THE ISOPERIMETRIC INEQUALITY FOR THE CAPILLARY ENERGY OUTSIDE CONVEX CYLINDERS

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Abstract. We study the isoperimetric problem for capillary surfaces with a general contact angle $\theta \in (0, \pi)$, outside convex infinite cylinders with arbitrary two-dimensional convex section. We prove that the capillary energy of any surface supported on any such convex cylinder is strictly larger than that of a spherical cap with the same volume and the same contact angle on a flat support, unless the surface is itself a spherical cap resting on a facet of the cylinder. In this class of convex sets, our result extends for the first time the well-known Choe-Ghomi-Ritoré relative isoperimetric inequality, corresponding to the case $\theta = \pi/2$, to general angles.

1. Introduction

Let $\mathbf{C} \subset \mathbb{R}^N$ be a closed convex set with nonempty interior. Given a set of finite perimeter $E \subset \mathbb{R}^N \setminus \mathbf{C}$ and $\lambda \in (-1, 1)$ we define the capillary energy as

$$J_{\lambda,\mathbf{C}}(E) := P(E; \mathbb{R}^N \setminus \mathbf{C}) - \lambda \mathcal{H}^{N-1}(\partial^* E \cap \partial \mathbf{C}).$$

Here, for any Borel set G, $P(E; G) = \mathcal{H}^{N-1}(\partial E^* \cap G)$ and $\partial^* E$ is the reduced boundary of E (for the definitions and the relevant properties see [1, 19]). The capillary energy has a natural physical motivation as it models a liquid drop supported on a given substrate and we refer to [11] for a comprehensive introduction to the topic.

For every $v > 0$ we consider the isoperimetric problem

$$I_{\mathbf{C}}(v) := \inf\{J_{\lambda,\mathbf{C}}(E) : \; E \subset \mathbb{R}^N \setminus \mathbf{C}, \, |E| = v\}. \tag{1.1}$$

When the convex set $\mathbf{C}$ is bounded the problem (1.1) has a minimizer, and if $\mathbf{C}$ is in addition smooth, then the minimizer is smooth up to a small singular set and the free boundary $\partial E \setminus \mathbf{C}$ meets the surface $\partial \mathbf{C}$ with an angle given the classical Young's law [23, 9]. We also mention the recent work related to Allard type regularity for critical sets [20]. When the convex set is unbounded the problem (1.1) might not admit a minimizer. This happens for instance when $\mathbf{C} = \mathcal{C} \times \mathbb{R} \subset \mathbb{R}^3$, with $\mathcal{C}$ the epigraph of a parabola. In this case, as a consequence of our main Theorem 1.1, minimizing sequences slide upwards to infinity along the boundary of $\mathbf{C}$ and the isoperimetric profile (1.1) agrees with the profile given by the half-space. In the case $\lambda = 0$ the problem for unbounded general convex sets $\mathbf{C}$ is studied in [13].

The issue we want to address here is to find the convex sets $\mathbf{C}$ for which the value of (1.1) is the smallest. In the case $\lambda = 0$ the problem reduces to the relative isoperimetric inequality outside convex sets proven by Choe-Ghomi-Ritoré in [7]: using the tools developed in [6] they show that the half-space gives the lowest value for (1.1) . On the other hand, rather surprisingly the capillary case with general $\lambda \neq 0$ has remained completely open until now as all the methods devised for the relative isoperimetric problem do not seem to be applicable to (1.1). Here, we solve it in the case of infinite convex cylinders. In fact, our result holds in every dimension for convex sets, whose normals span a two-dimensional plane.

In order to state our main result we denote the half-space by $\mathbf{H} = \{x \in \mathbb{R}^N : x_N < \lambda\}$ and by B_r^λ the solid spherical cap

$$B_r^\lambda = \{x \in B_r : x_N > \lambda\}.$$

Given $v > 0$, we let $B^\lambda[v] = B_r^\lambda$ denote the spherical cap with radius r such that $|B^\lambda[v]| = v$. Our main result is the following.

FIGURE 1. Droplets supported on a convex cylinder

Theorem 1.1. *Let $\lambda \in (-1, 1)$ and let $\mathbf{C}$ be of the form $\mathcal{C} \times \mathbb{R}^{N-2}$, where $\mathcal{C} \subset \mathbb{R}^2$ is a closed convex set of the plane with nonempty interior.* [1] *For every set of finite perimeter $E \subset \mathbb{R}^N \setminus \mathbf{C}$ such that $|E| = v$ we have*

$$J_{\lambda,\mathbf{C}}(E) \geq J_{\lambda,\boldsymbol{H}}(B^\lambda[v]). \tag{1.2}$$

Moreover the equality holds if and only if E sits on a facet of $\mathbf{C}$ and is isometric to $B^\lambda[v]$.

Note that in the physical case $N = 3$ the above theorem holds for all convex infinite cylinders with arbitrary two-dimensional section. We highlight also that we do not assume any regularity on $\mathbf{C}$. In particular, the theorem above applies to the case where $\mathbf{C}$ is an infinite wedge and shows that the capillary energy of a droplet sitting outside a wedge and wetting its ridge has energy strictly larger than a spherical cap lying on a flat surface, a fact that, to the best of our knowledge, was not proven before.

[1] For the case of a convex set with empty interior, see Remark 3.5.

Instead, the capillary isoperimetric problem *inside* a convex wedge for $\lambda = 0$ was studied in [18] where it is proved that the minimizer of the capillary energy is a portion of a ball centered at the ridge of the wedge. The same result holds also for critical points, as a consequence of the generalized Heinze-Karcher inequality proven in [15]. Instead, Theorem 1.1 implies that the capillary isoperimetric problem *outside* a convex wedge has the opposite behavior, since the minimizer is a spherical cap sitting on either facet of the wedge away from the ridge.

As we already mentioned, the case $\lambda = 0$ of Theorem 1.1 the problem (1.1) is the relative isoperimetric inequality due to Choe-Ghomi-Ritoré [7], see also [14] for the rigidity, i.e., the characterization of the equality case for general, possibly nonsmooth convex sets. We also refer to [17] for an alternative proof of the same inequality and to [16] for the problem in higher codimension.

In order to prove Theorem 1.1 we need to introduce some novel methods, that will be explained in more details in Section 1.1 below. Indeed, the approach based on normal cones developed in [6, 7] for the case $\lambda = 0$ (and further refined in [14]) gives only information on the free boundary $\partial E \setminus \mathbf{C}$, while the contact region $\partial E \cap \partial \mathbf{C}$ remains invisible. In order to overcome this, we adapt to the capillary problem the ABP-method, originally introduced by Cabré for the standard isoperimetric inequality. This approach already appears in [17] for $\lambda = 0$ and here we develop it in the case $\lambda \neq 0$. However, in order to apply the ABP-method one needs a subtle estimate on the set of subdifferentials of the function u solving problem (1.3) below, as we will explain in the next subsection. This difficulty appears already in the case $\lambda = 0$ and the authors in [17] overcome it by rewriting the aforementioned estimate in terms of suitable restricted normal cones to the graph of u and by using some of the results proved in [6]. This argument does not seem to generalize to the case $\lambda \neq 0$.

Instead of relying on normal cones, we first prove that the solution to the Neumann problem (1.3) is a viscosity supersolution of the same problem and exploit this property to study directly the geometry of its subdifferentials. Since the subdifferentials of a function are convex and their union is the whole space, the new point of view combined with a discretization procedure leads us to reformulate the aforementioned estimate in terms of an estimate on finite convex partitions of the space made up of subdifferentials of discrete functions. This seems to be a rather complicated combinatorial problem that for $\lambda \neq 0$ we can solve only in the planar case, allowing us to treat convex cylinders of the form $\mathcal{C} \times \mathbb{R}^{N-2}$. We note however that this method gives an easy proof of the inequality (1.5) and thus of (1.1) for all convex sets $\mathbf{C}$ when $\lambda = 0$. Finally, we conjecture that (1.2) holds for all convex sets.

We next give an overview of the ABP-argument which we use in the proof of Theorem 1.1.

1.1. **Overview of the proof.**

The proof of Theorem 1.1 is based on the ABP-method applied to Neumann problem (1.3). We note that in the context of isoperimetric problems this method was used first time by Cabré in [3, 4] and further generalized in [5]. A quantitative variant of this method in the spirit of [8] has been used recently in [22] to prove the stability of the isoperimetric inequality for the capillary energy in a half space.

In this paper we extend the ABP-method to the capillary isoperimetric problem outside convex cylinders. Let us sketch the proof and outline the main challenges of the argument.

By scaling we may reduce to the case $|E| = |B^\lambda|$. Assume for simplicity that the set E is regular in which case we denote it by $E = \Omega$. To be more precise we assume that $\Omega \subset \mathbb{R}^N \setminus \mathbf{C}$ is a Lipschitz regular open set with $|\Omega| = |B^\lambda|$, such that $\Sigma = \partial\Omega \setminus \mathbf{C}$ and $\Gamma = \partial\Omega \cap \mathbf{C}$ are smooth embedded manifolds with a common boundary denoted by γ. Let $u : \bar{\Omega} \to \mathbb{R}$ be the solution of the Neumann boundary problem

$$\begin{cases} \Delta u = c \quad \text{in } \Omega \\ \partial_\nu u = 1 \quad \text{on } \Sigma \\ \partial_\nu u = -\lambda \quad \text{on } \Gamma, \end{cases} \tag{1.3}$$

where $\lambda \in (-1, 1)$ and the constant

$$c = \frac{\mathcal{H}^{N-1}(\Sigma) - \lambda \mathcal{H}^{N-1}(\Gamma)}{|\Omega|} = \frac{J_{\lambda,\mathbf{C}}(\Omega)}{|\Omega|} \tag{1.4}$$

is the one prescribed by the compatibility condition. We point out that in the case $\Omega = B^\lambda$ up to an additive constant $u(x) = \frac{1}{2}|x|^2$ and $c = N$.

Let us denote the convex envelope of u by $\hat{u}$, i.e., the largest convex function that is below u. Note that if $x \in \Omega$ is such that $u(x) = \hat{u}(x)$, then the *subdifferential* of u at x is non-empty

$$J_{\bar{\Omega}} u(x) = \{\xi \in \mathbb{R}^N : u(y) - u(x) \geq \xi \cdot (y - x) \quad \text{for all } \ y \in \bar{\Omega}\} \neq \emptyset$$

and since u is smooth in Ω it holds $\xi = \nabla u(x)$ and $\nabla^2 u(x) \geq 0$. We denote the set of points for which $J_{\bar{\Omega}} u(x)$ is non-empty by Ω^+ and the union of subdifferentials by $\mathcal{A}_u$

$$\mathcal{A}_u = \bigcup_{x \in \Omega} J_{\bar{\Omega}} u(x).$$

In order to carry one the ABP-argument one needs to show the following crucial estimate

$$|\mathcal{A}_u| \geq |B^\lambda| \tag{1.5}$$

by somehow exploiting the Neumann boundary conditions in (1.3) and the geometry of $\mathbf{C}$.

Once (1.5) is proven, the claim then follows by using the Area Formula, the arithmetic-geometric mean inequality and recalling the value of c in (1.4)

$$|\Omega| = |B^\lambda| \leq |\mathcal{A}_u| = |\nabla u(\Omega^+)| \leq \int_{\Omega^+} \det \nabla^2 u \, dx \leq \int_{\Omega^+} \frac{(\Delta u)^N}{N^N} \, dx \leq \left(\frac{J_{\lambda,\mathbf{C}}(\Omega)}{|\Omega| N} \right)^N |\Omega|.$$

The above chain of inequalities gives the conclusion as

$$J_{\lambda,\mathbf{C}}(B^\lambda) = N|B^\lambda| = N|\Omega|.$$

The case of a general set of finite perimeter E follows by an approximation argument. In fact, a more refined approximation argument allows us also to characterize the case of equality, see Lemma 3.4.

It is then clear that the most relevant estimate is (1.5) which turns out to be challenging to prove. Indeed, as observed in [17] the inclusion $B^\lambda \subset \mathcal{A}_u$ does not hold in general and we need to develop a more subtle argument to overcome the problem. The same estimate was already proven in [17] in the case $\lambda = 0$ by reformulating the problem in terms of suitable restricted normal cones to the graph of u in the spirit of [6]. However their argument does not generalize to the case $\lambda \neq 0$.

In order to deal with the case of general λ's, we develop the following novel argument. We first show that the (variational) solution to (1.3) is a viscosity supersolution to the same problem, in the sense of Definition 2.1. Note that the latter definition is stronger than the one given in [10] and therefore the supersolution property in the above sense does not follow from known results. Using this property we are able to relate the subdifferentials of u in Ω with the subdifferentials of the restriction of u to Γ proving the following inclusion, see Lemma 2.4,

$$\mathcal{B}^\lambda_{u_\Gamma} \cap B_1 \subset \mathcal{A}_u \cap B_1,$$

where $\mathcal{B}^\lambda_{u_\Gamma} = \bigcup_{x\in\Gamma}\{\xi \in J_\Gamma u(x) : \xi \cdot \nu_{\mathbf{C}}(x) > \lambda\}$, $J_\Gamma u(x)$ is the subdifferential at x of the restriction of u to Γ and $\nu_{\mathbf{C}}(x)$ stands for the outer normal to $\mathbf{C}$ at x. This inclusion leads to the proof of (1.5) provided we are able to show that

$$|\mathcal{B}^\lambda_v \cap B_1| \geq |B^\lambda| \qquad \text{for any } v : K \to \mathbb{R}, \text{ with } K \subset \partial\mathbf{C} \text{ compact.} \tag{1.6}$$

In fact, it is enough to prove (1.6) only for discrete sets $K \subset \partial\mathbf{C}$, see Lemma 2.8. Note that property (1.6) has nothing to do with Neumann problem (1.3) and it only depends on the geometry of $\mathbf{C}$. We call this property *λ-ABP property*, see Definition 2.6. It can be easily shown that any convex sets satisfies the 0-ABP property, see Proposition 2.11. As already mentioned, in this way we obtain a new simple proof of (1.5) (and thus of the Choe-Ghomi-Ritoré isoperimetric inequality) in the case $\lambda = 0$.

