Global calibrations for the non-homogeneous Mumford-Shah functional

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Abstract
Using a calibration method we prove that, if $\Gamma \subset \Omega$ is a closed regular hypersurface and if the function g is discontinuous along Γ and regular outside, then the function u_β which solves

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is in turn discontinuous along Γ and it is the unique *absolute minimizer* of the non-homogeneous Mumford-Shah functional

$$\int_{\Omega \setminus S_u} |\nabla u|^2 \, dx + \mathcal{H}^{n-1}(S_u) + \beta \int_{\Omega \setminus S_u} (u-g)^2 \, dx,$$

over $SBV(\Omega)$, for β large enough. Applications of the result to the study of the gradient flow by the method of minimizing movements are shown.

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1 Introduction

The Mumford-Shah functional was introduced in [18] within the context of a variational approach in Image Segmentation. In the SBV setting proposed by De Giorgi (see [9]) it can be written as

$$F(u) = \int_{\Omega \setminus S_u} |\nabla u|^2 \, dx + \alpha \mathcal{H}^{n-1}(S_u) + \beta \int_{\Omega \setminus S_u} (u-g)^2 \, dx,$$

where $g : \Omega \to \mathbb{R}$ is the given input function, α and β are positive parameters, $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure, u is the unknown function in the space $SBV(\Omega)$ of special functions of bounded variation in Ω, S_u is the set of essential discontinuity points of u, while ∇u denotes its approximate gradient (see [5]).

In dimension two, the function u which minimizes F over $SBV(\Omega)$ (whose existence is stated in [4]) can be thought to represent a piecewise approximation of the input grey level function g, while S_u represents the set of relevant contours in the image. One of the mathematical features of the Mumford-Shah functional is a very strong lack of convexity, which produces, for example, non-uniqueness of the solution and makes the exhibition of explicit minimizers a very difficult task. Concerning this last point, the calibration method recently developed by Alberti, Bouchitté, and Dal Maso in [1] seems to be a powerful tool. For some applications of this method see [1], [8], [17], or [15]. Coming back to F, throughout the paper we keep the parameter α fixed (and, without loss of generality, equal to 1) and we are interested in minimizers of the functional

$$F_{\beta,g}(u) = \int_{\Omega \setminus S_u} |\nabla u|^2 \, dx + \mathcal{H}^{n-1}(S_u) + \beta \int_{\Omega \setminus S_u} (u-g)^2 \, dx, \tag{1.1}$$

with g piecewise smooth function. It is intuitive that taking β large means penalizing a lot the L^2-distance between g and the solution, which is therefore forced to be close to the input function. More precisely it is easy to see that, if for simplicity we take g belonging to $SBV(\Omega)$ such that

$$F_{\beta,g}(g) = \int_{\Omega\setminus S_g} |\nabla g|^2\,dx + \mathcal{H}^{n-1}(S_g) = C < +\infty, \tag{1.2}$$

then, denoting by u_β a minimum point of $F_{\beta,g}$, we have

$$\int_\Omega (u_\beta - g)^2\,dx \le \frac{F_{\beta,g}(u_\beta)}{\beta} \le \frac{F_{\beta,g}(g)}{\beta} = \frac{C}{\beta},$$

that is $u_\beta \to g$ in $L^2(\Omega)$ as $\beta \to +\infty$. This suggests that, in agreement with our expectations, if β is large, then u_β should be an accurate reconstruction of the original image g. Actually, T.J.Richardson in [20] has proved also the convergence of the discontinuity sets in dimension two: more precisely, he has shown that if g is a function of class $C^{0,1}$ outside any neighbourhood of the singular set S_g satisfying (1.2), and if S_g has no isolated points (i.e. for every $x \in S_g$ and for every $\rho > 0$, $\mathcal{H}^1(B_\rho(x)\cap S_g) > 0$), then, as $\beta \to +\infty$,

$$S_{u_\beta} \to S_g \text{ in the Hausdorff metric} \qquad \text{and} \qquad \mathcal{H}^1(S_{u_\beta}) \to \mathcal{H}^1(S_g).$$

In the main theorem of the paper (see Theorem 4.2), using the calibration method mentioned above, we are able to prove that, under suitable assumptions on the regularity of Ω, g, and S_g, a much stronger result holds true:
Suppose that Γ is a closed hypersurface of class $C^{2,\alpha}$ contained in the n-dimensional domain Ω (satisfying in turn some regularity assumptions), and let g a function belonging to $W^{1,\infty}(\Omega\setminus\Gamma)$, with $S_g = \Gamma$ and $\inf_{x\in\Gamma}(g^+(x) - g^-(x)) > 0$ (where g^+ and g^- denote the upper and the lower traces of g on Γ). Then there exists $\beta_0 > 0$ depending on Γ, on the $W^{1,\infty}$-norm of g, and on the size of the jump of g along Γ, such that, for $\beta \ge \beta_0$, $F_{\beta,g}$ has a unique minimizer u_β which satisfies

$$S_{u_\beta} = \Gamma.$$

Let us give now a short insight into some technical aspects of the proof; we start by recalling the theorem on which the calibration method is based. We shall consider the collection $\mathcal{F}(\Omega\times\mathbb{R})$ of all bounded vector fields $\phi = (\phi^x, \phi^z) : \Omega\times\mathbb{R} \to \mathbb{R}^n\times\mathbb{R}$ with the following property: there exists a finite family $(U_i)_{i\in I}$ of pairwise disjoint and Lipschitz open subsets of $\Omega\times\mathbb{R}$ whose closures cover $\Omega\times\mathbb{R}$, and a family $(\phi_i)_{i\in I}$ of vector fields in $Lip(\overline{U_i}, \mathbb{R}^n\times\mathbb{R})$ such that ϕ agrees at any point with one of the ϕ_i.

