A SPLITTING THEOREM FOR MANIFOLDS WITH A CONVEX BOUNDARY COMPONENT AND APPLICATIONS TO THE HALF-SPACE PROPERTY

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Abstract. We prove a warped product splitting theorem for manifolds with Ricci curvature bounded from below in the spirit of [Croke-Kleiner, *Duke Math. J.* (1992), [12]], but instead of asking that one boundary component is compact and mean convex, we require that it is parabolic and convex. The parabolicity assumption cannot be dropped as, otherwise, the catenoid in ambient dimension four would give a counterexample. As an application, we deduce a half-space theorem for mean convex sets in product manifolds (resp. for sets whose boundary has mean curvature bounded below by a definite constant, in warped products with negative curvature).

The results are obtained by combining glueing techniques for manifolds and optimal transport tools from synthetic Ricci curvature bounds.

1. Introduction

The celebrated Cheeger-Gromoll splitting theorem [10] states that a (complete, connected) Riemannian manifold with non-negative Ricci curvature containing a line is isometrically a product, where one of the factors is a real line. A variant of this result was proved by Kasue in [23]. There, it is shown that a Riemannian manifold (M, g) with mean convex compact boundary ∂M and non-negative Ricci curvature, containing a half line with initial point in ∂M, again is isometrically a product. In the latter case, the splitting factor is a half line. This result fails without the compactness assumption on the boundary, as one sees considering the epigraph of a strictly convex function in Euclidean space. In the same work, Kasue shows that a manifold with multiple mean convex boundary components and non-negative Ricci curvature is also a product, where one of the factors is a bounded interval of the real line. A generalization of Kasue's results to manifolds with Ricci curvature bounded below by a *negative* constant was given by Croke and Kleiner in [12], where they show the following.

Theorem ([12, Theorem 1]). *Let $\delta \in \{0, 1\}$ and let (M^n, g) be a Riemannian manifold with $\mathsf{Ric}_M \geq -(n-1)\delta$ and whose boundary consists of two connected components $\partial M = S_1 \cup S_2$, one of which is compact. Assume that the mean curvature of S_1 is $\geq \delta$ and the mean curvature of S_2 is $\geq -\delta$. Then, there exists $l > 0$ such that M is isometric to $S_1 \times [0, l]$ with the* *warped product metric* $ds^2 = e^{-2\delta t} g_1 + dt^2$, *where* g_1 *is the metric on* S_1. *Moreover,* $\mathsf{Ric}_{S_1} \geq 0$.

In this note, we prove a variant of this result. Instead of asking for the compactness of one of the boundary components, we ask for its parabolicity, which is a weaker condition. Unfortunately, we are not able to fully generalize the result from [12], since we also need to assume that the parabolic boundary component has a lower bound on the second fundamental form, rather than on the mean curvature. Anyway, this will suffice to obtain the desired applications to the half-space property.

We recall that a manifold is said to be *parabolic* if it admits no positive fundamental solution for the Laplacian. For example, $\mathbb{R}^2$ is parabolic, while $\mathbb{R}^n$ with $n \geq 3$ is not. More generally, a manifold (M, g) with non-negative Ricci curvature is parabolic if and only if

$$\int_1^\infty \frac{t}{\mathsf{Vol}(B_t(x))}\, dt = +\infty,$$

for some $x \in M$.

The main results of the note are Theorems 1 and 2 below. In a Riemannian manifold (M^n, g) with boundary, we denote by $\Pi^M_{\partial M}$ the second fundamental form of the boundary w.r.t. the inward pointing unit normal and we set $H^M_{\partial M} := (\mathrm{tr}\Pi^M_{\partial M})/(n-1)$ to be the associated mean curvature (the upper script M will be omitted if there is no ambiguity).

Theorem 1. *Let* $\delta \in \{0, 1\}$ *and let* (M^n, g) *be a Riemannian manifold, with* $\mathsf{Ric}_M \geq -(n-1)\delta$. *Let* S_1 *be a non-empty connected component of* ∂M, *and let* $S_2 := \partial M \setminus S_1$ *be non-empty as well. Assume that* S_1 *is parabolic, with* $\Pi_{S_1} \geq \delta$ *and* $\mathsf{Ric}_{S_1} \geq 0$, *while* $H_{S_2} \geq -\delta$. *Then, there exists* $l > 0$ *such that* M *is isometric to* $S_1 \times [0, l]$ *with the warped product metric* $ds^2 = e^{-2\delta t} g_1 + dt^2$, *where* g_1 *is the metric on* S_1.

Theorem 2. *Let* (M^n, g) *be a Riemannian manifold, with* $\mathsf{Ric}_M \geq -(n-1)$. *Let* ∂M *be the disjoint union* $\partial M = S_1 \cup S_2 \cup S_3$, *where each* S_i *is a union of connected components of* ∂M. *Assume that the following hold.*

1. S_1 *is non-empty with* $H_{S_1} \geq 1$.
2. S_2 *is non-empty, connected, parabolic, with* $\Pi_{S_2} \geq -1$, *and* $\mathsf{Ric}_{S_2} \geq 0$.
3. S_3 *has* $\Pi_{S_3} \geq -1$, *and* $\mathsf{Ric}_{S_3} \geq 0$.

Then, $S_3 = \emptyset$, *and there exists* $l > 0$ *such that* M *is isometric to* $S_1 \times [0, l]$ *with the warped product metric* $ds^2 = e^{-2\delta t} g_1 + dt^2$, *where* g_1 *is the metric on* S_1.

Remark (Comparing with Croke-Kleiner's result)**.** Theorem 1 follows from Croke-Kleiner's result, if we require that S_2 is connected and that S_1 is compact. On the other hand, Theorem 2 is independent of the aforementioned result even if we add these extra assumptions. Indeed, in Theorem 2, even if S_1 is connected and S_2 is compact, Croke-Kleiner's argument breaks in the presence of the additional boundary components contained in S_3. The reason for this is that all distance minimizing geodesics between S_1 and S_2 might 'touch' S_3. Hence, the Laplacians of the distance functions from S_1 and S_2 might not satisfy the usual inequalities (given by the Ricci curvature lower bounds) along these geodesics. This point, as well as the non-compactness of the connected components, is dealt with by combining glueing techniques for manifolds with optimal transport tools.

Remark (On the parabolicity assumption)**.** In Theorem 1, when $\delta = 0$, the parabolicity assumption cannot be dropped. To see this, consider the manifold $M \subset \mathbb{R}^4$ consisting of the portion of space bounded by a catenoid and a disjoint hyperplane. In this case, M satisfies the hypotheses of Theorem 1 with $\delta = 0$ (except for the parabolicity of the convex boundary component since $\mathbb{R}^3$ is not parabolic), but the conclusion fails. Similar constructions are likely to be possible for $\delta = 1$ in Theorem 1 and for Theorem 2. This is tied to the study of the Half Space Property (see [11] and the rest of the introduction) for warped products.