The case $\lambda \neq 0$ is considerably more difficult. In fact, proving the λ-APB property is equivalent to show that, given any convex partition $A_1, \dots, A_k$ of $\mathbb{R}^N$, with A_i the subdifferential at x_i of a function $v : \{x_1, \dots, x_k\} \subset \partial\mathbf{C} \to \mathbb{R}$, then

$$\sum_{i=1}^k \big|A_i \cap \{\xi \in B_1 : \ \xi \cdot \nu_i > \lambda\}\big| \geq |B^\lambda|, \tag{1.7}$$

where ν_i is the exterior normal to $\mathbf{C}$ at x_i. The proof of the above inequality is a difficult combinatorial problem, and we are able to show it only when all the normals to $\mathbf{C}$ lie in a 2-dimensional plane, that is in the case $\mathbf{C} = \mathcal{C} \times \mathbb{R}^{N-2}$. Indeed, in this case, by a slicing argument we are able to reduce the proof of (1.7) to a similar estimate for convex partitions in the plane. The proof of the latter, which is the content of Section 3.1, is still very complicated and we achieve it by studying the left hand side of (1.7) as a function of λ by means of geometrical and analytical arguments.

2. The ABP approach for the capillary energy

In this section we set up the tools that we need for the ABP argument. To this aim, in Section 2.1 we establish the crucial viscosity supersolution property for the variational solutions of the Neumann problem (1.3). In Section 2.2 we introduce a useful notion of restricted subdifferential which enables us to reduces the inequality (1.5) to a property of the convex set $\mathbf{C}$. As we already mentioned, we call it λ-ABP property and give its definition in Section 2.3 (Definition 2.6). At the end of the section we establish the relative isoperimetric inequality for the λ-capillary functional outside the convex sets that satisfy such a property.

As we mentioned in the introduction, we will first prove the relative isoperimetric inequality for regular sets. In order to emphasize this we always denote $E = \Omega$ when the set is assumed to be regular, i.e., it satisfies

$$\mathbf{C} \subset \mathbb{R}^N \text{ is a closed convex set of class } C^2 \tag{2.1}$$

and

$$\begin{aligned} &\Omega \subset \mathbb{R}^N \setminus \mathbf{C} \text{ is a bounded Lipschitz set such that } \Sigma := \partial\Omega \setminus \mathbf{C} \\ &\qquad\qquad \text{is a } (N-1)\text{-manifold with boundary of class } C^2. \end{aligned} \tag{2.2}$$

We call the boundary Σ the *free interface* and denote $\Gamma := \partial\Omega \cap \mathbf{C}$, which we call the *wetted region* which is also an embedded C^2-regular $(N-1)$-manifold with boundary. Note that Γ and Σ share the same boundary, which we denote by $\gamma := \overline{\Sigma} \cap \mathbf{C}$ and which by the assumption is a $(N-2)$-manifold of class C^2. We call γ the *contact set* of Σ with $\mathbf{C}$. Moreover, we will throughout the section assume $|\Omega| = |B^\lambda|$ if not otherwise mentioned.

We will denote by ν_Ω and $\nu_{\mathbf{C}}$ the outer unit normal to $\partial\Omega$ and to $\partial\mathbf{C}$, respectively. We also denote by $\nu_\Sigma = \nu_\Omega$ the outer unit normal field on Σ, which by our assumptions admits a continuous extension at γ, still denoted by ν_Σ. We also set $\nu_\Gamma = -\nu_{\mathbf{C}}$ on Γ. Note that the Lipschitz regularity of Ω yields

$$\nu_\Gamma \cdot \nu_\Sigma > -1 \qquad \text{on } \gamma.$$

Finally, we define the ε-neighborhood of a generic set $F \subset \mathbb{R}^N$ as $B_\varepsilon + F = \{x \in \mathbb{R}^N : \operatorname{dist}(x, F) < \varepsilon\}$.

2.1. **The Neumann problem.** In this section we consider the Neumann problem under the assumptions (2.1) and (2.2). We may also assume without loss of generality Ω to be connected. Note that even in this case Ω is still merely a Lipschitz domain and therefore the high regularity of u up to boundary is not granted. However, it turns out that we only need the solution to attain the boundary values in the viscosity sense, for which Hölder continuity up to the boundary is enough. To this aim we consider the variational solution of the problem (1.3) which by definition is a function $u \in H^1(\Omega)$ such that it holds

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx = -\int_\Omega c\varphi \, dx + \int_{\partial\Omega} g\varphi \, d\mathcal{H}^{N-1} \tag{2.3}$$

for all $\varphi \in H^1(\Omega)$, where c is given by (1.4),

$$g \equiv 1 \text{ on } \Sigma \quad \text{and} \quad g \equiv -\lambda \quad \text{on } \Gamma \setminus \gamma. \tag{2.4}$$

Since Ω is bounded and Lipschitz regular, g is bounded and we have the following compatibility condition

$$c|\Omega| = -\int_{\partial\Omega} g \, d\mathcal{H}^{N-1},$$

there exists a unique (up to an additive constant) variational solution of (1.3). Moreover, by standard elliptic regularity theory the variational solution is Hölder continuous up to the boundary, see for instance [21].

Let us proceed to the notion of viscosity solution. In fact, it turns out that we need only the concept of viscosity supersolution for the ABP-argument and therefore we reduce only to that. Here is the definition we need.

Definition 2.1. *A lower semicontinuous function $u : \overline{\Omega} \to \mathbb{R}$ is a* viscosity supersolution *of* (1.3) *if whenever $u - \varphi$ has a local minimum at $x_0 \in \overline{\Omega}$ for $\varphi \in C^2(\mathbb{R}^N)$, then*

$$\begin{cases} -\Delta\varphi(x_0) \geq -c & \text{if } x_0 \in \Omega, \\ \partial_{\nu_\Sigma}\varphi(x_0) - 1 \geq 0 & \text{if } x_0 \in \Sigma \setminus \gamma, \\ \partial_{\nu_\Gamma}\varphi(x_0) + \lambda \geq 0 & \text{if } x_0 \in \Gamma \setminus \gamma, \\ \max\{\partial_{\nu_\Sigma}\varphi(x_0) - 1, \partial_{\nu_\Gamma}\varphi(x_0) + \lambda\} \geq 0 & \text{if } x_0 \in \gamma, \end{cases} \tag{2.5}$$

where $\partial_{\nu_\Sigma}\varphi(x_0) = \nabla\varphi(x_0) \cdot \nu_\Sigma(x_0)$ and $\partial_{\nu_\Gamma}\varphi(x_0) = \nabla\varphi(x_0) \cdot \nu_\Gamma(x_0)$.

We can now prove the main result of the section.

Proposition 2.2. *Let Ω, $\mathbf{C}$ be as in* (2.1) *and* (2.2)*. Then the variational solution of* (1.3) *is a viscosity supersolution in the sense of Definition 2.1.*

Proof. Since the equation is satisfied classically in Ω and the Neumann boundary conditions are achieved in a classical sense at $\partial\Omega \setminus \gamma$, it will be enough to check the property on γ. To this aim, assume that $\varphi \in C^2(\mathbb{R}^N)$ and $x_0 \in \gamma$ are such that $(u - \varphi)(x) \geq 0$, with equality achieved only at x_0. We start by showing that

$$\max\{-\Delta\varphi(x_0) + c, \partial_{\nu_\Sigma}\varphi(x_0) - 1, \partial_{\nu_\Gamma}\varphi(x_0) + \lambda\} \geq 0. \tag{2.6}$$

We argue by contradiction, assuming that

$$\max\{-\Delta\varphi(x_0) + c, \partial_{\nu_\Sigma}\varphi(x_0) - 1, \partial_{\nu_\Gamma}\varphi(x_0) + \lambda\} < 0.$$

By continuity we may find a small ball $B_r(x_0)$ such that

$$-\Delta\varphi + c < 0 \quad \text{in } B_r(x_0) \qquad \text{and} \qquad \partial_\nu\varphi - g < 0 \quad \text{on } (\partial\Omega \cap B_r(x_0)) \setminus \gamma\,, \tag{2.7}$$

with g defined in (2.4). Then, setting $w := u - \varphi$, $h := -c + \Delta\varphi$, $f = \partial_\nu(u-\varphi) = g - \partial_\nu\varphi$, we have that w is a variational solution of

$$\begin{cases} -\Delta w = h & \text{in } \Omega \cap B_r(x_0), \\ \partial_\nu w = f & \text{in } \partial\Omega \cap B_r(x_0) \end{cases}$$

that is,

$$\int_\Omega \nabla w \cdot \nabla\psi\, dx = \int_\Omega h\psi\, dx + \int_{\partial\Omega} f\psi\, d\mathcal{H}^{N-1} \tag{2.8}$$

for all $\psi \in H^1(\Omega)$ with $\psi = 0$ in $\Omega \setminus B_r(x_0)$. Let us now choose $\psi := \min\{w - \varepsilon, 0\}$ and note that for $\varepsilon > 0$ small enough

$$\psi = \min\{w - \varepsilon, 0\} = 0 \quad \text{in } \Omega \setminus B_r(x_0)\,.$$

Then, (2.8) combined with the fact that $h > 0$ in $\Omega \cap B_r(x_0)$ and $f > 0$ in $\partial\Omega \cap B_r(x_0)$, yield

$$\int_\Omega \big|\nabla\big(\min\{w-\varepsilon,0\}\big)\big|^2 dx = \int_{\Omega\cap B_r(x_0)} h\psi\, dx + \int_{\partial\Omega\cap B_r(x_0)} f\psi\, d\mathcal{H}^{N-1} \le 0$$

and in turn $\min\{w - \varepsilon, 0\} = 0$ in Ω. This is impossible since $w - \varepsilon < 0$ in a neighborhood of x_0. Thus (2.6) is established.

The inequality (2.6) is not good enough, since we only want information on the boundary. We thus claim that in fact

$$\max\{\partial_{\nu_\Sigma}\varphi(x_0) - 1, \partial_{\nu_\Gamma}\varphi(x_0) + \lambda\} \ge 0. \tag{2.9}$$

To this aim we observe that for any $x_0 \in \gamma$ there exists a ball $B_r(\bar{x}) \subset \mathbb{R}^N \setminus \overline{\Omega}$ with $x_0 \in \partial B_r(\bar{x})$ (it is enough to take a ball contained in $\mathbf{C}$ and tangent to x_0, which is possible by the C^2 assumption on $\mathbf{C}$). By translating and dilating we may assume for simplicity that $\bar{x} = 0$ and that $r = 1$. We perturb the test function φ by a functions $\psi_q \in C^2(\mathbb{R}^n \setminus \{0\})$ that we define as

$$\psi_q(x) = \frac{1}{q^{3/2}}(|x|^{-q} - 1)$$

where $q > 0$ is a large number to be chosen. Then by the exterior ball condition we have that $\psi_q(x) \le 0$ for $x \in \Omega$ while since $x_0 \in \partial B_1$ it holds $\psi_q(x_0) = 0$. Moreover by a direct computation we see that

$$\Delta\psi_q(x_0) \ge \frac{\sqrt{q}}{2} \quad \text{and} \quad |\nabla\psi_q(x_0)| = \frac{1}{\sqrt{q}},$$

for q sufficiently large. We define a new test function

$$\varphi_q(x) = \varphi(x) + \psi_q(x).$$

By construction it holds $\varphi_q \le \varphi$ in Ω and $\varphi_q(x_0) = \varphi(x_0)$, hence x_0 is still a local minimum for $u - \varphi_q$. Thus

$$-\Delta\varphi_q(x_0) + c \le -\frac{\sqrt{q}}{2} - \Delta\varphi(x_0) + c < 0$$

when q is large. Therefore, for q large, from (2.6) we obtain (2.9) with φ replaced by φ_q. Finally, letting $q \to \infty$, since $\nabla\varphi_q(x_0) \to \nabla\varphi(x_0)$ we obtain (2.9). This concludes the proof. □

2.2. **Subdifferentials.** We need some notation in order to proceed. Given $X \subset \mathbb{R}^N$, a function $u : X \to \mathbb{R}$, a subset $Y \subset X$ and a point $x \in Y$ we define the subdifferential $J_Y u(x)$ as

$$J_Y u(x) := \big\{\xi \in \mathbb{R}^N :\ u(y) - u(x) \ge \xi \cdot (y - x)\ \text{ for all}\, y \in Y\big\}.$$

We note that Y may even be a discrete set.

Remark 2.3. *Note that if Y is bounded and u is bounded from below in Y, then*

$$\bigcup_{x\in Y} J_Y u(x) = \mathbb{R}^N.$$

Indeed, for any $\xi \in \mathbb{R}^N$, we may find $c < 0$ so negative that $\sup_{x\in Y}\big(-c + \xi\cdot x - u(x)\big) < 0$. Setting

$$t_0 := \sup\{t > 0 :\ -c + \xi \cdot x + s < u(x) \text{ for all } x \in Y \text{ and for } s \in (0,t)\},$$

we clearly have $-c+\xi\cdot\bar{x}+t_0 = u(\bar{x})$ for some $\bar{x} \in Y$ and $-c+\xi\cdot x+t_0 \le u(x)$ for all $x \in Y$; that is, $\xi \in J_Y u(\bar{x})$.

