Variational analysis of nonlocal Dirichlet problems in periodically perforated domains

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Abstract. In this paper we consider a family of non local functionals of convolution-type depending on a small parameter $\varepsilon > 0$ and Γ-converging to local functionals defined on Sobolev spaces as $\varepsilon \to 0$. We study the asymptotic behaviour of the functionals when the order parameter is subject to Dirichlet conditions on a periodically perforated domains, given by a periodic array of small balls of radius r_δ centered on a δ–periodic lattice, being $\delta > 0$ an additional small parameter and $r_\delta = o(\delta)$. We highlight differences and analogies with the local case, according to the interplay between the three scales ε, δ and r_δ. A fundamental tool in our analysis turns out to be a non local variant of the classical Gagliardo-Nirenberg-Sobolev inequality in Sobolev spaces which may be of independent interest and useful for other applications.

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1 Introduction

In the last decades there has been an increasing interest towards the analysis of variational models involving non local functionals of the form

$$\int_\Omega \int_\Omega f(x, y, u(y) - u(x))\, dx\, dy \tag{1.1}$$

where Ω is an open set of $\mathbb{R}^d$, in view of their relevance for applications in different directions, such as image processing [11, 16], population dynamics [15], continuum mechanics through the theory of perydinamics [7, 18, 25] and phase transition problems [1, 21].

The relation between non local functionals as in (1.1) when the energies concentrate on the diagonal $x = y$ and local functionals of the form

$$\int_\Omega f(x, \nabla u(x))\, dx \tag{1.2}$$

has been first investigated by Bourgain, Brezis and Mironescu in their seminal paper [8], where they study the asymptotic behaviour of Gagliardo seminorms $[u]_{W^{1-\varepsilon,p}(\Omega)}$ as $\varepsilon \to 0$, and in particular, in the case $p = 2$, show that

$$\varepsilon [u]_{W^{1-\varepsilon,2}(\Omega)} = \varepsilon \int_\Omega \int_\Omega \frac{|u(y) - u(x)|^2}{|y - x|^{d+2(1-\varepsilon)}}\, dx\, dy$$

approximate as $\varepsilon \to 0$ the square of the L^2 norm of ∇u, up to a multiplicative constant. The result has been subsequently extended in [20] in terms of Γ-convergence. A general asymptotic analysis as $\varepsilon \to 0$ of families of functionals of the form

$$\int_\Omega \int_\Omega f_\varepsilon(x, y, u(y) - u(x))\, dx\, dy, \tag{1.3}$$

under superlinear growth assumptions in the last variable and concentration of the energies on $x = y$, has been recently provided in [2], by using De Giorgi localization methods for Γ-convergence, leading to a general class of energies whose Γ-limits are of the form (1.2), with a number of applications, in particular to stochastic homogenization, to energies on point clouds and to gradient flows, which are just some of the potential directions of the theory.

Purpose of this paper is to investigate the asymptotic behaviour of energies as in (1.3) when the order parameter u is subject to pinning conditions, highlighting differences and analogies with the corresponding local case. Pinning sites are usually modelled as small zones where Dirichlet conditions are imposed. Here we consider the simplest case (but already presenting most of the main features) of periodically perforated domains where homogeneous Dirichlet conditions are imposed on a periodic array P_δ of small balls of radius r_δ centered on a δ-periodic lattice, being $\delta > 0$ an additional small parameter and $r_\delta = o(\delta)$. In the local case there is a wide literature devoted to the study of minimum problems involving energies as in (1.2) subject to this type of constraints and comprising a number of generalizations which cover also general non-periodic geometries. In particular, we refer to the celebrated paper by Cioranescu and Murat [12], where the authors study the asymptotic behaviour, as $\delta \to 0$, of solutions to the minima of the Dirichlet energy subject to the constraint above, and the paper [5], where the problem is set in the framework of Γ-convergence and extended to general vector energies. The asymptotic description of the problems becomes non trivial when the growth of f in the gradient variable in (1.2) is of order $p \leq d$ and leads to a critical size of the radii of the perforations, namely $r_\delta = O(\delta^{d/(d-p)})$, if $p < d$, and $\log(r_\delta) = O(\delta^{d/(d-1)})$, if $p = d$. Under this scaling the energetic contribution near each of the small balls can be decoupled from the others and from the diffused energy elsewhere and can be computed by means of a capacitary formula. For instance, in the model case $f(x, z) = |z|^p$ and given a forcing term $g \in L^{p'}(\Omega)$, one obtains that minimum problems

$$\min \left\{ \int_\Omega (|\nabla u|^p - gu)\, dx : \ u = 0 \text{ on } \ P_\delta \right\}$$

are approximated as $\delta \to 0$ by

$$\min \left\{ \int_\Omega (|\nabla u|^p\, dx + C_p |u|^p - gu)\, dx \right\},$$

where C_p is the p-capacity of the ball B_1 in $\mathbb{R}^d$ (see Section 4) and the middle term accounts for the energetic contribution near the perforations.

In this paper we focus on the case $p < d$ and consider energies defined on vector-valued functions $u \in L^p(\Omega; \mathbb{R}^m)$ and of the form

$$E_\varepsilon(u) = \frac{1}{\varepsilon^d} \int_\Omega \int_\Omega f\left(\frac{y-x}{\varepsilon}, \frac{u(y)-u(x)}{\varepsilon}\right) dx\, dy$$

where, for any $\xi \in \mathbb{R}^d$, $f(\xi, \cdot)$ is p-homogeneous, locally Lipschitz continuous and satisfies suitable decay assumptions as $|\xi| \to +\infty$, ensuring that interactions between points x and y at a long range distance are negligible as $\varepsilon \to 0$ (see hypotheses (**H**), (**G**) and (**L**) below). Under this assumptions, as a particular case of the asymptotic analysis provided in [2], the Γ-limit of E_ε is given by the local functional

$$E_0(u) = \int_\Omega f_{hom}(\nabla u)\, dx,$$

where $f_{hom}(S)$ is defined by a suitable homogenization formula (see (3.11)). We then impose the admissible functions u to satisfy the constraint $u = 0$ on P_δ and study the asymptotic behaviour of E_ε as ε and δ go to 0 according to the interplay between the three scales ε, δ and r_δ. A natural question is whether or not E_ε and E_0 share the same asymptotic behaviour under the imposed constraint on the admissible functions. We show that this is the case when ε goes to 0 faster then r_δ. Indeed, the first main result of the paper is Theorem 3.5, where we show that, if $\varepsilon = o(r_\delta)$ and $r_\delta = O(\delta^{d/(d-p)})$, E_ε and E_0, subject to the constraint $u = 0$ on P_δ, share the same Γ-limit, which is given by

$$\int_\Omega \left(f_{hom}(\nabla u) + \varphi(u)\right) dx, \tag{1.4}$$

where $\varphi(z)$ is described by the capacitary formula induced by $E_0(u)$, see (3.14). We point out that this kind of homogenization problems in the non local setting has been studied in [19], where the authors treat Dirichlet and Neumann boundary conditions for a non local equation in the scalar case and when $p = 2$.

A major difference is when ε scales like r_δ, since in this case the limit functional keeps memories of the non locality of the approximating energies. Indeed, the second main result of the paper is Theorem 3.6, where we show that, if $\varepsilon = O(r_\delta)$ and $r_\delta = O(\delta^{d/(d-p)})$, the Γ-limit of E_ε, subject to the constraint $u = 0$ on P_δ, is still of the form (1.4), but the density $\varphi(z)$ is now described by a non local capacitary formula, see (3.17). If r_δ does not scale like $\delta^{d/(d-p)}$, in both case $\varepsilon = o(r_\delta)$ and $\varepsilon = O(r_\delta)$ the asymptotic behaviour of E_ε is trivial and it is consistent with that of local energies of Dirichlet type in periodically perforated domains (see Remark 3.7). On the contrary, in Theorem 8.1 we show that if $\varepsilon \to 0$ slower than r_δ, then, for most of the choice of the scaling of r_δ with respect to δ, E_ε is not affected by the constraint $u = 0$ on P_δ and thus the Γ-limit is still given by E_0.

From a technical viewpoint, in order to prove Theorem 3.5 and Theorem 3.6 we mainly follow the strategy exploited in [5]. Nevertheless, the non local nature of our approximating energies does not allow us to simply adapt that argument to our case. The main difficulty we have encountered in the proof of Theorem 3.5 is to show the convergence of minimum problems on unbounded domains, defining the approximating capacitary densities, to the limit energy density φ, stated in Proposition 6.7. A crucial result that allowed us to overcome this difficulty is Theorem 5.1, which can be considered a non local variant of the classical Gagliardo-Nirenberg-Sobolev inequality in Sobolev spaces and may be of independent interest and useful for other applications. Specifically, we show that, fixed $r > 0$, the L^{p^*}-norm of suitable piecewise constant interpolations at scale ε of any admissible function is uniformly bounded from above, up to a multiplicative constant, by the energy

$$G_\varepsilon^{r,p}(u,\mathbb{R}^d) = \int_{B_r}\int_{\mathbb{R}}^d \left|\frac{u(x+\varepsilon\xi)-u(x)}{\varepsilon}\right|^p dx\,d\xi,$$

which in turn is controlled by $E_\varepsilon(u)$ and plays the role of $\int_{\mathbb{R}^d}|\nabla u|^p\,dx$ in the Gagliardo-Nirenberg-Sobolev inequality.

We conclude the introduction with some comments about future developments. In our model we refrain from maximal generality in order to emphasize the main features of the asymptotic process for our non local functionals under pinning conditions, but it would be worth extending our analysis to more general integrands. In particular in the critical regime, if one removes the p-homogeneity assumption on f, then, assuming the existence of

$$\lim_{\delta\to 0}\delta^{dp/(d-p)}f(\xi,\delta^{-d/(d-p)}z) =: f_{\infty,p}(\xi,z)$$

the limit energy should be still of the form (1.4), with f replaced by $f_{\infty,p}$ in the definition of the capacitary density φ. A natural follow-up of our results is also the extension to the case $p=d$ and to the critical regime $\log(r_\delta) = O(\delta^{d/(d-1)})$. We point out that a Γ-convergence analysis for local functionals in this setting has been provided in [24]. Furthermore, we believe that some of the techniques developed in this paper can also be used to extend the analysis to the case of perforations whose centres are randomly distributed according to a stationary point process (see [22] for results in this direction in the local case). We finally point out that another class of non local functionals, namely discrete functionals of the form

$$\frac{1}{\varepsilon^{p+d}}\sum_{i,j\in\mathcal{L}} f(i,j,u_j-u_i),$$

where $\mathcal{L}$ is a d-dimensional lattice, $u:\varepsilon\mathcal{L}\to\mathbb{R}^m$ and $u_i = u(\varepsilon i)$, have been widely investigated as a discrete approximation of integral functionals of p-growth (see e.g. [3]). In this setting, an asymptotic analysis similar to the one provided here when $\varepsilon = O(r_\delta)$ has been carried on in [23], where the main result can be considered the discrete analog of Theorem 3.6. It would be interesting to extend that analysis also to the case $\varepsilon = o(r_\delta)$.

2 Notation

In what follows $d,m\in\mathbb{N}$ will be fixed natural numbers denoting the dimension of the reference and target spaces of the functions we consider, respectively. Given $t\in\mathbb{R}$, $\lfloor t\rfloor$ denotes the integer part of t; for $x\in\mathbb{R}^d$, $r>0$, $B_r(x)$ (if $x=0$, simply B_r) stands for the open ball of centre x and radius r, $Q_r(x)$ (if $x=0$, simply Q_r) stands for the square $x+(-\frac{r}{2},\frac{r}{2})^d$. We denote by S^{d-1} the unit sphere in $\mathbb{R}^d$. If A is a subset of $\mathbb{R}^d$ then $\mathrm{dist}(x,A)=\inf\{|y-x| : y\in A\}$; $\mathcal{A}^{\mathrm{reg}}(A)$ is the subfamily of open subsets with Lipschitz boundary. By $A\subset\subset B$ we mean that the closure of A is a compact subset of B. If $A\subset\subset B$, a *cut-off function* between A and B is a (smooth) function φ with $0\le\varphi\le 1$, $\varphi=0$ on ∂B and $\varphi=1$ on A. Given a real function $h(\cdot)$, we use the symbols $o(h), O(h)$ respectively, to denote a generic function g such that $\lim_{t\to 0}\dfrac{g(t)}{h(t)}=0$, $\lim_{t\to 0}\frac{g(t)}{h(t)}=\gamma\in(0,+\infty)$. If E is a measurable subset of $\mathbb{R}^d$ we denote by $|E|$ its Lebesgue measure. We use standard notation for Lebesgue and Sobolev spaces. If u is an integrable function on a measurable set $E\subset\mathbb{R}^d$,

$$u_E := \frac{1}{|E|}\int_E u(x)dx$$

denotes the average of u on E. We use also standard notation for Γ-convergence [9, 13], indicating the topology with respect to which it is performed. Unless otherwise stated, the letter C denotes a generic strictly positive constant. Relevant dependencies on parameters will be as usual emphasised by putting them in parentheses.

3 Setting of the problem and main results

We fix a growth exponent $p \in (1, d)$ and we let $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary. Let δ, r_δ be given with $\delta > r_\delta > 0$ and set

$$P_\delta := \bigcup_{i \in \mathbb{Z}^d} B_{r_\delta}(\delta i), \tag{3.1}$$

$$L^p_\delta(\Omega; \mathbb{R}^m) := \{u \in L^p(\Omega; \mathbb{R}^m) : u \equiv 0 \text{ on } P_\delta \cap \Omega\}. \tag{3.2}$$

Given $\varepsilon > 0$ and $f : \mathbb{R}^d \times \mathbb{R}^m \to [0, +\infty)$ a positive Borel function, we introduce the non-local functionals $F_{\varepsilon,\delta} : L^p(\Omega; \mathbb{R}^m) \to [0, +\infty]$ defined as

$$F_{\varepsilon,\delta}(u) := \begin{cases} \displaystyle\int_{\mathbb{R}^d} \int_{\Omega_\varepsilon(\xi)} f\Big(\xi, D^\xi_\varepsilon u(x)\Big) dx\, d\xi & \text{if } u \in L^p_\delta(\Omega; \mathbb{R}^m), \\ +\infty & \text{otherwise,} \end{cases} \tag{3.3}$$

where

$$D^\xi_\varepsilon u(x) := \frac{u(x + \varepsilon\xi) - u(x)}{\varepsilon} \tag{3.4}$$

and for every subset A of $\mathbb{R}^d$ we set

$$A_\varepsilon(\xi) := \{x \in A \,|\, x + \varepsilon\xi \in A\}. \tag{3.5}$$

We consider the following set of assumptions on the function f above:

(H) **(p- homogeneity)** $f(\xi, tz) = t^p f(\xi, z)$ for every $(\xi, z) \in \mathbb{R}^d \times \mathbb{R}^m$ and $t > 0$;

(G) **(growth)** the functions $m(\xi) := \inf_{z \in S^{d-1}} f(\xi, z)$ and $M(\xi) := \sup_{z \in S^{d-1}} f(\xi, z)$ satisfy:

- (G0) there exist $\lambda_0, r_0 > 0$ such that $m(\xi) \geq \lambda_0$ if $|\xi| \leq r_0$;
- (G1) $\displaystyle\int_{\mathbb{R}^d} M(\xi)(|\xi|^p + 1)\, d\xi < +\infty$;

(L) **(p- Lipschitz continuity)** there exists $C > 0$ such that for every $\xi \in \mathbb{R}^d$

$$|f(\xi, w) - f(\xi, z)| \leq CM(\xi)(|z|^{p-1} + |w|^{p-1})|w - z| \quad \forall z, w \in \mathbb{R}^m.$$

We also introduce the 'truncated' functionals defined for every $T > 0$ as

$$F^T_{\varepsilon,\delta}(u) := \begin{cases} \displaystyle\int_{B_T} \int_{\Omega_\varepsilon(\xi)} f\Big(\xi, D^\xi_\varepsilon u(x)\Big) dx\, d\xi & \text{if } u \in L^p_\delta(\Omega; \mathbb{R}^m), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.6}$$

Note that $F^T_{\varepsilon,\delta}$ is of the form (3.3) with $f^T(\xi, z) := \chi_{B_T}(\xi) f(\xi, z)$ in place of $f(\xi, z)$.

Remark 3.1. Note that assumption (**H**) yields that $m(\xi)|z|^p \le f(\xi,z) \le M(\xi)|z|^p$ for every $(\xi,z) \in \mathbb{R}^d \times \mathbb{R}^m$.

Let us consider also the unconstrained family of functionals $\mathcal{F}_\varepsilon : L^p(\Omega;\mathbb{R}^m) \to [0,+\infty]$ defined by

$$\mathcal{F}_\varepsilon(u) := \int_{\mathbb{R}^d}\int_{\Omega_\varepsilon(\xi)} f\Big(\xi, D^\xi_\varepsilon u(x)\Big)dx\,d\xi. \tag{3.7}$$

We also introduce a localized version of such functionals by setting, for any open set $A \subset \mathbb{R}^d$ and $u \in L^p(A;\mathbb{R}^m)$,

$$\mathcal{F}_\varepsilon(u,A) := \int_{\mathbb{R}^d}\int_{A_\varepsilon(\xi)} f\Big(\xi, D^\xi_\varepsilon u(x)\Big)dx\,d\xi. \tag{3.8}$$

Moreover the *truncated* functionals are defined, for any $T > 0$, as

$$\mathcal{F}^T_\varepsilon(u,A) := \int_{B_T}\int_{A_\varepsilon(\xi)} f\Big(\xi, D^\xi_\varepsilon u(x)\Big)dx\,d\xi \tag{3.9}$$

and we drop the dependence on the set if $A = \Omega$, that is $\mathcal{F}^T_\varepsilon(u) := \mathcal{F}^T_\varepsilon(u,\Omega)$.
As a particular case of a more general result, in [2] it was proved the following Γ-convergence result (see [2, Theorem 6.1]).

