$\mathbf{W^{1/2}}$-maps into $\mathbf{S^1}$ and currents

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Let B^n be the unit ball in $\mathbb{R}^n$ and let $\mathcal{Y}$ be a smooth oriented Riemannian manifold of dimension larger than or equal to 1, compact, connected, and without boundary, isometrically embedded in some Euclidean space. Recently there has been quite some interest in the class of maps in the fractional Sobolev space $W^{1/2}$ from B^n with values into $\mathcal{Y}$, e.g. [3] [4] [5] [6] [7] [14] [19] for $\mathcal{Y}$ equal to the unit circle S^1 in $\mathbb{R}^2$, and [15] [16] for manifolds with 1-homology group without torsion.

Similarly to Sobolev maps in $W^{1,2}$ with values into a manifold $\mathcal{Y}$, compare e.g. [13], sequences of smooth maps with equibounded $W^{1/2}$-norms show concentration of $W^{1/2}$-energy, moreover there exist maps in $W^{1/2}(B^n, S^1)$ that cannot be approximated in the $W^{1/2}$-norm by functions in $C^\infty(B^n, S^1)$.

Two natural questions then arise: identify the *weak limits* of $W^{1/2}$-equibounded sequences of smooth maps, compute the $W^{1/2}$-energy of those limits and in particular the *relaxed* $W^{1/2}$-energy. It turns out that the natural setting in which we may answer to those questions is the geometric setting of *Cartesian currents*, see [13].

In this paper, by specializing the results of [16] to the case $\mathcal{Y} = S^1$, we shall recover and survey some of the known results.

1 $\mathbf{W^{1/2}}$-maps into $\mathbf{S^1}$ and their graphs

In this section we discuss some properties of currents carried by graphs of $W^{1/2}$-maps.

THE CLASS $W^{1/2}(B^n)$. We recall, see e.g. [1], that the fractional Sobolev space $W^{1/2}(B^n)$ is the Hilbert space of real valued functions u which have finite $W^{1/2}$-seminorm

$$|u|^2_{1/2,B^n} := \int_{B^n} \int_{B^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+1}} \, dx \, dy \tag{1.1}$$

endowed with the norm

$$\|u\|^2_{1/2,B^n} := \|u\|^2_{L^2(B^n)} + |u|^2_{1/2,B^n} \,.$$

Moreover, let

$$W^{1/2}(B^n, S^1) := \{u \in W^{1/2}(B^n, \mathbb{R}^2) \, : \, |u(x)| = 1 \quad \text{for a.e. } x \in B^n\} \,.$$

EXTENSION OF $W^{1/2}$-MAPS. Instead of working with the $W^{1/2}$-energy, given by (1.1), we may and shall work with the equivalent energy $\mathcal{E}_{1/2}(u)$ defined as follows. For $u \in W^{1/2}(B^n, S^1)$ we define the *extension* of u

$$U := \mathrm{Ext}(u) \in W^{1,2}(\mathcal{C}^{n+1}, \mathbb{R}^2) \,,$$

where $\mathcal{C}^{n+1}$ is the cylinder

$$\mathcal{C}^{n+1} := B^n \times I \,, \qquad I := [0,1] \,,$$

as the harmonic function U which minimizes the *Dirichlet integral*

$$\mathbf{D}(U) := \frac{1}{2} \int_{\mathcal{C}^{n+1}} |DU(x,t)|^2 \, dx \, dt$$

among all functions that agree with u on $B^n \times \{0\}$, and we set

$$\mathcal{E}_{1/2}(u) := \mathbf{D}(\mathrm{Ext}(u)) \,.$$

It is well-known that

$$\mathbf{D}(\mathrm{Ext}(u)) \simeq |u|_{1/2,B^n}\,.$$

Moreover, clearly the image of U is contained in the closed unit disk D^2,

$$U \in W^{1,2}(\mathcal{C}^{n+1}, D^2)\,.$$

GRAPHS OF $W^{1/2}$-MAPS. To any map $u \in W^{1/2} \cap L^\infty(B^n, \mathbb{R}^2)$ we can associate a current G_u in $\mathcal{D}_n(B^n \times \mathbb{R}^2)$, compare [14]. For this we recall the following facts:

(i) $W^{1/2} \cap L^\infty(B^n)$ is an algebra, since trivially

$$|uv|_{1/2} \le \|u\|_\infty |v|_{1/2} + \|v\|_\infty |u|_{1/2}\,;$$

(ii) if $u \in W^{1/2}(B^n)$, then $D_i u$ belongs to the dual space $W^{-1/2}(B^n)$ of $W^{1/2}(B^n)$ for every $i = 1, \dots, n$ and

$$|\langle Du, v\rangle| \le c\,|u|_{1/2}|v|_{1/2} \qquad \forall\, u, v \in W^{1/2}(B^n)\,;$$

(iii) if $u \in W^{1/2}(B^n)$ and $v \in W^{1/2} \cap L^\infty(B^n)$, then $v\,Du$ defines a distribution in B^n, indeed a linear continuous functional on $W^{1/2} \cap L^\infty$ by

$$\langle v\,Du, \varphi\rangle := \langle Du, v\,\varphi\rangle\,, \qquad \varphi \in W^{1/2} \cap L^\infty(B^n)\,;$$

in fact

$$|\langle v\,Du, \varphi\rangle| \le c\,|u|_{1/2}\Big(\|v\|_\infty |\varphi|_{1/2} + \|\varphi\|_\infty |v|_{1/2}\Big)\,.$$

Every form ω in $\mathcal{D}^n(B^n \times \mathbb{R}^2)$ with at most one vertical differential can be written as

$$\omega := \omega_0(x,y)\,dx + \sum_{i=1}^{n}\sum_{j=1}^{2} \omega_{i,j}(x,y)\,\widehat{dx_i} \wedge dy^j$$

where x and y are the variables in $\mathbb{R}^n$ and $\mathbb{R}^2$, respectively, ω_0 and $\omega_{i,j}$ are functions in $C^\infty_c(B^n \times \mathbb{R}^2)$, $dx := dx^1 \wedge \cdots \wedge dx^n$ and $\widehat{dx_i} := dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n$. If $u \in W^{1/2} \cap L^\infty(B^n, \mathbb{R}^2)$, then it is easily seen that also $\varphi(x, u(x))$ belongs to $W^{1/2} \cap L^\infty(B^n)$ for every $\varphi \in C^\infty_c(B^n \times \mathbb{R}^2)$ and hence, since $Du \in W^{-1/2}$, we can define the *graph current* associated to u, compare [14], acting in a distributional sense on forms with at most one vertical differential as

$$G_u(\omega) := \int_{B^n} \omega_0(x, u(x))\,dx + \sum_{i=1}^{n}\sum_{j=1}^{2} (-1)^{n-i} \langle \omega_{i,j}(x, u(x)),\, D_i u^j(x)\rangle\,.$$

Notice that, if u is smooth, we have $G_u = (\mathrm{Id} \bowtie u)_\# [\![B^n]\!]$, where $(Id \bowtie u)(x) := (x, u(x))$, i.e.

$$G_u(\omega) = \int_{B^n} (\mathrm{Id} \bowtie u)^\# \omega\,, \qquad \omega \in \mathcal{D}^n(B^n \times \mathbb{R}^2)\,.$$

Since every $u \in W^{1/2}(B^n, S^1)$ is in L^∞, then G_u is well defined as a current in $\mathcal{D}_n(B^n \times S^1)$. However, even in dimension $n = 1$ in general G_u is not an i.m. rectifiable current in $B^n \times S^1$. Moreover, if u_k converges to u strongly in $W^{1/2}$, then G_{u_k} converges to G_u weakly in the sense of currents. Finally, since $\mathrm{Ext}(u_k) \to U := \mathrm{Ext}(u)$ strongly in $W^{1,2}(\mathcal{C}^{n+1}, \mathbb{R}^2)$ we have, compare [14],

$$(-1)^{n-1} \partial G_U = G_u \qquad \text{on} \quad \mathcal{D}^n(B^n \times \{0\} \times S^1)\,. \tag{1.2}$$

BOUNDARY DATA. In the sequel we will denote by $\widetilde{B}^n$ a bounded domain in $\mathbb{R}^n$ such that $B^n \subset\subset \widetilde{B}^n$ and we let $\varphi : \widetilde{B}^n \to S^1$ be a given $W^{1/2}$-function, which will always be assumed to be smooth on $\widetilde{B}^n$, and we set

$$\begin{aligned} W^{1/2}_\varphi(\widetilde{B}^n, S^1) &:= \{u \in W^{1/2}(\widetilde{B}^n, S^1) \mid u = \varphi \quad \text{on } \widetilde{B}^n \setminus \overline{B}^n\} \\ C^\infty_\varphi(\widetilde{B}^n, S^1) &:= \{u \in C^\infty(\widetilde{B}^n, S^1) \mid u = \varphi \quad \text{on } \widetilde{B}^n \setminus \overline{B}^n\}\,. \end{aligned}$$

Density results for $W^{1/2}$-maps. If $n \geq 2$, let $R^\infty_{1/2}(B^n, S^1)$, respectively $R^0_{1/2}(B^n, S^1)$, be the set of all maps $u \in W^{1/2}(B^n, S^1)$ which are smooth, respectively continuous, except on a singular set $\Sigma(u)$ of the type

$$\Sigma(u) = \bigcup_{i=1}^{r} \Sigma_i\,, \qquad r \in \mathbb{N}\,,$$

where Σ_i is a smooth $(n-2)$-dimensional subset of B^n with smooth boundary, if $n \geq 3$, and Σ_i is a point if $n = 2$. Moreover, let

$$R^\infty_{1/2,\varphi}(\widetilde{B}^n, S^1) := \{u \in R^\infty_{1/2}(\widetilde{B}^n, S^1) \mid u = \varphi \quad \text{on } \widetilde{B}^n \setminus \overline{B}^n\}\,.$$

An argument similar to the one in [20] shows that $C^\infty_\varphi(\widetilde{B}^n, S^1)$ is dense in $W^{1/2}_\varphi(\widetilde{B}^n, S^1)$ if $n = 1$, however $C^\infty_\varphi(\widetilde{B}^n, S^1)$ is not dense in $W^{1/2}_\varphi(\widetilde{B}^n, S^1)$ if $n \geq 2$. The following density result was proved in [19] in the case $n = 2$, and in [15] in the case $n \geq 2$ and for more general target manifolds $\mathcal{Y}$.