Let Ω, Γ and Σ be as in (2.1) and (2.2). Recall that the crucial inequality (1.5) involves the set $\mathscr{A}_u$ which we define for any given function $u : \overline{\Omega} \to \mathbb{R}$ as

$$\mathscr{A}_u := \bigcup_{x\in\Omega}\{\xi \in \mathbb{R}^N : \xi \in J_{\overline{\Omega}} u(x)\}. \tag{2.10}$$

If the function u in (2.10) is the solution of (1.3), then it turns out that all relevant information is contained in its restriction to Γ. This leads us to define the following union of subdifferentials for functions defined on the boundary of $\mathbf{C}$, $v : K \subset \partial\mathbf{C} \to \mathbb{R}$

$$\mathscr{B}^\lambda_v := \bigcup_{x\in K}\{\xi \in J_K v(x) :\ \xi\cdot\nu_{\mathbf{C}}(x) > \lambda\}, \tag{2.11}$$

where $\lambda \in (-1, 1)$. If Ω and $u : \overline{\Omega} \to \mathbb{R}$ are as above then we denote the restriction of u to Γ by u_Γ and by the previous notation we have

$$\mathscr{B}^\lambda_{u_\Gamma} = \bigcup_{x\in\Gamma}\{\xi \in J_\Gamma u(x) :\ \xi\cdot\nu_{\mathbf{C}}(x) > \lambda\}. \tag{2.12}$$

We transform the information of the Neumann boundary problem (1.3) into the following information on the set $\mathscr{B}^\lambda_{u_\Gamma}$.

Lemma 2.4. *Let Ω, Σ and $\mathbf{C}$ be as in (2.1) and (2.2). Let u be the variational solution of (1.3). Then it holds*

$$\mathcal{B}^{\lambda}_{u_\Gamma} \cap B_1 \subset \mathcal{A}_u \cap B_1,$$

where $\mathcal{B}^{\lambda}_{u_\Gamma}$ is given by (2.12).

Proof. Recall that u is continuous up to the boundary and it is a viscosity supersolution of (1.3) by Proposition 2.2. Fix $\xi \in \mathcal{B}^{\lambda}_{u_\Gamma} \cap B_1$. Recall that this means that $\xi \in J_\Gamma u(x)$, for some $x \in \Gamma$ and that $\xi \cdot \nu_{\mathbf{C}}(x) > \lambda$ (and of course $|\xi| < 1$). We need to show that $\xi \in \mathcal{A}_u$, i.e., $\xi \in J_{\overline{\Omega}} u(\bar{x})$ for some $\bar{x} \in \Omega$.

First we claim that $\xi \notin J_{\overline{\Omega}} u(x)$, where $x \in \Gamma$ is the point for which $\xi \in J_\Gamma u(x)$. Indeed, assume the opposite, that is

$$u(y) \geq u(x) + \xi \cdot (y - x) =: \varphi(y) \qquad \text{for all } \; y \in \overline{\Omega}.$$

In particular, x is the minimum point of $u - \varphi$ and since u is a viscosity supersolution and $\nabla \varphi(x) = \xi$, see Definition 2.1, we have

$$\begin{cases} \xi \cdot \nu_\Gamma(x) + \lambda \geq 0 & \text{if } \; x \in \Gamma \setminus \gamma, \\ \max\{\xi \cdot \nu_\Sigma(x) - 1, \xi \cdot \nu_\Gamma(x) + \lambda\} \geq 0 & \text{if } x \in \gamma. \end{cases}$$

From the above condition, since $|\xi| < 1$ we have that $\xi \cdot \nu_\Gamma(x) + \lambda \geq 0$, which is impossible as $\xi \cdot \nu_\Gamma(x) = -\xi \cdot \nu_{\mathbf{C}}(x) < -\lambda$. Therefore $\xi \notin J_{\overline{\Omega}} u(x)$.

By the above it holds that the inequality $u(y) \geq \varphi(y)$ is not true for all $y \in \overline{\Omega}$. This means that $\bar{c} = \min_{y \in \overline{\Omega}} (u(y) - \varphi(y)) < 0$. In turn we have that the hyperplane $s = \varphi(y) + \bar{c}$ touches from below the graph of u at some point $\bar{x} \in \overline{\Omega}$. Clearly $\bar{x} \notin \Gamma$, since $\varphi(y) + \bar{c} < u(y)$ for all $y \in \Gamma$. On the other hand, if $\bar{x} \in \Sigma$, again by the supersolution property, we would have that $\xi \cdot \nu_\Sigma(\bar{x}) \geq 1$, which is impossible since $|\xi| < 1$. Therefore $\bar{x} \in \Omega$, which implies $\xi \in \mathcal{A}_u$. $\square$

We thus deduce from Lemma 2.4 that in order to show the inequality (1.5), it is enough to study the restriction of u on Γ. It turns out that, in terms of the inequality (1.5), it is not relevant that u_Γ is a restriction of the solution of the Neumann problem (1.3). Indeed, from now on we study generic functions $v : K \to \mathbb{R}$ which are defined on $K \subset \partial \mathbf{C}$. The following lemma provides an important property on the structure of the subdifferentials of such functions, which is due to the convexity of $\mathbf{C}$.

Lemma 2.5. *Let $\mathbf{C} \subset \mathbb{R}^N$ be a closed convex set of class C^1, $K \subset \partial \mathbf{C}$ and $v : K \to \mathbb{R}$. If $\xi \in J_K v(x)$ for some $x \in K$, then*

$$\xi + t\nu_{\mathbf{C}}(x) \in J_K v(x) \qquad \textit{for all } \; t > 0.$$

Proof. If $\xi \in J_K v(x)$, then

$$v(y) - v(x) \geq \xi \cdot (y - x) \qquad \text{for all } \; y \in K.$$

By convexity it holds $\nu_{\mathbf{C}}(x)\cdot(y-x)\le 0$ for all $y\in K$. Therefore for any $t>0$ it holds

$$v(y)-v(x)\ge(\xi+t\nu_{\mathbf{C}}(x))\cdot(y-x)\qquad\text{for all }\ y\in K,$$

that is $\xi+t\nu_{\mathbf{C}}(x)\in J_Kv(x)$. □

2.3. **The λ-ABP property and the isoperimetric inequality.** By Lemma 2.4 it is clear that if we would have $|\mathcal{B}^{\lambda}_{u_\Gamma}\cap B_1|\ge|B^\lambda|$, where u_Γ is the restriction of u on Γ, then we would have the inequality (1.5). As we discussed in the previous section, this property is related to generic functions $v:K\to\mathbb{R}$, where $K\subset\partial\mathbf{C}$. This in turn means that such a property is only related to the convex set $\mathbf{C}$ itself. For simplicity we restrict to functions defined on finite sets.

Definition 2.6. *Let $\mathbf{C}\subset\mathbb{R}^N$ be a closed convex set of class C^1 and $\lambda\in(-1,1)$. We say that $\mathbf{C}$ has the λ-*ABP property *if for any finite subset $K\subset\partial\mathbf{C}$ and for every $v:K\to\mathbb{R}$*

$$|\mathcal{B}^{\lambda}_{v}\cap B_1|\ge|B^\lambda|,$$

where we recall $B^\lambda=\{x\in B_1: x\cdot e_N>\lambda\}$ and $\mathcal{B}^{\lambda}_{v}$ is defined in (2.11).

Let us make a few remarks. First, it is not a priori clear why a convex set would satisfy Definition 2.6, but we show at the end of the section that every convex set satisfy λ-ABP property for $\lambda=0$. This follows rather easily from Lemma 2.5. In Section 3 we show that convex cylinders have the λ-ABP for every $\lambda\in(-1,1)$. This is much more involved, but the statement is again based on Lemma 2.5, which is the only property of the subdifferentials that we need.

We also remark that it follows immediately from the definition that for every $x\in K$ the subdifferential $J_Kv(x)$ is a closed convex set in $\mathbb{R}^N$. Moreover if $K\subset\partial\mathbf{C}$ is a finite set then there are only finitely many subdifferentials $J_Kv(x)$ and it is also immediate that they are essentially disjoint, i.e., they have disjoint interiors. Therefore by Remark 2.3, the subdifferentials $J_Kv(x)$ for $x\in K$ form a convex partition of the space $\mathbb{R}^N$, i.e., partition by convex sets. Hence, the property in Definition 2.6 is really a property related to convex partition of $\mathbb{R}^N$ by sets $J_Kv(x)$ which satisfy the statement of Lemma 2.5.

We proceed with the following remark.

Remark 2.7. *A simple scaling argument shows that $\mathbf{C}$ has the λ-ABP property if and only if $\eta\mathbf{C}$ has λ-ABP property, for any $\eta>0$. Moreover, if $\mathbf{C}$ has the λ-ABP property then it also holds*

$$|\mathcal{B}^{\lambda}_{v}\cap B_r|\ge|B^\lambda_r|,$$

for all $r\in(0,1)$.

We start by showing that the above ABP property stated on discrete sets is inherited by all compact subsets of $\partial\mathbf{C}$, provided that $\partial\mathbf{C}$ is smooth (C^1-regular is in fact enough).

Lemma 2.8. *Let $\mathbf{C} \subset \mathbb{R}^N$ be a closed convex set with C^1 boundary. Assume also that $\mathbf{C}$ satisfies the λ-ABP property, according to Definition 2.6, for some $\lambda \in (-1, 1)$. Then, for all compact subsets $K \subset \partial\mathbf{C}$ and for all bounded functions $v : K \to \mathbb{R}$ it holds*

$$|\mathcal{B}_v^\lambda \cap B_1| \geq |B^\lambda|, \tag{2.13}$$

where $\mathcal{B}_v^\lambda$ is defined in (2.11).

Proof. Fix K, which is a compact subset of $\partial\mathbf{C}$ and $v : K \to \mathbb{R}$ which is bounded. Note first that the convex envelope of v shares the same subdifferentials as v itself. Therefore we may assume that v is a restriction of a convex function, in particular, v is continuous and at every point in K its subdifferential is non-empty. Let us choose points $x_1, x_2, x_3, \dots$ in K which are dense in K. We define $K_n = \{x_1, \dots, x_n\}$ and the function $v_n : K_n \to \mathbb{R}$ as the restriction of v on K_n, i.e., $v_n(x_i) = v(x_i)$ for $i = 1, 2, \dots, n$. Note that then

$$\sup_{x \in K} \inf_{y \in K_n} |v(x) - v_n(y)| \to 0 \quad \text{as } n \to \infty. \tag{2.14}$$

Let us fix $\lambda' > \lambda$ close to λ and show first that when n is large it holds

$$\mathcal{B}_{v_n}^{\lambda'} \cap B_1 \subset \mathcal{B}_v^\lambda \cap B_1. \tag{2.15}$$

To this aim we fix $\xi \in \mathcal{B}_{v_n}^{\lambda'} \cap B_1$. This means that $\xi \in J_{K_n} v_n(x_n)$ for some $x_n \in K_n$ and $\xi \cdot \nu_{\mathbf{C}}(x_n) > \lambda'$. By Remark 2.3 there is $x \in K$ such that $\xi \in J_K v(x)$. By the convergence (2.14) for any $\varepsilon > 0$ it holds $|x - x_n| < \varepsilon$ when n is large. Therefore we have

$$\xi \cdot \nu_{\mathbf{C}}(x) \geq \xi \cdot \nu_{\mathbf{C}}(x_n) - |\nu_{\mathbf{C}}(x) - \nu_{\mathbf{C}}(x_n)| > \lambda' - O(\varepsilon) > \lambda$$

when ε is small and n is large. This yields (2.15).

By Definition 2.6 we have the estimate

$$|\mathcal{B}_{v_n}^\lambda \cap B_1| \geq |B^\lambda|. \tag{2.16}$$

Let us then prove that there exists a universal constant $C > 1$ such that

$$|\mathcal{B}_{v_n}^{\lambda'} \cap B_1| \geq |\mathcal{B}_{v_n}^\lambda \cap B_1| - C|\lambda' - \lambda|. \tag{2.17}$$

We define for $t \geq 0$

$$A_t := \bigcup_{x \in K_n} \{\xi \in J_{K_n} v_n(x) : \xi \cdot \nu_{\mathbf{C}}(x) = \lambda + t\}$$

and recall that by (2.11) $\mathcal{B}_{v_n}^{\lambda+t} = \bigcup_{x \in K_n} \{\xi \in J_{K_n} v_n(x) : \xi \cdot \nu_{\mathbf{C}}(x) > \lambda + t\}$. Clearly, the function $t \mapsto |\mathcal{B}_{v_n}^{\lambda+t} \cap B_r|$ is decreasing and Lipschitz continuous, and at points of differentiability it holds

$$\frac{d}{dt} |\mathcal{B}_{v_n}^{\lambda+t} \cap B_r| = -\mathcal{H}^{N-1}(A_t \cap B_r) \tag{2.18}$$

for all $r > 0$.