If $n = 3$, one can omit the parabolicity condition both in Theorems 1 and 2. Indeed, 2-dimensional manifolds with non-negative curvature are parabolic due to Bishop-Gromov's inequality.

Remark (On the $\mathsf{Ric}_{S_1} \geq 0$ assumption)**.** The assumption $\mathsf{Ric}_{S_1} \geq 0$ in Theorem 1 (resp. $\mathsf{Ric}_{S_2} \geq 0$ in Theorem 2) is necessary, in the following sense: if the conclusion of the theorem holds, namely $ds^2 = e^{-2\delta t} g_1 + dt^2$, then Gauss equations imply that $\mathsf{Ric}_{S_1} \geq 0$ (resp. $\mathsf{Ric}_{S_2} \geq 0$).

As an immediate consequence of Theorems 1 and 2, we obtain a slice theorem for warped products over parabolic manifolds with non-negative Ricci curvature, see Corollary 1 below. Let us give some context first. Following [32], a manifold (M, g) is said to have the Half Space Property if the only (properly embedded) minimal hypersurfaces of $M \times \mathbb{R}$ contained in a half-space are the horizontal slices $M \times \{t\}$. In recent years, several results have been obtained by different authors (see, among others, [14, 21, 17, 15, 11, 16]). Corollary 1 provides a result in the spirit of the Half Space Property for (possibly warped) products. The main differences with the aforementioned classical half-space results are that:

- Corollary 1 holds for mean convex boundaries in product manifolds, not only for minimal hypersurfaces;
- we obtain also a half-space result for sets whose boundary has mean curvature bounded below by 1 (or -1, depending wheter they lie in the lower or upper half space), in warped products with negative curvature.

Corollary 1. *Let $\delta \in \{0, 1\}$. Let (M^n, g) be a parabolic manifold with $\mathsf{Ric}_M \geq 0$ and let $M \times \mathbb{R}$ be equipped with $ds^2 = e^{-2t\delta} g + dt^2$. If $E \subset M \times (0, +\infty)$ is a smooth closed set with connected boundary and outward mean curvature $H_{\partial E} \geq -\delta$, then $E = M \times [a, +\infty)$, for some $a > 0$.*

If $E \subset M \times (-\infty, 0)$ is a smooth closed set with connected boundary and outward mean curvature $H_{\partial E} \geq \delta$, then $E = M \times (-\infty, a]$, for some $a < 0$.

Corollary 1 can be deduced from the previous results as follows. Let $E \subset M \times (0, +\infty)$ be with outward mean curvature satisfying $H_{\partial E} \geq -\delta$. Consider the manifold

$$N := M \times (0, +\infty) \setminus E.$$

Then, $\partial N = \partial E \cup M \times \{0\}$. Since the second fundamental form of $M \times \{0\}$ in $M \times (0, +\infty)$ is equal to δ, the result follows from Theorem 1. The case when $E \subset M \times (-\infty, 0)$ has outward mean cuvature satisfying $H_{\partial E} \geq \delta$ follows analogously, applying Theorem 2 instead of Theorem 1.

Remark (On a related result by Montiel)**.** In [29], Montiel proved that if (M, g) is compact with non-negative Ricci curvature, and $M \times \mathbb{R}$ is equipped with the metric $ds^2 = e^{-2t} g + dt^2$, then any hypersurface $\Sigma \subset M \times \mathbb{R}$ of constant mean curvature, that is locally a graph on M, must be a slice. Some generalizations of this result later appeared in [1, 2, 6]. Although similar in spirit, Montiel's theorem is independent of Corollary 1: indeed, it requires the mean curvature to be constant while we require the inequality $H \geq \pm 1$ and we do not assume the hypersurface to be locally a graph.

We briefly outline the strategy of the proof of Theorem 1, in the case $\delta = 0$. Consider the metric space (X, d) obtained by gluing M and $S_1 \times [0, +\infty)$ along their isometric boundaries. Consider the distance function d_{S_2} from S_2 in the glued space and its restriction to $S_1 \times [0, +\infty)$. We denote by d_{S_1} the distance from S_1 in $S_1 \times [0, +\infty)$. If we can show that d_{S_2} is constant on $S_1 \times \{0\}$, then the statement follows by standard arguments.

Thanks to the assumption on the fundamental form of S_1 (and previous results on the Laplacian of distance functions in metric measure spaces with synthetic Ricci curvature lower bounds, see [18, 8, 24, 28, 19]), it holds $\Delta \mathsf{d}_{S_2} \leq 0$ on $S_1 \times (0, +\infty)$ in distributional sense.
In particular, it holds $\Delta(\mathsf{d}_{S_2} - \mathsf{d}_{S_1}) \leq 0$ and, calling ν the exterior normal of $S_1 \times \{0\}$ in $S_1 \times [0, +\infty)$, it also (formally) holds $\nabla(\mathsf{d}_{S_2} - \mathsf{d}_{S_1}) \cdot \nu \geq 0$ on $S_1 \times \{0\}$. If $S_1 \times [0, +\infty)$ has sufficiently small volume growth at infinity, we then deduce that $\mathsf{d}_{S_2} - \mathsf{d}_{S_1}$ is constant by an integration by parts argument (see, for instance, [22] for this type of arguments in the smooth setting and, more generally, for parabolicity of manifolds with boundary). Since in general $S_1 \times [0, +\infty)$ does not satisfy the required volume growth at infinity, adapting a strategy that we recently devised in [13], we multiply the volume measure of $S_1 \times [0, +\infty)$ by a suitable weight. Making an appropriate choice of such a weight, we obtain that $\mathsf{d}_{S_2} - \mathsf{d}_{S_1}$ satisfies the previous Laplacian bound also in the weighted space, which, in addition, has the desired volume growth. We then carry out a weighted integration by parts argument to obtain that $\mathsf{d}_{S_2} - \mathsf{d}_{S_1}$ is constant. Hence, d_{S_2} is constant on $S_1 \times \{0\}$.

We conclude with an open question:

Open question. Is it possible to replace the bounds on the second fundamental forms:

- $\Pi_{S_1} \geq \delta$ in Theorem 1;
- $\Pi_{S_2} \geq -1$, $\Pi_{S_3} \geq -1$ in Theorem 2;

by the mean curvature bounds

- $H_{S_1} \geq \delta$ in Theorem 1;
- $H_{S_2} \geq -1$, $H_{S_3} \geq -1$ in Theorem 2.

