\int_{\mathbb{G}^N}\langle\nu_{z,x},(\underline{\Phi_1}-\underline{\Phi_2})(z,x,\lambda)\rangle\,dz\,dx\right|&\le|\Omega|\int_{\mathbb{G}^N}\|\underline{\Phi_1}(z,\cdot,\cdot)-\underline{\Phi_2}(z,\cdot,\cdot)\|_{C_0(\Omega\times K)}\,dz\\
&=|\Omega|\|\Phi_1-\Phi_2\|_{\mathrm{BAP}(\mathbb{R}^N;C_0(\Omega\times K))}.
\end{aligned}$$

Hence, since we know that (3.1) holds for $\Phi\in\mathrm{AP}(\mathbb{R}^N;C_0(\Omega\times K))$, and that any function in $\mathrm{BAP}(\mathbb{R}^N;C_0(\Omega\times K))$ may be approximated by functions in $\mathrm{AP}(\mathbb{R}^N;C_0(\Omega\times K))$ in the norm of $\mathrm{BAP}(\mathbb{R}^N;C_0(\Omega\times K))$, we obtain the validity of (3.1) from the above estimates by a simple passage to the limit.

As for the extension to $\Phi\in\mathrm{BAP}^p(\mathbb{R}^N;C(\bar\Omega\times K))$ with $p>1$, we have the following. Let $\varphi\in C_0(\Omega)$. We have, for $q=p/(p-1)$ and $\psi=1-\varphi$,

$$\begin{aligned}
\left|\int_\Omega\psi(x)\Phi(\frac{x}{\varepsilon_{i(d)}},x,u_{\varepsilon_{i(d)}}(x))\,dx\right|&\le\|\psi\|_q\left(\int_\Omega|\Phi(\frac{x}{\varepsilon_{i(d)}},x,u_{\varepsilon_{i(d)}}(x))|^p\,dx\right)^{1/p}\\
&\le\|\psi\|_q\left(\varepsilon_{i(d)}^N\int_{[-L/\varepsilon_{i(d)},L/\varepsilon_{i(d)}]^N\}}\|\Phi(z,\cdot,\cdot)\|_{C(\bar\Omega\times K)}^p\,dz\right)^{1/p},
\end{aligned}$$

and so, taking the $\limsup_D$ we obtain

$$\limsup_D \left| \int_\Omega \psi(x)\Phi(\frac{x}{\varepsilon_{i(d)}}, x, u_{\varepsilon_{i(d)}}(x))\, dx \right| \le (2L)^{N/p}\|\psi\|_q \|\Phi\|_{\mathrm{BAP}^p(\mathbb{R}^N;C(\bar\Omega\times K))}.$$

On the other hand, we have

$$\left| \int_\Omega \int_{\mathbb{G}^N} \langle \nu_{z,x}, \underline{\Phi}(z,x,\lambda)\rangle \psi(x)\, dz\, dx \right| \le \|\psi\|_q \|\Phi\|_{L^p(\mathbb{G};C(\bar\Omega\times K))}.$$

Hence, we may extend (3.1) to $\Phi \in \mathrm{BAP}^p(\mathbb{R}^N; C(\bar\Omega \times K))$ by multiplying Φ by $\varphi_n \in C_0(\Omega)$, with $\varphi_n \to 1$ in $L^q(\Omega)$, and taking the limit as $n \to \infty$ in the formula (3.1) for $\Phi\varphi_n$, which is valid since $\Phi\varphi_n \in \mathrm{BAP}(\mathbb{R}^N; C_0(\Omega \times K))$.

Finally, we prove the assertion for $\Phi \in L^1(\Omega; \mathrm{AP}(\mathbb{R}^N; C(K)))$. Indeed, since

$$\left| \int_\Omega \Phi_1(\frac{x}{\varepsilon_{i(d)}}, x, u_{\varepsilon_{i(d)}}(x))\, dx - \int_\Omega \Phi_2(\frac{x}{\varepsilon_{i(d)}}, x, u_{\varepsilon_{i(d)}}(x))\, dx \right| \le \int_\Omega \|\Phi_1(\cdot,x,\cdot) - \Phi_2(\cdot,x,\cdot)\|_{\mathrm{AP}(\mathbb{R}^N;C(K))}\, dx$$

and

$$\left| \int_\Omega \int_{\mathbb{G}^N} \langle \nu_{z,x}, \underline{\Phi_1}(z,x,\lambda)\rangle\, dz\, dx - \int_\Omega \int_{\mathbb{G}^N} \langle \nu_{z,x}, \underline{\Phi_2}(z,x,\lambda)\rangle\, dz\, dx \right| \le \int_\Omega \|\Phi_1(\cdot,x,\cdot) - \Phi_2(\cdot,x,\cdot)\|_{\mathrm{AP}(\mathbb{R}^N;C(K))}\, dx,$$

the validity of (3.1) even for functions $\Phi \in L^1(\Omega; \mathrm{AP}(\mathbb{R}^N; C(K)))$ follows by the density of $C_0(\Omega; \mathrm{AP}(\mathbb{R}^N; C(K)))$. □

Remark 3.1. A similar result holds, with minor adaptations in the proof, for families $\{u^\varepsilon\}_{\varepsilon>0} \subseteq L^1(\Omega; \mathbb{R}^m)$ that satisfy the condition

$$\lim_{R\to\infty} \limsup_{\varepsilon\to 0} |\{|u^\varepsilon| > R\}| = 0.$$

This happens, for instance, when a uniform bound in $L^p(\Omega; \mathbb{R}^m)$ is available. In this special case, the representation formula (3.1) is valid for all $\Phi(z,x,\lambda) \in \mathrm{AP}(\mathbb{R}^N; C_0(\Omega, C(\mathbb{R}^m)))$ such that

$$\lim_{|\lambda|\to\infty} \frac{|\Phi(z,x,\lambda)|}{|\lambda|^p} = 0 \qquad \text{uniformly as } (z,x) \in \mathbb{R}^N \times \Omega.$$

This extension is analogous to the well known one in the classical theory of Young measures (see, e.g., [7], [4], [46] etc.). In particular this gives as a corollary an extension to the almost periodic setting of the main result of the so called two-scale convergence introduced by Nguetseng [42] and further developed by Allaire [1].

As in the classical theory of Young measures (*cf.* [49]) we have the following consequence of Theorem 3.1 (cf. [22]).

Lemma 3.1 (Strong convergence)**.** *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set, let $\{u^\varepsilon\} \subseteq L^\infty(\Omega; \mathbb{R}^m)$ be uniformly bounded and let $\nu_{z,x}$ be an almost periodic two-scale Young measure generated by a subnet $\{u^{\varepsilon(d)}\}_{d\in D}$, according to Theorem 3.1. Assume that U belongs either to $L^1(\Omega; \mathrm{AP}(\mathbb{R}^N; \mathbb{R}^m)))$ or to $\mathrm{BAP}^p(\mathbb{R}^N; C(\bar\Omega; \mathbb{R}^m))$ for some $p > 1$. Then*

$$\nu_{z,x} = \delta_{\underline{U}(z,x)} \quad \textit{if and only if} \quad \lim_D \|u^{\varepsilon(d)}(x) - U(\frac{x}{\varepsilon(d)}, x)\|_{L^1(\Omega)} = 0. \tag{3.5}$$

Proof. If $\nu_{z,x} = \delta_{\underline{U}(z,x)}$, for some $U \in L^1(\Omega; \mathrm{AP}(\mathbb{R}^N; \mathbb{R}^m))$, we take in (3.1) $\Phi(z,x,\lambda) = |\lambda - U(z,x)|$ to conclude that $\|u^{\varepsilon(d)}(x) - U(\frac{x}{\varepsilon(d)}, x)\|_{L^1(\Omega)} \to 0$. On the other hand, if $\|u^{\varepsilon(d)}(x) - U(\frac{x}{\varepsilon(d)}, x)\|_{L^1(\Omega)} \to 0$ we must have $\langle \nu_{z,x}, |\lambda - \underline{U}(z,x)|\rangle = 0$ for a.e. (z,x), which implies $\nu_{z,x} = \delta_{\underline{U}(z,x)}$. The case when $U \in \mathrm{BAP}^p(\mathbb{R}^N; C(\bar\Omega; \mathbb{R}^m))$ for some $p > 1$ is similar. □

4. Flows Generated by Lipschitz Almost Periodic Vector Fields

Let $a \in \mathrm{AP} \cap W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)$, and let us assume that a is incompressible, i.e.

$$\nabla_z \cdot a(z) = 0. \tag{4.1}$$

Let us consider the Cauchy problem

$$\begin{cases} \dfrac{dX}{dt}(z,t) = a(X(z,t)), \\ X(z,0) = z. \end{cases} \tag{4.2}$$

In some occasions we will denote the map $t \mapsto X(z,t)$ by $X_t(z)$.

We are interested in the properties of the map $\boldsymbol{X}_t : \mathrm{BUC}(\mathbb{R}^N) \to \mathrm{BUC}(\mathbb{R}^N)$ defined by $g \mapsto g \circ X_t$.

Lemma 4.1. *$g \circ X_t \in \mathrm{AP}(\mathbb{R}^N)$ for any $g \in \mathrm{AP}(\mathbb{R}^N)$ and*

$$\fint_{\mathbb{R}^N} |g(X(z,t))|^2\,dz = \fint_{\mathbb{R}^N} |g(z)|^2\,dz. \tag{4.3}$$

Therefore $\boldsymbol{X}_t$ can be extended to an operator in $\mathrm{BAP}^2(\mathbb{R}^N)$ satisfying

$$\fint_{\mathbb{R}^N} |\boldsymbol{X}_t(g)|^2\,dz = \fint_{\mathbb{R}^N} |g(z)|^2\,dz \qquad \forall g \in \mathrm{BAP}^2(\mathbb{R}^N). \tag{4.4}$$

Proof. It suffices to prove that $\boldsymbol{X}_t(\mathrm{AP}(\mathbb{R}^N)) \subseteq \mathrm{AP}(\mathbb{R}^N)$ for each $t \in \mathbb{R}$. First, we show that if τ is a ε-period of $(a,g) \in \mathrm{AP}(\mathbb{R}^N;\mathbb{R}^2)$ then

$$|X_t(x+\tau) - X_t(x) - \tau| \le \frac{\varepsilon}{\mathrm{Lip}^2(a)}(e^{\mathrm{Lip}(a)|t|} - 1), \quad |g(X_t(x)+\tau) - g(X_t(x))| < \varepsilon. \tag{4.5}$$

Since the second inequality is trivial, we check the first one, in the case when $t > 0$. So, we have

$$\begin{aligned} |X_t(x+\tau) - X_t(x) - \tau| &= \left| \int_0^t (a(X_s(x+\tau)) - a(X_s(x)))\,ds \right| \\ \le \int_0^t |a(X_s(x+\tau)) - a(X_s(x)+\tau)|\,ds &+ \int_0^t |a(X_s(x)+\tau) - a(X_s(x))|\,ds \\ &\le \mathrm{Lip}(a) \int_0^t |X_s(x+\tau) - X_s(x) - \tau|\,ds + \varepsilon t. \end{aligned}$$

Applying Gronwall's lemma we arrive at the first inequality in (4.5).

Now, let $g \in \mathrm{AP}(\mathbb{R}^N)$ and let $p(s)$ be a modulus of continuity of g. By (4.5) and the triangle inequality we get

$$|g(X_t(z+\tau)) - g(X_t(z))| \le p\left(\frac{\varepsilon(e^{\mathrm{Lip}(a)|t|} - 1)}{\mathrm{Lip}^2(a)} \right) + \varepsilon$$

and since the right hand side is infinitesimal as $\varepsilon \to 0$ the proof of the almost periodicity of $\boldsymbol{X}_t(g)$ is achieved.

Now we prove (4.3). The incompressibility assumption (4.1) implies that the Jacobian determinant of X_t is a.e. equal to 1, and we have

$$\begin{aligned} \frac{1}{L^N} \int_{[0,L]^N} |g(X(z,t))|^2\,dz &= \frac{1}{L^N} \int_{X_t([0,L]^N)} |g(w)|^2\,dw \\ = \frac{1}{L^N} \int_{[0,L]^N} |g(w)|^2\,dw &- \frac{1}{L^N} \int_{[0,L]^N \setminus X_t([0,L]^N)} |g(w)|^2\,dw \\ &+ \frac{1}{L^N} \int_{X_t([0,L]^N) \setminus [0,L]^N} |g(w)|^2\,dw. \end{aligned}$$

Taking the limit as $L \to \infty$ and observing that the two last terms in the equality above go to 0 as $L \to \infty$ because

$$[\|a\|_\infty t, L - \|a\|_\infty t]^N \subseteq X_t([0,L]^N) \subseteq [-\|a\|_\infty t, L + \|a\|_\infty t]^N$$

we obtain (4.3). Relation (4.3) immediately implies that $\boldsymbol{X}_t$ can be extended to an operator in $\mathrm{BAP}^2(\mathbb{R}^N)$, and that $\boldsymbol{X}_t$ fulfils (4.4). ☐

Remark 4.1. The argument used in Lemma 4.1 can also be used to show that $g \circ W \in \mathrm{AP}(\mathbb{R}^N)$ whenever W can be written as $w_1 + w_2$, with w_1 affine and $w_2 \in \mathrm{AP}(\mathbb{R}^N;\mathbb{R}^N)$.

Corollary 4.1. *For any $t \in \mathbb{R}$ the flow map X_t can be uniquely extended to a homeomorphism $\underline{X}_t$ of $\mathbb{G}^N$ and $\boldsymbol{X}_t(g) = g(\underline{X}_t)$ for any $g \in L^2(\mathbb{G}^N)$.*

Proof. Since $C(\mathbb{G}^N)$ is isomorphic to $\mathrm{AP}(\mathbb{R}^N)$, it is a direct consequence of the invariance of $\mathrm{AP}(\mathbb{R}^N)$ under $\boldsymbol{X}_t$ and of Lemma 4.2 below, with $R = \mathbb{R}^N$, $X = \mathbb{G}^N$ and $W = X_t$. ☐

Lemma 4.2. *Let X be a compact Hausdorff topological space, $R \subseteq X$ dense and $W : R \to R$ a bijective map. Suppose that for all $g \in C(X)$ the map $g \circ W$ is the restriction to R of a unique $\tilde{g} \in C(X)$ and the same is true for $g \circ W^{-1}$. Then W can be extended to a homeomorphism $\underline{W} : X \to X$.*

Proof. Let $\omega \in X$. Any limit point $\bar{\omega}$ of $W(z)$ as $z \in X$ converges to ω must satisfy $\tilde{g}(\omega) = g(\bar{\omega})$ for any $g \in C(X)$. Since $C(X)$ separates the points of X (because X is a normal space, see [34]), we obtain that the limit $\underline{W}(\omega)$ of $W(z)$ as $z \in X \to \omega$ exists and satisfies

$$g(\underline{W}(\omega)) = \tilde{g}(\omega) \qquad \text{for all } g \in C(X). \tag{4.6}$$

The identity (4.6) implies that $\underline{W} : X \to X$ is continuous. Indeed, if $\omega_d \to \omega \in X$, we have

$$g(\underline{W}(\omega_d)) = \tilde{g}(\omega_d) \to \tilde{g}(\omega) = g(\underline{W}(\omega)) \tag{4.7}$$

and, again, this yields that any limit point $\bar{\omega}$ of $\underline{W}(\omega_d)$ must satisfy $g(\bar{\omega}) = g(\underline{W}(\omega))$ for any $g \in C(X)$, and therefore coincide with $\underline{W}(\omega)$.

