A SIMPLE PROOF OF THE 1-DIMENSIONAL FLAT CHAIN CONJECTURE

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Abstract. We provide a new proof of the fact that metric 1-currents with compact support in the Euclidean space correspond to Federer–Fleming flat chains; that is, the 1-dimensional case of the so-called *flat chain conjecture*. While previous proofs rely on the delicate task of constructing Lipschitz functions with small L^∞-norm but large derivative along certain directions at all points of a given Lebesgue null set (the so-called *width functions*), our approach is based primarily on Poincaré's lemma. This perspective allows us to identify a regularity question concerning the solvability of the equation $d\omega = \pi$ for differential k-forms, a question that is closely related to the general validity of the flat chain conjecture in higher dimensions.

Keywords: metric currents, flat chains, normal currents.

MSC (2010): 49Q15, 49Q20.

1 INTRODUCTION

In this note we give a new, elementary proof of the following result, see §2 for the relevant notation.

Theorem 1.1. *Let T be a metric 1-current with compact support in $\mathbb{R}^d (d \geq 2)$ and denote by $\widetilde{T}$ the classical current induced by T. Then, $\widetilde{T}$ is a flat chain.*

The fact that Ambrosio–Kirchheim metric 1-currents with compact support correspond to Federer–Fleming flat chains was proved by Schioppa in [18], by G. Alberti, D. Bate, and the first named author in [1], and by L. De Masi and the first named author in [10]. The analogous result for metric d-currents was proved in [12]. All these proofs rely, directly or indirectly, on the advances in understanding the structure of Lebesgue-null sets pioneered by G. Alberti, M. Csörnyei, and D. Preiss, see [3, 2], and in particular on the notion of *Alberti representations* of a measure μ, as well as the associated construction of *width functions*, that is, Lipschitz functions with small supremum norm and large derivative μ-a.e. along certain directions.

Our proof does not rely on any such notion or construction, and it sheds new light on the PDE challenges behind the possibility to prove the conjecture for k-dimensional currents, with $1 < k < d$. The proof is based on the observation that if Theorem 1.1 were false, then there would exist a nontrivial class of metric currents, which we call *purely non-flat* currents, on which the mass coincides with the flat norm (see Proposition 3.2), and for which the latter can be computed, by an applicaton of Hahn-Banach theorem, as the supremum of tests with *closed* forms of comass bounded by 1 (see Proposition 3.3). This observation is based on a recent result on the structure of flat chains of finite mass, see Theorem 2.2, which follows from a straightforward application of the polyhedral approximation theorem. An application of Poincaré's lemma would then imply the existence of a metric 1-current whose actions on an equi-Lipschitz and converging sequence of tests do not converge to the action on the limit test, contradicting the continuity axiom of metric currents.

The same strategy to prove that compactly supported metric k-currents in $\mathbb{R}^d$ correspond to Federer–Fleming flat chains for every $2 \leq k \leq d-1$ (the so-called *flat chain conjecture*) would go through if one were able to suitably extend to differential k-forms the L^∞–Lipschitz estimate for the equation $d\omega = \phi$ that Poincaré's lemma implies for 1-forms; see Remark 4.1 and [10, Conjecture 4.1].

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2 NOTATION AND PRELIMINARY RESULTS

2.1 *Classical currents in* $\mathbb{R}^d$

A *k-dimensional current* T in $\mathbb{R}^d$ is a continuous linear functional on the space of smooth and compactly supported differential k-forms on $\mathbb{R}^d$, endowed with the topology of test functions. The *boundary* of T, ∂T, is the $(k-1)$-current defined via $\langle \partial T, \omega \rangle := \langle T, d\omega \rangle$ for every smooth and compactly supported $(k-1)$-form ω. The *mass* of T, denoted by $\mathbb{M}(T)$, is the supremum of $\langle T, \omega \rangle$ over all k-forms ω such that $\|\omega\| \le 1$, where $\|\omega\|$ denotes the comass norm. We also use the symbol $\|\cdot\|$ to denote the comass norm of a k-covector and the mass norm of a k-vector. A current T is called *normal* if both T and ∂T have finite mass.