Theorem 3.2. *Let $\mathcal{F}_\varepsilon$ be defined by* (3.7)*, with f satisfying assumptions* (**H**) *and* (**G**)*. Then*

$$\Gamma(L^p) - \lim_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega f_{hom}(\nabla u)\,dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^m),\\ +\infty & \text{otherwise,}\end{cases} \tag{3.10}$$

where, for $S \in \mathbb{R}^{m\times d}$,

$$f_{hom}(S) := \lim_{R\to\infty}\frac{1}{R^d}\inf\Big\{\int_{Q_R}\int_{Q_R} f(y-x, v(y)-v(x))dx\,dy : v \in \mathcal{D}^S(Q_R)\Big\}, \tag{3.11}$$

with

$$\mathcal{D}^S(Q_R) := \big\{u \in L^p(\mathbb{R}^d;\mathbb{R}^m) : u(x) = Sx \text{ for a.e. } x \in \mathbb{R}^d\,,\ \mathrm{dist}(x,\mathbb{R}^d\backslash Q_R) < 1\big\}.$$

Remark 3.3. Note that the p-homogeneity assumption (**H**) is inherited by f_{hom}, that is

$$f_{hom}(tS) = t^p f_{hom}(S) \text{ for every } t > 0 \text{ and } S \in \mathbb{R}^{m\times d}.$$

Moreover assumption (**G**) easily yields that

$$m_0|S|^p \le f_{hom}(S) \le M_0|S|^p \text{ for every } S \in \mathbb{R}^{m\times d}$$

for two suitable strictly positive constants m_0, M_0.

Remark 3.4. Hypothesis (**L**) trivially holds if we assume that f satisfies (**H**), (**G**), and $f(\xi,\cdot)$ is convex for every $\xi \in \mathbb{R}^d$. Moreover, under this convexity assumption, the asymptotic formula (3.11) reduces to

$$f_{hom}(S) = \int_{\mathbb{R}^d} f(\xi, S\xi)\,d\xi$$

(see [2, Theorem 6.2]).

The main results of the paper are provided by the following two theorems.

Theorem 3.5 (Local capacitary term)**.** *Let* $F_{\varepsilon,\delta}$ *be defined by* (3.3)*, with* f *satisfying assumptions* (**H**)*,* (**G**) *and* (**L**)*. Assume moreover that*

$$\lim_{\delta\to 0}\frac{r_\delta}{\delta^{\frac{d}{d-p}}}=\beta \tag{3.12}$$

and that $\delta=\delta_\varepsilon$ *is such that*

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{r_{\delta_\varepsilon}}=0 \tag{3.13}$$

for some real number $\beta>0$*. Then*

$$\Gamma(L^p)\text{-}\lim_{\varepsilon\to 0}F_{\varepsilon,\delta_\varepsilon}(u)=\begin{cases}\displaystyle\int_\Omega f_{hom}(\nabla u)\,dx+\beta^{d-p}\int_\Omega\varphi(u)\,dx & \text{if } u\in W^{1,p}(\Omega;\mathbb{R}^m),\\ +\infty & \text{otherwise,}\end{cases}$$

where $f_{hom}(S)$ *is defined by* (3.11) *and for every* $z\in\mathbb{R}^m$

$$\varphi(z):=\inf\Big\{\int_{\mathbb{R}^d}f_{hom}(\nabla v)\,dx:\ v-z\in L^{p^*}(\mathbb{R}^d;\mathbb{R}^m),\ v\equiv 0\text{ in }B_1,\ v\in W^{1,p}_{\rm loc}(\mathbb{R}^d;\mathbb{R}^m)\Big\}. \tag{3.14}$$

Theorem 3.6 (Nonlocal capacitary term)**.** *Let* $F_{\varepsilon,\delta}$ *be defined by* (3.3)*, with* f *satisfying assumptions* (**H**)*,* (**G**) *and* (**L**)*. Assume moreover that*

$$\lim_{\delta\to 0}\frac{r_\delta}{\delta^{\frac{d}{d-p}}}=\beta$$

and that $\delta=\delta_\varepsilon$ *is such that*

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{r_{\delta_\varepsilon}}=\alpha \tag{3.15}$$

for some real numbers $\alpha,\beta>0$*. Then*

$$\Gamma(L^p)\text{-}\lim_{\varepsilon\to 0}F_{\varepsilon,\delta_\varepsilon}(u)=\begin{cases}\displaystyle\int_\Omega f_{hom}(\nabla u)\,dx+\beta^{d-p}\int_\Omega\varphi_{NL,\alpha}(u)\,dx & \text{if } u\in W^{1,p}(\Omega;\mathbb{R}^m),\\ +\infty & \text{otherwise,}\end{cases} \tag{3.16}$$

where $f_{hom}(S)$ *is defined by* (3.11) *and for every* $z\in\mathbb{R}^m$

$$\varphi_{NL,\alpha}(z):=\inf\left\{\mathcal{F}_\alpha(v,\mathbb{R}^d):\ v-z\in L^p(\mathbb{R}^d;\mathbb{R}^m),\ v\equiv 0\text{ in }B_1,\ v-z\text{ compactly supported}\right\}, \tag{3.17}$$

being $\mathcal{F}_\alpha$ *defined in* (3.8) *with* $\varepsilon=\alpha$*.*

Remark 3.7. Taking into account the non degeneracy of the capacitary densities (3.14), (3.17) proved in Proposition 6.4 below, and arguing by comparison, one easily infers that the results stated in Theorem 3.5 and Theorem 3.6 can be "continuously" extended to the case $\beta=0$ and $\beta=+\infty$. More in details, if $\beta=0$ and either (3.13) or (3.15) hold, then the functionals $F_{\varepsilon,\delta_\varepsilon}$ Γ-converge to the energy functional defined in (3.10), while, if $\beta=+\infty$ and either (3.13) or (3.15) hold, the Γ-limit of the functionals $F_{\varepsilon,\delta_\varepsilon}$ is trivially 0 if $u\equiv 0$ and $+\infty$ otherwise. This phenomenon is consistent with the asymptotic behaviour of local energies of Dirichlet type in periodically perforated domains. We will see in Section 8 that this is not the case if $\alpha=+\infty$.

For later use and reader's convenience we redefine the density functions we have introduced so far in the case $f(\xi, z)$ is replaced by $f^T(\xi, z) = \chi_{B_T}(\xi) f(\xi, z)$. More precisely we set

$$f^T_{hom}(S) := \lim_{R\to\infty} \frac{1}{R^d} \inf \Big\{ \int_{Q_R}\int_{Q_R} f^T(y-x, v(y)-v(x))dx\,dy : v \in \mathcal{D}^S(Q_R) \Big\}, \tag{3.18}$$

$$\varphi^T(z) := \inf \Big\{ \int_{\mathbb{R}^d} f^T_{hom}(\nabla v)\,dx : \; v - z \in L^{p*}(\mathbb{R}^d;\mathbb{R}^m), \; v \equiv 0 \text{ in } B_1, \; v \in W^{1,p}_{\rm loc}(\mathbb{R}^d;\mathbb{R}^m) \Big\}, \tag{3.19}$$

$$\varphi^T_{NL,\alpha}(z) := \inf \big\{ \mathcal{F}^T_\alpha(v, \mathbb{R}^d) : \; v - z \in L^p(\mathbb{R}^d;\mathbb{R}^m), \; v \equiv 0 \text{ in } B_1, \; v - z \text{ compactly supported} \big\} . \tag{3.20}$$

4 Preliminary results

In this section we collect some results that will be used in Section 6.

Capacity

Let us recall the notion of p-capacity for a given exponent $p \in (1, d)$ (see for instance [14],[17]). Given an open set $A \subset \mathbb{R}^d$ and an open set $E \subset\subset A$, the *relative p-capacity* of E in A is defined as

$$\mathrm{cap}_p(E, A) = \inf \left\{ \int_A |\nabla u|^p \, dx : u \in W^{1,p}_0(A), \, u = 1 \text{ a.e. in } E \right\} .$$

If $A = \mathbb{R}^d$, we simply write $\mathrm{cap}_p(E)$. It follows by the very definition that the set function $\mathrm{cap}_p(E, A)$ is increasing in the variable E and decreasing in the variable A. In addition, the following properties hold true

$$\begin{aligned} \mathrm{cap}_p(E) &= \inf \left\{ \int_{\mathbb{R}^d} |\nabla u|^p \, dx : u \in W^{1,p}_{\rm loc}(\mathbb{R}^d) \cap L^{p^*}(\mathbb{R}^d), \, u = 1 \text{ a.e. in } E \right\} \\ &= \lim_{R\to+\infty} \mathrm{cap}_p(E, B_R) = \inf_{R>0} \mathrm{cap}_p(E, B_R), \end{aligned} \tag{4.1}$$

where $p^* := \frac{pd}{d-p}$ is the conjugate exponent of p. It can be also proved that $\mathrm{cap}_p(E) > 0$ if $|E| > 0$.

Remark 4.1. One may also consider, for $z \in \mathbb{R}^m$, the vectorial infimum problems

$$\inf \left\{ \int_A |\nabla u|^p \, dx : u \in W^{1,p}_0(A;\mathbb{R}^m), \, u = z \text{ a.e. in } E \right\} . \tag{4.2}$$

Note that, thanks to the p-homogeneity and the rotational invariance of (4.2), it holds

$$\begin{aligned} &\inf \left\{ \int_A |\nabla u|^p \, dx : u \in W^{1,p}_0(A;\mathbb{R}^m), \, u = z \text{ a.e. in } E \right\} \\ &= |z|^p \inf \left\{ \int_A |\nabla u|^p \, dx : u \in W^{1,p}_0(A;\mathbb{R}^m), \, u = (1, 0, \dots, 0) \text{ a.e in } E \right\} , \end{aligned}$$

and the infimum in the last term can be in turn confined to functions $v \in W^{1,p}_0(A;\mathbb{R}^m)$ such that $v = (v_1, 0, \dots, 0)$. Hence, we can conclude that, for any $z \in \mathbb{R}^m$,

$$\inf \left\{ \int_A |\nabla u|^p \, dx : u \in W^{1,p}_0(A;\mathbb{R}^m), \, u = z \text{ a.e. in } E \right\} = \mathrm{cap}_p(E, A)|z|^p. \tag{4.3}$$

Convolution-type energies

The following results, contained in [2], extend corresponding results in Sobolev spaces to the case of convolution-type energies.
Let $r, \varepsilon > 0$, and $p > 1$. We set, for every open set $A \subset \mathbb{R}^d$ and $u \in L^p(A;\mathbb{R}^m)$,

$$G_\varepsilon^{r,p}(u,A) = \int_{B_r}\int_{A_\varepsilon(\xi)} \left|D_\varepsilon^\xi u(x)\right|^p \, dx\, d\xi, \tag{4.4}$$

where $D_\varepsilon^\xi u(x)$ and $A_\varepsilon(\xi)$ are defined by (3.4) and (3.5), respectively.
The next proposition rephrases Lemma 4.1 in [2] where the authors show that long-range energy contributions can be controlled by the short-range energy $G_\varepsilon^{r,p}$.

Proposition 4.2. *For every $r > 0$ there exists a positive constant C such that, for any open set $E \subset \Omega$, for every $\xi \in \mathbb{R}^d$ and $u \in L^p(\Omega;\mathbb{R}^m)$, there holds*

$$\int_E \Big|\frac{u(x+\varepsilon\xi)-u(x)}{\varepsilon}\Big|^p dx \leq C(|\xi|^p+1)G_\varepsilon^{r,p}(u, E + B_{\varepsilon(r+|\xi|)}).$$

for any $\varepsilon > 0$ such that

$$\varepsilon\, r < \operatorname{dist}(E + B_{\varepsilon(r+|\xi|)}, \Omega^c). \tag{4.5}$$

Remark 4.3. Note that if $\Omega = \mathbb{R}^d$ then (4.5) is satisfied for any $\varepsilon > 0$ and $\xi \in \mathbb{R}^d$, deriving in particular the following estimate, which will be useful later: for every $r > 0$, there exists $C = C(r)$ such that, for every $\xi \in \mathbb{R}^d$, $\varepsilon > 0$, and $u \in L^p_{loc}(\mathbb{R}^d;\mathbb{R}^m)$, with $u \equiv z$ on $\mathbb{R}^d \setminus B_{R-r\varepsilon}$, $z \in \mathbb{R}^m$, there holds

$$\int_{\mathbb{R}^d} \Big|\frac{u(x+\varepsilon\xi)-u(x)}{\varepsilon}\Big|^p dx \leq C(|\xi|^p+1)G_\varepsilon^{r,p}(u,\mathbb{R}^d). \tag{4.6}$$

As a consequence of Lemma 4.2 and a result concerning extension operators (see [2, Theorem 4.1]), the following estimate is derived.

Corollary 4.4. *[[2, Corollary 4.1]] For any open set $A \in \mathcal{A}^{\mathrm{reg}}(\Omega)$ and $r > 0$ there exist two positive constants $C = C(A)$ and $\varepsilon_0 = \varepsilon_0(r, A)$ such that for every $\xi \in \mathbb{R}^d$ and $u \in L^p(A;\mathbb{R}^m)$ there holds*

$$\int_{A_\varepsilon(\xi)} \left|\frac{u(x+\varepsilon\xi)-u(x)}{\varepsilon}\right|^p dx \leq C(|\xi|^p+1)\big(G_\varepsilon^{r,p}(u,A) + \|u\|^p_{L^p(A;\mathbb{R}^m)}\big),$$

for every $\varepsilon < \varepsilon_0$.

The following theorem states the analogue of the classical Poincaré-Wirtinger inequality for the functionals $G_\varepsilon^{r,p}$.

Theorem 4.5. *[[2, Proposition 4.2]] Let $r > 0$ and let A be a bounded connected open set of $\mathbb{R}^d$ with Lipschitz boundary. Then for every measurable set $E \subset A$ with $|E| > 0$ there exists a positive constant $C = C(A, E)$ such that for any $u \in L^p(A;\mathbb{R}^m)$ and $\varepsilon > 0$*

$$\int_A |u(x) - u_E|^p dx \leq C G_\varepsilon^{r,p}(u, A). \tag{4.7}$$

When we replace E and A with a translation of λE and λA, respectively, being $\lambda > 0$ a scaling factor, we have the following result.

Proposition 4.6. *Let $r > 0$ and let A be a bounded connected open set of $\mathbb{R}^d$ with Lipschitz boundary. Then for every measurable set $E \subset A$ with $|E| > 0$ there exists a positive constant $C = C(A, E)$ such that for every $x_0 \in \mathbb{R}^d$, $u \in L^p(\lambda A + x_0; \mathbb{R}^m)$ and $\varepsilon > 0$*

$$\int_{\lambda A + x_0} |u(x) - u_{\lambda E + x_0}|^p dx \leq C\, \lambda^p\, G^{r,p}_{\varepsilon}(u, \lambda A + x_0). \tag{4.8}$$

Proof. It is not restrictive to assume that $x_0 = 0$. If $u \in L^p(\lambda A; \mathbb{R}^m)$ the function $w(y) = u(\lambda y)$ belongs to $L^p(A; \mathbb{R}^m)$. Writing inequality (4.7) for w with ε replaced by $\frac{\varepsilon}{\lambda}$ we get

$$\int_A |w(y) - w_E|^p dy \leq C(A, E)\, G^{r,p}_{\frac{\varepsilon}{\lambda}}(w, A).$$

On the other hand $w_E = u_{\lambda E}$ and the change of variable $x = \lambda y$ gives the desired result. □

Eventually, the next result accounts for the compactness in the strong L^p-topology of sequences of functions with uniformly bounded energy on a regular bounded set of $\mathbb{R}^d$.

Theorem 4.7. *[[2, Theorem 4.2]] Let A be any bounded open Lipschitz set of $\mathbb{R}^d$ and let $\{u_\varepsilon\}_\varepsilon \subset L^p(A; \mathbb{R}^m)$ be such that for some $r > 0$*

$$\sup_{\varepsilon > 0} \left\{ \|u_\varepsilon\|_{L^p(A;\mathbb{R}^m)} + G^{r,p}_{\varepsilon}(u_\varepsilon, A) \right\} < +\infty.$$

Then, for any $\varepsilon_j \to 0$, $\{u_{\varepsilon_j}\}_j$ is relatively compact in $L^p(A; \mathbb{R}^m)$ and every limit of a converging subsequence lies in $W^{1,p}(A; \mathbb{R}^m)$.

5 Gagliardo-Nirenberg-Sobolev type inequality

In this section we state and prove a crucial result for our analysis which may be of independent interest and resembles the classical Gagliardo-Nirenberg-Sobolev inequality in Sobolev spaces. Its proof follows the lines of the proof of the corresponding result in Sobolev spaces. Such a result will allow us to prove the convergence of the infimum problems defining the approximating capacitary energy densities (see Proposition 6.7).
For $r, \sigma > 0$, set

$$PC_\sigma(\mathbb{R}^d; \mathbb{R}^m) := \{u : \mathbb{R}^d \to \mathbb{R}^m : \ u \text{ is constant on } \sigma k + [0, \sigma)^d \ \forall\, k \in \mathbb{Z}^d\}$$

and, given $p \geq 1$, let $T_\varepsilon : L^p(\mathbb{R}^d; \mathbb{R}^m) \to PC_{\tilde{r}\varepsilon}(\mathbb{R}^d; \mathbb{R}^m)$ be defined by

$$T_\varepsilon u(x) := \frac{1}{(\tilde{r}\varepsilon)^d} \int_{\tilde{r}\varepsilon k + [0, \tilde{r}\varepsilon)^d} u(y)\, dy \quad \text{on } \tilde{r}\varepsilon k + [0, \tilde{r}\varepsilon)^d,\ k \in \mathbb{Z}^d, \tag{5.1}$$

where

$$\tilde{r} := \frac{r}{\sqrt{d+3}}.$$

In the following result we extend the definition of $G^{r,p}_{\varepsilon}((u, \mathbb{R}^d)$ in (4.4) for $p = 1$.

Theorem 5.1 (Gagliardo-Nirenberg-Sobolev type inequality). *Let $p \in [1, d)$. Then there exists a constant $C = C(p, d, r) > 0$ such that for every $u \in L^p(\mathbb{R}^d; \mathbb{R}^m)$*

$$\left(\int_{\mathbb{R}^d} |T_\varepsilon u(x)|^{p^*}\, dx \right)^{\frac{p}{p^*}} \leq C G^{r,p}_{\varepsilon}(u, \mathbb{R}^d), \tag{5.2}$$

where $p^* := \frac{pd}{d-p}$.

