REGULARITY PROPERTIES OF EQUILIBRIUM CONFIGURATIONS OF EPITAXIALLY STRAINED ELASTIC FILMS

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Abstract. We consider a variational model introduced in the physical literature to describe the epitaxial growth of an elastic film over a rigid substrate, when a lattice mismatch between the two materials is present. We establish the regularity of volume constrained local minimizers of the total energy, proving in particular the so called zero contact-angle condition between the film and the substrate.

1. Introduction

In this paper we present some regularity results for equilibrium configurations of a variational energy modeling the epitaxial deposition of a film onto a rigid substrate in presence of a mismatch strain between the lattice parameters of the two crystalline solids.

The presence of such a strain is responsible of the so called Asaro-Grinfeld-Tiller instability of the flat configuration: to release some of the elastic energy due to the strain the atoms on the free surface of the film tend to rearrange into a more favorable configuration. Thus, the film profile becomes wavy or breaks into several material clusters, also known as *islands*, separated by a thin layer that wets the substrate. However, this phenomenon occurs only when the thickness of the film reaches a critical threshold. We refer to [2], [8] for a detailed account on this effect.

Among the several atomistic and continuum models available for the growth of epitaxially strained thin films, we follow here a variational approach contained in [14] (see also [12], [15]), which was first studied from an analytical point of view by Bonnetier and Chambolle in [3]. As in this paper we restrict to two-dimensional morphologies, corresponding to three-dimensional configurations with planar symmetry.

To describe the model studied in [3] we start by introducing the reference configuration of the film

$$\Omega_h := \left\{ z = (x,y) \in \mathbb{R}^2 : \, 0 < x < b, \, 0 < y < h\,(x) \right\} \,,$$

where $h : [0,b] \to [0,\infty)$ represents the free-profile of the film. Following the physical literature, the function h will be assumed to be b-periodic, i.e., $h(0) = h(b)$. Denoting by $u : \Omega_h \to \mathbb{R}^2$ the planar displacement of the film, we shall assume that $u(x,0) = e_0(x,0)$ at the interface between the film and the substrate, where the constant e_0 depends on the gap between the lattices of the two materials. In view of the periodicity assumption on h, we shall also impose the periodicity condition $u(b,y) = u(0,y) + e_0(b,0)$.

For a smooth configuration (h,u) the energy considered in [3] is given by

$$G(h,u) = \int_{\Omega_h} \left[\mu |E(u)|^2 + \frac{\lambda}{2} (\mathrm{div} u)^2 \right] dz + \sigma \mathcal{H}^1(\Gamma_h)\,, \tag{1.1}$$

where μ and λ represent the Lamé coefficients of the material, $E(u) := (\nabla u + \nabla^T u)/2$ is the linearized elastic strain, σ is the surface tension on the profile of the film, Γ_h is the graph of h, and $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure.

The equilibrium configurations are defined as the minimizers of the above energy under a volume constraint $|\Omega_h| = d$. However, as shown in [3], the minimizing sequences of the functional G may converge to more irregular profiles h, which are only lower semicontinuous functions of bounded variation. Denoting by $AP(0,b)$ (see (2.1)) the class of such admissible profiles, a standard relaxation procedure leads to the following representation of the energy associated to a configuration (h,u), where $h \in AP(0,b)$, $E(u) \in L^2(\Omega; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$, and u satisfies the Dirichlet and periodicity conditions stated above,

$$F(h,u) = \int_{\Omega_h} \Big[\mu|E(u)|^2 + \frac{\lambda}{2}(\mathrm{div} u)^2\Big]\, dz + \sigma\mathcal{H}^1(\Gamma_h) + 2\sigma\mathcal{H}^1(\Sigma_h)\,.$$

Here Γ_h denotes the extended graph of h and Σ_h is the union of all vertical cuts, whose length is counted twice, since they arise as limits of regular profiles. The existence of minimizers for F under the volume constraint $|\Omega_h| = d$ follows immediately as a result of its definition via relaxation.

In this paper we address the regularity of global and local constrained minimizers of F. Our approach follows very closely the one introduced by Fonseca, Leoni, Morini and the second author in [6] (see also [5] for a related problem), where a slightly different model and a stronger notion of local minimality were considered. As in [6] we show that the profile of a constrained local minimizer of F may have at most finitely many cut segments or singularities of cusp type, being C^1-regular away from these singular points (see Theorem 2.5 below). In particular, we show that the so called *zero contact-angle condition* also holds for the model studied here.

The notion of local minimality considered in this paper was introduced in [7] in connection with a thorough study of the local and global minimality properties of the flat configuration. As we have already mentioned, this notion is weaker than the one considered in [6]. As a consequence, the starting points of the regularity analysis, that is the equivalence between local minimality for the constrained problem and the penalized one and the interior ball condition, are more delicate to prove (see Proposition 3.1 and Theorem 3.4). Also the proof of the C^1-regularity at the interface between the film and the substrate presents additional technical difficulties because of the coupling of the Dirichlet condition on $\{y = 0\}$ and the Neumann condition on Γ_h that must be satisfied by the displacement u.

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2. Statement of the existence and regularity results

In this section we present the model studied by Bonnetier and Chambolle in [3] and the related functional setting. We also state the regularity theorem.

Throughout the paper we denote by $z = (x,y)$ the generic point in $\mathbb{R}^2$ and by $B_r(z)$ the open disc centered in z with radius r. Given two sets $A, B \subset \mathbb{R}^2$, their *Hausdorff distance* is defined as

$$d_{\mathcal{H}}(A,B) := \inf\{\varepsilon > 0 :\ A \subset \mathcal{N}_\varepsilon(B) \ \text{ and } \ B \subset \mathcal{N}_\varepsilon(A)\}\ ,$$

where $\mathcal{N}_\varepsilon(A)$ denotes the ε-neighborhood of A.

In the following we are going to consider lower semicontinuous (l.s.c.) periodic profiles. To this aim we introduce the class of admissible profiles

$$AP(0,b) := \big\{g : \mathbb{R} \to [0,+\infty) :\ g \text{ is l.s.c. and } b\text{-periodic, } \mathrm{Var}(g;0,b) < \infty\big\}\,, \tag{2.1}$$

where $\mathrm{Var}(g;0,b)$ denotes the *pointwise total variation* of g over the interval $(0,b)$

$$\mathrm{Var}(g;0,b)=\sup\Big\{\sum_{i=1}^{n}|g(x_i)-g(x_{i-1})|\,:\,0\le x_0<x_1<\dots x_n\le b\Big\}.$$

Since $g\in AP(0,b)$ is b-periodic, its pointwise total variation is finite over any bounded interval of $\mathbb{R}$. Therefore, it admits right and left limits at every $x\in\mathbb{R}$ denoted by $g(x+)$ and $g(x-)$, respectively. In the following we use the notation

$$g^+(x):=\max\{g(x+),g(x-)\}\,,\qquad g^-(x):=\min\{g(x+),g(x-)\}\,. \tag{2.2}$$

To represent the region occupied by the film, we set

$$\Omega_g:=\{(x,y):x\in(0,b),\,0<y<g(x)\},\quad \Omega_g^{\#}:=\{(x,y):x\in\mathbb{R},\,0<y<g(x)\}\,,$$

while the profile of the film is given by

$$\Gamma_g:=\{(x,y):x\in[0,b),\,g^-(x)\le y\le g^+(x)\}\,.$$

The set of vertical cracks (or cuts) is

$$\Sigma_g:=\{(x,y):x\in[0,b)\,,\,g(x)<g^-(x),\,g(x)\le y<g^-(x)\}\,.$$

We will also use the notation $\Gamma_g^{\#}:=\{(x,y)\in\mathbb{R}^2\,:\,x\in\mathbb{R},\,g^-(x)\le y\le g^+(x)\}$. The set $\Sigma_g^{\#}$ is defined similarly. Finally, we set $\widetilde{\Gamma}_g=\Gamma_g\cup\Sigma_g$, $\widetilde{\Gamma}_g^{\#}=\Gamma_g^{\#}\cup\Sigma_g^{\#}$.

Given $g\in AP(0,b)$, we denote

$$LD_{\#}(\Omega_g;\mathbb{R}^2):=\big\{v\in L^2_{\mathrm{loc}}(\Omega_g^{\#};\mathbb{R}^2)\,:\,v(x,y)=v(x+b,y)\text{ for }(x,y)\in\Omega_g^{\#},\ E(v)|_{\Omega_g}\in L^2(\Omega_g;\mathbb{M}^{2\times2}_{\mathrm{sym}})\big\},$$

where $E(v):=\frac{1}{2}(\nabla v+\nabla^T v)$, ∇v being the distributional gradient of v and $\nabla^T v$ its transpose. Given $e_0\neq 0$, we define

$$X(e_0;b):=\big\{(g,v)\,:\,g\in AP(0,b),\,v:\Omega_g^{\#}\to\mathbb{R}^2\text{ such that }v(\cdot,\cdot)-e_0(\cdot,0)\in LD_{\#}(\Omega_g;\mathbb{R}^2),$$
$$v(x,0)=(e_0x,0)\text{ for all }x\in\mathbb{R}\big\}\,.$$

We introduce the following convergence in $X(e_0;b)$.

Definition 2.1. *We say that $(h_n,u_n)\to(h,u)$ in $X(e_0;b)$ if the following two conditions hold:*

(i) $$\sup_n \mathrm{Var}(h_n;0,b)<+\infty\qquad\textit{and}\qquad d_{\mathcal{H}}(\mathbb{R}^2_+\setminus\Omega^{\#}_{h_n},\mathbb{R}^2_+\setminus\Omega^{\#}_h)\to 0\,,$$
where $\mathbb{R}^2_+:=\{(x,y)\in\mathbb{R}^2\,:\,y\ge 0\}$;

(ii) $$u_n\rightharpoonup u\ \textit{weakly in}\ W^{1,2}_{\mathrm{loc}}(\Omega_h^{\#};\mathbb{R}^2).$$

Notice that the definition is well posed since by (i) it follows that if $\mathrm{dist}(\Omega',\mathbb{R}^2_+\setminus\Omega_h^{\#})>0$ then $\mathrm{dist}(\Omega',\mathbb{R}^2_+\setminus\Omega^{\#}_{h_n})>0$ for n large enough. The notion of convergence just introduced is motivated by the following compactness result (see [3], [6]).

Theorem 2.2. *Let $(h_n,u_n)\in X(e_0;b)$ be such that*

$$\sup_n\Big\{\int_{\Omega_{h_n}}|E(u_n)|^2+\mathrm{Var}(h_n;0,b)+|\Omega_{h_n}|\Big\}<\infty\,.$$

Then there exist $(h,u)\in X(e_0;b)$ and a subsequence (h_{n_k},u_{n_k}) such that $(h_{n_k},u_{n_k})\to(h,u)$ in $X(e_0;b)$.

We work in the framework of linearized elasticity and we consider isotropic and homogeneous materials. Hence, the elastic energy density $W : \mathbb{M}^{2\times 2}_{\rm sym} \to [0,+\infty)$ takes the form

$$W(\xi) := \frac{1}{2}\mathbb{C}\xi : \xi = \mu|\xi|^2 + \frac{\lambda}{2}\big[\mathrm{tr}(\xi)\big]^2,$$

where

$$\mathbb{C}\xi = \begin{pmatrix} (2\mu+\lambda)\xi_{11} + \lambda\xi_{22} & 2\mu\xi_{12} \\ 2\mu\xi_{12} & (2\mu+\lambda)\xi_{22} + \lambda\xi_{11} \end{pmatrix}$$

and the *Lamé coefficients* μ and λ satisfy the ellipticity conditions

$$\mu > 0 \qquad \text{and} \qquad \lambda > -\mu\,. \tag{2.3}$$

Note that

$$W(\xi) \geq \min\{\mu, \mu+\lambda\}|\xi|^2 \qquad\qquad \text{for all } \xi \in \mathbb{M}^{2\times 2}_{\rm sym} \tag{2.4}$$

and thus condition (2.3) guarantees that W is coercive. If $(g,v) \in X(e_0; b)$ and g is Lipschitz, the total free energy is defined as

$$G(g,v) := \int_{\Omega_g} W(E(v))\,dz + \mathcal{H}^1(\Gamma_g)\,,$$

where we have set $\sigma = 1$ in (1.1). The following result, proved in [3] (see also [6]), gives a representation formula for the energy in the general case and an existence result for the corresponding constrained minimum problem. To this aim, we set for any $(g,v) \in X(e_0; b)$

$$F(g,v) := \inf\big\{\liminf_{n\to\infty} G(g_n, v_n) : (g_n, v_n) \to (g,v) \text{ in } X(e_0;b),\ g_n \text{ Lipschitz},\ |\Omega_{g_n}| = |\Omega_g|\big\}\,.$$

Theorem 2.3. *For any pair* $(g,v) \in X(e_0; b)$

$$F(g,v) = \int_{\Omega_g} W(E(v))\,dz + \mathcal{H}^1(\Gamma_g) + 2\,\mathcal{H}^1(\Sigma_g)\,.$$

Moreover, for any $d > 0$ *the minimum problem*

$$\min\{F(g,v) : (g,v) \in X(e_0;b),\ |\Omega_g| = d\} \tag{2.5}$$

has a solution, the minimum value in (2.5) *is equal to*

$$\inf\{G(g,v) : (g,v) \in X(e_0;b),\ |\Omega_g| = d,\ g \textit{ Lipschitz}\}$$

and the limit points of minimizing sequences are minimizers of (2.5).