By the Riesz and Radon–Nikodým theorems, a k-dimensional current T with finite mass can be written in an essentially unique way in the form $T = \tau_T \mu_T$ where μ_T is a finite positive measure and τ_T is a k-vector field with unit mass norm μ_T-a.e. In particular, the action of T on a smooth and compactly supported k-form ω is given by

$$\langle T, \omega \rangle = \int_{\mathbb{R}^d} \langle \omega(x), \tau_T(x) \rangle d\mu_T(x) \, .$$

Moreover, the restriction $T \llcorner E$ of such a current T to a Borel set $E \subset \mathbb{R}^d$ is defined as

$$\langle T \llcorner E, \omega \rangle = \int_E \langle \omega(x), \tau_T(x) \rangle d\mu_T(x) \, ,$$

for every smooth and compactly supported k-form ω.

2.2 *Flat chains*

On the space of smooth and compactly supported differential k-forms, for every compact set $K \subset \mathbb{R}^d$ we consider the flat seminorm, see [13, §4.1.12],

$$\mathbb{F}_K(\omega) := \max_{x \in K} \{ \|\omega(x)\|, \|d\omega(x)\| \}.$$

This induces a corresponding flat norm on currents, also denoted by $\mathbb{F}_K$, defined by

$$\mathbb{F}_K(T) := \sup\{ T(\omega) : \mathbb{F}_K(\omega) \le 1 \}.$$

Observe that, despite its name, the flat norm is not a norm. We recall that, by [13, §4.1.12], if $\mathbb{F}_K(T) < \infty$, then the support of T, denoted by $\mathrm{supp}(T)$, is contained in K; that is, $\mathrm{supp}(T) \subseteq K$ and moreover

$$\mathbb{F}_K(T) = \inf\{ \mathbb{M}(R) + \mathbb{M}(S) : T = R + \partial S, \, \mathrm{supp}(S) \subseteq K \}.$$

A k-dimensional current T is called a *flat chain* if there exists a compact set K and sequence of k-dimensional normal currents $(T_i)_{i \in \mathbb{N}}$ such that $\mathrm{supp}(T_i) \subseteq K$ and $\mathbb{F}_K(T_i - T) \to 0$ as $i \to \infty$. More information on currents and flat chains can be found in Federer's book [13, §4].

2.3 *Relations between normal currents and flat chains*

Normal currents and flat chains are closely related and share similar geometric properties. For instance, k-dimensional normal currents and flat chains with compact support cannot be supported on sets with zero integralgeometric k-dimensional measure; see [13, Theorem 4.2.14]. Moreover, as noted in [13, §4.1.17], every flat chain with compact support and finite mass can be approximated *in mass* by normal currents. Via a straightforward application of this fact and the polyhedral approximation theorem, see[13, Theorem 4.2.24], the following result was proved in [5]. For an analogue in dimension $k = 1$ in Carnot groups, see [11, Proposition 4.1].

Theorem 2.1 ([5, Theorem 1.1])**.** *Let $1 \leq k < d$, and let T be a k-dimensional flat chain with compact support and finite mass in $\mathbb{R}^d$. For every $\varepsilon > 0$, there exists a k-dimensional normal current T' in $\mathbb{R}^d$ and a Borel set $E \subset \mathbb{R}^d$ such that:*

(i) $\partial T' = 0$,

(ii) $T = T' \llcorner E$,

(iii) $\mathbb{M}(T') \leq (2 + \varepsilon)\, \mathbb{M}(T)$.

2.4 *k-Tangent Bundle*

We now introduce the main tool used in this paper. We begin with the following definition; see [5, Definition 4.1].

Definition 2.1 (k-tangent bundle)**.** Let μ be a Radon measure on $\mathbb{R}^d$. For every $k = 1, \ldots, d$ and for every point x in the support of μ, we denote by $V_k(\mu, x)$ the set of all k-vectors $v \in \Lambda_k(\mathbb{R}^d)$ for which there exists a k-dimensional normal current T with $\partial T = 0$ such that

$$\lim_{r \to 0} \frac{\mathbb{M}\big((T - v\mu) \llcorner B(x,r)\big)}{\mu(B(x,r))} = 0.$$

We set $V_k(\mu, x) := \{0\}$ when x does not belong to the support of μ.

Using a simple covering argument, one can deduce from Theorem 2.1 the following characterization of flat chains on $\mathbb{R}^d$ with compact support and finite mass.