We fix $x \in K_n$ and study the set

$$A_t(x) = \{\xi \in J_{K_n} v_n(x) : \xi \cdot \nu_{\mathbf{C}}(x) = \lambda + t\}.$$

Assume that $\xi \in A_t(x)$. In particular then $\xi \in J_{K_n} v_n(x)$. We use Lemma 2.5 to deduce that for every $s > 0$ it holds $\xi + s\nu_{\mathbf{C}}(x) \in J_{K_n} v_n(x)$. Therefore for every $s > 0$ it holds

$$A_t(x) + s\{\nu_{\mathbf{C}}(x)\} \subset A_{t+s}(x).$$

Since the map $\xi \mapsto \xi + s v_n(x)$ is an isometry, it holds $\mathcal{H}^{N-1}(A_t(x) \cap B_{1+t}) \le \mathcal{H}^{N-1}(A_{t+s}(x) \cap B_{1+t+s})$. Therefore we deduce that the function

$$t \mapsto \mathcal{H}^{N-1}(A_t \cap B_{1+t}) = \sum_{x \in K_n} \mathcal{H}^{N-1}(A_t(x) \cap B_{1+t})$$

is non-decreasing. Integrating (2.18) we have

$$\int_1^2 \mathcal{H}^{N-1}(A_t \cap B_{1+t})\, dt \le -\int_1^2 \frac{d}{dt} |\mathcal{B}^{\lambda+t}_{v_n} \cap B_3|\, dt \le |B_3|.$$

By the mean value theorem there is $\hat{t} \in [1, 2]$ such that $\mathcal{H}^{N-1}(A_{\hat{t}} \cap B_{1+\hat{t}}) \le C$. In turn, using the monotonicity obtained above, we have

$$\mathcal{H}^{N-1}(A_t \cap B_1) \le C \qquad \text{for all } t \in [0, 1].$$

The claim (2.17) then follows by integrating (2.18) from $t = 0$ to $t = \lambda' - \lambda$.

Finally the statement of the lemma follows from (2.15), (2.16) and (2.17) and letting $\lambda' \to \lambda$. $\square$

Proposition 2.9. *Let* $\mathbf{C}$ *be a closed convex set which satisfies* (2.1) *and assume it satisfies the* λ*-ABP property for some* $\lambda \in (-1, 1)$. *Then, for every open set* Ω *satisfying* (2.2) *and* $|\Omega| = v$ *we have*

$$J_{\lambda,\mathbf{C}}(\Omega) \ge J_{\lambda,\boldsymbol{H}}(B^\lambda[v]).$$

Moreover, if $J_{\lambda,\mathbf{C}}(\Omega) = J_{\lambda,\boldsymbol{H}}(B^\lambda[v])$, *then* Ω *coincides with a solid spherical cap isometric to* $B^\lambda[v]$, *supported on a facet of* $\mathbf{C}$.

Proof. By scaling and Remark 2.7, we may assume $v = |B^\lambda|$. The inequality follows from the argument from the introduction which we now repeat rigorously. Let $u : \bar{\Omega} \to \mathbb{R}$ be the variational solution of the Neumann boundary problem (1.3) and denote its restriction to $\Gamma \subset \partial\mathbf{C}$ by u_Γ. Let $\mathcal{A}_u$ be the set defined in (2.10) and let $\mathcal{B}^\lambda_{u_\Gamma}$ be the set defined in (2.12). Denote also by Ω^+ the set of points $x \in \Omega$ where $J_{\bar{\Omega}} u(x) \cap B_1$ is non-empty. Then by Lemma 2.4 it holds

$$|\mathcal{A}_u \cap B_1| \ge |\mathcal{B}^\lambda_{u_\Gamma} \cap B_1|.$$

On the other hand, Definition 2.6 of the λ-ABP-property and Lemma 2.8 yield

$$|\mathcal{B}^\lambda_{u_\Gamma} \cap B_1| \ge |B^\lambda|.$$

Therefore we have
(2.19)

$$|\Omega| = |B^\lambda| \le |\mathscr{A}_u \cap B_1| = |\nabla u(\Omega^+)| \le \int_{\Omega^+} \det \nabla^2 u \, dx \le \int_{\Omega^+} \frac{(\Delta u)^N}{N^N} \, dx \le \left(\frac{J_{\lambda,\mathbf{C}}(\Omega)}{|\Omega| N} \right)^N |\Omega^+|.$$

The inequality then follows from

$$J_{\lambda,\mathbf{C}}(B^\lambda) = N|B^\lambda| = N|\Omega| \ge N|\Omega^+|.$$

We now analyze the case of equality. Assume again without loss of generality that $v = |B^\lambda|$ and that Ω is an open set satisfying (2.2) for which $J_{\lambda,\mathbf{C}}(\Omega) = J_{\lambda,\mathbf{H}}(B^\lambda)$.

We begin by showing that Ω is connected. This follows from the fact that the isoperimetric function $v \mapsto J_{\lambda,\mathbf{H}}(B^\lambda[v])$ is strictly convex. Indeed, we argue by contradiction and assume there is a component $\Omega_1 \subset \Omega$, with $|\Omega_1| =: v_1 \in (0, |B^\lambda|)$ such that

$$J_{\lambda,\mathbf{C}}(\Omega) = J_{\lambda,\mathbf{C}}(\Omega_1) + J_{\lambda,\mathbf{C}}(\Omega \setminus \Omega_1).$$

Then by the isoperimetric inequality that we just proved, and by the strict concavity of the isoperimetric function it holds

$$J_{\lambda,\mathbf{C}}(\Omega) = J_{\lambda,\mathbf{C}}(\Omega_1) + J_{\lambda,\mathbf{C}}(\Omega \setminus \Omega_1) \ge J_{\lambda,\mathbf{H}}(B^\lambda[v_1]) + J_{\lambda,\mathbf{H}}(B^\lambda[|B^\lambda| - v_1]) > J_{\lambda,\mathbf{H}}(B^\lambda), \tag{2.20}$$

which is a contradiction. Hence, Ω is connected.

Let $u : \bar{\Omega} \to \mathbb{R}$ be the variational solution of the Neumann boundary problem (1.3). Since $J_{\lambda,\mathbf{C}}(\Omega) = J_{\lambda,\mathbf{H}}(B^\lambda)$, then all the inequalities in (2.19) become equalities and $|\Omega| = |\Omega^+|$. The equality in the arithmetic-geometric mean inequality means that

$$\det \nabla^2 u = \frac{(\Delta u)^N}{N^N} = 1 \text{ and thus all the eigenvalues of } \nabla^2 u \text{ are equal to 1 in } \Omega.$$

Hence, $\nabla^2 u = I$ in Ω and thus, by the connectedness of Ω, we infer that there exists $x_0 \in \mathbb{R}^N$ such that, up to an additive constant, $u(x) = \frac{1}{2}|x - x_0|^2$ in Ω. Moreover, it follows from the definition of the set Ω^+ that $|\nabla u(x)| < 1$ for all $x \in \Omega^+$. Since the equalities in (2.19) imply that $|\Omega \setminus \Omega^+| = 0$, we have $|\nabla u(x)| = |x - x_0| \le 1$ for all $x \in \Omega$. In other words, $\Omega \subset B_1(x_0)$. In turn, since $1 = \partial_\nu u(x) = (x - x_0) \cdot \nu_\Omega(x)$ for all $x \in \partial\Omega \setminus \mathbf{C}$, we have necessarily $\partial\Omega \setminus \mathbf{C} \subset \partial B_1(x_0)$.

Let now $\hat{x} \in \Gamma \setminus \gamma$ and consider the half-space

$$H := \{y \in \mathbb{R}^N : (y - x_0) \cdot \nu_{\mathbf{C}}(\hat{x}) > \lambda\}.$$

Since $-\lambda = \partial_\nu u(\hat{x}) = -(\hat{x} - x_0) \cdot \nu_{\mathbf{C}}(\hat{x})$, it follows that $\mathbf{C}$ is contained in $\mathbb{R}^N \setminus H$ and that ∂H is tangent to $\mathbf{C}$ at $\hat{x}$. On the other hand, for any nontangential direction $v \in \mathbb{S}^{N-1}$ such that $v \cdot \nu_{\mathbf{C}}(\hat{x}) > 0$, all the points of the half line $\hat{x} + tv$, $t > 0$, inside $B_1(x_0)$ are contained in Ω, since otherwise the same half line would hit a point of $\partial\Omega$ in $B_1(x_0) \setminus \mathbf{C}$, which is impossible since $\partial\Omega \setminus \mathbf{C} \subset \partial B_1(x_0)$. Thus, we may conclude that $B_1(x_0) \cap H \subset \Omega$. Since $B_1(x_0) \cap H$ is a spherical cap isometric to B^λ (and since $|\Omega| = |B^\lambda|$) we conclude that $\Omega = B_1(x_0) \cap H$, thus concluding the proof of the proposition.

□

Remark 2.10. *Note that if $N = 3$ and the relative isoperimetric problem*

$$\min\{J_{\lambda,\mathbf{C}}(E) : E \subset \mathbb{R}^N \setminus \mathbf{C} \text{ measurable, with } |E| = v\}$$

has a solution Ω, then, due to the the regularity results of [23], see also [9], Ω satisfies (2.2), provided $\mathbf{C}$ is of class $C^{1,1}$. Therefore if $\mathbf{C}$ satisfies the λ-ABP property Theorem 2.9 yields the relative isoperimetric inequality for all sets of finite perimeter and fully characterizes the case of equality.

We conclude this section by showing that the 0-ABP property holds for all convex sets with nonempty interior.

Proposition 2.11. *Let $\mathbf{C} \subset \mathbb{R}^N$ be a closed convex set of class C^1. Then $\mathbf{C}$ satisfies the 0-ABP property.*

Proof. Let $K = \{x_1, \dots, x_n\}$ be any discrete subset of $\partial\mathbf{C}$ and let $v : K \to \mathbb{R}$. Recall that $\bigcup_{i=1}^{n} J_K v(x_i) = \mathbb{R}^N$ (see Remark 2.3) and that the sets $J_K v(x_i)$ are a convex, in fact a finite intersection of half-spaces, and they have disjoint interiors. Moreover Lemma 2.5 implies that they have the property

$$\xi \in J_K v(x_i) \Longrightarrow \xi + t\nu_{\mathbf{C}}(x_i) \in J_K v(x_i) \text{ for all } t > 0. \tag{2.21}$$

Now, up to a set of Lebesgue measure zero, we may split $J_K v(x_i) = J_K v(x_i)^+ \cup J_K v(x_i)^-$, where

$$J_K v(x_i)^\pm := \{\xi \in J_K v(x_i) : \pm\xi \cdot \nu_{\mathbf{C}}(x_i) > 0\}.$$

Let us then fix $\xi \in J_K v(x_i)^-$ and denote $\lambda = -\xi \cdot \nu_{\mathbf{C}}(x_i) > 0$. Then by (2.21) the symmetric point $\hat{\xi} = \xi + 2\lambda\nu_{\mathbf{C}}(x_i)$ belongs to $J_K v(x_i)^+$. Hence $|J_K v(x_i)^+ \cap B_1| \geq \frac{1}{2}|J_K v(x_i) \cap B_1|$. In turn,

$$|\mathcal{B}_v^0| = \sum_{i=1}^{n} |J_K v(x_i)^+ \cap B_1| \geq \frac{1}{2}\sum_{i=1}^{n} |J_K v(x_i) \cap B_1| = \frac{1}{2}|B_1|, \tag{2.22}$$

which is the desired estimate. From the arbitrariness of K, we conclude that $\mathbf{C}$ satisfies the 0-ABP property. □

Remark 2.12. *By combining the previous proposition with Proposition 2.9 we recover an ABP-proof of the relative isoperimetric inequality outside convex sets obtained by Choe, Ghomi and Ritoré in [7]. Note that an ABP-argument for the same inequality has been already provided in [17]. However, in their argument our crucial estimate (2.22) is replaced by a geometric estimate based on normal cones and inspired by the techniques in [6], see [17,* Proposition 2.4].

3. The isoperimetric inequality outside convex cylinders

3.1. Convex cylinders have the λ-ABP property.

The purpose of this section is to prove the following:

Theorem 3.1. *Let $\mathbf{C}$ be of the form $\mathcal{C} \times \mathbb{R}^{N-2}$, where $\mathcal{C} \subset \mathbb{R}^2$ is a closed convex set of class C^1 with nonempty interior. Then $\mathbf{C}$ satisfies the λ-ABP property for every $\lambda \in (-1,1)$.*

In order to prove the above theorem we need to introduce some notation. In the following we denote generic vector $\xi \in \mathbb{R}^N$ by (z,w) where $z \in \mathbb{R}^2$, $w \in \mathbb{R}^{N-2}$. Given $E \subset \mathbb{R}^N$ and $w \in \mathbb{R}^{N-2}$ we set

$$E_w := \{z \in \mathbb{R}^2 : (z,w) \in E\}.$$

In the following, given $\lambda \in \mathbb{R}$, $\nu \in \mathbb{R}^2$, we set

$$H^\lambda_\nu = \{z \in \mathbb{R}^2 : z \cdot \nu > \lambda\}. \tag{3.1}$$

When $\nu = e_2$ we will simply write H^λ instead of $H^\lambda_{e_2}$. We denote by $D_r \subset \mathbb{R}^2$ the open disk of radius r centered at the origin and set $D = D_1$.