An *absolute calibration* for $u \in SBV(\Omega)$ in $\Omega\times\mathbb{R}$ is a vector field $\phi \in \mathcal{F}(\Omega\times\mathbb{R})$ which satisfies the following properties:

(a) $\mathrm{div}\phi = 0$ in U_i, for every $i \in I$;

(b) $\nu_{\partial U_i}\cdot\phi^+ = \nu_{\partial U_i}\cdot\phi^- = \nu_{\partial U_i}\cdot\phi$ $\mathcal{H}^{n-1}$-a.e in ∂U_i for every $i \in I$, where $\nu_{\partial U_i}(x)$ denotes the (unit) normal vector at x to ∂U_i, while ϕ^+ and ϕ^- denote the two traces of ϕ on the two sides of ∂U_i;

(c) $\dfrac{(\phi^x(x,z))^2}{4} \le \phi^z(x,z) + \beta(z - g(x))^2$ for almost every $x \in \Omega$ and every $z \in \mathbb{R}$;

(d) $\phi^x(x,u(x)) = 2\nabla u(x,y)$ and $\phi^z(x,u(x)) = |\nabla u(x)|^2 - \beta(g(x) - u(x))^2$ for almost every $x \in \Omega\setminus S_u$;

(e) $\displaystyle\int_{u^-(x)}^{u^+(x)} \phi^x(x,z)\,dz = \nu_u(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in S_u$, where $\nu_u(x)$ denotes the unit normal vector at x to S_u, which points toward u^+;

(f) $\left|\int_s^t \phi^x(x,z)\,dz\right| \le 1$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Omega$ and for every $s,\ t \in \mathbb{R}$;

(g) $\phi^x(x,z)\cdot\nu(x) = 0$ for $\mathcal{H}^n$-a.e. $(x,z) \in \partial(\Omega\times\mathbb{R})$, where $\nu(x)$ denotes the unit normal vector at x to $\partial\Omega$.

Note that conditions (a) and (b) imply that ϕ is divergence free in the sense of distributions in $\Omega\times\mathbb{R}$.

The following theorem is proved in [2].

Theorem 1.1 *If there exists an absolute calibration ϕ for u in $\Omega\times\mathbb{R}$, then u is an absolute minimizer of the Mumford-Shah functional (1.1) over $SBV(\Omega)$.*

Remark 1.2 If for a.e. $x \in \Omega$ the inequality in (b) is strict for $z \ne u(x)$, then u is the *unique* absolute minimizer of (1.1). The proof can be obtained arguing as in the last part of Paragraph 5.8 in [2].

The main difficulty in constructing the calibration comes from the fact that the candidate u_β, which is the solution of the Euler equation

$$\begin{cases} \Delta u_\beta = \beta(u_\beta - g) & \text{in } \Omega\setminus\Gamma \\ \partial_\nu u_\beta = 0 & \text{on } \partial(\Omega\setminus\Gamma), \end{cases} \tag{1.3}$$

presents, in general, a non vanishing gradient and a nonempty discontinuity set. We remark that the case of g equal to characteristic function of a regular set (i.e. with vanishing gradient) and the case of g regular in the whole Ω (i.e. with empty discontinuity set) have been already treated in [1] and require a simpler construction. From the point of view of calibrations, the interaction (actually the clash) between the (non vanishing) gradient and the (nonempty) discontinuity set is reflected in the fact that we have to guarantee simultaneously conditions (d) and (e), which push in opposite directions. Indeed condition (d) says that φ^x on the graph of u is tangential to Γ while (e) implies that φ^x must be on the average orthogonal to Γ for $x \in \Gamma$ and t between $u^-(x)$ and $u^+(x)$; so we have to "rotate" suitably φ^x, preserving at the same time condition (f). Another difficulty comes from the fact the we have to estimate how quickly the gradient of u_β changes direction; indeed if near Γ it becomes suddenly orthogonal to Γ and (e) holds true, it could happen that condition (f) is violated: this risk is overcome by carefully estimating the L^∞-norm of the Hessian matrix $\nabla^2 u_\beta$ with respect to β. In order to perform such an estimate we need to assume that Γ is of class $C^{2,\alpha}$, for some $\alpha > 0$. We underline that, at least in dimension two, the regularity assumption is close to optimal, since, by Bonnet Regularity Theorem (see [6]) (proved for $n = 2$) in a neighbourhood of any regular point the discontinuity set is of class $C^{1,1}$, for every $g \in L^\infty(\Omega)$.