Theorem 2.2 ([5, Theorem 1.2])**.** *Let μ be a positive Radon measure on a compact set $K \subset \mathbb{R}^d$. Let $\tau \in L^1(\mu)$ be a Borel k-vector field in $\mathbb{R}^d$, with $k < d$. The following statements are equivalent:*

(i) $\tau(x) \in V_k(\mu, x)$ *for μ-a.e. x;*

(ii) $\tau\mu$ *is a k-dimensional flat chain with finite mass.*

2.5 *Metric currents*

Let (X, d) be a complete metric space. We denote $\mathscr{D}^k(X) := \mathrm{Lip}_b(X, \mathbb{R}) \times \mathrm{Lip}(X, \mathbb{R})^k$, where $\mathrm{Lip}(X, \mathbb{R})$ is the space of Lipschitz functions on X and $\mathrm{Lip}_b(X, \mathbb{R})$ is the subspace of bounded Lipschitz functions.

Definition 2.2 (Metric currents)**.** A multilinear functional $T : \mathscr{D}^k(X) \to \mathbb{R}$ is said to be a k-dimensional *metric current* if the following conditions hold:

(i) *continuity*: for every $f \in \mathrm{Lip}_b(X, \mathbb{R})$, $(\pi_1^n)_{n \in \mathbb{N}}, \ldots, (\pi_k^n)_{n \in \mathbb{N}} \subset \mathrm{Lip}(X, \mathbb{R})$ with $\mathrm{Lip}(\pi_i^n) \leq C$ for every n, converging pointwise to $\pi_1, \ldots, \pi_k$,

$$T(f, \pi_1^n, \ldots, \pi_k^n) \to T(f, \pi_1, \ldots, \pi_k);$$

(ii) *locality*: if there exists $i \in \{1, \ldots, k\}$ such that $\pi_i \equiv c$ on a neighbourhood of supp$f$ then $T(f, \pi_1, \ldots, \pi_k) = 0$;

(iii) *finite mass*: there exists a finite Radon measure μ such that

$$|T(f, \pi_1, \ldots, \pi_k)| \leq \mathrm{Lip}(\pi_1) \cdots \mathrm{Lip}(\pi_k) \int_X |f| d\mu. \tag{1}$$

The minimal measure such that (1) holds is denoted by μ_T.

More information on metric currents can be found in [7] and [15].

Let us shift our attention to the case $X = \mathbb{R}^d$, equipped with the Euclidean distance. It's worth recalling that for every k-dimensional metric current T on $\mathbb{R}^d$ with compact support, there exists a corresponding "classical" k-dimensional current $\widetilde{T}$, see [7, Theorem 11.1]. Denoting by $\Lambda(k,d)$ the set of multi-indices $\alpha = (1 \le \alpha_1 < \cdots < \alpha_k \le d)$ of length k in $\mathbb{R}^d$, the condition defining $\widetilde{T}$ is that for every smooth and compactly supported k-form

$$\omega = \sum_{\alpha \in \Lambda(k,d)} \omega_\alpha dx_{\alpha_1} \wedge \cdots \wedge dx_{\alpha_k}$$

it holds

$$\langle \widetilde{T}, \omega \rangle = \sum_{\alpha \in \Lambda(k,d)} T(\omega_\alpha, x_{\alpha_1}, \ldots, x_{\alpha_k}). \tag{2}$$

Conversely, for every flat chain T with finite mass and compact support, there exists a corresponding metric current $\hat{T}$. These mappings are inverses of each other when restricted to normal currents, see [15, Theorem 5.5].

3 PROPERTIES OF PURELY NON-FLAT CURRENTS

By Theorem 2.1 and Theorem 2.2, many geometric properties valid for normal currents can be inferred also for flat chains with finite mass.

This note originates from the observation of some interesting properties enjoyed by those currents with finite mass whose tangent k-vector field is in some sense orthogonal to the tangent fields "allowed by flat chains", for which we give the following definitions.