Let $K = \{x_1, \dots, x_n\}$ be any finite subset of $\mathbf{C}$ and let $v : K \to \mathbb{R}$ be any function. The proof of Theorem 3.1 will follow if we show that

$$|\mathcal{B}^\lambda_v \cap B_1| \geq |B^\lambda|, \tag{3.2}$$

where $\mathcal{B}^\lambda_v$ is defined in (2.11). To prove (3.2) we will show that

$$\mathcal{H}^2((\mathcal{B}^\lambda_v)_w \cap D_r) \geq \mathcal{H}^2((B^{\lambda,e_2})_w \cap D_r) \tag{3.3}$$

for all $\lambda \in (-1,1)$, $w \in \mathbb{R}^{N-2}$ with $|w| < 1$ and $r = \sqrt{1-|w|^2}$. Since $\nu_{\mathbf{C}}(x) \in \{\xi \in \mathbb{R}^N : \xi = (z,0)\} \approx \mathbb{R}^2$ for all $x \in \partial\mathbf{C}$

$$(\mathcal{B}^\lambda_v)_w \cap D_r = \bigcup_{x\in K} \Big\{z \in D_{\sqrt{1-|w|^2}} : (z,w) \in J_K v(x),\, z \cdot \nu_{\mathbf{C}}(x) > \lambda\Big\},$$
$$(B^{\lambda,e_2})_w = H^\lambda \cap D_{\sqrt{1-|w|^2}}.$$

Indeed, we shall prove a stronger inequality than (3.3), that is that for any $|w| < 1$

$$\mathcal{H}^2\Big(\bigcup_{x\in K} \{z \in D_r : (z,w) \in J_K v(x),\, z \cdot \nu_{\mathbf{C}}(x) > \lambda\}\Big) \geq \mathcal{H}^2(H^\lambda \cap D_r),$$

for all $r \in (0,1)$. In turn, this inequality will follow from the inequality

$$\mathcal{H}^1\Big(\bigcup_{x\in K} \{z \in \partial D_\varrho : (z,w) \in J_K v(x),\, z \cdot \nu_{\mathbf{C}}(x) > \lambda\}\Big) \geq \mathcal{H}^1(H^\lambda \cap \partial D_\varrho),$$

for all $\varrho \in (0,1)$. By rescaling, this last inequality is equivalent to

$$\mathcal{H}^1\Big(\bigcup_{x\in K} \{z' \in \partial D : (z',w') \in J_K v_\varrho(x),\, z' \cdot \nu_{\mathbf{C}}(x) > \lambda'\}\Big) \geq \mathcal{H}^1(H^{\lambda'} \cap \partial D), \tag{3.4}$$

where we have set $w' = \frac{w}{\varrho} \in \mathbb{R}^{N-2}$, $\lambda' = \frac{\lambda}{\varrho} \in \mathbb{R}$, $u_\varrho = \frac{u}{\varrho}$. Note that since for all w'

$$\bigcup_{x\in K} \Big\{ z' \in \mathbb{R}^2 : (z', w') \in J_K v_\varrho(x) \Big\} = \mathbb{R}^2,$$

(3.4) is trivially satisfied if $\lambda' \geq 1$ or $\lambda' \leq -1$. Therefore, we are ultimately bound to show that for any function $v : K \to \mathbb{R}$, any $w \in \mathbb{R}^{N-2}$ and any $\lambda \in (-1, 1)$

$$\mathcal{H}^1\Big(\bigcup_{x\in K} \{z \in \partial D : (z, w) \in J_K v(x),\, z \cdot \nu_{\mathbf{C}}(x) > \lambda\}\Big) \geq \mathcal{H}^1(H^\lambda \cap \partial D). \tag{3.5}$$

The rest of the section is devoted to the proof of (3.5).

Given $w \in \mathbb{R}^{N-2}$, for all $i = 1, \dots, n$ we define the *cell* A_i and the normal ν_i associated with it as

$$A_i = \{z \in \mathbb{R}^2 : (z, w) \in J_K v(x_i)\} \quad \text{and} \quad \nu_i = \nu_{\mathbf{C}}(x_i). \tag{3.6}$$

Recall that each A_i is convex and more precisely is obtained as a finite intersection of half planes and that the A_i's have pairwise disjoint interiors and their union is the whole $\mathbb{R}^2$. Moreover, from Lemma 2.5 we have that if $\xi \in J_K v(x_i)$ then $\xi + t\nu_i \in J_\Gamma(x_i)$ for all $t > 0$. Therefore

$$z \in A_i,\, t > 0 \quad \Longrightarrow \quad z + t\nu_i \in A_i. \tag{3.7}$$

In what follows we say A_i and A_j are *neighboring cells* if $\mathcal{H}^1(\partial A_i \cap \partial A_j) > 0$.

We prove (3.5) by induction and to this aim we need the following deleting procedure. Remove a point x_j from K, set $\widetilde{K} = K \setminus x_j$ and denote by $\tilde{v}$ the restriction of v to $\widetilde{K}$ and by $J_{\widetilde{K}}\tilde{v}(x_i)$ the corresponding subdifferentials. Finally, define $\widetilde{A}_i$, for $i \neq j$ as in (3.6) with K and v replaced by $\widetilde{K}$ and $\tilde{v}$ respectively (and with w unchanged). Clearly, $J_K v(x_i) \subset J_{\widetilde{K}}\tilde{v}(x_i)$, hence $A_i \subset \widetilde{A}_i$, for all $i \neq j$. However, since also the interiors of the $\widetilde{A}_i$'s are mutually disjoint and convex and $\bigcup_{i\neq j} \widetilde{A}_i = \mathbb{R}^2$, we have necessarily that

$$\widetilde{A}_i = A_i \quad \text{if } A_i \text{ and } A_j \text{ are not neighboring cells.} \tag{3.8}$$

We will refer to this procedure as *deleting a cell.* Note that if $A_j \cap D = \emptyset$, we may delete A_j and this deletion does not affect the intersections of the remaining A_i's with D. Therefore, without loss of generality we may assume that all the A_i's intersect D.

We will say that A_i is a *disconnecting cell* if $D \setminus A_i$ is disconnected, see Figure 2. We will also say that a disconnecting cell A_i is *extremal* if at least one of the two connected components of $\overline{D \setminus A_i}$ contains no disconnecting cells.

We have the following lemma.

Lemma 3.2. *Let A_i be a disconnecting cell. Then either A_i is the only the disconnecting cell (hence extremal) or there are at least two extremal disconnecting cells.*

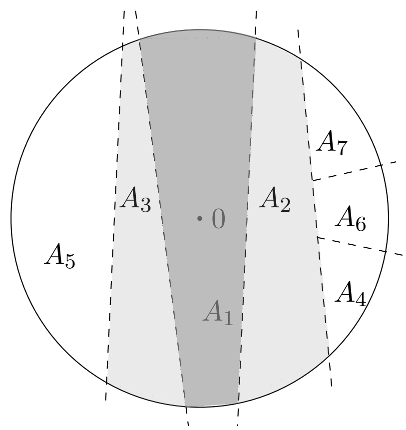

FIGURE 2. An example of a configuration with seven cells. The cells A_1, A_2 and A_3 are disconnecting and A_2 and A_3 are also extremal.

Proof. Denote by C_1 and C_2 the components of $\overline{D \setminus A_i}$ and assume that one of them, say C_1, contains a disconnecting cell A_j. Consider the component of $\overline{D \setminus A_j}$ that does not contain C_2. Either this component contains no disconnecting cells (which makes A_j extremal) or we iterate this procedure until we get after finitely many steps an extremal disconnecting cell contained in C_1. Now, if A_i is not extremal, we apply the same procedure in C_2 to find a second extremal disconnecting cell. □

We introduce the following useful quantities: for every $i = 1, \dots, n$, we set (see Figure 3)

$$\begin{aligned} \ell_i &:= \min\{z \cdot \nu_i : z \in A_i \cap \partial D\} \quad \text{(entry value of } A_i\text{)} \\ m_i &:= \max\{z \cdot \nu_i : z \in A_i \cap \partial D\} \quad \text{(exit value of } A_i\text{)}, \end{aligned} \tag{3.9}$$

Note that by (3.7) it is easily checked that if $0 \in A_i$ then $m_i = 1$ and that $m_j > 0$ for all j. By reordering the indices in what follows we may assume that $0 \in A_1$ and that

$$1 = m_1 \geq m_2 \geq \dots.. \geq m_n. \tag{3.10}$$

Note also that if A_i is not disconnecting, then it holds that

$$\lambda \in [\ell_i, m_i] \Longrightarrow \text{there exists } z \in A_i \cap \partial D \text{ s.t. } z \cdot \nu_i = \lambda. \tag{3.11}$$

Indeed, if A_i is not disconnecting, then the intersection of A_i with ∂D is a connected arc and therefore the function $z \mapsto z \cdot \nu_i$ takes all values in $[\ell_i, m_i]$ along such arc.

We also define

$$l := \max\{z \cdot \nu_1 : z \in \partial A_1 \cap \partial D\}$$

and note that $\ell_1 \leq l \leq 1$. Moreover, if $l < 1$ and $\lambda \in (l, 1)$ (recall that $m_1 = 1$), then the segment $D \cap \partial H^{\lambda}_{\nu_1}$ does not intersect ∂A_1 and therefore

$$\mathcal{H}^0(\partial D \cap \partial H^{\lambda}_{\nu_1} \cap A_1) = 2 \quad \text{for all } \lambda \in (l, 1). \tag{3.12}$$

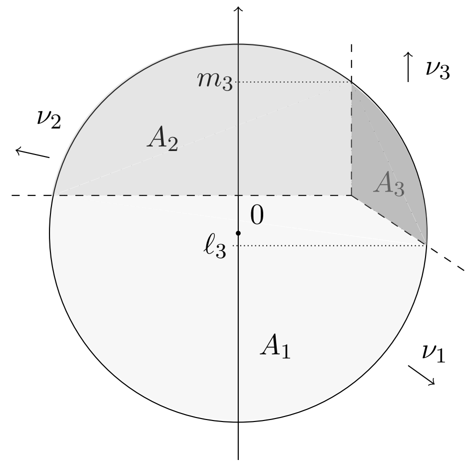

FIGURE 3. An example of a configuration with three cells with the associated normals ν_1, ν_2 and ν_3. The entry value of the third cell is ℓ_3 and the exit value is m_3.

Lemma 3.3. *It holds $m_2 \geq l$.*

Proof. We may assume that $l \geq 0$ since otherwise the claim is trivially true as $m_2 > 0$. Without loss of generality we may assume $\nu_1 = e_2$. Let $\bar{z} \in \partial A_1$ be such that $|\bar{z}| = \mathrm{dist}(0, \partial A_1) =: d$, see Figure 4. We claim that

$$d \leq \sqrt{1 - l^2}. \tag{3.13}$$

To this aim, let $\tilde{z} \in \partial A_1 \cap \partial D$ be such that $l = \tilde{z} \cdot e_2$. Let H be a (closed) half-plane such that $\tilde{z} \in \partial H$ and $A_1 \subset H$. Since A_1 contains the origin and $A_1 \subset H$ it holds $d = \mathrm{dist}(0, \partial A_1) \leq \mathrm{dist}(0, \partial H)$. Let ω be the unit vector normal to ∂H pointing towards A_1, i.e., $H = \{z : (z - \bar{z}) \cdot \omega \geq 0\}$. Then using the property (3.7) it holds $\bar{z} + te_2 \in A_1 \subset H$ for all $t > 0$, which implies $\omega \cdot e_2 \geq 0$. In turn, since $l \geq 0$, this implies that $\mathrm{dist}(0, \partial H)$ is less or equal than the distance of $\tilde{z}$ to the e_2-axis, which is given by $\sqrt{1 - l^2}$ (see Figure 4). Hence, the claim (3.13) follows.

Since $\bar{z} \in \partial A_1$ there is a neighboring cell, say A_i, such that $\bar{z} \in \partial A_i$. Then the choice of $\bar{z}$ implies that $d = |\bar{z}| = \mathrm{dist}(0, A_i)$. On the other hand by the property (3.7) it holds $t\nu_i + \bar{z} \in A_i$ for every $t > 0$. Therefore $|t\nu_i + \bar{z}|^2 \geq |\bar{z}|^2$ for $t > 0$ from which we deduce $\bar{z} \cdot \nu_i \geq 0$.

Denote by $\hat{t} \in [0, 1]$ the value such that $\hat{t}\nu_i + \bar{z} \in \partial D$. Then

$$1 = |\hat{t}\nu_i + \bar{z}|^2 = \hat{t}^2 + 2\hat{t}\bar{z} \cdot \nu_i + |\bar{z}|^2 = \hat{t}^2 + 2\hat{t}\bar{z} \cdot \nu_i + d^2.$$

By using this and (3.13) we have

$$\hat{t}^2 + 2\hat{t}\bar{z} \cdot \nu_i \geq l^2. \tag{3.14}$$

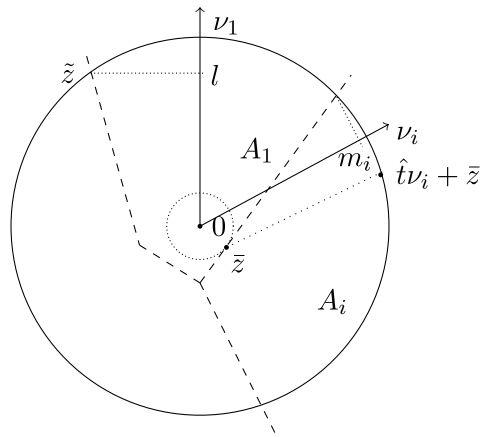

FIGURE 4. Situation as in the proof of Lemma 3.3. The point $\bar{z}$ is chosen as the closest to the origin on ∂A_1. The cell A_i is neighbour to A_1 such that $\bar{z} \in \partial A_i$.

On the other hand, since $\hat{t}\nu_i + \bar{z} \in A_i$, the definition of the exit value implies

$$m_i \geq (\hat{t}\nu_i + \bar{z}) \cdot \nu_i = \hat{t} + \bar{z} \cdot \nu_i.$$

Since $\bar{z} \cdot \nu_i \geq 0$, this implies

$$m_i^2 \geq \hat{t}^2 + 2\hat{t}\bar{z} \cdot \nu_i + (\bar{z} \cdot \nu_i)^2 \geq \hat{t}^2 + 2\hat{t}\bar{z} \cdot \nu_i$$

and the inequality of the lemma follows from (3.14) and from the fact that $m_2 \geq m_i$. □

Proof of Theorem 3.1. We let $K = \{x_1, \dots, x_n\} \subset \mathbf{C}$ and recall that we just need to show (3.5). We proceed by defining

$$\varphi_K(\lambda) = \sum_{i=1}^{n} \mathcal{H}^1(A_i \cap H^{\lambda}_{\nu_i} \cap \partial D), \quad \lambda \in [-1, 1],$$

where the cells A_i and the associated normal ν_i are defined in (3.6) (for a fixed $w \in \mathbb{R}^{N-2}$) and the half-spaces $H^{\lambda}_{\nu_i}$ in (3.1). Recall that the A_i's have mutually disjoint interiors, they are finite intersections of half-spaces and $\bigcup_i A_i = \mathbb{R}^2$. Setting also $\varphi_H(\lambda) = \mathcal{H}^1(H^{\lambda} \cap \partial D)$, the inequality (3.5) can be rewritten as

$$\varphi_K(\lambda) \geq \varphi_H(\lambda) \quad \text{for all } \lambda \in (-1, 1). \tag{3.15}$$

Note that

$$\begin{aligned} &\varphi_K(-1) = 2\pi = \varphi_H(-1), \\ &\varphi_K(1) = 0 = \varphi_H(1). \end{aligned} \tag{3.16}$$

We prove the inequality by induction on the cardinality n of K. Note that the conclusion is trivially true if $n = 1$. Therefore we assume the statement to be true for all $k \leq n - 1$.