Our regularity result applies not only to *global minimizer*, i.e. minimizers of (2.5), but also to *local minimizers*, which are defined as follows.

Definition 2.4. *We say that an admissible pair* $(h,u) \in X(e_0;b)$ *is a* local minimizer *for* F *if there exists* $\delta > 0$ *such that*

$$F(h,u) \leq F(g,v)$$

for all pairs $(g,v) \in X(e_0;b)$, *with* $|\Omega_g| = |\Omega_h|$ *and* $d_{\mathcal{H}}(\widetilde{\Gamma}^{\#}_h, \widetilde{\Gamma}^{\#}_g) < \delta$.

Notice that a (sufficiently regular) local minimizer $(h,u) \in X(e_0;b)$ satisfies the following set of Euler-Lagrange conditions:

$$\begin{cases} \mathrm{div}\,\mathbb{C}E(u) = 0 & \text{in } \Omega_h; \\ \mathbb{C}E(u)[\nu] = 0 & \text{on } \Gamma_h \cap \{y > 0\}; \\ \mathbb{C}E(u)(0,y)[\nu] = -\mathbb{C}E(u)(b,y)[\nu] & \text{for } 0 < y < h(0) = h(b); \\ k + W(E(u)) = \text{const} & \text{on } \Gamma_h \cap \{y > 0\}, \end{cases} \tag{2.6}$$

where ν denotes the outer unit normal to Ω_h and k is the curvature of Γ_h. Due to (2.4), equation $(2.6)_1$ is a linear elliptic system satisfying the Legendre-Hadamard condition.

Before stating the regularity result, we need to introduce the set of *cusp points* of a function $g \in AP(0,b)$

$$\Sigma_{g,c} := \{(x, g(x)) : x \in [0,b), g^-(x) = g(x), \text{ and } g'_+(x) = -g'_-(x) = +\infty\},$$

where g^- is defined in (2.2), and g'_+ and g'_- denote the right and left derivatives, respectively. As before, the set $\Sigma^\#_{g,c}$ is obtained by replacing $[0,b)$ by $\mathbb{R}$ in the previous formula and coincides with the b-periodic extension of $\Sigma_{g,c}$.

Theorem 2.5 (Regularity of local minimizers). *Let $(h,u) \in X(e_0;b)$ be a local minimizer for F. Then the following regularity results hold:*

(i) *there are at most finitely many cusp points and vertical cracks in* $[0,b)$*;*
(ii) *the curve $\Gamma^\#_h$ is of class C^1 away from $\Sigma^\#_h \cup \Sigma^\#_{h,c}$;*
(iii) *$\Gamma^\#_h \cap \{(x,y) : y > 0\}$ is of class $C^{1,\alpha}$ away from $\Sigma^\#_h \cup \Sigma^\#_{h,c}$ for all $\alpha \in (0,1/2)$;*
(iv) *let $A := \{x \in \mathbb{R} : h(x) > 0$ and h is continuous at $x\}$. Then A is an open set of full measure in $\{h > 0\}$ and h is analytic in A.*

Statement (ii) of Theorem 2.5 implies in particular that the zero contact-angle condition between film and substrate holds. On the other hand, if $h > 0$ is of class $C^{1,\alpha}$ for all $\alpha \in (0,1/2)$, and if $(h,u) \in X(e_0;b)$ satisfies the first three equations in (2.6), then classical elliptic regularity results (see [7, Proposition 8.9]) imply that $u \in C^{1,\alpha}(\overline{\Omega}_h)$ for all $\alpha \in (0,1/2)$. Moreover, if also $(2.6)_4$ holds in the distributional sense, then the results contained in [11, Subsection 4.2] imply that (h,u) is analytic.

3. The interior ball condition

The proof of Theorem 2.5 is quite long and we shall follow the path set in [6]. When dealing with the parts of $\widetilde{\Gamma}^\#_h$ above the x-axis, our argument is a bit simpler than the one followed in [6]. On the other hand, the proof of the zero contact-angle condition for the model considered here requires some new ideas and extra care.

The first step consists in removing the constraint $|\Omega_h| = d$ by showing that if (h,u) is a local minimizer, then (h,u) is also a local minimizer of the penalized functional

$$(g,v) \in X(e_0;b) \mapsto F(g,v) + \Lambda\big||\Omega_g| - d\big|,$$

for some $\Lambda > 0$ sufficiently large, depending also on u. This gives a much larger choice of variations and in particular allows us to prove that $\Omega^\#_h$ satisfies a uniform interior ball condition, namely that if $0 < \varrho < 1/\Lambda$ is sufficiently small (depending on u), then for all $z \in \widetilde{\Gamma}^\#_h$ there exists an open disk $B_\varrho(z_0) \subset \Omega^\#_h$ such that $\partial B_\varrho(z_0) \cap \widetilde{\Gamma}^\#_h = \{z_0\}$.

Proposition 3.1. *Let $(h,u) \in X(e_0;b)$ be a local minimizer for F. Then, there exist $\sigma_0, \Lambda_0 > 0$ such that*

$$F(h,u) = \min\big\{F(g,v) + \Lambda\big|d - |\Omega_g|\big| : (v,g) \in X(e_0;b),\, d_{\mathcal{H}}(\widetilde{\Gamma}_g, \widetilde{\Gamma}_h) \le \sigma\big\}, \tag{3.1}$$

for all $0 < \sigma \le \sigma_0$, $\Lambda \ge \Lambda_0$.

Proof. Observe that from Theorem 2.2 the existence of a minimizer (g, v) for the variational problem in (3.1) follows immediately. Moreover, since $F(g,v) + \Lambda\big|d - |\Omega_g|\big| \le F(h,u)$, we have

$$\big|d - |\Omega_g|\big| \le \frac{F(h,u)}{\Lambda}, \qquad \text{and} \qquad \mathcal{H}^1(\widetilde{\Gamma}_g) \le F(h,u)\,. \tag{3.2}$$

Next, given any function $k \in AP(0,b)$ and any positive number M, let us denote by $k^M = \min\{k, M\}$ the function obtained by truncating k at the level M. Clearly, $k^M \in AP(0,b)$. Moreover, it can be easily checked that if $k, l \in AP(0,b)$ and $M > 0$, then

$$d_{\mathcal{H}}(\widetilde{\Gamma}^{\#}_{k^M}, \widetilde{\Gamma}^{\#}_{l^M}) \le d_{\mathcal{H}}(\widetilde{\Gamma}^{\#}_{k}, \widetilde{\Gamma}^{\#}_{l}) \tag{3.3}$$

Let $\delta > 0$ be as in Definition 2.4. We now choose $\sigma_1 > 0$ with the property that

$$\|h - h^M\|_{L^1(0,b)} < 3\sigma_1 \implies d_{\mathcal{H}}(\widetilde{\Gamma}^{\#}_{h^M}, \widetilde{\Gamma}^{\#}_{h}) < \frac{\delta}{2}\,. \tag{3.4}$$

Next, we claim that there exists $\sigma_0 \in (0, \delta/2)$ such that if $g \in AP(0,b)$, then

$$\mathcal{H}^1(\widetilde{\Gamma}^{\#}_g) \le F(h,u), \quad \big|d - |\Omega_g|\big| \le F(h,u), \quad d_{\mathcal{H}}(\widetilde{\Gamma}^{\#}_{g}, \widetilde{\Gamma}^{\#}_{h}) \le \sigma_0 \quad \Longrightarrow \quad \|h - g\|_{L^1(0,b)} < \sigma_1\,. \tag{3.5}$$

To prove the claim we argue by contradiction, assuming that there exists a sequence $g_n \in AP(0,b)$ such that $\mathcal{H}^1(\widetilde{\Gamma}_{g_n}) \le F(h,u)$, $\big|d - |\Omega_{g_n}|\big| \le F(h,u)$ for all n, $d_{\mathcal{H}}(\widetilde{\Gamma}^{\#}_{g_n}, \widetilde{\Gamma}^{\#}_{h}) \to 0$ as $n \to \infty$, but $\|h - g_n\|_{L^1(0,b)} \ge \sigma_1$. Thus, we may assume that, up to a subsequence, g_n converge in $L^1(0,b)$ and a.e. to a function $g \in AP(0,b)$. Note that , since $d_{\mathcal{H}}(\widetilde{\Gamma}^{\#}_{g_n}, \widetilde{\Gamma}^{\#}_{h}) \to 0$, then for all $x \in (0,b)$ such that $g_n(x) \to g(x)$, we have $(x, g(x)) \in \widetilde{\Gamma}_h$. Therefore $g = h$ a.e. in $(0,b)$, thus contradicting the fact that $\|h - g_n\|_{L^1(0,b)} \ge \sigma_1$ for all n.

To prove the assertion it is enough to show that there exists $\Lambda_0 \ge 1$ such that for any $\sigma \in (0, \sigma_0]$ and any $\Lambda \ge \Lambda_0$, the minimizer (g,v) of the minimum problem in (3.1) satisfies the volume constraint $|\Omega_g| = d$. To this aim, we consider two cases.

If $|\Omega_g| > d$, we set $\tilde{g} = g^M$, where M is such that $|\Omega_{g^M}| = d$, and $\tilde{v} = v$. We have

$$F(\tilde{g}, \tilde{v}) < F(g,v) + \Lambda\big|d - |\Omega_g|\big| \le F(h,u)\,. \tag{3.6}$$

Moreover, from (3.2) and (3.5) it follows that $\|h - g\|_{L^1(0,b)} < \sigma_1$. Thus, from (3.2) we get

$$\begin{aligned}
\|h - h^M\|_{L^1(0,b)} &\le \|h - g\|_{L^1(0,b)} + \|g - g^M\|_{L^1(0,b)} + \|g^M - h^M\|_{L^1(0,b)} \\
&\le 2\|h - g\|_{L^1(0,b)} + \|g - g^M\|_{L^1(0,b)} \\
&\le 2\sigma_1 + \big|d - |\Omega_g|\big| \le 2\sigma_1 + \frac{F(h,u)}{\Lambda} < 3\sigma_1\,,
\end{aligned}$$

provided that we choose $\Lambda_0 > F(h,u)/\sigma_1$. Therefore, with such a choice of Λ_0, recalling (3.4) and that $\sigma_0 < \delta/2$ we have that

$$d_{\mathcal{H}}(\widetilde{\Gamma}^{\#}_{\tilde{g}}, \widetilde{\Gamma}^{\#}_{h}) \le d_{\mathcal{H}}(\widetilde{\Gamma}^{\#}_{g^M}, \widetilde{\Gamma}^{\#}_{h^M}) + d_{\mathcal{H}}(\widetilde{\Gamma}^{\#}_{h}, \widetilde{\Gamma}^{\#}_{h^M}) \le d_{\mathcal{H}}(\widetilde{\Gamma}^{\#}_{g}, \widetilde{\Gamma}^{\#}_{h}) + \frac{\delta}{2} < \delta\,,$$

where in the second inequality we have used (3.3). Hence, recalling (3.6), we get a contradiction to the minimality of (h,u). This proves that $|\Omega_g| \le d$.

Let us now show that also inequality $|\Omega_g| < d$ cannot hold if Λ_0 is large enough. In fact, if $|\Omega_g| < d$ we may define $\widetilde{g} = g + \frac{d - |\Omega_g|}{b}$. Note that with such a choice $|\Omega_{\tilde{g}}| = d$. Moreover, using (3.2) again,

$$d_{\mathcal{H}}(\widetilde{\Gamma}^{\#}_{\tilde{g}}, \widetilde{\Gamma}^{\#}_{h}) \le d_{\mathcal{H}}(\widetilde{\Gamma}^{\#}_{\tilde{g}}, \widetilde{\Gamma}^{\#}_{g}) + d_{\mathcal{H}}(\widetilde{\Gamma}^{\#}_{g}, \widetilde{\Gamma}^{\#}_{h}) < \frac{d - |\Omega_g|}{b} + \frac{\delta}{2} < \delta, \tag{3.7}$$

if we choose $\Lambda_0 > 2F(h,u)/(b\delta)$. Thus if we set

$$\tilde{v}(x,y) = \begin{cases} e_0(x,0) & \text{if } 0 < y < \frac{d-|\Omega_g|}{b} \\ v\big(x, y - \frac{d-|\Omega_g|}{b}\big) & \text{if } y > \frac{d-|\Omega_g|}{b}\,, \end{cases}$$

we get, using the minimality of (g,v),

$$\begin{aligned} F(\tilde{g},\tilde{v}) - F(h,u) &= F(g,v) + \frac{e_0^2}{2}(2\mu+\lambda)(d-|\Omega_g|) - F(h,u) \\ &\le \frac{e_0^2}{2}(2\mu+\lambda)(d-|\Omega_g|) - \Lambda(d-|\Omega_g|) < 0\,, \end{aligned}$$

provided we choose $\Lambda_0 > e_0^2(2\mu+\lambda)/2$. This inequality, together with (3.7), contradicts the local minimality of (h,u), thus proving the assertion. □

Before going on with the proof of the interior ball condition we state a simple approximation result for one-dimensional BV-functions.

