ON A DISCRETE MAX-PLUS TRANSPORTATION PROBLEM

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Dedicated to N.N. Uraltseva on the occasion of her 90th birthday

Abstract. We provide an explicit algorithm to solve the idempotent analogue of the discrete Monge-Kantorovich optimal mass transportation problem with the usual real number field replaced by the tropical (max-plus) semiring, in which addition is defined as the maximum and product is defined as usual addition, with $-\infty$ and 0 playing the roles of additive and multiplicative identities. Such a problem may be naturally called tropical or "max-plus" optimal transportation problem. We show that the solutions to the latter, called the optimal tropical plans, may not correspond to perfect matchings even if the data (max-plus probabilities) have all weights equal to zero, in contrast with the classical optimal transportation analogue, where perfect matching optimal plans always exist. Nevertheless, in some randomized situation the existence of perfect matching optimal tropical plans may occur rather frequently. At last, we prove that the uniqueness of solutions of the optimal tropical transportation problem is quite rare.

1. Introduction

In this paper we consider a discrete optimization problem that looks quite similar to the classical Monge-Kantorovich optimal mass transportation problem and in fact, as we argue later, is nothing else but the idempotent version of the latter.

Problem statement. Suppose we have m signal sources and n receivers regularly exchanging information between them. Each source $i \in \{1,\ldots,m\}$ may transmit an amount $h_{i,j}$ of information to $j \in \{1,\ldots,n\}$. The maximum amount of information the source i may send at one time is given by a number k_i, that is,

$$\max_{j\in\{1,\ldots,n\}} h_{i,j} = k_i. \tag{1}$$

Analogously, the maximum amount of information receiver j may get at one time is given by a number l_j, that is,

$$\max_{i\in\{1,\ldots,m\}} h_{i,j} = l_j. \tag{2}$$

Of course, (1) and (2) may only be simultaneously valid if

$$\max_{i\in\{1,\ldots,n\}} k_i = \max_{j\in\{1,\ldots,n\}} l_j. \tag{3}$$

The cost $C_{i,j}$ of transmitting between the source i and the receiver j depends affinely on the amount of transmitted information and takes into account the known fixed

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cost $g_{i,j}$ of using the communication channel between them, that is,

$$C_{i,j} = g_{i,j} + \gamma h_{i,j}$$

for some given coefficient $\gamma > 0$. The goal is to find the values $h_{i,j}$, $i = 1, \ldots, n$, $j = 1, \ldots, m$ (the respective matrix being further called the tropical transportation plan, the explanation of the terminology being given in the sequel) minimizing the maximum of $C_{i,j}$ over all i and j, that is, finding the

$$\inf\{\max_{i,j}(g_{i,j} + \gamma h_{i,j})\colon h_{i,j} \text{ satisfying (1) and (2)}\}.$$

Denoting $c_{i,j} := g_{i,j}/\gamma$, this amounts to solving

$$\inf\{\max_{i,j}(c_{i,j} + h_{i,j})\colon h_{i,j} \text{ satisfying (1) and (2)}\}. \tag{4}$$

Idempotent (max-plus or tropical) interpretation. Let us now completely change the point of view and look at the above problem as a version of the classical optimal mass transportation problem in the context of idempotent analysis, more precisely, analysis over the tropical (max-plus) semiring $\bar{\mathbb{R}}_- := \mathbb{R} \cup \{-\infty\}$ endowed with the operations

$$a \oplus b := \max\{a, b\}, \quad a \otimes b := a + b$$

which substitute the usual addition and multiplication of real numbers respectively. The value $-\infty$ is an identity with respect to $\oplus$ and 0 is an identity with respect to $\otimes$. Both operations are commutative, associative and $a \otimes (b \oplus c) = a \otimes b + a \otimes c$. Thus the roles of 0 and 1 on the usual real line are played here by $-\infty$ and 0 respectively. For the general overview of the idempotent analysis we refer the reader to the classical book [3].

The classical discrete Monge-Kantorovich optimal mass transportation problem (see, e.g. [4] for the comprehensive introduction into the subject) is that of finding the optimal plan of transportation (usually called just transport plan), that is, the matrix $\{\pi_{i,j}\}_{i,j=1}^{m,n}$ with each $\pi_{i,j} \in [0,1]$ and satisfying

$$\sum_{j=1}^{n} \pi_{i,j} = k_i, \tag{5}$$

$$\sum_{i=1}^{m} \pi_{i,j} = l_j \tag{6}$$

with given numbers k_i, l_j, $i = 1, \ldots, m$, $j = 1, \ldots, n$ and solving the minimization problem

$$\inf\{\sum_{i,j=1}^{m,n} c_{i,j}\pi_{i,j} \ : \pi_{i,j} \text{ satisfying (5) and (6)}\}. \tag{7}$$

This is usually interpreted as finding the way of optimally transporting the discrete measure

$$\mu := \sum_{i=1}^{m} k_i \delta_{x_i}$$

to another discrete measure

$$\nu := \sum_{j=1}^{n} l_j \delta_{y_i},$$

for some $x_i \in X$, $y_j \in Y$, $i = 1, \dots, m$, $j = 1, \dots, n$, with X and Y some sets and δ_z standing for the Dirac point mass at z. The value $\pi_{i,j}$ then stands for the amount of mass transported from x_i to y_j, or, in other words, one may see the whole matrix $\{\pi_{i,j}\}_{i,j=1}^{m,n}$ as representing the discrete measure $\sum_{i,j=1}^{m,n} \pi_{i,j}\delta_{(x_i,y_j)}$ over $X \times Y$, and $\sum_{i,j=1}^{m,n} c_{i,j}\pi_{i,j}$ is just the total transportation cost.

In the idempotent max-plus setting the role of the Dirac measure δ_x over a set X concentrated at $x \in X$ is played by the characteristic function (for which we retain the same notation as for the Dirac measure) $\delta_x \colon X \to \bar{\mathbb{R}}_-$ defined by

$$\delta_x(y) = \begin{cases} 0, & y = x, \\ -\infty, & y \neq x. \end{cases}$$

The measures μ and ν become then the functions $\mu \colon X \to \bar{\mathbb{R}}_-$ and $\nu \colon X \to \bar{\mathbb{R}}_-$ respectively defined by

$$\mu = \max_{i=1,\dots,m} (k_i + \delta_{x_i}), \qquad \nu = \max_{j=1,\dots,n} (l_j + \delta_{y_j}), \tag{8}$$

i.e. μ is the function taking the value k_i at each x_i and $-\infty$ elsewhere, and ν is the function taking the value l_j at each y_i and $-\infty$ elsewhere. We will be referring to the k_i's as the *weights* of μ and to the l_j's as the weights of ν. The tropical version of the total mass of a discrete measure becomes then the sum of its weights, that is

$$|\mu| := \max_{i=1,\dots,m} k_i, \qquad |\nu| := \max_{j=1,\dots,n} l_j.$$

We will assume, in complete analogy with the classical mass transportation theory, that $|\mu| = |\mu|$ (which is exactly the condition (3)), and for purely aesthetical reasons, which imply no loss of generality, that both total masses are zero, i. e. $|\mu| = |\nu| = 0$, so that μ and ν can be considered tropical versions of discrete probability measures. Such functions will be further called discrete *max-plus probabilities*, the set of such functions over a given set Z being denoted $\mathcal{M}(Z)$, so that $\mu \in \mathcal{M}(X)$ and $\nu \in \mathcal{M}(Y)$. Finally, $\pi_{i,j} \in [0, 1]$ is substituted by $h_{i,j} \in [-\infty, 0]$ and the Monge-Kantorovich problem (7) becomes (4).