Definition 3.1. [Purely non-flat current]

Given a vector subspace V of the space of k-vectors $\Lambda_k(\mathbb{R}^d)$, we denote

$$V^\perp := \{\tau \in \Lambda_k(\mathbb{R}^d) : \|\tau\| \le \|\tau + \sigma\|, \text{ for every } \sigma \in V\},$$

where $\|\cdot\|$ denotes the mass norm. We say that a k-current with finite mass $T = \tau_T \mu_T$ is *purely non-flat* if

$$\tau_T(x) \in (V_k(\mu_T, x))^\perp, \qquad \text{for } \mu_T\text{-almost every } x.$$

Lemma 3.1. *Given a vector subspace V in $\Lambda_k(\mathbb{R}^d)$, every element τ of $\Lambda_k(\mathbb{R}^d)$ can be written as $\tau = \tau_V + \tau_{V^\perp}$, where $\tau_V \in V$ and $\tau_{V^\perp} \in V^\perp$.*

Proof. Let $\tau_{V^\perp}$ be an element of the affine subspace $\tau + V$ minimizing the mass norm. Note that

$$\|\tau_{V^\perp}\| = \mathrm{dist}(0, \tau + V) = \mathrm{dist}(0, -\tau + V) = \mathrm{dist}(\tau, V).$$

Denote $\tau_V := \tau - \tau_{V^\perp}$. Obviously $\tau_V \in V$. Let us prove that $\tau_{V^\perp} \in V^\perp$. By an application of Hahn–Banach's theorem, see [17, Corollary 1.9.7], on the space $(\Lambda_k(\mathbb{R}^d), \|\cdot\|)$, there exists a k-covector $\omega \in \Lambda^k(\mathbb{R}^d)$ such that

$$\|\omega\| = 1, \qquad \langle \omega; \sigma \rangle = 0, \text{ for every } \sigma \in V, \qquad \text{and} \qquad \langle \omega, \tau \rangle = \mathrm{dist}(\tau, V) = \|\tau_{V^\perp}\|.$$

Since

$$\tau + V \subset \{\sigma \in \Lambda_k(\mathbb{R}^d) : \langle \omega, \sigma \rangle = \langle \omega, \tau \rangle\}, \tag{3}$$

then for every $\sigma \in V$ it holds

$$\|\tau_{V^\perp} + \sigma\| \overset{\|\omega\|=1}{\ge} \langle \omega, \tau_{V^\perp} + \sigma \rangle \overset{(3)}{=} \langle \omega, \tau \rangle = \|\tau_{V^\perp}\|,$$

that is, $\tau_{V^\perp} \in V^\perp$. This concludes the proof. □

Remark 3.1. The multimap $\tau \mapsto (\tau_V, \tau_{V^\perp})$ admits a Borel selection. Since $\tau_V = \tau - \tau_{V^\perp}$ it is sufficient to prove that we have a Borel selection $\tau \mapsto \tau_{V^\perp}$. We have shown in the proof of Lemma 3.1 that we can chose as $\tau_{V^\perp}$ any element of $\tau + V$ such that

$$\|\tau_{V^\perp}\| = \mathrm{dist}(\tau, V).$$

Thus, if we can prove that the multimap $P : \tau \mapsto \{z \in \tau + V : \|z\| = \mathrm{dist}(\tau, V)\}$ is closed-valued and Borel measurable, then it satisfies the hypotheses of the Kuratowski–Ryll-Nardzewski selection theorem (see [19, Theorem 5.2.1]), and the existence of the claimed Borel selection follows.

The fact that the multimap P takes closed values is an immediate consequence of the fact that $\tau \mapsto \mathrm{dist}(\tau, V)$ is continuous. Hence, one is left with checking that P is Borel measurable. This however can be proved with the same argument employed in [8, Lemma 3.1]. By definition of measurable multimap it is sufficient to show that for every closed set C in $\Lambda_k(\mathbb{R}^d)$ the set

$$\{z \in \Lambda_k(\mathbb{R}^d) : P(z) \cap C \neq \emptyset\}$$

is Borel, cf. [19, Lemma 5.1.2], and [19, Theorem 5.2.1]. Let us note that

$$\{z \in \Lambda_k(\mathbb{R}^d) : P(z) \cap C \neq \emptyset\} = V + \{z \in C : \|z\| = \mathrm{dist}(z, V)\} = \pi_V^{-1}(\pi_V(\{z \in C : \|z\| = \mathrm{dist}(z, V)\})),$$

where π_V denotes the orthogonal projection (with respect to the Euclidean inner product) onto the orthogonal complement of V (not to be confused with the set $V^\perp$ defined in 3.1). Since π_V is continuous and the maps $z \mapsto \|z\|$ and $z \mapsto \mathrm{dist}(z, V)$ are continuous, then $\pi_V(\{z \in C : \|z\| = \mathrm{dist}(z, V)\})$ is Borel, and therefore $\{z \in \Lambda_k(\mathbb{R}^d) : P(z) \cap C \neq \emptyset\}$ is Borel. This concludes the proof.