Proof. By a density argument and the L^p-continuity of $G_\varepsilon^{r,p}(\cdot,\mathbb{R}^d)$, it is enough to prove the inequality (5.2) for $u \in L^p(\mathbb{R}^d;\mathbb{R}^m)$ with compact support.
Let us first consider the case $p=1$ and fix such a u. We introduce some notation. For $k=(k_1,\dots,k_d)\in\mathbb{Z}^d$, set

$$Q_k^\varepsilon := \tilde{r}\varepsilon k + [0,\tilde{r}\varepsilon)^d$$

and, for every $j=1,\dots,d$, let $\hat{k}_j \in \mathbb{Z}^{d-1}$ defined by $\hat{k}_j=(k_1,\dots,k_{j-1},k_{j+1},\dots,k_d)$. Moreover, with fixed $k\in\mathbb{Z}^d$ and $j=1,\dots,d$, set $\hat{k}_j(h):=(k_1,\dots,k_{j-1},h,k_{j+1},\dots,k_d)$ for every $h\in\mathbb{Z}$, and denote by x_h an independent variable lying in the cube $Q^\varepsilon_{\hat{k}_j(h)}$. In particular, with the notation above, we may write $u(x_{k_1}) = \sum_{h=-\infty}^{k_1}(u(x_h)-u(x_{h-1}))$, being actually the latter a finite sum by the compactness of the support of u. Thus integrating in all the variables x_h with $h\le k_1$, we get

$$\left|\int_{Q_k^\varepsilon} u(x_{k_1})\,dx_{k_1}\right| \le \frac{1}{(\tilde{r}\varepsilon)^d}\sum_{h=-\infty}^{k_1}\int_{Q^\varepsilon_{\hat{k}_1(h)}}\int_{Q^\varepsilon_{\hat{k}_1(h-1)}} |u(x_h)-u(x_{h-1})|\,dx_{h-1}\,dx_h,$$

and, using the definition of T_ε given in (5.1),

$$\varepsilon^{d-1}|T_\varepsilon u(\varepsilon k)| \le C\frac{1}{\varepsilon^d}\sum_{h=-\infty}^{k_1}\int_{Q^\varepsilon_{\hat{k}_1(h)}}\int_{Q^\varepsilon_{\hat{k}_1(h)}}\left|\frac{u(x_h)-u(x_{h-1})}{\varepsilon}\right| dx_{h-1}\,dx_h,$$

where $C=C(r,d)$. For every $j=1,\cdots,d$, we define the stripe $S^\varepsilon_{\hat{k}_j}=\{(x_1,\cdots,x_d)\in\mathbb{R}^d : \tilde{r}_0\varepsilon k_i \le x_i < \tilde{r}_0\varepsilon(k_i+1), \forall i\neq j\}$, thus we have

$$\varepsilon^{d-1}|T_\varepsilon u(\varepsilon k)| \le C\, G_\varepsilon^{r,1}(u,S^\varepsilon_{\hat{k}_1}).$$

Analogously, for $j=2,\cdots,d$,

$$\varepsilon^{d-1}|T_\varepsilon u(\varepsilon k)| \le C\, G_\varepsilon^{r,1}(u,S^\varepsilon_{\hat{k}_j}),$$

which in turn implies

$$\varepsilon^{d(d-1)}|T_\varepsilon u(\varepsilon k)|^d \le C\prod_{j=1}^d G_\varepsilon^{r,1}(u,S^\varepsilon_{\hat{k}_j}),$$

or, equivalently,

$$\varepsilon^d|T_\varepsilon u(\varepsilon k)|^{\frac{d}{d-1}} \le C\prod_{j=1}^d \left(G_\varepsilon^{r,1}(u,S^\varepsilon_{\hat{k}_j})\right)^{\frac{1}{d-1}}.$$

Summing over $k_1\in\mathbb{Z}$ and using Hölder's inequality, we get

$$\begin{aligned}\sum_{k_1\in\mathbb{Z}}\varepsilon^d|T_\varepsilon u(\varepsilon k)|^{\frac{d}{d-1}} &\le C\left(G_\varepsilon^{r,1}(u,S^\varepsilon_{\hat{k}_1})\right)^{\frac{1}{d-1}}\sum_{k_1\in\mathbb{Z}}\prod_{j=2}^d\left(G_\varepsilon^{r,1}(u,S^\varepsilon_{\hat{k}_j})\right)^{\frac{1}{d-1}}\\ &\le C\left(G_\varepsilon^{r,1}(u,S^\varepsilon_{\hat{k}_1})\right)^{\frac{1}{d-1}}\prod_{j=2}^d\left(\sum_{k_1\in\mathbb{Z}}G_\varepsilon^{r,1}(u,S^\varepsilon_{\hat{k}_j})\right)^{\frac{1}{d-1}}.\end{aligned}$$

Next we sum over $k_2 \in \mathbb{Z}$ and we use again Hölder's inequality, obtaining

$$
\begin{aligned}
&\sum_{k_2\in\mathbb{Z}}\sum_{k_1\in\mathbb{Z}} \varepsilon^d |T_\varepsilon u(\varepsilon k)|^{\frac{d}{d-1}} \\
&\le C\left(\sum_{k_1\in\mathbb{Z}} G^{r,1}_\varepsilon(u, S^\varepsilon_{\hat{k}_2})\right)^{\frac{1}{d-1}} \sum_{k_2\in\mathbb{Z}}\left(G^{r,1}_\varepsilon(u, S^\varepsilon_{\hat{k}_1})\prod_{j=3}^d \sum_{k_1\in\mathbb{Z}} G^{r,1}_\varepsilon(u, S^\varepsilon_{\hat{k}_j})\right)^{\frac{1}{d-1}} \\
&\le C\left(\sum_{k_1\in\mathbb{Z}} G^{r,1}_\varepsilon(u, S^\varepsilon_{\hat{k}_2})\right)^{\frac{1}{d-1}}\left(\sum_{k_2\in\mathbb{Z}} G^{r,1}_\varepsilon(u, S^\varepsilon_{\hat{k}_1})\right)^{\frac{1}{d-1}}\prod_{j=3}^d\left(\sum_{k_2\in\mathbb{Z}}\sum_{k_1\in\mathbb{Z}} G^{r,1}_\varepsilon(u, S^\varepsilon_{\hat{k}_j})\right)^{\frac{1}{d-1}}.
\end{aligned}
$$

We iterate the procedure, finding out

$$
\sum_{i=1}^d\sum_{k_i\in\mathbb{Z}} \varepsilon^d |T_\varepsilon u(\varepsilon k)|^{\frac{d}{d-1}} \le C\prod_{j=1}^d\left(\sum_{\substack{i=1\\ i\neq j}}^d\sum_{k_i\in\mathbb{Z}} G^{r,1}_\varepsilon(u, S^\varepsilon_{\hat{k}_j})\right)^{\frac{1}{d-1}} \le C\left(G^{r,1}_\varepsilon(u,\mathbb{R}^d)\right)^{\frac{d}{d-1}}.
$$

On the other hand, we have

$$
\int_{\mathbb{R}^d} |T_\varepsilon u(y)|^{1^*}\,dy = \sum_{k\in\mathbb{Z}^d}\int_{Q^\varepsilon_k} |T_\varepsilon u(y)|^{1^*}\,dy = \sum_{i=1}^d\sum_{k_i\in\mathbb{Z}} (\tilde{r}\varepsilon)^d |T_\varepsilon u(\varepsilon k)|^{\frac{d}{d-1}};
$$

therefore

$$
\left(\int_{\mathbb{R}^d} |T_\varepsilon u(y)|^{1^*}\,dy\right)^{\frac{1}{1^*}} \le C\, G^{r,1}_\varepsilon(u,\mathbb{R}^d). \tag{5.3}
$$

Thus the theorem is proven for $p = 1$.

If $1 < p < d$, given $u \in L^p((\mathbb{R}^d;\mathbb{R}^m)$ with compact support, we use (5.3) with $|T_\varepsilon u|^\gamma$ in place of u, for some $\gamma > 1$ to be chosen later. We obtain

$$
\begin{aligned}
\left(\int_{\mathbb{R}^d} |T_\varepsilon u(y)|^{\gamma 1^*}\,dy\right)^{\frac{1}{1^*}} &\le C\int_{B_r}\int_{\mathbb{R}^d}\left|\frac{|T_\varepsilon u(x+\varepsilon\xi)|^\gamma - |T_\varepsilon u(x)|^\gamma}{\varepsilon}\right| dx\,d\xi \\
&\le C\int_{B_r}\int_{\mathbb{R}^d}(|T_\varepsilon u(x+\varepsilon\xi)|^{\gamma-1} + |T_\varepsilon u(x)|^{\gamma-1})\left|\frac{T_\varepsilon u(x+\varepsilon\xi) - T_\varepsilon u(x)}{\varepsilon}\right| dx\,d\xi,
\end{aligned}
$$

where now C depends on γ, d, r. By using Hölder's inequality with p and $p' = \frac{p}{p-1}$, we also get

$$
\begin{aligned}
\left(\int_{\mathbb{R}^d} |T_\varepsilon u(y)|^{\gamma 1^*}\,dy\right)^{\frac{1}{1^*}} &\le C\left(\int_{B_r}\int_{\mathbb{R}^d}(|T_\varepsilon u(x+\varepsilon\xi)|^{(\gamma-1)p'} + |T_\varepsilon u(x)|^{(\gamma-1)p'})\,dx\,d\xi\right)^{\frac{1}{p'}} \\
&\quad\times G^{r,p}_\varepsilon(T_\varepsilon u,\mathbb{R}^d)^{\frac{1}{p}} \le C\left(\int_{\mathbb{R}^d} |T_\varepsilon u(x)|^{(\gamma-1)p'}dx\right)^{\frac{1}{p'}} G^{r,p}_\varepsilon(T_\varepsilon u,\mathbb{R}^d)^{\frac{1}{p}},
\end{aligned}
$$

possibly for a different constant $C = C(\gamma, p, d, r)$. Choose γ so that $\gamma 1^* = \gamma\frac{d}{d-1} = (\gamma-1)p'$, and accordingly $\gamma = \frac{d-1}{d}p^*$. Thus

$$
\left(\int_{\mathbb{R}^d} |T_\varepsilon u(y)|^{p^*}\,dy\right)^{\frac{p}{p^*}} \le C\, G^{r,p}_\varepsilon(T_\varepsilon u,\mathbb{R}^d).
$$

We now observe that

$$G_\varepsilon^{r,p}(T_\varepsilon u,\mathbb{R}^d) \le C\int_{B_r}\int_{\mathbb{R}^d}\left|\frac{T_\varepsilon u(x)-u(x)}{\varepsilon}\right|^p dx\,d\xi + C\int_{B_r}\int_{\mathbb{R}^d}\left|\frac{T_\varepsilon u(x+\varepsilon\xi)-u(x+\varepsilon\xi)}{\varepsilon}\right|^p dx\,d\xi$$
$$+\, C\int_{B_r}\int_{\mathbb{R}^d}\left|\frac{u(x+\varepsilon\xi)-u(x)}{\varepsilon}\right|^p dx\,d\xi,$$

the last term being, up to a multiplicative constant, the functional $G_\varepsilon^{r,p}(u,\mathbb{R}^d)$. The first and the second term on the right hand side may be estimated as follows

$$\begin{aligned}\int_{B_r}\int_{\mathbb{R}^d}\left|\frac{T_\varepsilon u(x)-u(x)}{\varepsilon}\right|^p dx\,d\xi &= |B_r|\int_{\mathbb{R}^d}\left|\frac{T_\varepsilon u(x)-u(x)}{\varepsilon}\right|^p dx\\ &= |B_r|\sum_{k\in\mathbb{Z}^d}\int_{Q_k^\varepsilon}\left|\fint_{Q_k^\varepsilon}\frac{u(y)-u(x)}{\varepsilon}\,dy\right|^p dx\\ &\le |B_r|\sum_{k\in\mathbb{Z}^d}\int_{Q_k^\varepsilon}\fint_{Q_k^\varepsilon}\left|\frac{u(y)-u(x)}{\varepsilon}\right|^p dy\,dx\\ &= C\sum_{k\in\mathbb{Z}^d}\frac{1}{\varepsilon^d}\int_{Q_k^\varepsilon}\int_{Q_k^\varepsilon}\left|\frac{u(y)-u(x)}{\varepsilon}\right|^p dy\,dx \le C\,G_\varepsilon^{r,p}(u,\mathbb{R}^d),\end{aligned}$$

where we have used Jensen's inequality. This concludes the proof. □

By Theorem 5.1 and Theorem 4.7, we deduce the following compactness result.

Corollary 5.2. *Let $p\in(1,d)$ and let $\{u_\varepsilon\}_\varepsilon\subset L^p(\mathbb{R}^d;\mathbb{R}^m)$ be such that for some $r>0$*

(i) $\sup_{\varepsilon>0}\|u_\varepsilon\|_{L^p(K;\mathbb{R}^m)}<+\infty$ *for every compact set $K\subset\mathbb{R}^d$;*

(ii) $\sup_{\varepsilon>0} G_\varepsilon^{r,p}(u_\varepsilon,\mathbb{R}^d)<+\infty$.

Then, for any $\varepsilon_j\to 0$, $\{u_{\varepsilon_j}\}_j$ is relatively compact in $L^p_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^m)$ and every limit of a converging subsequence lies in $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^m)\cap L^{p^*}(\mathbb{R}^d;\mathbb{R}^m)$.

Proof. Theorem 4.7 yields that for any bounded open Lipschitz set $A\subset\mathbb{R}^d$ $\{u_{\varepsilon_j}\}_j$ is relatively compact in $L^p(A;\mathbb{R}^m)$ and any limit point lies in $W^{1,p}(A;\mathbb{R}^m)$. By a standard diagonalization argument, $\{u_j\}_j$ is also relatively compact in $L^p_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^m)$ and any limit point lies in $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^m)$. Let us consider a subsequence (not relabelled) u_{ε_j} and $u\in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^m)$ such that $u_j\to u$ strongly in $L^p_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^m)$ and pointwise in $\mathbb{R}^d$. Then it can be proved that also $T_{\varepsilon_j}u_{\varepsilon_j}\to u$ strongly in $L^p_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^m)$ and pointwise in $\mathbb{R}^d$ (see also [4, Lemma 2.11]). Thus, it is enough to apply Fatou's Lemma in (5.2) and use hypothesis (ii) to deduce that $u\in L^{p^*}(\mathbb{R}^d;\mathbb{R}^m)$. □

6 Supporting results

In this section we present some key results of technical flavor that we are later going to use for the proof of Theorem 3.5 and Theorem 3.6.

6.1 A joining lemma

Here we state and prove the analog of Lemma 3.1 in [5] for our non local functionals and we follow the lines of its proof. It allows to restrict our attention in our Γ-convergence analysis to sequences of converging functions that are constant on suitable annuli surrounding the perforations.

Lemma 6.1. *Let $\delta_j \to 0$ as $j \to +\infty$. Let $T \geq r_0$ be fixed and let $0 < \varepsilon_j < \rho_j < \frac{\delta_j}{2}$ with $\varepsilon_j = o(\rho_j)$ as $j \to +\infty$. Let u_j converge to u in $L^p(\Omega, \mathbb{R}^m)$ with $\sup_j \mathcal{F}^T_{\varepsilon_j}(u_j) < +\infty$. Set*

$$Z_j(\Omega) := \{i \in \mathbb{Z}^d : \operatorname{dist}(i\delta_j, \mathbb{R}^d \setminus \Omega) > \delta_j\}$$

and, for $h \in \mathbb{N}$ and $i \in Z_j(\Omega)$, set

$$A^{i,h}_j := \{x \in \Omega : 2^{-h-1}\rho_j < |x - i\delta_j| < 2^{-h}\rho_j\}, \tag{6.1}$$

$$u^{i,h}_j := \fint_{A^{i,h}_j} u_j \qquad \rho_{j,h} := \frac{3}{4}2^{-h}\rho_j. \tag{6.2}$$

Then, given $N \in \mathbb{N}$, for every $i \in Z_j(\Omega)$ there exists $k_i \in \{0, \cdots, N-1\}$ and a sequence w_j still converging to u in $L^p(\Omega, \mathbb{R}^m)$, such that for j sufficiently large

$$w_j = u_j \text{ on } \Omega \setminus \cup_{i \in Z_j(\Omega)} A^{i,k_i}_j, \tag{6.3}$$

$$w_j = u^{i,k_i}_j \text{ on } \partial B_{\rho_{j,k_i}}(i\delta_j) + B_{T\varepsilon_j}, \tag{6.4}$$

$$\left|\mathcal{F}^T_{\varepsilon_j}(w_j) - \mathcal{F}^T_{\varepsilon_j}(u_j)\right| \leq \frac{C}{N}. \tag{6.5}$$

Proof. Notice that, by (**H**) and (G0), $\sup_j G^{r_0,p}_{\varepsilon_j}(u_j, \Omega) < +\infty$. Let $\varphi = \varphi^{i,h}_j \in C^\infty_c(A^{i,h}_j)$ be such that $\varphi = 1$ on $\partial B_{\rho_{j,k_i}}(i\delta_j) + B_{T\varepsilon_j}$ and $|D\varphi| \leq \frac{C}{\rho_{j,h}}$ and define the function $w^{i,h}_j = \varphi u^{i,h}_j + (1-\varphi)u_j$. Adding and subtracting the quantity $\varphi(x + \varepsilon_j\xi)u_j(x)$ in the argument of f, and using (G1) and the convexity of the power function, we have

$$\begin{aligned}
&\mathcal{F}^T_{\varepsilon_j}(w^{i,h}_j, A^{i,h}_j) = \\
&= \int_{B_T}\int_{(A^{i,h}_j)_{\varepsilon_j}(\xi)} f\left(\xi, \frac{(u^{i,h}_j\varphi + (1-\varphi)u_j)(x+\varepsilon_j\xi) - (u^{i,h}_j\varphi + (1-\varphi)u_j)(x)}{\varepsilon_j}\right) dx\, d\xi \\
&\leq C\int_{B_T} M(\xi)\int_{(A^{i,h}_j)_{\varepsilon_j}(\xi)} \left|\frac{u_j(x+\varepsilon_j\xi) - u_j(x)}{\varepsilon_j}\right|^p + \left|u^{i,h}_j - u_j(x)\right|^p \left|\frac{\varphi(x+\varepsilon_j\xi) - \varphi(x)}{\varepsilon_j}\right|^p dx\, d\xi.
\end{aligned}$$