Theorem 1.1 *For every $n \geq 2$ the class $R^\infty_{1/2,\varphi}(\widetilde{B}^n, S^1)$ is dense in $W^{1/2}_\varphi(\widetilde{B}^n, S^1)$, and $R^\infty_{1/2}(B^n, S^1)$ is dense in $W^{1/2}(B^n, S^1)$.*

The currents $\mathbb{P}(u)$ and $\mathbb{D}(u)$. We shall denote by $\pi : \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}^n$ and $\widehat{\pi} : \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}^2$ the orthogonal projections onto the first and the second factor, respectively. Using the same notation, we shall also denote by $\pi : \mathbb{R}^n \times I \times \mathbb{R}^2 \to \mathbb{R}^n$, $\widetilde{\pi} : \mathbb{R}^n \times I \times \mathbb{R}^2 \to I$, and $\widehat{\pi} : \mathbb{R}^n \times I \times \mathbb{R}^2 \to \mathbb{R}^2$, where $I := [0, 1]$, the orthogonal projections onto the three factors, respectively. Also, we let ω_{S^1} denote the normalized volume 1-form in S^1,

$$\omega_{S^1} := \frac{1}{2\pi}\,(y^1 dy^2 - y^2 dy^1)\,.$$

Following [13, Vol. II, Sec. 4.2.5], we now define the current $\mathbb{P}(u) \in \mathcal{D}_{n-2}(B^n)$ by setting

$$\mathbb{P}(u)(\phi) = (-1)^n \partial G_u(\pi^\# \phi \wedge \widehat{\pi}^\# \omega_{S^1}) \qquad \forall\, \phi \in \mathcal{D}^{n-2}(B^n)\,.$$

We also define the current $\mathbb{D}(u) \in \mathcal{D}_{n-1}(B^n)$ as follows. We let $\widetilde{\omega}_{S^1} \in \mathcal{D}^1(\mathbb{R}^2)$ be a compactly supported smooth extension of ω_{S^1}, we consider a function $\eta \in C^\infty([0, 1], [0, 1])$ with $\eta \equiv 1$ and $\eta \equiv 0$ respectively in a neighborhood of 0 and 1, and we let $U = \mathrm{Ext}(u)$. We now define

$$\mathbb{D}(u)(\phi) := G_U(\pi^\# \phi \wedge d(\widetilde{\pi}^\# \eta \wedge \widehat{\pi}^\# \widetilde{\omega}_{S^1}))\,, \qquad \phi \in \mathcal{D}^{n-1}(B^n)\,.$$

We now show that

$$(-1)^n \partial\, \mathbb{D}(u) = \mathbb{P}(u)\,. \tag{1.3}$$

In fact, by (1.2) we have

$$\partial G_U(\pi^\# d\phi \wedge \widetilde{\pi}^\# \eta \wedge \widehat{\pi}^\# \widetilde{\omega}_{S^1}) = (-1)^{n-1} G_u(\pi^\# d\phi \wedge \widehat{\pi}^\# \omega_{S^1})$$

so that, since $d\omega_{S^1} = 0$, we compute for every $\phi \in \mathcal{D}^{n-2}(B^n)$

$$\begin{aligned} (-1)^n\, \mathbb{P}(u)(\phi) &= \partial G_u(\pi^\# \phi \wedge \widehat{\pi}^\# \omega_{S^1}) = G_u(\pi^\# d\phi \wedge \widehat{\pi}^\# \omega_{S^1}) \\ &= (-1)^{n-1} \partial G_U(\pi^\# d\phi \wedge \widetilde{\pi}^\# \eta \wedge \widehat{\pi}^\# \widetilde{\omega}_{S^1}) \\ &= G_U(\pi^\# d\phi \wedge d(\widetilde{\pi}^\# \eta \wedge \widehat{\pi}^\# d\widetilde{\omega}_{S^1})) = \mathbb{D}(u)(d\phi) = \partial\, \mathbb{D}(u)(\phi)\,. \end{aligned}$$

Moreover, we remark that $\mathbb{D}(u)$ is a current of finite mass in $\mathcal{D}_{n-1}(B^n)$ since $\mathrm{Ext}(u)$ is a $W^{1,2}$-function,

$$d(\widetilde{\pi}^\# \eta \wedge \widehat{\pi}^\# \widetilde{\omega}_{S^1}) = \widetilde{\pi}^\# d\eta \wedge \widehat{\pi}^\# \widetilde{\omega}_{S^1} + \widetilde{\pi}^\# \eta \wedge \widehat{\pi}^\# d\widetilde{\omega}_{S^1}\,,$$

and

$$\widetilde{\omega}_{S^1} \in \mathcal{D}^1(\mathbb{R}^2)\,, \qquad d\widetilde{\omega}_{S^1} \in \mathcal{D}^2(\mathbb{R}^2)\,. \tag{1.4}$$

Also, clearly $\mathbb{P}(u) = 0$ if u is smooth, say Lipschitz. Finally, taking into account Theorem 1.1, exactly as in [13, Vol. II, Sec. 5.4.2], where we take $p = 1$, by (1.3) we obtain that $\mathbb{P}(u)$ is an $(n-2)$-dimensional *real flat chain.*

Proposition 1.2 *Let* $u \in W^{1/2}_\varphi(\widetilde{B}^n, S^1)$ *and let* $\{u_k\} \subset R^\infty_{1/2,\varphi}(\widetilde{B}^n, S^1)$ *converge strongly in* $W^{1/2}$ *to* u. *Then* $\mathbb{P}(u)$ *is the real flat limit of the currents* $\mathbb{P}(u_k)$. *Moreover,* $\mathbb{P}(u_k)$ *is an i.m. rectifiable current in* $\mathcal{R}_{n-2}(\widetilde{B}^n)$, *with support contained in* $\overline{B}^n$*; in particular in case* $n = 2$ *we have* $\mathbb{P}(u_k) = \sum_i d_{i,k}\delta_{a_i^k}$, *where* $d_{i,k} \in \mathbb{Z}$ *are integer coefficients and the* $\delta_{a_i^k}$*'s are Dirac unit measures at points* $a_i^k \in \overline{B}^2$. *Finally, since the boundary data* φ *has a smooth extension from* $\widetilde{B}^n$ *into* S^1, *each* $\mathbb{P}(u_k)$ *is the boundary of an i.m. rectifiable current.*

We now show that $\mathbb{P}(u)$ is an *integral flat chain*, see [10].

Proposition 1.3 *Let* $u \in W^{1/2}_\varphi(\widetilde{B}^n, S^1)$ *and* $\{u_k\} \subset R^\infty_{1/2,\varphi}(\widetilde{B}^n, S^1)$ *converge strongly in* $W^{1/2}$ *to* u. *Then*

(i) $\mathbf{M}(\mathbb{D}(u_k) - \mathbb{D}(u)) \to 0$ *as* $k \to +\infty$;

(ii) *there exists an i.m. rectifiable current* $L \in \mathcal{R}_{n-1}(\widetilde{B}^n)$, *with support* $\operatorname{spt} L \subset \overline{B}^n$ *and finite mass,* $\mathbf{M}(L) < +\infty$, *such that* $\mathbb{P}(u) = \partial L$*; in particular* $\mathbb{P}(u)$ *is an integral flat chain;*

(iii) *if* $L_{u_k,u}$ *denotes an* $(n-1)$*-dimensional i.m. rectifiable current of least mass with support in* $\overline{B}^n$ *such that*
$$\partial L_{u_k,u} = \mathbb{P}(u) - \mathbb{P}(u_k)\,, \tag{1.5}$$
then $\mathbf{M}(L_{u_k,u}) \to 0$ *as* $k \to +\infty$*;*

(iv) *if* $n = 2$, *there exist points* $a_i, b_i \in \overline{B}^2$ *such that*
$$\mathbb{P}(u) = \sum_{i=1}^\infty (\delta_{a_i} - \delta_{b_i})\,, \qquad \sum_{i=1}^\infty |a_i - b_i| < +\infty\,.$$

Proof: Since $u_k \to u$ strongly in $W^{1/2}$, then $U_k := \operatorname{Ext}(u_k) \to \operatorname{Ext}(u) =: U$ strongly in $W^{1,2}$ and hence, by (1.4), the Lebesgue theorem yields (i). The rest of the theorem is proved as in [13, Vol. II, Sec. 4.2.5]. In fact, if Γ is an $(n-1)$-dimensional i.m. rectifiable current with compact support in $\mathbb{R}^n$ and
$$\begin{aligned} m_i(\Gamma) &:= \inf\{\mathbf{M}(T) \mid T \in \mathcal{R}_n(\mathbb{R}^n)\,, \quad \partial T = \Gamma\} \\ m_r(\Gamma) &:= \inf\{\mathbf{M}(T) \mid T \in \mathcal{D}_n(\mathbb{R}^n)\,, \quad \partial T = \Gamma\}\,, \end{aligned}$$
by Hardt-Pitts' theorem [17] we have that $m_i(\Gamma) = m_r(\Gamma)$. Therefore by Rmk. 1 in [13, Vol. II, Sec. 5.4.2] the claims follow. □

2 Cartesian currents with finite $\mathbf{W^{1/2}}$-energy

In this section we introduce the class of Cartesian currents with finite $W^{1/2}$-energy, see Definitions 2.1, 2.5 and 2.6, collecting some of their main properties. For the sake of clearness, all the proofs are postponed to the next section except for the proof of the closure theorem, Theorem 2.10.

Definition 2.1 *Let* $T \in \mathcal{D}_{n,1}(B^n \times S^1)$. *We say that* T *is a current in* $\mathcal{E}_{1/2}$-graph$(B^n \times S^1)$ *if*
$$\partial T = 0 \quad on \quad \mathcal{D}^{n-1}(B^n \times S^1) \tag{2.1}$$
and T *can be decomposed as*
$$T = G_{u_T} + S_T\,, \qquad S_T := \mathbb{L}(T) \times [\![\, S^1 \,]\!]\,, \tag{2.2}$$
where $u_T \in W^{1/2}(B^n, S^1)$ *and* $\mathbb{L}(T)$ *is an i.m. rectifiable current in* $\mathcal{R}_{n-1}(B^n)$.

Note that S_T is completely vertical, i.e. $S_T(\phi(x,y)\,dx) = 0$ for any $\phi \in C^\infty_c(B^n \times S^1)$. Moreover, the graph G_u of a $W^{1/2}$-map u is in $\mathcal{E}_{1/2}$-graph$(B^n \times S^1)$ if it has no inner boundary, i.e.,
$$\partial G_u = 0 \quad \text{on} \quad \mathcal{D}^{n-1}(B^n \times S^1)\,, \tag{2.3}$$

condition which is automatically satisfied in case of dimension $n=1$.

EXTENSION OF $\mathcal{E}_{1/2}$-GRAPHS. Following [14], we now extend currents in $\mathcal{E}_{1/2}\text{-graph}(B^n\times S^1)$ to suitable currents in $\mathcal{D}_{n+1,2}(\mathcal{C}^{n+1}\times\mathbb{R}^2)$.

Definition 2.2 *Let $T\in\mathcal{E}_{1/2}\text{-graph}(B^n\times S^1)$ be such that* (2.2) *holds. Then the* extension $\widetilde{T}:=\mathrm{Ext}(T)$ *is the current $\widetilde{T}\in\mathcal{D}_{n+1,2}(\mathcal{C}^{n+1}\times\mathbb{R}^2)$ defined by*

$$\widetilde{T}=(-1)^{n-1}\Big(G_{U_T}+\mathbb{L}(T)\times[\![D^2]\!]\Big)\,, \tag{2.4}$$

where $U_T:=\mathrm{Ext}(u_T)\in W^{1,2}(\mathcal{C}^{n+1},D^2)$ and $[\![D^2]\!]$ is the i.m. rectifiable current integration on the unit disk D^2, so that

$$\partial[\![D^2]\!]=[\![S^1]\!]\,.$$

Remark 2.3 Note that from Definition 2.2 and (1.2) we infer that the *boundary* of $\widetilde{T}$ over $B^n\times\{0\}\times S^1$ is equal to T. In fact,

$$\partial(\mathbb{L}(T)\times[\![D^2]\!])=\partial\,\mathbb{L}(T)\times[\![D^2]\!]+(-1)^{n-1}\,\mathbb{L}(T)\times\partial[\![D^2]\!]$$

and hence, since $\partial\,\mathbb{L}(T)\times[\![D^2]\!]=0$ on $\mathcal{D}^n(B^n\times S^1)$, we have

$$(-1)^{n-1}\partial(\mathbb{L}(T)\times[\![D^2]\!])=\mathbb{L}(T)\times[\![S^1]\!]\qquad\text{on}\qquad\mathcal{D}^n(B^n\times S^1)\,.$$

THE $\mathcal{E}_{1/2}$-ENERGY. We recall, compare [13], that the *Dirichlet energy* of a current T in $\mathcal{D}_{n+1}(\mathcal{C}^{n+1}\times\mathbb{R}^2)$ is defined in such a way that if $\widetilde{T}$ is given by (2.4) we have

$$\mathbf{D}(\widetilde{T})=\frac{1}{2}\int_{\mathcal{C}^{n+1}}|DU_T|^2\,dx\,dt+\pi\cdot\mathbf{M}(\mathbb{L}(T))\,,\qquad\pi=\mathbf{M}([\![D^2]\!])=\mathcal{L}^2(D^2)\,. \tag{2.5}$$

In particular, if $T=G_U$ for some $U\in W^{1,2}(\mathcal{C}^{n+1},\mathbb{R}^2)$, then $\mathbf{D}(G_U)=\mathbf{D}(U)$.