Results. In this paper we provide an explicit algorithm to solve the tropical transportation problem (4) and find an explicit formula for the tropical cost. As a consequence, we obtain some curious results on the solutions, i. e. optimal tropical plans. In particular, the optimal tropical plans naturally corresponding to perfect matchings may not exist even if the max-plus probabilities μ and ν have all the weights equal to zero (we henceforth call this case *fundamental*). This is in striking contrast with the classical optimal mass transportation, where an optimal transport plan corresponding to a perfect matching (i. e. a permutation matrix) between discrete measures which are sums of Dirac masses with equal weights always exists. Nevertheless, by introducing randomness of the cost, it turns out that the existence of perfect matching optimal tropical plan occurs rather frequently as the number of weights of both μ and ν becomes large. We also prove that the uniqueness of optimal tropical plan is quite rare. As for the optimal tropical cost, we prove that for Bernoulli cost matrices, it is asymptotically equal to the lowest value of the matrix (in the fundamental case).

2. Notation and preliminaries

In complete analogy with the classical optimal transportation theory, the matrix $\{h_{i,j}\}_{i,j=1}^{m,n}$ with each $h_{i,j} \in [-\infty, 0]$ satisfying (1) and (2) will be called discrete *max-plus (or tropical) plan* (or just a *plan* for brevity) for max-plus probabilities $\mu \in \mathcal{M}(X)$, $\nu \in \mathcal{M}(Y)$. Equivalently it can be seen as a max-plus probability $h \in \mathcal{M}(X \times Y)$ given by the formula

$$h = \max_{i=1,\ldots,m, i=1,\ldots,n} (h_{i,j} + \delta_{(x_i,y_j)}). \tag{9}$$

We denote by $\Pi(\mu, \nu)$ the set of all such plans (it is of course nonempty since $\mu \otimes \nu \in \Pi(\mu, \nu)$, where $(\mu \otimes \nu)_{i,j} := k_i + l_j$).

For the given cost matrix $\{c_{i,j}\}_{i,j=1}^{m,n}$ we define

$$d_c(\mu, \nu) := \inf \left\{ \max_{i=1,\ldots,m, i=1,\ldots,n} (c_{i,j} + h_{i,j}) \colon h \in \Pi(\mu, \nu) \right\}.$$

If we interpret h as an element of $h \in \mathcal{M}(X \times Y)$, i.e. as in (9), then we may write $h(x_i, y_j)$ and $c(x_i, y_j)$ instead of $h_{i,j}$ and $c_{i,j}$ respectively. Again for purely aesthetical reasons, to be able to easier interpret the numbers $c_{i,j}$ as a cost, it is convenient to assume $c_{i,j} \geq 0$ which always can be done without loss of generality. The minimizer $\pi \in \Pi(\mu, \nu)$ in the above problem will be called the *minimizing (or optimal) tropical plan*, the set of such minimizing plans being denoted by $\Pi^c(\mu, \nu)$. The number $d_c(\mu, \nu)$ will be called the *tropical cost* or *tropical distance* between μ and ν (the latter terminology is of course an abuse of the language since d_c is not necessarily a distance, nevertheless, it reminds of the optimal transportation distance between two measures).

In the sequel we assume the sequences of weights k_j and l_j to be ordered ordered in decreasing order with $k_1 = l_l = |\mu| = |\nu| = 0$, i.e.

$$k_n \leq k_{n-1} \leq \cdots \leq k_1 = 0, \quad l_n \leq l_{n-1} \leq \cdots \leq l_1 = 0. \tag{10}$$

We denote by $\Lambda(\mu)$ and $\Lambda(\nu)$ the sets of weights of μ and ν respectively.

For any $h \in \Pi(\mu, \nu)$, by the *support* of h, denoted $\operatorname{supp}(h)$, we will mean the subset of $X \times Y$ of points (x, y) where $h(x, y) > -\infty$, or, equivalently, with a slight abuse of the language, the set of pairs $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$ such that $h_{i,j} > -\infty$.

For a set X we denote by $\#X$ is cardinality. We also write sometimes $a \vee b$ for the maximum of the numbers a and b.

3. Reduced transportation plans and existence of minimizers

We start with the following definition.

Definition 3.1. *Given fixed discrete μ and ν, we will call a discrete plan $h \in \Pi(\mu, \nu)$ reduced if for each i, j such that $h_{i,j} := h(x_i, y_j) > -\infty$, the element $h_{i,j}$ is a* strict *maximum in its row* or *in its column, and denote by $\Pi_R(\mu, \nu)$ the set of reduced plans for discrete μ and ν.*

Without loss of generality for the tropical transportation problem, all the weights of μ and ν can be taken to be finite (i.e. different from $> -\infty$). In fact, if, say, $k_i = -\infty$ for some $i \in \{1, \ldots, m\}$, then the i-th row of h, for any $h \in \Pi(\mu, \nu)$ must consist only of $-\infty$. In this case, in the expression that defines $d_c(\mu,\nu)$, each of the elements over which the minimum is taken is

$$\begin{aligned}&\max_{(i,j)}(h_{i,j}+c_{i,j})\\&=\max\{\dots,h_{i,1}+c_{i,1},h_{i,2}+c_{i,2},\dots,h_{i,n}+c_{i,n},\dots\}\\&=\max\{\dots,-\infty,-\infty,\dots-\infty,\dots\},\end{aligned}$$

but the maximum is non-negative, so the the numbers $-\infty$ can be changed to sufficiently small negative numbers (negative but with large absolute value) without affecting the maximumm and then the weight $k_j=-\infty$ can be changed to $\max_i h_{i,j}$ where $h_{i,j}$ are the new numbers just mentioned.

The following assertion holds true.

Lemma 3.2. *For all discrete $\mu\in\mathcal{M}(X)$, $\nu\in\mathcal{M}(Y)$ one has*

$$d_c(\mu,\nu)=\inf\{\max_{(i,j)}(h_{i,j}+c_{i,j})\colon\ h\in\Pi_R(\mu,\nu)\}.$$

Moreover, for every minimizing plan h there is a reduced minimizing plan $\tilde h$ with $\operatorname{supp}\tilde h\subset\operatorname{supp}h$ and $\tilde h=h$ over the support of $\tilde h$.

Proof. If $h_{i,j}$ is not a strict maximum neither in its column nor in its row for some $i,j\in\{1,\dots,n\}$. Then changing $h_{i,j}$ to $-\infty$ (or to any number less than $h_{i,j}$) does not affect $\max_{(i,j)}(h_{i,j}+c_{i,j})$. Changing all such entries of the matrix $\{h_{i,j}\}$ will transform the plan to a reduced one, and thus

$$\begin{aligned}d_c(\mu,\nu)&=\inf\{\max_{(i,j)}(h_{i,j}+c_{i,j})\colon\ h\in\Pi(\mu,\nu)\}\\&=\inf\{\max_{(i,j)}(h_{i,j}+c_{i,j})\colon\ h\in\Pi_R(\mu,\nu)\}\end{aligned}$$

as claimed. □

As a consequence, the following existence result holds.

Theorem 3.3. *The discrete max-plus transportation problem admits a solution, namely,* inf *is actually a* min.