Definition 3.2 (Closed flat seminorm)**.** We define the *closed flat seminorm* of a current T as

$$\mathbb{F}_0(T) := \sup\{\langle T, \omega\rangle : \|\omega(x)\| \leq 1,\ d\omega(x) = 0\}.$$

In general the closed flat seminorm of a current T with $\mathrm{supp}(T) \subset K$ compact is not equivalent to the flat norm $\mathbb{F}_K$, indeed the former equals zero whenever $\partial T = 0$. However, we will show that the two quantities are equal for purely non-flat currents. We begin with the following

Proposition 3.2. *Assume that T is a purely non-flat k-current in $\mathbb{R}^d$ with $1 \leq k < d$, $\mathrm{supp}(T) \subset K$ compact. Then $\mathbb{F}_K(T) = \mathbb{M}(T)$.*

Proof. Since $\mathbb{M}(T)$ is finite and $\mathrm{supp}(T) \subset K$, then $\mathbb{F}_K(T) < \infty$. To prove that $\mathbb{F}_K(T) = \mathbb{M}(T)$, it is sufficient to check that $\mathbb{F}_K(T) \geq \mathbb{M}(T)$. Consider any decomposition of the form $T = R + \partial S$ with $\mathbb{M}(R) + \mathbb{M}(S) < \infty$ with $\mathrm{supp}(S) \subseteq K$. Note that when we endow the space $\bigcup_{i=0}^{\binom{d}{k}} Gr(i, \Lambda_k(\mathbb{R}^d))$ with its natural distance (see [4, Section 2.1]), the map $x \mapsto V_k(\mu_T, x)$ is universally measurable, that is, measurable w.r.t. the completion of the Borel σ-algebra according to any finite measure on $\mathbb{R}^d$. The proof follows mutatis mutandis that of [4, Lemma 6.9]. Writing $T = \tau_T \mu_T$ and $R = \tau_R \mu_R$, we define the Borel set

$$A := \{x \in \mathbb{R}^d : \tau_R(x) \in V_k(\mu_T, x)\}$$

and we decompose by Radon–Nikodým's theorem the measure μ_R as follows

$$\mu_R = \rho_R \mu_T + \mu_s, \qquad \text{with } \rho_R \in L^1(\mu_T) \text{ and } \mu_s \perp \mu_T.$$

We denote

$$R_s := \tau_R \mu_s, \qquad R_f := \tau_R \rho_R \mu_T \llcorner A, \qquad \text{and} \qquad R_n := \tau_R \rho_R \mu_T \llcorner (\mathbb{R}^d \setminus A),$$

so that $R = R_s + R_f + R_n$ and R_s, R_f and R_n are pairwise mutually singular as k-vector valued measures. Hence we have that

$$\mathbb{M}(R_s) + \mathbb{M}(R_f) + \mathbb{M}(R_n) = \mathbb{M}(R).$$

Note that by definition of A, Theorem 2.2 implies that R_f is a flat chain since the Radon–Nikodým derivative of R_f with respect to its total variation, is by definition is contained in $V_k(\mu_T, x)$, which coincides with $V_k(\mu_{R_f}, x)$, μ_{R_f}-a.e, by Besicovitch differentiation theorem, see [6, Theorem 2.22]. Now, since

$$T - R_n - R_s = R_f + \partial S,$$

we know that $T - R_n - R_s$ is a flat chain, since ∂S is a normal current, being a cycle with finite mass. Further, since R_s is singular with respect to both T and R_n, we deduce again from [5, Theorem 1.2] that R_s is also a flat chain. It follows that $T - R_n$ is a flat chain, so that, by [5, Theorem 1.2]

$$(\tau_T(x) - \tau_R(x)\rho_R(x)) \in V_k(\mu_T, x), \qquad \text{for } \mu_T \llcorner (\mathbb{R}^d \setminus A)\text{-a.e } x. \tag{4}$$