A similar reasoning gives that W^{-1} has a continuous extension $\underline{W^{-1}}$ as well, and it is easy to check that $\underline{W}$ is the inverse of $\underline{W^{-1}}$. ☐

Let $\mathcal{S}$ be the closed subspace of $\mathrm{BAP}^2(\mathbb{R}^N)$ defined as follows. Let us consider the equation

$$\nabla \cdot (a(z)v(z)) = 0. \tag{4.8}$$

We define a class of asymptotic solutions of (4.8) as follows:

$$\mathcal{S} := \left\{ v \in \mathrm{BAP}^2(\mathbb{R}^N) \ : \ \fint_{\mathbb{R}^N} v(z)a(z) \cdot \nabla\varphi(z)\, dz = 0 \quad \text{for all } \varphi \in \mathrm{AP}(\mathbb{R}^N) \text{ with } \nabla\varphi \in \mathrm{AP}(\mathbb{R}^N;\mathbb{R}^N) \right\} \tag{4.9}$$

and its subspace

$$\mathcal{S}^* := \left\{ v \in \mathrm{AP}(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N) :\ a \cdot \nabla v = 0 \quad \text{a.e.} \right\}. \tag{4.10}$$

Equivalently, recalling the canonical isomorphism between $\mathrm{BAP}^2(\mathbb{R}^N)$ and $L^2(\mathbb{G}^N)$, when v is viewed as a function in $L^2(\mathbb{G}^N)$, we can say that $v \in \mathcal{S}$ if

$$\int_{\mathbb{G}^N} v(\omega)\underline{a}(\omega) \cdot \underline{\nabla\varphi}(\omega)\, d\omega = 0 \qquad \text{for all } \varphi \in \mathrm{AP}(\mathbb{R}^N) \text{ with } \nabla\varphi \in \mathrm{AP}(\mathbb{R}^N;\mathbb{R}^N). \tag{4.11}$$

Given $g \in \mathrm{BAP}^2(\mathbb{R}^N)$, we denote by $\tilde{g} \in \mathcal{S}$ its orthogonal projection on $\mathcal{S}$. Accordingly, we denote by $\tilde{a}$ the vector field whose components $\tilde{a}_i$ are the projections on $\mathcal{S}$ of a_i.

By the properties of orthogonal projections, $\tilde{g}$ is characterized by

$$\int_{\mathbb{G}^N} gh\, d\omega = \int_{\mathbb{G}^N} \tilde{g}h\, dz, \qquad g \in L^2(\mathbb{G}^N),\ h \in \mathcal{S}. \tag{4.12}$$

Moreover, the Mean Ergodic Theorem (see [18], Theorem VIII.7.1), which is applicable due to (4.3) and to the fact that $\mathcal{S}$ is the invariant space of $\boldsymbol{X}_t$ (see Proposition 4.1 below) implies that for $g \in \mathrm{BAP}^2(\mathbb{R}^N)$

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \boldsymbol{X}_s g(z)\,ds = \tilde{g}(z) \qquad \forall g \in \mathrm{BAP}^2(\mathbb{R}^N), \tag{4.13}$$

in the sense of convergence in $\mathrm{BAP}^2(\mathbb{R}^N)$, and one can use this formula to link in a more explicit way $\tilde{g}$ to g.

We summarize some properties of $\mathcal{S}$ in the following proposition.

Proposition 4.1 (Characterization of $\mathcal{S}$)**.** *$\mathcal{S}$ is the invariant subspace under $\boldsymbol{X}_t$. Moreover, if $\mathcal{S}^*$ is dense in $\mathcal{S}$, then $\mathcal{S}\cap L^\infty(\mathbb{G}^N)$ is an algebra and*

$$\widetilde{gr} = g\tilde{r} \qquad \forall g \in \mathcal{S}\cap L^\infty(\mathbb{G}^N),\ r \in L^2(\mathbb{G}^N). \tag{4.14}$$

Proof. If v is invariant, for any almost periodic φ with an almost periodic gradient we have

$$0 = \int_{\mathbb{G}^N} (v\circ \underline{X}_t - v)\underline{\varphi}\,d\omega = \int_{\mathbb{G}^N} v(\underline{\varphi}\circ \underline{X}_{-t} - \underline{\varphi})\,d\omega.$$

Dividing both sides by t and letting $t \to 0$ we obtain (4.11). The converse implication is analogous: one can check that (4.11) implies that the time derivative of $t \mapsto \int_{\mathbb{G}^N} v\circ \underline{X}_t\underline{\varphi}\,d\omega$ vanishes identically.

Let now $g \in \mathcal{S}^*$ and $r \in \mathcal{S}$. Since $\underline{a}\cdot\underline{\nabla g} = 0$, by integrating the identity $rg\underline{a}\cdot\underline{\nabla\varphi} + r\varphi\underline{a}\cdot\underline{\nabla g} = r\underline{a}\cdot\underline{\nabla(\varphi g)}$ we obtain that $gr \in \mathcal{S}$. The density of $\mathcal{S}^*$ immediately gives that $\mathcal{S}\cap L^\infty(\mathbb{G}^N)$ is an algebra.

Finally, let g, r be as in (4.14). For any $h \in \mathcal{S}$ we have

$$\int_{\mathbb{G}^N} h\widetilde{gr}\,d\omega = \int_{\mathbb{G}^N} h(gr)\,d\omega = \int_{\mathbb{G}^N} hg\tilde{r}\,d\omega$$

because $hg \in \mathcal{S}$. Since $g\tilde{r} \in \mathcal{S}$ and $h \in \mathcal{S}$ is arbitrary this proves that $\widetilde{gr} = g\tilde{r}$. □

We conclude this section with some remarks on the density of $\mathcal{S}^*$ in $\mathcal{S}$. We still don't know whether this property holds in general, but we are able to give a positive answer in special cases.

Lemma 4.3. *Suppose that $W : \mathbb{R}^N_z \to \mathbb{R}^N_w$ is a bi-Lipschitz map satisfying $W(\mathrm{AP}(\mathbb{R}^N)) = \mathrm{AP}(\mathbb{R}^N)$ in the sense that, for all $g \in \mathrm{AP}(\mathbb{R}^N)$, $g\circ W \in \mathrm{AP}(\mathbb{R}^N)$ and $g\circ W^{-1} \in \mathrm{AP}(\mathbb{R}^N)$. Let $J = |\frac{\partial W}{\partial z}|$ and assume that $J \in \mathrm{AP}(\mathbb{R}^N)$ and $\kappa \le J \le \mathcal{K}$ for certain constants $0 < \kappa \le \mathcal{K}$. Assume that the vector field $a(z)$ satisfies*

$$\frac{1}{J(z)}a(z)\cdot\nabla_z(\varphi\circ W) = \left(\frac{\partial\varphi}{\partial w_N}\right)\circ W \qquad \forall\varphi \in C^1(\mathbb{R}^N_w).$$

Then $\mathcal{S}^*$ is dense in $\mathcal{S}$.

Proof. Let $H = \mathrm{BAP}^2(\mathbb{R}^N)$. Arguing as in Lemma 4.1 and using the fact that

$$[-L/C, L/C]^N \subseteq W([-L,L]^N) \subseteq [-LC, LC]^N$$

for a suitable constant C, it is immediate to check that $g\circ W \in H$ if and only if $g \in H$. Now, we consider on H the bilinear form

$$\begin{aligned} (f|g) &:= \lim_{L\to\infty}\frac{1}{(2L)^N}\int_{W([-L,L]^N)} f(w)g(w)\,dw \\ &= \lim_{L\to\infty}\frac{1}{(2L)^N}\int_{[-L,L]^N} f(W(z))g(W(z))J\,dz. \end{aligned} \tag{4.15}$$

The second equation in (4.15) shows that the limit in the definition of $(\cdot|\cdot)$ exists, due to the almost periodicity of J. Clearly, $(f|g)$ is also an inner product in H and, using the bounds on J, it is immediate to see that the norm $|||\cdot|||$ defined by by $|||f||| = (f|f)^{1/2}$ is equivalent to the norm induced by the usual inner product of H. Let us denote by $\tilde{H}$, the Hilbert space formed with the same elements of H, endowed with the inner product $(\cdot|\cdot)$, defined by (4.15). Using the Gauss-Green formula and the fact that the $(N-1)$-dimensional measure of the boundary of $W([-L,L]^N)$ goes like L^{N-1} as $L\to\infty$, it is easy to verify that $\{\cos\lambda\cdot w,\ \sin\lambda\cdot w,\ \lambda\in\mathbb{R}^N\}$ is also an orthogonal system of functions spanning $\tilde H$. Let $\tilde v(w)\in\tilde H$ satisfy

$$(\tilde v|\frac{\partial\varphi}{\partial w_N})=0 \quad \text{for all } \varphi\in \mathrm{AP}(\mathbb{R}^N) \text{ such that } \nabla\varphi\in\mathrm{AP}(\mathbb{R}^N). \tag{4.16}$$

Then we have

$$(\tilde v|\cos\lambda\cdot w)=(\tilde v|\sin\lambda\cdot w)=0 \quad \text{for all } \lambda\in\mathbb{R}^N \text{ such that } \lambda_N\neq 0. \tag{4.17}$$

Conversely, by the density of trigonometric polynomials one can immediately check that if $\tilde v\in\tilde H$ satisfies (4.17) then $\tilde v$ satisfies (4.16). Indeed, any function in $\mathrm{AP}(\mathbb{R}^N)$ which may be written as $\frac{\partial\varphi}{\partial w_N}$ for some $\varphi\in\mathrm{AP}(\mathbb{R}^N)$ with N-th partial derivative in $\mathrm{AP}(\mathbb{R}^N)$ is orthogonal to all function $\sin\lambda\cdot w$, $\cos\lambda\cdot w$ with $\lambda_N=0$, and so it can be uniformly approximated by finite combinations of functions in $\{\cos\lambda\cdot w,\ \sin\lambda\cdot w,\ \lambda\in\mathbb{R}^N,\ \lambda_N\neq 0\}$. In particular, the subspace $\mathcal{S}_W$ of the functions $\tilde v\in\tilde H$ satisfying (4.16) is spanned by $\mathcal{F}_W:=\{\cos\lambda\cdot w,\ \sin\lambda\cdot w,\ \lambda\in\mathbb{R}^N,\ \lambda_N=0\}$.

Now, by assumption, the map $g\mapsto g\circ W$ establishes a one-to-one correspondence between $\mathcal{S}_W$ and $\mathcal{S}$. Since $\mathcal{S}_W$ is spanned by $\mathcal{F}_W$ and the maps $\varphi\circ W$ with $\varphi\in\mathcal{F}_W$ are classical solutions of (4.8), we have that classical solutions are dense in $\mathcal{S}$ as desired. ☐

Notice that the last assumption of the lemma can be written as

$$a\cdot\nabla W^i=0 \quad (1\le i\le N-1), \qquad a\cdot\nabla W^N=J. \tag{4.18}$$

Before continuing our general discussion, we analyse the special two-dimensional case where $a(z_1,z_2)=(G'(z_2)+\gamma,1)$, with G, $G'\in\mathrm{AP}(\mathbb{R})$ and $\gamma\in\mathbb{R}$. In this case the map $W:\mathbb{R}^2\to\mathbb{R}^2$ defined by

$$W(z_1,z_2)=(z_1-G(z_2)+\gamma z_2,z_2),$$

is a bi-Lipschitz diffeomorphism with

$$W^{-1}(w_1,w_2)=(w_1+G(w_2)-\gamma w_2,w_2),$$

satisfying the hypotheses of Lemma 4.3 (see also Remark 4.1).

It is also easy to check that the flow $X_t:\mathbb{R}^2\to\mathbb{R}^2$ associated to this particular vector field a is given by

$$X_t(z_1,z_2)=(z_1+G(z_2+t)-G(z_2)+\gamma t,z_2+t),$$

hence $\mathcal{S}^*$ is dense in $\mathcal{S}$.

The general result for the two-dimensional case, $N=2$, is as follows.

Lemma 4.4. *Let $a(z)=(a_1(z),a_2(z))$ be a vector field in $\mathbb{R}^2$ satisfying the conditions:*
- (i) *$a\in\mathrm{AP}(\mathbb{R}^2)\cap W^{1,\infty}(\mathbb{R}^2)$ and $\nabla_z\cdot a=0$;*
- (ii) *there exist constants γ_1, γ_2 and functions $A_1\in\mathrm{AP}(\mathbb{R})$ and $A_2\in\mathrm{AP}(\mathbb{R}^2)$ such that*
$$a_2(z)=\frac{\partial A_2}{\partial z_1}(z)+\gamma_2 \qquad \textit{and} \qquad a_1(0,s)=\frac{dA_1}{ds}+\gamma_1;$$
- (iii) *there exists a function $\psi\in C^1(\mathbb{R}^2)$ of the form $\psi(z)=c_1z_1+c_2z_2+g(z)$, with c_1, $c_2\in\mathbb{R}$, $g\in\mathrm{AP}(\mathbb{R}^2)\cap W^{1,\infty}(\mathbb{R}^2)$, such that $\nabla\psi\cdot a\ge\kappa>0$ for some $\kappa>0$.*

Then, the hypotheses of Lemma 4.3 are satisfied and, in particular, $\mathcal{S}^*$ is dense in $\mathcal{S}$.

Proof. We must exhibit a diffeomorphism $W:\mathbb{R}^2\to\mathbb{R}^2$ with the properties stated in Lemma 4.3. We define $w_2(z)=\psi(z)$ and

$$w_1(z)=\int_0^{z_1}a_2(r,z_2)\,dr-\int_0^{z_2}a_1(0,s)\,ds=A_2(z)-A_2(0,z_2)-\gamma_2z_1-A_1(z_2)+A_1(0)+\gamma_1z_2,$$

and set $W(z)=(w_1(z),w_2(z))$. We observe that the Jacobian matrix of W is given by

$$DW(z)=\begin{bmatrix} a_2(z) & -a_1(z) \\ \frac{\partial\psi}{\partial z_1}(z) & \frac{\partial\psi}{\partial z_2}(z)\end{bmatrix}.$$

Hence, the Jacobian determinant is $J(z)=\nabla\psi(z)\cdot a(z)>\kappa>0$ and the inverse of DW is given by

$$DW(z)^{-1}=\frac{1}{J}\begin{bmatrix} \frac{\partial\psi}{\partial z_2}(z) & a_1(z) \\ \frac{-\partial\psi}{\partial z_1}(z) & a_2(z)\end{bmatrix}.$$

In particular, we have that $DW(z)$ and $DW(z)^{-1}$ are uniformly bounded in $\mathbb{R}^2$. Hence, from a classical theorem of Hadamard, it follows that W is a bi-Lipschitz diffeomorphism of $\mathbb{R}^2$ and $\frac{\partial}{\partial w_2}=J^{-1}a\cdot\nabla$ (see (4.18)). By Remark 4.1 we obtain that $g\circ W\in \mathrm{AP}(\mathbb{R}^2)$ and $g\circ W^{-1}\in\mathrm{AP}(\mathbb{R}^2)$ for any $g\in\mathrm{AP}(\mathbb{R}^2)$, and this concludes the proof. □

5. Application to Nonlinear Transport equations

In this section we study the homogenization problem for a nonlinear transport equation with an incompressible and autonomous velocity field. More specifically, let $a\in\mathrm{AP}\cap W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)$, and let us assume that a is incompressible, i.e.

$$\nabla_z\cdot a(z)=0. \tag{5.1}$$

We consider the equation

$$\partial_t u^\varepsilon+\nabla_x\cdot(a(\frac{x}{\varepsilon})f(u^\varepsilon))=0,\qquad t>0,\ x\in\mathbb{R}^N, \tag{5.2}$$

with $f\in C^1(\mathbb{R})$, and the initial data given by

$$u^\varepsilon(x,0)=U_0(\frac{x}{\varepsilon},x), \tag{5.3}$$

where $U_0(z,x)\in L^1_{loc}(\mathbb{R}^N;\mathrm{AP}(\mathbb{R}^N))$. For each $\omega\in\mathbb{G}^N$, we also consider the auxiliary initial value problem given by

$$U_t+\nabla_x\cdot(\tilde a(\omega)f(U))=0,\qquad t>0,\ x\in\mathbb{R}^N, \tag{5.4}$$

and the initial data

$$U(\omega,x,0)=U_0(\omega,x),\qquad x\in\mathbb{R}^N. \tag{5.5}$$

We keep the notation introduced in Section 4. The stability properties of entropy solutions to scalar conservation laws show (see the argument in Lemma 5.1 below) that, possibly modifying $\tilde a$ in a negligible set, U may be viewed as a Borel map from $\mathbb{G}^N$ into $L^1_{\rm loc}(\mathbb{R}^{N+1}_+)$, where $\mathbb{R}^{N+1}_+=\mathbb{R}^N\times(0,+\infty))$. Using this fact, one can for instance find a Borel function $\bar U$, setting

$$\bar U(\omega,x,t):=\liminf_{\varepsilon\to0}U(\omega,\cdot,t)*\rho_\varepsilon(x).$$

Hence, in the following we can assume with no loss of generality that U is a Borel map.