Lemma 3.2. *Let $h:[0,b]\to\mathbb{R}$ be a function with finite total variation. There exists a sequence of Lipschitz functions $g_n:[0,b]\to\mathbb{R}$ such that $g_n(0)=h(0)$, $g_n(b)=h(b)$, $g_n\to h$ in $L^1(0,b)$ and*

$$\mathcal{H}^1(\Gamma_{g_n}) \to \mathcal{H}^1(\Gamma_h \cap ((0,b)\times\mathbb{R})) + |h(0+)-h(0)| + |h(b-)-h(b)|\,.$$

Proof. Given $\varepsilon>0$, extend h to a function $h_\varepsilon:(-\varepsilon,b+\varepsilon)\to\mathbb{R}$, by setting $h_\varepsilon(x):=h(0)$ for $x<0$ and $h_\varepsilon(x):=h(b)$ if $x>b$. By a well-known property of BV functions, there exists $g_\varepsilon:(-\varepsilon,b+\varepsilon)\to\mathbb{R}$, Lipschitz continuous, such that

$$\|h_\varepsilon - g_\varepsilon\|_{L^1(-\varepsilon,b+\varepsilon)} < \varepsilon \quad \text{and} \quad |\mathcal{H}^1(\Gamma_{g_\varepsilon}) - H^1(\Gamma_{h_\varepsilon})| < \varepsilon\,,$$

where $\Gamma_{h_\varepsilon} := \{(x,y): x\in(-\varepsilon,b+\varepsilon), h_\varepsilon^-(x)\le y\le h_\varepsilon^+(x)\}$ and Γ_{g_ε} is the graph of g_ε. Moreover, by slightly modifying g_ε in a neighborhood of 0 and of b, if necessary, we may also assume that $g_\varepsilon(0)=h(0)$ and that $g_\varepsilon(b)=h(b)$. Then, $g_{\varepsilon|[0,b]}$ gives the required approximation of h in $[0,b]$. □

Next result contains a useful isoperimetric inequality, whose elementary proof is reproduced from [7, Lemma 6.6].

Lemma 3.3. *Let $k\in AP(0,b)$ and let $B_\rho(z_0)$ be a disk such that $B_\rho(z_0)\subset\{(x,y): x\in\mathbb{R}$ and $y<k(x)\}$ and $\partial B_\rho(z_0)\cap\widetilde{\Gamma}_k^\#$ contains $z_1=(x_1,y_1)$, $z_2=(x_2,y_2)$. Let γ be the shortest arc on $\partial B_\rho(z_0)$ connecting z_1 and z_2 (any of the two possible arcs if z_1 and z_2 are antipodal) and let γ' be the arc on $\widetilde{\Gamma}_k^\#$ connecting z_1 and z_2. Then*

$$\mathcal{H}^1(\gamma') - \mathcal{H}^1(\gamma) \ge \frac{1}{\rho}|D|\,,$$

where D is the region enclosed by $\gamma\cup\gamma'$.

Proof. Denote by h the function whose graph coincides with γ and observe that $\frac{1}{\rho} = -\big(\frac{h'}{\sqrt{1+h'^2}}\big)'$. If $k:[x_1,x_2]\to\mathbb{R}$ is any Lipschitz function such that $k(x_1)=y_1$ and $k(x_2)=y_2$, we have

$$\mathcal{H}^1(\gamma') - \mathcal{H}^1(\gamma) = \int_{x_1}^{x_2} \big(\sqrt{1+k'^2} - \sqrt{1+h'^2}\big)\,dx \ge \int_{x_1}^{x_2} \frac{(k'-h')h'}{\sqrt{1+h'^2}}\,dx = \frac{1}{\rho}\int_{x_1}^{x_2}(k-h)\,dx\,.$$

Assume now that $k \in AP(0,b)$. Without loss of generality, we may also assume that $k(x_i) = y_i$ for $i = 1,2$. Then, by approximating k in $[x_1, x_2]$ with a sequence k_n of Lipschitz functions according to Lemma 3.2 and passing to the limit in the above formula, we get

$$\mathcal{H}^1(\gamma') - \mathcal{H}^1(\gamma) \geq \frac{1}{\rho}\int_{x_1}^{x_2} (k-h)\,dx\,.$$

From this inequality, the result immediately follows by the assumption $k \geq h$. □

We are now in position to give the proof of the interior ball condition. Note that though the proof presented here is similar to the one given in [7, Lemma 6.7], the argument is more delicate since we are dealing with a more general notion of local minimizer.

Theorem 3.4. *Let $(h,u) \in X(e_0;b)$ be a b-periodic local minimizers for F. Then there exists $\rho_0 > 0$ such that for any $z \in \widetilde{\Gamma}_h^{\#}$ there exists a disk $B_{\rho_0}(z_0) \subset \{(x,y) : x \in \mathbb{R}$ and $y < h(x)\}$ with $\partial B_{\rho_0}(z_0) \cap \widetilde{\Gamma}_h^{\#} = \{z\}$.*

Proof. **Step 1.** Let us first prove that if ρ_0 is sufficiently small then the boundary of every disk contained in $\{(x,y) : x \in \mathbb{R}$ and $y < h(x)\}$ can intersect $\widetilde{\Gamma}_h^{\#}$ at most in one point.

Let σ_0, Λ_0 be as in Proposition 3.1 and assume that there exists $B_\rho(z_0) \subset \{(x,y) : x \in \mathbb{R}$ and $y < h(x)\}$ such that $\partial B_\rho(z_0) \cap \widetilde{\Gamma}_h^{\#}$ contains two distinct points z_1, z_2. Setting $z_i = (x_i, y_i)$, $i = 0,1,2$, we may assume that $x_1 < x_0 < x_2$. Let $\bar{z}$ be a point in $\widetilde{\Gamma}_h^{\#} \cap ([x_1,x_2]\times\mathbb{R})$ maximizing the distance from $\partial B_\rho(z_0)$. We claim that if

$$\rho < \frac{\sigma_0}{2(1+\Lambda_0\sigma_0)} \tag{3.8}$$

then $\mathrm{dist}(\bar{z}, \partial B_\rho(z_0)) \leq \sigma_0$.

In fact, if the claim were not true, setting $\bar{z} = (\bar{x}, \bar{y})$, the line $\{y = \bar{y} - \sigma_0/2\}$ would lie above the disk $B_\rho(z_0)$. Moreover, since $\widetilde{\Gamma}_h^{\#} \cap ([x_1,x_2]\times\mathbb{R})$ is a connected arc, this line would intersect it in at least two points. Let us denote by $\bar{z}_1 = (\bar{x}_1, \bar{y} - \sigma_0/2)$ and $\bar{z}_2 = (\bar{x}_2, \bar{y} - \sigma_0/2)$ the points in $\widetilde{\Gamma}_h^{\#} \cap ([x_1,x_2]\times\mathbb{R}) \cap \{y = \bar{y} - \sigma_0/2\}$, with the smallest and largest abscissa, respectively. Note that the projection over the y-axis of the subarc of $\widetilde{\Gamma}_h^{\#}$ connecting $\bar{z}_1$ to $\bar{z}$ is longer than $\sigma_0/2$ and the same is true for the projection of the arc connecting $\bar{z}_2$ to $\bar{z}$. Therefore we have that

$$\mathcal{H}^1\big(\widetilde{\Gamma}_h^{\#} \cap \big([\bar{x}_1, \bar{x}_2] \times (\bar{y} - \sigma_0/2, +\infty)\big)\big) \geq \sigma_0\,, \tag{3.9}$$

while $\bar{x}_2 - \bar{x}_1 \leq 2\rho$. Let us now define

$$\tilde{h}(x) = \begin{cases} \min\{h(x), \bar{y} - \frac{\sigma_0}{2}\} & \text{if } x \in [\bar{x}_1, \bar{x}_2] \\ h(x) & \text{otherwise,} \end{cases}$$

and set $\tilde{u} = u$. Then, we have that $d_{\mathcal{H}}(\widetilde{\Gamma}_{\tilde{h}}^{\#}, \widetilde{\Gamma}_h^{\#}) \leq \sigma_0$. Moreover, from (3.9) we easily get, recalling also (3.8),

$$F(\tilde{h},\tilde{u}) + \Lambda_0\big|d - |\Omega_{\tilde{h}}|\big| - F(h,u) \leq 2\rho + \rho\Lambda_0\sigma_0 - \sigma_0 < 0,$$

thus contradicting the fact that h minimizes the variational problem in (3.1).

Thus we have proved that if ρ satisfies (3.8) then every point in $\widetilde{\Gamma}_h^{\#} \cap ([x_1,x_2]\times\mathbb{R})$ lies at a distance from $\partial B_\rho(z_0)$ smaller than or equal to σ_0. Therefore, if we define $\tilde{h}$ to be the function obtained by replacing h in the interval $[x_1,x_2]$ with the function whose graph is the upper arc on $\partial B_\rho(z_0)$ connecting z_1 and z_2, we still have $d_{\mathcal{H}}(\widetilde{\Gamma}^{\#}_{\tilde h}, \widetilde{\Gamma}^{\#}_h) \le \sigma_0$. Defining $\tilde u$ as above, we then have, using Lemma 3.3,

$$\begin{aligned}
&F(\tilde h, \tilde u) + \Lambda_0 \big| d - |\Omega_{\tilde h}| \big| - F(h,u) \\
&\qquad \le \mathcal{H}^1\big(\partial B_\rho(z_0) \cap ([x_1,x_2]\times\mathbb{R}\big) - \mathcal{H}^1\big(\widetilde{\Gamma}^{\#}_h \cap ([x_1,x_2]\times\mathbb{R}\big) + \Lambda_0 |D| \\
&\qquad \le -\frac{1}{\rho}|D| + \Lambda_0|D|\,,
\end{aligned}$$

a quantity which is negative if we impose that ρ is also smaller than $1/\Lambda_0$. In this case we get again a contradiction to the minimality of (h,u).

Step 2. We argue here as in the proof of [6, Proposition 3.3], whose argument goes as follows.

Fix $\rho_0 > 0$ so that the conclusion of Step 1 applies and consider the union U of all balls of radius ρ_0 that are contained in $\Omega := \{(x,y) : x \in \mathbb{R} \text{ and } y < h(x)\}$. Our thesis is equivalent to showing that $\Omega \subset U$. Assume by contradiction that this inclusion doesn't hold. Then there exist $z_0' \in \Omega \cap \partial U$, a sequence of balls $B_{\rho_0}(z_n) \subset \Omega$, and $z_n' \in \partial B_{\rho_0}(z_n)$ such that $z_n' \to z_0'$. Up to extracting a subsequence we may assume that $B_{\rho_0}(z_n) \to B_{\rho_0}(z_0)$ in the Hausdorff distance, for some ball $B_{\rho_0}(z_0) \subset \Omega$ having z_0' at its boundary. Note that the intersection of $\partial B_{\rho_0}(z_0)$ with $\widetilde{\Gamma}^{\#}_h$ must be nonempty, since otherwise we could slightly translate the ball, still remaining in Ω and this would violate the fact that $z_0' \in \partial U$. Hence, by the previous step, $\partial B_{\rho_0}(z_0) \cap \widetilde{\Gamma}^{\#}_h = \{z'\}$. If z_0' and z' are antipodal, then we can find $\tau > 0$ such that $B_{\rho_0}(z_0 + \tau(z_0' - z')) \subset \Omega$, which would imply that $z_0' \in U$, a contradiction. If z_0' and z' are not antipodal, then we can rotate $B_{\rho_0}(z_0)$ around z_0', slightly away from z', to get a ball B' of radius ρ_0 such that $\overline{B}' \subset \Omega$ and $z_0' \in \partial B'$. Translating now B' towards z_0' we find a ball of the same radius containing z_0' and contained in Ω, which gives again $z_0' \in U$. This concludes the proof of the theorem. ☐

The interior ball condition gives a sort of 'unilateral bound' on the curvature of $\widetilde{\Gamma}^{\#}_h$. This is a key point in proving Theorem 2.5. More precisely we have the following result whose elementary, but delicate proof is given in [4, Lemma 3]. Indeed in that lemma an external uniform condition is assumed, but it can be checked that exactly the same arguments go through in our situation and lead to the following proposition, which we state without proof.

Proposition 3.5. *Let $(h,u) \in X(e_0;b)$ be a local b-periodic minimizer for the functional F. Then for any $z_0 \in \widetilde{\Gamma}^{\#}_h$ there exist an orthonormal basis e_1, $e_2 \in \mathbb{R}^2$, and a rectangle*

$$R := \{z_0 + se_1 + te_2 :\ -a' < s < a',\ -b' < t < b'\},$$

a', $b' > 0$, such that $\{(x,y) : x \in \mathbb{R}$ and $y < h(x)\} \cap R$ has one of the following two representations.