Since $T \llcorner A = (T - R_n) \llcorner A$ is also a flat chain, then [5, Theorem 1.2] and the fact that T is purely non-flat implies that $T \llcorner A = 0$. This implies that

$$\mathbb{M}(R) \geq \mathbb{M}(R_n) = \int_{\mathbb{R}^d \setminus A} \|\tau_R\| \rho_R d\mu_T \overset{(4)}{\geq} \int_{\mathbb{R}^d \setminus A} \|\tau_T\| d\mu_T \overset{T \llcorner A = 0}{=} \mathbb{M}(T),$$

where the inequality follows from the fact that $\tau_T \in (V_k(\mu_T, \cdot))^\perp$. Since the inequality $\mathbb{M}(R) \geq \mathbb{M}(T)$ holds for every decomposition of the form $T = R + \partial S$ as above, then [13, §4.1.12] implies that $\mathbb{F}_K(T) \geq \mathbb{M}(T)$. □

Proposition 3.3. *Assume that T is a purely non-flat k-current in $\mathbb{R}^d$ with $1 \leq k < d$, with* $\operatorname{supp}(T) \subset K$ *compact. Then* $\mathbb{F}_K(T) = \mathbb{F}_0(T)$.

Proof. The inequality $\mathbb{F}_0 \leq \mathbb{F}_K$ is true in general. Towards a proof by contradiction that $\mathbb{F}_0(T) = \mathbb{F}_K(T)$, assume that

$$f_0 := \mathbb{F}_0(T) < \mathbb{F}_K(T) =: f_1.$$

This implies that T, seen as a linear functional on the space Y of smooth and compactly supported k-forms ω with $d\omega = 0$, endowed with the comass norm, has operator norm equal to f_0 (note that $\mathbb{M} = \mathbb{F}_0$ on Y^*).

By Hahn–Banach theorem, T can be extended to a linear functional W on the space X of smooth and compactly supported k-forms, endowed with the comass norm, which coincides with T on Y. In particular, W is a k-current and $\partial T = \partial W$. Moreover $\mathbb{M}(W) = f_0$. Without loss of generality, we can assume that W is compactly supported, by possibly projecting W on a ball B compactly containing K. Notice that this operation does not increase the mass of W, while keeping $\partial W = \partial T$. Observe that, although the projection Φ on B is not a proper map, the push-forward of W is well-defined. This can be obtained as a limit in normal mass of the push-forward according to Φ of the normal currents $W_i := W \llcorner B_{r_i}$, for a suitable choice of the radii r_i with $r_i \to \infty$. The argument via slicing is standard, see e.g. the proof of [14, Lemma 3.10].

Now let us write $W = R + \partial S$, with $\mathbb{M}(R) \leq \mathbb{M}(W) = f_0$ and S is compactly supported. In particular, since $\partial T = \partial W$, then $\partial R - \partial T = \partial R - \partial W = 0$, that is $T - R = \partial N$ for some normal current N. We can write $R = R_s + R_f + R_n$ as in the proof of Proposition 3.2. As in the previous proof, using that T is purely non flat, we deduce that $\mathbb{M}(R) \geq \mathbb{M}(T)$, which is a contradiction because, by Proposition 3.2,

$$\mathbb{M}(T) = \mathbb{F}_K(T) = f_1 > f_0 \geq \mathbb{M}(W) \geq \mathbb{M}(R).$$

This concludes the proof. □

4 EQUIVALENCE BETWEEN COMPACTLY SUPPORTED METRIC 1-CURRENTS AND FLAT CHAINS WITH FINITE MASS IN THE EUCLIDEAN SPACE

We note that Proposition 3.2 and Proposition 3.3 yield an elementary proof of the fact that 1-dimensional metric currents with compact support in the Euclidean space correspond to Federer-Fleming flat chains. We begin with the following lemmas.

Lemma 4.1. *For every smooth 1-form ω on $\mathbb{R}^d$ such that $d\omega = 0$ and $\|\omega\| \leq 1$ there exists $\pi \in C^\infty(\mathbb{R}^d)$ such that*

$$d\pi = \omega \qquad \text{and} \qquad \operatorname{Lip}(\pi) \leq 1.$$

Proof. This is an immediate consequence of the fact that the exterior derivative coincides with the differential for smooth functions. □