Recalling that $|D\varphi| \leq \frac{C}{\rho_{j,h}}$ and using the Poincaré-Wirtinger inequality in Proposition 4.6 with $x_0 = i\delta_j$ and $\lambda = \frac{4}{3}\rho_{j,h}$, we have

$$\begin{aligned}
&\int_{B_T} M(\xi)\int_{(A^{i,h}_j)_{\varepsilon_j}(\xi)} \left|u^{i,h}_j - u_j(x)\right|^p \left|\frac{\varphi(x+\varepsilon_j\xi) - \varphi(x)}{\varepsilon_j}\right|^p dx\, d\xi \\
&\leq \frac{C}{\rho^p_{j,h}}\int_{B_T} M(\xi)|\xi|^p\, d\xi \int_{A^{i,h}_j} \left|u^{i,h}_j - u_j(x)\right|^p dx \leq C\, G^{r_0}_{\varepsilon_j}(u_j, A^{i,h}_j).
\end{aligned}$$

On the other hand, taking into account that $\varepsilon_j = o(\rho_j)$ we deduce that, for j sufficiently large, it holds

$$A^{i,h}_j + B_{\varepsilon_j(r_0+T)} \subset \bigcup_{\ell = h-1, h, h+1} A^{i,\ell}_j =: \tilde{A}^{i,h}_j, \quad h = 0, \cdots, N-1.$$

In addition, for j sufficiently large, thanks to the fact that $\mathrm{dist}(\tilde{A}_j^{i,h}, \Omega \setminus B_{\rho_j}(i\delta_j)) \sim \rho_j$, we also have

$$\varepsilon_j < r_0^{-1}\mathrm{dist}(\tilde{A}_j^{i,h}, \Omega \setminus B_{\rho_j}(i\delta_j)).$$

Therefore, using Proposition 4.2, we get

$$\begin{aligned}\int_{B_T} M(\xi) \int_{(A_j^{i,h})_{\varepsilon_j}(\xi)} & \left|\frac{u_j(x+\varepsilon_j\xi) - u_j(x)}{\varepsilon_j}\right|^p dx\, d\xi \\ & \leq C \int_{B_T} M(\xi)(|\xi|^p + 1)\, d\xi G_{\varepsilon_j}^{r_0,p}(u_j, \tilde{A}_j^{i,h}) \\ & \leq C\, G_{\varepsilon_j}^{r_0,p}(u_j, \tilde{A}_j^{i,h}).\end{aligned}$$

Hence, the previous estimates yield that

$$\mathcal{F}_{\varepsilon_j}^T(w_j^{i,h}, A_j^{i,h}) \leq C\, G_{\varepsilon_j}^{r_0,p}(u_j, \tilde{A}_j^{i,h}). \tag{6.6}$$

Since the sets $\tilde{A}_j^{i,h}$ overlap at most 3 times, with fixed $N \in \mathbb{N}$, we sum over $h = 0, \cdots, N-1$ and get

$$\sum_{h=0}^{N-1} G_{\varepsilon_j}^{r_0,p}(u_j, \tilde{A}_j^{i,h}) \leq 3\, G_{\varepsilon_j}^{r_0,p}(u_j, B_{\rho_j}(i\delta_j)).$$

Hence there exists $k_i \in \{0, \cdots, N-1\}$ such that

$$G_{\varepsilon_j}^{r_0,p}(u_j, \tilde{A}_j^{i,k_i}) \leq \frac{3}{N} G_{\varepsilon_j}^{r_0,p}(u_j, B_{\rho_j}(i\delta_j)),$$

which in turn yields

$$\mathcal{F}_{\varepsilon_j}^T(w_j^{i,k_i}, A_j^{i,k_i}) \leq \frac{C}{N} G_{\varepsilon_j}^{r_0,p}(u_j, B_{\rho_j}(i\delta_j)).$$

Noticing that estimate (6.6) holds even if we replace $w_j^{i,h}$ with u_j, we get

$$\begin{aligned}\left|\mathcal{F}_{\varepsilon_j}^T(u_j, A_j^{i,k_i}) - \mathcal{F}_{\varepsilon_j}^T(w_j^{i,k_i}, A_j^{i,k_i})\right| & \leq \mathcal{F}_{\varepsilon_j}^T(u_j, A_j^{i,k_i}) + \mathcal{F}_{\varepsilon_j}^T(w_j^{i,k_i}, A_j^{i,k_i}) \\ & \leq \frac{C}{N} G_{\varepsilon_j}^{r_0,p}(u_j, B_{\rho_j}(i\delta_j)).\end{aligned}$$

Then (6.3), (6.4)(6.5) are satisfied by w_j defined as

$$w_j(x) := \begin{cases} u_j(x) & \text{if } x \in \Omega \setminus \cup_{i \in Z_j(\Omega)} A_j^{i,k_i}, \\ \\ \varphi_j^{i,k_i}(x) u_j^{i,k_i}(x) + (1 - \varphi_j^{i,k_i}(x)) u_j(x) & \text{if } x \in A_j^{i,k_i},\ i \in Z_j(\Omega). \end{cases}$$

We finally prove the convergence of w_j to u in $L^p(\Omega; \mathbb{R}^m)$. We have

$$\begin{aligned}\int_\Omega |w_j - u_j|^p\, dx = & \sum_{i \in Z_j(\Omega)} \int_{A_j^{i,k_i}} \left|\varphi_j^{i,k_i} u_j^{i,k_i} + (1 - \varphi_j^{i,k_i}) u_j - u_j\right|^p dx \\ & \leq \sum_{i \in Z_j(\Omega)} \int_{A_j^{i,k_i}} |u_j^{i,k_i} - u_j|^p\, dx \leq C \sum_{i \in Z_j(\Omega)} (\rho_j^{k_i})^p G_{\varepsilon_j}^{r_0,p}(u_j, A_j^{i,k_i}) \\ & \leq C\, \rho_j^p \sum_{i \in Z_j(\Omega)} G_{\varepsilon_j}^{r_0,p}(u_j, A_j^{i,k_i}) \leq C\, \rho_j^p,\end{aligned}$$

where we used again Proposition 4.6 in the second line. Hence, passing to the limit as j tends to $+\infty$ we get the desired convergence.

□

6.2 Truncation Lemma

By the composition with a suitable lipschitz function, the following technical lemma allows us to replace a given sequence u_j with equibounded energies and L^p-norms, by a new sequence uniformly bounded in L^∞ and with a small gap in energy. The proof is strongly inspired by Lemma 3.5 in [10].

Lemma 6.2. *Let $\{u_j\}$ with $\sup_j(\mathcal{F}_{\varepsilon_j}(u_j) + \|u_j\|_{L^p(\Omega;\mathbb{R}^m)}) < +\infty$. Then for every $\eta > 0$ and $M > 1$ there exist $R_M > M > 0$ and a sequence of Lipschitz functions $\Phi_{j,M} : \mathbb{R}^m \to \mathbb{R}^m$ with $Lip(\Phi_{j,M}) = 1$, $\Phi_{j,M}(z) = z$ if $|z| < M$ and $\Phi_{j,M}(z) = 0$ if $|z| > R_M$, such that it holds*

$$\mathcal{F}_{\varepsilon_j}(\Phi_{j,M}(u_j)) \leq \mathcal{F}_{\varepsilon_j}(u_j) + \eta$$

for every $j \in \mathbb{N}$ such that $\varepsilon_j < \varepsilon_0$, with ε_0 depending on Ω. Moreover we can extract a subsequence (j_k) such that $\Phi_{j_k,M} =: \Phi_M$ do not depend on $k \in \mathbb{N}$.

Proof. Note that, by assumption (G0), $G^{r_0,p}_{\varepsilon_j}(u_j, \Omega)$ is uniformly bounded. Set

$$C_1 := \sup_j (G^{r_0,p}_{\varepsilon_j}(u_j, \Omega) + \|u_j\|_{L^p(\Omega;\mathbb{R}^m)}), \tag{6.7}$$

$$C_2 = 6\, C(\Omega, r_0) \int_{\mathbb{R}^d} M(\xi)(|\xi|^p + 1)\, d\xi, \tag{6.8}$$

where $C(\Omega, r_0)$ is the constant obtained by Corollary 4.4 applied with $A = \Omega$ and $r = r_0$.

Let $\eta > 0$ and $M > 0$ be fixed. Note that, once the statement is proved for a given positive constant M, then it holds true also for any $M' < M$, hence up to replace M with a bigger value it is not restrictive to assume that M is an integer and satisfies

$$M > \lfloor \frac{2\, C_1 C_2}{\eta} \rfloor + 2. \tag{6.9}$$

For $h = 1, \dots, M$ let $\Phi^h_M : \mathbb{R}^m \to \mathbb{R}^m$ be a Lipschitz function such that

$$\Phi^h_M(z) = \begin{cases} z & \text{if } |z| \leq M^h \\ 0 & \text{if } |z| > M^{h+1}. \end{cases}$$

and Φ^h_M connects linearly in the radial directions the values on the boundary of the annulus $\{z \in \mathbb{R}^m : M^h < |z| < M^{h+1}\}$. A quick computation shows that for any $h = 1, \dots, M$ $Lip(\Phi^h_M) \leq \frac{1}{M-1} < 1$ on the annulus, thus $Lip(\Phi^h_M) = 1$. Let $w^h_j = \Phi^h_M(u_j)$ and estimate $\mathcal{F}_{\varepsilon_j}(w^h_j)$ from above. Since $f(\xi, 0) = 0$ $\forall \xi \in \mathbb{R}^d$, we have that

$$\mathcal{F}_{\varepsilon_j}(w^h_j) = \int_{\mathbb{R}^d} \int_{\{x \in \Omega_{\varepsilon_j}(\xi) : |u_j(x)| \wedge |u_j(x+\varepsilon_j\xi)| \leq M^{h+1}\}} f\left(\xi, \frac{\Phi^h_M(u_j(x+\varepsilon_j\xi)) - \Phi^h_M(u_j(x))}{\varepsilon_j}\right) dx\, d\xi.$$

Now, for $\xi \in \mathbb{R}^d$, we distinguish in $\Omega_{\varepsilon_j}(\xi)$ the points where $|u_j(x)| \leq |u_j(x + \varepsilon_j\xi)|$ from those where $|u_j(x)| > |u_j(x + \varepsilon_j\xi)|$ and we perform a similar analysis in both the two sets.

To this end let us introduce the notation

$$\Omega^+_{\varepsilon_j}(\xi) = \{x \in \Omega_{\varepsilon_j}(\xi) : |u_j(x)| \leq |u_j(x + \varepsilon_j\xi)|\},$$

$$\Omega^-_{\varepsilon_j}(\xi) = \Omega_{\varepsilon_j}(\xi) \setminus \Omega^+_{\varepsilon_j}(\xi) = \{x \in \Omega_{\varepsilon_j}(\xi) : |u_j(x)| > |u_j(x + \varepsilon_j\xi)|\}.$$

The sets $\Omega^+_{\varepsilon_j}(\xi)\cap\{|u_j(x)|\wedge|u_j(x+\varepsilon_j\xi)|\leq M^{h+1}\}$ and $\Omega^-_{\varepsilon_j}(\xi)\cap\{|u_j(x)|\wedge|u_j(x+\varepsilon_j\xi)|\leq M^{h+1}\}$ can be in turn decomposed, respectively, as the union of the disjoint sets

$$\begin{aligned}
S^+_{1,h,j}(\xi) &= \{x\in\Omega^+_{\varepsilon_j}(\xi) : |u_j(x+\varepsilon_j\xi)| < M^h\}\\
S^+_{2,h,j}(\xi) &= \{x\in\Omega^+_{\varepsilon_j}(\xi) : |u_j(x)| < M^h, |u_j(x+\varepsilon_j\xi)|\geq M^{h+1}\}\\
S^+_{3,h,j}(\xi) &= \{x\in\Omega^+_{\varepsilon_j}(\xi) : |u_j(x)| < M^h \leq |u_j(x+\varepsilon_j\xi)| < M^{h+1}\}\\
S^+_{4,h,j}(\xi) &= \{x\in\Omega^+_{\varepsilon_j}(\xi) : M^h\leq |u_j(x)| \leq |u_j(x+\varepsilon_j\xi)| \leq M^{h+1}\}\\
S^+_{5,h,j}(\xi) &= \{x\in\Omega^+_{\varepsilon_j}(\xi) : M^h\leq |u_j(x)| < M^{h+1}\leq |u_j(x+\varepsilon_j\xi)|\}
\end{aligned}$$

and

$$\begin{aligned}
S^-_{1,h,j}(\xi) &= \{x\in\Omega^-_{\varepsilon_j}(\xi) : |u_j(x)| < M^h\}\\
S^-_{2,h,j}(\xi) &= \{x\in\Omega^-_{\varepsilon_j}(\xi) : |u_j(x+\varepsilon_j\xi)| < M^h, |u_j(x)|\geq M^{h+1}\}\\
S^-_{3,h,j}(\xi) &= \{x\in\Omega^-_{\varepsilon_j}(\xi) : |u_j(x+\varepsilon_j\xi)| < M^h \leq |u_j(x)| < M^{h+1}\}\\
S^-_{4,h,j}(\xi) &= \{x\in\Omega^-_{\varepsilon_j}(\xi) : M^h\leq |u_j(x+\varepsilon_j\xi)| \leq |u_j(x)| \leq M^{h+1}\}\\
S^-_{5,h,j}(\xi) &= \{x\in\Omega^-_{\varepsilon_j}(\xi) : M^h\leq |u_j(x+\varepsilon_j\xi)| < M^{h+1}\leq |u_j(x)|\}.
\end{aligned}$$

Hence, using the growth assumption on f and the Lipschitz continuity of $\Phi_{h,M}$, we have

$$\begin{aligned}
\mathcal{F}_{\varepsilon_j}(w^h_j) \leq{}& \int_{\mathbb{R}^d}\int_{S^{\pm}_{1,h,j}(\xi)} f\left(\xi,\frac{u_j(x+\varepsilon_j\xi)-u_j(x)}{\varepsilon_j}\right)dx\,d\xi + \int_{\mathbb{R}^d} M(\xi)\int_{S^+_{2,h,j}(\xi)}\left|\frac{u_j(x)}{\varepsilon_j}\right|^p dx\,d\xi\\
&+\int_{\mathbb{R}^d} M(\xi)\int_{S^-_{2,h,j}(\xi)}\left|\frac{u_j(x+\varepsilon_j\xi)}{\varepsilon_j}\right|^p dx\,d\xi + \sum_{i=3}^5\int_{\mathbb{R}^d} M(\xi)\int_{S^{\pm}_{i,h,j}(\xi)}\left|\frac{u_j(x+\varepsilon_j\xi)-u_j(x)}{\varepsilon_j}\right|^p dx\,d\xi,
\end{aligned}$$

where for the sake of notation we have set $S^{\pm}_{i,h,j}(\xi) = S^+_{i,h,j}(\xi)\cup S^-_{i,h,j}(\xi)$.

Let us now sum over $h = 1,\cdots,M$ and get

$$\begin{aligned}
\frac{1}{M}\sum_{h=1}^M \mathcal{F}_{\varepsilon_j}(w^h_j) \leq{}& \mathcal{F}_{\varepsilon_j}(u_j) + \frac{6}{M}\int_{\mathbb{R}^d} M(\xi)\int_{\Omega_{\varepsilon_j}(\xi)}\left|\frac{u_j(x+\varepsilon_j\xi)-u_j(x)}{\varepsilon_j}\right|^p dx\,d\xi\\
&+\frac{1}{M}\sum_{h=1}^M\int_{\mathbb{R}^d} M(\xi)\Big(\int_{S^+_{2,h,j}(\xi)}\left|\frac{u_j(x)}{\varepsilon_j}\right|^p dx + \int_{S^-_{2,h,j}(\xi)}\left|\frac{u_j(x+\varepsilon_j\xi)}{\varepsilon_j}\right|^p dx\Big)\,d\xi,
\end{aligned} \tag{6.10}$$

since the families $\{S^+_{i,h,j}(\xi)\}_{h\in\mathbb{N}}$ and $\{S^-_{i,h,j}(\xi)\}_{h\in\mathbb{N}}$, with $i = 3,4,5$, consist of pairwise disjoint sets. Using Corollary 4.4, (6.7) and (6.8), the second term in the right handside of (6.10) can be estimated from above by

$$\begin{aligned}
&\frac{6}{M}\int_{\mathbb{R}^d} M(\xi)\int_{\Omega_{\varepsilon_j}(\xi)}\left|\frac{u_j(x+\varepsilon_j\xi)-u_j(x)}{\varepsilon_j}\right|^p dx\,d\xi\\
&\qquad\leq \frac{6}{M}C(\Omega,r_0)\int_{\mathbb{R}^d} M(\xi)(|\xi|^p+1)\,d\xi\,\left(G^{r_0,p}_{\varepsilon_j}(u_j,\Omega) + \|u_j\|^p_{L^p(\Omega,\mathbb{R}^m)}\right)\\
&\qquad\leq \frac{1}{M}C_2\,\left(G^{r_0,p}_{\varepsilon_j}(u_j,\Omega) + \|u_j\|^p_{L^p(\Omega,\mathbb{R}^m)}\right) \leq \frac{1}{M}C_1C_2,
\end{aligned} \tag{6.11}$$

if $\varepsilon_j < \varepsilon_0$, with ε_0 as in Corollary 4.4. Since by (6.9) we have that $\dfrac{C_1\,C_2}{M} < \dfrac{\eta}{2}$, we get for $\varepsilon_j \in (0,\varepsilon_0)$

$$\frac{6}{M}\int_{\mathbb{R}^d} M(\xi)\int_{\Omega_{\varepsilon_j}(\xi)}\left|\frac{u_j(x+\varepsilon_j\xi)-u_j(x)}{\varepsilon_j}\right|^p dx\,d\xi < \frac{\eta}{2}. \tag{6.12}$$

