UNIVERSAL DIFFERENTIABILITY SETS IN LAAKSO SPACE

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Abstract. We show that there exists a family of mutually singular doubling measures on Laakso space with respect to which real-valued Lipschitz functions are almost everywhere differentiable. This implies that there exists a measure zero universal differentiability set in Laakso space. Additionally, we show that each of the measures constructed supports a Poincaré inequality.

1. Introduction

Rademacher's theorem states that Lipschitz functions between Euclidean spaces are differentiable Lebesgue almost everywhere. This result has many extensions and applications. One direction of research extends Rademacher's theorem to more general spaces, such as Banach spaces [3], Carnot groups [20], and metric measure spaces [7]. Another direction of research asks to what extent Rademacher's theorem is optimal [1, 8, 21, 24, 27]. The present paper contributes to this study in the setting of Laakso space, a metric measure space in which a version of Rademacher's theorem holds. Laakso space (Definition 2.3) is of the form $F := (I \times K)/\sim$, where K is the middle third Cantor set and $\sim$ is a suitable equivalence relation. It was first constructed by Laakso [18] who gave a family of spaces, depending on parameters, to show there exists an Ahlfors Q-regular metric measure space of any dimension $Q > 1$ which supports a Poincare inequality. See also [6] for a nice overview of the main properties of Laakso space.

It has been known for some time that Rademacher's theorem does not admit a converse for Lipschitz maps $f\colon \mathbb{R}^n \to \mathbb{R}$ for $n \geq 2$, see e.g. [10, 11]. To be more precise, there exists a Lebesgue measure zero set $N \subset \mathbb{R}^n$ containing a point of differentiability for every real-valued Lipschitz map on $\mathbb{R}^n$. Such a set N is often called a universal differentiability set (UDS). In addition to the Euclidean case, measure zero UDS are known to exist in some classes of Carnot groups, which include all step two Carnot groups and examples of arbitrarily high step [22, 19, 23]. The key technique used to prove these results is a refinement of the fact that existence of a maximal directional derivative implies differentiability [15]. Surprisingly, the second and third authors showed in [5] that this fact does not have a simple analogue in Laakso space. This left open the question of whether measure zero UDS exist in Laakso space. The present paper answers this question by showing that they do, by means of a different method.

Another natural way to study optimality of Rademacher's theorem is to ask whether Lipschitz functions can be differentiable almost everywhere with respect to another measure on the same space. It is known that in Euclidean spaces such a measure must be absolutely continuous with respect to Lebesgue measure [13] and in Carnot groups it must be absolutely continuous with respect to the natural Haar measure [12]. In this paper we show that these results do not extend to the Laakso space with its natural Hausdorff measure. In particular, there exists a family of mutually singular measures with respect to which real-valued Lipschitz functions are differentiable almost everywhere. It should be stressed that the idea behind the construction of the measures comes from work of Schioppa [26], who constructed a family of mutually singular measures on a metric measure space which are all doubling and support a Poincare inequality. The space Schioppa used is different from that of Laakso.

We now describe our main results. Our first main result gives a family of mutually singular doubling measures for which Rademacher's theorem holds.

Theorem 1.1. *There exist doubling measures μ_w on F for each $w \in (0,1)$ so that*
1. *μ_w and $\mu_{w'}$ are mutually singular whenever $w \neq w'$, and*
2. *for each $w \in (0,1)$, every Lipschitz map $f\colon F \to \mathbb{R}$ is differentiable almost everywhere with respect to μ_w.*

Each measure μ_w is the push forward of $\mathcal{H}^1 \times \nu_w$ under the quotient map, where ν_w is a suitable measure on K. The measure ν_w is defined by assigning a proportion w of the measure to the left similar copy of K and a proportion $1-w$ of the measure to the right similar copy of K at any stage in the construction of K. These measures are mutually singular for distinct w, while the measure $\nu_{1/2}$ is proportional to the natural Hausdorff measure on K. Due to the structure of μ_w (not least the Euclidean behavior in the I direction and that μ_w is doubling so the Lebesgue density theorem holds), one can adapt the explicit proof of Rademacher's theorem given in [5] to μ_w for any $w \in (0,1)$. This gives Theorem 1.1.

Our second main result deduces the existence of measure zero UDS in Laakso space. It is an immediate consequence of Theorem 1.1. To describe the UDS, first denote the left and right similar copies of the middle third Cantor set K by K_0 and K_1 respectively. Similarly we define K_a for any finite string a of 0's and 1's. For integer $n \geq 1$, define $X_n : K \to \mathbb{R}$ by $X_n = 1$ on K_{a0} and $X_n = 0$ on K_{a1} for any binary string a of length greater or equal to 0. We set $Q := 1 + (\log 2/\log 3)$, noting that F is Ahlfors Q-regular.

Theorem 1.2. *There exists a Borel set $N \subset F$ with $\mathcal{H}^Q(N) = 0$ such that every Lipschitz map $f\colon F \to \mathbb{R}$ is differentiable at a point of N.*

More precisely, we can choose N to be $q(I \times E_w)$ for any $w \neq 1/2$, where the map $q\colon I \times K \to F$ is the quotient mapping sending x to $[x]$ and

$$E_w = \{x \in K : \frac{1}{n}\sum_{k=1}^{n} X_n(x) \to w \text{ as } n \to \infty\}.$$

Historically, Rademacher's theorem for general metric spaces was discovered by Cheeger for spaces that satisfy a Poincaré inequality [7]. Despite this, a special feature of our work is that we do not use a Poincaré inequality to prove the above theorems. However, as recognized in [14], there is a partial converse and a Rademacher's theorem (for certain Banach-valued Lipschitz functions) implies a Poincaré inequality. Thus, it is relevant to study whether the presently studied spaces also satisfy a Poincaré inequality. This also draws a closer parallel to the work of Schioppa in [26], and shows that this work completely extends to Laakso spaces. Indeed, it shows that the Laakso space admits an uncountable family of mutually singular measures supporting a Poincaré inequality. We defer to Section 5 for definitions and a more detailed discussion.

Theorem 1.3. *For every $w \in (0,1)$ the space (F, d, μ_w) satisfies a $(1,1)$-Poincaré inequality.*

The organization of the paper is as follows. In Section 2 we review relevant background, including the definition of Laakso space and the notion of derivatives in this context. In Section 3 we construct the measures and prove their main properties. In Section 4 we show how the proof of Rademacher's theorem can be adapted from [5] and deduce the main results. We prove Theorem 1.3 in Section 5.

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2. Preliminaries

The terminology and construction in this paper follow that of [18]. Let $I = [0,1]$ and let $K \subset [0,1]$ the standard middle third Cantor set. Define $K_0 := (1/3)K$ and $K_1 := (1/3)K + (2/3)$ to be the left and right similar copies of K. We then define $K_{00} := (1/3)K_0 = (1/9)K$ and $K_{01} := (1/3)K_1 = (1/9)K + (2/9)$ to be the left and right similar copies of K_0. The set K_a is defined similarly when a is any finite string of 0's and 1's. We refer to such a string a as a binary string.

We define the height of a point $(x_1, x_2) \in I \times K$ by $h(x_1, x_2) := x_1$. If $n \in \mathbb{N}$ and $m_i \in \{0,1,2\}$ for $1 \leq i \leq n$, we define $w(m_1, \ldots, m_n) := \sum_{i=1}^{n} m_i/3^i$. A wormhole level of order n is a set of the form

$$\{w(m_1, \ldots, m_n)\} \times K \subset I \times K, \qquad m_n > 0.$$

Definition 2.1. We define an equivalence relation $\sim$ on $I \times K$ as follows. For each $n \in \mathbb{N}$ and wormhole level $\{w(m_1, \ldots, m_n)\} \times K$ of order n, identify pairwise $\{w(m_1, \ldots, m_n)\} \times K_{a0}$ and $\{w(m_1, \ldots, m_n)\} \times K_{a1}$ for each binary string a of length $n-1$. More precisely, a point $(x_1, x_2) \in \{w(m_1, \ldots, m_n)\} \times K_{a0}$ is identified with $(x_1, x_2 + (2/3^n)) \in \{w(m_1, \ldots, m_n)\} \times K_{a1}$. Such an identified point is called a wormhole of order n.

We denote the set of wormholes of order n by $J_n := \{w(m_1, \ldots, m_n) : m_i \in \{0,1,2\}, m_n > 0\} \times K$. Define $F := (I \times K)/\sim$. Let $q\colon I \times K \to F$ be given by $q(x_1, x_2) = [x_1, x_2]$, where $[x_1, x_2]$ denotes the equivalence class in F of $(x_1, x_2) \in I \times K$. We define the height $h\colon F \to I$ by $h[x_1, x_2] = x_1$. We define a metric d on F by

$$d(x, y) = \inf\{\mathcal{H}^1(p)\colon q(p) \text{ is a path joining } x \text{ and } y\},$$

where $p \subset I \times K$. In [18] it is shown that any pair of points can be connected by a path and so the metric d is well defined. The following proposition gives information about geodesics [18, Proposition 1.1].