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2. Proof of the results

We introduce a few optimal transport tools that are used in the proof of Lemma 1. If Lemma 1 and Lemma 2 (whose statements do not require optimal transport tools to be formulated) are assumed to be true, one can skip this part and read directly the proofs of Theorems 1 and 2.

Let X be an n-dimensional smooth manifold (possibly with boundary) equipped with a locally Lipschitz continuous metric g. The metric g induces a distance d and a volume measure, which coincides with the n-dimensional Hausdorff measure H^n induced by d. Given an open set $\Omega \subset \mathsf{X}$, we denote by $\mathsf{Lip}(\Omega)$ and $\mathsf{Lip}_c(\Omega)$ respectively Lipschitz continuous and compactly supported Lipschitz continuous functions on Ω.

Assume that the metric space (X, d) is complete. On the metric measure space $(\mathsf{X}, \mathsf{d}, \mathsf{H}^n)$ one can define the Wasserstein distance (w.r.t. to the distance squared cost) between two probability measures with finite second moment. The set of probability measures with finite second moment is denoted $\mathcal{P}_2(\mathsf{X})$. Given $N \in [1, +\infty)$ and $\mu = \rho \mathsf{H}^n \in \mathcal{P}_2(\mathsf{X})$, the Rényi entropy of μ with respect to H^n is defined as

$$U_N(\mu|\mathsf{H}^n) := -\int \rho^{1-1/N}\, d\mathsf{H}^n.$$

Given $K \in \mathbb{R}$, two measures $\mu_1 = \rho_1 \mathsf{H}^n, \mu_2 = \rho_2 \mathsf{H}^n \in \mathcal{P}_2(\mathsf{X})$, and an optimal plan π between them, we define

$$T^{(t)}_{K,N}(\pi|\mathsf{H}^n) := \int_{\mathsf{X}\times\mathsf{X}} [\tau^{1-t}_{K,N}(\mathsf{d}(x_1, x_2))\rho_1^{1/N}(x_1) + \tau^{t}_{K,N}(\mathsf{d}(x_1, x_2))\rho_2^{1/N}(x_2)]\, d\pi(x_1, x_2),$$

where, for every $s \in [0, 1]$, $\tau^s_{\cdot,\cdot}(\cdot)$ is an appropriate distortion coefficient (which is Lipschitz continuous in $(-\infty, 0] \times [2, +\infty) \times [0, +\infty)$, see [36]).

We say that X *satisfies the* $\mathrm{CD}(K,N)$ *condition in* $U \subset \mathsf{X}$, if the following holds. For every pair $\mu_1 = \rho_1 \mathsf{H}^n, \mu_2 = \rho_2 \mathsf{H}^n \in \mathcal{P}_2(\mathsf{X})$ of measures supported in U, there exists a Wasserstein geodesic $\{\xi_t\}_{t\in[0,1]} \subset \mathcal{P}_2(\mathsf{X})$ from μ_1 to μ_2 and an optimal coupling π of μ_1 and μ_2 such that, for every $t \in [0,1]$ and every $N' \geq N$, it holds

$$U_{N'}(\xi_t|\mathsf{H}^n) \leq -T^{(t)}_{K,N'}(\pi|\mathsf{H}^n).$$

This condition differs from the $\mathrm{CD}_{loc}(K,N)$ condition that appears in the literature (see for instance [3, 7]), since we are not requiring that the convexity of the entropy holds in a neighbourhood of every point. For more background on curvature dimension conditions in metric (measure) spaces, we refer to the foundational works [35, 36, 27].

We now turn our attention to Laplacians of functions on X, referring to [18, 8, 24, 28, 19] for more details and results on Laplacians in metric measure spaces with synthetic Ricci lower bounds. Let $\Omega \subset \mathsf{X}$ be an open set. We say that a Radon measure μ on Ω is the distributional Laplacian of $f \in \mathsf{Lip}(\Omega)$ (and we write $\Delta f = \mu$) if, for every $\phi \in \mathsf{Lip}_c(\Omega)$, it holds

$$-\int_\Omega \nabla f \cdot \nabla \phi \, d\mathsf{H}^n = \int \phi \, d\mu.$$

We remark that since we are working on a smooth manifold with a continuous metric, the product $\nabla f \cdot \nabla \phi$ is well defined H^n-almost everywhere. In particular, for our purposes, we do not need to consider more general notions of gradients on metric spaces.

We now recall some facts about the localization technique for $\mathrm{CD}(K,N)$ spaces. We refer to [9, 8] for the proofs, as well as for the definitions of transport set of a Lipschitz function, transport rays and disintegration of a measure.

Let $N > 1$ and $K \in \mathbb{R}$. Assume that X is $\mathrm{CD}(K,N)$ in a neighbourhood of every point and that it is non-branching (i.e. geodesics do not branch). Let $\phi \in \mathsf{Lip}(\mathsf{X})$ be a 1-Lipschitz function and consider the partition of its transport set into transport rays $\{r_\alpha\}_{\alpha\in Q}$, Q being a set of indexes with a measure q induced by the partition. Consider the associated disintegration of the measure H^n (restricted to the transport set of ϕ) into measures $\{h_\alpha\}_{\alpha\in Q}$, each concentrated on the corresponding transport ray.

Then, q-almost every measure h_α is absolutely continuous w.r.t. the Lebesgue measure of the corresponding transport ray, it admits a locally Lipschitz continuous density (that we still denote by h_α), and it satisfies in weak sense

$$(h_\alpha^{\frac{1}{N-1}})'' + \frac{K}{N-1} h_\alpha^{\frac{1}{N-1}} \leq 0.$$

It was shown in [8] that, if the function ϕ is the distance function from a closed set $E \subset \mathsf{X}$, we also have

$$\Delta \phi = (\log h_\alpha)' + [\Delta \phi]^{sing} \quad \text{in } \mathsf{X} \setminus E \text{ in distributional sense.}$$

In the previous equation, $[\Delta\phi]^{sing}$ is a negative singular measure (w.r.t. H^n), while the derivative $(\log h_\alpha)'$ is taken orienting the transport rays from E to $\mathsf{X}\setminus E$.