The starting point for the construction of φ is the following remark: if we fiber a neighbourhood of the graph of u_β by the graphs of a family of function $(v_t)_{t\in\mathbb{R}}$ all satisfying (1.3) and we let φ to be the vector field equal to

$$(2\nabla v_t, |\nabla v_t|^2 - \beta(v_t - g)^2)$$

on the graph of v_t, then φ is divergence free. Unfortunately this construction works only locally (i.e., in a neighbourhood of Γ), but we will see how to modify it suitably in order to get it working "globally".

As an application of our theorem, we give a proof of the following fact: if u_0 is regular enough outside a smooth singular set S_{u_0}, then the gradient flow $u(x,t)$ of u_0 (via minimizing movements, see next section for the definition) for the homogeneous functional

$$F_0(u) = \int_\Omega |\nabla u|^2\,dx + \mathcal{H}^{n-1}(S_u), \tag{1.4}$$

keeps, at least for small times, the singular set of $u(\cdot,t)$ equal to S_{u_0}, while u evolves in $\Omega\setminus S_{u_0}$ according to the heat equation with Neumann boundary conditions on $\partial(\Omega\setminus S_{u_0})$. This result was proved in dimension one by Gobbino (see [11]), with a slightly different definition of gradient flow.

The plan of the paper is the following. In Section 2 we recall some definitions, fix some notations, and collect some results which will be useful for the proof of our theorems. In Section 3 we provide the estimates we mentioned above on the norm of the solutions of (1.3). In Section 4 we give the proof of the main result and, in dimension two, we extend it to the case of Ω with piecewise smooth boundary (say a curvilinear polygon) and of Γ touching the boundary (orthogonally). The final section is devoted to the study of minimizing movements.

2 Preliminary Results

For fixed $R>0$, we introduce the following class of sets:

$$\mathcal{U}_R = \{ \; E \subset \mathbb{R}^n, E \text{ open} : \forall p \in \partial E \; \exists p', p'' : \\ p \in \partial B(p',R) \cap \partial B(p'',R),\, B(p',R) \subset E,\, B(p'',R) \subset \mathcal{C}E \; \}, \quad (2.1)$$

and

$$\mathcal{U}_R(\Omega) = \{E \in \mathcal{U}_R : \; E \subset \Omega,\, \mathrm{dist}(E, \partial\Omega) \geq R\}\,. \quad (2.2)$$

If E belongs to $\mathcal{U}_R$ and $p \in \partial E$, we denote the centers of the interior and exterior balls associated with p by p' and p'' respectively; moreover, we call $\mathcal{S}_E^p$ the class of all coordinate systems centred at p such that the vector $\frac{1}{2R}(p''-p')$ coincides with the n-th vector of the coordinate basis. The following proposition is proved in ([16])

Proposition 2.1 *There exists a constant $\rho > 0$ (depending only on R), such that for every $E \in \mathcal{U}_R(\Omega)$ and for every $p_0 \in \partial E$, if we call C the cylinder $\{x \in \mathbb{R}^{n-1} : |x| < \rho\} \times]-R, R[$ expressed with respect to a coordinate system belonging to $\mathcal{S}_E^{p_0}$, then $\partial E \cap C$ is the subgraph of a function f belonging to $W^{2,\infty}(\{x \in \mathbb{R}^{n-1} : |x| < \rho\})$. Moreover, the $W^{2,\infty}$-norm of f is bounded by a constant depending only on R (independent of p_0, of E and of the choice of the coordinate system in $\mathcal{S}_E^{p_0}$).*

Remark 2.2 Note that if Ω is bounded and of class C^2 then there exists $R>0$ such that $\Omega \in \mathcal{U}_R$.

For $E \subset \mathbb{R}^n$, we define the signed distance function

$$d_E(x) = \mathrm{dist}(x, E) - \mathrm{dist}(x, \mathcal{C}E).$$

Now we are going to state some basic properties of that function; for a proof see, for example, [10].

Lemma 2.3 *i) Let x be a point of $\mathbb{R}^n$. Then $d_E(x)$ is differentiable at x if and only if there exists a unique $y \in \partial E$ such that $|d_E(x)| = |x-y|$. In this case, we have*

$$\nabla d_E(x) = \frac{x-y}{d_E(x)}$$

and we can define the projection on ∂E $\pi_E(x) := y$.

ii) Let ∂E be a hypersurface of class C^k, $k \geq 2$. Then, for every $x \in \partial E$, there exists a neighbourhood V of x such that $d_E \in C^k(V)$ and $\pi_E \in C^{k-1}(V)$.

Lemma 2.4 *Let $E \subset \mathbb{R}^n$ be an open set whose boundary is a hypersurface of class $W^{2,\infty}$. Then for every $x \in \partial\Omega$, there exists a neighbourhood V of x where π_E is well defined and such that $d_E \in W^{2,\infty}(V(x))$. Moreover, denoting by $\lambda_1 \leq \cdots \leq \lambda_n$ the eigenvalues of $\nabla^2 d_E$ and by $k_1(y) \leq \cdots \leq k_{n-1}(y)$ the principal curvatures of ∂E at $\pi(y)$, we have*

$$\lambda_i := \begin{cases} 0 & \text{if } i = 1 \\ \frac{k_{i-1}(y)}{1+d_E(y)k_{i-1}(y)} & \text{if } i > 1. \end{cases}$$

Lemma 2.5 *Let E be an open set belonging to $\mathcal{U}_R$, for some $R > 0$. Then the projection π_E is well defined and of class $W^{1,\infty}$ in the $(R/2)$-neighbourhood of ∂E, and therefore d_E is of class $W^{2,\infty}$ in that neighbourhood. Moreover we have:*

$$\|d_E\|_{W^{2,\infty}} \leq C \qquad \textit{and} \qquad \|\pi_E\|_{W^{1,\infty}} \leq C,$$

where C is a positive constant depending only on R.