(i) *There exists a Lipschitz function $g : (-a',a') \to (-b',b')$ such that $g(0) = 0$ and*

$$\{(x,y) : x \in \mathbb{R} \text{ and } y < h(x)\} \cap R = \{z_0 + se_1 + te_2 :\ -a' < s < a',\ -b' < t < g(s)\}.$$

Moreover the function g admits left and right derivatives at every point, that are respectively left and right continuous.

(ii) *There exist two Lipschitz functions g_1, $g_2 : [0,a') \to (-b',b')$ such that $g_i(0) = (g_i)'_+(0) = 0$, for $i = 1, 2$, $g_1 \le g_2$, and*

$$\begin{aligned}
&\{(x,y) : x \in \mathbb{R} \text{ and } y < h(x)\} \cap \{z_0 + se_1 + te_2 :\ 0 < s < a',\ -b' < t < b'\} \\
&\qquad = \{z_0 + se_1 + te_2 :\ 0 < s < a',\ -b' < t < g_1(s) \ \text{ or } \ g_2(s) < t < b'\}\,.
\end{aligned}$$

Moreover the functions g_1, g_2 admit left and right derivatives at every point, that are respectively left and right continuous.

Remark 3.6. Note that in case (ii) the point z_0 is either a cusp point or a point in the vertical cut. Moreover, in this case, setting $z_0 = (x_0, y_0)$, the balls of radius ρ_0 centered in $(x_0 \pm \rho_0, y_0)$, are both tangent to $\widetilde{\Gamma}^\#_h$ and lay below $\widetilde{\Gamma}^\#_h$. This observation immediately implies that there are finitely many cusp points and vertical cuts, thus proving assertion (i) in Theorem 2.5.

4. A decay estimate for the gradient of u

Following [6] we are now going to show that given a point in $z_0 \in \widetilde{\Gamma}^\#_h \setminus \big(\Sigma^\#_h \cup \Sigma^\#_{h,c}\big)$, then in a neighborhood of z_0 the integral of $|\nabla u|^2$ over a disk of radius r decays faster than r. As in [6] the key ingredient to get such an estimate is a well known result of Grisvard ([10]) describing the behavior of solutions to the linear elasticity systems at corner points. Here, however, differently from [6], we have to distinguish between the case whether z_0 lies above the x-axis or is on the x-axis. The latter situation is more delicate to handle, while the former will be treated essentially as in [6], with some simplifications.

In this section we will consider open sets whose boundary can be decomposed into three curves

$$\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \,,$$

where Γ_1, Γ_2 are two segments meeting at the origin with an internal angle $\omega \in (0, 2\pi)$ and Γ_3 is a regular curve joining the two remaining endpoints of Γ_1 and Γ_2 in a smooth way. We shall refer to such an open set as to a *regular domain with corner angle* ω. If Ω is a such a domain and $u \in W^{1,2}(\Omega; \mathbb{R}^2)$, we set

$$\sigma(u)[\nu] := \mathbb{C}E(u)[\nu] = \big[\mu(\nabla u + \nabla^T u) + \lambda(\mathrm{div} u)I\big][\nu] \,,$$

where ν is the exterior normal to $\partial\Omega$ and I is the identity map. Moreover, following [10], for any complex number α we denote by S_α a complex valued function given in polar coordinate by

$$S_\alpha := r^\alpha \Phi_\alpha(r, \theta) \,,$$

where Φ_α is a smooth function depending on α and on the corner angle ω. Next result is proven in [10, Theorems I and 6.2].

Theorem 4.1 (Grisvard)**.** *Let Ω be a regular domain with corner and $f \in L^p(\Omega; \mathbb{R}^2)$, $1 < p \leq 2$.*
(a) Let $\omega \in (0, 2\pi)$ and let $w \in W^{1,2}(\Omega; \mathbb{R}^2)$ be a weak solution of the Neumann problem

$$\begin{cases} \mu\Delta w + (\lambda + \mu)\nabla(\mathrm{div} w) = f \;\; \text{in } \Omega \\ \sigma(w)[\nu] = 0 \;\; \text{on } \partial\Omega \,. \end{cases} \tag{4.1}$$

Then, there exist constants c_α, c'_α such that

$$w - \sum c_\alpha S_\alpha - \sum c'_\alpha \frac{\partial S_\alpha}{\partial \alpha} \in W^{2,p}(\Omega; \mathbb{R}^2) \,. \tag{4.2}$$

The first sum is extended to all simple complex roots of equation

$$\sin^2 \alpha\omega = \alpha^2 \sin^2 \omega \,, \tag{4.3}$$

contained in the strip $0 < \mathrm{Re}\,\alpha < 2 - 2/p$, while the second sum is extended to all double roots of (4.3) *contained in the same strip. However,* (4.2) *holds provided that* (4.3) *has no solutions on the line $\mathrm{Re}\,\alpha = 2 - 2/p$.*

(b) Moreover, if $\omega \in (0,\pi)$ and $w \in W^{1,2}(\Omega;\mathbb{R}^2)$ is a weak solution of the mixed problem

$$\begin{cases} \mu\Delta w + (\lambda+\mu)\nabla(\mathrm{div} w) = f \ \ \textit{in } \Omega \\ w = 0 \ \ \textit{on } \Gamma_1, \ \ \sigma(w)[\nu] = 0 \ \ \textit{on } \Gamma_2 \cup \Gamma_3 \,. \end{cases} \tag{4.4}$$

then (4.2) holds. In this case the first sum is extended to all simple complex roots of equation

$$\sin^2 \alpha\omega = \frac{(\lambda+2\mu)^2 - (\lambda+\mu)^2\alpha^2\sin^2\omega}{(\lambda+\mu)(\lambda+3\mu)}\,, \tag{4.5}$$

contained in the strip $0 < \operatorname{Re}\alpha < 2-2/p$, while the second sum is extended to all double roots of (4.5) contained in the same strip. As before, (4.2) holds provided that (4.5) has no solutions on the line $\operatorname{Re}\alpha = 2-2/p$.

Though this deep result gives no information about the roots of equations (4.3) and (4.5), it can be easily proved that the solutions to both equations contained in the strip $0 < \operatorname{Re}\alpha < 1$ are bounded. Hence, by analiticity, they are finitely many. A more precise information is provided by the following result, proved in [13, Theorem 2.2].

Theorem 4.2 (Nicaise)**.** *If $\omega \in (0,2\pi)$, then equation (4.3) has no root in the strip $0 < \operatorname{Re}\alpha \le \frac{1}{2}$. Similarly, equation (4.5) has no root in the same strip if $\omega \in (0,\pi)$.*

We are now going to combine the two previous results in order to get a useful estimate for the solutions of both problems (4.1) and (4.4). Before proving this estimate we need to recall the following well know Korn's inequality.

Theorem 4.3 (Korn's inequality)**.** *Let $M > 0$ and let $\Omega \subset \mathbb{R}^n$ be an open bounded domain starshaped with respect to a given ball $B_r(x_0) \subset \Omega$ and such that $\operatorname{diam}\Omega \le M$. Then there exists a constant $C = C(p,N,r,M) > 0$ such that for all $u \in W^{1,p}(\Omega;\mathbb{R}^n)$, $p > 1$,*

$$\int_\Omega |\nabla u|^p\,dx \le C\biggl(\int_\Omega |u|^p\,dx + \int_\Omega |E(u)|^p\,dx\biggr)\,. \tag{4.6}$$

Moreover, if

$$\int_{B_r(x_0)} u\,dx = 0 \quad \textit{and} \quad \int_{B_r(x_0)} (\nabla u - \nabla^T u)\,dx = 0\,,$$

then (4.6) holds in the stronger form

$$\int_\Omega |\nabla u|^p\,dx \le C\int_\Omega |E(u)|^p\,dx\,. \tag{4.7}$$

We can now pass to the proof of the a priori estimates. To this aim, we recall that an *infinitesimal rigid motion* is an affine displacement of the form $a + Ax$, where A is a skew symmetric 2×2 matrix and a is a constant vector.

Theorem 4.4. *Let Ω be as in Theorem 4.1. There exist $p \in (4/3,2)$ and $C > 0$ such that if $f \in L^p(\Omega;\mathbb{R}^2)$ and $w \in W^{1,2}(\Omega;\mathbb{R}^2)$ is a weak solution to problem (4.1), then*

$$\|w\|_{W^{2,p}(\Omega;\mathbb{R}^2)} \le C\bigl(\|w\|_{L^p(\Omega;\mathbb{R}^2)} + \|f\|_{L^p(\Omega;\mathbb{R}^2)}\bigr)\,. \tag{4.8}$$

Similarly, if $\omega \in (0,\pi)$ there exist $p \in (4/3,2)$ and C such that for every weak solution to problem (4.4)

$$\|w\|_{W^{2,p}(\Omega;\mathbb{R}^2)} \le C\|f\|_{L^p(\Omega;\mathbb{R}^2)}\,. \tag{4.9}$$

Proof. We start by proving (4.8).

As already observed, the strip $0 < \operatorname{Re}\alpha < 1$ contains only finitely many solutions to equation (4.3) and from Theorem 4.2 we may conclude that there exists ε such that they are all contained in the strip $\frac{1}{2} + \varepsilon < \operatorname{Re}\alpha < 1$. Therefore, if we choose $p > 4/3$ such that $2 - \frac{2}{p} < \frac{1}{2} + \varepsilon$, from statement (a) of Theorem 4.1 we get that any weak solution to (4.1), with $f \in L^p(\Omega;\mathbb{R}^2)$ is in $W^{2,p}(\Omega;\mathbb{R}^2)$.

To prove the estimate (4.8) let us set $V := \{u \in W^{2,p}(\Omega;\mathbb{R}^2) : \sigma(u)[\nu] = 0 \text{ on } \partial\Omega\}$, $\widetilde{V} := V/\sim$, where for every $u, v \in V$, we have set $u \sim v$ if and only if $u - v$ is an infinitesimal rigid motion. We define a norm in $\widetilde{V}$ setting for every equivalence class $[u]$, with $u \in V$,

$$\|[u]\|_{\widetilde{V}} := \||E(u)|\|_{L^p(\Omega)} + \||\nabla^2 u|\|_{L^p(\Omega)}\,.$$

Note that this definition is well posed, since if $u \sim v$, then $E(u) = E(v)$ and $\nabla^2 u = \nabla^2 v$. Note also that $\widetilde{V}$ is a Banach space. In fact, if $[u_h]$ is a Cauchy sequence in $\widetilde{V}$, set

$$a_h := \fint_{B_r(z_0)} u_h\,dz, \qquad A_h := \fint_{B_r(z_0)} \big(\nabla u_h - \nabla^T u_h\big)\,dz, \qquad v_h := u_h - a_h - A_h(z - z_0)\,,$$

where $B_r(z_0)$ is a fixed ball, such that Ω is starshaped with respect to this ball. Then, from (4.7) and the fact that $\int_{B_r(z_0)} v_h = 0$, it follows easily that the functions v_h converge in $W^{2,p}(\Omega;\mathbb{R}^2)$ to some v. Moreover, from the definition of v_h we have that $\sigma(v_h)[\nu] = 0$ on $\partial\Omega$. Hence, the same is true for v and this proves that $\widetilde{V}$ is a Banach space.

Consider now the operator $L : \widetilde{V} \to L^p(\Omega;\mathbb{R}^2)$ defined for any $[u] \in \widetilde{v}$ as

$$L[u] := \mu\Delta u + (\lambda + \mu)\nabla(\operatorname{div} u)\,.$$

From what we have observed in the first part of the proof it follows that L is a linear, continuous, surjective between two Banach spaces. Therefore to prove (4.8) we only need to show that L is injective. But this is easy since if $L[u] = 0$ for some $u \in V$, then, recalling that $\sigma(u)[\nu] = 0$, we get that

$$\int_\Omega \mathbb{C}E(u) : E(u)\,dz = 0$$

Therefore, from the coercivity condition (2.4) we deduce that $E(u) = 0$ and thus, Korn's inequality (4.7) yields that $u(z) = a + Az$ for some skew symmetric matrix A, hence $[u] = 0$. From the injectivity of L we then get that there exists $c > 0$ such that for all $u \in V$

$$\||E(u)|\|_{L^p(\Omega)} + \||\nabla^2 u|\|_{L^p(\Omega)} \le c\|L[u]\|_{L^p(\Omega;\mathbb{R}^2)}\,.$$

From this inequality and inequality (4.6) estimate (4.8) immediately follows.

The proof of (4.9) is even simpler. As before, from Theorem 4.2 we get that there exists $p > 4/3$ such that any weak solution to (4.4), with $f \in L^p(\Omega;\mathbb{R}^2)$ is in $W^{2,p}(\Omega;\mathbb{R}^2)$. Then, one may introduce the set $V := \{u \in W^{2,p}(\Omega;\mathbb{R}^2) :\ u = 0 \text{ on } \Gamma_1,\ \sigma(u)[\nu] = 0 \text{ on } \Gamma_2 \cup \Gamma_3\}$ and easily prove that it is a closed subspace of $W^{2,p}(\Omega;\mathbb{R}^2)$. Then, setting $Lu := \mu\Delta u + (\lambda + \mu)\nabla(\operatorname{div} u)$ for every $u \in W^{2,p}(\Omega;\mathbb{R}^2)$, the proof goes on as before. □

The above theorem gives a global estimate of solutions to the linear systems (4.1) and (4.4). However these solutions may have a singularity at the origin, being smooth in the remaining part of Ω. Next result provides the local estimates of the higher order derivatives away from the origin. The proof goes as in [6, Theorem 3.7].