We are left with the estimate of the last term in the right handside of (6.10). Arguing as in (6.11) and taking into account (6.8), we deduce that for $\varepsilon_j \in (0, \varepsilon_0)$

$$
\begin{aligned}
C_1 C_2 &\geq \int_{\mathbb{R}^d} M(\xi) \int_{S^{\pm}_{2,h,j}(\xi)} \left| \frac{u_j(x+\varepsilon_j\xi) - u_j(x)}{\varepsilon_j} \right|^p dx\, d\xi \\
&\geq \int_{\mathbb{R}^d} M(\xi) \int_{S^{\pm}_{2,h,j}(\xi)} \left| \left| \frac{u_j(x+\varepsilon_j\xi)}{\varepsilon_j} \right| - \left| \frac{u_j(x)}{\varepsilon_j} \right| \right|^p dx\, d\xi \\
&\geq \int_{\mathbb{R}^d} M(\xi) \int_{S^{\pm}_{2,h,j}(\xi)} \left(\frac{M^{h+1} - M^h}{\varepsilon_j} \right)^p dx\, d\xi = \frac{(M^{h+1} - M^h)^p}{\varepsilon_j^p} \int_{\mathbb{R}^d} M(\xi) |S^{\pm}_{2,h,j}(\xi)|\, d\xi,
\end{aligned}
$$

which in turn yields

$$
\int_{\mathbb{R}^d} M(\xi) |S^{\pm}_{2,h,j}(\xi)|\, d\xi \leq \frac{C_1 C_2 \varepsilon_j^p}{(M^{h+1} - M^h)^p}.
$$

Exploiting this last estimate and the very definition of $S^{+}_{2,h,j}(\xi), S^{-}_{2,h,j}(\xi)$, we also get for $\varepsilon_j \in (0, \varepsilon_0)$

$$
\begin{aligned}
&\int_{\mathbb{R}^d} M(\xi) \left(\int_{S^{+}_{2,h,j}(\xi)} \left| \frac{u_j(x)}{\varepsilon_j} \right|^p dx + \int_{S^{-}_{2,h,j}(\xi)} \left| \frac{u_j(x+\varepsilon_j\xi)}{\varepsilon_j} \right|^p dx \right) d\xi \\
&\leq \frac{M^{hp}}{\varepsilon_j^p} \int_{\mathbb{R}^d} M(\xi) |S^{\pm}_{2,h,j}(\xi)|\, d\xi \leq C_1\, C_2 \frac{M^{hp}}{(M^{h+1} - M^h)^p} = \frac{C_1\, C_2}{(M-1)^p}.
\end{aligned}
\tag{6.13}
$$

Since, by (6.9), $M \geq 2$, we have

$$
\frac{C_1\, C_2}{(M-1)^p} \leq \frac{C_1\, C_2}{(M-1)} \leq \frac{C_1\, C_2}{\lfloor \frac{2\, C_1 C_2}{\eta} \rfloor + 1} < \frac{\eta}{2}. \tag{6.14}
$$

Hence, by (6.10), (6.12), (6.13) and (6.14), we eventually deduce that for every $j \in \mathbb{N}$ such that $\varepsilon_j < \varepsilon_0$ there exists $h(j) \in \{1, \cdots, M\}$ satisfying

$$
\mathcal{F}_{\varepsilon_j}(w_j^{h(j)}) \leq \frac{1}{M} \sum_{h=1}^{M} \mathcal{F}_{\varepsilon_j}(w_j^h) \leq \mathcal{F}_{\varepsilon_j}(u_j) + \eta.
$$

We then define $\Phi_{j,M} = \Phi_M^{h(j)}$. Up to selecting a subsequence, we may also assume that $h(j)$ is a constant value in $\{1, \cdots, M\}$. □

6.3 Approximating capacitary energy densities

In this subsection we introduce and investigate the main properties of suitable energy densities defined through minimum problems of capacitary type involving the approximating energies $\mathcal{F}^T_\varepsilon$.

For any $\varepsilon > 0$, $T > r_0$, $R \geq 2 + T\varepsilon$, and $z \in \mathbb{R}^m$, set

$$
\varphi_{\varepsilon,T,R}(z) := \inf\{\mathcal{F}^T_\varepsilon(v, B_R) :\ v \in L^p_{\varepsilon,T,z}(B_R; \mathbb{R}^m)\}, \tag{6.15}
$$

where $\mathcal{F}^T_\varepsilon$ is defined in (3.9) and

$$
L^p_{\varepsilon,T,z}(B_R; \mathbb{R}^m) := \{v \in L^p(B_R; \mathbb{R}^m) :\ v \equiv 0 \text{ in } B_1,\ v \equiv z \text{ on } \partial^{\varepsilon T} B_R\}, \tag{6.16}
$$

with the notation $\partial^{\varepsilon T} A := \partial A + B_{\varepsilon T}$, for any $A \subset \mathbb{R}^d$.

We identify any $v \in L^p_{\varepsilon,T,z}(B_R;\mathbb{R}^m)$ with its extension in $L^p_{\rm loc}(\mathbb{R}^d;\mathbb{R}^m)$ such that $v \equiv z$ in $\mathbb{R}^d \setminus B_R$. Hence the function $\varphi_{\varepsilon,T,R}(z)$ can be also rewritten as

$$\varphi_{\varepsilon,T,R}(z) = \inf\{\mathcal{F}^T_\varepsilon(v,\mathbb{R}^d) : v - z \in L^p(\mathbb{R}^d;\mathbb{R}^m), v \equiv 0 \text{ in } B_1, v \equiv z \text{ in } \mathbb{R}^d \setminus B_{R-\varepsilon T}\}. \quad (6.17)$$

Note that the request $R \geq 2 + T\varepsilon$ is not restrictive, as we are interested in letting $R \to +\infty$; this assumption will be useful in Proposition 6.4.

Remark 6.3. Note that, if $f(\xi,\cdot)$ is convex for every $\xi \in \mathbb{R}^d$, the infimum defining $\varphi_{\varepsilon,T,R}(z)$ in (6.15) is actually a minimum. Indeed, by the convexity of $f(\xi,\cdot)$ also $\mathcal{F}^T_\varepsilon(v,B_R)$ is convex. Hence, taking into account Proposition 4.5 with $E = B_1$, $A = B_R$, $\mathcal{F}^T_\varepsilon(v,B_R)$ is lower semicontinuous and coercive with respect to the weak topology in $L^p(B_R;\mathbb{R}^m)$. As the constraints $v \equiv 0$ in B_1, $v \equiv z$ on $\partial^{\varepsilon T} B_R$ are convex and closed by the strong convergence in $L^p(B_R;\mathbb{R}^m)$ the existence of minimizers follows by the standard methods.

The properties of the densities $\varphi_{\varepsilon,T,R}$ we are going to state and prove will be instrumental in Subsection 6.4 in studying the pointwise and locally uniform limit of $\varphi_{\varepsilon,T,R}(\cdot)$ when the parameters R, T go to $+\infty$, and ε either goes to 0 or remains fixed equal to α. These results will allow us to estimate the energetic contribution near the perforations leading to the appearance of the density functions φ and $\varphi_{NL,\alpha}$ defined by (3.14) and (3.17), respectively.

The first result establishes growth conditions of order p of $\varphi_{\varepsilon,T,R}$.

Proposition 6.4. *Let f satisfy assumptions* (**H**) *and* (**G**)*, and let $T > r_0, \varepsilon_0 > 0$, and $R > 1$ be fixed such that $R - \varepsilon_0 T \geq 2$. Then, for every $0 < \varepsilon \leq \varepsilon_0$ there exists $c_1, c_2 > 0$ such that*

$$c_1|z|^p \leq \varphi_{\varepsilon,T,R}(z) \ \forall z \in \mathbb{R}^m \quad (6.18)$$

$$\varphi_{\varepsilon,T,R}(z) \leq c_2|z|^p \ \forall z \in \mathbb{R}^m. \quad (6.19)$$

In particular, the constant c_1 depends on $p, d, \lambda_0, r_0, \varepsilon_0$ and the constant c_2 on p, d, r_0.

Proof. We first prove (6.18). The proof relies on a suitable lower bound of $\mathcal{F}_\varepsilon(v,B_R)$ with discrete energies. In order to avoid too many technicalities, we restrict the proof to the case $d = 2$; the argument can be generalised to any dimension (see e.g. the proof of Theorem 2.6 in [26]). Let us introduce some notation. Given $\xi \in \mathbb{R}^2 \setminus \{0\}$, let $\mathcal{L}_\xi$ be the lattice in $\mathbb{R}^2$ defined by

$$\mathcal{L}_\xi = \mathbb{Z}\xi \oplus \mathbb{Z}\xi^\perp,$$

where $\xi^\perp := (-\xi_2, \xi_1)$.

Let $v \in L^p_{\varepsilon,T,z}(B_R;\mathbb{R}^m)$ and let $0 < \bar r < r_0$. By (**H**) and (G0), we get

$$\begin{aligned}\mathcal{F}^T_\varepsilon(v,B_R) &\geq \lambda_0\, G^{\bar r,p}_\varepsilon(v,B_R) = \lambda_0\, G^{\bar r,p}_\varepsilon(v,\mathbb{R}^2)\\ &= \frac{\lambda_0}{2}\int_{B_{\bar r}}\int_{\mathbb{R}^2}\sum_{\xi'\in\{\xi,\xi^\perp\}}\left|\frac{v(x+\varepsilon\xi')-v(x)}{\varepsilon}\right|^p dx\,d\xi\\ &= \frac{\lambda_0}{2}\int_{B_{\bar r}}\sum_{k\in\mathcal{L}_\xi}\int_{\varepsilon(k+Q^\xi)}\sum_{\xi'\in\{\xi,\xi^\perp\}}\left|\frac{v(x+\varepsilon\xi')-v(x)}{\varepsilon}\right|^p dx\,d\xi\end{aligned} \quad (6.20)$$

where

$$Q^\xi := [0,1)\xi \oplus [0,1)\xi^\perp.$$

Let $T^\xi_\varepsilon v$ the function which is constant on each square $\varepsilon(k+Q^\xi)$, $k \in \mathcal{L}_\xi$, and is defined by

$$T^\xi_\varepsilon v(x) = \frac{1}{(\varepsilon|\xi|)^2} \int_{\varepsilon(k+Q^\xi)} v(y)\,dy, \quad x \in \varepsilon(k+Q^\xi),\ k \in \mathcal{L}_\xi.$$

Then, by (6.20) and Jensen's inequality, we get

$$\begin{aligned} \mathcal{F}^T_\varepsilon(v, B_R) &\geq \frac{\lambda_0}{2} \int_{B_{\bar r}} \sum_{k\in\mathcal{L}_\xi} (\varepsilon|\xi|)^2 \sum_{\xi' \in \{\xi,\xi^\perp\}} \left| \frac{T^\xi_\varepsilon v(\varepsilon(k+\xi')) - T^\xi_\varepsilon v(\varepsilon k)}{\varepsilon} \right|^p d\xi \\ &= \frac{\lambda_0}{2} \int_{B_{\bar r}} |\xi|^p \sum_{k\in\mathcal{L}_\xi} (\varepsilon|\xi|)^2 \sum_{\xi' \in \{\xi,\xi^\perp\}} \left| \frac{T^\xi_\varepsilon v(\varepsilon(k+\xi')) - T^\xi_\varepsilon v(\varepsilon k)}{\varepsilon|\xi|} \right|^p d\xi. \end{aligned} \tag{6.21}$$

Let $\mathcal{T}^\pm_\xi$ be the triangles defined by

$$\mathcal{T}^\pm_\xi := \{x \in Q^\xi :\ \pm\langle x, \xi\rangle \leq \pm\langle x, \xi^\perp\rangle\ \},$$

and let w^ξ_ε the piecewise affine function obtained by linearly interpolating the values $\{T^\xi_\varepsilon v(\varepsilon k)\}_{k\in\mathcal{L}_\xi}$ on the triangles $\varepsilon(k+\mathcal{T}^\pm_\xi)$, $k \in \mathcal{L}_\xi$. Note that, for $\varepsilon \leq \varepsilon_0$, if $\bar r \leq \frac{1}{2\varepsilon_0}$ and $\xi \in B_{\bar r}$, then $w^\xi_\varepsilon \equiv 0$ on $B_{\frac12}$ and $w^\xi_\varepsilon \equiv z$ on $\mathbb{R}^2 \setminus B_{\frac32 R}$. Moreover, taking into account (4.3) and (4.1), we easily infer that

$$\begin{aligned} \sum_{k\in\mathcal{L}_\xi} (\varepsilon|\xi|)^2 \sum_{\xi' \in \{\xi,\xi^\perp\}} \left| \frac{T^\xi_\varepsilon v(\varepsilon(k+\xi')) - T^\xi_\varepsilon v(\varepsilon k)}{\varepsilon|\xi|} \right|^p &\geq C \int_{\mathbb{R}^2} |\nabla w^\xi_\varepsilon|^p\,dx \\ &\geq C\,\mathrm{cap}_p(B_{\frac12}, B_{\frac32 R})|z|^p \geq C\,\mathrm{cap}_p(B_{\frac12})|z|^p. \end{aligned} \tag{6.22}$$

In conclusion, selecting $\bar r := \max\{r_0, \frac{1}{2\varepsilon_0}\}$, we get

$$\mathcal{F}^T_\varepsilon(v, B_R) \geq \frac{\lambda_0}{2} \left(\int_{B_{\bar r}} |\xi|^p\,d\xi \right) C\,\mathrm{cap}_p(B_{\frac12})|z|^p = c_1|z|^p.$$

We now prove (6.19). To this aim, note that, by using a Fubini argument, one can easily shows that there exists $C = C(r_0)$ such that, for $\varepsilon \leq \varepsilon_0$, for any u such that $u - z \in C^1_c(B_{R-\varepsilon T}; \mathbb{R}^m)$, then

$$G^{r_0,p}_\varepsilon(u, \mathbb{R}^d) = G^{r_0,p}_\varepsilon(u, B_{R-\varepsilon T}) \leq C \int_{B_{R-\varepsilon T}} |\nabla u|^p\,dx.$$

By (**H**), (G1), (4.6), and the density of functions compactly supported in the capacitary problem, we then get

$$\begin{aligned} &\varphi_{\varepsilon,T,R}(z) \leq \mathcal{F}^T_\varepsilon(u, B_R) \leq C \left(\int_{\mathbb{R}^d} M(\xi)(|\xi|^p + 1)\,d\xi \right) G^{r_0,p}_\varepsilon(u, \mathbb{R}^d) \\ &\leq C \left(\int_{\mathbb{R}^d} M(\xi)(|\xi|^p + 1)\,d\xi \right) \inf\left\{ \int_{B_{R-\varepsilon T}} |\nabla u|^p\,dx :\ u \equiv 0 \text{ in } B_1,\ u - z \in C^1_c(B_{R-\varepsilon T}; \mathbb{R}^m) \right\} \\ &= C \left(\int_{\mathbb{R}^d} M(\xi)(|\xi|^p + 1)\,d\xi \right) \mathrm{cap}_p(B_1, B_{R-\varepsilon T})|z|^p \\ &\leq C \left(\int_{\mathbb{R}^d} M(\xi)(|\xi|^p + 1)\,d\xi \right) \mathrm{cap}_p(B_1, B_2)|z|^p = c_2|z|^p. \end{aligned}$$

□

In the next proposition we show that the functions $\varphi_{\varepsilon,T,R}$ are uniformly Lipschitz continuous on compact sets.

Proposition 6.5. *Let f satisfy assumptions* (**H**), (**G**) *and* (**L**). *Then there exist a constant $C>0$ independent of ε, T and R such that for every $z, w \in \mathbb{R}^m$ we have*

$$|\varphi_{\varepsilon,T,R}(w)-\varphi_{\varepsilon,T,R}(z)| \le C(|z|^{p-1}+|w|^{p-1}|)|w-z|. \tag{6.23}$$

Proof. Let us prove (6.23) for fixed z and w. Since the inequality is trivially true when $z=0$ or $w=0$, we may suppose both not null and consider the map $\phi:\mathbb{R}^m\to\mathbb{R}^m$ defined by

$$\phi(\zeta)=\frac{|w|}{|z|}\mathcal{R}^w_z(\zeta),$$

where $\mathcal{R}^w_z$ is a rotation that maps z into $\frac{|z|}{|w|}w$. Note that $\phi(0)=0$, $\phi(z)=w$ and

$$\|\nabla\phi\|_\infty \le C\frac{|w|}{|z|}, \quad \|\nabla\phi - I\|_\infty \le C\frac{|w-z|}{|z|}. \tag{6.24}$$

For $\eta>0$, let $v_z\in L^p_{\varepsilon,T,z}(B_R;\mathbb{R}^m)$ be such that $\mathcal{F}^T_\varepsilon(v_z,B_R)\le\varphi_{\varepsilon,T,R}(z)+\eta$ and set

$$v_w := \phi\circ v_z.$$

Note that $v_w\in L^p_{\varepsilon,T,w}(B_R;\mathbb{R}^m)$, hence

$$\varphi_{\varepsilon,T,R}(w)\le\mathcal{F}^T_\varepsilon(v_w,B_R)\le\varphi_{\varepsilon,T,R}(z)+\mathcal{F}^T_\varepsilon(v_w,B_R)-\mathcal{F}^T_\varepsilon(v_z,B_R)+\eta. \tag{6.25}$$

By hypothesis (**L**) and (6.24), we infer that for every $\xi\in\mathbb{R}^d$

$$\begin{aligned}|f(\xi,D^\xi_\varepsilon v_w(x))-f(\xi,D^\xi_\varepsilon v_z(x))| &\le CM(\xi)(|D^\xi_\varepsilon v_z(x)|^{p-1}+|D^\xi_\varepsilon v_w(x)|^{p-1})|D^\xi_\varepsilon v_z(x)-D^\xi_\varepsilon v_w(x)|\\ &\le CM(\xi)(|D^\xi_\varepsilon v_z(x)|^{p-1}+\|\nabla\phi\|^{p-1}_\infty|D^\xi_\varepsilon v_z(x)|^{p-1})\|\nabla\phi-I\|_\infty|D^\xi_\varepsilon v_z(x)|\\ &\le CM(\xi)\frac{|z|^{p-1}+|w|^{p-1}}{|z|^p}|w-z||D^\xi_\varepsilon v_z(x)|^p.\end{aligned}$$

Thus, by (6.25), we get

$$\varphi_{\varepsilon,T,R}(w)\le\varphi_{\varepsilon,T,R}(z)+C\frac{|z|^{p-1}+|w|^{p-1}}{|z|^p}|w-z|\int_{\mathbb{R}^d}M(\xi)\int_{(B_R)_\varepsilon(\xi)}|D^\xi_\varepsilon v_z(x)|^p\,dx\,d\xi+\eta. \tag{6.26}$$

By (**H**), (G0) and (6.19), we get

$$G^{ro,p}_\varepsilon(v_z,B_R)\le C\,\mathcal{F}^T_\varepsilon(v_z,B_R)\le C\,(\varphi_{\varepsilon,T,R}(z)+\eta)\le C(|z|^p+\eta). \tag{6.27}$$

Since, by (4.6), we have that for any $\xi\in\mathbb{R}^d$

$$\int_{(B_R)_\varepsilon(\xi)}|D^\xi_\varepsilon v_z(x)|^p\,dx\le C(|\xi|^p+1)G^{ro,p}_\varepsilon(v_z,B_R),$$

inequality (6.26) and (G1) yields that

$$\varphi_{\varepsilon,T,R}(w)\le\varphi_{\varepsilon,T,R}(z)+C(|z|^{p-1}+|w|^{p-1})|w-z|\frac{|z|^p+\eta}{|z|^p}+\eta.$$

Taking the limit as η tends to 0, and then reversing the role of z and w, (6.23) easily follows. $\square$

We conclude this subsection with a technical result, yielding that in the minimum problems defining $\varphi_{\varepsilon,T,R}$ we may reduce to admissible functions uniformly bounded in L^∞. The strategy of the proof is analogous to that of Lemma 6.2, hence we highlight only the main differences.