We will need the following theorem which provides a comparison principle between two parametrized families of measures satisfying a first-order differential inequality in conservation form, which extends a theorem of DiPerna [21].

Theorem 5.1. *Let* $\{\mu_{x,t}^i\}$, $(x,t) \in \mathbb{R}_+^{N+1}$, $i = 1,2$, *be two weakly measurable parametrized families of probability measures over a compact separable metric space* K. *Let* $\{\mu_{x,0}^i\}_{x\in\mathbb{R}^N}$, $i = 1,2$, *be two parametrized families of probability measures over* K *satisfying*

$$\begin{aligned} &\lim_{t\to 0}\frac{1}{t}\int_0^t\int_{\mathbb{R}^N}\langle\mu_{x,s}^1, g\rangle\phi(x)\,dx\,ds = \int_{\mathbb{R}^N}\langle\mu_{x,0}^1, g\rangle\phi(x)\,dx,\\ &\lim_{t\to 0}\int_{\{|x|<R\}}|\langle\mu_{x,s}^2, g\rangle - \langle\mu_{x,0}^2, g\rangle|\,dx = 0,\end{aligned}\tag{5.6}$$

for all $g \in C(K)$, $\phi \in C_c(\mathbb{R}^N)$ *and* $R > 0$. *Let* $I : K\times K \to \mathbb{R}$, $G : K \times K \to \mathbb{R}^N$ *be continuous functions with* $I \ge 0$ *and* $|G(\rho,\lambda)| \le C\,I(\rho,\lambda)$, *for some* $C > 0$. *Assume*

$$\begin{aligned} &\partial_t\langle\mu_{x,t}^1, I(\cdot,\lambda)\rangle + \nabla_x\cdot\langle\mu_{x,t}^1, G(\cdot,\lambda)\rangle \le 0,\\ &\partial_t\langle\mu_{x,t}^2, I(\rho,\cdot)\rangle + \nabla_x\cdot\langle\mu_{x,t}^2, G(\rho,\cdot)\rangle \le 0,\end{aligned}\tag{5.7}$$

in the sense of the distributions in $\mathbb{R}_+^{N+1}$. *Then, for a.e.* $t > 0$, *we have*

$$\int_{\{|x|<R\}}\langle\mu_{x,t}^1\otimes\mu_{x,t}^2, I(\cdot,\cdot)\rangle\,dx \le \int_{\{|x|<R+Ct\}}\langle\mu_{x,0}^1\otimes\mu_{x,0}^2, I(\cdot,\cdot)\rangle\,dx.\tag{5.8}$$

Proof. The proof follows as in [21] by using Kruzhkov's doubling variable method [36]. We sketch the proof as follows. From the first inequality in (5.7) we have

$$\int_{\mathbb{R}^{N+1}}\langle\mu_{x,t}^1, I(\cdot,\lambda)\rangle\xi_t(x,t;y,\tau) + \langle\mu_{x,t}^1, G(\cdot,\lambda)\rangle\cdot\nabla_x\xi(x,t;y,\tau)\,dx\,dt \ge 0,\quad \text{for all } (y,\tau)\in\mathbb{R}_+^{N+1},\tag{5.9}$$

for all $\xi \in C_c^\infty(\mathbb{R}_+^{N+1}\times\mathbb{R}_+^{N+1})$. We integrate in the variable $\lambda \in K$ by the measure $\mu_{y,\tau}^2$ and then integrate in $(y,\tau) \in \mathbb{R}_+^{N+1}$ to obtain

$$\int_{\mathbb{R}^{N+1}}\int_{\mathbb{R}_+^{N+1}}\langle\mu_{x,t}^1\otimes\mu_{y,\tau}^2, I(\cdot,\cdot)\rangle\xi_t + \langle\mu_{x,t}^1\otimes\mu_{y,\tau}^2, G(\cdot,\cdot)\rangle\cdot\nabla_x\xi\;dx\,dt\,dy\,d\tau \ge 0.\tag{5.10}$$

Analogously, starting with the second inequality in (5.7) and proceeding similarly, exchanging the roles of $\mu_{x,t}^1$ and $\mu_{y,\tau}^2$, we obtain

$$\int_{\mathbb{R}^{N+1}}\int_{\mathbb{R}_+^{N+1}}\langle\mu_{x,t}^1\otimes\mu_{y,\tau}^2, I(\cdot,\cdot)\rangle\xi_\tau + \langle\mu_{x,t}^1\otimes\mu_{y,\tau}^2, G(\cdot,\cdot)\rangle\cdot\nabla_y\xi\;dx\,dt\,dy\,d\tau \ge 0.\tag{5.11}$$

Adding (5.10) and (5.11), we obtain

$$\int_{\mathbb{R}^{N+1}}\int_{\mathbb{R}_+^{N+1}}\langle\mu_{x,t}^1\otimes\mu_{y,\tau}^2, I(\cdot,\cdot)\rangle(\xi_t+\xi_\tau) + \langle\mu_{x,t}^1\otimes\mu_{y,\tau}^2, G(\cdot,\cdot)\rangle\cdot(\nabla_x\xi+\nabla_y\xi)\;dx\,dt\,dy\,d\tau \ge 0.\tag{5.12}$$

From (5.12) we deduce

$$\int_{\mathbb{R}^{N+1}}\int_{\mathbb{R}_+^{N+1}}\langle\mu_{x,t}^1\otimes\mu_{x,t}^2, I(\cdot,\cdot)\rangle\phi_t + \langle\mu_{x,t}^1\otimes\mu_{x,t}^2, G(\cdot,\cdot)\rangle\cdot\nabla_x\phi\;dx\,dt \ge 0,\tag{5.13}$$

for all $\phi \in C_c^\infty(\mathbb{R}_+^{N+1})$, by taking

$$\xi(x,t;y,\tau) = \phi(\frac{x+y}{2},\frac{t+\tau}{2})\delta_n(\frac{x_1-y_1}{2})\cdots\delta_n(\frac{x_N-y_N}{2})\delta_n(\frac{t-\tau}{2}),$$

where $\delta_n(s) = n\delta(ns)$, $n \in \mathbb{N}$, with $\delta \in C_c^\infty(\mathbb{R})$, $\delta(-s) = \delta(s)$, $\delta \ge 0$, $\int_{\mathbb{R}}\delta(s)\,ds = 1$, and then making $n\to\infty$. Now, (5.13) implies that $\partial_t\langle\mu_{x,t}^1\otimes\mu_{x,t}^2, I(\cdot,\cdot)\rangle + \nabla_x\cdot\langle\mu_{x,t}^1\otimes\mu_{x,t}^2, G(\cdot,\cdot)\rangle$ is a (signed) Radon measure over $\mathbb{R}_+^{N+1}$, by a well known lemma of Schwartz [47]. We then integrate it over the set

$$\mathcal{K} = \{(x,t)\in\mathbb{R}_+^{N+1} \,:\, |x| < R + C(t-\tau),\ 0<\tau<t\},$$

use Gauss-Green formula and the properties of the normal traces of divergence-measure fields (see, e.g., [15]), to deduce that (5.8) holds for a.e. $t>0$, upon using (5.6) and the assumption $|G|\le CI$. More specifically, we consider $t>0$ which is Lebesgue point of the functions

$$\tau \mapsto \int_{\mathbb{R}^N} \langle \mu^1_{x,\tau}\otimes \mu_{x,\tau}, I(\cdot,\cdot)\rangle \phi(x)\,dx,$$

for all $\phi\in C_c(\mathbb{R}^N)$, a property verified for a.e. $t>0$. For those $t>0$, the normal trace on $\partial\mathcal{K}$ restricted to $\partial\mathcal{K}\cap\{\tau=t\}$ coincides with $\langle \mu^1_{x,t}\otimes\mu^2_{x,t}, I(\cdot,\cdot)\rangle$. On the other hand, the normal trace restricted to $\{|x|=R+c(t-\tau),\ 0<\tau<t\}$ is nonnegative due to $|G|\le C\,I$. Finally, (5.6) guarantees that the normal trace on $\partial\mathcal{K}\cap\{\tau=0\}$ coincides with $\langle\mu^1_{x,0}\otimes\mu^2_{x,0}, I(\cdot,\cdot)\rangle$, which concludes the proof.

□

In the following theorem we extend to the context of almost periodic velocity fields and initial data a result of W. E (*cf.* [22]), relative to the periodic case. We characterize the weak limit of u^ε and, under suitable additional regularity assumptions on U, we prove a strong correctors formula.

Theorem 5.2. *Let $a\in W^{1,\infty}\cap AP(\mathbb{R}^N;\mathbb{R}^N)$ and $U_0\in L^1_{loc}(\mathbb{R}^N;\mathrm{AP}(\mathbb{R}^N))$. Let $\{u^\varepsilon\}_{\varepsilon>0}$ be the sequence of entropy solutions of (5.2), (5.3). Assume that $f'(u)\neq 0$ for all $u\in\mathbb{R}$, that U_0 is bounded and satisfies*

$$U_0(\cdot,x)\in\mathcal{S} \text{ for a.e. } x\in\mathbb{R}^N, \text{ with } \mathcal{S} \text{ defined in (4.9)} \tag{5.14}$$

and finally that the set $\mathcal{S}^*$ defined in (4.10) is dense in $\mathcal{S}$.
Then u^ε weakly star converge in $L^\infty(\mathbb{R}^N\times(0,+\infty))$ to

$$u(x,t):=\int_{\mathbb{G}^N} U(\omega,x,t)\,d\omega, \tag{5.15}$$

where U is the solution of (5.4), (5.5). Suppose further that

$$\text{either } U\in L^1_{loc}(\mathbb{R}^N\times[0,T];C(\mathbb{G}^N)) \text{ or } U\in\bigcap_{R>0} L^2(\mathbb{G}^N;C(\overline{B}_R(0)\times[0,T])) \tag{5.16}$$

for some $T>0$. Then

$$\lim_{\varepsilon\to0} u^\varepsilon(x,t)-U(\frac{x}{\varepsilon},x,t)=0 \quad \text{in } L^1_{loc}(\mathbb{R}^N\times[0,T]). \tag{5.17}$$

Proof. We first observe that the entropy solutions u^ε of (5.2), (5.3) are uniformly bounded in $L^\infty(\Pi_T)$. Hence, taking into account Lemma 3.1, it suffices to show that any two-scale Young measure generated by a subnet of $\{u^\varepsilon\}_{\varepsilon>0}$ satisfies

$$\nu_{\omega,x,t}=\delta_{U(\omega,x,t)} \qquad \text{for a.e. } (\omega,x,t)\in\mathbb{G}^N\times\mathbb{R}^N\times(0,+\infty). \tag{5.18}$$

Let then $\nu_{\omega,x,t}$ be a two-scale Young measure generated by a subnet of $\{u^\varepsilon\}_{\varepsilon>0}$ which, for notational simplicity, we still denote by $\{u^\varepsilon\}$. For any nonnegative $\psi\in L^1(\mathbb{R}^N\times(0,+\infty))$ we set also

$$\sigma^\psi_\omega:=\int_{\mathbb{R}^N}\int_0^\infty \psi(x,t)\nu_{\omega,x,t}\,dxdt. \tag{5.19}$$

We use Theorem 5.1 to prove (5.18). So, let us consider the family of Kruzkhov's entropies

$$\eta(\lambda,k)=|\lambda-k|, \qquad q(\lambda,k)=\mathrm{sgn}(\lambda-k)(f(\lambda)-f(k)), \tag{5.20}$$

so that the entropy solution of (5.2) satisfies

$$\partial_t\eta(u^\varepsilon,k)+\nabla_x\cdot(a(\frac{x}{\varepsilon})q(u^\varepsilon,k))\le 0 \qquad \forall k\in\mathbb{R}, \tag{5.21}$$

in the sense of distributions: it means that for all nonnegative $\phi\in C_c^\infty(\mathbb{R}^{N+1})$ we have

$$\int_{\mathbb{R}^N}\int_0^\infty \{\eta(u^\varepsilon,k)\phi_t+q(u^\varepsilon,k)(a(\frac{x}{\varepsilon})\cdot\nabla_x\phi)\}\,dx\,dt+\int_{\mathbb{R}^N}\eta(U_0(\frac{x}{\varepsilon},x),k)\phi(x,0)\,dx\ge 0. \tag{5.22}$$

In (5.22) we take $\phi(x,t) = \varepsilon\varphi(\frac{x}{\varepsilon})\psi(x,t)$, where $\varphi \in \mathcal{B}\mathrm{AP}(\mathbb{R}^N)$ and $\psi \in C_c^\infty(\mathbb{R}^{N+1})$ are nonnegative, and let $\varepsilon \to 0$ to get

$$\int_{\mathbb{G}^N} \langle \sigma_\omega^\psi, q(\cdot,k)\rangle \underline{a}(\omega)\cdot \underline{\nabla\varphi}\, d\omega \geq 0. \tag{5.23}$$

By applying this inequality with $C \pm \varphi$, with $C = \|\varphi\|_\infty$, and using the arbitrariness of φ we get (recalling (4.11))

$$\omega \mapsto \langle \sigma_\omega^\psi, q(\cdot,k)\rangle \in \mathcal{S}. \tag{5.24}$$

As in [22], we now observe that equation (5.24) holds not only for functions of the type (5.20). Indeed, the same argument above works for any C^1 entropy-entropy flux pair $(\tilde\eta,\tilde q)$ with $\tilde\eta$ convex or concave. As a consequence, by approximation it holds for any Lipschitz entropy-entropy flux pair. Choosing $\tilde\eta$ such that $\tilde\eta' = \mathrm{sgn}(\cdot - k)/f'$ we obtain $\tilde q = \eta(\cdot,k)$, so that (5.24) gives

$$\omega \mapsto \langle \sigma_\omega^\psi, \eta(\cdot,k)\rangle \in \mathcal{S}. \tag{5.25}$$

By approximation (5.24) and (5.25) hold for any $\psi \in L^1(\mathbb{R}^N\times(0,+\infty))$.