Proof. It is enough to refer to Lemma 3.2 and observe that the set of reduced plans $\Pi_R(\mu,\nu)$ has finitely many elements. □

4. Algorithm to solve the discrete max-plus transportation problem

4.1. **Partition of the support of a plan.** Given discrete μ and ν, for each $i\in\{1,\dots,m\}$, $j\in\{1,\dots,n\}$, let

$$p_i=\max\{j\colon\ l_j\geq k_i\},\quad q_j=\max\{i\colon\ k_i\geq l_j\},$$
$$S_i=\{(i,1),\dots,(i,p_i)\},\quad T_j=\{(1,j),\dots,(q_j,j)\}.$$

The following assertion is valid.

Lemma 4.1. *Let $\mu\in\mathcal{M}(X)$, $\nu\in\mathcal{M}(Y)$ be discrete max-plus probabilities as in* (8)*; let $h\in\Pi_R(\mu,\nu)$.*

(1) *For each $i\in\{1,\dots,m\}$, at least one of the numbers $h_{i,1},\dots,h_{i,p_i}$ must be k_i, and the numbers $h_{i,p_i+1},\dots,h_{i,n}$ are all strictly less than k_i. Likewise, for each $j\in\{1,\dots,n\}$, at least one of the numbers $h_{1,j},\dots,h_{q_i,j}$ must be l_j, and the numbers $h_{q_{i+1},j},\dots,q_{n,j}$ are all strictly less than l_j.*

(2) *If the weights k_i and l_j are all distinct, with the exception of $k_1 = l_1 = 0$, then $S_i \cap T_j = \emptyset$ whenever $(i,j) \neq (1,1)$.*
(3) *$k_i = l_j$ for some $i,j \in \{1,\dots,m\} \times \{1,\dots,n\}$ if and only if $(i,j) \in S_i \cap T_j$.*

Proof. (1) Fix $i \in \{1,\dots,m\}$. As for the second part of the assertion: the maximum among $h_{i,1},\dots,h_{i,n}$ must be k_i. If $h_{i,p_i+\bar m} = k_i$ for some $\bar m \geq 0$, then the maximum among $h_{1,p_i+\bar m},\dots,h_{n,p_i+\bar m}$ is at least k_i. But this contradicts the definition of p_i, since the maximum among $h_{1,p_i+m},\dots,h_{n,p_i+\bar m}$ must be $\lambda_{p_i+\bar m}$. Thus the maximum occurs among $h_{i,1},\dots,h_{i,p_i}$, which proves the first part of the assertion. The claim about the numbers $h_{1,j},\dots,h_{n,j}$ is proved analogously.

(2) Suppose $(i,j) \neq (1,1)$ and $(q,p) \in S_i \cap T_j$. Then, by definition of p_i and q_j, $(q,p) = (i,j)$. Moreover, in this case, the Definition of p_i and q_j will contain only strict inequalities. Therefore, $l_1 > k_i, \dots, l_j > k_i$ and $k_1 > l_j, \dots, k_i > l_j$, which is a contradiction.

(3) Suppose $k_i = l_j$ for some $i,j \in \{1,\dots,m\} \times \{1,\dots,n\}$. Since $l_j \geq k_i$, we must have $(i,j) \in S_i$. Likewise, since $k_i \geq l_j$, then $(i,j) \in T_j$. Necessity is proven. Now suppose $(i,j) \in S_i \cap T_j$. Since $(i,j) \in S_i$, $j \leq p_i$ so $l_j \geq k_i$, and $(i,j) \in T_j$ gives $i \in T_i$, so $k_i \geq l_j$. This completes the proof. □

It follows from Lemma 4.1 that if none of the k_i is equal to any of the l_j, with the exception of $k_0 = l_0 = 0$, then $|\Pi_R(\mu,\nu)| = \Pi_{i=0}^{n-1} p_i q_i$.

Given discrete max-plus probabilities μ, ν and a real number λ, let

$$R_\lambda := \left(\bigcup_{\{i:\ k_i=\lambda\}} S_i \right) \cup \left(\bigcup_{\{j:\ l_j=\lambda\}} T_j \right), \tag{11}$$

which is a subset of $\{1,\dots,m\} \times \{1,\dots,n\}$. We call R_λ a *region* or λ-region to emphasize the dependence on λ. A region can look like a backwards L whose ends rest on the top and left edges of the grid, or a rectangle with its left side lying on the left edge of the grid, or a rectangle with its top side on the top edge of the grid, or a rectangle with both his left and top sides lying on the left and top sides of the grid, respectively.

Example 4.2. For $m = n = 6$ and the max-plus probabilities

$$\mu = \max\{0+\delta_{x_1}, 0+\delta_{x_2}, -2+\delta_{x_3}, -3+\delta_{x_4}, -4+\delta_{x_5}, -4+\delta_{x_6}\}$$
$$= \begin{cases} 0, & x \in \{x_1.x_2\}, \\ -2, & x = x_3, \\ -3, & x = x_4, \\ -4, & x \in \{x_5.x_6\} \end{cases}$$

and

$$\nu = \max\{0+\delta_{y_1}, 0+\delta_{y_2}, 0+\delta_{y_3}, -1+\delta_{y_4}, -2+\delta_{y_5}, -2+\delta_{y_6}\}$$
$$= \begin{cases} 0, & x \in \{y_1.y_2.y_3\}, \\ -1, & x = y_4, \\ -2. & x \in \{y_5.y_6\} \end{cases}$$

with x_j, $j = 1,\dots,m$ as well as y_i, $i = 1,\dots,n$ all distinct, the regions (each in a different color) and a plan are shown in Figure 1. △

We extend the notions of plan and reduced plan as follows. Fix μ, ν. Given a non-empty subregion R_λ of the $n \times n$ grid, a filling of it with $-\infty$ and real numbers

	0	0	0	-1	-2	-2
0	-∞	-∞	0	-1	-∞	-2
0	0	0	-∞	-∞	-2	-∞
-2	-∞	-∞	-2	-∞	-∞	-∞
-3	-∞	-∞	-∞	-∞	-3	-∞
-4	-∞	-∞	-∞	-4	-∞	-∞
-4	-4	-∞	-∞	-∞	-∞	-∞

FIGURE 1. Regions for the pair (μ, ν) of Example 4.2.

such that the maximum of each row and column of R_λ is λ is called a *plan* of R_λ. Let $\Pi(R_\lambda)$ be the set of plans of R_λ. Like above, a plan $h = (h_{i,j})_{(i,j)\in R_\lambda} \in \Pi(R_\lambda)$ is called *reduced* whenever $h_{i,j}$ is a strict maximum of its row or a strict maximum of its column, as long as $h_{i,j} > -\infty$. Thus, a reduced plan of a λ-region has no numbers other than $-\infty$ and λ. Denote by $\Pi_R(R_\lambda)$ the set of reduced plans of R_λ.

Given discrete μ, ν, a cost function c, we allow ourselves some abuse of notation by denoting

$$d_c(R_\lambda) := \min_{h\in\Pi(R_\lambda)} \max_{(i,j)\in R_\lambda} (h_{i,j} + c(x_i, y_j)).$$

The following assertion holds true.