Note that a simple application of the area formula implies that φ_K is locally Lipschitz continuous in $(-1,1)$ and that for a.e. $\lambda \in (-1,1)$

$$\begin{aligned}\varphi_K'(\lambda) &= -\sum_{i=1}^{n} \int_{A_i \cap \partial H^\lambda_{\nu_i} \cap \partial D} \frac{1}{\sqrt{1-(z\cdot\nu_i)^2}}\, d\mathcal{H}^0(z) \\ &= -\frac{1}{\sqrt{1-\lambda^2}} \sum_{i=1}^{n} \mathcal{H}^0(A_i \cap \partial H^\lambda_{\nu_i} \cap \partial D),\end{aligned} \tag{3.17}$$

while

$$\varphi_H'(\lambda) = -\frac{2}{\sqrt{1-\lambda^2}}. \tag{3.18}$$

Let us denote

$$\Lambda := \max_{i\in\{1,\dots,n\}} \ell_i, \tag{3.19}$$

where the ℓ_i's are the *entry values* defined in (3.9). We begin by proving the inequality (3.15) for all $\lambda \in [-1,\Lambda]$. Clearly we may assume that $\Lambda > -1$.

Let j be such that $\ell_j = \Lambda$ and $\lambda \in [-1,\Lambda)$. Since $\lambda < \ell_j$, by definition of the entry value we have $A_j \cap H^\lambda_{\nu_j} \cap \partial D = A_j \cap \partial D$, hence

$$\varphi_K(\lambda) = \sum_{i\neq j} \mathcal{H}^1(A_i \cap H^\lambda_{\nu_i} \cap \partial D) + \mathcal{H}^1(A_j \cap \partial D).$$

Denote by $\widetilde{A}_i$, $i \neq j$, the new partition obtained by deleting the cell A_j (see the definition of this deleting procedure before (3.8)). By the induction assumption we have

$$\sum_{i\neq j} \mathcal{H}^1(\widetilde{A}_i \cap H^\lambda_{\nu_i} \cap \partial D) \geq \varphi_H(\lambda) \qquad \text{for all } \lambda \in [-1,1].$$

In turn, we conclude that if $\lambda \in [-1,\Lambda)$

$$\begin{aligned}\varphi_K(\lambda) &= \sum_{i\neq j} [\mathcal{H}^1(A_i \cap \partial D) - \mathcal{H}^1((A_i \cap \partial D) \setminus H^\lambda_{\nu_i})] + \mathcal{H}^1(A_j \cap \partial D) \\ &= \sum_{i=1}^{n} \mathcal{H}^1(A_i \cap \partial D) - \sum_{i\neq j} \mathcal{H}^1((A_i \cap \partial D) \setminus H^\lambda_{\nu_i}) \\ &\geq 2\pi - \sum_{i\neq j} \mathcal{H}^1((\widetilde{A}_i \cap \partial D) \setminus H^\lambda_{\nu_i}) = \sum_{i\neq j} \mathcal{H}^1(\widetilde{A}_i \cap \partial D \cap H^\lambda_{\nu_i}) \geq \varphi_H(\lambda),\end{aligned}$$

where the first inequality follows from the fact the A_i's cover the whole plane and the fact that $A_i \subset \widetilde{A}_i$ for $i \neq j$. This proves (3.15) for $-1 \leq \lambda < \Lambda$, hence also for $1 \leq \lambda \leq \Lambda$ by continuity.

In order to prove (3.15) for $\lambda \in (\Lambda, 1)$ we distinguish two cases.

Case 1. We first assume that there are no disconnecting cells.

We will show that

$$\sum_{i=1}^{n} \mathcal{H}^0(A_i \cap \partial H^\lambda_{\nu_i} \cap \partial D) \geq 2 \quad \text{for } \lambda \in (\Lambda, 1). \tag{3.20}$$

Indeed, note that once this is established, then by (3.17) and (3.18) we get $\varphi'_K(\lambda) - \varphi'_H(\lambda) \leq 0$ for $\lambda \in (\Lambda, 1)$. That is, $\varphi_K - \varphi_H$ is non-increasing in $(\Lambda, 1)$. Recalling that $\varphi_K(1) - \varphi_H(1) = 0$ by (3.16), the conclusion would follow.

Let us therefore focus on the proof of (3.20). Recall that A_1 is the cell containing the origin of $\mathbb{R}^2$ and that the cells are indexed in such a way that the *exit values* defined in (3.9) satisfy (3.10). Recall also that by (3.12)

$$\mathcal{H}^0(\partial D \cap \partial H^\lambda_{\nu_1} \cap A_1) = 2 \quad \text{for } \lambda \in (l, 1)$$

and since A_1, A_2 are not disconnecting it holds by (3.11) that

$$\mathcal{H}^0(\partial D \cap \partial H^\lambda_{\nu_i} \cap A_i) \geq 1 \quad \text{for } \lambda \in (\ell_i, m_i),\ i = 1, 2.$$

The conclusion (3.20) now follows recalling that by Lemma 3.3 $m_2 \geq l$.

Step 2. Here we assume that there exists at least one disconnecting cell. By Lemma 3.2 we either have only one disconnecting cell or we can find two distinct extremal cells A_{j_1}, A_{j_2}. Let us assume from now on that the second case holds, the other one being analogous (in fact simpler). Let C_1 be the component of $\overline{D \setminus A_{j_1}}$, which does not contain A_{j_2}, and similarly let C_2 be the component of $\overline{D \setminus A_{j_2}}$, which does not contain A_{j_1}. Clearly C_1 and C_2 are disjoint. Let now J_1, J_2 be two disjoint set of indices such that

$$C_1 = \bigcup_{i \in J_1} (A_i \cap \overline{D}) \quad \text{and} \quad C_2 = \bigcup_{i \in J_2} (A_i \cap \overline{D}).$$

(To illustrate the situation the reader may consider the case in Figure 2, where $A_{j_1} = A_2$, $A_{j_2} = A_3$, $C_1 = (A_4 \cup A_6 \cup A_7) \cap \overline{D}$ and $C_2 = A_5 \cap \overline{D}$). Define also for $j = 1, 2$

$$M_j := \max_{i \in J_j} m_i \quad \text{and} \quad \overline{M} := \min\{M_1, M_2\}.$$

We claim that

$$\varphi_K(\lambda) \geq \varphi_H(\lambda) \quad \text{for all } \lambda \in [\overline{M}, 1]. \tag{3.21}$$

Indeed, without loss of generality we may assume $\overline{M} = M_1$ and $M_1 < 1$. By deleting the cells A_i for all $i \notin J_1$ and $i \neq j_1$, we generate a new partition of the form $\{A_i\}_{i \in J_1} \cup \{\tilde{A}\}$, with $\tilde{A} \supset A_{j_1}$. Note that we used the fact that by this deleting procedure the cells in C_1 are not affected due to (3.8). Note also that $\tilde{A}$ contains the origin and satisfies (3.7) with respect to ν_{j_1}. Indeed, otherwise the origin would be in C_1 and this in turn would imply that $M_1 = 1$. Let us define

$$\tilde{l} := \max\{z \cdot \nu_{j_1} \,:\, z \in \partial \tilde{A} \cap \partial D\}.$$

Then, it holds $\{z \in D : z \cdot \nu_{j_1} > \tilde{l}\} \subset \tilde{A}$ and thus

$$C_1 \subset \{z \in \overline{D} : z \cdot \nu_{j_1} \leq \tilde{l}\}. \tag{3.22}$$

Moreover, by Lemma 3.3 we have

$$M_1 \geq \tilde{l}. \tag{3.23}$$

Next we need another deleting procedure from the original partition $\{A_i\}_{i=1,\dots,n}$; namely, this time we delete all the cells A_i with $i \in J_1$. Using again (3.8), we see that newly generated partition is of the form $\{A_i\}_{i \notin J_1} \cup \{\tilde{A}_{j_1}\}$, where $\tilde{A}_{j_1} = A_{j_1} \cup C_1$. By the induction assumption it holds

$$\sum_{i \notin J_1} \mathcal{H}^1(A_i \cap H^\lambda_{\nu_i} \cap \partial D) + \mathcal{H}^1(\widetilde{A}_{j_1} \cap H^\lambda_{\nu_{j_1}} \cap \partial D) \geq \varphi_H(\lambda),$$

for all $\lambda \in [-1, 1]$. Note now that by (3.22) we have $\mathcal{H}^1(\widetilde{A}_{j_1} \cap H^\lambda_{\nu_{j_1}} \cap \partial D) = \mathcal{H}^1(A_{j_1} \cap H^\lambda_{\nu_{j_1}} \cap \partial D)$ for $\lambda > \tilde{l}$. Therefore, for all $\lambda > \tilde{l}$ we have

$$\begin{aligned} \varphi_K(\lambda) &\geq \sum_{i \notin J_1} \mathcal{H}^1(A_i \cap H^\lambda_{\nu_i} \cap \partial D) + \mathcal{H}^1(\widetilde{A}_{j_1} \cap H^\lambda_{\nu_{j_1}} \cap \partial D) \\ &= \sum_{i \notin J_1} \mathcal{H}^1(A_i \cap H^\lambda_{\nu_i} \cap \partial D) + \mathcal{H}^1(A_{j_1} \cap H^\lambda_{\nu_{j_1}} \cap \partial D) \geq \varphi_H(\lambda). \end{aligned}$$

Recalling (3.23), claim (3.21) follows.

Recalling the beginning of the proof and (3.21), we have proved the inequality (3.15) for $\lambda \in [-1, \Lambda] \cup [\overline{M}, 1]$, where Λ is defined in (3.19). To reach the conclusion, we need to fill the gap between Λ and $\overline{M}$ (assuming without loss of generality $\Lambda < \overline{M}$). To this aim, we will show that

$$\sum_{i=1}^{n} \mathcal{H}^0(A_i \cap \partial H^\lambda_{\nu_i} \cap \partial D) \geq 2 \quad \text{for } \lambda \in (\Lambda, \overline{M}). \tag{3.24}$$

Once the claim is established, arguing as in the previous case we may deduce that $\varphi_K - \varphi_H$ is non-increasing in $(\Lambda, \overline{M})$, and the conclusion then follows from (3.21).

Let now $i_1 \in J_1$, be such that $m_{i_1} = M_1$. Since A_{i_1} is non disconnecting, then by (3.11) we have

$$\mathcal{H}^0(\partial D \cap \partial H^\lambda_{\nu_{i_1}} \cap A_{i_1}) \geq 1 \quad \text{for } \lambda \in (\ell_{i_1}, M_1).$$

Similarly, there exist $i_2 \in J_2$ such that $m_{i_1} = M_1$ and

$$\mathcal{H}^0(\partial D \cap \partial H^\lambda_{\nu_{i_2}} \cap A_{i_2}) \geq 1 \quad \text{for } \lambda \in (\ell_{i_2}, M_2).$$

The previous two inequalities yield (3.24), recalling that ℓ_{i_1} and ℓ_{i_2} are less than or equal Λ, while $M_i \geq \overline{M}$ for $i = 1, 2$. This concludes the proof of (3.5) and in turn of Theorem 3.1. □

3.2. **Proof of the main result.** In this section we finally prove Theorem 1.1. We will need the following approximation lemma, whose proof is given in the Appendix.

To this aim we recall we recall that a sequence $\{C_n\}$ of closed sets of $\mathbb{R}^N$ converge in the *Kuratoswki sense* to a closed set C if the following conditions are satisfied:

(i) if $x_n \in C_n$ for every n, then any limit point of $\{x_n\}$ belongs to C;
(ii) any $x \in C$ is the limit of a sequence $\{x_n\}$ with $x_n \in C_n$.

One can easily see that $C_n \to C$ in the sense of Kuratowski if and only if $\mathrm{dist}(\cdot, C_n) \to \mathrm{dist}(\cdot, C)$ locally uniformly in $\mathbb{R}^N$.