Proposition 2.2. *Fix $x, y \in F$ with $h(x) \leq h(y)$. Let $[a,b] \subset I$ be an interval of minimum length that contains the heights of x and y and all the wormhole levels needed to connect those points with a path. Let p be any path starting from x, going down to height a, then up to height b, then down to y.*

Then p is a geodesic connecting x and y. All geodesics from x to y are of that form for some interval $[a', b']$ such that $b' - a' = b - a$.

Let $Q := 1 + (\log 2/\log 3)$. Note that K is Ahlfors $(Q-1)$-regular. It is shown in [18] that F is Ahflors Q-regular with respect to the metric d.

Definition 2.3. The Laakso space is the set of equivalence classes $F := (I \times K)/\sim$ equipped with the metric d and Hausdorff dimension $\mathcal{H}^Q$.

There are multiple ways that one can construct a Laakso space, and for simplicity we focus on the one particular construction that we gave, and call it the Laakso space.

Differentiability in F is meant with respect to the Lipschitz chart (F, h). This can be written explicitly as in the following definition.

Definition 2.4. Let $f\colon F \to \mathbb{R}$ and $x \in F$. We say that f is differentiable at x if there exists $Df(x) \in \mathbb{R}$ such that

$$\lim_{y \to x} \frac{f(y) - f(x) - Df(x)(h(y) - h(x))}{d(y, x)} = 0.$$

The Laakso space is known to be a PI space [18], hence admits a differentiable structure consisting of Lipschitz charts with respect to which Lipschitz functions are almost everywhere differentiable [7]. However, these results do not give the charts explicitly. The following theorem was proved explicitly in [5] with the Lipschitz chart is (F, h), so the notion of differentiability is as in Definition 2.4.

Theorem 2.5. *Every Lipschitz function $f\colon F \to \mathbb{R}$ is differentiable almost everywhere.*

Note that it seems likely Theorem 2.5 also follows from [9, Chapter 9, Theorem 9.1] on inverse limit spaces, once the Laakso space is recognized as such a space.

We will also make use of the directional derivatives defined below. As mentioned in [5], this is a weaker requirement than being differentiable.

Definition 2.6. Let $f\colon F \to \mathbb{R}$ and $x = [x_1, x_2] \in F$.

Suppose x is not a wormhole. Whenever the limit exists, we define

$$f_I(x) := \lim_{t \to 0} \frac{f[x_1 + t, x_2] - f[x_1, x_2]}{t}. \tag{2.1}$$

The limit is one-sided if $x_1 = 0$ or 1.

Suppose x is a wormhole of order n and $(x_1, x_2) \in I \times K$ is the representative of x with the smaller value of x_2. Whenever the limit exists, we define

$$f_L(x) := \lim_{t \to 0} \frac{f[x_1 + t, x_2] - f[x_1, x_2]}{t}$$
$$f_R(x) := \lim_{t \to 0} \frac{f[x_1 + t, x_2 + (2/3^n)] - f[x_1, x_2 + (2/3^n)]}{t}.$$

If $f_L(x)$ and $f_R(x)$ exist and are equal, we say that $f_I(x)$ exists and define it to be the common value. The limits are one-sided if $x_1 = 0, 1$.

3. Singular Doubling Measures on K and F

3.1. **Measures on K.** Given $w \in (0,1)$ and a binary string a, let $p_w(K_a) := w^s(1-w)^{N-s}$ where N is the length of the binary string a and s is the number of zeros in a. Intuitively, as the Cantor set is constructed by removing open middle thirds, p_w assigns mass w to the left similar copy and mass $1-w$ to the right similar copy.

Proposition 3.1. *For any $w \in (0,1)$, there is a unique Borel probability measure ν_w on K so that $\nu_w(K_a) = p_w(K_a)$ for each binary string a. For any Borel set $E \subset K$, we have*

$$\nu_w(E) = \inf\left\{\sum_{i=1}^{\infty} p_w(E_i) : E \subset \bigcup_{i=1}^{\infty} E_i,\ E_i \text{ similar copies of } K\right\}. \tag{3.1}$$

Proof. Let $\Sigma = \{0,1\}^{\mathbb{N}}$ be equipped with the product topology. Define a homeomorphism $\pi : \Sigma \to K$ by sending an infinite binary string $(a_i)_{i=1}^{\infty}$ to $\sum_{i=1}^{\infty} 2a_i 3^{-i} \in K$. Let $\nu_{w,\Sigma} = \prod_{i=1}^{\infty} \nu_0$ be a Bernoulli probability measure on Σ, that is the infinite product measure of binary probability measures ν_0 on $\{0,1\}$ where $\nu_0(\{0\}) = w$ and $\nu_0(\{1\}) = 1-w$. Let $\nu_w = \pi_*(\nu_{w,\Sigma})$. Then, $\nu_w(K_a) = p_w(K_a)$ follows directly from the definition of ν_w. Equation (3.1)follows since the similar copies of K generate the Borel σ-algebra of K, and uniqueness of ν_w follows for the same reason. □

We equip K with the induced Euclidean metric. Recall that a measure m on a metric space (X,d) is doubling if there is a constant $D > 1$ so that for all balls $B(x,r)$ in X it holds that $m(B(x,2r)) \le D\mu(B(x,r))$.

Proposition 3.2. *The probability measure ν_w is doubling on K for any $w \in (0,1)$.*

Proof. Let $x \in K$ and $0 < r \le 1/9$. Choose integer $N \ge 3$ with $\frac{1}{3^N} < r \le \frac{1}{3^{N-1}}$. Notice $B(x,r)$ must contain a similar copy of the Cantor set K_a which contains x and where the binary string a is of length N. If s is the number of zeros in a, then we have

$$\nu(B(x,r)) \ge \nu(K_a) = w^s(1-w)^{N-s}.$$

On the other hand, $2r \le \frac{2}{3^{N-1}} < \frac{1}{3^{N-2}}$. Since similar copies of level $N-2$ are separated by a distance $1/3^{N-2}$, this implies $B(x,2r)$ is contained inside K_b where b is the binary string equal to a except with the last two entries deleted. Hence

$$\nu(B(x,2r)) \le \nu(K_b) \le w^{s-2}(1-w)^{N-s-2}.$$

This implies $\nu(B(x,2r)) \le w^{-2}(1-w)^{-2}\nu(B(x,r))$. Hence ν is doubling. □

We next recall the strong law of large numbers from probability theory [2]. Recall that in the context of a probability space (Ω, Σ, P), where Σ is a σ-algebra and P is a probability measure, a random variable $X : \Omega \to [-\infty, \infty]$ is an extended real-valued measurable function on X. The mean or expectation $E(X)$ is simply the integral of X with respect to P if it exists. An event holds with probability 1 if it holds almost surely.

The distribution function of a random variable X is $F\colon \mathbb{R} \to [0,1]$ defined by $F(x) = P(X \le x) := P(\{w : X(w) \le x\})$. A collection of random variables is identically distributed if they have the same distribution function.

A finite collection of random variables $X_1, \dots, X_k$ is independent if

$$P(X_1 \leq x_1, \cdots, X_k \leq x_k) = P(X_1 \leq x_1) \cdots P(X_k \leq x_k).$$

An infinite collection of random variables is independent if each finite subcollection is independent.

Lemma 3.3 (Strong Law of Large Numbers)**.** *Let $X_1, X_2, \dots$ be random variables on a probability space. Assume they are independent and identically distributed and have finite mean. Then $\frac{1}{n}\sum_{k=1}^{n} X_k \to E(X_1)$ with probability 1.*

For each integer $n \geq 1$, define $X_n \colon K \to \mathbb{R}$ as follows. If $n = 1$, then X_1 is identically 1 on K_0 and identically 0 on K_1. If $n > 1$, then X_n is identically 1 on K_{a0} and identically 0 on K_{a1} for any binary string a of length $n-1$. Clearly X_n is Borel measurable for each $n \geq 1$. Denote $S_n = \sum_{k=1}^{n} X_k$. For each $w \in (0,1)$, define the Borel set

$$E_w = \{x \in K : S_n(x)/n \to w\}. \tag{3.2}$$

Proposition 3.4. *For any $w \in (0,1)$, we have $\nu_w(K \setminus E_w) = 0$ and $\nu_{w'}(E_w) = 0$ for all $w' \in (0,1) \setminus \{w\}$.*

In particular, for all $w, w' \in (0,1)$ with $w \neq w'$, the probability measures ν_w and $\nu_{w'}$ are mutually singular.