Proposition 4.3. *Suppose $\mu \in \mathcal{M}(X), \nu \in \mathcal{M}(Y)$ are discrete and a cost function $c : X \times Y \to [0,\infty)$ is given. Then*

$$d_c(\mu, \nu) = \max_{\lambda\in\Lambda(\mu)\cup\Lambda(\nu)} d_c(R_\lambda).$$

Proof. By definition,

$$d_c(\mu, \nu) = \min_{h\in\Pi(\mu,\nu)} \max_{(i,j)} (h_{i,j} + c(x_i, y_j)).$$

Let us look at

$$M = \max_{\lambda} \min_{h\in\Pi(R_\lambda)} \max_{(i,j)\in R_\lambda} (h_{i,j} + c(x_i, y_j)),$$

which is the right hand side of the inequality we wish to prove. For each one of the distinct λ's, we pick $h^\lambda \in R_\lambda$ for which $\max_{(i,j)\in R_\lambda}(h_{i,j} + c(x_i, y_j))$ takes the least possible value, i.e. we pick an optimal plan for the region R_λ for each λ. Further, let $\bar\lambda$ be the value of λ at which M is attained. Let $\bar h = \cup_\lambda h^\lambda$; then $\bar h \in \Pi(\mu, \nu)$ (i.e. it is a plan between μ and ν). We claim $\bar h$ is optimal for $d_c(\mu, \nu)$. Suppose it is not, and hence there is another $h^0 \in \Pi(\mu, \nu)$ such that

$$\max_{(i,j)} (h^0_{i,j} + c(x_i, y_j)) \le \max_{(i,j)} (h_{i,j} + c(x_i, y_j)) \quad \forall h \in \Pi(\mu, \nu).$$

In particular, if $h = \bar h$, then, by the assumption just made, the inequality must be strict, and

$$\max_{(i,j)\in R_{\bar\lambda}} (h^0_{i,j} + c(x_i, y_j)) \le \max_{(i,j)} (h^0_{i,j} + c(x_i, y_j)) < \max_{(i,j)} (\bar h_{i,j} + c(x_i, y_j)).$$

But the maximum value of the function $\lambda \mapsto \max_{(i,j)\in R_\lambda}(\bar{h}_{i,j} + c(x_i, y_j))$ is M and is attained at $\lambda = \bar{\lambda}$. Thus, it follows that

$$\max_{(i,j)\in R_{\bar\lambda}} (h^0_{i,j} + c(x_i, y_j)) < \max_{(i,j)\in R_{\bar\lambda}} (\bar{h}_{i,j} + c(x_i, y_j)) = \max_{(i,j)\in R_{\bar\lambda}} (h^{\bar\lambda}_{i,j} + c(x_i, y_j)),$$

which contradicts the definition of $h^{\bar\lambda}$. Therefore, $\bar{h}$ is optimal for $d_c(\mu, \nu)$, so $d_c(\mu, \nu) = M$. □

4.2. **Finding the optimal cost on a region.** By Proposition 4.3, to solve the original problem, it is enough to find the optimal plan for each λ-region R_λ, hence also finding the respective optimal costs $d_c(R_\lambda)$; the optimal plan for the original problem will then coincide over each R_λ with the optimal plan for this region.

To find the optimal plan for the given region R_λ, suppose the cost function c be given (we assume all the points x_i and y_i to be fixed beforehand), and number the values of c over R_λ in an increasing order. Namely, suppose that $s \in \mathbb{Z}^+$ is the number of distinct values that c takes on the region R_λ and denote these values, in increasing order, by

$$c_1 < \cdots < c_s. \tag{12}$$

For each $m \in \{1, 2, \ldots, s\}$ we define the function $h^m_c \colon R_\lambda \to \{-\infty, \lambda\}$ by the formula

$$h^m_c(i,j) := \begin{cases} \lambda, & \text{if } c(x_i, y_j) \le c_m, \\ -\infty, & \text{otherwise.} \end{cases}$$

That is, h^m_c is a filling of the region R_λ with $-\infty$ and λ (i.e. a mapping from R_λ into $\{\lambda, -\infty\}$) such that the λ's appear in the cells that host one of the smallest m values of c on the region, while $-\infty$ appears in the other cells. In particular, for $m = s$, h^s_c fills all the cells in the region R_λ with λ, and hence is a plan for R_λ, that is, $h^s_c \in \Pi(R_\lambda)$. This motivates the following definition.

Definition 4.4. *Given λ, a λ-region R_λ, and a cost function c, let m be the smallest integer for which the filling h^m_c of the region R_λ constitutes a plan, i. e.*

$$m_c(\lambda) = \min\{m\colon\ h^m_c \in \Pi(R_\lambda)\}.$$

It is convenient to assign to each $(i,j) \in R_\lambda$ the number (from 1 to s) that the value $c(x_i, y_j)$ occupies in the list (12). Such an assignment is given by a function $f : R_\lambda \to \{1, 2, \ldots, s\}$ satisfying:

$$f(i_1, j_1) < f(i_2, j_2) \quad \text{if and only if} \quad c(x_{i_1}, y_{j_1}) < c(x_{i_2}, y_{j_2}) \quad \text{for } (i_1, j_1), (i_2, j_2) \in R_\lambda. \tag{13}$$

We illustrate the above definitions with the following example.

Example 4.5. Suppose the region is $\{1, 2, 3\}^2$ and the cost function (restricted to this region) is, in matrix form,

$$[c(x_i, y_j)]^2_{i,j=1} = \begin{pmatrix} 2 & 4 & 8 \\ 8 & 2 & 0 \\ 2 & 0 & 5 \end{pmatrix}.$$

Then $f(2,3)=f(3,2)=1$, $f(1,1)=f(2,2)=f(3,1)=2$, $f(1,2)=3$, $f(1,3)=f(2,1)=4$, and

$$h_c^1=\begin{pmatrix}-\infty & -\infty & -\infty\\ -\infty & -\infty & \lambda\\ -\infty & \lambda & -\infty\end{pmatrix},\quad h_c^2=\begin{pmatrix}\lambda & -\infty & -\infty\\ -\infty & \lambda & \lambda\\ \lambda & \lambda & -\infty\end{pmatrix},$$
$$h_c^3=\begin{pmatrix}\lambda & -\infty & -\infty\\ -\infty & \lambda & \lambda\\ \lambda & \lambda & \lambda\end{pmatrix},\quad h_c^4=\begin{pmatrix}\lambda & \lambda & \lambda\\ \lambda & \lambda & \lambda\\ \lambda & \lambda & \lambda\end{pmatrix}.$$

Here $m_c(\lambda)=2$, $h_c^{m_c(\lambda)}=h_c^2$. △

Lemma 4.6. *Let R_λ be a λ-region, c be a cost function. Let $h\in\Pi(R_\lambda)$ be a minimizer for $d_c(R_\lambda)$. Then the support of h is included in the support of $h_c^{m_c(\lambda)}$ and, with the notation of* (12),

$$d_c(R_\lambda)=\lambda+c_{m_c(\lambda)}.$$

Moreover, $h_c^{m_c(\lambda)}$ is itself a minimizing plan.