PROOF. The fact that π_E is well defined in the $(R/2)$-neighbourhood of ∂E (denoted by $(\partial E)_{R/2}$) is an easy consequence of the definition of $\mathcal{U}_R$: indeed let x be a point of $(\partial E)_{R/2} \cap \complement E$ and let $p \in \partial E$ such that $d_E(x) = |x - p|$. We claim that such a p is unique. Indeed let $B(p'', R) \subset \complement E$ be the exterior ball associated with p (see the definition (2.1)); since the vector $p'' - p$ is parallel to $x - p$ (indeed both vectors are normal to ∂E at p), it is clear that $\overline{B(x, d_E(x))} \setminus \{p\} \subset B(p'', R) \subset \complement E$ and so p is the unique minimum point.

Concerning the smoothness, it is enough to prove that d_E is of class $W^{2,\infty}$, then we conclude by the equality

$$\pi_E(x) = x - d_E(x)\nabla d_E(x).$$

Exploiting the definition of $\mathcal{U}_R$ in a way similar to the one we did above, we can easily see that, for every $\varepsilon \in (0, R/2)$,

$$(E)_\varepsilon \in \mathcal{U}_{R-\varepsilon} \qquad \text{and} \qquad d_{(E)_\varepsilon} = d_E - \varepsilon, \tag{2.3}$$

implying that $\partial((E)_\varepsilon)$ is in turn of class $W^{2,\infty}$. So if $x \in (\partial E)_{R/2}$, then $x \in \partial((E)_\varepsilon)$ for $\varepsilon = d_E(x)$. By Lemma 2.4 there exists a neighbourhood V of x where $d_{(E)_\varepsilon}$ is of class $W^{2,\infty}$ and $\|d_{(E)_\varepsilon}\|_{W^{2,\infty}} \leq C$, with C depending only on R. Recalling (2.3), we are done. $\square$

For the proof of the announced estimates on the norm of the solutions of (1.3), we will use some technical results coming from sectorial operators theory and from interpolation theory.

First let us recall what a sectorial operator is.

Let X a complex Banach space and $A : D(A) \to X$ a closed linear operator with not necessarily dense domain; call $\rho(A)$ the resolvent set of A and for $\lambda \in \rho(A)$ denote by $R(\lambda, A)$ the resolvent operator $(\lambda I - A)^{-1}$ belonging to $L(X)$.

Definition 2.6 *A is said to be sectorial (in X) if the following two conditions are satisfied:*

i) there exist $\omega \in \mathbb{R}$ and $\theta \in (\frac{\pi}{2}, \pi)$ such that

$$S_{\theta,\omega} := \{\lambda \in \mathbb{C} : |\arg(\lambda - \omega)| \leq \theta\} \subset \rho(A);$$

ii) there exists a positive constant M such that, for every $\lambda \in S_{\theta,\omega}$, there holds

$$\|R(\lambda, A)\|_{L(x)} \leq \frac{M}{|\lambda - \omega|}.$$

We recall that $D(A)$, endowed with the norm

$$\|x\|_{D(A)} = \|x\|_X + \|Ax\|_X$$

is a Banach space continuously embedded in X.

Let Ω be either $\mathbb{R}^n$ or $\mathbb{R}^n_+$ and let $A : \Omega \to \mathbb{R}^{n\times n}$ be a matrix with coefficients belonging to $W^{1,\infty}(\Omega)$ and uniformly elliptic, i.e., satisfying

$$A(x)\xi \cdot \xi \geq \lambda|\xi|^2 \qquad \forall x \in \Omega,\ \forall \xi \in \mathbb{R}^n,$$

where λ_0 is a suitable positive constant; set

$$D(\mathcal{A}_0) := \left\{ u \in L^\infty(\Omega) : u \in \bigcap_{p\geq 1} W^{2,p}_{loc}(\Omega),\ \operatorname{div}(A\nabla u) \in L^\infty(\Omega) \text{ and } A\nabla u \cdot \nu = 0 \text{ on } \partial\Omega \right\}$$

$$D(\mathcal{A}_1) := \left\{ u \in D(\mathcal{A}_0) : \operatorname{div}(A\nabla u) \in W^{1,\infty}(\Omega) \text{ and } A\nabla u \cdot \nu = 0 \text{ on } \partial\Omega \right\},$$

where $\nu(x)$ denotes the outer unit normal vector at x to Ω, and define the operators

$$\begin{array}{rcl} \mathcal{A}_0 : D(\mathcal{A}_0) & \to & L^\infty(\Omega) \\ u & \mapsto & f\operatorname{div}(A\nabla u), \end{array} \tag{2.4}$$

and

$$\begin{array}{rcl} \mathcal{A}_1 : D(\mathcal{A}_1) & \to & W^{1,\infty}(\Omega) \\ u & \mapsto & f\operatorname{div}(A\nabla u), \end{array} \tag{2.5}$$

where $f : \Omega \to (0, +\infty)$ is a positive function of class $W^{1,\infty}$ satisfying:

$$f(x) \geq \lambda_1 > 0 \qquad \forall x \in \Omega.$$

The following fact is proved in [14] (see Theorem 3.1.6, page 77, Theorem 3.1.7, page 78, and 3.1.26, page 103).