Proposition 4.5. *Let Ω be as in Theorem 4.1 and let $0 < r_0 < r_1$ be such that the open set $A := \{z \in \Omega : r_0 < |z| < r_1\}$ has positive distance from Γ_3. For any integer $k \in \mathbb{N}$ there exists a constant C_k such that, if $w \in W^{1,2}(\Omega;\mathbb{R}^2)$ is a weak solution to either* (4.1) *or* (4.4) *, then*

$$\sup_A |\nabla^k w|^2 \leq C_k \int_\Omega |\nabla w|^2 \, dz \, .$$

Proof. By the Sobolev embedding theorem it is enough to show that for all $k \geq 2$ one has

$$\int_A |\nabla^k w|^2 \, dz \leq C'_k \int_\Omega |\nabla w|^2 \, dz \, . \tag{4.10}$$

for some constant C'_k independent of w, a weak solution to either (4.1) or (4.4). To this aim let us fix a point $z_0 \in \Gamma_1$ and $r > 0$ such that the ball $B_{2r}(z_0)$ has positive distance both from the origin and Γ_3, and denote by τ and ν the tangential and normal unit vector, respectively. By applying the standard difference quotient argument in the direction τ (note that this argument works under both Neumann and Dirichlet conditions) and then Korn's inequality (4.7), one easily gets that $\frac{\partial w}{\partial \tau}$ belongs $W^{1,2}(\Omega \cap B_{2r}(z_0);\mathbb{R}^2)$ and satisfies the equation

$$\int_{\Omega \cap B_{2r}(z_0)} \mathbb{C} E\Big(\frac{\partial w}{\partial \tau}\Big) : E(\varphi) \, dz = 0$$

for all $\varphi \in W^{1,2}(\Omega \cap B_{2r}(z_0);\mathbb{R}^2)$ vanishing in a neighborhood of $\overline{\Omega} \cap \partial B_{2r}(z_0)$. Thus, choosing in the above equation $\varphi := \eta^2 \frac{\partial w}{\partial \tau}$, where η is a smooth cut-off function with compact support in $B_{3r/2}(z_0)$, $\eta \equiv 1$ in $B_r(z_0)$, $\|\nabla \eta\|_\infty \leq c/r$, we get from (2.4)

$$\int_{\Omega \cap B_{2r}(z_0)} \eta^2 \Big| E\Big(\frac{\partial w}{\partial \tau}\Big)\Big|^2 dz \leq c \int_{\Omega \cap B_{2r}(z_0)} |\nabla \eta|^2 |\nabla w|^2 \, dz \, .$$

Applying (4.7) once more we have

$$\begin{aligned}\int_{\Omega \cap B_r(z_0)} \Big|\nabla\Big(\frac{\partial w}{\partial \tau}\Big)\Big|^2 dz &\leq \int_{\Omega \cap B_{2r}(z_0)} \Big|\nabla\Big(\eta \frac{\partial w}{\partial \tau}\Big)\Big|^2 dz \leq c \int_{\Omega \cap B_{2r}(z_0)} \Big|E\Big(\eta \frac{\partial w}{\partial \tau}\Big)\Big|^2 dz \\ &\leq c \int_{\Omega \cap B_{2r}(z_0)} |\nabla \eta|^2 |\nabla w|^2 \, dz \leq \frac{c}{r^2} \int_{\Omega \cap B_{2r}(z_0)} |\nabla w|^2 \, dz \, .\end{aligned}$$

Covering $\partial A \cap \Gamma_1$ with a finite number of such balls $B_r(z_0)$, we get that there exist a neighborhood U of $\partial A \cap \Gamma_1$ and a positive constant c such that

$$\int_U \Big|\nabla\Big(\frac{\partial w}{\partial \tau}\Big)\Big|^2 dz \leq c \int_\Omega |\nabla w|^2 \, dz \, . \tag{4.11}$$

We have thus estimated the L^2 norm in U of $\frac{\partial^2 w}{\partial \tau^2}$ and $\frac{\partial^2 w}{\partial \tau \partial \nu}$. In order to estimate $\frac{\partial^2 w}{\partial \nu^2}$ we use the Lamé system by rewriting it in terms of $\frac{\partial^2 w}{\partial \tau^2}$ and $\frac{\partial^2 w}{\partial \tau \partial \nu}$. Setting

$$(1,0) = \alpha\tau + \beta\nu, \quad (0,1) = \beta\tau - \alpha\nu \, ,$$

where $\alpha^2 + \beta^2 = 1$, we have

$$\frac{\partial w}{\partial x} = \alpha \frac{\partial w}{\partial \tau} + \beta \frac{\partial w}{\partial \nu}, \quad \frac{\partial w}{\partial y} = \beta \frac{\partial w}{\partial \tau} - \alpha \frac{\partial w}{\partial \nu}$$

and the Lamé system becomes

$$\begin{aligned}\frac{\partial^2 w_1}{\partial \nu^2}\left[\mu + (\mu+\lambda)\,\beta^2\right] - \frac{\partial^2 w_2}{\partial \nu^2}\,(\mu+\lambda)\,\alpha\beta &= f_1, \\ -\frac{\partial^2 w_1}{\partial \nu^2}\,(\mu+\lambda)\,\alpha\beta + \frac{\partial^2 w_2}{\partial \nu^2}\left[\mu + (\mu+\lambda)\,\alpha^2\right] &= f_2,\end{aligned}$$

where f_1 and f_2 are linear combinations of the remaining second order derivatives of w_1 and w_2 with coefficients depending only on λ, μ, α and β. Hence

$$\frac{\partial^2 w_1}{\partial \nu^2} = \frac{f_1\left[\mu + (\mu+\lambda)\,\alpha^2\right] + f_2\,(\mu+\lambda)\,\alpha\beta}{(2\mu+\lambda)\,\mu},$$
$$\frac{\partial^2 w_2}{\partial \nu^2} = \frac{f_2\left[\mu + (\mu+\lambda)\,\beta^2\right] + f_1\,(\mu+\lambda)\,\alpha\beta}{(2\mu+\lambda)\,\mu},$$

which by (4.11) gives

$$\int_U |\nabla^2 w|^2\,dz \le c\int_\Omega |\nabla w|^2\,dz\,.$$

Since also $\frac{\partial w}{\partial \tau}$ satisfies the Lamé system with the same boundary condition of w on Γ_1 we get that for any $k \ge 2$

$$\int_U |\nabla^k w|^2\,dz \le c_k\int_\Omega |\nabla w|^2\,dz\,.$$

A similar estimate can be obtained in a neighborhood U' of $\partial A \cup \Gamma_2$ and in a neighborhood of $\overline{A} \setminus (U \cup U')$, thus proving (4.10). □

Next result is a technical lemma concerning the lifting of a normal trace on the boundary of a regular domain with corner to a $W^{2,2}$ function of the whole domain. In the case of Neumann condition it was proved in [6, Lemma 3.12]. However, in order to apply it to mixed Dirichlet-Neumann problems, we also need to treat the case when both the trace of the normal derivative and of the functions are assigned.

Lemma 4.6. *Let Ω be a regular domain with corner angle $\omega \in (0,2\pi)$.*
(a) Let $g \in W^{\frac12,2}(\partial\Omega,\mathbb{R}^2)$ be a function vanishing in a neighborhood of the origin. There exists $v \in W^{2,2}(\Omega,\mathbb{R}^2)$ such that

$$\sigma(v)[\nu] = \left[\mu(\nabla v + \nabla^T v) + \lambda(\operatorname{div} v)I\right][\nu] = g \qquad \text{on } \partial\Omega$$

and

$$||v||_{W^{2,2}(\Omega;\mathbb{R}^2)} \le c(\Omega)||g||_{W^{\frac12,2}(\partial\Omega;\mathbb{R}^2)}\,.$$

(b) Let $g \in W^{\frac12,2}(\Gamma_2 \cup \Gamma_3,\mathbb{R}^2)$ be a function vanishing in a neighborhood of the origin. There exists $v \in W^{2,2}(\Omega,\mathbb{R}^2)$ such that

$$v = e_0(x,0) \quad \text{on } \Gamma_1, \qquad \sigma(v)[\nu] = g \quad \text{on } \Gamma_2 \cup \Gamma_3 \tag{4.12}$$

and

$$||v||_{W^{2,2}(\Omega;\mathbb{R}^2)} \le c(\Omega)\left[||g||_{W^{\frac12,2}(\Gamma_2\cup\Gamma_3;\mathbb{R}^2)} + |e_0|\right]. \tag{4.13}$$

Proof. **Step 1.** It is easily checked that the the condition $\sigma(v)[\nu] = g$ is equivalent to

$$\begin{cases} \dfrac{\partial v_1}{\partial x}(2\mu+\lambda)\nu_1 + \dfrac{\partial v_1}{\partial y}\mu\nu_2 + \dfrac{\partial v_2}{\partial x}\mu\nu_2 + \dfrac{\partial v_2}{\partial y}\lambda\nu_1 = g_1 \\[2ex] \dfrac{\partial v_1}{\partial x}\lambda\nu_2 + \dfrac{\partial v_1}{\partial y}\mu\nu_1 + \dfrac{\partial v_2}{\partial x}\mu\nu_1 + \dfrac{\partial v_2}{\partial y}(2\mu+\lambda)\nu_2 = g_2 \end{cases} \tag{4.14}$$

Since

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial \nu}\nu_1 - \frac{\partial v}{\partial \tau}\nu_2, \qquad \frac{\partial v}{\partial y} = \frac{\partial v}{\partial \nu}\nu_2 + \frac{\partial v}{\partial \tau}\nu_1,$$

where ν is the exterior normal and τ is the tangent versor to $\partial\Omega$ oriented counterclockwise, system (4.14) can be rewritten as

$$\begin{cases} \dfrac{\partial v_1}{\partial \nu}[\mu+(\mu+\lambda)\nu_1^2]+\dfrac{\partial v_2}{\partial \nu}(\mu+\lambda)\nu_1\nu_2=\tilde{g_1} \\[2ex] \dfrac{\partial v_1}{\partial \nu}(\mu+\lambda)\nu_1\nu_2+\dfrac{\partial v_2}{\partial \nu}[\mu+(\mu+\lambda)\nu_2^2]=\tilde{g_2}, \end{cases}$$

where

$$\tilde{g_1}=g_1+\frac{\partial v_1}{\partial \tau}(\mu+\lambda)\nu_1\nu_2+\frac{\partial v_2}{\partial \tau}(\mu\nu_2^2-\lambda\nu_1^2) \tag{4.15}$$

and

$$\tilde{g_2}=g_2+\frac{\partial v_1}{\partial \tau}(\lambda\nu_2^2-\mu\nu_1^2)-\frac{\partial v_2}{\partial \tau}(\mu+\lambda)\nu_1\nu_2. \tag{4.16}$$

Therefore we have

$$\begin{cases} \dfrac{\partial v_1}{\partial \nu}=\dfrac{\tilde{g_1}[\mu+(\mu+\lambda)\nu_2^2]-\tilde{g_2}(\mu+\lambda)\nu_1\nu_2}{(2\mu+\lambda)\mu}\equiv\dfrac{h_1}{(2\mu+\lambda)\mu} \\[2ex] \dfrac{\partial v_2}{\partial \nu}=\dfrac{\tilde{g_2}[\mu+(\mu+\lambda)\nu_1^2]-\tilde{g_1}(\mu+\lambda)\nu_1\nu_2}{(2\mu+\lambda)\mu}\equiv\dfrac{h_2}{(2\mu+\lambda)\mu}\,. \end{cases} \tag{4.17}$$

Let us now impose that $\frac{\partial v_1}{\partial \tau}=\frac{\partial v_2}{\partial \tau}=0$. With such a choice, even if ν is discontinuous at the origin, by the assumption on g the functions h_1, h_2 at the right-hand sides of (4.17) are zero in a neighborhood of the origin, hence they are both in $W^{\frac{1}{2},2}(\partial\Omega)$. Therefore we may apply Theorem 1.5.2.8 of [9] to get a function $v\in W^{2,2}(\Omega;\mathbb{R}^2)$ with $v=0$ and normal derivatives $\frac{\partial v_i}{\partial \nu}=h_i$ on $\partial\Omega$, and such that the $W^{2,2}(\Omega;\mathbb{R}^2)$ norm of v is bounded by a constant times the $W^{\frac{1}{2},2}(\partial\Omega;\mathbb{R}^2)$ norm of (h_1,h_2). Hence assertion (a) follows.

Step 2. We now deal with case (b). Note that, up to a rotation, we may always assume that Γ_1 lies on the positive x semi-axis.