Lemma 3.4. *Let $\mathbf{C} \subset \mathbb{R}^N$ be of the form $\mathcal{C} \times \mathbb{R}^{N-2}$, where $\mathcal{C} \subset \mathbb{R}^2$ is a closed convex set of the plane with nonempty interior. Let $E \subset \mathbb{R}^N \setminus \mathbf{C}$ be a set of finite perimeter. Then there exist a sequence of closed convex sets $\mathbf{C}_n$ of the form $\mathbf{C}_n = \mathcal{C}_n \times \mathbb{R}^{N-2}$, $\mathcal{C}_n \subset \mathbb{R}^2$, which satisfy (2.1) and a sequence of open sets $\Omega_n \subset \mathbb{R}^N \setminus \mathbf{C}_n$ which satisfy (2.2), in fact $\partial \mathbf{C}_n$ and $\Sigma_n := \partial\Omega_n \setminus \mathbf{C}_n$ are C^∞, such that the following hold:*

(i) *$\mathbf{C}_n \to \mathbf{C}$ in the Kuratowski sense, with $\mathbf{C} \subset \mathbf{C}_n$ for all n;*

(ii) *$|\Omega_n \triangle E| \to 0$ as $n \to \infty$ and $\partial\Omega_n \subset \{x : \mathrm{dist}(x, \partial E) < \frac{1}{n}\}$;*

(iii) *$P(\Omega_n; \mathbb{R}^N \setminus \mathbf{C}_n) \to P(E; \mathbb{R}^N \setminus \mathbf{C})$;*

(iv) *$\mathcal{H}^{N-1}(\partial\Omega_n \cap \partial\mathbf{C}_n) \to \mathcal{H}^{N-1}(\partial^* E \cap \partial\mathbf{C})$.*

Moreover, if E coincides with a bounded open set of finite perimeter $\Omega \subset \mathbb{R}^N \setminus \mathbf{C}$ such that $\partial\Omega \setminus \mathbf{C}$ is smooth, then we may construct the approximating sets in such a way that in addition it holds:

(v) *$\Omega_n \setminus \mathbf{C}_n = (\Omega)_{\varepsilon_n} \setminus \mathbf{C}_n$ for a suitable sequence $\varepsilon_n \to 0$, where $(\Omega)_{\varepsilon_n}$ denotes the ε_n-neighborhood of Ω.*

Proof of Theorem 1.1. We begin by observing that by Proposition 2.9 and Theorem 3.1 it follows that the inequality (1.2) holds if $\mathbf{C}$ and $\Omega \subset \mathbb{R}^N \setminus \mathbf{C}$ satisfy (2.1) and (2.2) respectively. In the general case, the inequality (1.2) follows by approximation, see Lemma 3.4.

We now analyse the case of equality. Assume that E is a set of finite perimeter with $|E| = v$ for which equality in (1.2) holds. Thus, in particular, E is a minimizer of the isoperimetric problem (1.1). Without loss of generality, we may assume $v = |B^\lambda|$.

We now apply to E the Steiner symmetrisation of codimension $N-2$ with respect to the plane containing the section $\mathcal{C}$ of the cylinder $\mathbf{C}$, see [2]. Recall that we write $x = (z, w)$, with $z \in \mathbb{R}^2$ (the plane containing $\mathcal{C}$) and $w \in \mathbb{R}^{N-2}$. For any $z \in \mathbb{R}^2$ we let $E_z := \{w \in \mathbb{R}^{N-2} : (z, w) \in E\}$ be the corresponding $(N-2)$-slice of E and define the function

$$m(z) := \mathcal{H}^{N-2}(E_z). \tag{3.25}$$

and, correspondingly, we let $r(z)$ be the radius of the $(N-2)$-dimensional ball having $(N-2)$-measure equal to $m(z)$. Setting also

$$\pi(E) := \{z \in \mathbb{R}^2 : m(z) > 0\},$$

we can define the Steiner symmetral of E as

$$E^S := \{(z,w) \in \mathbb{R}^N : z \in \pi(E) \text{ and } |w| < r(z)\}.$$

Clearly $|E^S| = |E|$. Since $\partial^* E^S \cap \partial \mathbf{C}$ is equivalent to $(\partial^* E \cap \partial \mathbf{C})^S$ we also have $\mathscr{H}^{N-1}(\partial^* E^S \cap \mathbf{C}) = \mathscr{H}^{N-1}(\partial^* E \cap \mathbf{C})$. Finally, $P(E^S; \mathbb{R}^N \setminus \mathbf{C}) \le P(E; \mathbb{R}^N \setminus \mathbf{C})$ by [2, Theorem 1.1] and thus $J_{\lambda,\mathbf{C}}(E^S) = J_{\lambda,\mathbf{C}}(E) = J_{\lambda,\mathbf{C}}(B^\lambda)$. In particular, also E^S is a minimizer of (1.1). Then E^S is equivalent to a bounded open set (see for instance [12, Theorem 3.2], which is proven for $N = 3$ but the same arguments holds in every dimension). It is also well known that volume-constrained minimizers of the perimeter are (Λ, r_0)-minimizers of the perimeter (see for instance [19, Example 21.3]). Thus the regularity theory for (Λ, r_0)-minimizers (see for instance [19, Theorems 21.8 and 28.1]) applies and yields that $\partial E^S \setminus \mathbf{C}$ is a smooth constant mean curvature manifold up to a closed singular set $\Sigma_{sing} \subset \partial\Omega \setminus \mathbf{C}$ of Hausdorff dimension less than or equal to $N-8$. But note now that if Σ_{sing} is nonempty, then by the rotational symmetry of E^S in the w-plane it follows immediately that its Hausdorff dimension is at least $N-2$, which is impossible. Thus we have shown that $\partial\Omega \setminus \mathbf{C}$ is smooth. Moreover, arguing as in the proof of Proposition 2.9, see (2.20), we deduce that Ω is connected.

Let Ω_n and $\mathbf{C}_n$ be the two approximating sequences provided by Lemma 3.4. Note that we may enforce that the approximating sets Ω_n additionally satisfy property (v) of the lemma. Let u_n be the variational solution of

$$\begin{cases} \Delta u_n = c_n & \text{in } \Omega_n \\ \partial_\nu u_n = 1 & \text{on } \Sigma_n \\ \partial_\nu u_n = -\lambda & \text{on } \Gamma_n, \end{cases}$$

where $\Sigma_n = \partial\Omega_n \setminus \mathbf{C}_n$, $\Gamma_n := \partial\Omega_n \cap \mathbf{C}_n$ and

$$c_n = \frac{\mathscr{H}^{N-1}(\Sigma_n) - \lambda \mathscr{H}^{N-1}(\Gamma_n)}{|\Omega_n|} = \frac{J_{\lambda,\mathbf{C}_n}(\Omega_n)}{|\Omega_n|}.$$

By Theorem 3.1 the convex sets $\mathbf{C}_n$ satisfy the λ-ABP property, therefore as in the proof of Proposition 2.9, see (2.19), we have

$$\begin{aligned} |\Omega_n| \le |B^\lambda| \le |\mathscr{A}_{u_n} \cap B_1| \le |\nabla u_n(\Omega_n^+)| \le \int_{\Omega_n^+} \det \nabla^2 u_n \, dx \\ \le \int_{\Omega_n^+} \frac{(\Delta u_n)^N}{N^N} \, dx \le \frac{(J_{\lambda,\mathbf{C}_n}(\Omega_n))^N}{N^N |\Omega_n|^N} |\Omega_n^+| \le \frac{(J_{\lambda,\mathbf{C}_n}(\Omega_n))^N}{N^N |\Omega_n|^N} |\Omega_n|, \end{aligned} \tag{3.26}$$

where

$$\Omega_n^+ := \big\{ x \in \Omega_n : J_{\overline{\Omega}_n} u_n(x) \neq \emptyset \text{ and } \nabla u_n(x) \in B_1 \big\}.$$

Note that by properties (ii), (iii) and (iv) of Lemma 3.4 we have that $|\Omega_n| \to |\Omega| = |B^\lambda|$ and

$$J_{\lambda,\mathbf{C}_n}(\Omega_n) \to J_{\lambda,\mathbf{C}}(\Omega) = J_{\lambda,\mathbf{C}}(B^\lambda) = N|B^\lambda|. \tag{3.27}$$

and thus

$$c_n \to c := \frac{J_{\lambda,\mathbf{C}}(B^\lambda)}{|B^\lambda|} = N. \tag{3.28}$$

Let us observe that up to adding constants, we may assume that each u_n vanishes at some point x_n of Ω_n^+. Thus, setting $\xi_n = \nabla u_n(x_n) \in B_1$, we have

$$u_n(y) \geq u_n(x_n) + \xi \cdot (y - x_n) = \xi \cdot (y - x_n) \geq -\mathrm{diam}\,(\Omega_n) \quad \text{for all } y \in \Omega_n$$

and thus the u_n's are uniformly bounded below. Since they solve the equation $\Delta u_n = c_n$, with c_n in turn uniformly bounded, it follows from a standard Harnack inequality argument that

$$\sup_n \|u_n\|_{\Omega'} < +\infty \quad \text{for all } \Omega' \subset\subset \Omega.$$

In turn, recalling also (3.28) and by standard elliptic regularity, we may assume that there exists $u \in C^\infty(\Omega)$ such that up to extracting a (non relabelled) subsequence

$$\Delta u = N \ \text{ in } \Omega \quad \text{ and } \quad u_n \to u \in C^\infty(\overline{\Omega}') \ \text{ for all } \Omega' \subset\subset \Omega.$$

Note now that by (3.27), the inequalities in (3.26) become equalities in the limit for u. In particular, $|\Omega_n^+| \to |\Omega| = |B^\lambda|$ and since $|\Omega_n \Delta \Omega| \to 0$, we have (up to a non relabelled subsequence)

$$\chi_{\Omega_n^+} \to \chi_\Omega \text{ a.e. }.$$

We may now argue as in the proof of Proposition 2.9 to deduce that $|\nabla u(x)| \leq 1$ and $\nabla^2 u = I$ in Ω and thus, by the connectedness of Ω, there exist $x_0 \in \mathbb{R}^N$ and $b \in \mathbb{R}$ such that

$$u(x) = \frac{1}{2}|x - x_0|^2 + b \quad \text{for all } x \in \Omega.$$

We now study the boundary conditions satisfied by u. To this aim, let $B_r(x) \subset\subset \mathbb{R}^N \setminus \mathbf{C}$. By (v) of Lemma 3.4 we have that $\partial\Omega_n \cap B_r(x)$ converge in C^∞ to $\partial\Omega \cap B_r(x)$. Since

$$\int_{B_r(x)\cap\Omega_n} \nabla u_n \cdot \nabla\varphi = -c_n \int_{B_r(x)\cap\Omega_n} \varphi\, dx + \int_{\partial\Omega_n \cap B_r(x)} \varphi\, d\mathcal{H}^{N-1} \quad \text{for all } \varphi \in H_0^1(B_r(x)), \tag{3.29}$$

by a standard Caccioppoli Inequality argument and exploiting that Trace Theorem holds on $\partial\Omega_n \cap B_r(x)$ with uniformly bounded constants, we deduce $\sup_n \|u_n\|_{H^1(\Omega_n \cap B_r(x))} < +\infty$ and thus, in particular, $u_n \rightharpoonup u$ weakly in $H^1(\Omega \cap B_r(x))$. Therefore, we can pass to the limit in (3.29) to get

$$\int_{B_r(x)\cap\Omega} \nabla u \cdot \nabla\varphi = -N \int_{B_r(x)\cap\Omega} \varphi\, dx + \int_{\partial\Omega \cap B_r(x)} \varphi\, d\mathcal{H}^{N-1} \quad \text{for all } \varphi \in C_c^\infty(B_r(x)),$$

which yields $\partial_\nu u = 1$ on $\partial\Omega \cap B_r(x)$ and thus on $\partial\Omega \setminus \mathbf{C}$ by the arbitrariness of $B_r(x)$.

Note now that as $\mathbf{C}_n \to \mathbf{C}$ in the Kuratowski sense we have that the boundaries $\partial\mathbf{C}_n$ are locally equi-Lipschitz. Thus for any ball $B_r(x)$ such that $\overline{B}_r(x) \cap \partial\Omega = (\Gamma \setminus \gamma) \cap \overline{B}_r(x)$, we have eventually $\overline{B}_r(x) \cap \partial\Omega_n = (\Gamma_n \setminus \gamma_n) \cap \overline{B}_r(x)$ and we may extend each u_n to a function $\tilde{u}_n \in H^1(B_r)$ with equibounded H^1-norms. Therefore, up to a non ralebelled subsequence, we may assume $\tilde{u}_n \rightharpoonup \tilde{u}$ in $H^1(B_r(x))$, with $\tilde{u} = u$ in $\Omega \cap B_r(x)$. We may now argue similarly as before, to deduce that $\partial_\nu u = -\lambda$ a.e. in $\Gamma \setminus \gamma$. We may now argue as in the final part of the proof of Proposition 2.9 to conclude that Ω is a spherical cap isometric to B^λ sitting on a facet of $\mathbf{C}$.

To conclude the theorem, we observe that the function m defined in (3.25) is smooth in $\mathbb{R}^2 \setminus \mathcal{C}$ (as it coincides with that of the spherical cap). Hence, by [2, Theorem 1.2 and Proposition 3.5] we may conclude that E is equivalent to a translate of $E^S = \Omega$. This concludes the proof of the theorem. ☐

Remark 3.5. *We observe here that Theorem 1.1 extends to the case convex cylinders with empty interior of the form $\mathbf{C} = I \times \mathbb{R}^{N-2}$, where $I \subset \mathbb{R}$ is any closed interval. In this case the capillary energy must be defined as follows*

$$J_{\lambda,\mathbf{C}}(E) := P(E;\mathbb{R}^N \setminus \mathbf{C}) - \lambda \int_{\mathbf{C}} \left(\mathrm{Tr}^+(\chi_E) + \mathrm{Tr}^-(\chi_E)\right) d\mathcal{H}^{N-1}, \tag{3.30}$$

where $\mathrm{Tr}^\pm(\chi_E)$ denote the traces of the characteristic function χ_E on both sides of $\mathbf{C}$, see for instance [1, Theorem 3.77]*. Indeed, this follows easily after observing that the right-hand side of (3.30) is the limit of $J_{\lambda,\mathbf{C}_n}(E \setminus \mathbf{C}_n)$, where $\mathbf{C}_n$ denotes the closed $\frac{1}{n}$-neighborhood of $\mathbf{C}$.*

4. Appendix

Proof of Lemma 3.4. Assume first that $E = \Omega$, where Ω is a bounded open set of finite perimeter. Let B_R be a ball such that $\Omega \subset\subset B_R$ and assume without loss of generality that $\mathcal{C}$ contains the origin of $\mathbb{R}^2$.