Next, we take in (5.22) $\phi(x,t) = \varphi(\frac{x}{\varepsilon})\psi(x,t)$, where $\varphi\in\mathcal{S}^*$ and $\psi\in C_c^\infty(\mathbb{R}^{N+1})$ are nonnegative. Passing to the limit as $\varepsilon\to 0$ we get

$$\begin{aligned}\int_{\mathbb{R}^N}\int_0^\infty\int_{\mathbb{G}^N} \{\langle \nu_{\omega,x,t},\eta(\cdot,k)\rangle \underline{\varphi}\psi_t + \langle \nu_{\omega,x,t}, q(\cdot,k)\rangle \underline{\varphi}(\underline{a}\cdot\nabla_x\psi)\}\, d\omega\, dx\, dt&\\ +\int_{\mathbb{R}^N}\int_{\mathbb{G}^N} \eta(U_0(\omega,x),k)\underline{\varphi}(\omega)\psi(x,0)\, d\omega\, dx \geq 0.&\end{aligned} \tag{5.26}$$

By Proposition 4.1 and (5.24) the maps $\omega\mapsto \underline{\varphi}(\omega)\langle\sigma_\omega^{\partial_i\psi}, q(\cdot,k)\rangle$ belong to $\mathcal{S}$. Therefore, taking (4.12) into account, we can rewrite (5.26) as

$$\begin{aligned}\int_{\mathbb{R}^N}\int_0^\infty\int_{\mathbb{G}^N} \{\langle \nu_{\omega,x,t},\eta(\cdot,k)\rangle \underline{\varphi}\psi_t + \langle \nu_{\omega,x,t}, q(\cdot,k)\rangle \underline{\varphi}(\tilde a\cdot\nabla_x\psi)\}\, d\omega\, dx\, dt&\\ +\int_{\mathbb{R}^N}\int_{\mathbb{G}^N} \eta(U_0(\omega,x),k)\underline{\varphi}(\omega)\psi(x,0)\, d\omega\, dx \geq 0,&\end{aligned} \tag{5.27}$$

for all nonnegative $\varphi\in\mathcal{S}^*$ and all nonnegative $\psi\in C_c^\infty(\mathbb{R}^{N+1})$. But then, using also the fact that $\omega\mapsto \langle\sigma_\omega^{\psi_t},\eta(\cdot,k)\rangle$ (see (5.25)) and $\tilde a_i\langle\sigma_\omega^{\partial_i\psi}, q(\cdot,k)\rangle$ belong to $\mathcal{S}$ and assumption (5.14) on U_0, we obtain that (5.27) holds for all $\varphi\in L^2(\mathbb{G}^N)$ (here we use the density of $\mathcal{S}^*$ in $\mathcal{S}$). In particular, for each fixed nonnegative $\psi\in C_c^\infty(\mathbb{R}^{N+1})$, inequality (5.27) can be strengthened to an inequality a.e. on $\omega\in\mathbb{G}^N$. A density argument on the class of test functions ψ then gives that for a.e. $\omega\in\mathbb{G}^N$ the following property is fulfilled:

$$\begin{aligned}\int_{\mathbb{R}^N}\int_0^\infty \{\langle \nu_{\omega,x,t},\eta(\cdot,k)\rangle \psi_t + \langle \nu_{\omega,x,t}, q(\cdot,k)\rangle (\tilde a(\omega)\cdot\nabla_x\psi)\}\, dx\, dt&\\ +\int_{\mathbb{R}^N} \eta(U_0(\omega,x),k)\psi(x,0)\, dx \geq 0,&\end{aligned} \tag{5.28}$$

for all nonnegative $\psi\in C_c^\infty(\mathbb{R}^{N+1})$.

We are going to apply Theorem 5.1 to show that $\nu_{\omega,x,t}$ is a Dirac measure for almost every $(\omega,x,t)\in\mathbb{G}^N\times\mathbb{R}^N\times(0,+\infty)$. To do this, first we observe that (5.28) implies

$$\lim_{t\to 0}\frac{1}{t}\int_0^t\int_{\mathbb{R}^N}\langle\nu_{\omega,x,\tau},g\rangle\phi(x)\,dx\,d\tau = \int_{\mathbb{R}^N}\langle\delta_{U_0(\omega,x)},g\rangle\phi(x)\,dx, \tag{5.29}$$

for all $g \in C(\mathbb{R})$ and $\phi \in C_c(\mathbb{R}^N)$. Indeed, choosing $\psi(x,t) = \delta_h(t)\phi(x)$, with $\delta_h(t) = \max\{h^{-1}(h-t), 0\}$, for $t \ge 0$, $h > 0$, $\phi \in C_c^\infty(\mathbb{R}^N)$, $\phi \ge 0$, in (5.28), we obtain

$$\lim_{h\to 0} \frac{1}{h}\int_0^h \int_{\mathbb{R}^N} \langle \nu_{\omega,x,t}, |\cdot - k|\rangle \phi(x)\,dx\,dt \le \int_{\mathbb{R}^N} |U_0(\omega,x) - k|\phi(x)\,dx, \tag{5.30}$$

for all $\phi \in C_c^\infty(\mathbb{R}^N)$, $\phi \ge 0$, and a fortiori also for all nonnegative $\phi \in L^1(\mathbb{R}^N)$. Taking advantage of the flexibility given by the presence of $\phi \in L^1(\mathbb{R}^N)$ in (5.30), we may replace k by any function $k(x)$ in $L^\infty(\mathbb{R}^N)$, in particular, $k(x) = U_0(\omega,x)$. This proves (5.29).

Now, let $U(\omega,x,t)$ be the solution of (5.4), (5.5). The entropy condition states that

$$\partial_t \eta(\lambda, U) + \nabla_x \cdot (\tilde{a}(\omega) q(\lambda, U)) \le 0 \qquad \text{for all } \lambda \in \mathbb{R},\ \omega \in \mathbb{G}^N. \tag{5.31}$$

and

$$\lim_{t\to 0} \int_{\{|x|<R\}} |U(\omega,x,t) - U_0(\omega,x)|\,dx = 0, \qquad \text{for all } R > 0. \tag{5.32}$$

Therefore, we can apply Theorem 5.1 with $\mu^1_{x,t} = \nu_{\omega,x,t}$, $\mu^2_{x,t} = \delta_{U(\omega,x,t)}$, $I = \eta$ and $G = \tilde{a}(\omega)q$, for a.e. $\omega \in \mathbb{G}^N$. From this we easily deduce that $\nu_{\omega,x,t} = \delta_{U(\omega,x,t)}$, for a.e. $(\omega,x,t) \in \mathbb{G}^N \times \mathbb{R}^N \times [0,T]$.

To prove the weak convergence $u^\varepsilon \rightharpoonup u$, with $u(x,t)$ given by (5.15), we argue as follows. Let $U^\delta \in C(\mathbb{G}^N \times \mathbb{R}^N \times (0,+\infty))$ be bounded. Using (3.1) with test function

$$\Phi(\lambda,\omega,x,t) := |\lambda - U^\delta(\omega,x,t)|\psi(x,t)$$

with $\psi \in C_c(\mathbb{R}^N \times (0,+\infty))$ nonnegative, we obtain

$$\limsup_{\varepsilon\to 0} \int_{\mathbb{R}^N}\int_0^\infty \psi(x,t)|u^\varepsilon(x) - U^\delta(\frac{x}{\varepsilon},x,t)|\,dxdt = \int_{\mathbb{R}^N}\int_0^\infty\int_{\mathbb{G}^N} |U\psi - U^\delta\psi|\,d\omega dx dt.$$

On the other hand, the continuity of U_δ gives

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^N}\int_0^\infty U^\delta(\frac{x}{\varepsilon},x,t)\psi(x,t)\,dxdt = \int_{\mathbb{R}^N}\int_0^\infty\int_{\mathbb{G}^N} U^\delta(\omega,x,t)\,d\omega\psi(x,t)\,dxdt.$$

Hence, combining the previous two formulas, we get

$$\limsup_{\varepsilon\to 0}\left|\int_{\mathbb{R}^N}\int_0^\infty u^\varepsilon(x)\psi(x,t) - \bar{U}^\delta(x,t)\psi(x,t)\,dxdt\right| \le \|U^\delta\psi - U\psi\|_{L^1},$$

with $\bar{U}^\delta(x,t) := \int_{\mathbb{G}^N} U^\delta(\omega,x,t)\,d\omega$. By a density argument we obtain the weak star convergence of u^ε to $\lim_\delta \bar{U}^\delta$, i.e. $\int_{\mathbb{G}^N} U(\omega,x,t)\,d\omega$. Finally the fact that $u^\varepsilon(x,t) - U(\frac{x}{\varepsilon},x,t) \to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N \times [0,T])$ as $\varepsilon \to 0$, under assumption (5.16), follows directly by Lemma 3.1. □

Remark 5.1 (Convergence of translates). We can easily show that the same convergence statement, with the same limit u, holds for the solutions u^ε_ω associated to the vectorfields $a_\omega(z) := \underline{a}(z+\omega)$ with the initial data $U_{0\omega}(z,x) := U_0(\omega+z,x)$, for any $\omega \in \mathbb{G}^N$. We may obtain these translates by uniform continuity as functions in $\mathrm{AP} \cap W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)$ and $\mathrm{AP}(\mathbb{R}^N; L^1_{loc}(\mathbb{R}^N))$, respectively. Let $\mathcal{S}_\omega$ be defined as $\mathcal{S}$ with $a(z)$ replaced by $a(\omega+z)$ and let Π_ω be the orthogonal projection of $\mathrm{BAP}^2(\mathbb{R}^N)$ onto $\mathcal{S}_\omega$. We have $\Pi_\omega(v(\omega+\cdot)) = \Pi(v)(\omega+\cdot)$ for all $v \in \mathrm{AP}(\mathbb{R}^N)$. Indeed, $w(z) = \Pi(v)(\omega+z)$ satisfies $\nabla\cdot a(\omega+z)w(z) = 0$, in the sense of distributions, that is, $w \in \mathcal{S}_\omega$ and for any $w' \in \mathcal{S}_\omega$ we have

$$⨍_{\mathbb{R}^N} (v(\omega+z) - w(z))w'(z)\,dz = \int_{\mathbb{G}} v(\omega+z)w'(z)\,dz - \int_{\mathbb{G}} \Pi(v)(z)w'(z-\omega)\,dz = 0,$$

since $w'(\cdot-\omega) \in \mathcal{S}$. In particular, $\Pi(a(\omega+\cdot)) = \tilde{a}(\omega+\cdot)$ for all $\omega \in \mathbb{G}^N$. The fact that $u^\varepsilon_\omega(x,t) - U(\omega + \frac{x}{\varepsilon},x,t) \to 0$ in $L^1_{loc}(\mathbb{R}^N \times [0,T])$ is then proved as above. In particular, $u^\varepsilon_\omega(x,t)$ converges weakly star in

$L^\infty(\mathbb{R}^N \times [0,T])$ to

$$w - \lim_{\varepsilon\to 0} U(\omega + \frac{x}{\varepsilon}, x, t) = \int_{\mathbb{G}} U(\omega + z, x, t)\, dz = \int_{\mathbb{G}} U(z, x, t)\, dz = \bar{U}(x,t).$$

Concerning (5.16), ensuring the existence of strong correctors, we observe that the first alternative is trivially satisfied if U_0 and $\tilde{a}$ are independent of z, in which case we may take any $T > 0$. A simple example is provided, for $N = 2$, by the incompressible vector field $a(z) = (g(z_2), \beta)$ with $g \in \mathrm{AP}(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ and $\beta \neq 0$. In this case $\tilde{a}(z) = (⨍ g, \beta)$, which follows easily from (4.13). The following lemma gives sufficient conditions for the verification of the second alternative in (5.16).

Lemma 5.1. *If the range of a is contained in a closed convex set $\mathcal{P}$, then $U \in L^2(\mathbb{G}^N; C(\overline{B}(0,R) \times [0,T]))$ for any $R > 0$, for any $T > 0$ such that the entropy solutions V_b of*

$$\partial_t V_b + \nabla_x \cdot (b f(V_b)) = 0, \qquad t > 0,\ x \in \mathbb{R}^N, \tag{5.33}$$

$$V_b(x,0) = U_0(z,x), \qquad x \in \mathbb{R}^N, \tag{5.34}$$

have locally uniformly bounded Lipschitz constant in $\mathbb{R}^N \times [0,T]$, with respect to $b \in \mathcal{P}$ and $z \in \mathbb{R}^N$.

Proof. By applying (4.13) we obtain that also the range of $\tilde{a}$ is contained in $\mathcal{P}$. We will prove that $U(z,x,t) \in L^2(\mathbb{G}; C(\overline{B}_R(0) \times [0,T]))$ for any $R > 0$. Since U is bounded we need only to check its measurability. This follows by the fact that for any $\delta > 0$ it is possible to find a compact $K_\delta \subseteq \mathbb{G}$ such that $U(z,x,t) \in C(K_\delta; C(\mathbb{R}^N \times [0,T]))$. Indeed, given $\delta > 0$ we may find K_δ such that the restriction of $\tilde{a}$ to $\mathcal{K}_\delta$ is continuous. Now, the stability properties of entropy solutions tell us that $\omega \mapsto U(\omega, \cdot, \cdot)$ is continuous from $K_\delta \subseteq \mathbb{G}^N$ into $L^1_{loc}(\mathbb{R}^N \times [0, +\infty))$. The local uniform Lipschitz bound then gives continuity with respect to the stronger topology. ☐

An example where Lemma 5.1 applies is provided by the case in which all the components of a are nonnegative, $f''(u) \geq 0$ for all $u \in \mathbb{R}$ and $\frac{\partial U_0}{\partial x_i}(z,x) \geq 0$ for all $(z,x) \in \mathbb{R}^N \times \mathbb{R}^N$, $i = 1, l \ldots, N$. In this case, if $b \in \mathcal{P} = [0,M]^N$ for some $M > 0$, then it is then well known that the entropy solution U_b of (5.33), (5.34) can be constructed by the method of characteristics in such a way that $U_b \in W^{1,\infty}_{loc}(\mathbb{R}^N \times [0,+\infty))$ if the initial datum is a Lipschitz function. We remark that, in general, entropy solutions are discontinuous.

6. Application to Homogenization of Conservation Laws with Oscillatory External Forces

We consider the homogenization problem for a scalar conservation law in several space variables with a one-space variable oscillatory external force:

$$\begin{cases} \partial_t u + \displaystyle\sum_{k=1}^N \partial_k f^k(u) = \frac{1}{\varepsilon} h(\frac{x_1}{\varepsilon}), & (x,t) \in \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(\frac{x_1}{\varepsilon}, x), & x \in \mathbb{R}^N. \end{cases} \tag{6.1}$$

Here we assume $h = V'$ with h, $V \in \mathrm{AP}(\mathbb{R})$ and $⨍ h = 0$. We assume with no loss of generality that also $⨍ V = 0$.

The following result extends to the almost periodic and multidimensional context a previous result of W. E and D. Serre [23].

Theorem 6.1. *Assume that $u_0(\cdot, x) \in \mathrm{AP}(\mathbb{R})$ for any $x \in \mathbb{R}^N$, that $\|u_0\|_\infty < +\infty$, that*

$$\partial_{z_1} f^1(u_0(z_1, x)) = h(z_1) \quad \text{for a.e. } x \in \mathbb{R}^N \tag{6.2}$$

and that the functions f^k in (6.1) are bi-Lipschitz and monotone. Denote $g = (f^1)^{-1}$ and let $\bar f^k$ be defined implicitly by

$$p = \fint_{\mathbb{R}} g(\bar f^1(p) + V(z_1))\,dz_1, \tag{6.3}$$

$$\bar f^k(p) := \fint_{\mathbb{R}} f^k \circ g(\bar f^1(p) + V(z_1))\,dz_1 \qquad 2 \le k \le N. \tag{6.4}$$

Let $\bar u(x,t)$ be the unique entropy solution of

$$\begin{cases} \partial_t \bar u + \sum_{k=1}^N \partial_k \bar f^k(\bar u) = 0, & (x,t) \in \mathbb{R}^d \times (0,\infty), \\ \\ \bar u(x,0) = \fint_{\mathbb{R}} u_0(z_1,x)\,dz_1 & x \in \mathbb{R}^N, \end{cases} \tag{6.5}$$

and set

$$U(z_1,x,t) = g(V(z_1) + \bar f^1(\bar u(x,t))). \tag{6.6}$$

Then, as $\varepsilon \to 0$, we have $u^\varepsilon \to \bar u$ in the weak star topology of $L^\infty(\mathbb{R}^N \times (0,+\infty))$ and

$$\|u^\varepsilon - U(\frac{x_1}{\varepsilon}, x, t)\|_{L^1_{loc}(\mathbb{R}^N \times [0,+\infty))} \to 0. \tag{6.7}$$

Proof. We can assume with no loss of generality that all functions f^k are increasing. First we observe that the solutions u^ε of (6.1) are bounded in $L^\infty(\mathbb{R}^N \times [0,+\infty))$ uniformly with respect to ε. Indeed, for any $\alpha \in \mathbb{R}$, let $\Psi_\alpha(y) = g(V(y) + \alpha)$, and notice that $\Psi_\alpha(x_1/\varepsilon)$ is a stationary solution of (6.1). Since u_0 is bounded and $g(s) \to \pm\infty$ as $s \to \pm\infty$, we have

$$g(V(x_1/\varepsilon) - C) \le u_0(x_1/\varepsilon, x) \le g(V(x_1/\varepsilon) + C),$$

for some constant C and so, by the monotonicity of the solution operator of (6.1), we get

$$g(V(x_1/\varepsilon) - C) \le u^\varepsilon(x,t) \le g(V(x_1/\varepsilon) + C).$$

In the sequel we denote by K a closed interval containing the image of all functions u^ε.