Definition 2.4 *Let T be in $\mathcal{E}_{1/2}\text{-graph}(B^n\times S^1)$, so that* (2.2) *holds. The $\mathcal{E}_{1/2}$-*energy $\mathcal{E}_{1/2}(T)$ *of T is defined as the Dirichlet energy $\mathbf{D}(\widetilde{T})$ of the extension $\widetilde{T}:=\mathrm{Ext}(T)$, see* (2.4) *and* (2.5).

If $T=G_u$ for some $u\in W^{1/2}(B^n,S^1)$ and $U=\mathrm{Ext}(u)$, we also define

$$\mathrm{Ext}(G_u):=(-1)^{n-1}G_U\,,\qquad\mathcal{E}_{1/2}(G_u):=\mathbf{D}(G_U)=\mathbf{D}(U)\simeq|u|_{1/2}\,.$$

Finally, if $A\subset B^n$ is a Borel set, and $T\in\mathcal{E}_{1/2}\text{-graph}(B^n\times S^1)$, we will denote

$$\mathcal{E}_{1/2}(T,A\times S^1):=\mathbf{D}(\mathrm{Ext}(T\llcorner A\times\mathbb{R}^2))$$

and if $u\in W^{1/2}(B^n,S^1)$

$$\mathcal{E}_{1/2}(u,A):=\mathbf{D}(\mathrm{Ext}(u_{|A}),A\times I)=\frac{1}{2}\int_{A\times I}|D\,\mathrm{Ext}(u_{|A})|^2\,dx\,dt\,.$$

We now give the following

Definition 2.5 *Let $T\in\mathcal{D}_n(B^n\times S^1)$. We say that T is a* Cartesian current *in* $\mathrm{cart}^{1/2}(B^n\times S^1)$ *if T belongs to $\mathcal{E}_{1/2}\text{-graph}(B^n\times S^1)$ and the $\mathcal{E}_{1/2}$-energy $\mathcal{E}_{1/2}(T)$ of T is finite, see Definitions* 2.1 *and* 2.4.

Definition 2.6 *We say that a map $u\in W^{1/2}(B^n,S^1)$ is in* $\mathrm{cart}^{1/2}(B^n,S^1)$ *if the current G_u associated to its graph is in* $\mathrm{cart}^{1/2}(B^n\times S^1)$.

Therefore, a $W^{1/2}$-map u is in $\mathrm{cart}^{1/2}(B^n, S^1)$ if its graph G_u has no inner boundary, i.e. (2.3) holds true. In particular, any smooth map $u : B^n \to S^1$ with finite $W^{1/2}$-energy belongs to $\mathrm{cart}^{1/2}(B^n, S^1)$.

Finally, if $\varphi : \widetilde{B}^n \to S^1$ is a given $W^{1/2}$-function, which is assumed to be smooth on $\widetilde{B}^n$, in the sequel we will denote

$$\begin{aligned} \mathrm{cart}^{1/2}_\varphi(\widetilde{B}^n, S^1) &:= \{u \in \mathrm{cart}^{1/2}(\widetilde{B}^n, S^1) \mid u = \varphi \quad \text{on } \widetilde{B}^n \setminus \overline{B}^n\} \\ \mathrm{cart}^{1/2}_\varphi(\widetilde{B}^n \times S^1) &:= \{T \in \mathrm{cart}^{1/2}(\widetilde{B}^n \times S^1) \mid (T - G_\varphi) \llcorner (\widetilde{B}^n \setminus \overline{B}^n) \times \mathbb{R}^2 = 0\}\,. \end{aligned}$$

The weak convergence. We say that $\{T_k\} \subset \mathrm{cart}^{1/2}(B^n \times S^1)$ converges to $T \in \mathcal{D}_{n,1}(B^n \times S^1)$ weakly in $\mathrm{cart}^{1/2}$ if $T_k \rightharpoonup T$ weakly in $\mathcal{D}_n(B^n \times S^1)$ and $\sup_k \mathcal{E}_{1/2}(T_k) < +\infty$.

The 1-dimensional case. Definition 2.5 is motivated by the following

Theorem 2.7 *Let $\{u_k\} \subset C^1(B^1, S^1)$ be a sequence of smooth maps with $\sup_k |u_k|_{1/2} < +\infty$. Then, possibly passing to a subsequence, G_{u_k} converges weakly in $\mathcal{D}_1(B^1 \times S^1)$ to some current $T \in \mathrm{cart}^{1/2}(B^1 \times S^1)$.*

Therefore the class $\mathrm{cart}^{1/2}(B^1 \times S^1)$ contains the weak limits in $\mathrm{cart}^{1/2}$ of sequences of graphs of smooth maps with equibounded $\mathcal{E}_{1/2}$-energy. Moreover we have the following lower semicontinuity property.

Proposition 2.8 *Let $\{u_k\} \subset C^1(B^1, S^1)$ be such that $\sup_k |u_k|_{1/2} < +\infty$ and $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{D}_1(B^1 \times S^1)$ to some current $T \in \mathcal{E}_{1/2}$-graph$(B^1 \times S^1)$. Then $T \in \mathrm{cart}^{1/2}(B^1 \times S^1)$ and*

$$\mathcal{E}_{1/2}(T) \le \liminf_{k\to+\infty} \mathcal{E}_{1/2}(G_{u_k})\,.$$

Finally the following closure theorem holds true.

Theorem 2.9 *The classes $\mathrm{cart}^{1/2}(B^1 \times S^1)$ and $\mathrm{cart}^{1/2}_\varphi(\widetilde{B}^1 \times S^1)$ are closed under weak convergence in $\mathrm{cart}^{1/2}$.*

The n-dimensional case. Taking into account Theorem 2.9 and the density result of Theorem 6.1 below, we also have in any dimension $n \ge 2$ the following

Theorem 2.10 (Closure theorem). *The classes $\mathrm{cart}^{1/2}(B^n \times S^1)$ and $\mathrm{cart}^{1/2}_\varphi(\widetilde{B}^n \times S^1)$ are closed under weak convergence in $\mathrm{cart}^{1/2}$.*

Moreover,

Proposition 2.11 *Let $\{T_k\} \subset \mathrm{cart}^{1/2}(B^n \times S^1)$ be such that $T_k \rightharpoonup T$ weakly in $\mathrm{cart}^{1/2}$ to some current $T \in \mathcal{E}_{1/2}$-graph$(B^n \times S^1)$. Then $T \in \mathrm{cart}^{1/2}(B^n \times S^1)$ and*

$$\mathcal{E}_{1/2}(T) \le \liminf_{k\to+\infty} \mathcal{E}_{1/2}(T_k)\,. \tag{2.6}$$

Proposition 2.12 *Let $\{T_k\} \subset \mathrm{cart}^{1/2}(B^n \times S^1)$ be such that $\sup_k \mathcal{E}_{1/2}(T_k) < +\infty$. Then, possibly passing to a subsequence we have that $T_k \rightharpoonup T$ weakly in $\mathrm{cart}^{1/2}$ to some current $T \in \mathrm{cart}^{1/2}(B^n \times S^1)$.*

The rest of this section is dedicated to outline the proof of Theorem 2.10.

Let $\{T_k\} \subset \mathrm{cart}^{1/2}(B^n \times S^1)$ be such that $T_k \rightharpoonup T$ weakly in $\mathrm{cart}^{1/2}$ to some current $T \in \mathcal{D}_{n,1}(B^n \times S^1)$. We have to show that $T \in \mathrm{cart}^{1/2}(B^n \times S^1)$. To this aim, we first write T_k as

$$T_k = G_{u_k} + \mathbb{L}(T_k) \times [\![\, S^1 \,]\!]\,, \tag{2.7}$$

where $u_k \in W^{1/2}(B^n, S^1)$ and $\mathbb{L}(T_k) \in \mathcal{R}_{n-1}(B^n)$. If $\widetilde{T}_k := \mathrm{Ext}(T_k)$, we have

$$\widetilde{T}_k = (-1)^{n-1}\Big(G_{U_k} + \mathbb{L}(T_k) \times [\![\, D^2 \,]\!]\Big)\,, \tag{2.8}$$

where $U_k = \mathrm{Ext}(u_k) \in W^{1,2}(\mathcal{C}^{n+1}, D^2)$. Moreover, since $\sup_k \mathcal{E}_{1/2}(T_k) < +\infty$ we have

$$\sup_k \mathbf{D}(U_k) < \infty\,, \qquad \sup_k \mathbf{M}(\mathbb{L}(T_k)) < \infty\,. \tag{2.9}$$

Therefore, from [13], possibly passing to a subsequence we have $\widetilde{T}_k \rightharpoonup \widetilde{T}$ to some current $\widetilde{T} \in \mathcal{D}_{n+1}(\mathcal{C}^{n+1} \times \mathbb{R}^2)$ such that $\widetilde{T} = (-1)^{n-1}\Big(G_U + \widetilde{S}_T\Big)$ for some $U \in W^{1,2}(\mathcal{C}^{n+1}, \mathbb{R}^2)$ and some $\widetilde{S}_T \in \mathcal{R}_{n+1}(\mathcal{C}^{n+1} \times \mathbb{R}^2)$ which vanishes on forms with no completely vertical differentials. Now we check that

$$T = \partial\widetilde{T} = G_u + S_T \qquad \text{on} \qquad \mathcal{D}^n(B^n \times \{0\} \times \mathbb{R}^2)\,, \tag{2.10}$$

where $u \in W^{1/2}(B^n, S^1)$ is the trace of U on $B^n \times \{0\}$ and $S_T \in \mathcal{D}_n(B^n \times S^1)$ is completely vertical, i.e. $S_T(\phi(x, y)\, dx) = 0$ for all $\phi \in C^\infty_c(B^n \times S^1)$. Moreover, due to the weak convergence we also infer that T satisfies (2.1). To show that T decomposes as in (2.2), we argue as follows.

Structure of the weak limit current. According to [13], the weak limit current $T \in \mathcal{D}_n(B^n \times S^1)$ decomposes as

$$T = G_u + \mathbb{L}(T) \times [\![\, S^1 \,]\!] \qquad \text{on} \qquad \mathcal{D}^n(B^n \times S^1)\,, \tag{2.11}$$

where $\mathbb{L}(T) \in \mathcal{D}_{n-1}(B^n)$ is defined by

$$\mathbb{L}(T)(\phi) := S_T(\pi^\# \phi \wedge \widehat{\pi}^\# \omega_{S^1})\,, \qquad \phi \in \mathcal{D}^{n-1}(B^n)\,, \tag{2.12}$$

so that it remains to show that $\mathbb{L}(T)$ is an i.m. rectifiable currents in $\mathcal{R}_{n-1}(B^n)$.

To this aim, following [11], taking into account the density of smooth graphs in $\mathrm{cart}^{1/2}(B^n \times S^1)$, see Theorem 6.1 below, one first shows

Theorem 2.13 *The current $\mathbb{L}(T)$ is a flat chain in B^n.*

Slicing by lines. Let P be an oriented straight line in $\mathbb{R}^n$ and $P_\lambda := P + \sum_{i=1}^{n-1} \lambda_i \nu_i$ the family of oriented lines parallel to P, where $\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \in \mathbb{R}^{n-1}$, $\mathrm{span}(\nu_1, \ldots, \nu_{n-1})$ being the orthogonal hyperspace to P. Since T satisfies (2.10), where $\mathbf{D}(\widetilde{T}) < +\infty$, we infer that for $\mathcal{H}^{n-1}$-a.e. λ the *slice* $T \llcorner \pi^{-1}(P_\lambda)$ of T over $\pi^{-1}(P_\lambda)$ is a well defined 1-dimensional flat chain in $(B^n \cap P_\lambda) \times S^1$ and $T_k \llcorner \pi^{-1}(P_\lambda)$ belongs to $\mathrm{cart}^{1/2}((B^n \cap P_\lambda) \times S^1)$ for every k.