Proof. Fix $w \in (0,1)$. Recall the construction of ν_w from the proof of Proposition 3.1. The measure $\nu_{w,\Sigma}$ is a Bernoulli probability measure and the random variables $Y_i := X_i \circ \pi$ are of the form $1 - Z_i$, where Z_i are the independent and identically distributed projections onto the i'th component of Σ. In particular, Y_i are independent and identically distributed. Hence, in Σ, $\frac{1}{n}\sum_{i=1}^{n} Y_i \to w$ almost surely with respect to $\nu_{w,\Sigma}$ by Lemma 3.3. Since π is a measure preserving bijection, also $\frac{1}{n}\sum_{i=1}^{n} X_i(x) \to w$ for ν_w-almost every $x \in K$. Thus, $\nu_w(E_w) = 1$. Further, by the same argument, $\nu_{w'}(E_w) = 0$ for any $w' \in (0,1) \setminus \{w\}$. Thus, ν_w and $\nu_{w'}$ are pairwise singular. □

3.2. **Measures on F.** Recall that $\mathcal{H}^1$ denotes the Hausdorff measure on I with respect to the Euclidean distance. For any $w \in (0,1)$, we define

$$\mu_w = q_*(\mathcal{H}^1 \times \nu_w) \tag{3.3}$$

and

$$N_w = q(I \times E_w). \tag{3.4}$$

To prove that μ_w is doubling, the following simple lemma will be useful.

Lemma 3.5. *Suppose $x, y \in F$ with $d(x,y) < 1$. Let $N \geq 1$ be the unique integer satisfying $1/3^N \leq d(x,y) < 1/3^{N-1}$. Then any geodesic joining x to y can pass through at most one wormhole of level less than or equal to $N-1$.*

In particular, suppose $x \in F$ and $0 < r < 1$. Let $N \geq 1$ be the unique integer satisfying $1/3^N \leq r < 1/3^{N-1}$. Then at vertical distance at most r above and below x, one can find at most one wormhole with a level less than or equal to $N-2$.

Proof. The first part of the Lemma was proved in [5]. To prove the second part notice that $(h(x) - r, h(x) + r)$ has length $2r$ and

$$2r < 2/3^{N-1} < 1/3^{N-2}.$$

Since wormholes of level less than or equal to $N-2$ are spaced apart by a distance $1/3^{N-2}$, the conclusion follows. $\square$

Proposition 3.6. *For every $w \in (0,1)$, μ_w is a doubling measure with respect to the metric d on F.*

Proof. We denote $\nu = \nu_w$ and $\mu = \mu_w$ for convenience. That μ is Borel follows from continuity of q. Fix $x = [x_1, x_2] \in F$ and $0 < r < 1/3$. Fix an integer N such that $1/3^N \leq r < 1/3^{N-1}$.

We estimate $\mu(B(x,r))$ from below. Without loss of generality assume $x_1 < 2/3$, since otherwise one can apply a similar argument with upwards and downwards reversed. For each $M \geq 1$, wormholes of level M are spaced apart by a distance at most $2/3^M$. If $M \geq N+2$ then $r/2 \geq 2/3^M$. Hence, starting at x, one can reach by a curve of length at most r any point $y = [y_1, y_2]$ satisfying both:

- $x_1 \leq y_1 \leq x_1 + r/2$, and
- y_2 is reached from x_2 by wormholes of level $M \geq N+2$.

Note that if x is a wormhole level then either representative of x_2 may be used here. This shows that $q^{-1}(B(x,r))$ contains a set of the form $[x_1, x_1 + r/2] \times K_{N+1}$, where $K_{N+1} \subset K$ is a similar copy of K obtained after splitting $N+1$ times which contains x_2. Hence

$$\begin{aligned}\mu(B(x,r)) &= (\mathcal{H}^1 \times \nu)(q^{-1}(B(x,r)) \\ &\geq (r/2)\nu(K_{N+1}).\end{aligned}$$

In particular, balls have strictly positive measure.

We next estimate $\mu(B(x,2r))$ from above. Notice $2r < 1/3^{N-2}$. By Lemma 3.5, at vertical distance at most $2r$ above and below x, one can find at most one wormhole with a level less than or equal to $N-3$. Hence $q^{-1}(B(x,2r))$ is contained in a set of the form

$$\Big([x_1 - 2r, x_1 + 2r] \times K^1_{N-2}\Big) \cup \Big([x_1 - 2r, x_1 + 2r] \times K^2_{N-2}\Big),$$

where K^1_{N-2}, K^2_{N-2} are similar copies of K obtained after splitting $N-2$ times. Note that one of the two similar copies (temporarily denoted K_a for some binary string a) contains x_2, while the other is obtained by switching one of the entries of K_a at a coordinate less than or equal to $N-3$. This leads to the estimate

$$\begin{aligned}\mu(B(x,2r)) &= (\mathcal{H}^1 \times \nu)(q^{-1}(B(x,2r))) \\ &\leq 4r(\nu(K^1_{N-2}) + \nu(K^2_{N-2})).\end{aligned}$$

Combining the two estimates yields

$$\frac{\mu(B(x,2r))}{\mu(B(x,r))} \leq 8\frac{(\nu(K^1_{N-2}) + \nu(K^2_{N-2}))}{\nu(K_{N+1})}.$$

The result follows because $\nu(K^1_{N-2})/\nu(K_{N+1})$ and $\nu(K^2_{N-2})/\nu(K_{N+1})$ are both bounded above by $\max(w^{-4}, (1-w)^{-4})$. $\square$

Remark 3.7. If ν is an arbitrary doubling measure on K, it does not necessarily follow that $\mu = q_*(\mathcal{H}^1 \times \nu)$ is a doubling measure on F. For instance, given $0 < \lambda < 1$ and $0 < \widehat{\lambda} < 1$, define a measure ν on K as follows. First assign measure $1/2$ to both the left and right similar copies K_0 and K_1 of K. Then, at any given stage, if K_a is a similar copy for which a starts with 0 we assign a proportion λ

of the measure to K_{a0} and $1-\lambda$ to K_{a1}, while if K_a is a similar copy for which a starts with 1 we assign a proportion $\widehat{\lambda}$ of the measure to K_{a0} and $1-\widehat{\lambda}$ to K_{a1}. Then, using a similar argument to that of Proposition 3.2, it is not difficult to see ν is a doubling measure on K. We claim $\mu := q_*(\mathcal{H}^1 \times \nu)$ is not doubling on F. To see this we consider for any $m \geq 1$ the open balls

$$B_m = B\left(\left[\frac{1}{3}+\frac{1}{3^m}, 0\right], \frac{1}{3^m}\right), \qquad 2B_m = B\left(\left[\frac{1}{3}+\frac{1}{3^m}, 0\right], \frac{2}{3^m}\right).$$

Then $p^{-1}(B_m)$ is contained in a set of the form $(1/3, 1/3+2/3^m) \times K_b$ where $|b| = m-1$ and b begins with a 0. Hence

$$\mu(B_m) \leq \frac{2}{3^m}\frac{1}{2}\lambda^{m-2} = \frac{\lambda^{m-2}}{3^m}.$$

On the other hand, $p^{-1}(2B_m)$ contains a set $(1/3-1/3^m, 1/3+1/3^m) \times (K_b \cup K_{b'})$ where b' agrees with b except the first entry is 1 rather than 0. Hence

$$\mu(2B_m) \geq \frac{2}{3^m}\left(\frac{1}{2}\lambda^{m-2} + \frac{1}{2}\widehat{\lambda}^{m-2}\right) = \frac{1}{3^m}\left(\lambda^{m-2} + \widehat{\lambda}^{m-2}\right).$$

Hence

$$\frac{\mu(2B_m)}{\mu(B_m)} \geq 1 + \left(\frac{\widehat{\lambda}}{\lambda}\right)^{m-2}.$$

If $\widehat{\lambda} > \lambda$, letting $m \to \infty$ shows μ is not doubling. A similar argument applies if instead $\widehat{\lambda} < \lambda$, changing the center of the balls to the point $[\frac{1}{3}+\frac{1}{3^m}, 1]$.

Proposition 3.8. *For any $w \in (0,1)$, we have $\mu_w(F \setminus N_w) = 0$ and $\mu_{w'}(N_w) = 0$ for all $w' \in (0,1) \setminus \{w\}$.*

In particular, for all $w, w' \in (0,1)$ with $w \neq w'$, the probability measures μ_w and $\mu_{w'}$ are mutually singular.