Lemma 4.2. *Let η be a positive and finite Radon measure on $\mathbb{R}^d$ and let μ be a Radon measure singular with respect to the Lebesgue measure $\mathscr{L}^d$. Then, there exists a set of full measure of vectors $v \in \mathbb{R}^d$ such that η and $(\tau_v)_\sharp \mu$ are mutually singular, where τ_v denotes the translation map $\tau_v(x) := v + x$.*

Proof. Without loss of generality we can assume that μ is supported in $B(0,1)$ and that $\eta \perp \mathscr{L}^d$. Let $A \subset \mathbb{R}^d$ be a Borel set such that $\mathscr{L}^d(A) = 0$ and $\eta(\mathbb{R}^d \setminus A) = 0$ and observe that by Tonelli's theorem we have

$$\begin{aligned}\int_{B(0,1)} (\tau_v)_\sharp \mu(A) d\mathscr{L}^d(v) &= \int_{B(0,1)} \int \mathbb{1}_A(z+v) d\mu(z) d\mathscr{L}^d(v) \\ &= \int \int_{B(0,1)} \mathbb{1}_A(z+v) d\mathscr{L}^d(v) d\mu(z) = \int \mathscr{L}^d(A - z) d\mu(z) = 0.\end{aligned} \tag{5}$$

The above computation implies that $(\tau_v)_\sharp \mu(A) = 0$ for $\mathscr{L}^d$-almost every $v \in B(0,1)$, so that for those v's the measures η and $(\tau_v)_\sharp \mu$ are mutually singular. □

Proposition 4.3. *Let T be a metric 1-current in $\mathbb{R}^d$, for $d > 1$, with support contained in the interior of a compact set K, such that $\widetilde{T}$ is purely non-flat. Let $\{v_i\}_{i \in \mathbb{N}} \subset \mathbb{R}^d$ be such $v_i \to 0$ as $i \to \infty$. Assume moreover that $(\tau_{v_i})_\sharp \widetilde{T}$ and $\widetilde{T}$ are mutually singular as vector valued measures for every $i \in \mathbb{N}$. Then*

$$\limsup_{i \to \infty} \mathbb{F}_K(\widetilde{T} - (\tau_{v_i})_\sharp \widetilde{T}) = 0.$$

Proof. Towards a proof by contradiction of the proposition, assume that there exist a metric current T and a sequence $\{v_i\}_{i \in \mathbb{N}}$ as above for which

$$\limsup_{i \to \infty} \mathbb{F}_K(\widetilde{T} - (\tau_{v_i})_\sharp \widetilde{T}) > c > 0. \tag{6}$$

Since $\widetilde{T}$ is purely non flat, and since $(\tau_{v_i})_\sharp \widetilde{T}$ and $\widetilde{T}$ are mutually singular for every $i \in \mathbb{N}$, we deduce from [5, Theorem 1.2] that $\widetilde{T} - (\tau_{v_i})_\sharp \widetilde{T}$ is also purely non flat for every $i \in \mathbb{N}$. By Proposition 3.3, we deduce that

$$\mathbb{F}_K(\widetilde{T} - (\tau_{v_i})_\sharp \widetilde{T}) = \mathbb{F}_0(\widetilde{T} - (\tau_{v_i})_\sharp \widetilde{T}), \qquad \text{for every } i \in \mathbb{N}.$$

From (6), we infer the existence of a sequence of smooth, closed 1-forms $\{\omega_i\}_{i \in \mathbb{N}}$ with $\|\omega_i\| \leq 1$ for every $i \in \mathbb{N}$ and a non-relabeled subsequence of vectors $v_i \to 0$ such that

$$\langle \widetilde{T} - (\tau_{v_i})_\sharp \widetilde{T}, \omega_i \rangle > c, \qquad \text{for every } i \in \mathbb{N}.$$

By Lemma 4.1 and the latter, for every $i \in \mathbb{N}$ we can find $\pi_i \in C^\infty(\mathbb{R}^d)$ such that $\mathrm{Lip}(\pi_i) \leq 1$ and $\omega_i = d\pi_i$. Possibly subtracting a constant, we can assume that $\pi_i(0) = 0$, for every $i \in \mathbb{N}$. Hence we can find a 1-Lipschitz function π_∞ such that, up to non-relabeled subsequences,

$$\pi_i \to \pi_\infty \quad \text{locally uniformly.} \tag{7}$$

We deduce that for every $i \in \mathbb{N}$ we have

$$\begin{aligned} c <& \langle \widetilde{T} - (\tau_{v_i})_\sharp \widetilde{T}, \omega_i \rangle = \langle \widetilde{T}, \omega_i \rangle - \langle (\tau_{v_i})_\sharp \widetilde{T}, \omega_i \rangle \\ =& \langle \widetilde{T}, \omega_i \rangle - \langle \widetilde{T}, (\tau_{v_i})^\sharp \omega_i \rangle \overset{(2)}{=} T(1, \pi_i) - T(1, \pi_i \circ \tau_{v_i}). \end{aligned}$$