Since $T_k \rightharpoonup T$ weakly in $\mathrm{cart}^{1/2}$, for $\mathcal{H}^{n-1}$-a.e. λ, passing to a subsequence we have $T_k \llcorner \pi^{-1}(P_\lambda) \rightharpoonup T \llcorner \pi^{-1}(P_\lambda)$ weakly in $\mathrm{cart}^{1/2}$. Therefore, from the closure result of Theorem 2.9 we infer that the slice $T \llcorner \pi^{-1}(P_\lambda) \in \mathrm{cart}^{1/2}((B^n \cap P_\lambda) \times S^1)$ and hence that $\mathbb{L}(T) \llcorner \pi^{-1}(P_\lambda) = \mathbb{L}(T \llcorner \pi^{-1}(P_\lambda))$ is 0-dimensional and rectifiable. Since $\mathbb{L}(T)$ is a flat chains, the rectifiability criterion of B. White [21] yields that $\mathbb{L}(T)$ is an i.m. rectifiable currents in $\mathcal{R}_{n-1}(B^n)$. Similarly to [11], we then conclude that T decomposes as in (2.2) and hence that $T \in \mathcal{E}_{1/2}\text{-graph}(B^n \times S^1)$. Finally, by lower semicontinuity, Proposition 2.11, we have $\mathcal{E}_{1/2}(T) < +\infty$ and hence $T \in \mathrm{cart}^{1/2}(B^n \times S^1)$. The closure of the class $\mathrm{cart}^{1/2}_\varphi(\widetilde{B}^n \times S^1)$ is obtained in a similar way.

3 Proofs

In this section we collect for the reader's convenience the proofs of the results stated in Sec. 2.

Proof of Theorem 2.7: Let $\mathbb{R}^2 \simeq \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$, let S^2 be the unit 2-sphere in $\mathbb{R}^3$ and let

$$S^2_\pm := \{\widetilde{y} = (y, \lambda) \in S^2 \,:\, y \in \mathbb{R}^2,\ \pm\lambda \geq 0\}\,.$$

Moreover, let

$$Q^+ := B^1 \times]0, 1[\,, \qquad Q^- := B^1 \times]-1, 0[\,, \qquad Q := B^1 \times]-1, 1[$$

and $U^+_k : Q^+ \to S^2_+$ be the energy minimizing map with boundary condition $U^+_{k\,|B^1 \times \{0\}} = u_k$. Of course

$$\mathbf{D}(U^+_k, Q^+) \simeq |u_k|_{1/2}\,.$$

Also, let $U_k^- : Q^- \to S^2_-$ be given by

$$U_k^-(x,t) := \Phi \circ U_k^+(x,-t)\,, \qquad \Phi(y,\lambda) := (y,-\lambda)\,.$$

Finally let $U_k : Q \to S^2$ be defined by $U_k(x,t) := U_k^\pm(x,t)$ if $\pm t \in [0,1[$. We notice that $U_k^\pm$ is Hölder continuous because of Morrey's theorem [18].

Since $\sup_k |u_k|_{1/2} < +\infty$, we have $\sup_k \mathbf{D}(U_k, Q) < +\infty$. Moreover G_{U_k} belongs to the class $\text{cart}^{2,1}(Q \times S^2)$. Therefore, possibly passing to a subsequence, from [13] we infer that $G_{U_k} \rightharpoonup \widetilde{T}$ weakly in $\mathcal{D}_2(Q \times S^2)$ to some current $\widetilde{T} \in \text{cart}^{2,1}(Q \times S^2)$. As a consequence, $\widetilde{T}$ may be decomposed as

$$\widetilde{T} = G_U + \sum_{i=1}^{j_0} \delta_{x_i} \times [\![\, S^2 \,]\!]\,, \tag{3.1}$$

where $U \in W^{1,2}(Q, S^2)$ and δ_{x_i} is the unit Dirac mass at the point $x_i \in Q$. Moreover, since $U_k \rightharpoonup U$ weakly in $W^{1,2}(Q, S^2)$, and $\partial G_{U_k} \llcorner B^1 \times \{0\} = G_{u_k}$, then the trace of U on $B^1 \times \{0\}$ is a function u in $W^{1/2}(B^1, S^1)$ and $\partial G_U \llcorner (B^1 \times \{0\} \times S^1) = G_u$.

Possibly reordering the indices, we may and will suppose that the points x_i in (3.1) belong to $B^1 \times \{0\}$ if and only if $i \in \{1, \ldots, i_0\}$ for some given $i_0 \le j_0$. Due to the symmetry of the functions U_k, we infer that

$$G_{u_k} = \partial G_{U_k^+} \llcorner (B^1 \times \{0\} \times S^1) \rightharpoonup G_u + \sum_{i=1}^{i_0} \delta_{x_i} \times [\![\, S^1 \,]\!] \qquad \text{in} \quad \mathcal{D}_1(B^1 \times S^1)\,,$$

so that $T \in \mathcal{E}_{1/2}$-graph$(B^1 \times S^1)$, see Definition 2.1. To conclude that $\mathcal{E}_{1/2}(T) < +\infty$, and hence that $T \in \text{cart}^{1/2}(B^1 \times S^1)$ according to Definition 2.5, we argue as in the proof of Proposition 2.8. □

Proof of Proposition 2.8: As in the proof of Proposition 2.11, with $T_k = G_{u_k}$. □

Proof of Theorem 2.9: Let $\{T_k\}$ be a sequence in $\text{cart}^{1/2}(B^1 \times S^1)$ such that $\sup_k \mathcal{E}_{1/2}(T_k) < \infty$ and $T_k \rightharpoonup T$. In the following section, Theorem 4.1, we will show that for every T_k there exists a sequence of smooth maps $\{u_h^{(k)}\}$ in $C^\infty(B^1, S^1)$ such that $G_{u_h^{(k)}} \rightharpoonup T_k$ in $\text{cart}^{1/2}$ and $\mathcal{E}_{1/2}(G_{u_h^{(k)}}) \to \mathcal{E}_{1/2}(T_k)$ as $h \to \infty$. Therefore, by a diagonal argument we may and will assume that $T_k = G_{u_k}$ for some smooth map $u_k \in W^{1/2}(B^n, S^1)$. By Theorem 2.7 we then conclude that $T \in \text{cart}^{1/2}(B^1 \times S^1)$. □

Proof of Proposition 2.11: If T_k and T are given by (2.7) and (2.2), and $\widetilde{T}_k := \text{Ext}(T_k)$ and $\widetilde{T} := \text{Ext}(T)$ by (2.8) and (2.4), respectively, possibly passing to a subsequence we may and will suppose that, on one side, the lower limit in (2.6) is a finite limit and, on the other side, that $\widetilde{T}_k \rightharpoonup \widehat{T}$ to some current $\widehat{T} \in \mathcal{D}_{n+1}(\mathcal{C}^{n+1} \times \mathbb{R}^2)$. Due to weak convergence $T_k \rightharpoonup T$, we infer that $T = \partial \widehat{T}$ on $\mathcal{D}^n(B^n \times \{0\} \times S^1)$. Moreover, since U_k is the harmonic extension of u_k, the energy of U_k does not concentrate in the interior of $\mathcal{C}^{n+1}$ as $k \to +\infty$. As a consequence we obtain that $\widehat{T} = \widetilde{T}$ and hence that $\mathcal{E}_{1/2}(T) := \mathbf{D}(\widetilde{T}) = \mathbf{D}(\widehat{T})$. Finally, by lower semicontinuity of the Dirichlet energy w.r.t. the weak convergence in $\mathcal{D}_{n+1}(\mathcal{C}^{n+1} \times \mathbb{R}^2)$, we get $\mathbf{D}(\widehat{T}) \le \liminf_k \mathbf{D}(\widetilde{T}_k)$ and hence the assertion, as $\mathbf{D}(\widetilde{T}_k) = \mathcal{E}_{1/2}(T_k)$. □

Proof of Proposition 2.12: If $\widetilde{T}_k := \text{Ext}(T_k)$ then, as in the proof of Theorem 2.10, possibly passing to a subsequence we infer that $\widetilde{T}_k \rightharpoonup \widetilde{T}$ weakly in $\mathcal{D}_{n+1}(\mathcal{C}^{n+1} \times \mathbb{R}^2)$ to some current $\widetilde{T}$ such that if $T := \partial \widetilde{T}$ on $\mathcal{D}^n(B^n \times \{0\} \times S^1)$, then $T \in \text{cart}^{1/2}(B^n \times S^1)$ and $T_k \rightharpoonup T$ weakly in $\mathcal{D}_n(B^1 \times S^1)$, as required. □

Proof of Theorem 2.13: In Theorem 6.1 we will show that for every T_k in $\text{cart}^{1/2}(B^n \times S^1)$, respectively in $\text{cart}^{1/2}_\varphi(\widetilde{B}^n \times S^1)$, there exists a sequence of smooth maps $\{u_h^{(k)}\}$ in $C^\infty(B^n, S^1)$, respectively in $C^\infty_\varphi(\widetilde{B}^n, S^1)$, such that $G_{u_h^{(k)}} \rightharpoonup T_k$ in $\text{cart}^{1/2}$ and $\mathcal{E}_{1/2}(G_{u_h^{(k)}}) \to \mathcal{E}_{1/2}(T_k)$ as $h \to \infty$. Therefore, since T is the weak limit of a sequence $\{T_k\}$ in $\text{cart}^{1/2}(B^n \times S^1)$ with equibounded $\mathcal{E}_{1/2}$-energy, by a diagonal argument we may and will assume that $T_k = G_{u_k}$ for some smooth map $u_k \in W^{1/2}(B^n, S^1)$.

Let $\mathbb{L}(G_{u_k}) \in \mathcal{D}_{n-1}(B^n)$ be given by

$$\mathbb{L}(G_{u_k})(\phi) := G_{u_k}(\pi^\# \phi \wedge \widehat{\pi}^\# \omega_{S^1})\,, \qquad \phi \in \mathcal{D}^{n-1}(B^n).$$

Since $u_k \in W^{1/2}(B^n, S^1)$ is smooth, we infer that $\mathbb{L}(G_{u_k})$ is a flat chain. In fact, since $du_k^{\#}\omega_{S^1} = u_k^{\#}d\omega_{S^1} = 0$, then $u_k^{\#}\omega_{S^1}$ is a closed 1-form in $\mathcal{D}^1(B^n)$ and hence $u_k^{\#}\omega_{S^1} = dg_k$ for some $g_k \in C_c^\infty(B^n)$. As a consequence, since for every $\phi \in \mathcal{D}^{n-1}(B^n)$ one has $d(\phi \wedge g_k) = d\phi \wedge g_k + (-1)^{n-1}\phi \wedge dg_k$, whereas $G_{u_k} = (Id \bowtie u_k)_\# [\![B^n]\!]$, and G_{u_k} has no boundary in $B^n \times S^1$, we infer

$$\mathbb{L}(G_{u_k})(\phi) = \int_{B^n} \phi \wedge u_k^{\#}\omega_{S^1} = \int_{B^n} \phi \wedge dg_k = (-1)^n \int_{B^n} d\phi \wedge g_k$$

and finally, by the definition of *flat norm*, see [10],

$$\mathbf{F}\big(\mathbb{L}(G_{u_k})\big) := \sup\{\mathbb{L}(G_{u_k})(\phi) \mid \phi \in \mathcal{D}^{n-1}(B^n)\,,\ \mathbf{F}(\phi) \leq 1\} \leq \int_{B^n} |g_k|\, dx < \infty\,,$$

where

$$\mathbf{F}(\phi) := \max\{\sup_{x\in B^n} \|\phi(x)\|\,,\ \sup_{x\in B^n} \|d\phi(x)\|\}\,.$$

We now show that $\{\mathbb{L}(G_{u_k})\}_k$ is a Cauchy sequence w.r.t. the flat norm, i.e., that

$$\mathbf{F}\big(\mathbb{L}(G_{u_k}) - \mathbb{L}(G_{u_h})\big) := \sup\{\big(\mathbb{L}(G_{u_k}) - \mathbb{L}(G_{u_h})\big)(\phi) \mid \phi \in \mathcal{D}^{n-1}(B^n)\,,\ \mathbf{F}(\phi) \leq 1\}$$

is small for k, h large. Similarly to Sec. 1, we choose a smooth extension $\widetilde{\omega}_{S^1} \in \mathcal{D}^1(\mathbb{R}^2)$ of ω_{S^1} and a function $\eta \in C^\infty([0,1],[0,1])$ with $\eta \equiv 1$ and $\eta \equiv 0$ respectively in a neighborhood of 0 and 1, and we let $U_k = \mathrm{Ext}(u_k)$. For every $\phi \in \mathcal{D}^{n-1}(B^n)$ with $\mathbf{F}(\phi) \leq 1$, by (1.2) we have

$$(-1)^{n-1}\,\mathbb{L}(G_{u_k})(\phi) = \partial G_{U_k}(\pi^\# \phi \wedge \widetilde{\pi}^\# \eta \wedge \widehat{\pi}^\# \widetilde{\omega}_{S^1}) = \int_{\mathcal{C}^{n+1}} \big(d(\phi \wedge \eta) \wedge U_k^{\#}\widetilde{\omega}_{S^1} + (-1)^{n-1}\phi \wedge \eta \wedge U_k^{\#} d\widetilde{\omega}_{S^1}\big)\,.$$