Theorem 2.7 *The operators $\mathcal{A}_0$ and $\mathcal{A}_1$ are sectorial in $L^\infty(\Omega)$ and $W^{1,\infty}(\Omega)$ respectively. In particular there exist two positive constants β_0 and K, depending on the constants λ_0, λ_1, on $W^{1,\infty}$-norm of A and f, such that the problem*

$$\begin{cases} f\operatorname{div}(A\nabla u) = \beta(u-g) & \textit{in } \Omega, \\ A\nabla u \cdot \nu = 0 & \textit{in } \partial\Omega, \end{cases} \tag{2.6}$$

admits a unique solution $u \in D(\mathcal{A})$, for every $\beta \geq \beta_0$ and for every $g \in L^\infty(\Omega)$. Moreover u satisfies

$$\|u\|_\infty + \beta^{-\frac{1}{2}}\|\nabla u\|_\infty \leq K\|g\|_\infty; \tag{2.7}$$

if g belongs to $W^{1,\infty}(\Omega)$ then the following estimate actually holds

$$\|u\|_{W^{1,\infty}} + \beta^{-\frac{1}{2}}\|f\operatorname{div}(A\nabla u)\|_\infty + \sup_{x_0\in\Omega} \beta^{\frac{n}{2p}}\|\nabla^2 u\|_{L^p\left(B(x_0,\frac{1}{\sqrt{\beta}})\cap\Omega\right)} \leq K\|g\|_{W^{1,\infty}}. \tag{2.8}$$

Given a sectorial operator $A : D(A) \to X$ there is a natural way to construct a family of intermediate spaces between $D(A)$ and X, by setting for $\theta \in (0,1)$

$$D(A,\theta,\infty) = \left\{ x \in X : \sup_{t>2\omega\vee 1} \left(t^\theta \|AR(t,A)x\|_{L(X)}\right) < +\infty \right\},$$

where ω is the real number appearing in i) of Definition 2.6. Setting

$$[x]_{D(A,\theta,\infty)} = \sup_{t>2\omega\vee 1} \left(t^\theta \|AR(t,A)x\|_{L(X)}\right), \tag{2.9}$$

one sees that $[x]_{D(A,\theta,\infty)}$ is a seminorm and $D(A,\theta,\infty)$ endowed with the norm

$$\|x\|_{D(A,\theta,\infty)} = \|x\|_X + [x]_{D(A,\theta,\infty)} \tag{2.10}$$

is a Banach space. Moreover, for $0 \leq \theta_1 < \theta_2 \leq 1$,

$$Y \subseteq D(A,\theta_2,\infty) \subset D(A,\theta_1,\infty) \subseteq X,$$

with continuous embeddings. An important fact is stated in the following proposition

Proposition 2.8 *(see Proposition 2.2.7, page 50 of [14])*

$$\begin{array}{rcl} A_\theta : D(A,\theta+1,\infty) := \{x \in D(A) :\ Ax \in D(A,\theta,\infty)\} & \to & D(A,\theta,\infty) \\ x & \mapsto & Ax, \end{array}$$

is sectorial in $D(A,\theta,\infty)$*; moreover*

$$\|R(\lambda, A_\theta)\|_{L(D(A,\theta,\infty))} \leq \|R(\lambda, A)\|_{L(X)}. \tag{2.11}$$

Next theorem gives a useful characterization of the intermediate spaces $D(A,\theta,\infty)$ in the case of elliptic operators.

Theorem 2.9 *(see Theorem 3.1.30, page 108 of [14]) Let* $\mathcal{A}_0$ *be the operator defined in* (2.4)*. Then for every* $\theta \in (0, \frac{1}{2})$*,*

$$D(\mathcal{A}_0, \theta, \infty) = C^{0,2\theta}(\overline{\Omega}),$$

with equivalence of the respective norms. In particular there exists two constants C_1 *and* C_2 *depending only on the* $W^{1,\infty}$*-norm of* A *and* f *and on the constants* λ_0 *and* λ_1*, such that*

$$C_1\|g\|_{D(\mathcal{A}_0,\theta,\infty)} \leq \|g\|_{C^{0,2\theta}(\overline{\Omega})} \leq C_2\|g\|_{D(\mathcal{A}_0,\theta,\infty)}. \tag{2.12}$$