Proof. Fix any $w, w' \in (0,1)$ with $w \neq w'$. By Proposition 3.4, we know that $\nu_w(K \setminus E_w) = \nu_{w'}(E_w) = 0$. Recall $N_w = q(I \times E_w) \subset F$. Note that the symmetric difference of $q^{-1}(q(I \times E_w))$ and $I \times E_w$ is contained in a set of the form $C \times K$ where C is countable, hence has measure zero with respect to $\mathcal{H}^1 \times \nu_w$. Hence

$$\begin{aligned}
\mu_w(F \setminus N_w) &= (\mathcal{H}^1 \times \nu_w)((I \times K) \setminus q^{-1}(q(I \times E_w))) \\
&= (\mathcal{H}^1 \times \nu_w)((I \times K) \setminus (I \times E_w)) \\
&= (\mathcal{H}^1 \times \nu_w)(I \times (K \setminus E_w)) \\
&= \nu_w(K \setminus E_w) \\
&= 0.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\mu_{w'}(N_w) &= (\mathcal{H}^1 \times \nu_{w'})(q^{-1}(q(I \times E_w))) \\
&= (\mathcal{H}^1 \times \nu_{w'})(I \times E_w) \\
&= \nu_{w'}(E_w) \\
&= 0.
\end{aligned}$$

This proves the first part of the proposition. The second is then an immediate consequence. This concludes the proof. □

We now briefly describe how $\mu_{1/2}$ is related to $\mathcal{H}^Q$ which is the measure normally used on F. Note, since K is $(Q-1)$-Ahlfors regular, we have $0 < \mathcal{H}^{Q-1}(K) < \infty$.

Proposition 3.9. *For every Borel set $E \subset K$ we have*

$$\nu_{1/2}(E) = \frac{\mathcal{H}^{Q-1}(E)}{\mathcal{H}^{Q-1}(K)}.$$

Hence for every Borel set $E \subset F$ we have

$$\mu_{1/2}(E) = \frac{q_*(\mathcal{H}^1 \times \mathcal{H}^{Q-1})(E)}{\mathcal{H}^{Q-1}(K)}.$$

Consequently, there exists a constant $C \geq 1$ such that for every Borel set $E \subset F$

$$C^{-1}\mathcal{H}^Q(E) \leq \mu_{1/2}(E) \leq C\mathcal{H}^Q(E),$$

where $\mathcal{H}^Q$ denotes the Hausdorff measure of dimension Q on F with metric d.

Proof. For any Borel set $E \subset K$, denote $\widetilde{\nu}(E) = \frac{\mathcal{H}^{Q-1}(E)}{\mathcal{H}^{Q-1}(K)}$. Then $\widetilde{\nu}$ is a probability measure on Borel subsets of K. It also holds that $\widetilde{\nu}(K_a) = (1/2)^N = p_{1/2}(K_a)$ for any binary string a of length N. Since $\nu_{1/2}$ was a unique extension of $p_{1/2}$, it follows that $\widetilde{\nu}(E) = \nu_{1/2}(E)$ for any Borel set $E \subset K$. Hence the first part of the proposition follows.

The second part follows by definition of product measure and the definition of μ_w in (3.3). The third part follows by combining the second part with the fact that $q_*(\mathcal{H}^1 \times \mathcal{H}^{Q-1})$ is bounded within constant multiples of $\mathcal{H}^Q$ since both are Ahlfors Q-regular, as explained in [5]. □

4. Rademacher's Theorem for a Singular Measure

In this section we prove Theorem 1.1. Before doing so, we describe how it can be combined with the results of the previous section to prove our main result Theorem 1.2.

Proof of Theorem 1.2 from Theorem 1.1. Fix any $w \neq 1/2$ and consider the Borel set N_w. By Proposition 3.8, $\mu_w(N_w) > 0$. Hence, by Theorem 1.1, each Lipschitz function $f\colon F \to \mathbb{R}$ is differentiable at some point of N_w (with the point possibly depending on f). Since $w \neq 1/2$, applying Proposition 3.8 implies $\mu_{1/2}(N_w) = 0$. Hence, by Proposition 3.9, $\mathcal{H}^Q(N_w) = 0$. □

We use the rest of this section to prove Theorem 1.1. We divide the proof into several steps, following [5] with adjustments to account for the fact μ_w is doubling instead of Q-Ahlfors regular. For the remainder of this section fix $w \in (0,1)$ and denote $\nu = \nu_w$, $\mu = \mu_w$.

4.1. **Measure Theoretic Preliminaries.** The following lemma follows by Tonelli's theorem. The proof is the same as in [5], up to replacing $\mathcal{H}^Q$ with $\mu = q_*(\mathcal{H}^1 \times \nu)$.

Lemma 4.1. *Suppose $A \subset F$ is Borel with respect to the metric d and*

$$\mathcal{L}^1\{t \in I : [t,z] \in A\} = 0 \text{ for every } z \in K.$$

Then $\mu(A) = 0$.

The next lemma is as in [5], except the measure $\mathcal{H}^Q$ is replaced by μ. The proof is the same, since we may apply Lemma 4.1 with the measure μ instead of $\mathcal{H}^Q$.

Lemma 4.2. *The following statements hold for every Lipschitz map $f\colon F \to \mathbb{R}$.*

(1) *For every $z \in K$, the set*

$$D_z := \{t \in I : \text{the directional derivative } f_I[t,z] \text{ exists}\}$$

is Borel with respect to the Euclidean metric on I and has full $\mathcal{L}^1$ measure.

(2) *For every $z \in K$, the map from D_z to $\mathbb{R}$ defined by $t \mapsto f_I[t,z]$ is Borel measurable with respect to the Euclidean metric on I.*

(3) *The set*

$$D := \{x \in F : \text{the directional derivative } f_I(x) \text{ exists}\}$$

is Borel measurable with respect to d on F and has full μ measure.

(4) *The map $f_I\colon D \to \mathbb{R}$ defined by $x \mapsto f_I(x)$ is Borel measurable.*

Recall that if (X,m) is a doubling metric measure space than the Lebesgue density theorem holds and Borel functions are approximately continuous almost everywhere. I.e. if $g\colon X \to \mathbb{R}$ is Borel, then for almost every $x \in X$

$$\lim_{r\to 0} \frac{m\{y \in B(x,r) : |g(y)-g(x)| > \varepsilon\}}{m(B(x,r))} = 0 \qquad \text{for every } \varepsilon > 0.$$

We will use these facts in $\mathbb{R}$ equipped with Euclidean distance and Lebesgue measure and in F equipped with the metric d and measure μ.

4.2. **Auxilliary Sets.** Let $f\colon F \to \mathbb{R}$ be a Lipschitz function and let $D \subset F$ denote the set of points where the directional derivative of f exists. Denote $L = \mathrm{Lip}(f)$. Let $C_\mu \geq 1$ be the doubling constant of μ. By iterating the doubling condition, there exists $Q > 0$ and $C_Q \geq 1$ both depending only on C_μ such that for all $x \in F$, $0 < r \leq R$, and $y \in B(x,R)$, we have

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \geq C_Q^{-1}\left(\frac{r}{R}\right)^Q.$$

Note that wormholes have measure zero with respect to μ. To see this notice that if $W \subset F$ is the set of wormholes, then $q^{-1}(W)$ is contained in a set of the form $C \times K$ where C is countable. Hence

$$\begin{aligned}\mu(W) &= (\mathcal{H}^1 \times \nu)(q^{-1}(W)) \\ &\leq (\mathcal{H}^1 \times \nu)(C \times K) \\ &= \mathcal{H}^1(C) \\ &= 0.\end{aligned}$$

Definition 4.3. Let D' be the set of all points $x \in D$ which are not a wormhole. We define several sets as follows.

(1) For each $\varepsilon > 0$ and $x \in D$,

$$D_\varepsilon(x) := \{y \in D : |f_I(y) - f_I(x)| \leq \varepsilon\}.$$

(2) For each $\varepsilon > 0$ and integer $k \geq 1$, let $E_k^1(\varepsilon)$ be the collection of all points $x = [x_1, x_2] \in D'$ such that

$$\mathcal{L}^1\{t \in (x_1 - r, x_1 + r) \cap I : [t, x_2] \notin D_\varepsilon(x)\} \leq \varepsilon r$$

for every $0 < r < 1/k$.

(3) For each $\varepsilon > 0$ and integer $k \geq 1$, let $E_k^2(\varepsilon)$ be the collection of all points $x \in D'$ for which

$$\mu\Big(B(x,r) \setminus (D_\varepsilon(x) \cap E_k^1(\varepsilon))\Big) \leq \frac{C_Q^{-1}\varepsilon^Q \min(w, 1-w)}{2^{Q+2}}\mu(B(x,r)) \tag{4.1}$$

for every $0 < r < 1/k$.

The proof of the next lemma is the same as in [5] with $\mathcal{H}^Q$ replaced by μ, which is possible by applying Lemma 4.1.

Lemma 4.4. *For all $\varepsilon > 0$ and integer $k \geq 1$, $E_k^1(\varepsilon)$ is Borel with respect to d and*

$$\mu\left(F \setminus \bigcup_{k=1}^{\infty} E_k^1(\varepsilon)\right) = 0.$$

The proof of the next lemma requires minor adaptations from [5] to account for the change in measure. We first make two remarks.

Remark 4.5. First we claim that $\lim_{s\to s_0} \mu(B(x,s)) = \mu(B(x,s_0))$ for any $s_0 > 0$. To see this, note μ is a doubling measure on a length space so satisfies a δ-annular decay property for some $0 < \delta \leq 1$ depending only on the doubling constant of μ [4]. To be more specific, there is $K \geq 1$ such that for all $x \in F$, $r > 0$, $0 < \varepsilon < 1$,

$$\mu(B(x,r) \setminus B(x, r(1-\varepsilon))) \leq K\varepsilon^\delta \mu(B(x,r)).$$

From this, the claim clearly follows.