Proof. Let $\{(x_{i_1},y_{j_1}),\dots,(x_{i_p},y_{j_p})\}$ be the support of h. Then

$$d_c(R_\lambda)=\max_{1\le k\le p}\{c(x_{i_k},y_{j_k})+\lambda\}.$$

With the notation of (12), let c_m be the largest of the $c(x_{i_k},y_{j_k})$; then $d_c(R_\lambda)=\lambda+c_m$. But then the filling h_c^m, by definition, must have a λ in every cell (i,j) such that $c(x_i,y_j)\in\{c_1,\dots,c_m\}$. Thus, the support of h is included in the support of h_c^m, and h_c^m is a plan, so $m_c(\lambda)\le m$ and

$$d_c(R_\lambda)=\lambda+c_{m_c(\lambda)}\le\lambda+c_m=d_c(R_\lambda).$$

On the other hand, since $h_c^{m_c(\lambda)}$ is a plan, we must have

$$d_c(R_\lambda)\le\lambda+c_{m_c(\lambda)}.$$

Combining the last two inequalities, we obtain that $d_c(R_\lambda)=\lambda+c_{m_c(\lambda)}$, as desired, and $m=m_c(\lambda)$, so the support of h is included in the support of $h_c^{m_c(\lambda)}$. This means $h_c^{m_c(\lambda)}$ is itself a minimizing plan, and the last assertion follows. □

We collect the preceding conclusions in the following:

Theorem 4.7. *Let $\mu\in\mathcal{M}(X)$, $\nu\in\mathcal{M}(Y)$ be discrete, i. e. $\mu=\max_{i=1}^n(k_i+\delta_{x_i})$, $\nu=\max_{j=1}^n(k_j+\delta_{y_j})$. and $c:X\times Y\to[0,\infty)$ be a given cost function. Then to get a minimizing plan h one considers for every $\lambda\in\Lambda(\mu)\cup\Lambda(\nu)$ (i. e. for each distinct weight of either μ and ν) the respective region R_λ and a minimizing plan h_λ for each R_λ (e.g. $h_\lambda:=h_c^{m_c(\lambda)}$), setting then $h:=h_\lambda$ over each R_λ. Furthermore,*

$$d_c(\mu,\nu)=\max_{\lambda\in\Lambda(\mu)\cup\Lambda(\nu)}(\lambda+c(x_{i_\lambda},y_{j_\lambda})),$$

where each $(i_\lambda,j_\lambda)\in f^{-1}(m_c(\lambda))$, f standing for the numbering function satisfying (13). *In particular, if all the weights k_i and l_j are distinct, except $k_1=l_1=0$, then*

$$d_c(\mu,\nu)=\max_{0\le i\le n-1}\min_{j\le p_i}(k_i+c(x_i,y_j))\vee\max_{0\le j\le n-1}\min_{i\le q_j}(l_j+c(x_i,y_j)).$$

Proof. It is a direct consequence of combining Lemma 4.6 with Proposition 4.3. □

4.3. **Remarks on uniqueness of plans on a region.** As we see fom Example 4.5,, the filling $h_c^{m_c(\lambda)}$ (i.e. the first filling, in going from $m=1$ to $m=s$, that results in a plan) is *not* necessarily a *reduced* plan. Another, simpler, example of such a situation is

$$[c(x_i,y_j)]_{i,j=1}^2 = \begin{pmatrix} 1 & 3 \\ 3 & 3 \end{pmatrix};$$

supposing $\{(1,1),(1,2),(2,1),(2,2)\}$ is a region R_λ, then here $m_c(\lambda)=2$ and $h^{m_c(\lambda)}$ is the 2×2 matrix with λ in every entry.

We can state the following about reduced minimized plans and uniqueness of minimizing plans of a region.

Proposition 4.8. *Let $\lambda \le 0$, R_λ be a λ-region, $c : X\times Y \to [0,\infty)$ be a cost function. If $h_c^{m_c(\lambda)}$ is a reduced plan, then it is the* unique *reduced minimizing plan for $d_c(R_\lambda)$. Vice versa, if a minimizing plan for $d_c(R_\lambda)$ contains only $-\infty$ and λ and is unique among minimizing plans with this property, then it is reduced and must coincide with $h_c^{m_c(\lambda)}$.*

Proof. To prove the first assertion;, suppose that $h_c^{m_c(\lambda)}$ is a reduced plan for $d_c(R_\lambda)$. It is minimizing by Lemma 4.6. If there is another reduced minimizing plan h for $d_c(R_\lambda)$, then by Lemma 4.6 its support is a subset of the support of $h_c^{m_c(\lambda)}$. Hence if $h \ne h_c^{m_c(\lambda)}$, then for some (x_i,y_j) one has $h(x_i,y_j)=-\infty$ and $h_c^{m_c(\lambda)}(x_i,y_j)=\lambda$. But since $h_c^{m_c(\lambda)}$ is reduced, then the matrix $\{h_c^{m_c(\lambda)}(x_k,y_l)\}_{k,l}$ has either in the i-th column or in the j-th row all the entries except the entry (i,j) strictly less than λ, hence $-\infty$. Therefore the matrix $\{h(x_k,y_l)\}_{k,l}$ has either all the i-th column or all the j-th row made of entries $-\infty$, contradicting the fact that h is a plan for R_λ, hence proving the assertion.

To prove the second assertion, let h be the unique minimizing plan for $d_c(R_\lambda)$ among minimizing plans containing only $-\infty$ and λ. It has to be reduced by Lemma 3.2. On the other hand, also $h_c^{m_c(\lambda)}$ contains only $-\infty$ and λ and is a minimizing plan for $d_c(R_\lambda)$. by Lemma 4.6. Thus $h = h_c^{m_c(\lambda)}$ as claimed. □

We remark that the latter Proposition 4.8 asserts that having a unique plan (among all plans containing only $-\infty$ and λ) is equivalent to $h_c^{m_c(\lambda)}$ being reduced, but this is *not* equivalent to the existence of a unique reduced minimizing plan as the following example shows.

Example 4.9. Suppose $\lambda=0$.

(1) If the cost function is

$$[c(x_i,y_j)]_{i,j=1}^2 = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix},$$

then $h_c^{m_c(\lambda)}$ is not reduced; there are two minimizing plans (containing only 0 and $-\infty$), with one of them the only reduced minimizing plan:

$$h_c^{m_c(\lambda)} = \begin{pmatrix} 0 & 0 \\ -\infty & 0 \end{pmatrix}, \quad h_1 = \begin{pmatrix} 0 & -\infty \\ -\infty & 0 \end{pmatrix}.$$

(2) If the cost function is

$$[c(x_i,y_j)]_{i,j=1}^2 = \begin{pmatrix} 1 & 4 & 2 \\ 6 & 7 & 8 \\ 5 & 9 & 3 \end{pmatrix},$$

then $h_c^{m_c(\lambda)}$ is not reduced, and there are at least two reduced minimizing plans:

$$h_c^{m_c(\lambda)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\infty & -\infty \\ 0 & -\infty & 0 \end{pmatrix}, \quad h_1 = \begin{pmatrix} -\infty & 0 & 0 \\ 0 & -\infty & -\infty \\ 0 & -\infty & -\infty \end{pmatrix}, \quad h_2 = \begin{pmatrix} -\infty & 0 & -\infty \\ 0 & -\infty & -\infty \\ -\infty & -\infty & 0 \end{pmatrix}.$$

△

4.4. **A remark on perfect matchings.** Of particular interest, as in the classical mass transportation problem, are minimizing plans supported on subsets of the type $\{(x_1, y_{\sigma(1)}), \ldots, (x_n, y_{\sigma(n)})\}$, where $\sigma\colon \{1\ldots, n\} \to \{1\ldots, n\}$. We will call them *perfect matching* plans. The plan h_1 in Example 4.9(1) and the plan h_3 and In Example 4.9(2) are perfect matchings, while the other plans in these examples are not. The example below shows that for some data one might have no perfect matching minimizing plans.