Observing that $\pi_i \circ \tau_{v_i}$ are 1-Lipschitz and, by (7), they converge locally uniformly to π_∞, we reach a contradiction to the continuity of metric currents. □

The conclusion of the proof is now a simple consequence of the fact that if T is a metric current with compact support such that $\widetilde{T}$ is purely non-flat, then $\mu_{\widetilde{T}}$ is singular with respect to $\mathscr{L}^d$.

Theorem 4.4. *Let T be a metric 1-current in $\mathbb{R}^d$ with compact support and $d \geq 2$. Then $\widetilde{T}$ is a flat chain.*

Proof. Assume by contradiction that there exists a metric 1-current T with compact support in $\mathbb{R}^2$ ($d \geq 2$) such that $\widetilde{T}$ is not a flat chain. Consider the flat chain $T_f := (\tau_{\widetilde{T}})_{V(\mu_{\widetilde{T}},\cdot)}\mu_{\widetilde{T}}$, which is well defined thanks to the measurability of $(\tau_{\widetilde{T}})_{V(\mu_{\widetilde{T}},\cdot)}$, see Remark 3.1, and the associated metric current $\widehat{T_f}$. We deduce that $T_n := T - \widehat{T_f}$ is a non trivial metric current such that $\widetilde{T_n}$ is purely non flat.

We claim that the total variation of $\widetilde{T_n}$ is singular with respect to $\mathscr{L}^d$. Indeed, suppose that there exists a Borel set B of positive $\mu_{\widetilde{T_n}}$-measure such that $\mu_{\widetilde{T_n}} \llcorner B \ll \mathscr{L}^d$. In particular, it follows from the Lebesgue density theorem that the current $\widetilde{T_n} \llcorner B$ is a limit in mass of currents of the form

$$T_N := \sum_{i=1}^{N} v_i \mathscr{L}^d \llcorner B_i,$$

where B_i are balls and v_i are constant on each B_i. It is easy to check that such T_N is a normal current, which proves that $\widetilde{T_n} \llcorner \mathbb{1}_B$ is a nontrivial flat chain: a contradiction which proves the claim.

By Lemma 4.2 we can find arbitrarily small vectors $v_i \in \mathbb{R}^d$ such that $\mu_{\widetilde{T_n}}$ and $(\tau_{v_i})_\sharp \mu_{\widetilde{T_n}}$ are mutually singular, so that, observing that $\widetilde{T_n} - (\tau_{v_i})_\sharp \widetilde{T_n}$ is purely non-flat, we have

$$\mathbb{F}_K(\widetilde{T_n} - (\tau_{v_i})_\sharp \widetilde{T_n}) = \mathbb{M}(\widetilde{T_n} - (\tau_{v_i})_\sharp \widetilde{T_n}) = 2\mathbb{M}(\widetilde{T_n}),$$

where the first equality follows from Proposition 3.2. This however contradicts Proposition 4.3. □

Remark 4.1. We remark that we used that T is 1-dimensional only in Lemma 4.1. Providing a suitable generalization of this lemma does not seem feasible, see for instance [9] and [16, Theorem 2.1]. This obstruction is deeply connected to the fact that Schauder estimates for the Laplacian fail for continuous data. However, when we apply Lemma 4.1 in Proposition 4.3, we do not really need the equality $d\omega = \pi$, but only the equality between the action of $\widetilde{T}$ on ω and the action of T on $(1, \pi)$ when $\widetilde{T}$ is purely non-flat. Therefore, in principle, the obstruction above does not exclude the possibility to adapt our strategy to prove the flat chain conjecture for k-dimensional currents, with $1 < k < d$.

CONFLICT OF INTEREST AND DATA AVAILABILITY

On behalf of all authors, the corresponding author states that there is no conflict of interest.

The manuscript has no associated data.

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