Since $\{U_k\}$ is equibounded in $W^{1,2}$, possibly passing to a subsequence $U_k^{\#}\widetilde{\omega}_{S^1} \to U^{\#}\widetilde{\omega}_{S^1}$ and $U_k^{\#}d\widetilde{\omega}_{S^1} \rightharpoonup U^{\#}d\widetilde{\omega}_{S^1}$ weakly in L^1 for some $U \in W^{1,2}(\mathcal{C}^{n+1}, \mathbb{R}^2)$, so that we infer that $\{\mathbb{L}(G_{u_k})(\phi)\}_k$ is a Cauchy sequence. As a consequence, if $\mathcal{F}^{n-1}(B^n)$ denotes a countable dense subset of smooth forms ϕ in $\mathcal{D}^{n-1}(B^n)$ satisfying $\mathbf{F}(\phi) \leq 1$, by a diagonal argument we infer that

$$\sup\{\big(\mathbb{L}(G_{u_k}) - \mathbb{L}(G_{u_h})\big)(\phi) \mid \phi \in \mathcal{F}^{n-1}(B^n)\}$$

is small for k, h large. By density of $\mathcal{F}^{n-1}(B^n)$, we obtain that $\{\mathbb{L}(G_{u_k})\}_k$ is a Cauchy sequence w.r.t. the flat norm and hence, due to weak convergence of G_{u_k} to T, that the current $R_T \in \mathcal{D}_{n-1}(B^n)$

$$R_T(\phi) := T(\pi^\# \phi \wedge \widehat{\pi}^\# \omega_{S^1})\,, \qquad \phi \in \mathcal{D}^{n-1}(B^n),$$

is a flat chain. Moreover, since $T = G_u + \mathbb{L}(T) \times [\![S^1]\!]$, where $u \in W^{1/2}(B^n, S^1)$, as a consequence of the strong density result for Cartesian currents in $\mathrm{cart}^{1/2}$, in the next sections we will obtain the sequential weak density of smooth maps in $W^{1/2}(B^n, S^1)$, see Corollary 6.2. Therefore, setting

$$D_T(\phi) := G_u(\pi^\# \phi \wedge \widehat{\pi}^\# \omega_{S^1})\,, \qquad \phi \in \mathcal{D}^{n-1}(B^n),$$

by repeating the previous argument we easily infer that D_T is a flat chain and hence the assertion, being $\mathbb{L}(T) = R_T - D_T$, see (2.12). □

4 The density result in dimension 1

In this section we prove the following

Theorem 4.1 *Let $n = 1$. For every $T \in \mathrm{cart}^{1/2}(B^1 \times S^1)$ there exists a sequence of smooth maps $u_k : B^1 \to S^1$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathrm{cart}^{1/2}$ and*

$$\lim_{k\to+\infty} \mathcal{E}_{1/2}(u_k, B^1) = \mathcal{E}_{1/2}(T, B^1 \times S^1)\,. \tag{4.1}$$

Similarly we have

Corollary 4.2 *For every $T \in \mathrm{cart}^{1/2}_\varphi(\widetilde{B}^1 \times S^1)$ there exists a sequence of smooth maps $\{u_k\} \subset C^\infty_\varphi(\widetilde{B}^1, S^1)$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathrm{cart}^{1/2}$ and $\mathcal{E}_{1/2}(u_k, \widetilde{B}^1) \to \mathcal{E}_{1/2}(T, \widetilde{B}^1 \times S^1)$ as $k \to +\infty$.*

We recall by Sec. 2 that every $T \in \mathrm{cart}^{1/2}(B^1 \times S^1)$ has the form

$$T = G_{u_T} + \sum_{i=1}^{i_0} \delta_{x_i} \times [\![S^1]\!] , \tag{4.2}$$

where δ_{x_i} is the Dirac mass in $x_i \in B^1$. Moreover the $\mathcal{E}_{1/2}$-energy of T is defined as the Dirichlet energy of its extension $\widetilde{T} = \mathrm{Ext}(T) := G_{U_T} + \sum_{i=1}^{i_0} \delta_{x_i} \times [\![D^2]\!]$, where $U_T := \mathrm{Ext}(u_T) \in W^{1,2}(\mathcal{C}^2, D^2)$, so that

$$\mathcal{E}_{1/2}(T) := \mathbf{D}(\widetilde{T}) = \frac{1}{2} \int_{\mathcal{C}^2} |Du_T|^2 \, dx + \pi \cdot i_0 ,$$

compare Definition 2.2.

Remark 4.3 For future use, if $0 < \delta < 1$ we denote

$$D^2_\delta := \{ y \in D^2 \, : \, \mathrm{dist}(y, S^1) \le \delta \} , \qquad S^1_\delta := \{ y \in D^2 \, : \, |y| = \varepsilon \}$$

the δ-neighborhood of S^1 in D^2 and the circle of radius δ. Also, let Π_δ denote the nearest point projection of D^2_δ onto S^1. Note that Π_δ is a well defined Lipschitz map with Lipschitz constant $L_\delta \to 1^+$ as $\delta \to 0^+$.

In the sequel we will also denote

$$B^+_r := \overline{B}^2_r \cap \mathcal{C}^2 , \qquad \partial^+ B_r := \partial B^2_r \cap \{ (x,t) \in \mathcal{C}^2 \mid t > 0 \} , \qquad J_r := \partial B^+_r \setminus \partial^+ B_r = [-r, r] \times \{0\} , \tag{4.3}$$

where $B^2_r := \{ (x,t) \in \mathbb{R}^2 \mid x^2 + t^2 < r^2 \}$, and

$$B^+ := B^+_1 , \qquad \partial^+ B := \partial^+ B_1 , \qquad J := J_1 .$$

We will also write

$$\mathbf{T}(U) = u$$

if $u \in W^{1/2}(B^n, \mathbb{R}^2)$ is the trace of a function $U \in W^{1,2}(\mathcal{C}^{n+1}, \mathbb{R}^2)$ on $B^n \times \{0\}$.

The proof of Theorem 4.1 relies on the following density result.

Proposition 4.4 *Let U be a smooth $W^{1,2}$-map from $\mathcal{C}^2$ into D^2 with trace $\mathbf{T}(U) \in W^{1/2}(B^1, D^2_\delta)$. Then there exist a sequence $\{U_k\}$ of smooth maps from $\mathcal{C}^2$ into D^2, with traces $u_k := T(U_k) \in W^{1/2}(B^1, D^2_\delta)$ for every k, and a sequence of radii $r_k \searrow 0$ such that $U_k = U$ outside $B^+_{r_k}$ and $G_{U_k} \rightharpoonup G_U + \delta_0 \times [\![D^2]\!]$ weakly in $\mathcal{D}_2(\mathcal{C}^2 \times \mathbb{R}^2)$ with*

$$\lim_{k \to +\infty} \mathbf{D}(U_k, \mathcal{C}^2) = \mathbf{D}(U, \mathcal{C}^2) + \pi .$$

To prove Proposition 4.4 we make use of the following

Proposition 4.5 *Let $P \in S^1_\delta$ be a given point, $\delta \in [1/2, 1]$. For every $\varepsilon > 0$ there exists a Lipschitz function $f_\varepsilon : B^+ \to D^2$ such that $f_{\varepsilon | \partial^+ B} \equiv P$, $f_\varepsilon(J) \subset D^2_\delta$, $f_{\varepsilon \#} [\![B^+]\!] = [\![D^2]\!]$, $f_{\varepsilon \#} [\![J]\!] = [\![S^1]\!]$, and*

$$\mathbf{D}(f_\varepsilon, B^+) \le \pi + \varepsilon .$$

PROOF: By slightly modifying the identity map from B^2 onto D^2, for any $\varepsilon > 0$ we first define a Lipschitz map $g_\varepsilon : B^2 \to D^2$ such that $g_{\varepsilon \#} [\![B^2]\!] = [\![D^2]\!]$, the mapping area $A(g_\varepsilon, B^2) \le \pi + \varepsilon$, and such that g_ε maps the upper part $\partial^+ B$ of the boundary of B^2 constantly into the north pole $P_N = (0,1) \in S^1$. Secondly, by connecting P_N with P in D^2_δ, we slightly modify g_ε in such a way that g_ε maps $\partial^+ B$ constantly into the given point $P \in S^1_\delta$, whereas $g_\varepsilon(\partial B^2) \subset D^2_\delta$, and $g_{\varepsilon\#}[\![\partial B^2]\!] = [\![S^1]\!]$. Let $\psi : B^+ \to \overline{B}^2$ be a bilipschitz homeomorphism such that ψ is the identity on $\partial^+ B$. Then $\widetilde{f}_\varepsilon := g_\varepsilon \circ \psi : B^+ \to D^2$ is a Lipschitz continuous function satisfying $\widetilde{f}_{\varepsilon|\partial^+ B} \equiv P$, $\widetilde{f}_\varepsilon(J) \subset D^2_\delta$, $\widetilde{f}_{\varepsilon\#}[\![B^+]\!] = [\![D^2]\!]$, and $\widetilde{f}_{\varepsilon\#}[\![J]\!] = [\![S^1]\!]$, whereas

$$A(\widetilde{f}_\varepsilon, B^+) = A(g_\varepsilon, B^2) \le \pi + \varepsilon\,.$$

We now apply Morrey's ε-conformality theorem [18, Thm. 2.1] and define an orientation preserving diffeomorphism $\Psi_\varepsilon : B^+ \to B^+$ such that, if $f_\varepsilon := \widetilde{f}_\varepsilon \circ \Psi_\varepsilon$, we have

$$\mathbf{D}(f_\varepsilon, B^+) \le (1+\varepsilon)\, A(f_\varepsilon, B^+) = (1+\varepsilon)\, A(\widetilde{f}_\varepsilon, B^+) \le (1+\varepsilon)\,(\pi+\varepsilon)\,.$$

Finally, due to the *three points condition*, we may and do define Ψ_ε so that it maps $\partial^+ B$ onto $\partial^+ B$ and J onto J. The assertion easily follows. ☐

Proof of Proposition 4.4: For $\varepsilon > 0$ and $r \in (0, 1/2)$, we let $U_{\varepsilon,r} : B^+_r \to D^2$ be given by

$$U_{\varepsilon,r}(z) := \begin{cases} U\Big(\dfrac{2|z|-r}{|z|}\, z\Big) & \text{if} \quad r/2 \le |z| \le r \\[2mm] f_\varepsilon(2z/r) & \text{if} \quad |z| < r/2 \end{cases} \qquad z = (x,t) \in B^+_r\,,$$

where f_ε is given by Proposition 4.5, with $P = U(0) \in D^2_\delta$, and we set $U_{\varepsilon,r}(z) \equiv U$ on $\mathcal{C}^2 \setminus B^+_r$. We have

$$\int_{B^+_r \setminus B^+_{r/2}} |DU_{r,\varepsilon}|\, dz \le C \int_{B^+_r} |DU|\, dz < \varepsilon\,,$$

by absolute continuity, if $r = r(\varepsilon)$ is small. Moreover,

$$\frac{1}{2} \int_{B^+_{r/2}} |DU_{r,\varepsilon}|\, dz = \mathbf{D}(f_\varepsilon, B^+)\,,$$

so that the claim follows from Proposition 4.5, letting $\varepsilon_k \searrow 0$. ☐

Proof of Theorem 4.1: Since $n = 1$, adapting an argument by [20], as in [3, Sec. 2.1] we can find a sequence of smooth maps $U_k : \mathcal{C}^2 \to D^2$ such that $U_k \to \mathrm{Ext}(u_T)$ strongly in $W^{1,2}(\mathcal{C}^2, \mathbb{R}^2)$ and for which there exists a positive number $t_0 > 0$ such that $U_k(B^1 \times [0, t_0]) \subset D^2_{1/2}$ for every k. In particular the traces $u_k := \mathbf{T}(U_k)$ belong to $W^{1/2}(B^1, D^2_{1/2})$ and $u_k \to u$ in $W^{1/2}(B^1)$.