Proposition 6.6. *Let $T > r_0$, $\alpha > 0$, $R \geq 2 + T\alpha$ and $\bar{C}, M_0 > 0$ be fixed. Then for every $\eta > 0$ there exists $M > M_0$ such that for every $z \in B_{M_0}$, given $v \in L^p_{\alpha,T,z}(B_R;\mathbb{R}^m)$ such that*

$$\mathcal{F}^T_\alpha(v, B_R) \leq \bar{C}|z|^p,$$

then there exists $v_M \in L^p_{\alpha,T,z}(B_R;\mathbb{R}^m)$, with $\|v_M\|_{L^\infty(B_R;\mathbb{R}^m)} \leq M$, such that

$$\mathcal{F}^T_\alpha(v_M, B_R) \leq \mathcal{F}^T_\alpha(v, B_R) + \eta.$$

Proof. Given $z \in B_{M_0}$ and $R > 0$, by (**H**) and (G0) we have that

$$G^{r_0,p}_\alpha(v, B_R) \leq C|z|^p,$$

where the constant C depends only on $\bar{C}, r_0, \lambda_0$. So now it suffices to retrace the steps of the proof of Lemma 6.2, replacing the constants in (6.7) and (6.8) with

$$C_1 := \sup_R G^{r_0,p}_\alpha(v, B_R) < +\infty,$$

and

$$C_2 = 6C(r_0) \int_{\mathbb{R}^d} M(\xi)(|\xi|^p + 1)\, d\xi,$$

respectively, where $C(r_0)$ is the constant obtained in Remark 4.3, and using Remark 4.3 instead of Corollary 4.4. The function v_M is obtained through $\Phi_M(v)$ with a suitable choice of M. □

6.4 Asymptotics of the approximating capacitary energy density

We now show that, if $R_\varepsilon \to +\infty$ as $\varepsilon \to 0$ the functions $\varphi_{\varepsilon,T,R_\varepsilon}(z)$ approximate the energy density $\varphi^T(z)$ defined in (3.19). A crucial role in the proof is played by Corollary 5.2.

Proposition 6.7. *Let φ^T and $\varphi_{\varepsilon,T,R}$ be defined by* (3.19) *and* (6.15)*, respectively. Then, if $R_\varepsilon \to +\infty$ as $\varepsilon \to 0$, it holds*

$$\lim_{\varepsilon\to 0} \varphi_{\varepsilon,T,R_\varepsilon}(z) = \varphi^T(z) \tag{6.28}$$

uniformly on compact sets.

Proof. By Proposition 6.5 it suffices to prove that (6.28) holds pointwise. We will show that it is a consequence of Theorem 3.2, and Corollary 5.2. With fixed $z \in \mathbb{R}^m$, let $v_\varepsilon \in L^p_{\varepsilon,T,z}(B_{R_\varepsilon};\mathbb{R}^m)$ be such that

$$\varphi_{\varepsilon,T,R_\varepsilon}(z) = \mathcal{F}^T_\varepsilon(v_\varepsilon, B_{R_\varepsilon}) + o(\varepsilon),$$

and let $u_\varepsilon \in L^p(\mathbb{R}^d;\mathbb{R}^m)$ equal to $v_\varepsilon - z$ on B_{R_ε} and $u_\varepsilon \equiv 0$ on $\mathbb{R}^d \setminus B_{R_\varepsilon}$. By (**H**), (G0) and (6.19), it holds

$$\sup_\varepsilon G^{r_0,p}_\varepsilon(u_\varepsilon, \mathbb{R}^d) < +\infty. \tag{6.29}$$

By Theorem 4.5 applied with A any bounded open Lipschitz set in $\mathbb{R}^d$ and $E = B_1$, we get

$$\sup_\varepsilon \|u_\varepsilon\|_{L^p(A;\mathbb{R}^m)} < +\infty.$$

Thus, using Corollary 5.2 we get that, up to a subsequence (not relabelled), u_ε converge in $L^p_{\rm loc}(\mathbb{R}^d;\mathbb{R}^m)$ to a function $u\in W^{1,p}_{\rm loc}(\mathbb{R}^d;\mathbb{R}^m)\cap L^{p^*}(\mathbb{R}^d;\mathbb{R}^m)$ such that $u=-z$ on B_1. Moreover, with fixed $R>0$, by Theorem 3.2, we deduce that

$$\liminf_{\varepsilon\to 0}\varphi_{\varepsilon,T,R_\varepsilon}(z)=\liminf_{\varepsilon\to 0}\mathcal{F}^T_\varepsilon(v_\varepsilon,B_{R_\varepsilon})\geq\liminf_{\varepsilon\to 0}\mathcal{F}^T_\varepsilon(v_\varepsilon,B_R)\geq\int_{B_R}f^T_{hom}(\nabla u)\,dx.$$

Letting $R\to+\infty$ we obtain

$$\liminf_{\varepsilon\to 0}\varphi_{\varepsilon,T,R_\varepsilon}(z)\geq\int_{\mathbb{R}^d}f^T_{hom}(\nabla u)\,dx\geq\varphi^T(z).$$

We now claim that

$$\varphi^T(z)=\inf\Big\{\int_{\mathbb{R}^d}f^T_{hom}(\nabla u)\,dx : u\equiv -z \text{ in } B_1,\ u\in W^{1,p}(\mathbb{R}^d;\mathbb{R}^m)\text{ compactly supported}\Big\}.$$

To this aim, let us us fix a cut-off function ζ between B_1 and B_2 and $u\in L^{p*}(\mathbb{R}^d;\mathbb{R}^m)\cap W^{1,p}_{\rm loc}(\mathbb{R}^d;\mathbb{R}^m)$, $u\equiv -z$ in B_1, with $\int_{\mathbb{R}^d}f^T_{hom}(\nabla u)\,dx<+\infty$. Note that, taking into account Remark 3.3, $\nabla u\in L^p(\mathbb{R}^d;\mathbb{R}^{d\times m})$. We now set, for any $n\in\mathbb{N}$, $u_n(x)=\zeta(x/n)u(x)$. An easy computation shows that $u_n\in W^{1,p}(\mathbb{R}^d;\mathbb{R}^m)$, u_n is compactly supported and $u_n\equiv -z$ in B_1. Moreover it holds that $\nabla u_n\to\nabla u$ strongly in $L^p(\mathbb{R}^d;\mathbb{R}^{d\times m})$. Indeed, by Hölder inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^d}|\nabla u_n-\nabla u|^p\,dx&\leq C\int_{\mathbb{R}^d\setminus B_n}|\nabla u|^p\,dx+\frac{C}{n^p}\int_{B_{2n}\setminus B_n}|u|^p\,dx\\&\leq C\int_{\mathbb{R}^d\setminus B_n}|\nabla u|^p\,dx+\frac{C}{n^p}|B_{2n}\setminus B_n|^{1-\frac{p}{p^*}}\Big(\int_{B_{2n}\setminus B_n}|u|^{p^*}\,dx\Big)^{\frac{p}{p^*}}\\&\leq C\int_{\mathbb{R}^d\setminus B_n}|\nabla u|^p\,dx+C\Big(\int_{B_{2n}\setminus B_n}|u|^{p^*}\,dx\Big)^{\frac{p}{p^*}},\end{aligned}$$

and the last two terms tend to 0 as $n\to+\infty$. The claim follows by using the dominated convergence theorem together with Remark 3.3.

Thus, taking the claim into account and using a density argument, given $\eta>0$, we may assume that there exists $u\in C^\infty_c(\mathbb{R}^d;\mathbb{R}^m)$ such that $u\equiv -z$ on B_1 and

$$\int_{\mathbb{R}^d}f^T_{hom}(\nabla u)\,dx\leq\varphi^T(z)+\eta.$$

Let $\bar R>0$ such that $\operatorname{supp}u\subseteq B_{\bar R}$. Then, by [2, Proposition 5.3] applied with $A=B_{\bar R}\setminus B_1$, there exists a family of functions $u_\varepsilon\in L^p(B_{\bar R};\mathbb{R}^m)$, with $u_\varepsilon\equiv -z$ on B_1 and $u_\varepsilon\equiv 0$ on $\partial^{\varepsilon T}B_{\bar R}$ such that

$$\lim_{\varepsilon\to 0}\mathcal{F}^T_\varepsilon(u_\varepsilon,\mathbb{R}^d)=\int_{\mathbb{R}^d}f^T_{hom}(\nabla u)\,dx.$$

Hence, set $v_\varepsilon:=u_\varepsilon+z$, we get that $v_\varepsilon\in L^p_{\varepsilon,T,z}(B_{R_\varepsilon};\mathbb{R}^m)$ for ε small enough, thus

$$\limsup_{\varepsilon\to 0}\varphi_{\varepsilon,T,R_\varepsilon}(z)\leq\limsup_{\varepsilon\to 0}\mathcal{F}^T_\varepsilon(v_\varepsilon,B_{R_\varepsilon})\leq\int_{\mathbb{R}^d}f^T_{hom}(\nabla u)\,dx\leq\varphi^T(z)+\eta$$

and the thesis follows by the arbitrariness of $\eta>0$. □

Note that for any $\alpha > 0$, $T > r_0$ and $z \in \mathbb{R}^m$, the function $R \in [2 + \alpha T, +\infty) \mapsto \varphi_{\alpha,T,R}(z)$ is decreasing, as it is easily seen by (6.17). Hence, for $z \in \mathbb{R}^m$, it is well defined $\lim_{R\to+\infty} \varphi_{\alpha,T,R}(z)$ and this convergence is also locally uniform by Proposition 6.5. One can easily shows that

$$\varphi^T_{NL,\alpha}(z) = \lim_{R\to+\infty} \varphi_{\alpha,T,R}(z),$$

where $\varphi^T_{NL,\alpha}$ is defined by (3.20).

The properties of the densities $\varphi_{\varepsilon,T,R}$ obtained so far allow to prove the following result about the L^1-convergence of suitable Riemann sums to the capacitary densities φ^T and $\varphi^T_{NL,\alpha}$.

Proposition 6.8. *Let $\varepsilon_j \to 0$ and $R_j \to +\infty$ as $j \to +\infty$ and let (u_j) be a bounded sequence in $L^\infty(\Omega;\mathbb{R}^m)$ such that $\sup_j \mathcal{F}^T_{\varepsilon_j}(u_j) < +\infty$ and $u_j \to u$ in $L^p(\Omega;\mathbb{R}^m)$, for some $u \in W^{1,p}(\Omega;\mathbb{R}^m)$. Let A^{i,k_i}_j and u^{i,k_i}_j, $i \in Z_j(\Omega)$, be as in (6.1) and (6.2), respectively, with $\rho_j = O(\delta_j)$ and $h = k_i$, for an arbitrary choice of k_i.*

(i) Let $\Psi^T_j : \Omega \to \mathbb{R}$ be defined by

$$\Psi^T_j(x) = \sum_{i\in Z_j(\Omega)} \varphi_{\varepsilon_j,T,R_j}(u^{i,k_i}_j)\chi_{Q_{\delta_{\varepsilon_j}}(i)}(x),$$

where

$$Q_{\delta_{\varepsilon_j}}(i) := \delta_{\varepsilon_j} i + \delta_{\varepsilon_j} Q_1.$$

Then $\Psi^T_j \to \varphi^T(u)$ in $L^1(\Omega)$.

(ii) Let $\Psi^T_{j,\alpha} : \Omega \to \mathbb{R}$ be defined by

$$\Psi^T_{j,\alpha}(x) = \sum_{i\in Z_j(\Omega)} \varphi_{\alpha,T,R_j}(u^{i,k_i}_j)\chi_{Q_{\delta_{\varepsilon_j}}(i)}(x),$$

Then $\Psi^T_{j,\alpha} \to \varphi^T_{NL,\alpha}(u)$ in $L^1(\Omega)$.

Proof. We prove (i), the proof of (ii) being analogous. We have the following estimate

$$\begin{aligned}\int_\Omega |\Psi^T_j(x) - \varphi^T(u(x))|\,dx \le& \sum_{i\in Z_j(\Omega)} \int_{Q_{\delta_{\varepsilon_j}}(i)} \left|\varphi_{\varepsilon_j,T,R_j}(u^{i,k_i}_j) - \varphi_{\varepsilon_j,T,R_j}(u_j(x))\right|\,dx\\ &+ \sum_{i\in Z_j(\Omega)} \int_{Q_{\delta_{\varepsilon_j}}(i)} \left|\varphi^T(u(x)) - \varphi_{\varepsilon_j,T,R_j}(u_j(x))\right|\,dx\\ &+ \int_{\Omega\setminus\bigcup_{i\in Z_j(\Omega)} Q_{\delta_{\varepsilon_j}}(i)} |\varphi^T(u(x))|\,dx =: I^1_j + I^2_j + I^3_j.\end{aligned}$$

By Proposition 6.7, we easily deduce that $I^2_j \to 0$. Since $|\Omega \setminus \bigcup_{i\in Z_j(\Omega)} Q_{\delta_{\varepsilon_j}}(i)| \to 0$, we also infer that $I^3_j \to 0$. Finally, by Proposition 6.5, we may estimate I^1_j as follows

$$I^1_j \le C \sum_{i\in Z_j(\Omega)} \int_{Q_{\delta_{\varepsilon_j}}(i)} \left|u^{i,k_i}_j - u_j(x)\right|\,dx.$$

By Hölder's inequality and Proposition 4.6, we have

$$\int_{Q_{\delta_{\varepsilon_j}(i)}} \left| u_j^{i,k_i} - u_j(x) \right| dx \le \delta_{\varepsilon_j}^{\frac{d(p-1)}{p}} \left(\int_{Q_{\delta_{\varepsilon_j}(i)}} \left| u_j^{i,k_i} - u_j(x) \right|^p dx \right)^{\frac{1}{p}}$$
$$\le C \delta_{\varepsilon_j}^{\frac{d(p-1)}{p}} \delta_{\varepsilon_j} \left(G_{\varepsilon_j}^{r_0,p}(u_j, Q_{\delta_{\varepsilon_j}(i)}) \right)^{\frac{1}{p}},$$

Hence

$$I_j^1 \le C \delta_{\varepsilon_j} \left(G_{\varepsilon_j}^{r_0,p}(u_j, \Omega) \right)^{\frac{1}{p}} \le C \delta_{\varepsilon_j} \left(\mathcal{F}_{\varepsilon_j}^T(u_j) \right)^{\frac{1}{p}} \to 0.$$

□

We conclude this subsection showing the convergence of $\varphi_{NL,\alpha}^T$ to $\varphi_{NL,\alpha}$ as $T \to +\infty$.

Proposition 6.9. *For any $z \in \mathbb{R}^m$, it holds*

$$\lim_{T \to +\infty} \varphi_{NL,\alpha}^T(z) = \sup_{T > r_0} \varphi_{NL,\alpha}^T(z) = \varphi_{NL,\alpha}(z),$$

where $\varphi_{NL,\alpha}(z)$ is defined in (3.17).