Let $\nu_{\omega,x,t} \in \mathcal{M}(K)$ be the two-scale space-time Young measure associated with a subnet of $\{u^\varepsilon\}_{\varepsilon>0}$ with test functions oscillating only on the first variable x_1; the theorem will be proved if we can show that $\nu_{\omega,x,t}$ is a Dirac measure for almost every $(\omega,x,t) \in \mathbb{G} \times \mathbb{R}^N \times (0,+\infty)$, where, in this section, $\mathbb{G}$ denotes the Bohr compactification of $\mathbb{R}$. This will be achieved, as in [23], by adapting DiPerna's method in [21], that is, with the application of Theorem 5.1.

Observe that, for every $\alpha \in \mathbb{R}$, the entropy solutions of (6.1) satisfy

$$|u^\varepsilon - \Psi_\alpha(x_1/\varepsilon)|_t + \sum_{k=1}^N |f^k(u^\varepsilon) - f^k(\Psi_\alpha(x_1/\varepsilon)))|_{x_k} \le 0 \tag{6.8}$$

because the monotonicity of f^k gives

$$\operatorname{sgn}(u^\varepsilon - \Psi_\alpha(x_1/\varepsilon))(f^k(u^\varepsilon) - f^k(\Psi_\alpha(x_1/\varepsilon))) = |f^k(u^\varepsilon) - f^k(\Psi_\alpha(x_1/\varepsilon))|.$$

Hence for any nonnegative $\phi \in C_c^\infty(\mathbb{R}^N \times \mathbb{R})$ we have

$$\int_{\mathbb{R}^N}\int_0^\infty \Big\{ |u^\varepsilon - \Psi_\alpha(x_1/\varepsilon)|\phi_t + |f^1(u^\varepsilon) - f^1(\Psi_\alpha(x_1/\varepsilon))|\phi_{x_1} \tag{6.9}$$
$$+ \sum_{k=2}^N |f^k(u^\varepsilon) - f^k(\Psi_\alpha(x_1/\varepsilon))|\phi_{x_k} \Big\}\,dx\,dt + \int_{\mathbb{R}^N} |u_0^\varepsilon - \Psi_\alpha(x_1/\varepsilon)|\phi(x,0)\,dx \ge 0.$$

Setting $\phi(x,t) = \varepsilon\varphi(x_1/\varepsilon)\psi(x,t)$, where $\varphi \in \mathrm{AP}(\mathbb{R})$ and $\psi \in C_c^\infty(\mathbb{R}^N \times (0,+\infty))$ are nonnegative, and letting $\varepsilon \to 0$, we get

$$\int_{\mathbb{R}^N}\int_0^\infty\int_{\mathbb{G}} \psi(x,t)\langle \nu_{\omega,x,t}, |f^1(\lambda) - f^1(\Psi_\alpha(\omega))|\rangle \underline{\varphi'}(\omega)\, dz dx dt \geq 0.$$

Now apply the inequality above to $C \pm \varphi$, with $C = \|\varphi\|_\infty$, to obtain

$$\int_{\mathbb{R}^N}\int_0^\infty\int_{\mathbb{G}} \psi(x,t)\langle \nu_{\omega,x,t}, |f^1(\lambda) - f^1(\Psi_\alpha(\omega))|\rangle \underline{\varphi'}(\omega)\, dz dx dt = 0. \tag{6.10}$$

As in [23], we define a new family of parameterized measures $\mu_{\omega,x,t}$ by

$$\langle \mu_{\omega,x,t}, \theta\rangle = \langle \nu_{\omega,x,t}, \theta(f^1(\lambda) - V(\omega))\rangle \qquad \text{for any } \theta \in C(\mathbb{R}),$$

so that $\mu_{\omega,x,t}$ are the image of $\nu_{\omega,x,t}$ under the map $\lambda \mapsto f^1(\lambda) - V(\omega)$ and are supported in a compact set K' containing all points $f^1(\lambda) - V(\omega)$ with $\lambda \in K$ and $\omega \in \mathbb{G}$. Equation (6.10) can also be rephrased as

$$\int_{\mathbb{R}^N}\int_0^\infty\int_{\mathbb{G}} \psi(x,t)\langle \mu_{\omega,x,t}, \theta\rangle\, \underline{\varphi'}(\omega) dz dx dt = 0 \tag{6.11}$$

where $\theta(\rho) = |\rho - \alpha|$, $\alpha \in \mathbb{R}$. On the other hand, using the same test function as above in the integral equation defining weak solution of (6.1), we get in a similar way that the same holds when θ is any affine function. Now, since any continuous function may be locally uniformly approximated in $\mathbb{R}$ by finite linear combinations of affine functions and functions of the form $|\rho - \alpha|$ (because these combinations generate the piecewise affine functions), we conclude that (6.11) holds for any continuous function θ. Then, we can apply Proposition 2.3 to obtain that $\mu_{\omega,x,t}$ is independent of ω in the following (weak) sense: for any ψ, θ as above the function

$$\omega \mapsto \int_{\mathbb{R}^N}\int_0^\infty \psi(x,t)\langle \mu_{\omega,x,t}, \theta\rangle\, dx dt$$

is equivalent to a constant (since $\mathbb{G}$ is not separable this does not imply in principle that $\omega \mapsto \mu_{\omega,x,t}$ is constant for a.e. (x,t)!). Using this fact, and defining

$$\mu_{x,t} := \int_{\mathbb{G}} \mu_{\omega,x,t}\, d\omega \in \mathcal{M}(K'),$$

we have, in particular,

$$\int_{\mathbb{R}^{N+1}_+} \psi(x,t)\langle \mu_{\omega,x,t}, \theta\rangle\, dx\, dt = \int_{\mathbb{R}^{N+1}_+} \psi(x,t)\langle \mu_{x,t}, \theta\rangle\, dx\, dt, \qquad \text{for a.e. } \omega \in \mathbb{G}.$$

Hence,

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_0^\infty \langle \mu_{x,t}, \int_{\mathbb{G}} w(\omega,\cdot)\, d\omega\rangle \psi(x,t)\, dx dt \\
&= \sum_i m(G_i)\int_{\mathbb{R}^N}\int_0^\infty \langle \mu_{x,t}, \theta_i\rangle \psi(x,t)\, dx dt = \sum_i \int_{\mathbb{R}^N}\int_0^\infty\int_{G_i} \langle \mu_{\omega,x,t}, \theta_i\rangle\, d\omega\, \psi(x,t)\, dx dt \\
&= \int_{\mathbb{R}^N}\int_0^\infty\int_{\mathbb{G}} \langle \mu_{\omega,x,t}, w(\omega,\cdot)\rangle \psi(x,t)\, d\omega\, dx dt
\end{aligned} \tag{6.12}$$

for any simple function $w = \sum_i \theta_i \chi_{G_i}$ from $\mathbb{G}$ to $C(K')$. By approximation (6.12) holds for any $w \in C(\mathbb{G} \times K')$.

Next, we take any nonnegative $\phi \in C_c^1(\mathbb{R}^{N+1})$ in (6.9) and take the limit as $\varepsilon \to 0$, passing to a subnet if necessary, to get

$$\int_{\mathbb{R}^N}\int_0^\infty \int_{\mathbb{G}} \Big\{ \langle \nu_{\omega,x,t}\,, |\lambda - \Psi_\alpha(\omega)|\rangle \phi_t + \langle \nu_{\omega,x,t}, |f^1(\lambda) - f^1(\Psi_\alpha(\omega))|\rangle \phi_{x_1} + \sum_{k=2}^N \big\langle \nu_{\omega,x,t}\,, |f^k(\lambda) - f^k(\Psi_\alpha(\omega))|\big\rangle\, \phi_{x_k} \Big\}\, d\omega\, dx\, dt + \int_{\mathbb{R}^N}\int_{\mathbb{G}} |u_0(\omega,x) - \Psi_\alpha(\omega)|\phi(x,0)\, d\omega\, dx \geq 0. \tag{6.13}$$

We extend $\nu_{\omega,t,x}$ and $\mu_{\omega,t,x}$ to negative times with the 0 value. Then, using the substitution formulas $\lambda = g(\rho + V(\omega))$, $\Psi_\alpha(z_1) = g(\alpha + V(\omega))$, and taking into account (6.12), we can rewrite (6.13) as

$$\partial_t \langle \mu_{x,t}, I(\cdot,\alpha)\rangle + \nabla_x \cdot \langle \mu_{x,t}, G(\cdot,\alpha)\rangle \leq \delta_\alpha \qquad \forall \alpha \in \mathbb{R} \tag{6.14}$$

in the sense of distributions in $\mathbb{R}^N \times \mathbb{R}$, with

$$I(\rho,\alpha) := \int_{\mathbb{G}} |g(\rho + V(\omega)) - g(\alpha + V(\omega)|\, d\omega, \tag{6.15}$$

$$G^k(\rho,\alpha) := \int_{\mathbb{G}} |m^k(\rho + V(\omega)) - m^k(\alpha + V(\omega)|\, d\omega, \quad (1 \leq k \leq N) \tag{6.16}$$

$$\delta_\alpha(\phi) := \int_{\mathbb{R}^N}\int_{\mathbb{G}} |u_0(\omega,x) - g(\alpha + V(\omega))|\phi(x,0)\, d\omega\, dx, \tag{6.17}$$

with $m^k = f^k \circ g$ (notice that $m^1(t) = t$). By (6.14) and the definition of δ_α we obtain

$$\operatorname{ess} - \limsup_{t\downarrow 0} \int_{\mathbb{R}^N} \chi(x)\langle \mu_{x,t}, I(\cdot,\alpha)\rangle\, dx \leq \int_{\mathbb{R}^N} \chi(x) \int_{\mathbb{G}} |u_0(\omega,x) - g(\alpha + V(\omega))|\, d\omega\, dx \tag{6.18}$$

for any nonnegative $\chi \in C_c^\infty(\mathbb{R}^N)$ and any $\alpha \in \mathbb{R}$, which easily extends to all nonnegative $\chi \in L^1(\mathbb{R}^N)$. Using the flexibility provided by $\chi \in L^1(\mathbb{R}^N)$ in (6.18), we deduce that the same inequality still holds if α is a bounded measurable function $\alpha(x)$; in particular, for $\alpha(x) = f^1(u_0(\omega,x)) - V(\omega) = \bar f^1(\bar u(x,0))$, where the last equality follows from (6.3) and (6.5). For this choice of $\alpha(x)$ we have $g(\alpha(x) + V(\omega)) = u_0(\omega,x)$ and so (6.18) implies

$$\lim_{t\to 0} \frac{1}{t}\int_0^t \int_{\mathbb{R}^N} \langle \mu_{x,\tau}, g\rangle \phi(x)\, dx\, d\tau = \int_{\mathbb{R}^N} \langle \delta_{\bar f^1(\bar u(x,0))}, g\rangle \phi(x)\, dx, \tag{6.19}$$

for all $g \in C(\mathbb{R})$ and $\phi \in C_c(\mathbb{R}^N)$.

The idea now is to apply Theorem 5.1 to show that $\nu_{\omega,x,t}$ is a Dirac measure for almost every (ω,x,t). Let $\bar u(x,t)$ be the entropy solution of (6.5). We have

$$\partial_t |\gamma - \bar u| + \sum_{k=1}^N \partial_{x_k} |\bar f^k(\gamma) - \bar f^k(\bar u)| \leq 0 \tag{6.20}$$

in the sense of distributions in $\mathbb{R}^N \times (0,+\infty)$ for all $\gamma \in \mathbb{R}$, and also

$$\lim_{t\to 0} \int_{\{|x|<R\}} |\bar u(x,t) - \bar u_0(x)|\, dx = 0, \qquad \text{for all } R > 0. \tag{6.21}$$

Let $\xi(x,t) = \bar f^1(\bar u(x,t))$, $\rho = \bar f^1(\gamma)$. Then the definitions of $\bar f^k$ give

$$\bar u(x,t) = \int_{\mathbb{G}} g(\xi(x,t) + V(\omega))\, d\omega, \qquad \gamma = \int_{\mathbb{G}} g(\rho + V(\omega))\, d\omega,$$

$$\bar f^k(\bar u(x,t)) = \int_{\mathbb{G}} m^k(\xi(x,t) + V(\omega))\, d\omega, \qquad \bar f^k(\gamma) = \int_{\mathbb{G}} m^k(\rho + V(\omega))\, d\omega.$$

Moreover, since g and all m^k are monotone, we have

$$\left|\int_{\mathbb{G}}(g(\rho+V(\omega))-g(\xi+V(\omega)))\,d\omega\right| = \int_{\mathbb{G}}|g(\rho+V(\omega))-g(\xi+V(\omega))|\,d\omega,$$

$$\left|\int_{\mathbb{G}}(m^k(\rho+V(\omega))-m^k(\xi+V(\omega)))\,d\omega\right| = \int_{\mathbb{G}}|m^k(\rho+V(\omega))-m^k(\xi+V(\omega))|\,d\omega.$$

Hence, we can write (6.20) as

$$\partial_t I(\rho,\xi(x,t))+\nabla\cdot G(\rho,\xi(x,t))\le 0 \qquad \forall\rho\in\mathbb{R} \tag{6.22}$$

in the distribution sense in $\mathbb{R}^{N+1}_+$. From (6.21) we also have

$$\lim_{t\to 0}\int_{\{|x|<R\}}|\xi(x,t)-\xi(x,0)|\,dx=0, \qquad \text{for all } R>0. \tag{6.23}$$

We can now apply Theorem 5.1 with $\mu^1_{x,t}=\mu_{x,t}$, $\mu^2_{x,t}=\delta_{\xi(x,t)}$, I and G as given by (6.15) and (6.16), to conclude that $\mu_{t,x}$ is the Dirac mass concentrated at $\xi(x,t)$ for a.e. (x,t). Recalling the definition of $\mu_{t,x}$ we have also that $\mu_{\omega,x,t}$ is the Dirac mass at $\xi(x,t)$ for a.e. (ω,t,x), and finally that $\nu_{\omega,t,x}$ is the Dirac mass concentrated at $g(V(\omega)+\bar f(\xi(x,t)))$ for a.e. (ω,t,x). Taking into account Lemma 3.1, this proves (6.7) and it remains only to show the weak convergence of u^ε to $\bar u$.