Second we claim that $\lim_{x\to x_0} \mu(B(x,s)) = \mu(B(x_0,s))$ for any $x_0 \in F$. To see this notice that $|\mu(B(x,s)) - \mu(B(x_0,s))|$ is bounded by the maximum of

$$\mu(B(x_0, s + d(x,x_0))) - \mu(B(x_0,s))$$

and

$$\mu(B(x, s + d(x,x_0))) - \mu(B(x,s)).$$

Both of these converge to zero as $x \mapsto x_0$, so the claim follows.

Lemma 4.6. *For $\varepsilon > 0$ and integer $k \geq 1$, $E_k^2(\varepsilon)$ is Borel with respect to d and*

$$\mu\left(F \setminus \bigcup_{k=1}^{\infty} E_k^2(\varepsilon)\right) = 0.$$

Proof. Fix $\varepsilon > 0$. We first show that $E_k^2(\varepsilon)$ is Borel with respect to d. Note that (4.1) holds for all $0 < r < 1/k$ if and only if it holds for all rational $0 < r < 1/k$. This follows by choosing a rational sequence $0 < r_n < 1/k$ with $r_n \downarrow r$, applying (4.1) for each n, and using Remark 4.5. Hence it suffices to show that both sides of the estimate defining $E_k^2(\varepsilon)$ are Borel measurable functions of x. For the right side we simply note that $x \mapsto \mu(B(x,r))$ is continuous for each r by Remark 4.5. For the left side, consider $D' \to \mathbb{R}$ given by $x \mapsto \mu(B(x,r) \setminus (D_\varepsilon(x) \cap E_k^1(\varepsilon)))$. Notice that for every $\alpha > 0$, the set

$$\big\{x \in D' : \mu\{y \in B(x,r) : y \notin E_k^1(\varepsilon) \text{ or } |f_I(y) - f_I(x)| > \varepsilon\} > \alpha\big\}$$

can be written as

$$\bigcup_{\substack{\eta>\varepsilon\\ \eta\in\mathbb{Q}}} \bigcap_{n=1}^{\infty} \bigcup_{q\in\mathbb{Q}} \Big(\{x \in D' : |f_I(x) - q| < 1/n\} \\ \cap \{x \in D' : \mu\{y \in B(x,r) : |f_I(y) - q| > \eta \text{ or } y \notin E_k^1(\varepsilon)\} > \alpha\}\Big).$$

The first set inside the decomposition above is Borel by Lemma 4.2. The second is an open subset of D', using Remark 4.5, hence Borel. Hence $E_k^2(\varepsilon)$ is Borel.

Using Lemma 4.4, almost every point of F is a density point of $E_k^1(\varepsilon)$ with respect to μ for some $k \geq 1$. Also, since f_I is Borel, almost every point of F is a point of approximate continuity of f_I with respect to μ. Hence for almost every x there exists $k \in \mathbb{N}$ and $R > 0$ such that

$$\mu\Big(B(x,r) \setminus (D_\varepsilon(x) \cap E_k^1(\varepsilon))\Big) < \frac{C_Q^{-1}\varepsilon^Q \min(w, 1-w)}{2^{Q+2}} \mu(B(x,r))$$

for every $0 < r < R$. Choose $K \in \mathbb{N}$ such that $K \geq k$ and $1/K < R$. Then using the fact $E_k^1(\varepsilon) \subset E_K^1(\varepsilon)$ it follows

$$\mu\Big(B(x,r) \setminus (D_\varepsilon(x) \cap E_K^1(\varepsilon))\Big) < \frac{C_Q^{-1}\varepsilon^Q \min(w, 1-w)}{2^{Q+2}} \mu(B(x,r))$$

for every $0 < r < 1/K$. Hence $x \in E_K^2(\varepsilon)$. This shows $\mu(F \setminus \bigcup_{k=1}^{\infty} E_k^2(\varepsilon)) = 0$. □

4.3. **Choice of Suitable Line Segments.** Fix $0 < \varepsilon < 1$. Let $x \in \bigcup_{k=1}^{\infty} E_k^2(\varepsilon)$ and fix $K \geq 1$ such that $x \in E_K^2(\varepsilon)$. Let $y \in F$ with $d(y,x) < 1/(2K)$. Let $N \geq 1$ be the unique integer such that $1/3^N \leq d(x,y) < 1/3^{N-1}$.

Assume that infinitely many wormhole levels are needed to connect x to y by a geodesic. It will be clear how the following argument can be simplified if only finitely many wormhole levels or even no wormhole levels are needed. Denote $T = d(x,y)$ and choose $\gamma\colon [0,T] \to F$ such that

- γ is a geodesic from x to y with $\gamma(0) = x$ and $\gamma(T) = y$.
- γ is a concatenation of countably many lines in the I direction parameterized at unit speed.

By Lemma 3.5, any geodesic joining x to y must pass through at most one wormhole of level less than or equal to $N-1$. We enumerate the wormhole levels needed to connect x to y by a strictly increasing sequence N_i for integer $i \geq 0$, where possibly $N_0 \leq N-1$, but necessarily $N_i \geq N$ for $i \geq 1$. Since $N_1 \geq N$ and N_i are strictly increasing, it follows that $N_i \geq N + i - 1$ for $i \geq 1$.

For each $i \geq 0$, let λ_i be the point in the interval $[0,T]$ where γ jumps using the wormhole of order N_i. Geodesics in F can be chosen so that they change their direction (up or down) in the I component at most twice (Proposition 2.2). Hence, during any subinterval of $[0,T]$ of length t, the geodesic spends at least a time $t/3$ following the same direction (either up or down but not changing between them) in the I component. Since in any direction wormhole levels of order N_i are spaced apart by at most a distance $2/3^{N_i}$, we can additionally choose γ so that it satisfies:

- $\lambda_0 \leq d(x,y)$, and
- $\lambda_i \leq 2/3^{N_i-1}$ for $i \geq 1$.

Using $N_i \geq N+i-1$ for $i \geq 1$ and the definition of N, we estimate as follows:

$$\begin{aligned}\sum_{i=0}^{\infty} \lambda_i &\leq d(x,y) + \sum_{i=1}^{\infty} \frac{2}{3^{N_i-1}} \\ &\leq d(x,y) + \sum_{i=1}^{\infty} \frac{2}{3^{N+i-2}} \\ &= d(x,y) + \frac{1}{3^{N-2}} \\ &\leq 10d(x,y).\end{aligned}$$

Let $(\mu_i)_{i=0}^{\infty}$ be a strictly decreasing rearrangement of $\{\lambda_i : i \geq 0\} \cup \{T\}$. Thus $\mu_0 = T$, $\mu_i \to 0$ as $i \to \infty$, $\gamma|_{[\mu_{i+1},\mu_i]}$ is a line segment for each $i \geq 0$, and

$$\sum_{i=0}^{\infty} \mu_i = \sum_{i=0}^{\infty} \lambda_i + T \leq 11d(x,y). \tag{4.2}$$

Denote $p_i = \gamma(\mu_i)$ for $i \geq 0$. Notice $p_0 = y$ and $p_i \to x$ as $i \to \infty$. It follows that

$$f(y) - f(x) = \sum_{i=0}^{\infty} (f(p_i) - f(p_{i+1})). \tag{4.3}$$

Since $\gamma|_{[\mu_{i+1},\mu_i]}$ is a line segment in the I direction, it follows p_i is reached from p_{i+1} by travelling a displacement $h(p_i) - h(p_{i+1})$ in the I direction.

4.4. **Estimate Along Line Segments.** Our aim is to show that $f(p_i) - f(p_{i+1})$ is well approximated by $f_I(x)(h(p_i) - h(p_{i+1}))$ for every $i \geq 0$. Fix $i \geq 0$ until otherwise stated.