Example 4.10. Consider the cost matrix

$$[c(x_i, y_j)]_{i,j=0}^2 = \begin{pmatrix} 5 & 1 & 5 \\ 5 & 2 & 5 \\ 3 & 5 & 4 \end{pmatrix}.$$

If $k_2 = k_1 = k_0 = l_2 = l_1 = l_0 = 0$, then

$$h = \begin{pmatrix} -\infty & 0 & -\infty \\ -\infty & 0 & -\infty \\ 0 & -\infty & 0 \end{pmatrix}$$

is the unique minimizing plan (among plans containing only 0 and $-\infty$), but is not a perfect matching. △

We stress that the nonexistence of the optimal tropical plans even when the max-plus probabilities μ and ν have all the weights equal to zero (we will call this case *fundamental*) is in a striking contrast with the classical optimal mass transportation. The latter always admits an optimal transport plan corresponding to a perfect matching (i. e. a permutation matrix) between discrete measures which are sums of Dirac masses with equal weights, by virtue of the Birkhoff-von Neumann theorem which states that the set of extreme points of the Birkhoff polytope of bistochastic matrices in $\mathbb{R}^{n^2}$ is exactly the set of permutation matrices (and hence a linear functional on this polytope always attains its minimum on a permutation matrix).

The following assertion holds true.

Proposition 4.11. *Let* $\mu = \max_{j=1}^n (k_j + \delta_{x_j})$, $\nu = \max_{j=1}^n (l_j + \delta_{y_j})$, *with the elements arranged as in* (10) *as usual. If there is* $j \in \{1, \ldots, n\}$ *such that* $k_j \neq l_j$, *then there is no plan that would correspond to a perfect matching.*

Proof. If $h \in \Pi(\mu, \nu)$ is not reduced, then it does not correspond to a perfect matching, so assume that $h \in \Pi_R(\mu, \nu)$. Recall the definition 11 and consider the disjoint regions R_{λ_k}, $k = 1, \ldots, r$ determined by the plan h, where λ_k, $k = 1, \ldots, r$ are all the distinct weights of the max-plus probabilities μ and ν. Suppose that the set $\{i\colon k_i = \lambda_k\}$ has $m_{k,1}$ elements, and the set $\{j\colon\ l_j = \lambda_k\}$ has $m_{k,2}$ elements; at least one of these two numbers must be positive. Observe that the plan h must have at least $\max\{m_{k,1}, m_{k,2}\}$ finite (i.e. different from $-\infty$) entries on the region R_{λ_k}. Thus, the plan h has at least

$$m = \max\{m_{1,1}, m_{1,2}\} + \cdots + \max\{m_{r,1}, m_{r,2}\}$$

finite entries in total. Keep in mind that

$$\sum_{k=1}^{r} m_{k,1} = \sum_{k=1}^{r} m_{k,2} = n.$$

The plan will correspond to a perfect matching only if there are n finite entries in total. The only way to have $m = n$ is if $m_{k,1} = m_{k,2}$ for every $k = 1, \dots, r$. Given that the weights are arranged as in (10) as usual, the conclusion follows. □

5. Uniqueness of solution and perfect matchings for random costs

In this section, we will try to elucidate some questions regarding the optimal cost, perfect matchings and uniqueness when we introduce some randomness in the cost function. We will limit ourselves to the fundamental case (i.e. when all the weights of the discrete max-plus measures are zero) and with $m = n$, i. e.:

$$\begin{aligned} \mu_{\mathbf{0}}^n &= \max\{0 + \delta_{x_1}, \dots 0 + \delta_{x_m}\}, \\ \nu_{\mathbf{0}}^n &= \max\{0 + \delta_{y_1}, \dots, 0 + \delta_{y_n}\}. \end{aligned}$$

with x_j, $j = 1, \dots, n$ as well as y_i, $i = 1, \dots, m$ all distinct. In what follows the sequences of max-plus probabilities $\mu_{\mathbf{0}}^n$ and $\nu_{\mathbf{0}}^n$ as above are fixed, while the cost function is random, i. e. is represented by a Bernoulli random matrix, i. e. each entry in the $n \times n$ cost matrix is independent from the others and takes the value c_1 with probability p and c_2 with probability $q = 1 - p$, where $c_1 < c_2$.

5.1. Random tropical cost.

The following statement holds true.

Theorem 5.1. *Let $c_1 < c_2$, and suppose that for each n, $\mu_{\mathbf{0}}^n$ and $\nu_{\mathbf{0}}^n$ are discrete max-plus probabilities with all their weights equal to zero, and c^n is a Bernoulli cost matrix: $\mathbb{P}(c^n(x_i, y_j) = c_1) = p$, $\mathbb{P}(c^n(x_i, y_j) = c_2) = q = 1 - p$ for $i, j \in \{1, \dots, n\}$, where $x_1, \dots, x_n$ and $y_1, \dots, y_n$ are the points of the support of $\mu_{\mathbf{0}}^n$ and $\nu_{\mathbf{0}}^n$. If $q < 1$, then*

$$\mathbb{P}(d_{c^n}(\mu_{\mathbf{0}}^n, \nu_{\mathbf{0}}^n) = c_1) \to 1 \quad \text{as} \quad n \to \infty$$

Proof. Even though a very short argument can be provided, we will derive a formula for the probability under question. Referring to Lemma 4.6 (and recall definition 4.4) the tropical distance d_{c^n} between $\mu_{\mathbf{0}}^n = \max_{i=1}^n (0 + \delta_{x_i})$ and $\nu_{\mathbf{0}}^n = \max_{j=1}^n (0 + \delta_{y_j})$ will be c_1 or c_2 depending on whether $m_{c^n}(0)$ is 1 or 2 respectively. It is 1 if and only if in the matrix for c^n there is at least one c_1 in every row *and* in every column. Denote by F_i the event that there is at least one c_1 in the i-th row of the matrix, and by C_j the event that there is at least one c_1 in the j-th column of the matrix.In the calculation that follows we retain for the sake of clarity the notation m for the number of rows and n for the number of columns in the cost matrix, although one has $m = n$. Therefore for the indices i and j one has $i \in \{1, \dots, m\}$, $j \in \{1, \dots, n\}$. Thus

$$\mathbb{P}(d_{c^n}(\mu_{\mathbf{0}}^n, \nu_{\mathbf{0}}^n) = c_1) = \mathbb{P}((\cap_{i=1}^m F_i) \cap (\cap_{j=1} C_j)) = 1 - \mathbb{P}((\cup_{i=1}^m F_i^c) \cup (\cup_{j=1} C_j^c)),$$

where the upper index c denotes the complement of the event. We have

$$\begin{aligned}
&\mathbb{P}((\cup_{i=1}^m F_i^c) \cup (\cup_{j=1}^n C_j^c)) = \\
&= \sum_{s=1}^{m+n} (-1)^{s+1} \sum_{\substack{a+b=s \\ (a,b)\neq(0,0)}} \binom{m}{a}\binom{n}{b} \mathbb{P}(F_1^c \cap \cdots \cap F_a^c \cap C_1^c \cap \cdots \cap C_b^c) \\
&= \sum_{s=1}^{m+n} (-1)^{s+1} \sum_{a+b=s} \binom{m}{a}\binom{n}{b} q^{mn-(m-a)(m-b)} \\
&= -q^{mn} \sum_{\substack{0\le a\le m \\ 0\le b\le n \\ (a,b)\neq(0,0)}} (-1)^{a+b} \binom{m}{a}\binom{n}{b} q^{-(m-a)(m-b)}.
\end{aligned}$$