On small half-disks $x_i + B^+_{r_{k,h}}$ around each x_i and contained in $\mathcal{C}^2$, see (4.2), we then apply Proposition 4.4 to each U_k and find a sequence of smooth maps $\{U_{k,h}\}_h$ from $\mathcal{C}^2$ into $\mathbb{R}^2$, with traces $u_{k,h} := \mathbf{T}(U_{k,h}) \in W^{1/2}(B^1, D^2_{1/2})$ for every h, and a sequence of radii $r_{k,h} \searrow 0$ as $h \to +\infty$ such that $U_{k,h} = U_k$ outside $x_i + B^+_{r_{k,h}}$ and

$$G_{U_{k,h}} \rightharpoonup G_{U_k} + \sum_{i=1}^{i_0} \delta_{x_i} \times [\![D^2]\!]\,, \qquad \lim_{h\to+\infty} \mathbf{D}(U_{k,h}, \mathcal{C}^2) = \mathbf{D}(U_k, \mathcal{C}^2) + \pi\,.$$

By a diagonal procedure we therefore find a smooth sequence $\{V_k\} \subset C^1(\mathcal{C}^2, \mathbb{R}^2)$, again with traces $v_k := \mathbf{T}(V_k) \in W^{1/2}(B^1, D^2_{1/2})$, such that $G_{V_k} \rightharpoonup \widetilde{T}$ weakly in $\mathcal{D}_2(\mathcal{C}^2 \times \mathbb{R}^2)$ and $\mathbf{D}(V_k, \mathcal{C}^2) \to \mathbf{D}(\widetilde{T})$ as $k \to +\infty$. Finally, setting $u_k := \Pi_{1/2} \circ v_k$, compare Remark 4.3, we obtain the assertion. ☐

Proof of Corollary 4.2: Since u is smooth on $\widetilde{B}^1 \setminus B^1$, we may and do define the sequence $U_k : \mathcal{C}^2 \to \mathbb{R}^2$ so that in particular $(G_{u_k} - G_\varphi) \llcorner (\widetilde{B}^1 \setminus \overline{B}^1) \times \mathbb{R}^2 = 0$ for every k. Moreover, since the points x_i in (4.2) can be taken distant from the boundary of B^1, we apply Proposition 4.4 by taking the radii $r_{k,h}$ small so that in particular $U_{k,h}$ coincides with U_k in a small neighborhood of $\partial B^1 \times I$, as required. ☐

5 The dipole construction

In this section we provide the approximation of dipoles, see [8] [12] [13, Vol. II, Sec. 4.2.3], for $W^{1/2}$-maps with values in S^1. We first fix some notation. We set

$$\widetilde{\mathcal{C}}^{n+1} := \widetilde{B}^n \times I \,, \qquad I = [0,1]\,.$$

Let Δ denote the $(n-1)$-simplex in B^n given by the convex hull

$$\Delta := \mathrm{co}\left(\{0_{\mathbb{R}^n}, l\,e_1, l\,e_2, \ldots, l\,e_{n-1}\}\right), \qquad 0 < l < 1\,,$$

$(e_1,\ldots,e_n)$ being the standard basis in $\mathbb{R}^n$. We will denote by

$$z = (x,t) = (\widetilde{x}, x_n, t)\,, \qquad \widetilde{x} := (x_1,\ldots,x_{n-1})\,,$$

a generic point z in $\widetilde{\mathcal{C}}^{n+1}$. Moreover, for $\delta > 0$ and $0 < m \ll 1$, in the sequel we let

$$\varphi^m_\delta(y) := \min\{my, \delta\}\,, \qquad y \geq 0\,,$$

we denote by

$$y(\widetilde{x}) := \mathrm{dist}(\widetilde{x}, \partial\Delta)$$

the distance of $\widetilde{x}$ from the boundary of the $(n-1)$-simplex Δ and we set

$$\phi^m_\delta(z) := (\widetilde{x}, \varphi^m_\delta(y(\widetilde{x}))x_n, \varphi^m_\delta(y(\widetilde{x}))t)\,,$$

so that if

$$\Omega^m_\delta := \phi^m_\delta(\Delta \times B^+)\,, \qquad B^+ := \{(x_n,t) \in B^2 \mid t > 0\}\,,$$

then Ω^m_δ is a small "neighbor" of the simplex Δ in $\mathcal{C}^{n+1}$, compare (4.3). Finally, for every non-zero integer $q \in \mathbb{Z} \setminus \{0\}$ we will denote by $q\,[\![\, S^1 \,]\!]$ and $q\,[\![\, D^2 \,]\!]$ the currents integration of forms on S^1 and D^2, respectively, with integer multiplicity $|q|$ and orientation induced by the sign of q.

Proposition 5.1 *Let $U : \widetilde{\mathcal{C}}^{n+1} \to D^2$ be a $W^{1,2}$-map which is smooth in the interior of $\Omega^{m_0}_{\delta_0}$, for some fixed small $m_0, \delta_0 > 0$, and such that $u := \mathbf{T}(U) \in W^{1/2}_\varphi(\widetilde{B}^n, S^1)$. Let $q \in \mathbb{Z}$. Then for every $\varepsilon > 0$, $0 < \delta < \delta_0$ and $0 < m < m_0$ there exists a map $U_\varepsilon : \widetilde{\mathcal{C}}^{n+1} \to D^2$, with trace $\mathbf{T}(U_\varepsilon) \in W^{1/2}_\varphi(B^n, S^1)$, such that U_ε is smooth in the closure of Ω^m_δ, except for the boundary of Δ. Moreover $G_{U_\varepsilon} \rightharpoonup G_U + [\![\, \Delta \,]\!] \times q\,[\![\, D^2 \,]\!]$ weakly in $\mathcal{D}_{n+1}(\widetilde{\mathcal{C}}^{n+1} \times \mathbb{R}^2)$ as $\varepsilon \to 0^+$ and*

$$\mathbf{D}(U_\varepsilon, \widetilde{\mathcal{C}}^{n+1}) \leq \mathbf{D}(U, \widetilde{\mathcal{C}}^{n+1}) + \mathcal{H}^{n-1}(\Delta) \cdot |q|\,\pi + \varepsilon\,. \tag{5.1}$$

To prove Proposition 5.1 we make use of

Lemma 5.2 *Let $V : \Delta \times B^+ \to \mathbb{R}^2$ be a $W^{1,2}$-function and let*

$$V^m_\delta(z) := V \circ (\phi^m_\delta)^{-1}(z)\,, \qquad z \in \Omega^m_\delta\,.$$

Then there exists an absolute constant $c > 0$ such that

$$\begin{aligned} \int_{\Omega^m_\delta} |DV^m_\delta|^2\,dz \;&\leq\; \int_{\Delta\times B^+} |D_{(x_n,t)}V|^2\,dz + c\,\delta^2 \int_{\Delta\times B^+} |D_{\widetilde{x}}V|^2\,dz \\ &+\; c\,m^2 \int_{\{\widetilde{x}\in\Delta \mid y(\widetilde{x}) \leq \delta/m\}\times B^+} |D_{(x_n,t)}V|^2\,dz\,. \end{aligned} \tag{5.2}$$

Moreover, by adapting the proof of Proposition 4.5, we easily obtain the following

Proposition 5.3 *For every $q \in \mathbb{Z}\setminus\{0\}$, $P \in S^1$ and $\varepsilon > 0$, there exists a Lipschitz function $f^P_\varepsilon : B^+ \to D^2$ such that $f^P_{\varepsilon|\partial^+ B} \equiv P$, $f^P_\varepsilon(J) \subset S^1$, $f^P_{\varepsilon\#}[\![B^+]\!] = q\,[\![D^2]\!]$, $f^P_{\varepsilon\#}[\![J]\!] = q\,[\![S^1]\!]$, and*

$$\mathbf{D}(f^P_\varepsilon, B^+) \leq |q|\,\pi + \varepsilon\,.$$

Finally, for every $\varepsilon > 0$ we may and do define f^P_ε in such a way that for every $\sigma > 0$ there exists $\eta > 0$ such that

$$\forall P_1, P_2 \in S^1\,, \quad |P_1 - P_2| < \eta \implies \|f^{P_1}_\varepsilon - f^{P_2}_\varepsilon\|_{L^\infty(B^+)} < \sigma\,.$$

Proof: We first slightly modify the map from B^2 onto D^2 given in complex variables by

$$z = \rho\, e^{i\theta} \mapsto \rho^{|q|}\, e^{iq\theta}, \qquad q \in \mathbb{Z}\setminus\{0\}\,,$$

and we define a Lipschitz map $g^P_\varepsilon : B^2 \to D^2$ such that $g^P_{\varepsilon\#}[\![B^2]\!] = q\,[\![D^2]\!]$, the mapping area $A(g^P_\varepsilon, B^2) \leq |q|\,\pi + \varepsilon$, and such that g^P_ε maps the upper part $\partial^+ B$ of the boundary of B^2 constantly into the point $P \in S^1$. We then let $\widetilde{f}_\varepsilon := g_\varepsilon \circ \psi : B^+ \to D^2$, where $\psi : B^+ \to \overline{B}^2$ is a bilipschitz homeomorphism which is the identity on $\partial^+ B$. We then proceed as in Proposition 4.5, by means of Morrey's ε-conformality theorem. The assertion easily follows, since g^P_ε may be defined so that it continuously depend on the point P. $\square$

Proof of Proposition 5.1: We give the details of the proof in the case $n = 2$. We refer to [16] for the more general case $n \geq 3$. We first introduce the cylindrical coordinates

$$z = (x_1, x_2, t) = F(\rho, \theta, x_1) := (x_1, \rho\cos\theta, \rho\sin\theta)\,, \qquad \rho > 0, \quad \theta \in [0, \pi]\,,$$

so that $\rho = \sqrt{{x_2}^2 + t^2}$. In the sequel we will also denote

$$\widehat{W}(\rho, \theta, x_1) := W(F(\rho, \theta, x_1))\,.$$

Since U is smooth in the interior of $\Omega^{m_0}_{\delta_0}$, possibly taking a barycentric subdivision of Δ, without loss of generality we may and will assume that the oscillation of U is smaller than ε in the interior of $\Omega^{m_0}_{\delta_0}$. If $n = 2$ we have that Δ is the line segment connecting the points $a_+ := (l, 0, 0)$ and $a_- = 0_{\mathbb{R}^3}$. Similarly to [2], we can choose small half-balls of radius r around $a_\pm$ and replace U there by the radial maps

$$U_r(z) := U\left(a_\pm + r\frac{z - a_\pm}{|z - a_\pm|}\right) \tag{5.3}$$

so that

$$\mathbf{D}(U_r, B^3_r(a_\pm) \cap \mathcal{C}^3) = \frac{r}{2}\int_{\partial B^3_r(a_\pm)\cap\mathcal{C}^3} |D_\tau U|^2\, d\mathcal{H}^2 = O(r)\,,$$

where τ is an orthonormal frame of $\partial B^3_r(a_\pm)$ and $O(r_j) \to 0$ for some sequence $r_j \searrow 0$. Moreover, due to the smoothness of U, without loss of generality we may and do choose m so that

$$\int_{K^m_{a_\pm}} |DU|^2\, d\mathcal{H}^2 < \infty\,, \tag{5.4}$$

where $K^m_{a_\pm}$ is the cone of vertex $a_\pm$ and angle $\arctan m$

$$K^m_{p_i} := \{z = F(\rho, \theta, y) \in \mathcal{C}^3 \,:\, 0 < \rho = m\,|y - a_\pm|\}\,.$$