Lemma 1. *Let $\delta\in\{0,1\}$ and let (M^n,g) be a Riemannian manifold with $\mathsf{Ric}_M\geq-(n-1)\delta$. Let S_1 be a union of connected components of ∂M. Assume that $\partial M\setminus S_1\neq\emptyset$ and set $S_2:=\partial M\setminus S_1$. Let $(N,\tilde g)$ be a second manifold with $\mathsf{Ric}_N\geq-(n-1)\delta$, and whose boundary ∂N is isometric to S_1. Let (X,d) be the metric space obtained by gluing M and N along the two isometric copies of S_1, and let d_{S_2} be the distance from S_2 in X. Assume that $\Pi^M_{S_1}+\Pi^N_{\partial N}\geq 0$.*

1. *If $H_{S_2}\geq-\delta$, then $\Delta\mathsf{d}_{S_2}\leq(n-1)\delta$ in $\mathsf{X}\setminus S_2$ in distributional sense.*
2. *If $H_{S_2}\geq\delta$, then $\Delta\mathsf{d}_{S_2}\leq-(n-1)\delta$ in $\mathsf{X}\setminus S_2$ in distributional sense.*
3. *Let S be a connected component of S_1 and let $D\geq 0$ be the distance between S and S_2 in X. If $H_S\geq-\delta$, then $\Delta\mathsf{d}_S\leq(n-1)\delta$ in the D-neighbourhood of S in $\mathsf{X}\setminus S$ in distributional sense.*

Proof. From now on, we refer to S_1, S_2, M and N as subsets of X. Balls $B_r(x)\subset\mathsf{X}$ are always defined w.r.t. the distance d on X. The space X admits a smooth structure such that the metric g^{X} on X defined by $g^{\mathsf{X}}_{|M}=g$ and $g^{\mathsf{X}}_{|N}=\tilde g$ is locally Lipschitz continuous and it induces the distance d. In particular, the definitions that were previously given for smooth manifolds with a continuous metric apply to this setting.

Step 1. We show that, for every $x\in\mathsf{X}\setminus S_2$, there exists $r>0$ such that X satisfies $\mathrm{CD}(-(n-1)\delta,n)$ in $B_r(x)$.

Suppose first that the point $x\in\mathsf{X}\setminus S_2$ does not belong to S_1. In this case, there exists a convex ball $B_r(x)$ not intersecting $S_1\cup S_2$. Consider two probability measures $\mu_1=\rho_1\mathsf{H}^n,\mu_2=\rho_2\mathsf{H}^n\in\mathcal{P}_2(B_r(x))$. Any Wasserstein geodesic $\{\xi_t\}_{t\in[0,1]}\subset\mathcal{P}_2(\mathsf{X})$ between μ_1 and μ_2 is concentrated on geodesics connecting points in $B_r(x)$. These geodesics are themselves contained in $B_r(x)$ by convexity, so that they lie in the region where X is a smooth manifold with Ricci bounded below by $-(n-1)\delta$. Hence, $\{\xi_t\}_{t\in[0,1]}$ satisfies the required convexity condition of the entropy by [34].

Suppose now that $x\in S_1$. If $R>0$ is small enough, using the same arguments of [33] (here we use that $\Pi^M_{S_1}+\Pi^N_{\partial N}\geq 0$, see also [30]), we obtain the following. There exists a sequence of smooth Riemannian metrics g_k on $\bar B_R(x)\subset\mathsf{X}$, converging uniformly to g^{X} on $\bar B_R(x)$, such that $\mathsf{Ric}_{(B_R(x),g_k)}\geq-(n-1)\delta-1/k$. We set $r:=R/8$. If k is large enough, geodesics in $(\bar B_R(x),g_k)$ connecting points in $B_r(x)$ are contained in $B_{R/2}(x)$. In particular, $(\bar B_R(x),g_k)$ satisfies $\mathrm{CD}(-(n-1)\delta-1/k,n)$ in $B_r(x)$ by the argument used in the previous case. We now consider $(\bar B_R(x),g^{\mathsf{X}})$.

We denote by V and V_k the volume measures relative to g^{X} and g_k in $\bar B_R(x)$. Fix two probability measures $\mu_1,\mu_2\in\mathcal{P}(B_r(x))$ that are absolutely continuous w.r.t. V (or, equivalently, V_k) and have continuous densities μ_1/V and μ_2/V. Thanks to [37, Corollary 29.22], it is enough to verify the convexity property of the $\mathrm{CD}(-(n-1)\delta, n)$ condition under this extra continuity assumption.

For every k, let $\{\xi_t^k\}_{t\in[0,1]}$ be a Wasserstein geodesic between μ_1 and μ_2 in $(\bar{B}_R(x), g_k)$, and let π_k be an optimal plan between the same measures such that, for every $t \in [0,1]$ and every $N \geq n$, it holds

$$U_N(\xi_t^k|V_k) \leq -T^{(t)}_{-(n-1)\delta-1/k,N}(\pi_k|V_k).$$

By [37, Theorem 28.9], the following holds. There exists a Wasserstein geodesic $\{\xi_t\}_{t\in[0,1]}$ between μ_1 and μ_2 in $(\bar{B}_R(x), g^{\mathsf{X}})$ which arises as limit of $\{\xi_t^k\}_{t\in[0,1]}$. The measures π_k converge weakly to an optimal plan π between μ_1 and μ_2 in $(\bar{B}_R(x), g^{\mathsf{X}})$.

Since $g_k \to g_{\mathsf{X}}$ uniformly on $\bar{B}_R(x)$, we also have the following. The Riemannian distances d_k induced by g_k on $\bar{B}_R(x)\times\bar{B}_R(x)$ converge uniformly to the distance d' induced by g^{X} on $\bar{B}_R(x)\times\bar{B}_R(x)$, which coincides with d on $B_r(x)\times B_r(x)$ by the choice of r. The densities of V_k w.r.t. V converge uniformly to 1 on $\bar{B}_R(x)$.

Hence, for every $t \in [0,1]$ and every $N \geq n$, we have uniform convergence

$$\begin{aligned}
&\tau^t_{-(n-1)\delta-1/k,N} \circ \mathsf{d}_k \to \tau^t_{-(n-1)\delta,N} \circ \mathsf{d} \quad \text{in } B_r(x)\times B_r(x),\\
&(\mu_1/V_k)^{1/N} \to (\mu_1/V)^{1/N} \quad \text{in } B_r(x),\\
&(\mu_2/V_k)^{1/N} \to (\mu_2/V)^{1/N} \quad \text{in } B_r(x).
\end{aligned}$$

Combining this with the weak convergence of π_k to π, and the continuity of μ_1/V and μ_2/V, we deduce

$$T^{(t)}_{-(n-1)\delta-1/k,N}(\pi_k|V_k) \to T^{(t)}_{-(n-1)\delta,N}(\pi|V) \quad \text{for all } t\in[0,1] \text{ and } N \geq n.$$

Hence, by [37, Theorem 29.20], it holds

$$U_N(\xi_t|V) \leq -T^{(t)}_{-(n-1)\delta,N}(\pi|V) \quad \text{for every } t\in[0,1] \text{ and every } N\geq n.$$

Finally, since geodesics in $(\bar{B}_R(x), g^{\mathsf{X}})$ connecting points in $B_r(x)$ are contained in $B_{R/2}(x)$, the Wasserstein geodesic $\{\xi_t\}_{t\in[0,1]}$ in $(\bar{B}_R(x), g^{\mathsf{X}})$ is also a Wasserstein geodesic in X. Hence, for every $x \in \mathsf{X}\setminus S_2$, there exists $r > 0$ such that X satisfies $\mathrm{CD}(-(n-1)\delta, n)$ in $B_r(x)$.