By (6.7) it suffices to show the convergence of $U(x_1/\varepsilon,x,t)$: for any $\phi\in C_0(\mathbb{R}^N\times(0,+\infty))$, we have

$$\begin{aligned}
\lim_{\varepsilon\to0}\int_{\mathbb{R}^N}\int_0^\infty U(\frac{x}{\varepsilon},x,t)\phi(x,t)\,dx\,dt &= \int_{\mathbb{R}^N}\int_0^\infty \phi(x,t)\left(\int_{\mathbb{G}}U(z,x,t)\,dz\right)dx\,dt\\
&= \int_{\mathbb{R}^N}\int_0^\infty \phi(x,t)\left(\int_{\mathbb{G}}g(V(z_1)+\bar f^1(\bar u(x,t)))\,dz_1\right)dx\,dt\\
&= \int_{\mathbb{R}^N}\int_0^\infty \phi(x,t)\bar u(x,t)\,dx\,dt,
\end{aligned}$$

by the definitions of $\bar f^1$ and U. □

Remark 6.1 (Convergence of translates). The convergence statement also applies to the solutions u^ε_ω of the PDE associated to the functions $h_\omega(z_1)=\underline{h}(\omega+z_1)$, $\omega\in\mathbb{G}$, with the initial condition $u_{0\omega}(z_1,x)=\underline{u(\cdot,x)}(\omega+z_1)$, that (by the averaging properties of almost periodic functions) induce the same initial datum $\bar u(x,0)$ and therefore the same solution $\bar u$ of the limit scalar conservation law. Hence

$$\|u^\varepsilon_\omega(x,t)-U_\omega(\frac{x_1}{\varepsilon},x,t)\|_{L^1_{loc}(\mathbb{R}^d\times[0,+\infty))}\to 0, \tag{6.24}$$

where $U_\omega(x,t)$ is given by (6.6) with $V(z_1)$ replaced by $V_\omega(z_1)=\underline{V}(\omega+z_1)$, and $u^\varepsilon_\omega\to\bar u$ in the weak star topology of $L^\infty(\mathbb{R}^N\times(0,+\infty))$.

In closing this section we would like to mention that some extension of the theorem of W. E and D. Serre [23] to equations with non-monotone flux functions has recently been obtained by D. Amadori [3]. We take the opportunity to thank Debora Amadori for bringing [23] to our attention.

7. Application to homogenization of Hamilton-Jacobi equations

In this section we apply the results of the previous sections to the homogenization problem for the Hamilton-Jacobi equation

$$u(x)+H(\frac{x}{\varepsilon},x,Du(x))=0, \qquad x\in\mathbb{R}^N, \tag{7.1}$$

where ε is a positive constant. We assume that the Hamiltonian $H(z,x,p)$ is almost periodic in z. The homogenization of (7.1) in the periodic case was addressed by Lions, Papanicolaou and Varadhan [37] (see also [24], [11]). In the almost periodic context, the first homogenization result was obtained by Arizawa [5] for Hamiltonians which are convex in p; subsequently, this result was extended to more general Hamiltonians by Ishii [31]. We follow the approach in [31], adding to it further observations that are derived from the results of the previous sections, in particular, the existence of multiscale Young measures. We also provide a global statement involving all translates of the Hamiltonian by $\omega \in \mathbb{G}$.

As is well-known, equation (7.1) does not have a classical solution in general, and we adopt the notion of viscosity solutions as weak solutions of (7.1). We refer to viscosity solutions, viscosity subsolutions and viscosity supersolutions simply as solutions, subsolutions and supersolutions (see, e.g., [16] for a detailed account on the theory of viscosity solutions).

Given $x, p \in \mathbb{R}^N$ fixed, together with (7.1), we consider the following set of auxiliary equations

$$\varepsilon v_\varepsilon(z) + H(z, x, p + Dv_\varepsilon(z)) = 0, \qquad z \in \mathbb{R}^N. \tag{7.2}$$

We make the following assumptions on H (cf. [31]):

(H0) $H \in C(\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N)$.
(H1) For each $R > 0$, $H(z, x, p) \in \mathrm{AP}(\mathbb{R}^N; \mathrm{BUC}(B(0; R) \times B(0; R)))$.
(H2) $\lim_{R\to\infty} \inf\{H(z, x, p) : z, x, p \in \mathbb{R}^N,\ |p| \geq R\} = +\infty$.
(H3) For each $R > 0$ there is a function $\omega_R \in C([0, \infty))$, with $\omega_R(0) = 0$, such that

$$|H(z, x, p) - H(z, x, q)| \leq \omega_R(|p - q|), \qquad z, x \in \mathbb{R}^N,\ p, q \in B(0; R).$$

(H4) $\sup\{|H(z, x, p)| : z, x \in \mathbb{R}^N,\ p \in B(0, R)\} < \infty$.

The following theorem has been proved by H. Ishii in [31], and we reproduce with minor variants his proof for the reader's convenience.

Theorem 7.1. *Under assumptions* (H0), (H1), (H2), (H3) *we have the following:*

(i) *For each $\varepsilon > 0$, there is a unique solution $u^\varepsilon \in \mathrm{BUC}(\mathbb{R}^N)$ of* (7.1) *and $\|u^\varepsilon\|_\infty + \|Du^\varepsilon\|_\infty \leq A$ for some number $A > 0$ independent of ε.*
(ii) *For each $\varepsilon > 0$ and $x, p \in \mathbb{R}^N$ fixed, there is a unique solution $v_\varepsilon \in BUC(\mathbb{R}^N)$ of* (7.2), *$v_\varepsilon(z) = v_\varepsilon(z; x, p)$, and $\|\varepsilon v_\varepsilon\|_\infty + \|Dv_\varepsilon\|_\infty \leq A(x, p)$ for some number $A(x, p) > 0$ depending on (x, p) but independent of ε.*
(iii) *For each x, p fixed $\varepsilon v_\varepsilon(z)$ are almost periodic in $\mathbb{R}^N$ and converge uniformly in $\mathbb{R}^N$ to a constant $-\bar{H}(x, p)$.*
(iv) *$\bar{H} \in C(\mathbb{R}^N \times \mathbb{R}^N)$ and, for each $R > 0$, there is a function $\nu_R \in C([0, \infty))$, with $\nu_R(0) = 0$, such that*

$$|\bar{H}(x, p) - \bar{H}(x, q)| \leq \nu_R(|p - q|), \qquad x \in \mathbb{R}^N,\ p, q \in B(0; R).$$

(v) *$u_\varepsilon \to u$, as $\varepsilon \to 0$, locally uniformly in $\mathbb{R}^N$, where u is the unique solution in* $\mathrm{BUC}(\mathbb{R}^N)$ *of*

$$u(x) + \bar{H}(x, Du(x)) = 0, \qquad x \in \mathbb{R}^N. \tag{7.3}$$

Proof. 1. The solution of (7.1) is obtained by Perron's method (*cf.* [29]), observing that, for $\tilde{A} = \sup\{|H(z, x, 0)| : z, x \in \mathbb{R}^N\}$, we have that the functions $\tilde{u}(x) := \tilde{A}$ and $\tilde{v}(x) = -\tilde{A}$ are supersolution and subsolution of (7.1), respectively. In particular, $\|u^\varepsilon\|_\infty \leq \tilde{A}$. The fact that $\|Du^\varepsilon\|_\infty$ is uniformly bounded follows from (H2) and the uniform boundedness of u^ε. Uniqueness follows by the standard comparison principle. Analogously, we prove the existence and the uniqueness of the solution of (7.2), and the bounds in (ii).

2. Assertion (iii) is the decisive point in the whole theorem. Fix $x, p \in \mathbb{R}^N$, and denote by $v_\varepsilon(z) = v_\varepsilon(z; x, p)$ the solution of (7.2). First, we observe that $\varepsilon v_\varepsilon(z) - \varepsilon v_\varepsilon(0)$ converges locally uniformly to 0, which easily follows from the uniform boundedness of Dv_ε, so item (i) of Lemma 2.2 is satisfied. Also, item (ii) of Lemma 2.2 is satisfied. Indeed, by assumption (H1), given $\delta > 0$ there are $t_1, \ldots, t_r \in \mathbb{R}^N$ such that, given any $t \in \mathbb{R}^N$, there is a $t_j \in \{t_1, \ldots, t_r\}$, $|H(z + t, x, q) - H(z + t_j, x, q)| < \delta$ for all $z, q \in \mathbb{R}^N$ with $|q| \leq A + |p|$. For such $t \in \mathbb{R}^N$, we have that $v_\varepsilon(z + t) \pm \frac{\delta}{\varepsilon}$ is a supersolution (subsolution) of

$$\varepsilon v + H(z + t_j, x, p + Dv) = 0,$$

and, consequently, by comparison, $\|\varepsilon v_\varepsilon(\cdot+t)-\varepsilon v_\varepsilon(\cdot+t_j)\|_\infty<\delta$, uniformly in ε, as was to be shown. Hence, we have $\|\varepsilon v_\varepsilon-\varepsilon v_\varepsilon(0)\|_\infty\to 0$. Since $\varepsilon v_\varepsilon(0)$ is bounded, we may extract a subsequence $\varepsilon_i\to 0$ such that $\varepsilon_i v_{\varepsilon_i}(0)$ converges to some $-\lambda\in\mathbb{R}$.

3. We claim that $\varepsilon v_\varepsilon(0)\to-\lambda$. Indeed, suppose there is another subsequence $\varepsilon_i'\to 0$ such that $\varepsilon_i' v_{\varepsilon_i'}(0)\to -\mu$, with $\mu\neq\lambda$, say, $\mu<\lambda$. Let $\delta<(\lambda-\mu)/2$. In this case, for ε sufficiently small, it would be possible to find solutions w, w' to

$$H(z,x,p+Dw)\geq\lambda-\delta,\quad\text{and}\quad H(z,x,p+Dw')\leq\mu+\delta,$$

which contradicts the following result (cf. [31], Proposition 6).

4. *Let $G\in C(\mathbb{R}^N\times\mathbb{R}^N)$ satisfy the condition: for each $R>0$ there is a function $\nu_R\in C([0,\infty))$, with $\nu_R(0)=0$, such that*

$$|G(x,p)-G(x,q)|\leq\nu_R(|p-q|),\qquad x\in\mathbb{R}^N,\ p,q\in B(0,R).$$

Let $\lambda,\mu\in\mathbb{R}$. Suppose there are bounded Lipschitz functions, $v(x)$ and $w(x)$, satisfying $G(x,Dv(x))\geq\lambda$ and $G(x,Dw(x))\leq\mu$ in $\mathbb{R}^N$. Then $\lambda\leq\mu$.

We reproduce here the proof of this proposition as in [31] for the sake of completeness. Suppose, by absurd, that $\lambda>\mu$ and let v and w be as above. Set $L=\|Dw\|_\infty$. Let $\rho\in(0,1)$, and define the function $\tilde w\in C(\mathbb{R}^N)$ by $\tilde w(x)=w(x)-\rho(|x|^2+1)^{1/2}$. Then we have $\|D\tilde w-Dw\|_\infty\leq\rho$ and hence $\tilde w$ is a solution of

$$G(x,D\tilde w(x))\leq\mu+\nu_{L+1}(\rho)\qquad x\in\mathbb{R}^N.$$

Now, we choose $\rho\in(0,1)$ such that $\mu+\nu_{L+1}(\rho)<\lambda$ and consider the function for any $\sigma>0$ the function

$$\tilde w(x)-v(y)-\sigma|x-y|^2\qquad (x,y)\in\mathbb{R}^N\times\mathbb{R}^N.$$

Since $\tilde w(x)\to-\infty$ as $|x|\to\infty$, it is easy to check that this function attains a maximum at some point $(x_\sigma,y_\sigma)\in\mathbb{R}^N\times\mathbb{R}^N$ with $|x_\sigma|\leq C(\rho)$, and we have

$$G(x_\sigma,2\sigma(x_\sigma-y_\sigma))\leq\mu+\nu_{L+1}(\rho)$$

and

$$G(y_\sigma,2\sigma(x_\sigma-y_\sigma))\geq\lambda.$$

Moreover, the Lipschitz condition on $\tilde w$ and v yields that $\sigma(x_\sigma-y_\sigma)$ is bounded. Hence, sending $\sigma\to\infty$, we see that for some $\hat x,\hat p\in\mathbb{R}^N$, $\lambda\leq G(\hat x,\hat p)\leq\mu+\nu_{L+1}(\rho)$, which gives a contradiction and proves the claim.

6. Hence, we set $\bar H(x,p)=-\lim_{\varepsilon\to0}\varepsilon v_\varepsilon(0;x,p)$, which is well defined.

7. The fact that $\bar H(x,p)$ is continuous in x follows from the continuity in x of $H(z,x,p)$, the uniform boundedness of $\varepsilon v_\varepsilon(z;x,p)$ for $|p|<R$, which in turn implies the uniform boundedness of $Dv_\varepsilon(z;x,p)$ for $|p|<R$, and comparison principle. These facts imply that for $\tilde x$ arbitrarily close to x, $\varepsilon v(0;\tilde x,p)$ will be as close as we wish to $\varepsilon v_\varepsilon(0;x,p)$, uniformly in ε, and so we will have $\bar H(\tilde x,p)$ arbitrarily close to $\bar H(x,p)$, by passing to the limit when $\varepsilon\to 0$. Following the same line of reasoning we prove that $|\bar H(x,p)-\bar H(x,p)|\leq\nu_R(|p-q|)$, uniformly for $x\in\mathbb{R}^N$ and $|p|,|q|<R$, for some $\nu_R\in C([0,\infty))$ with $\nu_R(0)=0$.

8. Since the solutions u^ε of (7.1) satisfy $\|u^\varepsilon\|_\infty+\|Du^\varepsilon\|_\infty\leq C$ for some $C>0$ independent of ε, the family $\{u^\varepsilon\,:\,\varepsilon>0\}$ is relatively compact with respect to local uniform convergence in $\mathbb{R}^N$. We fix any sequence $\varepsilon_j\to0$ such that $u^{\varepsilon_j}(x)\to\bar u(x)$ locally uniformly on $\mathbb{R}^N$ as $j\to\infty$. We will show that $\bar u$ is a solution of (7.3). By the uniqueness of the bounded solution u of (7.3), we conclude that $\bar u=u$, which implies that $u^\varepsilon(x)\to u(x)$ locally uniformly in $\mathbb{R}^N$.

9. Let $\varphi\in C^1(\mathbb{R}^N)$ and assume that $\bar u-\varphi$ has a strict maximum at $\hat x$. For simplicity we write u_j for u^{ε_j}. Possibly replacing φ by $\varphi(x)+|x-\hat x|^2$ we can assume that $\varphi(x)\to+\infty$ as $|x|\to+\infty$.