Let now $W_\varepsilon : \Delta \times B^+ \to \mathbb{R}^2$ be given by

$$W_\varepsilon(x_1, x_2, t) := f^{P(x_1)}_\varepsilon(x_2, t)\,,$$

where $f^{P(x_1)}_\varepsilon$ is given by Proposition 5.3 in correspondence to the point $P(x_1) := U(x_1, 0, 0)$. Setting

$$\Phi_\varepsilon(z) := W_\varepsilon \circ (\phi^m_\delta)^{-1}(z)\,, \qquad z \in \Omega^m_\delta\,,$$

by Lemma 5.2 we estimate

$$\mathbf{D}(\Phi_\varepsilon,\Omega_\delta^m)\le\int_\Delta \mathbf{D}(f_\varepsilon^{P(x_1)},B^+)\,d\mathcal{H}^1(x_1)+\varepsilon\le\mathcal{H}^1(\Delta)\cdot(|q|\,\pi+\varepsilon)+\varepsilon$$

if we choose δ sufficiently small. Define $V_\varepsilon:\Omega_\delta^m\to\mathbb{R}^2$ in cylindrical coordinates by

$$\widehat{V}_\varepsilon(\rho,\theta,x_1):=\begin{cases}\widehat{\Phi}_\varepsilon(2\rho,\theta,\widetilde{y}) & \text{if}\quad 0\le\rho<\varphi_\delta^m(\widetilde{y})/2\\ \widehat{\Psi}_\delta^m(\rho,\theta,\widetilde{y}) & \text{if}\quad \varphi_\delta^m(\widetilde{y})/2\le\rho<\varphi_\delta^m(\widetilde{y})\end{cases}\qquad \theta\in[0,\pi]\,,\quad x_1\in\operatorname{int}(\Delta)\,,$$

where $y(x_1):=\operatorname{dist}(x_1,\partial\Delta)$ and

$$\widehat{\Psi}_\delta^m(\rho,\theta,\widetilde{y}):=\left(\frac{2\rho}{\varphi_\delta^m(\widetilde{y})}-1\right)\cdot\widehat{U}(\varphi_\delta^m(\widetilde{y}),\theta,\widetilde{y})+\left(2-\frac{2\rho}{\varphi_\delta^m(\widetilde{y})}\right)\cdot\widehat{U}(0,\theta,\widetilde{y})\,,$$

so that $\widehat{\Psi}_\delta^m(\varphi_\delta^m(\widetilde{y}),\theta,\widetilde{y})=\widehat{U}(\varphi_\delta^m(\widetilde{y}),\theta,\widetilde{y})$ and $\widehat{\Psi}_\delta^m(\varphi_\delta^m(\widetilde{y})/2,\theta,\widetilde{y})=P(x_1)=\widehat{\Phi}_\varepsilon(\varphi_\delta^m(\widetilde{y}),\theta,\widetilde{y})$. We have

$$\mathbf{D}(V_\varepsilon,\{0\le\rho<\varphi_\delta^{m,i}(\widetilde{y})/2\,,\ \theta\in[0,\pi]\,,\ x_1\in\Delta\})=\mathbf{D}(\Phi_\varepsilon,\Omega_\delta^m)\,.$$

Moreover, by (5.3) we estimate

$$\begin{aligned}&\mathbf{D}(V_\varepsilon,\{\varphi_\delta^{m,i}(\widetilde{y})/2\le\rho<\varphi_\delta^{m,i}(\widetilde{y})\,,\ \theta\in[0,\pi]\,,\ x_1\in\Delta\})\le\\ &\qquad c\left(\frac{\delta}{m}\,\varepsilon^2+l\varepsilon^2+\delta\int_{(0,l)\times\partial B_\delta^+}|DU|^2\,d\mathcal{H}^2+m\int_{K_{a_\pm}^m\cap B^3_{r_{\delta,m}}(a_\pm)}|DU|^2\,d\mathcal{H}^2\right),\end{aligned}$$

where $r_{\delta,m}:=\delta\sqrt{1+m^2}/m$. Now, since

$$\liminf_{\rho\to0^+}\rho\int_{(0,l)\times\partial B_\rho^+}|DU|^2\,d\mathcal{H}^2=0\,,$$

if we first choose m small, and then $\delta=\delta(V_\varepsilon,\varepsilon,m)$ sufficiently small, by (5.4) we have

$$\mathbf{D}(V_\varepsilon,\Omega_\delta^m)\le\mathcal{H}^1(\Delta)\cdot(|q|\,\pi+\varepsilon)+2\varepsilon\,.$$

We finally let $U_\varepsilon\equiv U$ on $\widetilde{\mathcal{C}}^3\setminus\Omega_\delta^m$, and $U_\varepsilon:=\Pi_\varepsilon\circ V_\varepsilon$ on Ω_δ^m, see Remark 4.3. □

6 The density result in higher dimension

In this section we prove in any dimension the following

Theorem 6.1 *Let $n\ge2$. Also, let $\varphi:\widetilde{B}^n\to S^1$ be a given smooth $W^{1/2}$-function. For every $T\in\operatorname{cart}^{1/2}(B^n\times S^1)$, respectively $T\in\operatorname{cart}_\varphi^{1/2}(\widetilde{B}^n\times S^1)$, there exists a sequence of smooth maps $\{u_k\}$ in $C^\infty(B^n,S^1)$, respectively in $C_\varphi^\infty(\widetilde{B}^n,S^1)$, such that $G_{u_k}\rightharpoonup T$ weakly in $\operatorname{cart}^{1/2}$ and*

$$\lim_{k\to+\infty}\mathcal{E}_{1/2}(u_k)=\mathcal{E}_{1/2}(T)\,.$$

As a consequence, compare [13], Vol. II, Sec. 4.2.5, we infer the following density results for $W^{1/2}$-maps.

Corollary 6.2 *Every map u in $W^{1/2}(B^n,S^1)$, respectively in $W_\varphi^{1/2}(\widetilde{B}^n,S^1)$, can be approximated weakly in $W^{1/2}$ by a sequence of maps in $C^1(B^n,S^1)\cap W^{1/2}(B^n,S^1)$, respectively in $C_\varphi^1(\widetilde{B}^n,S^1)\cap W^{1/2}(\widetilde{B}^n,S^1)$. Moreover, u can be approximated strongly in $W^{1/2}$ by a sequence of smooth maps provided that $\mathbb{P}(u)=0$.*

Proof of Theorem 6.1: We divide the proof in five steps.

Step 1: Reduction to finite mass singularities. We first recall that if $T \in \mathrm{cart}^{1/2}_{\varphi}(\widetilde{B}^n, S^1)$, then T decomposes as

$$T = G_{u_T} + \mathbb{L}(T) \times [\![S^1]\!] , \tag{6.1}$$

where $u_T \in W^{1/2}_{\varphi}(\widetilde{B}^n, S^1)$ and $\mathbb{L}(T) \in \mathcal{R}_{n-1}(\widetilde{B}^n)$, with $\mathrm{spt}\, \mathbb{L}(T) \subset \overline{B}^n$. As a consequence of the results proved in Sec. 2, we have

Proposition 6.3 *There exists a sequence $\{u_k\}$ in $R^{\infty}_{1/2,\varphi}(\widetilde{B}^n, S^1)$, strongly converging to u_T in $W^{1/2}$, such that if L_{u_k,u_T} is given by (1.5), then*

$$T_k := G_{u_k} + ((-1)^n L_{u_k,u_T} + \mathbb{L}(T)) \times [\![S^1]\!]$$

belongs to $\mathrm{cart}^{1/2}_{\varphi}(\widetilde{B}^n \times S^1)$, and for every k the mass $\mathbf{M}(\partial((-1)^n L_{u_k,u_T} + \mathbb{L}(T)))$ is finite in $\widetilde{B}^n$, whereas $T_k \rightharpoonup T$ and $\mathcal{E}_{1/2}(T_k) \to \mathcal{E}_{1/2}(T)$ as $k \to +\infty$.

Proof: Since $\partial T = 0$ on $\mathcal{D}^{n-1}(\widetilde{B}^n \times S^1)$, by (6.1) we infer that

$$\partial\, \mathbb{L}(T) = (-1)^{n-1}\, \mathbb{P}(u_T) .$$

Due to Theorem 1.1 and Proposition 1.3, by (1.5) we have

$$\partial((-1)^n L_{u_k,u_T} + \mathbb{L}(T) = (-1)^{n-1}\, \mathbb{P}(u_k) ,$$

which is an i.m. rectifiable $(n-2)$-current (a finite sum of unit Dirac masses if $n = 2$), see Proposition 1.2. Moreover

$$\partial G_{u_k} = (-1)^n\, \mathbb{P}(u_k) \times [\![S^1]\!] \qquad \text{on} \qquad \mathcal{D}^{n-1}(\widetilde{B}^n \times S^1) , \tag{6.2}$$

whence $T_k \in \mathrm{cart}^{2,1}_{\varphi}(\widetilde{B}^n \times S^1)$, with $T_k \rightharpoonup T$. Now, writing

$$T_k = G_{u_k} + \mathbb{L}(T_k) \times [\![S^1]\!] , \qquad \mathbb{L}(T_k) := (-1)^n L_{u_k,u_T} + \mathbb{L}(T)$$

we have $\mathbf{M}(\mathbb{L}(T_k) - \mathbb{L}(T)) = \mathbf{M}(L_{u_k,u_T}) \to 0$, which yields that $\mathcal{E}_{1/2}(T_k) \to \mathcal{E}_{1/2}(T)$. □

By Proposition 6.3, arguing as in [11] we may and do suppose

$$T = G_{u_T} + \sum_{q \in \mathbb{Z}} \mathbb{L}_q \times q\, [\![S^1]\!] , \qquad \widetilde{T} := \mathrm{Ext}(T) = (-1)^{n-1} \Big(G_{U_T} + \sum_{q \in \mathbb{Z}} \mathbb{L}_q \times q\, [\![D^2]\!] \Big) ,$$

where $U_T := \mathrm{Ext}(u_T)$ and the $\mathbb{L}_q$'s are i.m. rectifiable currents in $\mathcal{R}_{n-1}(\widetilde{B}^n)$ with multiplicity 1, pairwise disjoint supports contained in $\overline{B}^n$, and finite boundary mass, $\sum_q \mathbf{M}(\partial\, \mathbb{L}_q) < \infty$.

Step 2: Approximation by polyhedral chains. Since the $\mathbb{L}_q$'s are integral currents with pairwise disjoint supports, using Federer's polyhedral approximation theorem [9], for every $q \in \mathbb{Z}$ we find an integral polyhedral $(n-1)$-chain P^{ε}_q with support contained in a small neighborhood of radius $c\,\varepsilon$ in $\overline{B}^n$ of the support of $\mathbb{L}_q$, and a function $U_\varepsilon \in C^\infty(\widetilde{\mathcal{C}}^{n+1}, D^2)$, with trace $u_\varepsilon := \mathbf{T}(U_\varepsilon) \in R^{\infty}_{1/2,\varphi}(\widetilde{B}^n, S^1)$, such that if

$$\widetilde{T}_\varepsilon := G_{U_\varepsilon} + \sum_{q \in \mathbb{Z}} P^{\varepsilon}_q \times q\, [\![D^2]\!] ,$$

$\widetilde{T}_\varepsilon$ converges weakly in $\mathcal{D}_{n+1}(\widetilde{\mathcal{C}}^{n+1} \times \mathbb{R}^2)$ to $\widetilde{T}$ as $\varepsilon \to 0$ and

$$\mathbf{D}(U_\varepsilon, \widetilde{\mathcal{C}}^{n+1}) + \sum_{q \in \mathbb{Z}} q\, \pi\, \mathbf{M}(P^{\varepsilon}_q) \to \mathbf{D}(U_T, \widetilde{\mathcal{C}}^{n+1}) + \sum_{q \in \mathbb{Z}} q\, \pi\, \mathbf{M}(\mathbb{L}_q) ,$$

which yields $\mathbf{D}(\widetilde{T}_\varepsilon) \to \mathbf{D}(\widetilde{T})$ as $\varepsilon \to 0$. Moreover, since the $\mathbb{L}_q$'s have disjoint supports, we may and do choose the P^{ε}_q's so that for every small $\varepsilon > 0$ they have pairwise disjoint supports.