Step 2. We show that, for every $x \in \mathsf{X}\setminus S_2$, there exists $r > 0$ such that geodesics of X contained in $B_r(x)$ do not branch.

If $x \in \mathsf{X}\setminus(S_1\cup S_2)$, there exists a smooth neighbourhood of x, implying the claim. Hence, suppose that $x \in S_1$. Since M and N are smooth manifolds, there exists $k' \in \mathbb{R}$ such that both their sectional curvatures in a neighbourhood of x are bounded below by k'. Hence, using the same arguments of [25] (and using again the condition on the second fundamental forms), we obtain the following. There exist $R > 0$ and a sequence of Riemannian metrics g_k on $\bar{B}_R(x) \subset \mathsf{X}$, converging uniformly to g^{X} on $\bar{B}_R(x)$, such that $\mathsf{Sec}_{(B_R(x),g_k)} \geq k'-1$.

Hence, for k fixed, every point $y \in B_R(x)$ has a neighbourhood $U_y \subset B_R(x)$, depending on k, where triplets of points satisfy the comparison property of Alexandrov spaces with curvature bounded below by $k' - 1$ (w.r.t. the distance induced by g_k in $\bar{B}_R(x)$). This implies, by the proof of Toponogov's Theorem (see [4, Theorem 10.3.1]), the existence of $0 < r \ll R$ (this time independent of k), such that triplets of points in $B_r(x)$ satisfy the comparison property of Alexandrov spaces with curvature bounded below by $k' - 1$ (w.r.t. the distance induced by g_k in $\bar{B}_R(x)$).

It is easy to check that since $g_k \to g^{\mathsf{X}}$ uniformly, then also $(\bar{B}_R(x), g^{\mathsf{X}})$ satisfies the same triangle comparison property for points contained in $B_r(x)$. This implies that geodesics in $B_r(x)$ do not branch (see [5]).

Step 3. We first prove the Laplacian bounds 1 and 2 simultaneously. Then we prove point 3.

Assume that $H_{S_2} \geq \mp\delta$. Since $\mathsf{Ric}_M \geq -(n-1)\delta$, it holds

$$\Delta \mathsf{d}_{S_2} \leq \pm(n-1)\delta$$

in a neighbourhood of S_2 in classical sense (see, for instance, [12]). We now use a globalization technique that was previously used in [28] and [24].

Consider the partition of the transport set of d_{S_2} in transport rays $\{r_\alpha\}_{\alpha \in Q}$, Q being a set of indexes with a measure q induced by the partition. By [8, Theorem 3.4], there exists a disintegration of the measure H^n into measures $\{h_\alpha\}_{\alpha \in Q}$, each concentrated on the corresponding transport ray. By [9, Theorem 4.2] (the proof works in our setting since the space X is non-branching and $\mathrm{CD}(-(n-1)\delta, n)$ around every point $x \in \mathsf{X} \setminus S_2$), q-almost every measure h_α is absolutely continuous w.r.t. the Lebesgue measure of the corresponding transport ray, it admits a locally Lipschitz continuous density (that we still denote by h_α), and it satisfies in weak sense

$$(h_\alpha^{\frac{1}{n-1}})'' - \delta h_\alpha^{\frac{1}{n-1}} \leq 0. \tag{1}$$

By [8, Corollary 4.16] (as before, the proof works in our setting thanks to the previous steps), it holds

$$\Delta \mathsf{d}_{S_2} = (\log h_\alpha)' + [\Delta \mathsf{d}_{S_2}]^{sing} \quad \text{on } \mathsf{X} \setminus S_2$$

in distributional sense, where $[\Delta \mathsf{d}_{S_2}]^{sing}$ is a negative singular measure and the transport rays of d_{S_2} are now oriented from S_2 to $\mathsf{X} \setminus S_2$.

Hence, q-almost every density h_α satisfies $(\log h_\alpha)' \leq \pm(n-1)\delta$ near S_2. Setting

$$f_\alpha := h_\alpha^{\frac{1}{n-1}},$$

we obtain that on the corresponding transport ray r_α, thanks to (1), it holds in weak sense

$$\begin{cases} f_\alpha'' - \delta f_\alpha \leq 0 & \text{on } r_\alpha \\ f_\alpha' \leq \pm\delta f_\alpha & \text{in a neighbourhood of } S_2 \cap r_\alpha \text{ in } r_\alpha. \end{cases}$$

By Riccati Comparison (see, for instance, [24, Lemma 3.9]), f_α then satisfies $(\log f_\alpha)' \leq \pm\delta$. In particular,

$$\Delta \mathsf{d}_{S_2} \leq (\log h_\alpha)' = (n-1)(\log f_\alpha)' \leq \pm(n-1)\delta.$$

We now consider point 3. As before, we consider the partition of the transport set of d_S in transport rays $\{r_\alpha\}_{\alpha\in Q}$, and the disintegration of the measure H^n into measures $\{h_\alpha\}_{\alpha\in Q}$, each concentrated on the corresponding transport ray. By definition of D, the intersection of a transport ray r_α with the D-neighbourhood of S is contained in $\mathsf{X}\setminus S_2$. Hence, the same argument that we used for points 1 and 2 can be repeated. □

The next lemma is standard and it follows from [12] and [23]. We sketch a proof for the sake of completeness.

Lemma 2. *Let $\delta \in \{0,1\}$ and let (M^n, g) be a Riemannian manifold with $\mathsf{Ric}_M \geq -(n-1)\delta$. Let S_1 be a union of connected components of ∂M and let $S_2 := \partial M \setminus S_1 \neq \emptyset$. Let $H_{S_1} \geq \delta$, $H_{S_2} \geq -\delta$ and let $\mathsf{d}_{S_1} + \mathsf{d}_{S_2}$ be constant. Then, there exists $l > 0$ such that M is isometric to $S_1 \times [0, l]$ with the warped product metric $ds^2 = e^{-2\delta t} g_1 + dt^2$, where g_1 is the metric on S_1.*

Proof. Since $\mathsf{d}_{S_1} + \mathsf{d}_{S_2} \equiv D$ is constant, every point of M lies on a unique geodesic realizing the distance from S_1 to S_2. In particular, d_{S_1} and d_{S_2} are both smooth and they satisfy

$$\Delta \mathsf{d}_{S_1} \leq -(n-1)\delta \quad \text{and} \quad \Delta \mathsf{d}_{S_2} \leq (n-1)\delta.$$

Hence, $\Delta \mathsf{d}_{S_1} = -(n-1)\delta$. Using $t = \mathsf{d}_{S_1}$ as a coordinate on M, we can write $g = dt^2 + g^\perp(t)$, where $g^\perp(t)$ is a metric on the t enlargement of S_1 in M.