10. Take $\delta\in(0,1)$ and let $v^\delta\in\mathrm{BUC}(\mathbb{R}^N)$ be a solution of (7.2) with $x=\hat x$, $p=D\varphi(\hat x)$ and $\delta>0$ sufficiently small, so that

$$H(z,\hat x,D\varphi(\hat x)+Dv^\delta(z))\geq\bar H(\hat x,D\varphi(\hat x))-\delta,\qquad z\in\mathbb{R}^N.$$

Consider the function $u_j(x) - \varphi(y) - \varepsilon_j v^\delta(\frac{y}{\varepsilon_j}) - \sigma|x-y|^2$ on $\mathbb{R}^N \times \mathbb{R}^N$ and let $(x_\sigma, y_\sigma) \in \mathbb{R}^N \times \mathbb{R}^N$ be one of its maximum points, whose existence is obviously ensured. We have

$$u_j(x_\sigma) + H(\frac{x_\sigma}{\varepsilon_j}, x_\sigma, 2\sigma(x_\sigma - y_\sigma)) \le 0,$$

$$H(\frac{y_\sigma}{\varepsilon_j}, \hat{x}, D\varphi(\hat{x}) - D\varphi(y_\sigma) + 2\sigma(x_\sigma - y_\sigma)) \ge \bar{H}(\hat{x}, D\varphi(\hat{x})) - \delta.$$

Since φ has a quadratic growth at infinity and u_j is bounded we have $|y_\sigma| \le C$ with C depending only on j. Since $\|Du_j\|_\infty \le L$ and hence $2\sigma|x_\sigma - y_\sigma| \le L$, sending $\sigma \to \infty$ and passing to a subsequence if necessary we get

$$u_j(\tilde{x}_j) + H(\frac{\tilde{x}_j}{\varepsilon_j}, \tilde{x}_j, p_j) \le 0,$$

$$H(\frac{\tilde{x}_j}{\varepsilon_j}, \hat{x}, D\varphi(\hat{x}) - D\varphi(\tilde{x}_j) + p_j) \ge \bar{H}(\hat{x}, D\varphi(\hat{x})) - \delta$$

for some $p_j \in \overline{B}(0; L)$ and some maximizer $\tilde{x}_j$ of $u_j - \varphi - \varepsilon_j v^\delta(\cdot/\varepsilon_j)$. Taking the difference of the above inequalities and letting $j \to \infty$ we can use the fact that $\tilde{x}_j \to \hat{x}$ to obtain that $\bar{u}(\hat{x}) + \bar{H}(\hat{x}, D\varphi(\hat{x})) \le \delta$, and, since $\delta > 0$ is arbitrary, it follows that $\bar{u}$ is a subsolution of (7.3).
Arguing in an entirely similar way we also obtain that $\bar{u}$ is a supersolution of (7.3). □

In the following theorem we analyze the convergence of u_ε to u from the Young measure viewpoint, deriving some relation between the effective Hamiltonian $\bar{H}$, the Young measures $\nu_{z,x}$ generated by u_ε and the asymptotic mean value of H. Notice that it is still an open problem (to our knowledge, even in the *periodic* case) to give a characterization of $\nu_{z,x}$ and to show full convergence as $\varepsilon \to 0$ (i.e. not along subnets). We are able to characterize $\nu_{z,x}$ only when the Hamiltonian is strictly convex (or concave) and the PDE for correctors admits exact solutions. Nevertheless, concerning the latter, we emphasize that we do not require sublinear growth at infinity for these solutions $v(z)$ (see (7.5) below), since we only impose conditions on their gradients $D_z v(z)$.

Theorem 7.2 (Strong convergence and correctors). *Let H, u_ε, u, v_ε as in Theorem 7.1. If $\nu_{z,x}$ is any two-scale Young measure associated with $D(u_\varepsilon - u)$, then*

$$\bar{H}(x, Du) = \int_{\mathbb{G}^N} \langle \nu_{z,x}, H(z, x, Du + \cdot)\rangle\, dz. \tag{7.4}$$

Consequently, in case $H(z,x,p)$ is convex in p, for each z, $x \in \mathbb{R}^N$, we have

$$\bar{H}(x, Du) \ge \int_{\mathbb{G}^N} H(z, x, Du)\, dz,$$

and, in case of strict convexity, equality holds if and only if $Du^\varepsilon \to Du$ in $L^1_{loc}(\mathbb{R}^N; \mathbb{R}^N)$.
More yet, if $H(z,x,\cdot)$ is strictly convex, then the following two conditions are equivalent:

(a) *for a.e. $x \in \mathbb{R}^N$ there is a solution $v(z) = v(z;x)$ of*

$$H(z, x, Du(x) + D_z v(z)) = \bar{H}(x, Du(x)) \tag{7.5}$$

such that $x \mapsto D_z v(z;x) \in L^1_{loc}(\mathbb{R}^N; \mathrm{AP}(\mathbb{R}^N; \mathbb{R}^N))$;

(b) *$Du^\varepsilon(x) - D_z v(\frac{x}{\varepsilon}; x) - Du(x) \to 0$ in $L^1_{loc}(\mathbb{R}^N)$ with $x \mapsto D_z v(z;x) \in L^1_{loc}(\mathbb{R}^N; \mathrm{AP}(\mathbb{R}^N; \mathbb{R}^N))$.*

Both conditions imply that $\nu_{z,x} = \delta_{Du(x) + D_z v(z;x)}$ for a.e. $(z,x) \in \mathbb{G}^N \times \mathbb{R}^N$.

Proof. Since $u^\varepsilon(x) \to u(x)$ locally uniformly in $\mathbb{R}^N$ and u is a solution of (7.3) we conclude that

$$H(\frac{x}{\varepsilon}, x, Du^\varepsilon(x)) \to \bar{H}(x, Du(x)) \qquad \text{locally uniformly in } \mathbb{R}^N.$$

Moreover, we must have $Du^\varepsilon \to Du$ in the weak star topology of $L^\infty(\mathbb{R}^N;\mathbb{R}^N)$. Hence, given any two-scale Young measure $\nu_{z,x}$ generated in $\Omega = B(0,R)$ by the sequence of uniformly bounded functions $D(u^\varepsilon - u)$ (with $K = \overline{B}(0,2A)$), choosing as a test function

$$\Phi(z,x,\lambda) := H(z,x,Du(x)+\lambda)\varphi(x) \in L^1(\Omega;\mathrm{AP}(\mathbb{R}^N;C(K))), \qquad \varphi \in C_c^\infty(\mathbb{R}^N)$$

we must have

$$\int_{\Omega\times\mathbb{G}^N} \varphi(x)\langle \nu_{z,x}, \underline{H}(z,x,Du(x)+\cdot)\rangle\, dzdx = \int_\Omega \bar{H}(x,Du(x))\varphi(x)\,dx.$$

Since φ is arbitrary we get

$$\int_{\mathbb{G}^N} \langle \nu_{z,x}, \underline{H}(z,x,Du(x)+\cdot)\rangle\, dz = \bar{H}(x,Du(x)) \qquad \text{for a.e. } x \in B(0,R).$$

When $H(z,x,\cdot)$ is convex, Jensen's inequality implies

$$\int_{\mathbb{G}^N} \underline{H}(z,x,Du(x))\, dz \le \bar{H}(x,Du(x)) \qquad \text{for a.e. } x \in B(0,R),$$

and, in case $H(z,x,\cdot)$ is strictly convex, equality holds if and only if $\nu_{z,x}$ is the Dirac measure concentrated in $Du(x)$, that is, $Du^\varepsilon \to Du$ strongly in $L^1(B(0,R);\mathbb{R}^N)$.

Now we prove the equivalence of (a) and (b). If $v(z;x)$ satisfies (7.5) and $g(x,z) \in L^1_{loc}(\mathbb{R}^N;\mathrm{AP}(\mathbb{R}^N;\mathbb{R}^N))$, with $g(x,z) := D_z v(z;x)$, we have

$$u^\varepsilon + H(\frac{x}{\varepsilon},x,Du^\varepsilon(x)) - H(\frac{x}{\varepsilon},x,Du(x)+D_z v(\frac{x}{\varepsilon};x)) = -\bar{H}(x,Du(x)).$$

Multiplying by $\phi(x)\varphi(\frac{x}{\varepsilon})$ with $\phi \in C_0(\mathbb{R}^N)$, $\varphi \in \mathrm{AP}(\mathbb{R}^N)$, integrating in $\mathbb{R}^N$ and taking the limit along a suitable subnet $\varepsilon(d)$, $d \in D$, we obtain

$$\int_{\mathbb{R}^N}\int_{\mathbb{G}^N} \{\langle \nu_{z,x}, H(z,x,\cdot)\rangle - H(z,x,Du(x)+D_z v(z;x))\}\,\phi(x)\underline{\varphi}(z)\,dz\,dx = 0,$$

where $\nu_{z,x}$ is the two-scale Young measure associated with Du^ε along the subnet $\varepsilon(d)$. Since φ and ϕ are arbitrary, we have

$$\langle \nu_{z,x}, H(z,x,\lambda)\rangle - H(z,x,Du(x)+D_z v(z;x)) = 0 \qquad \text{for a.e. } (z,x) \in \mathbb{G}^N \times \mathbb{R}^N.$$

By the strict convexity of $H(z,x,\cdot)$ we conclude that

$$\nu_{z,x} = \delta_{Du(x)+D_z v(z;x)} \qquad \text{for a.e. } (z,x) \in \mathbb{G}^N \times \mathbb{R}^N, \tag{7.6}$$

and this implies that $Du^\varepsilon(x) - D_z v(\frac{x}{\varepsilon};x) - Du(x) \to 0$ in $L^1_{loc}(\mathbb{R}^N)$ by Lemma 3.1.

Conversely, if $Du^\varepsilon(x) - D_z v(\frac{x}{\varepsilon};x) - Du(x) \to 0$ in $L^1_{loc}(\mathbb{R}^N)$ and $D_z v(z;x) \in L^1_{loc}(\mathbb{R}^N;\mathrm{AP}(\mathbb{R}^N))$, again by Lemma 3.1 we must have (7.6) for a.e. $(z,x) \in \mathbb{G}^N \times \mathbb{R}^N$. Then, multiplying by $\phi(x)\varphi(\frac{x}{\varepsilon})$, with $\phi \in C_0(\mathbb{R}^N)$, $\varphi \in \mathrm{AP}(\mathbb{R}^N)$, the equation

$$u^\varepsilon + H(\frac{x}{\varepsilon},x,Du(x)+D_z v(\frac{x}{\varepsilon};x)) = H(\frac{x}{\varepsilon},x,Du(x)+D_z v(\frac{x}{\varepsilon};x)) - H(\frac{x}{\varepsilon},x,Du^\varepsilon(x)),$$

which holds for a.e. $x \in \mathbb{R}^N$, integrating in $\mathbb{R}^N$, passing to a suitable subnet $\{\varepsilon(d)\}_{d\in D}$ and using the fact that $u^\varepsilon \to u(x) = -\bar{H}(x,Du(x))$ for a.e. $x \in \mathbb{R}^N$, we obtain

$$\int_{\mathbb{R}^N}\int_{\mathbb{G}^N} H(z,x,Du(x)+D_z v(z;x))\phi(x)\varphi(z)\,dz\,dx = \int_{\mathbb{R}^N}\int_{\mathbb{G}^N} \bar{H}(x,Du(x))\phi(x)\varphi(z)\,dz\,dx,$$

which implies that $v(\cdot;x)$ satisfies a.e. in $\mathbb{R}^N$ the PDE (7.5) for a.e. $x \in \mathbb{R}^N$. $\square$

An example is provided in the unidimensional case $N = 1$ by an equation of the form

$$u + H(D_x u) = V(\frac{x}{\varepsilon}), \tag{7.7}$$

where $V \in \mathrm{AP}(\mathbb{R})$ and $H(p)$ is strictly monotone (increasing or decreasing) and strictly convex (or concave). In this special case the boundedness of $D_x u^\varepsilon$ is still guaranteed by the boundedness of u^ε, which is again a consequence of Perron's method: instead of hypothesis (H2) we use the fact that H is invertible. Clearly, for each $p \in \mathbb{R}$, there is a unique value $\tilde{H}(p)$ defined by

$$p = \fint_{\mathbb{R}} H^{-1}(V(z) + \tilde{H}(p))\, dz.$$

We claim that $\tilde{H}(p) = \bar{H}(p)$. Indeed, if $v_\varepsilon = v_\varepsilon(x; z, p)$ is the solution of

$$\varepsilon v + H(p + D_z v) = V(z), \tag{7.8}$$

we have

$$D_z v_\varepsilon = H^{-1}(-\varepsilon v_\varepsilon(z) + V(z)) - p,$$

and, since $-\varepsilon v_\varepsilon \to \bar{H}(p)$ uniformly in $\mathbb{R}$, after taking the average in $\mathbb{R}$ of both sides of the above equation and sending $\varepsilon \to 0$, we conclude that

$$p = \fint_{\mathbb{R}} H^{-1}(\bar{H}(p) + V(z))\, dz,$$

and, so, $\bar{H}(p)$ coincides with $\tilde{H}(p)$. Hence, the equation

$$H(p + D_z v) = \bar{H}(p) + V(z) \tag{7.9}$$

may be easily solved, and we find

$$D_z v(z; p) = H^{-1}(\bar{H}(p) + V(z)) - p. \tag{7.10}$$

So, v is determined up to a constant, and $D_z v \in \mathrm{AP}(\mathbb{R})$. Therefore, we may apply the above observation to conclude that $Du^\varepsilon - Du(x) - Dv(\frac{x}{\varepsilon}; Du(x)) \to 0$ strongly in L^1_{loc}.

In the following remark we consider, together with (3.3) and (3.4), also the families of their translates,

$$u_{\varepsilon\omega}(x) + H_\omega(\frac{x}{\varepsilon}, x, Du_{\varepsilon\omega}(x)) = 0, \qquad x \in \mathbb{R}^N,\ \omega \in \mathbb{G}^N, \tag{7.11}$$

and

$$\varepsilon v_{\varepsilon\omega}(z) + H_\omega(z, x, p + Dv_{\varepsilon\omega}(z)) = 0, \qquad z \in \mathbb{R}^N,\ \omega \in \mathbb{G}^N, \tag{7.12}$$

with $H_\omega(z, x, p) := H(\omega + z, x, p)$. Here we keep the same notation $H(\omega + z, x, p)$ understanding that, if $\omega \in \mathbb{G}^N$, we are considering the extension of H to $\mathbb{G}^N \times \mathbb{R}^N \times \mathbb{R}^N$. A continuity argument shows that the maps H_ω still satisfy (H0)-(H4) (even uniformly with respect to the parameter ω).

Remark 7.1 (Convergence of translates). The solutions $u^{\varepsilon\omega}(x)$ of (7.11) converge to $u(x)$, as $\varepsilon \to 0$, locally uniformly in $\mathbb{R}^N$, for all $\omega \in \mathbb{G}^N$, where u is still the unique solution in $\mathrm{BUC}(\mathbb{R}^N)$ of (7.3). It suffices to notice that the effective Hamiltonian $\bar{H}_\omega(x, p)$ given by Theorem 7.1 is the (constant) uniform limit of $\varepsilon v_{\varepsilon\omega}(\cdot; x, p)$ as $\varepsilon \to 0$. In the case when $\omega \in \mathbb{R}^N$ we have the obvious relation

$$v_{\varepsilon\omega}(z; x, p) = v_\varepsilon(z + \omega; x, p) \qquad \forall z \in \mathbb{R}^N,$$

that, keeping ε fixed, can be extended to $v_{\varepsilon\omega}(\cdot; x, p) = \underline{v}_\varepsilon(\cdot + \omega; x, p)$ for any $\omega \in \mathbb{G}^N$. Passing to the limit as $\varepsilon \to 0$ we obtain that $\bar{H}_\omega(x, p) = \bar{H}(x, p)$.

We conclude by recalling briefly how these results fit in the more general framework of stationary ergodic processes. Given a probability space $(\Omega, \mathcal{F}, \mu)$, for $x \in \mathbb{R}^N$ let $\tau_x : \Omega \to \Omega$ be a measure preserving and ergodic transformation group (i.e. $\tau_x A = A$ for all $x \in \mathbb{R}^N$ implies that either $\mu(A) = 0$ or $\mu(A) = 1$). We say that a measurable function $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is stationary if

$$f(\omega; y + z) = f(\tau_z \omega; y) \qquad \text{for all } y, z \in \mathbb{R}^N \text{ and } \omega \in \Omega.$$

The random variable $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is stationary ergodic if it is stationary and the underlying transformation group is ergodic. In view of Theorem 2.8, for each x, p fixed, the process $f(\omega; z, x, p) = H(\omega + z, x, p)$ is clearly stationary ergodic (*cf.* [33], Ch. 7).