Let us recall now the definition of gradient flow for the homogeneous Mumford-Shah functional (1.4) via minimizing movements (see for instance [7] or [3]). Let Ω be a bounded open subset of $\mathbb{R}^n$ and consider an initial datum $u_0 \in L^\infty(\Omega)$. For fixed $\delta > 0$ (which is the time discretization parameter) we can define the δ*-approximate evolution* $u_\delta(\cdot) : [0,+\infty) \to SBV(\Omega)$ as the affine interpolation of the discrete function

$$\begin{array}{rcl} \delta\mathbb{N} & \to & SBV(\Omega) \\ \delta i & \mapsto & u_{\delta,i}, \end{array}$$

where $u_{\delta,i}$ is inductively defined as follows: $u_{\delta,0} = u_0$ and $u_{\delta,i}$ is a solution of

$$\min_{v \in SBV(\Omega)} \int_\Omega |\nabla v|^2\, dx + \mathcal{H}^{n-1}(S_v) + \frac{1}{\delta}\int_\Omega |v - u_{\delta,i-1}|^2\, dx.$$

The existence of a solution of the problem above is guaranteed by the Ambrosio theorem (see [4]). We call *minimizing movement* for F_0 with initial datum u_0, the set of all functions $v : [0,+\infty) \to SBV(\Omega)$ such that, for a suitable subsequence $\delta_n \downarrow 0$, $u_{\delta_n}(t) \to v(t)$ in $L^2(\Omega)$, for every $t > 0$.

3 Technical Estimates

3.1 Estimates in smooth domains

Given a hypersurface Γ of class $C^{2,\alpha}$ we can define

$$\Lambda^\alpha(\Gamma) := \sup_{x,y\in\Gamma} \frac{|\nabla_\tau \nu(x) - \nabla_\tau \nu(y)|}{|x-y|^\alpha}, \tag{3.1}$$

where ν is a smooth unit normal vector field to Γ and ∇_τ denotes the tangential gradient along Γ.

Lemma 3.1 *Let* Ω *be either* $\mathbb{R}^n$ *or* $\mathbb{R}^n_+$ *and* $\mathcal{A}_0$ *be the operator defined in* (2.4)*. Then for every* $\gamma \in (0, \frac{1}{2})$ *there exist two positive constants* K_0 *and* β_0*, depending only on the constants of ellipticity* λ_0*,* λ_1*, on* γ*, and on the* $W^{1,\infty}$*-norm of the matrix* A *and of the function* f*, such that for every* $\beta \geq \beta_0$ *and for every* $g \in C^{0,1-\gamma}(\overline{\Omega})$ *the solution* u *of* (2.6) *satisfies*

$$\beta^{\frac{1}{2}-\gamma}\|u - g\|_{C^{0,\gamma}(\overline{\Omega})} \leq K_0\|g\|_{C^{0,1-\gamma}(\overline{\Omega})}. \tag{3.2}$$

PROOF. Recall that $u-g=\mathcal{A}_0R(\beta,\mathcal{A}_0)g$: in order to obtain the thesis we have to estimate the quantity $\beta^{\frac{1}{2}-\gamma}\|\mathcal{A}_0R(\beta,\mathcal{A}_0)g\|_{C^{0,\gamma}(\overline{\Omega})}$. By Theorems 2.7 and 2.9, by (2.9) and (2.10), there exist $C_0>0$, $C_1>0$, and $\beta_0>0$, depending only on λ_0, λ_1, on γ, and on the $W^{1,\infty}$-norm of A and f, such that

$$\begin{aligned}\|\mathcal{A}_0R(\beta,\mathcal{A}_0)g\|_{C^{0,\gamma}(\overline{\Omega})} &\leq C_0\|\mathcal{A}_0R(\beta,\mathcal{A}_0)g\|_{D(\mathcal{A}_0,\frac{\gamma}{2},\infty)}\\ &= C_0\left(\|\mathcal{A}_0R(\beta,\mathcal{A}_0)g\|_\infty+\sup_{t\geq 2\beta_0\vee 1}t^{\frac{\gamma}{2}}\|\mathcal{A}_0R(t,\mathcal{A}_0)\mathcal{A}_0R(\beta,\mathcal{A}_0)g\|_\infty\right), \qquad (3.3)\end{aligned}$$

and

$$\sup_{2\beta_0\vee 1\leq t}t^{\frac{1-\gamma}{2}}\|\mathcal{A}_0R(t,\mathcal{A}_0)g\|_\infty\leq C_1\|g\|_{C^{0,1-\gamma}(\overline{\Omega})}. \tag{3.4}$$

We observe that (2.7) implies the existence of two positive constants β_0 and C_2, depending in turn on λ_0, λ_1 and on the $W^{1,\infty}$-norm of A and f, such that