Using Cauchy-Schwarz inequality first, and then plugging d_{S_1} into the Bochner Formula, we obtain

$$(n-1)\delta = \frac{(\Delta \mathsf{d}_{S_1})^2}{n-1} \leq |\mathsf{Hess}(\mathsf{d}_{S_1})|^2 = -\mathsf{Ric}_M(\nabla \mathsf{d}_{S_1}, \nabla \mathsf{d}_{S_1}) \leq (n-1)\delta.$$

In particular, all inequalities in the previous expression are equalities, forcing $\mathsf{Hess}(\mathsf{d}_{S_1}) = -\delta g^\perp$ everywhere on M. Denoting by $L_{\partial t}$ the Lie derivative in the direction of $\nabla \mathsf{d}_{S_1}$, the previous equality reads

$$L_{\partial t}\, g^\perp = -2\delta g^\perp. \tag{2}$$

Consider now the map $\phi : S_1 \times (0, D) \to M\setminus(S_1 \cup S_2)$ which sends (x, t) to the point at distance t from x along the flow line of $\nabla \mathsf{d}_{S_1}$. This map is bijective since every point of M lies on a unique geodesic realizing the distance from S_1 to S_2. Equipping $S_1 \times (0, D)$ with the metric $ds^2 = dt^2 + e^{-2\delta} g_1$, we obtain that ϕ is an isometry by (2) and the definition of Lie derivative, concluding the proof. □

We now prove Theorems 1 and 2. We recall that a manifold (M, g) without boundary is said to be *parabolic* if it has no positive fundamental solution for the Laplacian. This is equivalent to asking that for every $x \in M$ and $r > 0$, it holds

$$\inf \left\{ \int_M |\nabla \phi|^2 \, d\mathsf{Vol} : \phi \in \mathsf{Lip}_c(M) : \phi \equiv 1 \text{ on } B_r(x) \right\} = 0.$$

If $\mathsf{Ric}_M \geq 0$, parabolicity is equivalent to requiring that

$$\int_1^{+\infty} \frac{t}{\mathsf{Vol}(B_t(x))} \, dt = +\infty,$$

for some $x \in M$. For the proofs of these facts, we refer to [20] and the monograph [26].

Proof of Theorem 1. We first introduce some tools. Consider the manifold $S_1 \times (0, +\infty)$ equipped with the warped product metric $\tilde{g} = e^{2\delta t} g_1 + dt^2$. We denote by Vol the Riemannian volume in $S_1 \times (0, +\infty)$ w.r.t. the metric $\tilde{g}$ and by Vol_{S_1} the Riemannian volume on S_1 w.r.t. the metric g_1.

Let $\mathsf{d}_{S_1 \times \{0\}}$ be the distance from $S_1 \times \{0\}$ in $S_1 \times (0, +\infty)$ and let d_{S_1} be the distance from S_1 in M. By standard computations (see [31]), it holds

$$\Delta \mathsf{d}_{S_1 \times \{0\}} = (n-1)\delta, \quad \text{on } S_1 \times (0, +\infty).$$

Similarly, the second fundamental form of $S_1 \times \{0\}$ in $S_1 \times (0, +\infty)$ satisfies $\Pi_{S_1 \times \{0\}} = -\delta$. Moreover, $\mathsf{Ric}_{S_1} \geq 0$ together with Gauss' equation implies that $\mathsf{Ric}_{S_1 \times (0, +\infty)} \geq -(n-1)\delta$.

Consider the metric space obtained by glueing M and $S_1 \times (0, +\infty)$ along the two boundaries isometric to S_1. On this metric space, we consider the distance function from S_2 and we denote it by d_{S_2}. In particular, this allows us to define d_{S_2} on $S_1 \times (0, +\infty)$ (seen as a subset of the previously defined metric space). We now show that the function $\bar{\mathsf{d}} := \mathsf{d}_{S_2} - \mathsf{d}_{S_1 \times \{0\}}$ is constant on $S_1 \times (0, +\infty)$. This implies that $\mathsf{d}_{S_2} + \mathsf{d}_{S_1}$ is constant on M, so that the statement follows by Lemma 2.

By Lemma 1, it holds $\Delta \mathsf{d}_{S_2} \leq (n-1)\delta$ in distributional sense on $S_1 \times (0, +\infty)$. Hence,

$$\Delta(\mathsf{d}_{S_2} - \mathsf{d}_{S_1 \times \{0\}}) \leq 0 \quad \text{on } S_1 \times (0, +\infty) \text{ in distributional sense.} \tag{3}$$

Consider the smooth function $V \in C^\infty(S_1 \times (0, +\infty))$ given by $V = n^\delta \mathsf{d}_{S_1 \times \{0\}}$. We denote by $\Delta_w \bar{\mathsf{d}}$ the weighted Laplacian of $\bar{\mathsf{d}}$ w.r.t. V, i.e. the measure $\Delta_w \bar{\mathsf{d}} := \Delta \bar{\mathsf{d}} - \nabla V \cdot \nabla \bar{\mathsf{d}}$. Thanks to (3) and the definition of V, it holds

$$\Delta_w \bar{\mathsf{d}} \leq 0 \quad \text{on } S_1 \times (0, +\infty) \text{ in distributional sense.} \tag{4}$$

Let $\phi \in \mathsf{Lip}_c(S_1 \times [0, +\infty))$ be a positive function. Consider the functions $\phi_\epsilon \in \mathsf{Lip}_c(S_1 \times [0, +\infty))$ defined by $\phi_\epsilon(x, t) := (1 - \epsilon^{-1} t) \vee 0$. Integrating by parts, it holds

$$\begin{aligned}
&\int_{S_1\times(0,+\infty)} (\nabla \bar{\mathsf{d}}\cdot\nabla\phi)\, e^{-V} d\mathsf{Vol}\\
&= \int_{S_1\times(0,+\infty)} [\nabla \bar{\mathsf{d}}\cdot\nabla(\phi\phi_\epsilon)]\, e^{-V} d\mathsf{Vol} + \int_{S_1\times(0,+\infty)} [\nabla \bar{\mathsf{d}}\cdot\nabla(\phi(1-\phi_\epsilon))]\, e^{-V} d\mathsf{Vol}\\
&= \int_{S_1\times(0,+\infty)} [\nabla \bar{\mathsf{d}}\cdot(\phi\nabla\phi_\epsilon+\phi_\epsilon\nabla\phi)]\, e^{-V} d\mathsf{Vol} - \int_{S_1\times(0,+\infty)} \phi(1-\phi_\epsilon)\, e^{-V} d\Delta_w \bar{\mathsf{d}}.
\end{aligned}$$