Assuming that $p < 1$ (otherwise $\mathbb{P}(d_c(\mu_{\mathbf{0}}, \nu_{\mathbf{0}}) = c_1) = 1$ for any n so that there is nothing to prove). Then

$$\begin{aligned}
&\mathbb{P}((\cup_{i=1}^m F_i^c) \cup (\cup_{j=1}^n C_j^c)) \\
&= -q^{mn} \left(\sum_{\substack{0\le a\le m \\ 0\le b\le n}} (-1)^{a+b} \binom{m}{a}\binom{n}{b} q^{-(m-a)(m-b)} - q^{-mn} \right) \\
&= -q^{mn}(-1)^n \sum_{a=0}^{m} \binom{m}{a} (-1)^a \sum_{b=0}^{n} (-1)^{n-b} \binom{n}{b} (q^{-(m-a)})^{n-b} + 1 \\
&= -q^{mn}(-1)^{m+n} \sum_{a=0}^{b} \binom{m}{a} (-1)^{m-a} (1-q^{-(m-a)})^n + 1.
\end{aligned}$$

Recalling that $m = n$, we get

$$\mathbb{P}(d_{c^n}(\mu_{\mathbf{0}}^n, \nu_{\mathbf{0}}^n) = c_1) = q^{n^2} \sum_{j=0}^{n} (-1)^j \binom{n}{j} (1-q^{-j})^n. \tag{14}$$

Thus,

$$\mathbb{P}(d_c^n(\mu_{\mathbf{0}}^n, \nu_{\mathbf{0}}^n) = c_1) \to 1 \quad \text{as} \quad n \to \infty,$$

if $q < 1$, proving the claim. $\square$

It is convenient to introduce a special notation for the expression in the right hand side of (14), namely, we set

$$\mathfrak{s}(n;p) := \begin{cases} (1-p)^{n^2} \sum_{j=0}^n (-1)^j \binom{n}{j} (1-(1-p)^{-j})^n. & \text{if } p \in [0,1), \\ 1, & \text{if } p = 1. \end{cases}$$

Remark 5.2. The relationship (14) reads

$$\lim_n \mathfrak{s}(n;p) = 1, \quad 0 < p \le 1.$$

It is also easy to show that

$$\lim_{p\to 0} \mathfrak{s}(n;p) = 0, \quad \lim_{p\to 1} \mathfrak{s}(n;p) = 1, \qquad n \in \mathbb{N},$$

so that $p \mapsto \mathfrak{s}(n;p)$ is continuous over $[0,1]$. The asymptotics of $\mathfrak{s}$, hence that of a probability that the random tropical cost be equal to the minimum value of the cost function, may be interesting also for the more general cases when p is not constant but depends on n. For instance, one has $\lim_{n\to\infty} \mathfrak{s}(n, 1/n^\gamma) = 0$ for all $\gamma \geq 1$ and $\lim_{n\to\infty} \mathfrak{s}\big(n, 1/n^{1/2}\big) = 1$. $\diamond$

Remark 5.3. A quite similar situation occurs not only when the cost is given not necessarily by a Bernoulli random matrix, but, say, by a binomial one. Namely, suppose now that $s \in \mathbb{N}$ is fixed, and each entry in the cost matrix c^n can take one of the values $c_1 < \cdots < c_s$ (as in (12)), with c_1 appearing with probability p_1. Let $q := 1 - p_1$. Then the lower bound for $\mathbb{P}(d_{c^n}(\mu_{\mathbf{0}}^n, \nu_{\mathbf{0}}^n) = c_1)$ can be obtained in the same way as in the proof of the Theorem 5.1. Therefore

$$\lim_{n\to\infty} \mathbb{P}(d_c(\mu_{\mathbf{0}}^n, \nu_{\mathbf{0}}^n) = c_1) = 1.$$

Thus, even if the available choices for the entries of the cost matrix for c^n is a large but fixed number, the tropical distance between $\mu_{\mathbf{0}}^n$ and $\nu_{\mathbf{0}}^n$ is equal to the the smallest value c_1 of the cost with large probability for large n (with probability of this event tending to one as $n \to \infty$). Moreover, if p_j is the probability of c_j appearing in any given entry of the cost matrix, then it follows from the calculation that

$$\mathbb{P}\left(d_{c^n}(\mu_{\mathbf{0}}^n, \nu_{\mathbf{0}}^n) = c_j\right) = \mathfrak{s}\left(n, \sum_{p=1}^{j} p_k\right) - \mathfrak{s}\left(n, \sum_{p=1}^{j-1} p_k\right), \tag{15}$$

which tends to zero as $n \to \infty$, the above equality (15) giving the rate of convergence. $\diamond$

5.2. **Presence of perfect matching optimal plans.** We consider the following definition.

Definition 5.4. *Let μ and ν be discrete max-plus probabilities and let h be a plan for a* square *region R_λ. We will say that h* contains a perfect matching*, if there is a perfect matching plan $\tilde{h}$ for the same region with support contained in the support of h.*

In other words, h is a perfect matching plan for a region R_λ if it can be "simplified" by substituting some of its entries equal to λ by $-\infty$ to get a perfect matching plan for a R_λ.

We will again discuss the case of a random cost provided by a Bernoulli cost matrix, and restrict ourselves to the fundamental case. To simplify the discussion, let $c_1 = 0$ and $c_2 = 1$. If there is a zero in every row and every column of the matrix, then, as we know, the tropical cost is 0, but if we look at the corresponding plan (represented by the matrix h), it may be impossible to "simplify" it (change some of the entries equal to 0 to $-\infty$) so as to produce a perfect matching plan (see Example 4.10), that is, it does not contain a perfect matching. In the opposite direction, if the corresponding optimal plan contains a perfect matching, then the tropical cost is 0. Summing up, there are the following possibilities.

- The tropical cost is 1. This occurs exactly when some row or column of the cost matrix fails to have a 0. Then there is always a perfect matching plan. In fact, the absence of a 0 in some row or column of the cost matrix means that $h_c^{m_c(0)}$ is the matrix with 1 in all the entries, which contains any perfect matching plan. For instance, if the cost matrix is

$$[c(x_i, y_j)]_{i,j=1}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

then a possible perfect matching minimizing plan is

$$[h_{i,j}]_{i,j=1}^{2} = \begin{pmatrix} 0 & -\infty \\ -\infty & 0 \end{pmatrix}.$$

- The tropical cost is 0, but the optimal plan does not contain a perfect matching.
- The tropical cost is 0, and the optimal plan contains a perfect matching.

For the following theorem we give here a random graph argument based on the strong and remarkable result of Bollobás and Thomason (see [2, theorem 7.11]) that will also be used in the proof of Theorem 5.7 below.

Theorem 5.5. *Let $c_1 < c_2$, and suppose that for each n, $\mu_{\mathbf{0}}^n$ and $\nu_{\mathbf{0}}^n$ are discrete max-plus probabilities with all their weights equal to zero, and c^n is a Bernoulli cost matrix: $\mathbb{P}(c^n(x_i, y_j) = c_1) = p_n$, $\mathbb{P}(c^n(x_i, y_j) = c_2) = q_n = 1 - p_n$ for $i, j \in \{1, \dots, n\}$, where $x_1, \dots, x_n$ and $y_1, \dots, y_n$ are the points of the support of $\mu_{\mathbf{0}}^n$ and $\nu_{\mathbf{0}}^n$. If $p_n \geq (\log n)/n$ for all but finitely many n, then*

$$\lim_{n\to\infty} \mathbb{P}(\exists h \in \Pi^{c^n}(\mu_{\mathbf{0}}^n, \nu_{\mathbf{0}}^n) : h \text{ contains a perfect matching}) = 1.$$