Proof. We first observe that the function $\varphi_{NL,\alpha}^T$ is increasing in T, hence it is well defined the limit

$$\lim_{T \to +\infty} \varphi_{NL,\alpha}^T(z) = \sup_{T > r_0} \varphi_{NL,\alpha}^T(z),$$

for any $z \in \mathbb{R}^m$. Since $\mathcal{F}_\alpha^T(v, \mathbb{R}^d) \le \mathcal{F}_\alpha(v, \mathbb{R}^d)$ for every $T > 0$,

$$\sup_{T > r_0} \varphi_{NL,\alpha}^T(z) \le \varphi_{NL,\alpha}(z).$$

Let v be an admissible function in the minimum problem defining $\varphi_{NL,\alpha}(z)$. In particular, $v - z$ satisfies (4.6), that is

$$\int_{\mathbb{R}^d} \left| \frac{v(x + \varepsilon\xi) - v(x)}{\varepsilon} \right|^p dx \le C(|\xi|^p + 1) G_\varepsilon^{r_0,p}(v, \mathbb{R}^d),$$

where the constant C depends on r_0. We now multiply each side of the previous inequality by the growth function $M(\xi)$, we apply Remark 3.1, integrate on $\mathbb{R}^d \setminus B_T$, and finally obtain

$$\int_{\{|\xi| \ge T\}} \int_{\mathbb{R}^d} f\left(\xi, \frac{v(x + \varepsilon\xi) - v(x)}{\varepsilon}\right) dx\, d\xi \le C \int_{\{|\xi| \ge T\}} M(\xi)(|\xi|^p + 1) d\xi\, G_\varepsilon^{r_0,p}(v, \mathbb{R}^d).$$

Thanks to (G1), this gives

$$\mathcal{F}_\alpha(v, \mathbb{R}^d) \le \mathcal{F}_\alpha^T(v, \mathbb{R}^d) + o(T)\, G_\varepsilon^{r_0,p}(v, \mathbb{R}^d).$$

We now choose a function v_T such that $v_T \equiv 0$ in B_1, $v_T - z \in L^p(\mathbb{R}^d; \mathbb{R}^m)$, $v_T - z$ is compactly supported, and

$$\mathcal{F}_\alpha^T(v_T, \mathbb{R}^d) \le \varphi_{NL,\alpha}^T(z) + o(T).$$

Thus, in particular

$$\varphi_{NL,\alpha}(z) \le \mathcal{F}_\alpha(v_T, \mathbb{R}^d) \le \mathcal{F}_\alpha^T(v_T, \mathbb{R}^d) + o(T)\, G_\varepsilon^{r_0,p}(v_T, \mathbb{R}^d)$$
$$\le \varphi_{NL,\alpha}^T(z) + o(T) + o(T)\, G_\varepsilon^{r_0,p}(v_T, \mathbb{R}^d). \tag{6.30}$$

By finally using that

$$G_\varepsilon^{r_0,p}(v_T, \mathbb{R}^d) \le C\, \mathcal{F}_\alpha^T(v_T, \mathbb{R}^d) \le C\, \varphi_{NL,\alpha}^T(z) + o(T) \le C(|z|^p + 1),$$

the desired conclusion follows letting T tend to $+\infty$ in (6.30). □

7 Proof of Theorem 3.5 and Theorem 3.6

We will prove the two statements simultaneously, distinguishing the two regimes provided by assumption (3.13) and (3.15), respectively, only when necessary.
Dividing the proof into three steps, we first show that it suffices to prove the theorems for the truncated functionals $F^T_{\varepsilon,\delta_\varepsilon}$, and then we deal separately with the Γ-$\liminf$ and the Γ-$\limsup$ inequalities.

Step 1. It is not restrictive to prove both theorems under the additional assumption that there exists $T > 0$ such that $f(\xi, z) = 0$ if $|\xi| > T$.
Indeed, under the hypotheses of Theorem 3.5, assume that for every $T > 0$

$$\Gamma(L^p) - \lim_{\varepsilon\to 0} F^T_{\varepsilon,\delta_\varepsilon}(u) = \begin{cases} \int_\Omega f^T_{hom}(\nabla u)\,dx + \int_\Omega \varphi^T(u)\,dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^m), \\ +\infty & \text{otherwise,} \end{cases} \tag{7.1}$$

where $F^T_{\varepsilon,\delta}$ is defined by (3.6) and f^T_{hom} and φ^T are defined by (3.18) and (3.19), respectively. By [2, Lemma 5.1],

$$\Gamma(L^p) - \lim_{\varepsilon\to 0} F_{\varepsilon,\delta_\varepsilon}(u) = \lim_{T\to+\infty} \Gamma(L^p) - \lim_{\varepsilon\to 0} F^T_{\varepsilon,\delta_\varepsilon}(u),$$

hence, by Monotone Convergence Theorem, the statement follows once we prove that for every $S \in \mathbb{R}^{d\times m}$ and $z \in \mathbb{R}^m$

$$\lim_{T\to+\infty} f^T_{hom}(S) = f_{hom}(S), \quad \lim_{T\to+\infty} \varphi^T(z) = \varphi(z).$$

The equalities above are, in turn, again a straightforward consequence of Monotone Convergence Theorem. We may argue analogously in the setting of Theorem 3.6, taking into account Proposition 6.9.

Step 2. With fixed $T > r_0$, we now prove the validity of the Γ-$\liminf$ inequality for $F^T_{\varepsilon,\delta_\varepsilon}$ for both theorems.
Given $\varepsilon_j \to 0^+$ as $j \to +\infty$, let $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ and let $u_j \to u$ in $L^p(\Omega;\mathbb{R}^m)$ be such that $\sup_j F^T_{\varepsilon_j,\delta_{\varepsilon_j}}(u_j) < +\infty$. Up to passing to a subsequence (not relabelled), given $\eta > 0$ and $M > 0$, we may apply Lemma 6.2 and find $R_M > M$ and a Lipschitz function $\Phi_M : \mathbb{R}^m \to \mathbb{R}^m$, with $Lip(\Phi_M) = 1$, $\Phi_M(z) = z$ if $|z| < M$ and $\Phi_M(z) = 0$ if $|z| > R_M$ such that

$$F^T_{\varepsilon_j,\delta_{\varepsilon_j}}(u_j) > F^T_{\varepsilon_j,\delta_{\varepsilon_j}}(\Phi_M(u_j)) - \eta. \tag{7.2}$$

Notice that $\phi_M(u_j) \to \Phi_M(u)$ in $L^p(\Omega;\mathbb{R}^m)$. Given $N \in \mathbb{N}$, let $\{w^M_j\}_j$ the sequence constructed in Lemma 6.1 applied with $\rho_j = \delta_{\varepsilon_j}/4$ and $\{\Phi_M(u_j)\}_j$ in place of $\{u_j\}_j$. Set

$$E_j = \bigcup_{i\in Z_j(\Omega)} B_{\rho_{j,k_i}}(\delta_{\varepsilon_j} i),$$

where ρ_{j,k_i} is defined in Lemma 6.1, and define

$$v^M_j(x) := \begin{cases} w^M_j(x) & \text{if } x \in \Omega \setminus E_j \\ (\Phi_M(u_j))^{i,k_i} & \text{if } x \in B_{\rho_{j,k_i}}(\delta_{\varepsilon_j} i). \end{cases}$$

Notice that, by (**H**), (G0) and Lemma 6.1, $\sup_j G^{r_0,p}_{\varepsilon_j}(v^M_j,\Omega) < +\infty$, Hence, by Theorem 4.7, $\{v^M_j\}_j$ is relatively compact in $L^p(\Omega;\mathbb{R}^m)$. Arguing as in [6], we now show that $v^M_j \to \Phi_M(u)$ in $L^p(\Omega;\mathbb{R}^m)$. Specifically, for $h \in \{0,\dots,N-1\}$ set

$$r_h := \frac{3}{4}2^{-h-2}, \quad \chi^h_j(x) := \chi^h\left(\frac{x}{\delta_{\varepsilon_j}}\right),$$

where χ^h coincides with $\chi_{Q_1\setminus B_{r_h}}$ on Q_1 and is extended Q_1-periodically in $\mathbb{R}^d$,

$$Z^h_j := \{i \in Z_j(\Omega) : k_i = h\}, \quad D^h_j := \bigcup_{i\in Z^h_j} \delta_{\varepsilon_j}(i+Q_1), \quad \psi^h_j(x) := \chi_{D^h_j}(x).$$

Recall that

$$\chi^h_j \overset{*}{\rightharpoonup} m_h := |Q_1 \setminus B_{r_h}| > 0 \quad \text{weakly* in } L^\infty(\mathbb{R}^d)$$

and note that,

$$\sum_{h=0}^{N-1} \psi^h_j \to 1 \text{ strongly in } L^1(\Omega).$$

Moreover, since $\chi^h_j \geq \chi^0_j$ for every $h \in \{0,\dots,N-1\}$, we have that

$$\chi_{\Omega\setminus E_j} = \sum_{h=0}^{N-1} \psi^h_j\chi^h_j \geq \chi^0_j \sum_{h=0}^{N-1} \psi^h_j \overset{*}{\rightharpoonup} m_0 > 0. \tag{7.3}$$

Let us consider a subsequence (not relabelled) such that $\chi_{\Omega\setminus E_j} \overset{*}{\rightharpoonup} g$ in $L^\infty(\Omega)$ and $v^M_j \to v$ strongly in $L^p(\Omega;\mathbb{R}^m)$. Hence

$$\chi_{\Omega\setminus E_j} v^M_j \rightharpoonup g\,v, \quad \chi_{\Omega\setminus E_j} w^M_j \rightharpoonup g\,\Phi_M(u) \quad \text{weakly in } L^p(\Omega;\mathbb{R}^m).$$

Taking into account that $\chi_{\Omega\setminus E_j} v^M_j = \chi_{\Omega\setminus E_j} w^M_j$, we conclude that $v = \Phi_M(u)$ thanks to the lower bound on g ensured by (7.3).
By Lemma 6.1 and (7.2), we have

$$\begin{aligned} F^T_{\varepsilon_j,\delta_{\varepsilon_j}}(u_j) &\geq F^T_{\varepsilon_j,\delta_{\varepsilon_j}}(\Phi_M(u_j)) - \eta \geq \mathcal{F}^T_{\varepsilon_j}(w^M_j) - \frac{C}{N} - \eta \\ &\geq \mathcal{F}^T_{\varepsilon_j}(w^M_j,\Omega\setminus E_j) + \mathcal{F}^T_{\varepsilon_j}(w^M_j,E_j) - \eta - \frac{C}{N} \\ &\geq \mathcal{F}^T_{\varepsilon_j}(v^M_j) + \mathcal{F}^T_{\varepsilon_j}(w^M_j,E_j) - \eta - \frac{C}{N}, \end{aligned} \tag{7.4}$$

since by definition $w^M_j = v^M_j$ in $\Omega\setminus E_j$, and v^M_j is constant on each $\partial^{\varepsilon_j T} B_{\rho_{j,k_i}}(\delta_{\varepsilon_j} i)$.
By Theorem 3.2, it holds

$$\liminf_{j\to+\infty} \mathcal{F}^T_{\varepsilon_j}(v^M_j) \geq \int_\Omega f^T_{hom}(\nabla\Phi_M(u))\,dx. \tag{7.5}$$

We now turn to the estimate of the contribution on E_j. At this point we need to distinguish whether (3.13) or (3.15) holds.
Case $\varepsilon = o(r_{\delta_\varepsilon})$. With fixed $i \in Z_j(\Omega)$, let

$$v^M_{j,i}(y) := w^M_j(\delta_{\varepsilon_j} i + r_{\delta_{\varepsilon_j}} y)$$

be defined on the ball $B_{R_j^i}$, with $R_j^i := \rho_{j,k_i} r_{\delta_{\varepsilon_j}}^{-1}$ and extended to $(\Phi_M(u_j))^{i,k_i}$ outside this ball. Setting $s_j := \varepsilon_j r_{\delta_{\varepsilon_j}}^{-1}$, we get

$$\mathcal{F}_{\varepsilon_j}^T(w_j^M, B_{\rho_{j,k_i}}(\delta_{\varepsilon_j} i)) = r_{\delta_{\varepsilon_j}}^{d-p} \mathcal{F}_{s_j}^T(v_{j,i}^M, B_{R_j^i}) \geq r_{\delta_{\varepsilon_j}}^{d-p} \varphi_{s_j,T,R_j^i}(\Phi_M(u_j))^{i,k_i}). \tag{7.6}$$

We take

$$\Psi_j^T(x) = \sum_{i \in Z_j(\Omega)} \varphi_{s_j,T,R_j^i}((\Phi_M(u_j))^{i,k_i}) \chi_{Q_{\delta_{\varepsilon_j}}(i)}(x).$$

By (7.6), we get

$$\mathcal{F}_{\varepsilon_j}^T(w_j^M, E_j) \geq \frac{r_{\delta_{\varepsilon_j}}^{d-p}}{\delta_{\varepsilon_j}^d} \int_\Omega \Psi_j^T(x)\, dx = \left(\frac{r_{\delta_{\varepsilon_j}}}{\delta_{\varepsilon_j}^{\frac{d}{d-p}}} \right)^{d-p} \int_\Omega \Psi_j^T(x)\, dx. \tag{7.7}$$

By Proposition 6.8 (i), applied to $(\Phi_M(u_j))$, with s_j, R_j^i in place of ε_j, R_j, respectively, we have

$$\Psi_j^T \to \varphi^T(\Phi_M(u)) \text{ in } L^1(\Omega). \tag{7.8}$$

Hence, by (7.7), we deduce that

$$\liminf_{j \to +\infty} \mathcal{F}_{\varepsilon_j}^T(w_j^M, E_j) \geq \beta^{d-p} \int_\Omega \varphi^T(\Phi_M(u))\, dx. \tag{7.9}$$

Case $\boldsymbol{\varepsilon = O(r_{\delta_\varepsilon})}$. With fixed $i \in Z_j(\Omega)$, we now set

$$v_{j,i}^M(y) := w_j^M\Big(\delta_{\varepsilon_j} i + \frac{\varepsilon_j}{\alpha} y\Big)$$

on the ball $B_{R_j^i}$, with $R_j^i := \alpha \rho_{j,k_i} \varepsilon_j^{-1}$, and extend it, as in the previous case, to $(\Phi_M(u_j))^{i,k_i}$ outside this ball. We get

$$\mathcal{F}_{\varepsilon_j}^T(w_j^M, B_{\rho_{j,k_i}}(\delta_{\varepsilon_j} i)) = \Big(\frac{\varepsilon_j}{\alpha}\Big)^{d-p} \mathcal{F}_\alpha^T(v_{j,i}^M, B_{R_j^i}). \tag{7.10}$$

Define $t_j := \alpha \frac{r_{\delta_{\varepsilon_j}}}{\varepsilon_j}$ and note that, by (3.15), $t_j \to 1$. We take

$$\tilde{v}_{j,i}^M(y) := \begin{cases} 0 & \text{if } y \in B_1 \\ v_{j,i}^M(y) & \text{if } y \in B_{R_j^i} \setminus B_1. \end{cases}$$

Notice that, if $t_j > 1$, then $\tilde{v}_{j,i}^M$ coincides with $v_{j,i}^M$. A straightforward computation shows that

$$\mathcal{F}_\alpha^T(v_{j,i}^M, B_{R_j^i}) \geq \mathcal{F}_\alpha^T(\tilde{v}_{j,i}^M, B_{R_j^i}) - C(M)|1 - t_j|. \tag{7.11}$$

Since $\tilde{v}_{j,i}^M \in L^p_{\alpha,T,(\Phi_M(u_j))^{i,k_i}}(B_{R_j^i}; \mathbb{R}^m)$, by (7.10) and (7.11), we get

$$\begin{aligned} \mathcal{F}_{\varepsilon_j}^T(w_j^M, B_{\rho_{j,k_i}}(\delta_{\varepsilon_j} i)) &\geq \Big(\frac{\varepsilon_j}{\alpha}\Big)^{d-p} (\mathcal{F}_\alpha^T(\tilde{v}_{j,i}^M, B_{R_j^i}) - C(M)|1 - t_j|) \\ &\geq \Big(\frac{\varepsilon_j}{\alpha}\Big)^{d-p} (\varphi_{\alpha,T,R_j^i}((\Phi_M(u_j))^{i,k_i}) - C(M)|1 - t_j|). \end{aligned} \tag{7.12}$$

By taking

$$\Psi^T_{j,\alpha}(x) = \sum_{i\in Z_j(\Omega)} \varphi_{\alpha,T,R^i_j}((\Phi_M(u_j))^{i,k_i})\chi_{Q_{\delta_{\varepsilon_j}}(i)}(x),$$

using (7.12), we obtain

$$\mathcal{F}^T_{\varepsilon_j}(w^M_j, E_j) \geq t_j^{p-d}(\beta^{d-p} + o(1)) \int_\Omega \Psi^T_{j,\alpha}(x)\,dx + o(1). \tag{7.13}$$

Thanks to Proposition 6.8 (*ii*), applied to $(\Phi_M(u_j))$, with R^i_j in place of R_j, we have

$$\Psi^T_{j,\alpha} \to \varphi^T_{NL,\alpha}(\Phi_M(u)) \text{ in } L^1(\Omega). \tag{7.14}$$

Hence, by (7.13), we deduce that

$$\liminf_{j\to+\infty} \mathcal{F}^T_{\varepsilon_j}(w^M_j, E_j) \geq \beta^{d-p} \int_\Omega \varphi^T_{NL,\alpha}(\Phi_M(u))\,dx, \tag{7.15}$$

which is the analogue of (7.9) in the previous case.

By (7.4), (7.5), together with (7.9) and (7.15), and by the arbitrariness of $\eta > 0$ and $N \in \mathbb{N}$, we infer that

$$\liminf_{j\to+\infty} F^T_{\varepsilon_j,\delta_{\varepsilon_j}}(u_j) \geq \int_\Omega f^T_{hom}(\nabla\Phi_M(u))\,dx + \beta^{d-p}\int_\Omega \varphi^T(\Phi_M(u))\,dx,$$

under the assumption (3.13), and

$$\liminf_{j\to+\infty} F^T_{\varepsilon_j,\delta_{\varepsilon_j}}(u_j) \geq \int_\Omega f^T_{hom}(\nabla\Phi_M(u))\,dx + \beta^{d-p}\int_\Omega \varphi^T_{NL,\alpha}(\Phi_M(u))\,dx,$$

if (3.13) is replaced by (3.15). Letting $M \to +\infty$, we conclude that, under the assumptions of Theorem 3.5,

$$\liminf_{j\to+\infty} F^T_{\varepsilon_j,\delta_{\varepsilon_j}}(u_j) \geq \int_\Omega f^T_{hom}(\nabla u)\,dx + \beta^{d-p}\int_\Omega \varphi^T(u)\,dx,$$

and, under the assumption of Theorem 3.6,

$$\liminf_{j\to+\infty} F^T_{\varepsilon_j,\delta_{\varepsilon_j}}(u_j) \geq \int_\Omega f^T_{hom}(\nabla u)\,dx + \beta^{d-p}\int_\Omega \varphi^T_{NL,\alpha}(u)\,dx.$$

Step 3. With fixed $T \geq r_0$, we now prove the validity of the Γ-lim sup inequality for $F^T_{\varepsilon,\delta_\varepsilon}$. By a density argument it suffices to prove the inequality for $u \in C^\infty_c(\mathbb{R}^d;\mathbb{R}^m)$. For such a u, fixed an open set $\Omega' \in \mathcal{A}^{\rm reg}(\mathbb{R}^d)$ such that $\Omega' \supset\supset \Omega$, and given $\varepsilon_j \to 0$ as $j \to +\infty$, by Theorem 3.2 there exists a sequence $(\tilde u_j)$, converging in $L^p(\Omega';\mathbb{R}^m)$ to u, such that

$$\lim_{j\to+\infty} \mathcal{F}^T_{\varepsilon_j}(\tilde u_j, \Omega') = \int_{\Omega'} f^T_{hom}(\nabla u)\,dx. \tag{7.16}$$

Taking into account Lemma 6.2, up to replacing $\tilde u_j$ with a suitable truncation, we may also assume that $\sup_j \|\tilde u_j\|_{L^\infty(\Omega';\mathbb{R}^m)} < +\infty$. Thus, given $N \in \mathbb{N}$, we consider the sequence (w_j) constructed in Lemma 6.1 applied with Ω' in place of Ω, $\rho_j = \delta_{\varepsilon_j}/4$ and $\tilde u_j$ in place of u_j, so that

$$\mathcal{F}^T_{\varepsilon_j}(w_j, \Omega') \leq \mathcal{F}^T_{\varepsilon_j}(\tilde u_j, \Omega') + \frac{C}{N}. \tag{7.17}$$

We now pass to the estimate of the energetic contribution on the set

$$E_j = \bigcup_{i\in Z_j(\Omega')} B_{\rho_{j,k_i}}(\delta_{\varepsilon_j} i).$$

As previously done, we distinguish whether (3.13) or (3.15) holds.