$$\|\beta R(\beta,\mathcal{A}_0)\|_{L(L^\infty(\Omega))}\leq C_2, \tag{3.5}$$

for every $\beta\geq\beta_0$. Using (3.5) and (3.4), we can estimate

$$\begin{aligned}\sup_{2\beta_0\vee 1\leq\beta\leq t}\beta^{\frac{1}{2}-\gamma}t^{\frac{\gamma}{2}}\|\mathcal{A}_0R(t,\mathcal{A}_0)\mathcal{A}_0R(\beta,\mathcal{A}_0)g\|_\infty &= \sup_{2\beta_0\vee 1\leq\beta\leq t}\beta^{\frac{1}{2}-\gamma}t^{\frac{\gamma}{2}}\|\mathcal{A}_0R(\beta,\mathcal{A}_0)\mathcal{A}_0R(t,\mathcal{A}_0)g\|_\infty\\ &= \sup_{2\beta_0\vee 1\leq\beta\leq t}\left(\frac{\beta}{t}\right)^{\frac{1}{2}-\gamma}t^{\frac{1-\gamma}{2}}\|(\beta R(\beta,\mathcal{A}_0)-I)\mathcal{A}_0R(t,\mathcal{A}_0)g\|_\infty\\ &\leq (C_2+1)\sup_{2\beta_0\vee 1\leq t}t^{\frac{1-\gamma}{2}}\|\mathcal{A}_0R(t,\mathcal{A}_0)g\|_\infty\\ &\leq (C_2+1)C_1\|g\|_{C^{0,1-\gamma}}, \qquad (3.6)\end{aligned}$$

and analogously

$$\sup_{2\beta_0\vee 1\leq t\leq\beta}\beta^{\frac{1}{2}-\gamma}t^{\frac{\gamma}{2}}\|\mathcal{A}_0R(t,\mathcal{A}_0)\mathcal{A}_0R(\beta,\mathcal{A}_0)g\|_\infty\leq(C_2+1)C_1\|g\|_{C^{0,1-\gamma}}. \tag{3.7}$$

Combining (3.6), (3.7), (3.3), and using again (3.4), we finally obtain

$$\begin{aligned}\sup_{\beta\geq 2\beta_0\vee 1}\beta^{\frac{1}{2}-\gamma}\|\mathcal{A}_0R(\beta,\mathcal{A}_0)g\|_{C^{0,\gamma}(\overline{\Omega})} &\leq C_0\Bigg(\sup_{\beta\geq 2\beta_0\vee 1}\beta^{\frac{1}{2}-\gamma}\|\mathcal{A}_0R(\beta,\mathcal{A}_0)g\|_\infty\\ &\qquad+\sup_{\beta,t\geq 2\beta_0\vee 1}\beta^{\frac{1}{2}-\gamma}t^{\frac{\gamma}{2}}\|\mathcal{A}_0R(t,\mathcal{A}_0)\mathcal{A}_0R(\beta,\mathcal{A}_0)g\|_\infty\Bigg)\\ &\leq C_0(C_1+C_2+1)\|g\|_{C^{0,1-\gamma}}.\end{aligned}$$

□

The following theorem provides the preannounced estimate on the Hessian ∇^2u of the function u which solves (1.3); we recall that $(\partial\Omega')_R$ denotes the R-neighbourhood of $\partial\Omega'$.

Theorem 3.2 *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain of class $C^{1,1}$.*

i) For every $R>0$, we can find two positive constants $\beta_0=\beta_0(R)$ and $K=K(R)$ with the property that if Ω' is a domain belonging to $\mathcal{U}_R(\Omega)$, then for every $\beta\geq\beta_0$ and for every $g\in W^{1,\infty}\left(\Omega\setminus\overline{\Omega'}\right)$ the solution u of

$$\begin{cases}\Delta u=\beta(u-g) & \text{in } \Omega\setminus\overline{\Omega'},\\ \partial_\nu u=0 & \text{on } \partial\left(\Omega\setminus\overline{\Omega'}\right),\end{cases} \tag{3.8}$$

satisfies

$$\|\nabla u\|_\infty + \beta^{-\frac{1}{2}}\|\Delta u\|_\infty + \beta^{\frac{n}{2p}-1} \sup_{x_0\in\Omega\setminus\overline{\Omega'}} \|\nabla^2 u\|_{L^p\left(B(x_0,\frac{1}{\sqrt{\beta}})\cap\Omega\setminus\overline{\Omega'}\right)} \le K\|g\|_{W^{1,\infty}}. \tag{3.9}$$

A similar conclusion holds for the solution of

$$\begin{cases} \Delta u = \beta(u-g) & \text{in } \Omega', \\ \partial_\nu u = 0 & \text{on } \partial\Omega'. \end{cases} \tag{3.10}$$

ii) For every $R > 0$, for every $\overline{\Lambda} > 0$, and for every $\gamma \in (0,\alpha)$ (with $\alpha \in (0,1)$), there exist two positive constants $\beta_0 = \beta_0(R,\overline{\Lambda},\gamma)$ and $K = K(R,\overline{\Lambda},\gamma)$ with the property that if Ω' is a domain of class $C^{2,\alpha}$ belonging to $\mathcal{U}_R(\Omega)$, and $\Lambda^\alpha(\partial\Omega') \le \overline{\Lambda}$, then, for every $\beta \ge \beta_0$ and for every $g \in W^{1,\infty}\left(\Omega\setminus\overline{\Omega'}\right)$, the solution u of (3.8) satisfies

$$\|\nabla^2 u\|_{L^\infty\left((\partial\Omega')_R\cap(\Omega\setminus\overline{\Omega'})\right)} \le K\beta^{\frac{1}{2}+\gamma}\|g\|_{W^{1,\infty}}.$$

A similar conclusion holds for the solution of problem (3.10).