We start by fixing v equal to $e_0(x,0)$ on the whole boundary $\partial\Omega$. Next, we extend g to a function $\bar{g}:\partial\Omega\mapsto\mathbb{R}^2$. To this aim, if U is a neighbourhood of the origin such that $g\equiv 0$ on $A:=U\cap\Gamma_2$, we set $B:=U\cap\Gamma_1$ and define $\bar{g}$ as follows:

$$\bar{g}(z)=\begin{cases} (\alpha,\beta)\in\mathbb{R}^2 & \text{if } z\in B \\ g(z) & \text{if } z\in\Gamma_1\cup\Gamma_2 \\ \text{any smooth interpolation between } (\alpha,\beta) \text{ and } g & \text{otherwise.} \end{cases}$$

We now impose that $\frac{\partial v}{\partial \nu}$ satisfies (4.17) where $\tilde{g_1}$, $\tilde{g_2}$ are defined as in (4.15), (4.16), with g replaced by $\bar{g}$. Then, in order to prove (4.12) it is enough to show that α and β can be chosen in such a way that the functions h_1, h_2 in (4.17) belong to $W^{\frac{1}{2},2}(\partial\Omega)$ and satisfy

$$||h_1||_{W^{\frac{1}{2},2}(\partial\Omega)}+||h_2||_{W^{\frac{1}{2},2}(\partial\Omega)}\le c[||g||_{W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3;\mathbb{R}^2)}+|e_0|]. \tag{4.18}$$

Once this inequality is proved, [10, Theorem 1.5.2.8] again will imply that there exists $v\in W^{2,2}(\partial\Omega,\mathbb{R}^2)$, satisfying (4.17) and $v=e_0(x,0)$ on Γ_1, hence (4.12), such that (4.13) holds.

To prove (4.18) note that

$$\frac{\partial v}{\partial \tau}=\begin{cases} e_0(1,0) & \text{on } \Gamma_1 \\ e_0(-\nu_2,0) & \text{on } \Gamma_2\,. \end{cases} \tag{4.19}$$

In order to choose α, β so that $h_1, h_2 \in W^{\frac{1}{2},2}(\partial\Omega)$ we only need to worry about the regularity of these functions in $A \cup B$. From (4.15), (4.16) and (4.19), we have that

$$h_1 \equiv -2e_0\mu(\mu+\lambda)\nu_1\nu_2^2 \qquad \text{on } A \tag{4.20}$$

while $h_1 \equiv \alpha(2\mu+\lambda)$ on B. Thus, we set α so that $\alpha(2\mu+\lambda)$ is equal to the right hand side of (4.20). Similarly, observing that

$$h_2 = e_0\mu\nu_2[\nu_1^2(2\mu+\lambda) - \lambda\nu_2^2] \qquad \text{on A} \tag{4.21}$$

and that $h_2 = \beta\mu + \lambda\mu e_0$ on B, we choose β so that $\beta\mu+\lambda\mu e_0$ coincides with the right hand-side of (4.21). Note that, with such choice of α and β the functions h_1, h_2 belong to $W^{\frac{1}{2},2}(\partial\Omega)$ (while in general $\bar g_1, \bar g_2$ do not) and that (4.18) is satisfied. □

We are now in position to prove a decay estimate near the origin for the gradient of solutions either of the Neumann problem (4.1) or of the mixed problem (4.4).

Theorem 4.7. *Let Ω be a regular domain with corner and let $u \in W^{1,2}(\Omega;\mathbb{R}^2)$ be either a weak solution to the Neumann problem*

$$\begin{cases} \mu\Delta u + (\lambda+\mu)\nabla(\mathrm{div} u) = 0 & \text{in } \Omega \\ \sigma(w)[\nu] = 0 & \text{on } \Gamma_1 \cup \Gamma_2 \,. \end{cases}$$

or a weak solution to the mixed problem

$$\begin{cases} \mu\Delta u + (\lambda+\mu)\nabla(\mathrm{div} u) = 0 & \text{in } \Omega \\ u = e_0(x,0) \ \text{ on } \Gamma_1, \ \ \sigma(u)[\nu] = 0 & \text{on } \Gamma_2 \end{cases}$$

(in the latter case assume also $\omega \in (0,\pi)$). Let $r_0 > 0$ such that $\overline{B}_{r_0} \cap \Gamma_3 = \emptyset$. Then, there exist $C > 0$, $\alpha > 1/2$, depending only on λ, μ and Ω, such that for all $r \in (0, r_0)$

$$\int_{B_r\cap\Omega} |\nabla u|^2 \, dz \le Cr^{2\alpha} \int_\Omega \big(|u|^2 + |\nabla u|^2\big)\, dz \,. \tag{4.22}$$

Proof. We shall prove the assertion only for the mixed problem, the other case being similar and actually simpler (see also [6, Theorem 3.11]).

Fix $t \in (0,1)$ so that $\overline{B}_{r_0} \cap t\,\Gamma_3 = \emptyset$ and set $\Omega' := t\,\Omega$. Clearly, Ω' is a regular domain with the same corner angle ω of Ω and boundary $\partial\Omega' = t(\Gamma_1 \cup \Gamma_2 \cup \Gamma_3)$. By Proposition 4.5 we get that $u \in C^\infty(\Omega' \setminus \overline{B}_{r_1})$, where $0 < r_1 < r_0$, and

$$\int_{\Omega'\setminus\overline{B}_{r_1}} |\nabla^2 u|^2 \, dz \le c \int_\Omega |\nabla u|^2 \, dz \,.$$

From this estimate and the assumption $\sigma(u)[\nu] = 0$ on Γ_2, setting $\Gamma := t\,\Gamma_2 \cup t\,\Gamma_3$, we have

$$\|\sigma(u)[\nu]\|^2_{W^{\frac{1}{2},2}(\Gamma;\mathbb{R}^2)} \le c_0 \int_{\Omega'\setminus\overline{B}_{r_1}} \big(|\nabla u|^2 + |\nabla^2 u|^2\big)\, dz \le C_0 \int_\Omega |\nabla u|^2 \, dz \,,$$

for some constant C_0 independent of u. By applying Lemma 4.6 to the function $g := \sigma(u)[\nu] \in W^{\frac{1}{2},2}(t\Gamma_2 \cup t\Gamma_3; \mathbb{R}^2)$ we get that there exists a function $v \in W^{2,2}(\Omega';\mathbb{R}^2)$ such that

$$v = e_0(x,0) \quad \text{on } t\,\Gamma_1, \qquad \sigma(v)[\nu] = g \quad \text{on } t(\Gamma_2 \cup \Gamma_3)\,, \tag{4.23}$$

$$\|v\|_{W^{2,2}(\Omega';\mathbb{R}^2)} \le c\big[\|g\|_{W^{\frac{1}{2},2}(\partial\Omega';\mathbb{R}^2)} + |e_0|\big] \le c\|u\|_{W^{1,2}(\Omega;\mathbb{R}^2)} \,. \tag{4.24}$$

Defining $w := u - v$, we get from (4.23) that w solves the mixed problem

$$\begin{cases} \mu\Delta w + (\lambda+\mu)\nabla(\mathrm{div} w) = -\mu\Delta v - (\lambda+\mu)\nabla(\mathrm{div} v) \ \text{ in } \Omega' \\ w = 0 \ \text{ on } t\Gamma_1, \ \ \sigma(w)[\nu] = 0 \ \text{ on } t(\Gamma_2 \cup \Gamma_3)\,. \end{cases}$$

Thus, (4.9) implies that

$$\|w\|_{W^{2,p}(\Omega';\mathbb{R}^2)} \le c\|v\|_{W^{2,p}(\Omega';\mathbb{R}^2)}\,, \tag{4.25}$$

for some $p > 4/3$ depending only on the Lamé constants and on Ω. Thus, if $0 < r < r_0$, using the Sobolev imbedding theorem, (4.24) and (4.25), we have

$$\begin{aligned} \int_{B_r\cap\Omega} |\nabla u|^2\,dz &\le 2\int_{B_r\cap\Omega} \big(|\nabla w|^2 + |\nabla v|^2\big)\,dz \le c\bigg(\int_{B_r\cap\Omega}\Big[|\nabla w|^{\frac{2p}{2-p}} + |\nabla v|^{\frac{2p}{2-p}}\Big]\,dz\bigg)^{\frac{2-p}{p}} r^{\frac{4(p-1)}{p}} \\ &\le cr^{\frac{4(p-1)}{p}}\big(\|w\|^2_{W^{2,p}(\Omega';\mathbb{R}^2)} + \|v\|^2_{W^{2,p}(\Omega';\mathbb{R}^2)}\big) \le cr^{\frac{4(p-1)}{p}}\int_{B_r\cap\Omega}\big(|u|^2 + |\nabla u|^2\big)\,dz\,, \end{aligned}$$

hence (4.22) follows with $\alpha := 2(p-1)/p$, which is strictly greater than $1/2$, since $p > 4/3$. □

Let (h,u) be a local minimizer of F. We are now going to describe the behavior of ∇u near a point $z_0 = (x_0, y_0) \in \Gamma_h^{\#} \setminus (\Sigma_h^{\#} \cup \Sigma_{h,c}^{\#})$, showing that the integral of $|\nabla u|^2$ in a ball $B_r(z_0)$ decays faster than r.

In order to prove this decay estimate we need a slightly different version of the Korn inequality whose proof can be found in [6, Theorem 4.2].

Theorem 4.8 (Korn's Inequality in subgraphs of Lipschitz functions)**.** *Let B' be the unit ball in $\mathbb{R}^{n-1}$ and let $h : B' \to [-L, L]$ be a Lipschitz function with* $\mathrm{Lip}\, h \le L$ *for some $L > 0$. Define*

$$C_h := \{(x', x_n) \in B' \times \mathbb{R} : \, -4L < x_n < h(x')\}.$$

Then there exists a constant C depending only on N, p, and L such that

$$\int_{C_h} |\nabla u|^p\,dx \le C\bigg(\int_{C_h} |u|^p dx + \int_{C_h} |E(u)|^p dx\bigg)$$

for all $u \in W^{1,p}(C_h;\mathbb{R}^n)$, $p > 1$. Moreover for any ball B compactly contained in $B' \times (-4L, -3L)$ there exists a constant C_1 depending only on N, p, L and on the radius of B such that

$$\int_{C_h} |\nabla u|^p\,dx \le C_1\int_{C_h} |E(u)|^p\,dx$$

for all $u \in W^{1,p}(C_h;\mathbb{R}^n)$ with

$$\int_B \big(\nabla u - \nabla^T u\big)\,dx = 0.$$

As a consequence of previous result and Proposition 3.5 we have that if $z_0 \in \Gamma_h^{\#} \setminus (\Sigma_h^{\#} \cup \Sigma_{h,c}^{\#})$ then there exists a neighborhood U of z_0 such that $u \in W^{1,2}(\Omega^{\#} \cap U;\mathbb{R}^2)$.

Theorem 4.9. *Let (h,u) be a a local minimizer of F and $z_0 \in \Gamma_h^{\#} \setminus (\Sigma_h^{\#} \cup \Sigma_{h,c}^{\#})$. Then, there exist $\alpha > 1/2$, $C_0 > 0$, $r_0 > 0$, such that for all $r \in (0, r_0)$*

$$\int_{B_r(z_0)\cap\Omega_h^{\#}} |\nabla u|^2\,dz \le C_0 r^{2\alpha}\,. \tag{4.26}$$

Proof. As in [6, Theorem 3.13] we will prove the result by a blow-up argument. We shall discuss only the case $z_0 = (x_0, y_0)$ with $y_0 = 0$, whose proof is similar but more difficult than the proof needed in the case $y_0 > 0$, which goes exactly as in [6].

We are going to show that there exist $\alpha > 1/2$, $C_0 > 0$, $r_0 > 0$, such that for all $r \in (0, r_0)$

$$\int_{B_r(z_0)\cap\{x>x_0\}\cap\Omega_h^\#} |\nabla u|^2\, dz \le C_0 r^{2\alpha}\,. \tag{4.27}$$

Then, combining (4.27) with a similar estimate in $B_r(z_0) \cap \{x < x_0\} \cap \Omega_h^\#$, we get (4.26).

Step 1. We start by assuming first that $h'_+(x_0) < \infty$. We may also assume that h is not identically zero in a right neighborhood of x_0, since otherwise there is nothing to prove. By Proposition 3.5 there exists $r_1 > 0$ such that $h_{|(x_0,x_0+r_1)}$ is a Lipschitz function with Lipschitz constant L. For any $0 < r < r_1$ set

$$T_r(z_0) := \{(x, y) \in (x_0, x_0 + r) \times \mathbb{R} :\ 0 < y < h(x)\}\,.$$

Set also

$$v(x, y) := u(x, y) - e_0(x, 0) \qquad \text{for all } (x, y) \in \Omega_h^\#\,.$$

We claim that there exist $C_1 > 0$, $\beta > 1/2$ such that for all $\tau \in (0, 1/2]$ there exists $r_\tau > 0$ with the property that

$$\int_{T_{\tau r}(z_0)} |\nabla v|^2\, dz \le C_1\tau^{2\beta} \int_{T_r(z_0)} \big(1 + |\nabla v|^2\big)\, dz \qquad \text{for all } r \in (0, r_\tau)\,. \tag{4.28}$$

We argue by contradiction assuming that (4.28) does not hold, i.e., there exist $\tau_0 \in (0, 1/2]$ and a sequence of radii r_n converging to zero such that

$$\int_{T_{\tau_0 r_n}(z_0)} |\nabla v|^2\, dz > C_1\tau_0^{2\beta} \int_{T_{r_n}(z_0)} \big(1 + |\nabla v|^2\big)\, dz \qquad \text{for all } n \in \mathbb{N} \tag{4.29}$$

(C_1 and β will be determined at the end of Step 3).