Proof. Every c^n is associated with one and only one random bipartite (undirected) graph $G_n(c^n)$ with the sets $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ as the two disjoint sets of vertices in the following way: $c^n(x_i, y_j) = c_1$ if $x_i y_j$ is an edge, and $c^n(x_i, y_j) = c_2$ otherwise. The plan $h_{c^n}^{m_{c^n}(0)}$ contains a perfect matching plan if and only if the bipartite graph $G_n(c^n)$ contains a perfect matching. In the proof of [2, theorem 7.11], it is shown that the probability that the random bipartite graph contains a perfect matching approaches 1 as $n \to \infty$. Thus, the probability that $h_{c^n}^{m_{c^n}(0)}$ contains a perfect matching also approaches 1 as $n \to \infty$. Since $h_{c^n}^{m_{c^n}(0)}$ is always an optimal plan, the result follows. $\square$

Remark 5.6. An alternative proof of Theorem 5.5 can be made as follows. Regardless of whether the tropical cost is c_1 or c_2, for the plan $h_c^{m_c(0)}$ (which is always minimizing), the property of containing a perfect matching plan is characterized by the fact that, for some permutation $\sigma \in S_n$, the product

$$\Pi_{j=1}^{n} |c_2 - c(x_j, y_{\sigma(j)})|$$

is different from zero (necessarily then it is equal to $(c_2 - c_1)^n$). The latter is guaranteed, for instance, when the matrix $[c_2 - c(x_i, y_j)]_{i,j=1}^{n}$ is not singular (i.e. has nonzero determinant). By a theorem of Basak and Rudelson [1], this probability approaches 1, for every $0 < p < 1$. $\diamond$

5.3. **Uniqueness of minimizing plans.** We show now that in the fundamental case (when all the weights of the discrete max-plus masure are zero), when the uniform probability is put on the space of the cost matrices, the uniqueness of a minimizing plan containing only 0 and $-\infty$ is an asymptotically rare event in the sense that its probability tends to zero as the number of weights approaches infinity. Namely, the following result is valid.

Theorem 5.7. *Fix any positive real number $M > 0$ and let $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$ be sequences of subsets of X and Y respectively, with $\#X_n = \#Y_n = n$ for all n. For each $n \in \mathbb{N}$, let*

$$\mu_{\mathbf{0}}^n := \max\{0 + \delta_{x_1}, \dots, 0 + \delta_{x_n}\}, \quad \nu_{\mathbf{0}}^n := \max\{0 + \delta_{y_1}, \dots, 0 + \delta_{y_n}\},$$

where $x_1, \dots, x_n$ are the elements of X_n and $y_1, \dots, y_n$ those of Y_n. For each n let P_n be the uniform probability measure over $[0, M]^{X_n \times Y_n}$. Define $C_n \subset [0, M]^{X_n \times Y_n}$ as the set of functions c such that there is a unique, among plans containing only 0 and $-\infty$, minimizing plan for $d_c(\mu_n, \nu_n)$. Then

$$\lim_{n\to\infty} P_n(C_n) = 0.$$

Proof. In order to apply the theory from [2], let us introduce the notion of *bipartite graph process*, specifically, on the set of vertices $X_n \cup Y_n$. Any given bijective function $f\colon \{1, \dots, n^2\} \to \{1, \dots, n\}^2$ we define determines a sequence of n^2+1 graphs in the following way: at time step $t = 0$ there are no edges and at step $t \in \{1, \dots, n^2\}$ the edge $(i, j) := f(t)$ is added. At the n^2-th time step we obtain the complete bipartite graph. Note that the set of bijective functions $f\colon \{1, \dots, n^2\} \to \{1, \dots, n\}^2$ is in one-to-one correspondence with the set of permutations of $\{1, \dots, n^2\}$, i. e. with the symmetric group S_{n^2} of order n^2; in fact, each f^{-1} is an enumeration of the cells of an $n \times n$ matrix. If the function f (or equivalently the respective permutation $\sigma \in S_{n^2}$) is chosen randomly, with uniform probability, then we have a *random bipartite graph process*, which coincides with the one described in [2] (see pp. 42 and 171 therein). Let

$$\Omega_n := \{\omega : X_n \times Y_n \to [0, M] \ : \ \omega \text{ takes } n^2 \text{ distinct values } \},$$

and for each $\omega \in \Omega_n$ define the mapping $f_\omega\colon \{1, \dots, n^2\} \to \{1, \dots, n\}^2$ by setting $f_\omega(t) := (i, j)$, where (i, j) is the unique pair of indices such that $\omega(x_i, y_j)$ is the t-th largest value among the n^2 distinct values $\omega(x_1, y_1), \dots, \omega(x_n, y_n)$. Thus, each $\omega \in \Omega_n$ determines an ordering of the matrix cells f_ω which, in turn, gives the above described graph process with $f := f_\omega$.

Since P_n is the uniform measure on $[0, M]^{X_n \times Y_n}$, we have $P_n(\Omega_n) = 1$. Moreover, since P_n is uniform, for each bijective $g\colon \{1, \dots, n^2\} \to \{1, \dots, n\}^2$, the set $\{\omega \in \Omega_n \colon f_\omega = g\}$ has the same P_n-measure, namely, $1/(n^2)!$. Hence, these sets form a partition of the probability space

$$([0, M]^{X_n \times Y_n}, \mathcal{B}([0, M]^{X_n \times Y_n}), P_n)$$

into $(n^2)!$ equiprobable events, where $\mathcal{B}([0, M]^{X_n \times Y_n})$ stands for the Borel σ-algebra of $[0, M]^{X_n \times Y_n}$. Thus, the bipartite random graph process can be equivalently sampled from this probability space, rather than directly from the set of bijective $g\colon \{1, \dots, n^2\} \to \{1, \dots, n\}^2$ (or equivalently, from S_{n^2}) endowed with the uniform probability. Let us denote by $\{G_t\}_{t=0}^{n^2}$ a generic realization of our bipartite random graph process on $X_n \cup Y_n$, and let τ be the stopping time

$$\tau := \min\{t \ : \ G_t \text{ has degree } 1\}.$$

That is, τ is the first instance t such that every x_i belongs to an edge and also every y_j belongs to an edge. Recalling now Definition 4.4 and the algorithm of section 4.2, we have:

$$\tau(\omega) = m_\omega(0) \tag{16}$$

for P_n-a.e. $\omega \in \Omega_n$. Denote by D_n the event that G_τ contains a perfect matching. By [2, theorem. 7.11],

$$\lim_{n\to\infty} \mathbb{P}_n(D_n) = 1, \tag{17}$$

which means, in words, that by the time the bipartite graph achieves degree 1 (this is exactly the time when the minimizing plan $h_\omega^{m_\omega(0)}$ is formed, by (16)), the graph contains a perfect matching. Let

$$H_n := \{\omega \in \Omega_n \ : \ h_\omega^{m_\omega(0)} \text{ is not reduced}\}.$$

By Proposition 4.8, we will be done if we show that $P_n(H_n) \to 1$ as $n \to \infty$. Now, the event D_n is the disjoint union of F_n and E_n, where F_n is the event that G_τ is *exactly* a perfect matching, and E_n is the event that G_τ has a perfect matching and at least one more edge. As can easily be argued, $P_n(F_n) \to 0$ as $n \to \infty$ (in fact, for F_n to hold, at the last step of forming G_τ only one possibility of forming an edge, or equivalently only one way of placing a zero in the respective row of the matrix, results in a perfect matching). Thus, by (17), $P_n(E_n) \to 1$ as $n \to \infty$. On the other hand, the event E_n is included in H_n: indeed, a graph in E_n corresponds to a plan in the support of which there is triple of indices, two of which are in the same column and two of which are in the same row, thereby violating Definition 3.1. Therefore, $\lim_{n\to\infty} P_n(H_n) = 1$ hence concluding the proof. □

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