Lemma 4.7. *There exist points $q_i, q_{i+1} \in F$ with the following properties:*

1. *$d(q_{i+1}, p_{i+1}) \leq \varepsilon\mu_{i+1}$,*
2. *$d(q_i, p_i) \leq 6\varepsilon\mu_{i+1}$,*
3. *$q_{i+1} \in E_K^1(\varepsilon) \cap D_\varepsilon(x)$,*
4. *q_i is reached from q_{i+1} by travelling a vertical displacement $h(p_i) - h(p_{i+1})$ in the I direction.*

Proof. Using $0 < \varepsilon < 1$ and $\mu_{i+1} \leq T < 1/2K$ gives $\mu_{i+1} + \varepsilon\mu_{i+1} < 1/K$. Hence the fact that $x \in E_K^2(\varepsilon)$ gives,

$$\mu\Big(B(x, \mu_{i+1} + \varepsilon\mu_{i+1}) \setminus (D_\varepsilon(x) \cap E_K^1(\varepsilon))\Big) < \frac{C_Q^{-1}\varepsilon^Q \min(w, 1-w)}{2^{Q+2}} \mu(B(x, \mu_{i+1} + \varepsilon\mu_{i+1})).$$

Since γ is a geodesic, $d(x, p_{i+1}) = d(\gamma(0), \gamma(\mu_{i+1})) = \mu_{i+1}$. Hence

$$B(p_{i+1}, \varepsilon\mu_{i+1}) \subset B(x, \mu_{i+1} + \varepsilon\mu_{i+1}).$$

It follows that

$$\mu\Big(B(p_{i+1}, \varepsilon\mu_{i+1}) \setminus (D_\varepsilon(x) \cap E_K^1(\varepsilon))\Big) < \frac{C_Q^{-1}\varepsilon^Q \min(w, 1-w)}{2^{Q+2}} \mu(B(x, \mu_{i+1} + \varepsilon\mu_{i+1})).$$

However, using the doubling property and $0 < \varepsilon < 1$,

$$\frac{\mu(B(p_{i+1}, \varepsilon\mu_{i+1}))}{\mu(B(x, \mu_{i+1} + \varepsilon\mu_{i+1}))} \geq C_Q^{-1} \left(\frac{\varepsilon}{1+\varepsilon}\right)^Q > C_Q^{-1} \frac{\varepsilon^Q}{2^Q}.$$

Combining the previous two steps gives

$$\mu\Big(B(p_{i+1},\varepsilon\mu_{i+1})\setminus(D_\varepsilon(x)\cap E_K^1(\varepsilon))\Big)<\frac{\min(w,1-w)}{4}\mu(B(p_{i+1},\varepsilon\mu_{i+1})). \tag{4.4}$$

Fix an integer $B\geq 1$ such that $1/3^B\leq\varepsilon\mu_{i+1}<1/3^{B-1}$. This implies that within a vertical distance $\varepsilon\mu_{i+1}$ of p_{i+1}, there is at most two wormhole levels of order less than or equal to $B-1$. Let S be the set of points $z\in B(p_{i+1},\varepsilon\mu_{i+1})$ such that z can be connected to p_{i+1} using only wormhole levels of order $n\geq B$. Then we obtain

$$\mu(S\cap B(p_{i+1},\varepsilon\mu_{i+1}))\geq\frac{\min(w,1-w)}{4}\mu(B(p_{i+1},\varepsilon\mu_{i+1})). \tag{4.5}$$

Indeed, the ball $B(p_{i+1},\varepsilon\mu_{i+1})$ intersects at most three sets $K_{a_j}\times I$, where a_j are finite strings of length $B-1$, $j=1,2,3$. Without loss of generality, assume $p_{i+1}\in K_{a_1}\times I$ and note that each a_j differs from a_1 in at most one entry. Define the sets $Q_j=B(p_{i+1},\varepsilon\mu_{i+1})\cap(K_{a_j}\times I)$. Note $S\cap B(p_{i+1},\varepsilon\mu_{i+1})=Q_1$. For each j we have, as a consequence of Fubini's theorem,

$$\mu(Q_j)\leq\frac{\mu(Q_1)}{\min(w,1-w)}.$$

Hence

$$B(p_{i+1},\varepsilon\mu_{i+1})=\sum_{j=1}^{3}\mu(Q_j)\leq\frac{3\mu(Q_1)}{\min(w,1-w)}.$$

Hence

$$\mu(Q_1)\geq\frac{\min(w,1-w)}{3}\mu(B(p_{i+1},\varepsilon\mu_{i+1}))$$

which gives the desired inequality.

Using (4.4) and (4.5) we can choose a point q_{i+1} with

$$q_{i+1}\in S\cap B(p_{i+1},\varepsilon\mu_{i+1})\cap D_\varepsilon(x)\cap E_K^1(\varepsilon).$$

Clearly by definition q_{i+1} satisfies (1) and (3).

Next, define q_i from q_{i+1} as stated in (4). Then q_i can be reached from p_i from a vertical displacement at most $2\varepsilon\mu_{i+1}$ and wormhole levels of order $n\geq B$. Such jump levels are spaced by at most $2/3^B$ in the vertical direction. Hence

$$d(q_i,p_i)\leq 2\varepsilon\mu_{i+1}+(4/3^B)\leq 6\varepsilon\mu_{i+1}.$$

This shows that q_i satisfies (2) and completes the proof. □

Using Lemma 4.7 and the same steps as in [5] yields the estimate

$$\begin{aligned}&|f(p_i)-f(p_{i+1})-f_I(x)(h(p_i)-h(p_{i+1}))|\\&\qquad\leq(2L+2)\varepsilon|h(p_i)-h(p_{i+1})|+7L\varepsilon\mu_{i+1}.\end{aligned}$$

Adding these estimates over all $i\geq 0$ gives

$$\begin{aligned}&|f(y)-f(x)-f_I(x)(h(y)-h(x))|\\&\qquad\leq(2L+2)\varepsilon d(x,y)+77L\varepsilon d(x,y).\end{aligned}$$

Using a similar argument to that of [5] then concludes the proof of Theorem 1.1.

5. Poincaré inequality

In this section, we give an argument that (F, d, μ) satisfies also a $(1,1)$-*Poincaré inequality*: This means exists constants $C > 0, \lambda \geq 1$ so that

$$\frac{1}{\mu(B(x,r))}\int_{B(x,r)} |f - f_B| d\mu \leq Cr\frac{1}{\mu(B(x,\lambda r))}\int_{B(x,\lambda r)} \mathrm{Lip}[f] d\mu,$$

holds for all Lipschitz functions $f : F \to \mathbb{R}$ and all balls $B = B(x,r) \subset F$. Here, $\mathrm{Lip}[f](x) = \limsup_{y\to x} \frac{|f(y)-f(x)|}{d(x,y)}$, and $f_A = \frac{1}{\mu(A)}\int_A f d\mu$ for Borel sets $A \subset X$ with $\mu(A) > 0$. See [7] and [25] for further background on the Poincaré inequality, and [17, Theorem 2] for relationships between equivalent formulations of the Poincaré inequality. In [7], it was shown that a doubling metric measure space satisfying a Poincaré inequality satisfies a notion of differentiability with respect to a collection of charts. Theorem 2.5 shows that the chart constructed in [7] can be chosen as (F, h). Thus, by proving the Poincaré inequality, we establish a closer connection between this work and [7]. Further, we show that the present examples are similar to the ones by Schioppa in [26].

The argument is based on using a pointwise version of the Poincaré inequality: There exist constants $C, \lambda \geq 1$ so that for all Lipschitz functions $f : F \to \mathbb{R}$ and all points $p, q \in X$ we have:

$$|f(p) - f(q)| \leq Cd(x,y)(M_{\lambda d(x,y)}\mathrm{Lip}[f](p) + M_{\lambda d(x,y)}\mathrm{Lip}[f](q)),$$

where

$$M_R h(x) = \sup_{r\in(0,R)} \frac{1}{\mu(B(x,r))}\int_{B(x,r)} |h| d\mu$$

is the Hardy-Littlewood maximal function. By a result from [25, Theorem 8.1.7] a $(1,1)$-Poincaré inequality is equivalent to a pointwise $(1,1)$-Poincaré inequality. This result is originally due to Hajłasz and Koskela, see e.g. [16]. We will prove the pointwise version of the Poincaré inequality by a "chaining of balls"-type argument, although in our case, we will chain rectangles of the form $J \times K_x$. This type of argument is also due to Hajłasz and Koskela.

Our argument will use the following simple one-dimensional result. Let $J \subset \mathbb{R}$ be an interval, $f : J \to \mathbb{R}$ a Lipschitz function, and let $A, B \subset J$ be positive measure subsets. Then we have the following inequality:

$$\left|\frac{1}{|A|}\int_A f(t)dt - \frac{1}{|B|}\int_B f(t)dt\right| \leq \int_J \mathrm{Lip}[f](t)dt, \tag{5.1}$$

where $|A|$ is the Lebesgue measure of the set A. Indeed, for every $a \in A, b \in B$, we have $|f(a) - f(b)| \leq \int_J \mathrm{Lip}[f](t)dt$. Taking an average integral in both $a \in A$ and $b \in B$ yields (5.1).

To simplify the presentation of the proof below, we will use $A \lesssim B$ to indicate that there is a constant C so that $A \leq CB$. The constant C in all instances will only depend on the space F and not on the function f, or other variables in the proof.

Proof. In the proof, we denote by μ the measure μ_w and by ν the measure ν_w. First, we prove three inequalities, (5.2), (5.3), (5.4), where one can control the differences of averages over sets of the form $q(J \times K_x)$, called rectangles, which lie "near" to each other in specific ways. The three cases are showed in Figure 2.