Step 3: Approximation by well-intersecting polyhedral chains. By Step 2 we may suppose

$$T := G_{u_T} + \sum_{q\in\mathbb{Z}} P_q \times q\,[\![\, S^1 \,]\!] \in \mathrm{cart}^{1/2}_\varphi(\widetilde{B}^n \times S^1)\,, \tag{6.3}$$

where the P_q's are polyhedral $(n-1)$-chains with multiplicity 1 and pairwise disjoint supports $\mathrm{spt}\, P_q \subset \overline{B}^n$, and $u_T \in R^\infty_{1/2,\varphi}(\widetilde{B}^n, S^1)$ is locally Lipschitz on $\widetilde{B}^n \setminus \bigcup_q \mathrm{spt}\,\partial P_q$. Moreover, possibly dividing the simplices of a triangulation of P_q, we may and will suppose that every P_q is the union of a finite number of $(n-1)$-simplices Δ which only intersect at the boundary points.

Step 4: Approximating the dipoles. Now we first approximate the dipoles $\Delta \times q\,[\![\, S^1 \,]\!]$ by means of Proposition 5.1. In fact, by taking m_0 and δ_0 small we may and will assume that the neighborhoods $\Omega^{m_0}_{\delta_0}$ corresponding to different simplices Δ are pairwise interiorly disjoint.

By a diagonal argument, we then find a sequence $\{U_\varepsilon\}$ such that $u_\varepsilon := \mathbf{T}(U_\varepsilon) \in R^\infty_{1/2,\varphi}(\widetilde{B}^n \times S^1)$ and the graphs G_{u_ε} weakly converge to T with $\mathcal{E}_{1/2}(G_{u_\varepsilon}) \to \mathcal{E}_{1/2}(T)$. However, u_ε is smooth except on a singular set Σ_ε of B^n given by the $(n-2)$-skeleton of the union of the polyhedral $(n-1)$-chains P_q, but

$$\partial G_{u_\varepsilon} = 0 \qquad \text{on } \mathcal{D}^{n-1}(B^n)\,. \tag{6.4}$$

To remove the singular set Σ_ε, we finally make use of the following variant of a result from [15].

Proposition 6.4 *Under the previous hypotheses, for $\varepsilon > 0$ small enough there exists a sequence of smooth maps $\{u^{(\varepsilon)}_m\} \subset C^\infty_\varphi(\widetilde{B}^n, S^1_\varepsilon)$ which converges to u_ε strongly in $W^{1/2}$ as $m \to +\infty$.*

We refer to [16] for the proof of Proposition 6.4 in dimension $n \geq 3$, which relies on (6.4) and on the smoothness of the boundary datum $\varphi : \widetilde{B}^n \to S^1$ on the whole of $\widetilde{B}^n$. In the case $n = 2$, we do not need to assume that the boundary datum is smooth, since we actually reduce to remove *homologically trivial* point singularities, as follows.

Proposition 6.5 (Removing point singularities). *Let $u \in R^\infty_{1/2}(B^2, S^1)$ be in $\mathrm{cart}^{1/2}(B^2, S^1)$, so that (6.4) holds, with $u = u_\varepsilon$ and $n = 2$. Then there exists a sequence of smooth maps $u_k \subset C^\infty(B^2, S^1)$ which converges to u strongly in $W^{1/2}$.*

Since we use a local argument, we may assume that u has only one singularity at the origin, i.e., $u \in C^\infty(B^2 \setminus \{0\}, S^1)$. For $0 < r < 1$ we denote

$$Q_r := B^3_r \cap \mathcal{C}^3\,, \qquad \partial^+ Q_r := \partial B^3_r \cap \{z = (x,t) \in \mathcal{C}^3 \mid t > 0\}\,, \qquad F_r := Q_r \cap (B^2 \times \{0\})\,.$$

Let $U \in W^{1,2}(\mathcal{C}^3, D^2)$ be the harmonic extension of u. For every fixed $\varepsilon > 0$ let $0 < R = R(\varepsilon) \ll 1$ be such that

$$\mathbf{D}(U, Q_R) \leq \varepsilon\,.$$

Since

$$\mathbf{D}(U, Q_R \setminus Q_{R/2}) = \frac{1}{2}\int_{R/2}^R dr \int_{\partial^+ Q_r} |DU|^2\, d\mathcal{H}^2\,,$$

there exists $r = r_\varepsilon \in [R/2, R]$ such that

$$\mathbf{D}(U, \partial^+ Q_r) := \frac{1}{2}\int_{\partial^+ Q_r} |DU|^2\, d\mathcal{H}^2 \leq \frac{4}{R}\mathbf{D}(U, Q_R \setminus Q_{R/2}) \leq \frac{4\varepsilon}{R}\,. \tag{6.5}$$

To remove the singularity of u, we have to show that

$$\{w \in W^{1/2}(B^2_r, \mathbb{R}^2) \cap C^0(\overline{B}^2_r, S^1) \mid w_{|\partial B^2_r} = u_{|\partial B^2_r}\} \neq \emptyset\,, \tag{6.6}$$

i.e., $u_{|\partial B^2_r}$ is homotopic to a constant map in S^1. Therefore, it suffices to show that $d\, u_{|\partial B^2_r}{}^\# \omega_{S^1} = 0$, i.e., that $u_{|\partial B^2_r}$ has zero degree. This follows from condition (6.4) with $u = u_\varepsilon$, as

$$\int_{\partial B^2_r} u_{|\partial B^2_r}{}^\# \omega_{S^1} = G_{u_{|\partial B^2_r}}(\widehat{\pi}^\# \omega_{S^1}) = \partial G_{u_{|B^2_r}}(\widehat{\pi}^\# \omega_{S^1}) = G_{u_{|B^2_r}}(d\widehat{\pi}^\# \omega_{S^1}) = G_{u_{|B^2_r}}(\widehat{\pi}^\# d\omega_{S^1}) = 0\,.$$

As a consequence there exists a smooth extension $u_r : B^2_r \to S^1$ of $u_{|\partial B^2_r}$ with finite $W^{1/2}$-energy.

Let now $V_r : Q_r \to D^2$ be the solution of the Dirichlet problem on Q_r with boundary condition

$$\begin{cases} V_r = U & \text{on} \quad \partial^+ Q_r \\ V_r = u_r & \text{on} \quad F_r \, . \end{cases}$$

Let $0 < \delta < r$ to be fixed later. Define $U_r : \mathcal{C}^3 \to D^2$ by

$$U_r(z) := \begin{cases} V_r\left(\dfrac{r}{\delta} z\right) & \text{if} \quad |z| \le \delta \\ U\left(r\dfrac{z}{|z|}\right) & \text{if} \quad \delta \le |z| \le r \\ U(z) & \text{if} \quad |z| \ge r \end{cases}$$

so that $U_r \in W^{1,2}(\mathcal{C}^3, \mathbb{R}^2)$ is continuous and with trace $\mathbf{T}(U_r) \in W^{1/2}(B^2, S^1)$. We easily estimate

$$\mathbf{D}(U_r, \mathcal{C}^3) \le \mathbf{D}(U, \mathcal{C}^3) + c\, r\, \mathbf{D}(U, \partial^+ Q_r) + \frac{\delta}{r} \mathbf{D}(V_r, Q_r)$$

for some absolute constant $c > 0$. Therefore, since $r < R$, by (6.5) we have

$$\mathbf{D}(U_r, \mathcal{C}^3) \le \mathbf{D}(U, \mathcal{C}^3) + 4c\,\varepsilon + \frac{\delta}{r} \mathbf{D}(V_r, Q_r) \le \mathbf{D}(U, \mathcal{C}^3) + (4c+1)\,\varepsilon \, ,$$

taking $\delta = \delta(\varepsilon)$ sufficiently small. Letting $\varepsilon \to 0$ we infer that $U_{r_\varepsilon} \to U$ in $W^{1,2}(\mathcal{C}^3, \mathbb{R}^2)$ and finally that $\mathbf{T}(U_{r_\varepsilon}) \to u$ in $W^{1/2}(B^2, S^1)$, with $\mathbf{T}(U_{r_\varepsilon}) \in W^{1/2}(B^2, S^1)$ continuous. By a standard argument, we now approximate $\mathbf{T}(U_{r_\varepsilon})$ by smooth functions, as required. ☐

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References

[1] ADAMS R.A., *Sobolev spaces*, Academic Press, New York 1975.

[2] BETHUEL F., The approximation problem for Sobolev maps between manifolds. *Acta Math.* **167** (1992) 153-206.

[3] BETHUEL F., Approximations in the trace spaces defined between manifolds. *Nonlinear Analysis* **24** (1995) 121-130.

[4] BOURGAIN J., BREZIS H., MIRONESCU P., On the structure of the Sobolev space $H^{1/2}$ with values into the circle, *C.R. Acad. Sci. Paris* **331** (2000) 119-124.

[5] BOURGAIN J., BREZIS H., MIRONESCU P., Lifting in Sobolev spaces, *J. Anal. Math* **80** (2000) 37-86.

[6] BOURGAIN J., BREZIS H., MIRONESCU P., $H^{1/2}$ maps with values into the circle: minimal connections, lifting, and the Ginzburg Landau equation. *Publ. Math. Ist. Hautes Études Sci.* **99** (2004) 1-115.

[7] BOURGAIN J., BREZIS H., MIRONESCU P., Lifting, degree and distributional Jacobian revisited. *Preprint* 2003.

[8] BREZIS H., CORON J.M., LIEB E.H., Harmonic maps with defects. *Comm. Math. Phys.* **107** (1986) 649-705.

[9] FEDERER H., *Geometric measure theory.* Grundlehren math. Wissen. 153, Springer, Berlin (1969).

[10] FEDERER H., Real flat chains, cochains and variational problems. *Indiana Univ. Math. J.* **24** (1974) 351-407.

[11] GIAQUINTA M., MODICA G., On sequences of maps with equibounded energies. *Calc. Var.* **12** (2001) 213-222.

[12] GIAQUINTA M., MODICA G., SOUČEK J., Liquid crystals: relaxed energies, dipoles, singular lines and singular points. *Ann. Sc. Norm. Sup. Pisa* **17** (1990) 415-437.

[13] GIAQUINTA M., MODICA G., SOUČEK J., *Cartesian currents in the calculus of variations,* I, II. Ergebnisse Math. Grenzgebiete (III Ser), 37, 38, Springer, Berlin (1998).

[14] GIAQUINTA M., MODICA G., SOUČEK J., On sequences of maps into S^1 with equibounded $W^{1/2}$ energies. *Selecta Math.* (N. S.) **10** n. 3 (2004) 359-375.

[15] GIAQUINTA M., MUCCI D., Density results for the $W^{1/2}$ energy of maps into a manifold. *Math. Z.* To appear.

[16] GIAQUINTA M., MUCCI D., On sequences of maps into a manifold with equibounded $\mathbf{W^{1/2}}$-energies. *J. Funct. Anal.* To appear.

[17] HARDT B., PITTS J., Solving the Plateau's problem for hypersurfaces without the compactness theorem for integral currents. In *Geometric Measure Theory and the Calculus of Variations*, edited by Allard W.K. and Almgren F.J.. Proc. Symp. Pure Math. **44**. Am. Math. Soc., Providence (1996) 255-295.

[18] MORREY C.B., The problem of Plateau on a Riemannian manifold. *Ann. Math.* **49** (1948) 807-851.

[19] RIVIÈRE T., Dense subsets of $H^{1/2}(S^2; S^1)$, *Annals of Global Analysis and Geometry* **18** (2000) 517-528.

[20] SCHOEN R., UHLENBECK, K., Boundary regularity and the Dirichlet problem for harmonic maps. *J. Diff. Geom.* **18** (1983) 253-268.

[21] WHITE B., Rectifiability of flat chains. *Ann. Math.* (2) **150** no. 1 (1999) 165-184.

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