Define the sets

$$T_n := \frac{1}{r_n}(T_{r_n}(z_0) - z_0) = \Big\{(s, t) \in \mathbb{R}^2 :\ 0 < s < 1,\ 0 < t < \frac{h(x_0 + r_n s)}{r_n}\Big\}\,.$$

Setting $g_n(s) := h(x_0 + r_n s)/r_n$, $g_\infty(s) := h'_+(x_0)s$ for $s \in [0, 1]$, since by Proposition 3.5 h'_+ is right continuous in 0, it is easily checked that

$$g_n \to g_\infty \ \text{ in } C([0, 1]), \qquad g'_n \to g'_\infty \ \text{ in } L^p(0, 1) \text{ for all } p \ge 1\,, \tag{4.30}$$

In particular, χ_{T_n} converges almost everywhere to χ_{T_∞}, where

$$T_\infty := \{(s, t) \in \mathbb{R}^2 :\ 0 < s < 1,\ 0 < t < g_\infty(s)\}\,.$$

Note that $T_\infty = \emptyset$ if $h'_+(x_0) = 0$. We rescale also v by setting

$$v_n(z) := \frac{v(z_0 + r_n z)}{\lambda_n r_n} \qquad \text{for all } z \in T_n\,,$$

where $\lambda_n > 0$ and

$$\lambda_n^2 := \frac{1}{|T_{r_n}(z_0)|} \int_{T_{r_n}(z_0)} |\nabla v|^2\, dz\,.$$

Up to a (not relabeled) subsequence, we may assume that $\lambda_n \to \lambda_\infty$ as $n \to \infty$. Note that by (4.29) we have that $\lambda_\infty \in (0, \infty]$. Since

$$\frac{1}{|T_n|} \int_{T_n} |\nabla v_n|^2\, dz = 1\,, \tag{4.31}$$

and $v_n = 0$ on $\partial T_n \cap \{y = 0\}$, observing that the functions g_n are are Lipschitz continuous with Lipschitz constants bounded by L, we may extend each function v_n to a function (still denoted by v_n) defined in the rectangle $R = (0, 1) \times (0, 2L)$ so that

$$\|v_n\|_{W^{1,2}(R;\mathbb{R}^2)} \leq c(L)\|\|\nabla v_n\|\|_{L^2(T_n)} \leq c\,.$$

Therefore, up to a subsequence, we may also assume that there exists $v_\infty \in W^{1,2}(R; \mathbb{R}^2)$, with $v_\infty = 0$ on $(0, 1) \times \{0\}$, such that $v_n \rightharpoonup v_\infty$ weakly in $W^{1,2}(R; \mathbb{R}^2)$. Moreover, from the definition of v_n we have that for every $\varphi \in C^1(\overline{R}; \mathbb{R}^2)$ vanishing on $[0, 1] \times \{0\}$ and on $\{1\} \times [0, 2L]$

$$\int_{T_n} \mathbb{C}E(v_n) : E(\varphi)\, dz = -\frac{e_0}{\lambda_n} \int_{\partial T_n} \big[(2\mu + \lambda)\nu_1\varphi_1 + \lambda\varphi_2\nu_2\big]\, d\mathcal{H}^1\,. \tag{4.32}$$

If $h'_+(x_0) > 0$, letting $n \to \infty$ in (4.32), we get

$$\int_{T_\infty} \mathbb{C}E(v_\infty) : E(\varphi)\, dz = -\frac{e_0}{\lambda_\infty} \int_{\partial T_\infty} \big[(2\mu + \lambda)\nu_1\varphi_1 + \lambda\varphi_2\nu_2\big] d\mathcal{H}^1 \tag{4.33}$$

for every $\varphi \in C^1(\overline{R}; \mathbb{R}^2)$ vanishing on $[0, 1] \times \{0\}$ and on $\{1\} \times [0, 2L]$. In fact, the convergence of the left-hand side of (4.32) to the left-hand side of (4.33) is obvious, while the convergence of the right-hand side follows from (4.30) observing that

$$\begin{aligned}\lim_{n\to\infty} \int_{\partial T_n} \big[(2\mu + \lambda)\nu_1\varphi_1 + \lambda\varphi_2\nu_2\big] d\mathcal{H}^1 &= \lim_{n\to\infty} \int_0^1 \big[-(2\mu + \lambda)g'_n(s)\varphi_1(s, g_n(s)) + \lambda\varphi_2(s, g_n(s))\big] ds\\ &= \int_0^1 \big[-(2\mu + \lambda)g'_\infty(s)\varphi_1(s, g_\infty(s)) + \lambda\varphi_2(s, g_\infty(s))\big] ds\\ &= \int_{\partial T_\infty} \big[(2\mu + \lambda)\nu_1\varphi_1 + \lambda\varphi_2\nu_2\big] d\mathcal{H}^1\,.\end{aligned}$$

Note that if $h'_+(x_0) = 0$, we get no limit equation, since both sides of (4.32) converge to zero.

Step 2. Let us now fix a function $\psi \in C^1(\overline{R}; \mathbb{R}^2)$ vanishing on $\{1\} \times [0, 2L]$.

If $h'_+(x_0) > 0$, from (4.32) and (4.33) we get, setting $\varphi = \psi^2 v_n$ and $\varphi = \psi^2 v_\infty$, respectively,

$$\begin{aligned}\int_{T_n} \psi^2\, \mathbb{C}E(v_n) : E(v_n)\, dz &= -\int_{T_n} \psi\, \mathbb{C}E(v_n) : \big(v_n \otimes \nabla\psi + (v_n \otimes \nabla\psi)^T\big)\, dz\\ &\quad - \frac{e_0}{\lambda_n} \int_0^1 \big[-(2\mu + \lambda)g'_n(s)v_{n,1}(s, g_n(s))\psi_1^2(s, g_n(s)) + \lambda v_{n,2}(s, g_n(s))\psi_2^2(s, g_n(s))\big] ds\\ \int_{T_\infty} \psi^2\, \mathbb{C}E(v_\infty) : E(v_\infty)\, dz &= -\int_{T_\infty} \psi\, \mathbb{C}E(v_\infty) : \big(v_\infty \otimes \nabla\psi + (v_\infty \otimes \nabla\psi)^T\big)\, dz\\ &\quad - \frac{e_0}{\lambda_\infty} \int_0^1 \big[-(2\mu + \lambda)g'_\infty(s)v_{\infty,1}(s, g_\infty(s))\psi_1^2(s, g_\infty(s)) + \lambda v_{\infty,2}(s, g_\infty(s))\psi_2^2(s, g_\infty(s))\big] ds\,.\end{aligned}$$

Therefore, from the two previous equations we may conclude that

$$\lim_{n\to\infty} \int_{T_n} \psi^2\, \mathbb{C}E(v_n) : E(v_n)\, dz = \int_{T_\infty} \psi^2\, \mathbb{C}E(v_\infty) : E(v_\infty)\, dz\,, \tag{4.34}$$

provided that we show that the right-hands side converge to the corresponding right-hand side. To this aim, note that the convergence

$$\int_{T_n} \psi\, \mathbb{C}E(v_n) : \big(v_n \otimes \nabla\psi + (v_n \otimes \nabla\psi)^T\big) \longrightarrow \int_{T_\infty} \psi\, \mathbb{C}E(v_\infty) : \big(v_\infty \otimes \nabla\psi + (v_\infty \otimes \nabla\psi)^T\big)$$

is an easy consequence of the weak convergence in $W^{1,2}(R;\mathbb{R}^2)$ of v_n to v_∞. Let us now prove that

$$\int_0^1 g_n'(s)v_{n,1}(s,g_n(s))\psi_1^2(s,g_n(s))\,ds \longrightarrow \int_0^1 g_\infty'(s)v_{\infty,1}(s,g_\infty(s))\psi_1^2(s,g_\infty(s))\,ds\,. \tag{4.35}$$

The proof of the convergence

$$\int_0^1 v_{n,2}(s,g_n(s))\psi_2^2(s,g_n(s))\,ds \longrightarrow \int_0^1 g_\infty'(s)v_{n,2}(s,g_\infty(s))\psi_2^2(s,g_\infty(s))\,ds\,,$$

is similar and actually simpler. To prove (4.35) observe that, since $v_n(s,0)=v_\infty(s,0)=0$ for $s\in(0,1)$, we have

$$\begin{aligned}\lim_{n\to\infty}\int_0^1 g_n'(s)v_{n,1}(s,g_n(s))\psi_1^2(s,g_n(s))\,ds &= \lim_{n\to\infty}\int_0^1 g_n'(s)\psi_1^2(s,g_n(s))\int_0^{g_n(s)}\frac{\partial v_{n,1}}{\partial y}(s,t)\,dsdt\\ &= \lim_{n\to\infty}\int_{T_n} g_n'(s)\psi_1^2(s,g_n(s))\frac{\partial v_{n,1}}{\partial y}(s,t)\,dsdt = \int_{T_\infty} g_\infty'(s)\psi_1^2(s,g_\infty(s))\frac{\partial v_{\infty,1}}{\partial y}(s,t)\,dsdt\\ &= \int_0^1 g_\infty'(s)v_{\infty,1}(s,g_\infty(s))\psi_1^2(s,g_\infty(s))\,ds\,.\end{aligned}$$

Now that we have established (4.34) observe that it implies that

$$\lim_{n\to\infty}\int_{T_n}\psi^2\,\mathbb{C}E(v_n-v_\infty):E(v_n-v_\infty)\,dz=0$$

for all $\psi\in C^1(\overline{R};\mathbb{R}^2)$ vanishing on $\{1\}\times[0,2L]$. Finally, from this last equation, recalling that $v_n=v_\infty=0$ on $(0,1)\times\{0\}$ and using Theorem 4.8, we conclude that

$$\lim_{n\to\infty}\int_{\tau T_n}|\nabla v_n-\nabla v_\infty|^2\,dz=0 \qquad \text{for all } 0<\tau<1\,. \tag{4.36}$$

Let us now show that if $h_+'(x_0)=0$, we have

$$\lim_{n\to\infty}\int_{\tau T_n}|\nabla v_n|^2\,dz=0 \qquad \text{for all } 0<\tau<1\,. \tag{4.37}$$

In fact, to prove this equality we choose, as before, $\varphi=\psi^2 v_n$ in (4.32), thus getting

$$\begin{aligned}\int_{T_n}\psi^2\,\mathbb{C}E(v_n):E(v_n)\,dz &= -\int_{T_n}\psi\,\mathbb{C}E(v_n):\big(v_n\otimes\nabla\psi+(v_n\otimes\nabla\psi)^T\big)\,dz\\ &+\frac{e_0}{\lambda_\infty}\int_0^1\big[-(2\mu+\lambda)g_n'(s)v_{n,1}(s,g_n(s))\psi_1^2(s,g_n(s))+\lambda v_{n,2}(s,g_n(s))\psi_2^2(s,g_n(s))\big]ds\,.\end{aligned}$$

Then we conclude that the right-hand side of this equality tends to zero. In fact

$$\lim_{n\to\infty}\int_{T_n}\psi\,\mathbb{C}E(v_n):\big(v_n\otimes\nabla\psi+(v_n\otimes\nabla\psi)^T\big)\,dz=0\,,$$

since $v_n\to v_\infty$ weakly in $W^{1,2}(R;\mathbb{R}^2)$ and $\chi_{T_n}\to0$ a.e. in R, while, arguing as before, we have

$$\begin{aligned}&\lim_{n\to\infty}\int_0^1\big[-(2\mu+\lambda)g_n'(s)v_{n,1}(s,g_n(s))\psi_1^2(s,g_n(s))+\lambda v_{n,2}(s,g_n(s))\psi_2^2(s,g_n(s))\big]ds\\ &\quad=\int_0^1\big[-(2\mu+\lambda)g_\infty'(s)v_{\infty,1}(s,g_\infty(s))\psi_1^2(s,g_\infty(s))+\lambda v_{\infty,2}(s,g_\infty(s))\psi_2^2(s,g_\infty(s))\big]ds=0\,,\end{aligned}$$

since $g_\infty\equiv0$ and $v_\infty(s,0)=0$ for $s\in(0,1)$. Therefore, we may conclude that

$$\lim_{n\to\infty}\int_{T_n}\psi^2\,\mathbb{C}E(v_n):E(v_n)\,dz=0\,,$$

hence, (4.37) follows.