8. Application to Homogenization of Fully Nonlinear Elliptic Equations

In this section we consider the homogenization problem for a fully nonlinear elliptic equation of the form

$$u + F(\frac{x}{\varepsilon}, x, D^2 u) = 0, \qquad x \in \mathbb{R}^N. \tag{8.1}$$

Here, we make the following basic assumptions on $F(z, x, M)$:

- (F1) $F(z, x, M) \in C(\mathbb{R}^N \times \mathbb{R}^N \times \mathcal{S}^N)$, where $\mathcal{S}^N$ is the space of symmetric $N \times N$ matrices;
- (F2) $F(z, x, M) \in \mathrm{AP}(\mathbb{R}^N; \mathrm{BUC}(B(0; R) \times \mathcal{S}^N_R))$, where $\mathcal{S}^N_R := \{M \in \mathcal{S}^N \, : \, \|M\| < R\}$;
- (F3) $\lambda \|N\| \le F(z, x, M - N) - F(z, x, M) \le \Lambda \|N\|$, for certain $\lambda, \Lambda > 0$, for all $N \ge 0$;
- (F4) $\sup\{|F(z, x, M)| \, : \, z, x \in \mathbb{R}^N, \; M \in \mathcal{S}^N_R\} < \infty$.

Here, for $M \in \mathcal{S}$, we denote $\|M\| = \sup_{|y|=1} |My|$.

As in the last section, for each $x, M \in \mathbb{R}^N \times \mathcal{S}$ fixed, we consider the auxiliary equation

$$\varepsilon v + F(z, x, M + D^2 v) = 0, \qquad z \in \mathbb{R}^N, \tag{8.2}$$

as well as all translates, for $\omega \in \mathbb{G}$, of both equations:

$$u + F(\omega + \frac{x}{\varepsilon}, x, D^2 u) = 0, \qquad x \in \mathbb{R}^N, \tag{8.3}$$

and

$$\varepsilon v + F(\omega + z, x, M + D^2 v) = 0, \qquad z \in \mathbb{R}^N. \tag{8.4}$$

Before we state our result for the fully nonlinear elliptic equations (8.1), let us recall some facts about the regularity theory for solutions of (8.1) (*cf.* [13], Ch. 7). Let us denote

$$G^\varepsilon(x, M) = F(\frac{x}{\varepsilon}, x, M) - F(\frac{x}{\varepsilon}, x, 0),$$

and for a uniformly elliptic operator G with ellipticity constants λ, Λ we set

$$\beta(x, x_0) = \sup_{M \in \mathcal{S}^N \setminus \{0\}} \frac{|G(x, M) - G(x_0, M)|}{\|M\|}.$$

Let B_1 denote the unit ball centered at 0 and let $x_0 \in B_1$. We say that $G(x_0, D^2 w) = 0$ has $C^{1,1}$ interior estimates (with constant c_e) if for any $w_0 \in C(\partial B_1)$ there exists a solution $w \in C^2(B_1) \cap C(\bar{B}_1)$ of

$$\begin{cases} G(x_0, D^2 w(x)) = 0 & \text{if } x \in B_1, \\ w(x) = w_0(x) & \text{if } x \in \partial B_1 \end{cases}$$

such that

$$\|w\|_{C^{1,1}(\bar{B}_{1/2})} \le c_e \|w_0\|_{L^\infty(\partial B_1)}.$$

If $G(x_0, M)$ is concave (or convex) in $M \in \mathcal{S}$ for any $x_0 \in B_1$, then $G(D^2 w, x_0) = 0$ has $C^{1,1}$ estimates with a universal c_e, by Theorem 6.6 of [13] (see, in particular, (6.14), p. 57 of [13]). We recall the following theorem of Caffarelli [12] (*cf.* [13], Theorem 7.1).

Theorem 8.1 (cf. Theorem 7.1, [13]). *Let u be a bounded solution in B_1 of*

$$G(x, D^2u) = f(x).$$

Assume that $G(x, 0) \equiv 0$ in B_1 and that $G(x_0, D^2w)$ has $C^{1,1}$ interior estimates (with constant c_e) for any $x_0 \in B_1$. Let $N < p < \infty$ and suppose that $f \in L^p(B_1)$. Then there exist positive constants β_0 and C depending only on N, λ, Λ, c_e and p such that if

$$\left(|B_r(x_0)| \int_{B_r(x_0)} \beta(x, x_0)^n \, dx\right)^{1/n} \le \beta_0 \tag{8.5}$$

for any ball $B_r(x_0) \subseteq B_1$, then $u \in W^{2,p}(B_{1/2})$ and

$$\|u\|_{W^{2,p}(B_{1/2})} \le C(\|u\|_{L^\infty(B_1)} + \|f\|_{L^p(B_1)}).$$

The following result is the analogue of Theorem 7.1 for nonlinear elliptic equation and its proof follows a similar line of arguments. Assertion (vii) is motivated by the more general discussion developed in [14]. We also refer to [6] for a related result.

Theorem 8.2. *We have the following:*

(i) *For each $\varepsilon > 0$ there is a unique solution $u^\varepsilon \in \mathrm{BUC}(\mathbb{R}^N)$ of (8.1) and $\|u^\varepsilon\|_\infty \le A_0$, for some $A_0 > 0$ independent of ε,and, for any compact $K \subseteq \mathbb{R}^N$, $[u^\varepsilon]_{C^{0,\alpha}(K)} \le A_1(K)$, for some $0 < \alpha < 1$ and $A_1 > 0$ possibly depending on K but independent of ε.*

(ii) *For each $\varepsilon > 0$ and $x \in \mathbb{R}^N$, $M \in \mathcal{S}^N$ fixed, there is a unique solution $v_\varepsilon \in \mathrm{BUC}(\mathbb{R}^N)$ of (8.2), $\varepsilon\|v_\varepsilon\|_\infty \le A_0(x, M)$ for some number $A_0(x, M) > 0$, depending on (x, M) but independent of ε and, for any compact $K \subseteq \mathbb{R}^N$, $[v_\varepsilon]_{C^{0,\alpha}(K)} \le A_1(x, M, K)$ for some $0 < \alpha < 1$ and $A_1 > 0$ depending on K, x, M but independent of ε.*

(iii) *For each x, M fixed, $\varepsilon v_\varepsilon(z; x, M)$ converges uniformly for $z \in \mathbb{R}^N$ to a constant $-\bar{F}(x, M)$.*

(iv) *$\bar{F}(x, M)$ satisfies* (F1)-(F4).

(v) *$u_\varepsilon \to u$ as $\varepsilon \to 0$ locally uniformly in $\mathbb{R}^N$, where u is the unique solution in $\mathrm{BUC}(\mathbb{R}^N)$ of*

$$u(x) + \bar{F}(x, D^2u(x)) = 0, \qquad x \in \mathbb{R}^N. \tag{8.6}$$

(vi) *Let all $G^\varepsilon(x, M)$, $\varepsilon > 0$, satisfy (8.5) of Theorem 8.1 and assume that $F(z, x, M)$ has polynomial growth as $\|M\| \to \infty$, uniformly for $(z, x) \in \mathbb{R}^N \times \mathbb{R}^N$. Then u^ε is uniformly bounded in $W^{2,p}_{loc}$, for any $N < p < \infty$. Let $\nu_{z,x}$ be any two-scale Young measure associated with the sequence $D^2(u_\varepsilon - u)$. Then*

$$\bar{F}(x, D^2u) = \int_{\mathbb{G}^N} \langle \nu_{z,x}, F(z, x, D^2u + \cdot)\rangle \, dz. \tag{8.7}$$

Consequently, if $F(z, x, \cdot)$ is convex, we have

$$\bar{F}(x, D^2u) \ge \int_{\mathbb{G}^N} F(z, x, D^2u) \, dz,$$

and, in case of strict convexity, equality holds if and only if $D^2u^\varepsilon \to D^2u$ in $L^1_{loc}(\mathbb{R}^N; \mathbb{R}^N)$. Analogous conclusions hold when $F(z, x, \cdot)$ is concave for each $z, x \in \mathbb{R}^N$.

(vii) *More yet, if $F(z, x, \cdot)$ is strictly convex and has polynomial growth as $\|M\| \to \infty$, uniformly for $(z, x) \in \mathbb{R}^N \times \mathbb{R}^N$, then the following two conditions are equivalent:*

(a) *for a.e. $x \in \mathbb{R}^N$ there is a solution $v(z) = v(z; x)$ of*

$$F(z, x, D^2u(x) + D_z^2v(z)) = \bar{F}(x, Du^2(x)) \tag{8.8}$$

such that $x \mapsto D_z^2v(z; x) \in L^1_{loc}(\mathbb{R}^N; \mathrm{AP}(\mathbb{R}^N; \mathcal{S}^N))$;

(b) *$D^2u^\varepsilon(x) - D_z^2v(\frac{x}{\varepsilon}; x) - D^2u(x) \to 0$ in $L^1_{loc}(\mathbb{R}^N)$ with $x \mapsto D_z^2v(z; x) \in L^1_{loc}(\mathbb{R}^N; \mathrm{AP}(\mathbb{R}^N; \mathcal{S}^N))$.*

Both conditions imply that $\nu_{z,x} = \delta_{D^2u(x) + D_z^2v(z;x)}$ for a.e. $(z, x) \in \mathbb{G}^N \times \mathbb{R}^N$.

(viii) *More generally, we have for the solutions $u_\omega^\varepsilon(x)$ of* (8.3) *that $u_\omega^\varepsilon(x) \to u(x)$ as $\varepsilon \to 0$ locally uniformly in $\mathbb{R}^N$, for all $\omega \in \mathbb{G}^N$, where u is the unique solution in* $\mathrm{BUC}(\mathbb{R}^N)$ *of* (8.6).

Proof. The proof follows closely the lines of the proof of Theorem 7.1, the main difference being the usual changes required in the comparision arguments for Hamilton-Jacobi equations when transported to second order elliptic equations. In fact, these changes involve crucial ideas that allowed the extension of the theory of viscosity solutions to second order fully nonlinear elliptic equations, which were primarily due to Jensen [32] (see also [31] and [16]). Since the rigorous arguments require technical but nowadays standard procedures, here we will limit ourselves in giving these arguments in a formal way, assuming smoothness of the functions involved.

1. Existence of a solution of (8.1) is again provided by Perron's method, through comparison principle, in view of (F4), and uniqueness also follows by comparison (see, e.g., [30], [16]). Now we do not have a uniform estimate for Du^ε, but, instead, we have a local uniform estimate for the Hölder continuity of u^ε, by a well known regularity result for fully nonlinear elliptic equations which follows from Harnack inequality (see [13], Ch. 4). This proves (i). The proof of (ii) is analogous.

2. Concerning assertion (iii), as for the compactness of $\varepsilon v_\varepsilon$, it follows again by the compactness criterion in AP, Lemma 2.2, assumption (F2), and the uniform boundedness of $[v_\varepsilon]_{C^{0,\alpha}(K)}$, for any compact $K \subseteq \mathbb{R}^N$.

3. The uniqueness of $\bar{F}(x, M)$ and, hence, the convergence of the whole sequence $\varepsilon v_\varepsilon$ follows from the following analogue of 4. of the proof of Theorem 7.1.

4. *Let $G \in C(\mathbb{R}^N \times \mathbb{R}^N)$ satisfy the ellipticity condition (F3). Let μ, $\nu \in \mathbb{R}$. Suppose there are bounded continuous functions, $v(x)$ and $w(x)$, satisfying $G(x, D^2v(x)) \ge \mu$ and $G(x, D^2w(x)) \le \nu$ in $\mathbb{R}^N$. Then $\mu \le \nu$.*

We only give the formal argument assuming smoothness of v and w. These arguments may be made rigorous using regularizations (by means of the so called inf and sup convolutions) of the type introduced in [32]. So, assume on the contrary that $\mu > \nu$ and let us try to get a contradiction. We may assume, without loss of generality that $v(0) = w(0) = 0$. We consider the function $w(x) - v(x) - \rho|x|^2$, and let x_ρ be a point of maximum, which certainly exists. We have, $D^2w(x_\rho) - \rho I \le D^2v(x_\rho)$ and so, by ellipticity,

$$\mu \le G(x_\rho, D^2v(x_\rho)) \le G(x_\rho, D^2w(x_\rho) - \rho I) \le G(x_\rho, D^2w(x_\rho)) + \Lambda\rho \le \nu + \Lambda\rho.$$

Hence, taking $\Lambda\rho < \mu - \nu$ we arrive at a contradiction.

5. The proof of (iv) proceeds again by comparison. For instance, let us prove (F3). We have

$$\bar{F}(x, M - N) - \bar{F}(x, M) = \lim_{\varepsilon \to 0} \big(\varepsilon v_\varepsilon(z, x, M) - \varepsilon v_\varepsilon(z, x, M - N)\big) \le \Lambda\|N\|,$$

by comparision, since, by (F3), $v_\varepsilon(z, x, M - N)$ satisfies

$$\varepsilon v_\varepsilon(z, x, M - N) + F(z, x, M + D^2v_\varepsilon(z, x, M - N)) \ge -\Lambda\|N\|.$$

The other inequality follows similarly.

6. Assertion (v) again requires extreme value arguments and, as above, we here only give their formal version. Again, by compactness, we have a subsequence $u_j := u^{\varepsilon_j}$ converging to a certain u, locally uniformly in $\mathbb{R}^N$. We will show that u is solution (8.6) and, so, by uniqueness, we will have that the whole sequence u^ε converges locally uniformly to u. So, let us fix $\hat{x} \in \mathbb{R}^N$ and, for some $\delta > 0$, consider the function $v_\delta(z)$ satisfying

$$F(z, \hat{x}, D^2u(\hat{x}) + D^2v_\delta(z)) \ge \bar{F}(\hat{x}, D^2u(\hat{x})) - \delta.$$

Take $\rho > 0$, and let x_j be a point of maximum of

$$u_j(x) - u(x) - \varepsilon_j^2 v_\delta(\frac{x}{\varepsilon_j}) - \rho|x - \hat{x}|^2 + \rho,$$

which certainly exists. We clearly have $x_j \to \hat{x}$ as $j \to \infty$. We have

$$u_j(x_j) + F(x_j, \frac{x_j}{\varepsilon_j}, D^2u(x_j) + D^2v_\delta(\frac{x_j}{\varepsilon_j}) + \rho I) \le 0,$$

and

$$F(\hat{x}, \frac{x_j}{\varepsilon_j}, D^2u(\hat{x}) + D^2v_\delta(\frac{x_j}{\varepsilon_j})) \geq \bar{H}(\hat{x}, D^2u(\hat{x})) - \delta,$$

which, after subtraction, gives

$$u_j(x_j) + \bar{F}(\hat{x}, D^2u(\hat{x})) \leq O(|x_j - \hat{x}|) + O(\rho) + \delta.$$

Hence, letting $j \to \infty$ we arrive at

$$u(\hat{x}) + \bar{F}(\hat{x}, D^2u(\hat{x})) \leq O(\rho) + \delta,$$

from which it follows that u is a solution of

$$u(x) + \bar{F}(x, D^2u(x)) \leq 0.$$

The other inequality can be proved similarly.

7. Concerning assertion (vi), the fact that $\|u^\varepsilon\|_{W^{2,p}_\infty} < \infty$, uniformly in ε, follows from Theorem 8.1 by using assumption (F4) and the uniform bound for $\|u^\varepsilon\|_\infty$. The assumption that $F(z, x, M)$ has polynomial growth in M allows the application of the two-scale Young measure to $F(z, x, M)$. Assertions (vi), (vii) are proved exactly as the corresponding assertions in Theorem 7.2. Finally, the convergence of translates in (viii) can be proved by the same argument used in Remark 7.1. □

Remark 8.1. Concerning item (vii) of Theorem 8.2, first, it is important to observe that we do not require the solution of (8.8) to have subquadratic growth at infinity, since we impose conditions only on $D^2_z v(z)$. Second, it is easy to see that also in the above context of fully nonlinear elliptic equations, the one-dimensional case provides us with an example analogous to the one given in Section 7 after Theorem 7.2 for Hamilton-Jacobi equations.

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