First, Case A), where two rectangles are connected through a wormhole: Let $q(J \times K_x)$ and $q(J \times K_{x'})$ be sets with $|x| = |x'|$, $J \subset [0,1]$ is a sub-interval and for which the strings x and x' differ only at the n'th bit for some $1 \leq n \leq |x|$, and $t_n \in J$ for some wormhole level $t_n \times K$ of order n. Let s be a point in K_x, which via its trinary expansion can be identified by an infinite trinary string, and let $s' \in K_{x'}$ be the infinite trinary string obtained from s with the n'th bit flipped. Equivalently s' is obtained from s via the identification in Definition 2.1, i.e. a translation to the right or left by $2(3^{-n})$, depending on if the n'th bit of s is a 0 or 2, respectively. Then, for $t \in [0, 3|J|]$ we define

$$\gamma_s(t) = \begin{cases} [(\max(J) - t, s)] & t \in [0, |J|] \\ [(\min(J) + (t - |J|), s)] & t \in [|J|, |J| + t_n - \min(J)] \\ [(\min(J) + (t - |J|), s')] & t \in [|J| + t_n - \min(J), 2|J|] \\ [(\max(J) - (t - 2|J|), s')] & t \in [2|J|, 3|J|] \end{cases}$$

Notice $\gamma_s(|J|) = [(\min(J), s)]$ from both the first and second line, $\gamma_s(|J| + t_n - \min(J)) = [(t_n, s)] = [(t_n, s')]$ from the second and third line and $\gamma_s(2|J|) = [(\max(J), s')]$ from the final two lines. See Figure 1 for a figure of the curve $\gamma_s(t)$.

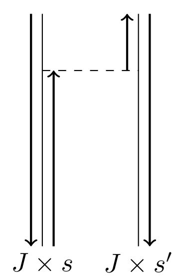

FIGURE 1. The path γ_s. The dashed line shows where the wormhole is used.

The curve γ_s is a unit speed curve, and $f \circ \gamma_s(t)$ is Lipschitz. First, by using the fact that $\mu = q_*(\mathcal{H}^1 \times \nu)$, we get by a change of variables and the fact $\mu(q(J \times K_x)) = |J|\mu(K_x)$ that

$$\begin{aligned} \frac{1}{\mu(q(J \times K_x))} \int_{q(J\times K_x)} f d\mu &= \frac{1}{|J|\nu(K_x)} \int_{J\times K_x} f(q(t,s)) dt d\nu(s) \\ &= \frac{1}{|J|\nu(K_x)} \int_{K_x} \int_0^{|J|} f(\gamma_s(t)) dt d\nu(s). \end{aligned}$$

Next, we use fact that the map $T(s) = s', T : K_x \to K_{x'}$ is a translation, and the push-forward is given by $T_*(\nu|_{K_x}) = \frac{\nu(K_x)}{\nu(K_{x'})}\nu|_{K_{x'}}$. This follows from the definition of ν and Proposition 3.1, since if a is any finite string then $T_*\nu(K_{x'a}) = \nu(K_{xa}) = \nu(K_x)\nu(K_a)$ and $\nu(K_{x'a}) = \nu(K_{x'})\nu(K_a)$. Since the sets K_{xa} generate the sigma-algebra, this shows that the measures $T_*(\nu|_{K_x})$ and $\nu|_{K_{x'}}$ differ only by the factor

$\nu(K_x)/\nu(K_{x'})$. This, together with the same change of variables, yields

$$\frac{1}{\mu(q(J\times K_{x'}))}\int_{q(J\times K_{x'})} f d\mu = \frac{1}{|J|\nu(K_{x'})}\int_{J\times K_{x'}} f(q(t,s'))dtd\nu(s') = \frac{1}{|J|\nu(K_x)}\int_{K_x}\int_{2|J|}^{3|J|} f(\gamma_s(t))dtd\nu(s).$$

Thus, (5.1) and integration over s gives

(5.2)

$$\begin{aligned}&\left|\frac{1}{\mu(q(J\times K_x))}\int_{q(J\times K_x)} f d\mu - \frac{1}{\mu(q(J\times K_{x'}))}\int_{q(J\times K_{x'})} f d\mu\right| \\ &\leq \frac{1}{|J|\nu(K_x)}\int_{K_x}\left|\int_0^{|J|} f(\gamma_s(t))dt - \int_{2|J|}^{3|J|} f(\gamma_s(t))dt\right| d\nu(s) \\ &\leq 2|J|\frac{1}{\mu(q(J\times K_{x'}))}\int_{q(J\times K_{x'})} \mathrm{Lip}[f](t)d\mu + 2|J|\frac{1}{\mu(q(J\times K_x))}\int_{q(J\times K_x)} \mathrm{Lip}[f](t)d\mu.\end{aligned}$$

Next, we consider Case B), where the two rectangles are of the form $q(J\times K_x)$ and $q(J'\times K_x)$ with $J'\subset J$ and J', J are subintervals of $[0,1]$. Then (5.1) implies

$$\begin{aligned}&\left|\frac{1}{\mu(q(J\times K_x))}\int_{J\times K_x} f d\mu - \frac{1}{\mu(q(J'\times K_x))}\int_{q(J'\times K_x)} f d\mu\right| \\ &\leq |J|\frac{1}{\mu(q(J\times K_x))}\int_{q(J\times K_x)} \mathrm{Lip}[f] d\mu. \end{aligned} \tag{5.3}$$

Finally, consider the Case C) consisting of rectangles $q(J\times K_x)$ and $q(J'\times K_{x'})$, where $J'\subset J$, x' is obtained from the string x by adding one bit and J contains a wormhole at level $|x|+1$. By symmetry consider only the case, where $w' = w0$ and consider three averages $Q_1 = \frac{1}{\mu(q(J'\times K_{x0}))}\int_{q(J'\times K_{x0})} f d\mu$, $Q_2 = \frac{1}{\mu(q(J\times K_{x0}))}\int_{q(J\times K_{x0})} f d\mu$ and $Q_3 = \frac{1}{\mu(q(J\times K_{x1}))}\int_{q(J\times K_{x1})} f d\mu$. Then

$$\begin{aligned}&\left|\frac{1}{\mu(q(J\times K_x))}\int_{q(J\times K_x)} f d\mu - \frac{1}{\mu(q(J'\times K_{x'}))}\int_{q(J'\times K_{x'})} f d\mu\right| \\ &= |(wQ_2 + (1-w)Q_3) - Q_1| \\ &\leq (1-w)|Q_2 - Q_3| + |Q_2 - Q_1| \\ &\lesssim |J|\frac{1}{\mu(q(J\times K_x))}\int_{q(J\times K_x)} \mathrm{Lip}[f] d\mu. \end{aligned} \tag{5.4}$$

In the last line, we used (5.3) to estimate the difference $|Q_2 - Q_1|$ and (5.2) to estimate the difference $|Q_2 - Q_3|$.

With these simple estimates given, we construct a chain of sets of the form $q(J_i \times K_{x_i})$, $i \in \mathbb{Z}$, which connect every pair of points, and where consecutive sets are related to each other as in one of the three cases considered above. Let $p = [x_1, x_2], q = [y_1, y_2] \in F$ be two distinct points. Choose $n \in \mathbb{N}$ so that $2(3^{-n-1}) < d(p,q) \leq 2(3^{-n})$. Let J_0 be a shortest interval containing x_1, y_1 and which contains all the wormhole levels needed to connect p to q and which has length comparable to $d(p,q)$, with a constant independent of p and q. Let $m \leq 0$ and define K_m^1 for to be the $n+m-1$'th level interval in the Cantor set containing x_2, and for

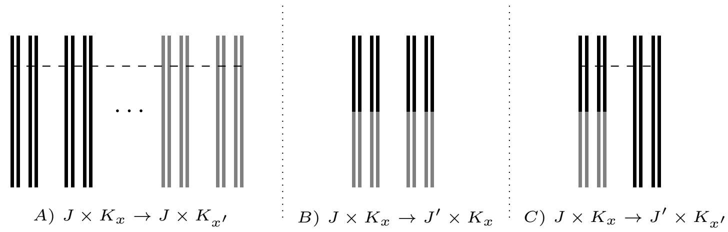

Figure 2. The three cases for rectangles that we consider. One of the sets is shaded gray and the other set is shaded black. In the middle case, the gray shading overlaps with the black shading. The dashed lines show wormhole levels. Notice how Case C can be decomposed into parts, which are related via Case A and Case B.

$m > 0$ let K_m^2 to be the $n+m-2$'th level interval in the Cantor set containing y_2. First, let $J_1^1 = J_0 = J_1^2$. Further, let $\{J_m^1\}_{m=2}^\infty$ be an a nested sequence of intervals of lengths 3^{-n-m} which contains x_1 and where each interval is contained in J_0. Symmetrically, let $\{J_m^2\}_{m=2}^\infty$ be an a nested sequence of intervals of lengths 3^{-n-m} which contains y_1 and where each interval is contained in J_0.