Since $\nabla \bar{\mathsf{d}}\cdot\nabla\phi_\epsilon \geq 0$ in $S_1\times(0,+\infty)$, passing to the limit in the previous equation as $\epsilon\to 0$ and taking into account (4), it holds

$$\int_{S_1\times(0,+\infty)} (\nabla \bar{\mathsf{d}}\cdot\nabla\phi)\, e^{-V} d\mathsf{Vol} \geq 0. \tag{5}$$

Let $\psi \in \mathsf{Lip}_c(S_1\times[0,+\infty))$ be a positive function, and set $v := (1+\bar{\mathsf{d}})^{-1}$. We repeat a computation from [38] (see also [26, Lemma 7.1]). By (5), it holds

$$\begin{aligned}
0 &\geq \int_{S_1\times(0,+\infty)} [\nabla(\psi^2 v)\cdot\nabla v]\, e^{-V} d\mathsf{Vol}\\
&= 2\int_{S_1\times(0,+\infty)} \psi v(\nabla\psi\cdot\nabla v)\, e^{-V} d\mathsf{Vol} + \int_{S_1\times(0,+\infty)} \psi^2|\nabla v|^2\, e^{-V} d\mathsf{Vol},
\end{aligned}$$

so that

$$\int_{S_1\times(0,+\infty)} \psi^2|\nabla v|^2\, e^{-V} d\mathsf{Vol} \leq 2\int_{S_1\times(0,+\infty)} |\psi v\nabla\psi\cdot\nabla v|\, e^{-V} d\mathsf{Vol}. \tag{6}$$

By Young's inequality, we then obtain

$$\begin{aligned}
2\int_{S_1\times(0,+\infty)} |\psi v\nabla\psi\cdot\nabla v|\, e^{-V} d\mathsf{Vol} &\leq \frac{1}{2}\int_{S_1\times(0,+\infty)} \psi^2|\nabla v|^2\, e^{-V} d\mathsf{Vol}\\
&\quad + 8\int_{S_1\times(0,+\infty)} |\nabla\psi|^2 v^2\, e^{-V} d\mathsf{Vol},
\end{aligned}$$

which, combining with (6), gives

$$\int_{S_1\times(0,+\infty)} \psi^2|\nabla v|^2\, e^{-V} d\mathsf{Vol} \leq 16\int_{S_1\times(0,+\infty)} |\nabla\psi|^2 v^2\, e^{-V} d\mathsf{Vol}.$$

Recalling the definition of v, this reads

$$\int_{S_1\times(0,+\infty)} \psi^2\frac{|\nabla\bar{\mathsf{d}}|^2}{(1+\bar{\mathsf{d}})^2}\, e^{-V} d\mathsf{Vol} \leq 16\int_{S_1\times(0,+\infty)} |\nabla\psi|^2\, e^{-V} d\mathsf{Vol}. \tag{7}$$

Let $x\in S_1$ and $r>0$. Let $\phi_1 \in \mathsf{Lip}_c(S_1)$ be such that $\phi\equiv 1$ in $B_r(x)\subset S_1$ and $\phi_2\in\mathsf{Lip}_c([0,+\infty)$ be such that $\phi_2\equiv 1$ in $[0,r)$, both taking values in $[0,1]$. Let $\psi\in\mathsf{Lip}_c(S_1\times[0,+\infty))$ be defined by $\psi(y,t) := \phi_1(y)\phi_2(t)$. We denote by $\nabla^t\phi_1$ the gradient of ϕ_1 w.r.t. the metric $e^{2\delta t}g_1$ on S_1 and we note that $\nabla^t\phi_1 = e^{-\delta t}\nabla^0\phi_1$. In particular, it holds

$$|\nabla\psi|^2(y,t) \leq |\nabla^0\phi_1|^2(y) + \phi_1^2(y)(\phi_2'(t))^2.$$

Hence, with this choice of ψ, the r.h.s. in (7) is bounded from above by

$$\begin{aligned}(8)\qquad & 16\int_0^{+\infty} e^{-n^\delta t+\delta(n-1)t}\,dt\int_{S_1}|\nabla^0\phi_1|^2\,d\mathsf{Vol}_{S_1}\\ &+16\int_{S_1}\phi_1^2\,d\mathsf{Vol}_{S_1}\int_0^{+\infty}|\phi_2'|^2e^{-n^\delta t+\delta(n-1)t}\,dt.\end{aligned}$$

Taking an infimum over all the admissible $\phi_2 \in \mathsf{Lip}_c([0,+\infty))$ such that $\phi_2 \equiv 1$ on $[0,r)$, we deduce

$$(9)\qquad \int_{B_r(x)\times[0,r]}\frac{|\nabla\bar{\mathsf{d}}|^2}{(1+\bar{\mathsf{d}})^2}\,e^{-V}d\mathsf{Vol}\le 16\int_0^{+\infty}e^{-t}\,dt\int_{S_1}|\nabla^0\phi_1|^2\,d\mathsf{Vol}_{S_1}.$$

Since S_1 is parabolic, taking the infimum of the r.h.s. over all the functions $\phi_1 \in \mathsf{Lip}_c(S_1)$ such that $\phi_1 \equiv 1$ on $B_r(x) \subset S_1$, we deduce $|\nabla\bar{\mathsf{d}}| \equiv 0$ on $B_r(x) \times [0, r]$. The constancy of $\bar{\mathsf{d}}$ follows from the arbitrariness of $r > 0$, concluding the proof. □

In the previous proof, the weight V was needed to compensate for the excessive volume growth of the space $S_1 \times (0, +\infty)$ equipped with

$$ds^2 = e^{2\delta t}g_1 + dt^2.$$

In the proof of Theorem 2, we consider $S_2 \times (0, +\infty)$ equipped with

$$ds^2 = e^{-2t}g_2 + dt^2,$$

g_2 being the metric on S_2. Hence, the volume growth will be easier to control even without a weight.

Nevertheless, given $f \in C^\infty(S_2)$, its gradients w.r.t. $e^{-2t}g_2$ increases as t increases. Hence, also in the proof of Theorem 2, a weight is needed, this time to compensate for the previously described gradient growth.