Step 3. Assume first $h'_+(x_0) > 0$ and set

$$u_\infty := \frac{e_0}{\lambda_\infty}(x,0) + v_\infty \,.$$

From (4.33) we have that u_∞ solves the problem

$$\begin{cases} \mu\Delta u_\infty + (\lambda+\mu)\nabla(\operatorname{div} u_\infty) = 0 \text{ in } T_\infty \\ u_\infty = \dfrac{e_0}{\lambda_\infty}(x,0) \text{ on } (0,1)\times\{0\}, \;\; \sigma(u_\infty)[\nu] = 0 \text{ on } \{(s, g_\infty(s)) : s \in (0,1)\}\,. \end{cases}$$

Therefore, from Theorem 4.7 we get that there exist $C > 0$, $\beta \in (1/2, 1]$, depending only on λ, μ and $h'_+(0)$, but not on e_0, λ_∞, such that

$$\int_{\tau T_\infty} |\nabla u_\infty|^2\, dz \le C\tau^{2\beta} \int_{T_\infty} \big(|u_\infty|^2 + |\nabla u_\infty|^2\big)\, dz \qquad \text{for all } 0 < \tau \le 1/2\,.$$

From this inequality, recalling that $v_\infty = 0$ on $(0,1)\times\{0\}$, we have in particular that

$$\int_{\tau_0 T_\infty} |\nabla v_\infty|^2\, dz \le C\tau_0^{2\beta} \int_{T_\infty} \Big(\frac{e_0^2}{\lambda_\infty^2} + |\nabla v_\infty|^2\Big)\, dz\,,$$

for some C again depending only on λ, μ and $h'_+(x_0)$, but not on e_0, λ_∞. Recalling (4.31) and (4.36) we have

$$\int_{\tau_0 T_\infty} |\nabla v_\infty|^2\, dz = \lim_{n\to\infty} \int_{\tau_0 T_n} |\nabla v_n|^2\, dz = \lim_{n\to\infty} \frac{1}{r_n^2\lambda_n^2} \int_{T_{\tau_0 r_n}(z_0)} |\nabla v|^2\, dz \le C_2\tau_0^{2\beta} \lim_{n\to\infty}\Big(1 + \frac{e_0^2}{\lambda_n^2}\Big)\,,$$

for some C_2 independent of e_0 and λ_∞. Therefore, for n large,

$$\int_{T_{\tau_0 r_n}(z_0)} |\nabla v|^2\, dz < 2C_2\tau_0^{2\beta}(r_n^2\lambda_n^2 + r_n^2 e_0^2)\,,$$

hence,

$$\int_{T_{\tau_0 r_n}(z_0)} |\nabla v|^2\, dz < C_3\tau_0^{2\beta} \int_{T_{r_n}(z_0)} \big(1 + |\nabla v|^2\big)\, dz\,,$$

for some C_3 depending only on λ, μ, $h'_+(x_0)$, e_0 and L, but not on λ_∞. This inequality contradicts (4.29), if we choose $C_1 > C_3$, thus proving (4.28). Therefore we may conclude that there exists $C_4 > 0$ such that for all $\tau \in (0, 1/2]$ there exists $r_\tau > 0$ with the property that

$$\int_{T_{\tau r}(z_0)} \big(1 + |\nabla u|^2\big)\, dz \le C_4\tau^{2\beta} \int_{T_r(z_0)} \big(1 + |\nabla u|^2\big)\, dz \qquad \text{for all } r \in (0, r_\tau]\,.$$

Fix now $\alpha \in (1/2, \beta)$, $\tau \in (0, 1/2]$ such that $C_4\tau^{2\beta} \le \tau^{2\alpha}$. Iterating the previous inequality we then get

$$\int_{T_{\tau^k r_\tau}(z_0)} \big(1 + |\nabla u|^2\big)\, dz \le \tau^{2k\alpha} \int_{T_{\tau^{k-1} r_\tau}(z_0)} \big(1 + |\nabla u|^2\big)\, dz \qquad \text{for all } k \in \mathbb{N}\,.$$

From this inequality (4.27) easily follows.

If $h'_+(x_0) = 0$, the argument is easier. In fact, from (4.37) we have

$$\lim_{n\to\infty} \int_{\tau_0 T_n} |\nabla v_n|^2\, dz = 0\,,$$

hence

$$\lim_{n\to\infty} \frac{1}{r_n^2\lambda_n^2} \int_{T_{\tau_0 r_n}(z_0)} |\nabla v|^2\, dz = 0$$

and the conclusion follows as before.

Step 4. We are left with the case $h'_+(x_0) = +\infty$. In this case, by Proposition 3.5 there exists a rectangle $R = (x_0, x_0 + b') \times (0, a')$ such that $\Omega_h^\# \cap R$ lies above the graph of a Lipschitz function $g : [0, a') \to [x_0, x_0 + b')$ with right and left derivatives at every point, which are right and left continuous respectively and such that $g'(0) = 0$. The proof then goes exactly as before. □

5. Proof of Theorem 2.5

By Proposition 3.5, for each point $z_0 \in \Gamma_h^\# \setminus (\Sigma_h^\# \cup \Sigma_{h,c}^\#)$ there exists a rectangular neighborhood R such that $\Gamma_h^\# \cap R$ is the graph of a Lipschitz function g having right and left derivatives at every point, which are right and left continuous, respectively. We shall say that z_0 is a *corner point* if the corresponding right and left derivatives of g are not equal. Next result shows that $\Gamma_h^\# \setminus (\Sigma_h^\# \cup \Sigma_{h,c}^\#)$ has not corner points, hence it is of class C^1.

Proposition 5.1. *Let (h, u) be a local minimizer of F. Then, $\Gamma_h^\# \setminus (\Sigma_h^\# \cup \Sigma_{h,c}^\#)$ is of class C^1.*

Proof. As we have observed above, it is enough to show that if $z_0 = (x_0, y_0) \in \Gamma_h^\# \setminus (\Sigma_h^\# \cup \Sigma_{h,c}^\#)$, then z_0 is not a corner point. To this aim we assume $y_0 = 0$, the case $y_0 > 0$ being similar and actually simpler (see [6, Theorem 3.14]).

Let $r \in (0, r_0)$, where r_0 is as in Theorem 4.9, and let $z'_r = (x'_r, y'_r)$ the point in $\Gamma_h^\# \cap \partial B_r(z_0)$, with the smallest abscissa. Similarly, let $z''_r = (x''_r, y''_r)$ the point in $\Gamma_h^\# \cap \partial B_r(z_0)$, with the largest abscissa. Set

$$\tilde{h}(x) = \begin{cases} h(x) & \text{if } x \notin [x'_r, x''_r] \\ s(x) & \text{if } x \in [x'_r, x''_r], \end{cases}$$

where $s : [x'_r, x''_r] \to \mathbb{R}$ is the affine function whose graph is the segment connecting z'_r and z''_r. Set $\widetilde{\Omega}_h^\# = \Omega_h^\# \cup (\mathbb{R} \times (-\infty, 0])$ and extend u to $\widetilde{\Omega}_h^\#$ so that the new function, still denoted by u, satisfies $u(x, y) = e_0(x, 0)$ for all $x \in \mathbb{R}$, $y \le 0$. From (4.26) we have that

$$\int_{B_r(z_0) \cap \widetilde{\Omega}_h^\#} |\nabla u|^2 \, dz \le C r^{2\alpha}$$

for some $\alpha > 1/2$ and $C > 0$ independent of r. Then, since $\widetilde{\Omega}_h^\# \cap B_r(z_0)$ is a Lipschitz domain, we may extend u to a function defined on $B_r(z_0)$ and still denoted by u such that

$$\int_{B_r(z_0)} |\nabla u|^2 \, dz \le C' r^{2\alpha}$$

where C' is independent of r. Then, from Proposition 3.1 we have

$$\begin{aligned} 0 &\ge F(h, u) - F(\tilde{h}, u) - \Lambda \big| |\Omega_{\tilde{h}}| - |\Omega_h| \big| \\ &\ge \mathcal{H}^1(\Gamma_h^\# \cap B_r(z_0)) - |z'_r - z''_r| - \int_{B_r(z_0)} W(E(u)) \, dz - \Lambda \pi r^2 \\ &\ge |z'_r - z_0| + |z''_r - z_0| - |z'_r - z''_r| - C' r^{2\alpha} - \Lambda \pi r^2 . \end{aligned}$$

Then, from the previous chain of inequalities we obtain that

$$2r - |z'_r - z''_r| \le c r^{2\alpha},$$

for some $c > 0$ independent of r. Therefore, dividing both sides of the inequality above by r, we have

$$2 - \frac{|z'_r - z''_r|}{r} \le c r^{2\alpha - 1}.$$

Letting $r \to 0$, since $\alpha > 1/2$, we obtain $\lim_{r\to 0^+} \frac{|z'_r - z''_r|}{r} = 2$, thus showing that the left and right tangent lines at z_0 coincide. This concludes the proof. □

Let us now prove that $\Gamma_h^\# \setminus (\Sigma_h^\# \cup \Sigma_{h,c}^\#) \cap \{y > 0\}$ is of class $C^{1,\alpha}$ for all $\alpha \in (0, 1/2)$. For the sake of completeness, we reproduce here the proof given in [6], with some small simplifications. To this aim we recall the following decay estimate, which is more or less classic and that can be obtained arguing as in Theorem 4.9 (see also [6, Theorem 3.16]).

Theorem 5.2. *Let (h, u) be a local minimizer for the functional F. Then for every closed subarc $\Gamma \subset \Gamma_h^\# \setminus (\Sigma_h^\# \cup \Sigma_{h,c}^\#) \cap \{y > 0\}$ and for every $0 < \sigma < 1$ there exist a constant $C > 0$ and a radius r_0 such that for all $0 < r < R_0$ and for all $z_0 \in \Gamma$*

$$\int_{B_r(z_0)\cap\Omega_h^\#} |\nabla u|^2 \, dz \le C r^{2\sigma} \,.$$

We conclude by proving statement (iii) of Theorem 2.5.

Proposition 5.3. *Let (h, u) be a local minimizer of F. Then, $\Gamma_h^\# \setminus (\Sigma_h^\# \cup \Sigma_{h,c}^\#) \cap \{y > 0\}$ is of class $C^{1,\alpha}$ for all $\alpha \in (0, 1/2)$.*

Proof. Let us fix an open subarc $\Gamma \subset \Gamma_h^\# \setminus (\Sigma_h^\# \cup \Sigma_{h,c}^\#) \cap \{y > 0\}$ such that Γ is the graph of a C^1 function. To fix the ideas (the general case being similar), let us assume that $\Gamma = \{(x, h(x)) : x \in (a', b')\}$ and that $h \in C^1([a', b'])$. Set $M := \max\{|h'(x)| : x \in [a', b']\}$ and fix $\sigma \in (1/2, 1)$. Then, from Theorem 5.2 there exist $r_1 > 0$ and $C > 0$ such that for all $x_0 \in (a', b')$ and $r \in (0, r_0)$

$$\int_{C_r(x_0)\cap\Omega_h^\#} |\nabla u|^2 \, dz \le C r^{2\sigma} \,,$$

where $C_r(x_0) := (x_0 - r, x_0 + r) \times (h(x_0) - Mr, h(x_0) + Mr)$. As in the proof of Proposition 5.1 we may extend u to $C_r(x_0)$ so that the resulting extension, still denoted by u, satisfies the estimate (see Theorem 5.2)

$$\int_{C_r(z_0)} |\nabla u|^2 \, dz \le C r^{2\sigma} \,, \tag{5.1}$$

for some C independent of x_0 and r. We set now

$$\tilde h(x) = \begin{cases} h(x) & \text{if } x \notin [x_0, x_0 + r] \\ s(x) & \text{if } x \in [x_0, x_0 + r] \,, \end{cases}$$

where $s : [x_0, x_0 + r] \to \mathbb{R}$ is the piecewise affine function whose graph connects $(x_0, h(x_0))$ to $(x_0 + r, h(x_0 + r))$. Then, from Proposition 3.1 and (5.1), arguing as in the proof of Proposition 5.1, we easily get that, for r is sufficiently small (but not depending on x_0),

$$\int_{x_0}^{x_0+r} \sqrt{1 + h'^2} \, dx - \sqrt{(h(x_0 + r) - h(x_0))^2 + r^2} \le c r^{2\sigma} \,. \tag{5.2}$$

Using the elementary inequality

$$\sqrt{1 + b^2} - \sqrt{1 + a^2} \ge \frac{a(b - a)}{\sqrt{1 + a^2}} + \frac{(b - a)^2}{2(1 + \max\{a^2, b^2\})^{3/2}}$$

with $a := \fint_{x_0}^{x_0+r} h'\,dx$ and $b := h'(x)$, and integrating the result in (x_0, x_0+r), we get

$$\frac{1}{2(1+M^2)^{3/2}} \fint_{x_0}^{x_0+r} \Big(h'(x) - \fint_{x_0}^{x_0+r} h'\,ds\Big)^2 dx$$
$$\leq \frac{1}{r}\int_{x_0}^{x_0+r} \sqrt{1+h'^2}\,dx - \frac{1}{r}\sqrt{(h(x_0+r)-h(x_0))^2+r^2} \leq cr^{2\sigma-1}\,,$$

where we also used (5.2). Thus, in particular,

$$\fint_{x_0}^{x_0+r} \Big| h'(x) - \fint_{x_0}^{x_0+r} h'\,ds \Big|\,dx \leq cr^{\sigma-\frac{1}{2}}\,.$$

A similar inequality holds also in the interval $(x_0 - r, x_0)$. Hence, by [1, Theorem 7.51] we conclude that h is in $C^{1,\sigma-\frac{1}{2}}([a', b'])$. This proves the assertion. □

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