Case $\boldsymbol{\varepsilon = o(r_{\delta_\varepsilon})}$. Set

$$R^i_j := \frac{\rho_{j,k_i}}{r_{\delta_{\varepsilon_j}}}, \quad s_j = \frac{\varepsilon_j}{r_{\delta_{\varepsilon_j}}},$$

where ρ_{j,k_i} is defined in Lemma 6.1. For $i \in Z_j(\Omega')$, let $\tilde v_{j,i} \in L^p_{s_j,T,u_j^{i,k_i}}(B_{R^i_j};\mathbb{R}^m)$ be such that

$$\mathcal{F}^T_{s_j}(\tilde v_{j,i}, B_{R^i_j}) = \varphi_{s_j,T,R^i_j}(u_j^{i,k_i}) + o(\varepsilon_j).$$

Then set

$$v_{j,i}(x) := \tilde v_{j,i}\left(\frac{x-\delta_{\varepsilon_j} i}{r_{\delta_{\varepsilon_j}}}\right), \quad x \in B_{\rho_{j,k_i}}(\delta_{\varepsilon_j} i),$$

and

$$u_j(x) := \begin{cases} w_j(x) & \text{if } x \in \Omega' \setminus E_j \\ v_{j,i}(x) & \text{if } x \in B_{\rho_{j,k_i}}(\delta_{\varepsilon_j} i). \end{cases}$$

Note that

$$\mathcal{F}^T_{\varepsilon_j}(v_{j,i}, B_{\rho_{j,k_i}}(\delta_{\varepsilon_j} i)) = r^{d-p}_{\delta_{\varepsilon_j}} \mathcal{F}^T_{s_j}(\tilde v_{j,i}, B_{R^i_j}) = r^{d-p}_{\delta_{\varepsilon_j}}(\varphi_{s_j,T,R^i_j}(u_j^{i,k_i}) + o(\varepsilon_j)). \tag{7.18}$$

Moreover $u_j \in L^p_{\delta_{\varepsilon_j}}(\Omega';\mathbb{R}^m)$ and, arguing as in **Step 2**, we also deduce that $u_j \to u$ in $L^p(\Omega;\mathbb{R}^m)$. We finally pass to the estimate of the energy. By (7.18), we get

$$\begin{aligned} \mathcal{F}^T_{\varepsilon_j}(u_j) &\le \mathcal{F}^T_{\varepsilon_j}(w_j,\Omega') + \sum_{i\in Z_j(\Omega')} \mathcal{F}^T_{\varepsilon_j}(u_j, B_{\rho_{j,k_i}}(\delta_{\varepsilon_j} i)) \\ &= \mathcal{F}^T_{\varepsilon_j}(w_j,\Omega') + \sum_{i\in Z_j(\Omega')} r^{d-p}_{\delta_{\varepsilon_j}}(\varphi_{s_j,T,R^i_j}(u_j^{i,k_i}) + o(\varepsilon_j)) \end{aligned} \tag{7.19}$$

Applying Proposition 6.8 (i) as in *Step 2*, we deduce that

$$\sum_{i\in Z_j(\Omega')} r^{d-p}_{\delta_{\varepsilon_j}} \varphi_{s_j,T,R^i_j}(u_j^{i,k_i}) \to \beta^{d-p}\int_{\Omega'} \varphi^T(u)\,dx. \tag{7.20}$$

Case $\boldsymbol{\varepsilon = O(r_{\delta_\varepsilon})}$. We now set

$$R^i_j := \alpha\,\frac{\rho_{j,k_i}}{\varepsilon_j},$$

where ρ_{j,k_i} is defined in Lemma 6.1. By applying Proposition 6.6 with $M_0 = \sup_j \|\tilde u_j\|_{L^\infty(\Omega';\mathbb{R}^m)}$, given $\eta > 0$ there exist $M > \sup_j \|\tilde u_j\|_{L^\infty(\Omega';\mathbb{R}^m)}$ such that for every $i \in Z_j(\Omega')$ there exists $\tilde v_{j,i} \in L^p_{\alpha,T,u_j^{i,k_i}}(B_{R^i_j};\mathbb{R}^m)$ such that $\|\tilde v_{j,i}\|_{L^\infty(\Omega';\mathbb{R}^m)} \le M$ and

$$\mathcal{F}^T_\alpha(\tilde v_{j,i}, B_{R^i_j}) \le \varphi_{\alpha,T,R^i_j}(\tilde u_j^{i,k_i}) + \eta.$$

Set $t_j = \alpha \frac{r_{\delta_{\varepsilon_j}}}{\varepsilon_j}$ and note that, by (3.15), $t_j \to 1$. Then define

$$\hat{v}_{j,i}(y) := \begin{cases} 0 & \text{if } y \in B_{t_j} \\ \tilde{v}_{j,i}(y) & \text{if } y \in B_{R^i_j} \setminus B_{t_j}. \end{cases}$$

Notice that $\hat{v}_{j,i}$ coincides with $\tilde{v}_{j,i}$ if $t_j \leq 1$. As in *Step 2*, a straightforward computation shows that

$$\mathcal{F}^T_\alpha(\hat{v}_{j,i}, B_{R^i_j}) \leq \mathcal{F}^T_\alpha(\tilde{v}_{j,i}, B_{R^i_j}) + C(M)|1 - t_j|$$

We take

$$v_{j,i}(x) := \hat{v}_{j,i}\left(\alpha \frac{x - \delta_{\varepsilon_j} i}{\varepsilon_j}\right) \; x \in B_{\rho_{j,k_i}}(\delta_{\varepsilon_j} i),$$

and

$$u_j(x) := \begin{cases} w_j(x) & \text{if } x \in \Omega' \setminus E_j \\ v_{j,i}(x) & \text{if } x \in B_{\rho_{j,k_i}}(\delta_{\varepsilon_j} i). \end{cases}$$

Note that

$$\mathcal{F}^T_{\varepsilon_j}(v_{j,i}, B_{\rho_{j,k_i}}(\delta_{\varepsilon_j} i)) = t_j^{p-d} r^{d-p}_{\delta_{\varepsilon_j}} \mathcal{F}^T_\alpha(\hat{v}_{j,i}, B_{R^i_j}) \leq t_j^{p-d} r^{d-p}_{\delta_{\varepsilon_j}} (\varphi_{\alpha,T,R^i_j}(\tilde{u}^{i,k_i}_j) + C(M)|1 - t_j| + \eta). \tag{7.21}$$

We notice that $u_j \in L^p_{\delta_{\varepsilon_j}}(\Omega; \mathbb{R}^m)$ and, as in the previous case, we deduce that $u_j \to u$ in $L^p(\Omega; \mathbb{R}^m)$. By (7.21), we get the counterpart of (7.19)

$$\begin{aligned} \mathcal{F}^T_{\varepsilon_j}(u_j) &\leq \mathcal{F}^T_{\varepsilon_j}(w_j, \Omega') + \sum_{i \in Z_j(\Omega')} \mathcal{F}^T_{\varepsilon_j}(u_j, B_{\rho_{j,k_i}}(\delta_{\varepsilon_j} i)) \\ &\leq \mathcal{F}^T_{\varepsilon_j}(w_j, \Omega') + t_j^{p-d} \sum_{i \in Z_j(\Omega')} r^{d-p}_{\delta_{\varepsilon_j}} (\varphi_{\alpha,T,R^i_j}(\tilde{u}^{i,k_i}_j) + C|1 - t_j| + \eta). \end{aligned} \tag{7.22}$$

Applying Proposition 6.8 (*ii*) as in **Step 2**, we deduce that

$$\sum_{i \in Z_j(\Omega')} r^{d-p}_{\delta_{\varepsilon_j}} \varphi_{\alpha,T,R^i_j}(\tilde{u}^{i,k_i}_j) \to \beta^{d-p} \int_{\Omega'} \varphi^T_{NL,\alpha}(u)\, dx. \tag{7.23}$$

which corresponds to (7.20).
Hence, by (7.16), (7.17), together with (7.19), (7.20), (7.22), and (7.23), we get that

$$\limsup_{j \to +\infty} \mathcal{F}^T_{\varepsilon_j}(u_j) \leq \int_{\Omega'} f^T_{hom}(\nabla u)\, dx + \beta^{d-p} \int_{\Omega'} \varphi^T(u)\, dx + \frac{C}{N},$$

under the assumptions of Theorem 3.5, and

$$\limsup_{j \to +\infty} \mathcal{F}^T_{\varepsilon_j}(u_j) \leq \int_{\Omega'} f^T_{hom}(\nabla u)\, dx + \beta^{d-p} \int_{\Omega'} \varphi^T_{NL,\alpha}(u)\, dx + \frac{C}{N},$$

under the assumptions of Theorem 3.6. The conclusion follows by the arbitrariness of $N \in \mathbb{N}$ and letting $\Omega' \to \Omega$.

8 The scaling regime $r_{\delta_\varepsilon} = o(\varepsilon)$

Theorems 3.5, 3.6 and Remark 3.7 provide a complete description of the asymptotic behaviour of $F_{\varepsilon,\delta_\varepsilon}$ when $\lim_{\varepsilon\to 0} \varepsilon r_{\delta_\varepsilon}^{-1} = \alpha \in [0,+\infty)$. In this section we consider the case when $\varepsilon \to 0$ slower than r_{δ_ε}, showing that, for most of the choice of the scaling of r_δ with respect to δ, $F_{\varepsilon,\delta_\varepsilon}$ is not affected by the constraint $u \in L^p_{\delta_\varepsilon}(\Omega;\mathbb{R}^m)$ and then $\Gamma(L^p) - \lim_{\varepsilon\to 0} F_{\varepsilon,\delta_\varepsilon}(u) = \Gamma(L^p) - \lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u)$.

Assume from now on

$$\lim_{\delta\to 0} \frac{r_\delta}{\delta} = 0, \tag{8.1}$$

$$\lim_{\varepsilon\to 0} \frac{\varepsilon}{r_{\delta_\varepsilon}} = +\infty. \tag{8.2}$$

First of all let us note that, since $F_{\varepsilon,\delta_\varepsilon}(u) \geq \mathcal{F}_\varepsilon(u)$, we have that

$$\Gamma(L^p) - \liminf_{\varepsilon\to 0} F_{\varepsilon,\delta_\varepsilon}(u) \geq \Gamma(L^p) - \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u). \tag{8.3}$$

Given $\varepsilon_j \to 0$ as $j \to +\infty$, let $u \in C^\infty_c(\mathbb{R}^d;\mathbb{R}^m)$ and let $\tilde{u}_j \to u$ in $L^p(\Omega;\mathbb{R}^m)$ such that

$$\lim_{j\to+\infty} \mathcal{F}_{\varepsilon_j}(\tilde{u}_j) = \int_\Omega f_{\text{hom}}(\nabla u)\,dx.$$

Arguing as in Step 3 of the proof of Theorems 3.5 and 3.6, we may assume $\tilde{u}_j$ bounded in $L^\infty(\Omega;\mathbb{R}^m)$.

Set then

$$u_j(x) := \begin{cases} \tilde{u}_j(x) & \text{if } x \in \Omega \setminus P_{\delta_{\varepsilon_j}} \\ 0 & \text{if } x \in \Omega \cap P_{\delta_{\varepsilon_j}}. \end{cases}$$

Clearly $u_j \in L^p_{\delta_{\varepsilon_j}}(\Omega;\mathbb{R}^m)$ and, by (8.1), $u_j \to u$ in $L^p(\Omega;\mathbb{R}^m)$ as $j \to +\infty$. By assumption (ii), we have

$$\mathcal{F}_{\varepsilon_j}(u_j) \leq \mathcal{F}_{\varepsilon_j}(\tilde{u}_j) + \frac{\|u_j\|^p_\infty}{\varepsilon_j^p} \sum_{i\in\mathbb{Z}^d} \int_{\mathbb{R}^d} M(\xi)|S^\xi_{i,j}|\,d\xi, \tag{8.4}$$

where

$$S^\xi_{i,j} := \Omega \cap \Big((B_{r_{\delta_{\varepsilon_j}}}(\delta_{\varepsilon_j} i) \cap ((B_{r_{\delta_{\varepsilon_j}}}(\delta_{\varepsilon_j} i))^c - \varepsilon_j\xi)) \cup ((B_{r_{\delta_{\varepsilon_j}}}(\delta_{\varepsilon_j} i))^c \cap (B_{r_{\delta_{\varepsilon_j}}}(\delta_{\varepsilon_j} i) - \varepsilon_j\xi)) \Big).$$

Note that, since $S^\xi_{i,j} \subseteq B_{r_{\delta_{\varepsilon_j}}}(\delta_{\varepsilon_j} i) \cup (B_{r_{\delta_{\varepsilon_j}}}(\delta_{\varepsilon_j} i) - \varepsilon_j\xi)$, then

$$|S^\xi_{i,j}| \leq C r^d_{\delta_{\varepsilon_j}}.$$

Thus, by (8.4) and (G1), we get

$$\mathcal{F}_{\varepsilon_j}(u_j) \leq \mathcal{F}_{\varepsilon_j}(\tilde{u}_j) + C\frac{r^d_{\delta_{\varepsilon_j}}}{\varepsilon_j^p}\frac{1}{\delta^d_{\varepsilon_j}}. \tag{8.5}$$

By (8.3),(8.5) and a density argument, we infer that $\Gamma(L^p) - \lim_{\varepsilon\to 0} F_{\varepsilon,\delta_\varepsilon}(u) = \Gamma(L^p) - \lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u)$ under the additional condition

$$\lim_{\varepsilon\to 0} \frac{r^d_{\delta_\varepsilon}}{\varepsilon^p}\frac{1}{\delta^d_\varepsilon} = 0, \tag{8.6}$$

which can be written as

$$\lim_{\varepsilon\to 0} \frac{\varepsilon}{\left(\frac{r_{\delta_\varepsilon}}{\delta_\varepsilon}\right)^{d/p}} = +\infty.$$

Note that

$$\frac{r^d_{\delta_\varepsilon}}{\varepsilon^p}\frac{1}{\delta^d_\varepsilon} = \left(\frac{r_{\delta_\varepsilon}}{\varepsilon}\right)^p \frac{r^{d-p}_{\delta_\varepsilon}}{\delta^d_\varepsilon},$$

which, thanks to (8.2), yields that (8.6) is satisfied if $r_\delta \leq C\delta^{\frac{d}{d-p}}$. We may then conclude that the following Γ-convergence result holds.

Theorem 8.1. *Let $F_{\varepsilon,\delta}$ be defined by* (3.3)*, with f satisfying assumptions* (**H**)*,* (**G**) *and* (**L**) *and $1 < p < d$. Assume moreover that* (8.2) *and one of the following two assumptions hold*

a) $\limsup_{\delta\to 0} \frac{r_\delta}{\delta^{\frac{d}{d-p}}} < +\infty$

b) $\lim_{\varepsilon\to 0} \frac{\varepsilon}{\left(\frac{r_{\delta_\varepsilon}}{\delta_\varepsilon}\right)^{d/p}} = +\infty.$

Then

$$\Gamma(L^p)\text{-}\lim_{\varepsilon\to 0} F_{\varepsilon,\delta_\varepsilon}(u) = \begin{cases} \displaystyle\int_\Omega f_{hom}(\nabla u)\,dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^m), \\ +\infty & \text{otherwise,} \end{cases}$$

where $f_{hom}(S)$ is defined by (3.11)*.*

We have summarized our Γ-convergence results in the following table (see Theorem 3.5, Theorem 3.6, Remark 3.7, and Theorem 8.1), schematising how the interplay between the various parameters affects the Γ-limit of the non-local functionals $F_{\varepsilon,\delta}$ defined in (3.3). The domain of the Γ-limit is $W^{1,p}(\Omega;\mathbb{R}^m)$ if not specified.

ε \ δ	$\lim_{\delta\to 0} \frac{r_\delta}{\delta^{\frac{d}{d-p}}} = \beta \geq 0$	$\lim_{\delta\to 0} \frac{r_\delta}{\delta^{\frac{d}{d-p}}} = +\infty$
$\lim_{\varepsilon\to 0} \frac{\varepsilon}{r_{\delta_\varepsilon}} = 0$	$\int_\Omega f_{hom}(\nabla u)\,dx + \beta^{d-p}\int_\Omega \varphi(u)\,dx$	0 iff $u \equiv 0$
$\lim_{\varepsilon\to 0} \frac{\varepsilon}{r_{\delta_\varepsilon}} = \alpha > 0$	$\int_\Omega f_{hom}(\nabla u)\,dx + \beta^{d-p}\int_\Omega \varphi_{NL,\alpha}(u)\,dx$	0 iff $u \equiv 0$
$\lim_{\varepsilon\to 0} \frac{\varepsilon}{r_{\delta_\varepsilon}} = +\infty$	$\int_\Omega f_{hom}(\nabla u)\,dx$	if $\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(\frac{r_{\delta_\varepsilon}}{\delta_\varepsilon}\right)^{\frac{d}{p}} = 0$ $\int_\Omega f_{hom}(\nabla u)\,dx$

Table 1

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