Given $\sigma > 0$ we begin by constructing a sequence of smooth convex sets $\mathcal{C}^k_\sigma \subset \mathbb{R}^2$, with $\mathcal{C} \subset \mathcal{C}^k_\sigma$, converging to $(1+\sigma)\mathcal{C}$ in the Kuratowski sense as $k \to \infty$ and such that $(1+\sigma)\mathcal{C} \cap D_R \subset \mathcal{C}^k_\sigma \cap D_R$, where D_R is the two-dimensional disk with radius R. Up to slightly dilating $\mathcal{C}^k_\sigma$ if needed, we may always assume that, setting $\mathbf{C}^k_\sigma = \mathcal{C}^k_\sigma \times \mathbb{R}^{N-2}$, we have

$$\mathcal{H}^{N-1}(\partial\Omega \cap \partial\mathbf{C}^k_\sigma) = 0 \qquad \text{for all } k, \sigma. \tag{4.1}$$

We consider the signed distance function $\mathrm{sd}_{\mathbf{C}^k_\sigma}(x)$ from $\partial\mathbf{C}^k_\sigma$, which is a C^∞ function in $O^k_\sigma = \{x : \mathrm{sd}_{\mathbf{C}^k_\sigma}(x) > -\eta^k_\sigma\}$ for some $\eta^k_\sigma > 0$. Consider the smooth convex sets $\mathbf{C}^k_{\sigma,s} := \{x : \mathrm{sd}_{\mathbf{C}^k_\sigma(x)} \le s\}$ for $s > -\eta^k_\sigma$.

To approximate Ω we first extend $\chi_\Omega\big|_{\mathbb{R}^N \setminus \mathbf{C}}$ to a function $u \in BV(\mathbb{R}^N)$, with compact support, such that $|Du|(\partial\mathbf{C}) = 0$, $0 \le u \le 1$, see [1, Proposition 3.21]. Note that for all $t \in (0,1)$, $\{u > t\} \setminus \mathbf{C} = \Omega$. For any $\varepsilon > 0, t \in (0,1)$ we set $U_{\varepsilon,t} = \{x : u_\varepsilon(x) > t\}$, where

$u_\varepsilon = \varrho_\varepsilon * u$, for a standard mollifier ϱ_ε. Note that for a.e. $t \in (0,1)$ there exists a sequence ε_n converging to zero such that and

$$
\begin{aligned}
&\lim_{n\to\infty} |U_{\varepsilon_n,t}\triangle\{u>t\}| = 0, \qquad \lim_{n\to\infty} P(U_{\varepsilon_n,t}) = P(\{u>t\}),\\
&\partial U_{\varepsilon_n,t} \subset \Big\{x:\, \mathrm{dist}(x,\partial\{u>t\}) < \frac{1}{n}\Big\},
\end{aligned}
\tag{4.2}
$$

see [1, Theorem 3.42].

Consider now the C^∞ map $x \mapsto (\mathrm{sd}_{\mathbf{C}^k_\sigma}(x), u_\varepsilon(x))$ defined for all $x \in O^k_\sigma$. By Sard's theorem we have that

$$
\mathrm{rank}\begin{pmatrix}\nabla \mathrm{sd}_{\mathbf{C}^k_\sigma}(x)\\ \nabla u_{\varepsilon_n}(x)\end{pmatrix} = 2 \quad \text{on } \big\{x:\, \mathrm{sd}_{\mathbf{C}^k_\sigma}(x)=s,\ u_{\varepsilon_n}(x)=t\big\} \text{ for a.e. } (s,t)\in(0,\infty)\times(0,1).
$$

Hence, we may fix from now on $t \in (0,1)$ satisfying (4.2) and such that the or a.e. $s > 0$ the above rank condition holds for all n. Therefore for a.e. $s > 0$ the open set $\Omega^k_{\sigma,\varepsilon_n,s} = U_{\varepsilon_n,t}\setminus \mathbf{C}^k_{\sigma,s}$ is a Lipschitz domain such that $\partial\Omega^k_{\sigma,\varepsilon_n,s}\setminus \mathbf{C}^k_{\sigma,s}$ is a C^∞ manifold with boundary. Note that for any σ and k we have that for a.e. s, $\mathcal{H}^{N-1}(\partial\Omega\cap\partial\mathbf{C}^k_{\sigma,s}) = 0$. Therefore for all such s, we have

$$
\begin{aligned}
\lim_{n\to\infty} P(\Omega^k_{\sigma,\varepsilon_n,s};\mathbb{R}^N\setminus\mathbf{C}^k_{\sigma,s}) &= \lim_{n\to\infty} P(U_{\varepsilon_n,t};\mathbb{R}^N\setminus\mathbf{C}^k_{\sigma,s})\\
&= P(\Omega;\mathbb{R}^N\setminus\mathbf{C}^k_{\sigma,s}) = P(\Omega\setminus\mathbf{C}^k_{\sigma,s};\mathbb{R}^N\setminus\mathbf{C}^k_{\sigma,s}).
\end{aligned}
\tag{4.3}
$$

From the above convergence and the continuity of the trace Theorem for BV functions, see [1, Theorem 3.88] we have that

$$
\lim_{n\to\infty}\mathcal{H}^{N-1}(\partial\Omega^k_{\sigma,\varepsilon_n,s}\cap\partial\mathbf{C}^k_{\sigma,s})) = \mathcal{H}^{N-1}(\partial^*(\Omega\setminus\mathbf{C}^k_{\sigma,s})\cap\partial\mathbf{C}^k_{\sigma,s}) = \mathcal{H}^{N-1}(\Omega\cap\partial\mathbf{C}^k_{\sigma,s}),
\tag{4.4}
$$

where the last equality follows from the fact that $\mathcal{H}^{N-1}(\partial\Omega\cap\partial\mathbf{C}^k_{\sigma,s}) = 0$. Observe now that, since $\mathbf{C}^k_{\sigma,s}$ converge to $\mathbf{C}^k_\sigma$ in the Kuratowski sense, as $s \to 0$, we have in particular that $\mathcal{H}^{N-1}\llcorner\partial\mathbf{C}^k_{\sigma,s} \overset{*}{\rightharpoonup} \mathcal{H}^{N-1}\llcorner\partial\mathbf{C}^k_\sigma$, see for instance [12, Remark 2.2]. Therefore, thanks to (4.1) we conclude that $\mathcal{H}^{N-1}(\Omega\cap\partial\mathbf{C}^k_{\sigma,s}) \to \mathcal{H}^{N-1}(\Omega\cap\partial\mathbf{C}^k_\sigma) = \mathcal{H}^{N-1}(\partial^*(\Omega\setminus\mathbf{C}^k_\sigma)\cap\partial\mathbf{C}^k_\sigma)$. Thus we have

$$
\begin{aligned}
&\lim_{s\to0} P(\Omega\setminus\mathbf{C}^k_{\sigma,s};\mathbb{R}^N\setminus\mathbf{C}^k_{\sigma,s}) = P(\Omega\setminus\mathbf{C}^k_\sigma;\mathbb{R}^N\setminus\mathbf{C}^k_\sigma)\\
&\lim_{s\to0}\mathcal{H}^{N-1}(\partial^*(\Omega\setminus\mathbf{C}^k_{\sigma,s})\cap\partial\mathbf{C}^k_{\sigma,s}) = \mathcal{H}^{N-1}(\partial^*(\Omega\setminus\mathbf{C}^k_\sigma)\cap\partial\mathbf{C}^k_\sigma).
\end{aligned}
\tag{4.5}
$$

By a similar argument, if $\sigma > 0$ is such that $\mathcal{H}^{N-1}(\partial\Omega\cap\partial(1+\sigma)\mathbf{C}) = 0$, setting $\mathbf{C}_\sigma = (1+\sigma)\mathbf{C}$ we have

$$
\begin{aligned}
&\lim_{k\to\infty} P(\Omega\setminus\mathbf{C}^k_\sigma;\mathbb{R}^N\setminus\mathbf{C}^k_\sigma) = P(\Omega\setminus\mathbf{C}_\sigma;\mathbb{R}^N\setminus\mathbf{C}_\sigma),\\
&\lim_{k\to\infty}\mathcal{H}^{N-1}(\partial^*(\Omega\setminus\mathbf{C}^k_\sigma)\cap\partial\mathbf{C}^k_\sigma) = \mathcal{H}^{N-1}(\partial^*(\Omega\setminus\mathbf{C}_\sigma)\cap\partial\mathbf{C}_\sigma).
\end{aligned}
\tag{4.6}
$$

Finally, we note that by monotone convergence

$$
\lim_{\sigma\to0} P(\Omega\setminus\mathbf{C}_\sigma;\mathbb{R}^N\setminus\mathbf{C}_\sigma) = \lim_{\sigma\to0} P(\Omega;\mathbb{R}^N\setminus\mathbf{C}_\sigma) = P(\Omega;\mathbb{R}^N\setminus\mathbf{C}).
\tag{4.7}
$$

By scaling, this is equivalent to say that

$$\lim_{\sigma\to 0} P\big(\big((1+\sigma)^{-1}\Omega\big)\setminus \mathbf{C};\mathbb{R}^N\setminus\mathbf{C}\big)=P(\Omega;\mathbb{R}^N\setminus\mathbf{C}).$$

Therefore, the trace theorem again implies that

$$\begin{aligned}&\lim_{\sigma\to 0}\mathcal{H}^{N-1}(\partial^*(\Omega\setminus\mathbf{C}_\sigma)\cap\partial\mathbf{C}_\sigma)\\ &=\lim_{\sigma\to 0}(1+\sigma)^{N-1}\mathcal{H}^{N-1}\big(\partial^*\big((1+\sigma)^{-1}\Omega\big)\setminus\mathbf{C}\big)\cap\partial\mathbf{C}\big)=\mathcal{H}^{N-1}(\partial^*\Omega\cap\partial\mathbf{C}).\end{aligned}$$

From this equality, together with (4.3)-(4.7) we conclude, by a diagonal argument, that there exist sequences $s_n\to 0^+$, $k_n\to\infty$ and $\sigma_n\to 0^+$ such that, setting $\Omega_n=\Omega^{k_n}_{\sigma_n,\varepsilon_n,s_n}$, $\mathbf{C}_n=\mathbf{C}^{k_n}_{\sigma_n,s_n}$, (i)–(iv) hold.

If E is now a general set of finite perimeter, the conclusion follows through a further diagonal procedure by approximating E with a sequence of bounded open sets $\{\Omega_n\}$ contained in $\mathbb{R}^N\setminus\mathbf{C}$ in such way that $|\Omega_n\Delta E|\to 0$ and $P(\Omega_n;\mathbb{R}^N\setminus\mathbf{C})\to P(E;\mathbb{R}^N\setminus\mathbf{C})$, which implies $\mathcal{H}^{N-1}(\partial\Omega_n\cap\partial\mathbf{C}_n)\to\mathcal{H}^{N-1}(\partial^*E\cap\partial\mathbf{C})$ by the continuity of the trace Theorem.

Let us now consider the case where E coincides with an open set of finite perimeter $\Omega\subset\mathbb{R}^N\setminus\mathbf{C}$ such that $\partial\Omega\setminus\mathbf{C}$ is smooth. In this case we can simplify the approximation, by considering the signed distance function sd_Ω to the boundary of Ω. By the smoothness of $\partial\Omega\setminus\mathbf{C}$ and the properties of signed distance functions, we have that for all $\sigma>0$ there exists $\varepsilon(\sigma)>0$ such that sd_Ω is smooth in $(\partial\Omega)_{\varepsilon(\sigma)}\setminus(1+\sigma)\mathbf{C}$. Here $(\partial\Omega)_{\varepsilon(\sigma)}$ denotes the $\varepsilon(\sigma)$-neighborhood of $\partial\Omega$. The idea is now to proceed as before, with u_ε replaced by sd_Ω, the set $U_{\varepsilon,t}$ replaced by $(\Omega)_\varepsilon:=\{x:\mathrm{sd}_\Omega\le\varepsilon\}$, and with $\Omega^k_{\sigma,\varepsilon,s}:=(\Omega)_\varepsilon\setminus\mathbf{C}^k_{\sigma,s}$. Note that again by Sard's theorem we have

$$\operatorname{rank}\begin{pmatrix}\nabla\mathrm{sd}_{\mathbf{C}^k_\sigma}(x)\\ \nabla\mathrm{sd}_\Omega(x)\end{pmatrix}=2\quad\text{on }\big\{x:\,\mathrm{sd}_{\mathbf{C}^k_\sigma}(x)=s,\ \mathrm{sd}_\Omega(x)=\varepsilon\big\}\text{ for a.e. }(s,\varepsilon)\in(0,\infty)\times(0,\varepsilon(\sigma)).$$

Hence, for all such (s,ε) the set $\Omega^k_{\sigma,\varepsilon,s}$ is a Lipschitz domain such that $\partial\Omega^k_{\sigma,\varepsilon,s}\setminus\mathbf{C}^k_{\sigma,s}$ is a C^∞ manifold with boundary. Moreover, for a.e. $s>0$ and for all k we can always find a sequence $\varepsilon_n\to 0$ such that (s,ε_n) satisfies the above rank condition for all n. Along this subsequence, we easily get that $\big|\Omega^k_{\sigma,\varepsilon_n,s}\Delta(\Omega\setminus\mathbf{C}^k_{\sigma,s})\big|\to 0$ and that (4.3) and (4.4) hold. We may now proceed as before to reach the conclusion. □

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