Proof of Theorem 2. We argue as in Theorem 1. Let

$$S := S_2 \cup S_3.$$

Consider the manifold $S\times(0, +\infty)$ equipped with the warped product metric

$$\tilde{g} = e^{-2t}g_S + dt^2,$$

where g_S is the metric induced by M on S. By standard computations, the second fundamental form of $S \times \{0\}$ in $S \times (0, +\infty)$ satisfies $\Pi_{S\times\{0\}} = 1$. Moreover, since $\mathsf{Ric}_S \ge 0$, Gauss' equations imply that

$$\mathsf{Ric}_{S\times(0,+\infty)} \ge -(n-1).$$

In a similar fashion, we denote by g_2 the restriction of g_S to S_2 and we consider $S_2 \times (0, +\infty)$, which is a connected component of $S \times (0, +\infty)$. We denote by Vol the Riemannian volume in $S_2 \times (0, +\infty)$ w.r.t. the metric $\tilde{g}$ and by Vol_{S_2} the Riemannian volume on S_2 w.r.t. the metric g_2. Let $\mathsf{d}_{S_2\times\{0\}}$ be the distance from $S_2 \times \{0\}$ in $S_2 \times (0, +\infty)$. It holds

$$\Delta\mathsf{d}_{S_2\times\{0\}} = -(n-1) \quad \text{on } S_2 \times (0, +\infty).$$

Consider the metric space (X,d) obtained by gluing M and $S\times(0,+\infty)$ along the two boundaries isometric to S. On this metric space, we consider the distance function from S_1 and we denote it by d_{S_1}. In particular, this allows us to define d_{S_1} on $S_2\times(0,+\infty)$ (seen as a subset of the previously defined metric space). We first show that the function

$$\bar{\mathsf{d}}:=\mathsf{d}_{S_1}-\mathsf{d}_{S_2\times\{0\}}$$

is constant on $S_2\times(0,+\infty)$.

By Lemma 1, it holds $\Delta\mathsf{d}_{S_1}\leq-(n-1)$ in distributional sense on $S_2\times(0,+\infty)$. Hence,

$$\Delta(\mathsf{d}_{S_1}-\mathsf{d}_{S_2\times\{0\}})\leq 0 \quad \text{on } S_2\times(0,+\infty) \text{ in distributional sense.} \tag{10}$$

Consider the smooth function $V\in C^\infty(S_2\times(0,+\infty))$ given by

$$V=2\mathsf{d}_{S_2\times\{0\}}.$$

Let $\phi\in\mathsf{Lip}_c(S_2\times[0,+\infty))$ be a positive function. Arguing as in Theorem 1 (derivation of (5)), one obtains

$$\int_{S_2\times(0,+\infty)}(\nabla\bar{\mathsf{d}}\cdot\nabla\phi)\,e^{-V}d\mathsf{Vol}\geq 0. \tag{11}$$

If $\psi\in\mathsf{Lip}_c(S_2\times[0,+\infty))$ is a positive function, arguing again as in Theorem 1 (derivation of (9)), one has

$$\int_{S_2\times(0,+\infty)}\psi^2\frac{|\nabla\bar{\mathsf{d}}|^2}{(1+\bar{\mathsf{d}})^2}\,e^{-V}d\mathsf{Vol}\leq 16\int_{S_2\times(0,+\infty)}|\nabla\psi|^2\,e^{-V}d\mathsf{Vol}. \tag{12}$$

Let $x\in S_2$ and $r>0$. Let $\phi_1\in\mathsf{Lip}_c(S_2)$ be such that $\phi\equiv1$ in $B_r(x)$ and $\phi_2\in\mathsf{Lip}_c([0,+\infty)$ be such that $\phi_2\equiv1$ in $[0,r)$, both taking values in $[0,1]$. Let $\psi\in\mathsf{Lip}_c(S_2\times[0,+\infty))$ be defined by $\psi(y,t):=\phi_1(y)\phi_2(t)$. We denote by $\nabla^t\phi_1$ the gradient of ϕ_1 w.r.t. the metric $e^{-2t}g_2$ on S_2 and we note that $\nabla^t\phi_1=e^t\nabla^0\phi_1$. In particular, it holds

$$|\nabla\psi|^2(y,t)\leq e^{2t}|\nabla^0\phi_1|^2(y)+\phi_1^2(y)(\phi_2'(t))^2.$$

With this choice of ψ, the r.h.s. in (12) is bounded from above by

$$\begin{aligned}&16\int_0^{+\infty}e^{-(n-1)t}\,dt\int_{S_2}|\nabla^0\phi_1|^2\,d\mathsf{Vol}_{S_2}\\&\quad+16\int_{S_2}\phi_1^2\,d\mathsf{Vol}_{S_2}\int_0^{+\infty}(\phi_2')^2e^{-(n-1)t}\,dt.\end{aligned} \tag{13}$$

We now conclude that $\bar{\mathsf{d}}$ is constant on $S_2\times(0,+\infty)$ as in Theorem 1. Hence, d_{S_1} is constant on S_2 with constant value $D>0$.

Let now d_{S_2} be the distance from S_2 in $\mathsf{X}\setminus S_2\times[0,+\infty)$. By Lemma 1, it holds $\Delta(\mathsf{d}_{S_1}+\mathsf{d}_{S_2})\leq0$ in the D-neighbourhood of S_2 in $\mathsf{X}\setminus S_2\times[0,+\infty)$. We claim that the set

$$C:=\{x\in\mathsf{X}\setminus(S_1\cup S_2\times[0,+\infty)):\mathsf{d}_{S_1}+\mathsf{d}_{S_2}=D\}$$

is open and closed in $\mathsf{X} \setminus (S_1 \cup S_2 \times [0, +\infty))$. Closedness is trivial. Let $p \in C$. We show that there exists a neighbourhood of p in X contained in C. The set C is contained in the D-neighbourhood of S_2, while $\mathsf{d}_{S_1} + \mathsf{d}_{S_2} \geq D$ on X. Hence, by the maximum principle, $\mathsf{d}_{S_1} + \mathsf{d}_{S_2} = D$ in a neighbourhood of p. Since C is open and closed in $\mathsf{X} \setminus (S_1 \cup S_2 \times [0, +\infty))$, it coincides with $\mathsf{X} \setminus (S_1 \cup S_2 \times [0, +\infty))$ itself. Hence, $S_3 = \emptyset$, since otherwise $\mathsf{d}_{S_1} + \mathsf{d}_{S_2}$ would attain arbitrarily large values on $S_3 \times (0, +\infty) \subset \mathsf{X} \setminus (S_1 \cup S_2 \times [0, +\infty))$. The conclusion then follows from Lemma 2. □

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