Now, consider the sets Q_i defined as follows. $Q_0 = q(J_0 \times K_0^1)$, and $Q_m = q(J_m^1 \times K_{|m|}^1)$ for $m < 0$ and $Q_m = q(J_m^2 \times K_{|m|}^2)$ for $m > 0$. By construction we can write $Q_m = q(J_m \times K_{x_m})$ with $|J_m| \leq \mathrm{diam}(Q_m) \lesssim d(p,q)3^{-|m|}$. Notice that for all $m \in \mathbb{Z}$, the sets Q_m and Q_{m+1} relate to each other as one of the three cases considered in the beginning of the proof. First, for each $i < 0$, we have that Q_i is obtained from Q_{i+1} as in case C), since $J_i^1 \subset J_{i+1}^1$ and K_{i-1}^1 is the left or right half of K_i^1. Similarly for $i > 0$, we have that Q_i and Q_{i+1} are related as in C). We are left to consider Q_0 and Q_1. We have $J_1^2 = J_0$ and $K_0^1 = K_a$ and $K_1^2 = K_b$ for some finite strings a, b with $|a| = |b|$. By Lemma 3.5, the strings a, b can differ from each other by at most one bit of index $\leq n-1$, and there is a wormhole level $t \in J_0$ corresponding to this level. Thus, we can apply Case A) to estimate the difference between f_{Q_0} and f_{Q_1}.

Now, by combining continuity of f with a telescoping sum argument as well as (5.4), (5.3) and (5.2), we get

$$\begin{aligned}
|f(p) - f(q)| &\leq \sum_{i \in \mathbb{Z}} \left| \frac{1}{\mu(Q_i)} \int_{Q_i} f d\mu - \frac{1}{\mu(Q_{i+1})} \int_{Q_{i+1}} f d\mu \right| \\
&\lesssim \sum_{i \in \mathbb{Z}} \mathrm{diam}(Q_i) \frac{1}{\mu(Q_i)} \int_{Q_i} \mathrm{Lip}[f] d\mu + \mathrm{diam}(Q_{i+1}) \frac{1}{\mu(Q_{i+1})} \int_{Q_{i+1}} \mathrm{Lip}[f] d\mu \\
&\lesssim \sum_{i \in \mathbb{Z}} \mathrm{diam}(Q_i) \frac{1}{\mu(Q_i)} \int_{Q_i} \mathrm{Lip}[f] d\mu. \qquad (5.5)
\end{aligned}$$

In the last line, we observed that the sum over i of the two terms on the second line are equal, since they are obtained by shifting indices $i \to i+1$. Notice that for $i \leq 0$ we have $d(Q_i, p) \lesssim 3^{-|i|} d(p,q)$, and that $\mu(Q_i) \gtrsim \mu(B(p, 3^{-|i|} d(p,q)))$ by doubling since Q_i contains a ball with radius comparable to $3^{-|i|} d(p,q)$. Thus, we see that there is a constant $\lambda > 1$ independent of p and q for which

$$\frac{1}{\mu(Q_i)}\int_{Q_i}\mathrm{Lip}[f]d\mu \lesssim M_{\lambda d(p,q)}\mathrm{Lip}[f](p) \text{ for } i \leq 0.$$

Similarly,

$$\frac{1}{\mu(Q_i)}\int_{Q_i}\mathrm{Lip}[f]d\mu \lesssim M_{\lambda d(p,q)}\mathrm{Lip}[f](q) \text{ for } i > 0.$$

Thus, from this, (5.5) and $\mathrm{diam}(Q_i) \lesssim 3^{-|i|}d(p,q)$ we get some constant $C > 1$ independent of the points p, q and the function f for which

$$|f(p) - f(q)| \leq Cd(p,q)(M_{\lambda d(p,q)}\mathrm{Lip}[f](p) + M_{\lambda d(p,q)}\mathrm{Lip}[f](q)).$$

This is the pointwise Poincaré inequality, and by [25, Theorem 8.1.7] this implies the $(1,1)$-Poincaré inequality. □

References

[1] Alberti, G., Csornyei, M., Preiss, D.: *Differentiability of Lipschitz Functions, Structure of Null Sets, and Other Problems*, Proc. Int. Congress Math. III, (2010), 1379–1394.

[2] Billingsley, P.: *Probability and Measure*, 3rd Edition, John Wiley & Sons, Inc., 1995, ISBN 0-471-00710-2.

[3] Benyamini, Y., Lindenstrauss, J.: *Geometric nonlinear functional analysis. Vol. 1.*, American Mathematical Society Colloquium Publications, 48. American Mathematical Society, Providence, RI, 2000. xii+488 pp. ISBN: 0-8218-0835-4.

[4] Buckley, S.: *Is the Maximal Function of a Lipschitz Function Continuous?* Annales Academiae Scientiarum Fennicae Mathematica 24 (1999), 519–528.

[5] Capolli, M., Pinamonti, A., Speight, G.: *Maximal Directional Derivatives in Laakso Space*, To appear in Comm. Contemp. Math. https://doi.org/10.1142/S0219199724500172

[6] Capolli, M.: *An Overview on Laakso Spaces*, preprint available at arXiv:2211.05681

[7] Cheeger, J.: *Differentiability of Lipschitz Functions on Metric Measure Spaces*, Geometric and Functional Analysis 9(3) (1999), 428–517.

[8] Csornyei, M., Jones, P.: *Product formulas for measures and applications to analysis and geometry*, announcement of result is in slides available at: http://www.math.sunysb.edu/Videos/dfest/PDFs/38-Jones.pdf.

[9] Cheeger, J., Kleiner, B.: *Inverse Limit Spaces Satisfying a Poincare Inequality*, Analysis and Geometry in Metric Spaces 2015; 1:15–39.

[10] Doré, M., Maleva, O.: *A compact null set containing a differentiability point of every Lipschitz function*, Mathematische Annalen 351(3) (2011), 633–663.

[11] Dymond, M., Maleva, O.: *Differentiability inside sets with upper Minkowski dimension one*, Michigan Mathematical Journal 65 (2016), 613–636.

[12] De Philippis, G., Marchese, A., Merlo, A., Pinamonti, A., Rindler, F.: *On the converse of Pansu's Theorem*, preprint available at arXiv:2211.06081.

[13] De Philippis, G., Rindler, F.: *On the structure of A-free measures and applications*, Annals of Mathematics 184 (2016), 1017–1039.

[14] Eriksson-Bique, S.: *Characterizing spaces satisfying Poincaré inequalities and applications to differentiability, Geom. Funct. Anal.* 29 (1) (2019), 119–189.

[15] Fitzpatrick, S.: *Differentiation of Real-Valued Functions and Continuity of Metric Projections*, Proceedings of the American Mathematical Society 91(4) (1984), 544–548.

[16] Hajłasz, P., Koskela, P., *Sobolev met Poincaré*, Mem. Am. Math. Soc. 688, 101 p. (2000).

[17] Stephen, K. *Modulus and the Poincaré inequality on metric measure spaces*, Mathematische Zeitschrift 245(2): 255-292, (2003).

[18] Laakso, T. J., *Ahlfors Q-Regular Spaces with Arbitrary $Q > 1$ Admitting Weak Poincaré Inequality*, Geometric and Functional Analysis 10(1), 111–123, 2000.

[19] Le Donne, E., Pinamonti, A., Speight, G.: *Universal Differentiability Sets and Maximal Directional Derivatives in Carnot Groups*, Journal de Mathématiques Pures et Appliquées 121 (2019), 83–112.

[20] Pansu, P.: *Metriques de Carnot-Caratheodory et quasiisometries des espaces symwtriques de rang un*, Annals of Mathematics 129(1) (1989), 1-60.

[21] Preiss, D., Speight, G.: *Differentiability of Lipschitz Functions in Lebesgue Null Sets*, Inventiones Mathematicae 199(2) (2015), 517–559.

[22] Pinamonti, A., Speight, G.: *A Measure Zero Universal Differentiability Set in the Heisenberg Group*, Mathematische Annalen 368, no. 1-2, (2017) 233–278.

[23] Pinamonti, A., Speight, G.: *Universal Differentiability Sets in Carnot Groups of Arbitrarily High Step*, Israel Journal of Mathematics, (2019).

[24] Preiss, D.: *Differentiability of Lipschitz Functions on Banach spaces*, Journal of Functional Analysis 91(2) (1990), 312–345.

[25] Heinonen, J., Koskela, P., Shanmugalingam, N., and Tyson, J. T., *Sobolev spaces on metric measure spaces*, vol. 27 of *New Mathematical Monographs.* Cambridge University Press, Cambridge, 2015.

[26] Schioppa, A.: *Poincare Inequalities for Mutually Singular Measures*, Analysis and Geometry in Metric Spaces 2015; 3:40-45.

[27] Zahorski, Z.: *Sur l'ensemble des points de non-dérivabilité d'une fonction continue*, Bulletin de la Société Mathématique de France 74 (1946), 147–178.

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