Proof. We will prove in details only *ii)*. Fix $p \in \partial\Omega'$. By Proposition 2.1 there exist two positive constants η and M_1, the former depending only on R while the latter also on $\Lambda^\alpha(\partial\Omega)$, such that the cylinder $C^\eta := \{x \in \mathbb{R}^{n-1} : |x| < \eta\}\times]-R,R[$ (expressed with respect to a coordinate system belonging to $\mathcal{S}^p_{\Omega'}$), intersected with Ω' is the subgraph of a function f belonging to $C^{2,\alpha}(S)$ $(S := C^\eta \cap \{x_n = 0\})$ and satisfying

$$\|f\|_{C^{2,\alpha}} \le M_1. \tag{3.11}$$

Let $\theta \in C^{2,\alpha}_0(C^\eta)$, $0 \le \theta \le 1$ and $\theta \equiv 1$ in $2^{-1}C^\eta$, such that

$$\partial_\nu\theta = 0 \text{ on } \partial\Omega'\cap C^\eta \qquad \text{and} \qquad \|\theta\|_{C^{2,\alpha}} \le M_2, \tag{3.12}$$

where M_2 depends only on R.

Set $v = \theta u$ and note that v solves

$$\begin{cases} \Delta v = \beta(v-h) & \text{in } \Omega'\cap C^\eta, \\ \partial_\nu v = 0 & \text{on } \partial(\Omega'\cap C^\eta), \end{cases}$$

where $h := \theta g + \beta^{-1}(\Delta\theta u + 2\nabla u\nabla\theta)$; finally, denoting by ψ the map

$$\begin{array}{rcl} C^\eta & \to & \psi(C^\eta) \\ (x_1,\dots,x_{n-1},x_n) & \mapsto & (x_1,\dots,x_{n-1},x_n - f(x_1,\dots,x_{n-1})), \end{array}$$

and setting $\tilde v := v\circ\psi^{-1}$ and $\tilde h := h\circ\psi^{-1}$, one sees that (recall that $\tilde v$ and $\tilde h$ have compact support in $\psi(C^\eta)$)

$$\begin{cases} \tilde f \mathrm{div}(\tilde A\nabla\tilde v) = \beta(\tilde v - \tilde h) & \text{in } \mathbb{R}^n_+, \\ \tilde A\nabla\tilde v\cdot\nu = 0 & \text{on } \partial(\mathbb{R}^n_+), \end{cases}$$

where $\tilde A$ and $\tilde f$ are $W^{1,\infty}$-extensions to $\mathbb{R}^n_+$ of the matrix-valued function $A := \left[\frac{D\psi(D\psi)^*}{|\det\psi|}\right]\circ\psi^{-1}$ and of the function $f := |\det\psi|\circ\psi^{-1}$ respectively , satisfying

$$\|\tilde A\|_{W^{1,\infty}(\mathbb{R}^n_+)} = \|A\|_{W^{1,\infty}(\psi(C^\eta))}, \qquad \|\tilde f\|_{W^{1,\infty}(\mathbb{R}^n_+)} = \|f\|_{W^{1,\infty}(\psi(C^\eta))}$$

and

$$\tilde{A}(x)\xi\cdot\xi \geq \frac{1}{2}|\xi|^2 \ \forall x \in \mathbb{R}^n_+,\ \forall \xi \in \mathbb{R}^n, \qquad \tilde{f}(x) \geq \frac{1}{2}\ \forall x \in \mathbb{R}^n_+$$

(since $A(0) = I$ and $f(0) = 1$, by (3.11), we can choose η depending only on R such that the property above holds true in $\psi(C^\eta)$).

The solution $\tilde{v}$ can be suitably decomposed as $\tilde{v} = \tilde{v}_1 + \tilde{v}_2 + \tilde{v}_3$ in the following way: set $h_1 = \theta g$, $h_2 = \beta^{-1}\nabla u \nabla\theta$, $h_3 := \beta^{-1}\Delta\theta u$, and $\tilde{h}_i = h_i \circ \psi^{-1}$ $(i = 1,2,3)$ and choose $\tilde{v}_i$ as the solution of

$$\begin{cases} \mathrm{div}(\tilde{A}\nabla\tilde{v}_i) = \beta(\tilde{v}_i - \tilde{h}_i) & \text{in } \mathbb{R}^n_+,\\ \tilde{A}\nabla\tilde{v}_i\cdot\nu = 0 & \text{on } \partial(\mathbb{R}^n_+),\end{cases}$$

for $i = 1,2,3$.

Applying Lemma 3.1 we have, for $i = 1,2,3$,

$$\beta^{\frac{1}{2}-\gamma}\|\tilde{v}_i - \tilde{h}_i\|_{C^{0,\gamma}} \leq K_0\|g\|_{C^{0,1-\gamma}}, \tag{3.13}$$

where K_0 is a constant depending only on γ and on the norm of $\tilde{A}$, therefore (by definition of A and by (3.11)) only on γ and R.

Estimate for $\tilde{v}_1$. ¿From (3.13), (3.11), (3.12), and the definition of $\tilde{h}_1$ we deduce

$$\beta^{\frac{1}{2}-\gamma}\|\tilde{v}_1 - \tilde{h}_1\|_{C^{0,\gamma}} \leq K_0K_1(\|g\|_{C^{0,1-\gamma}} + \beta^{-1}\|u\|_{C^{0,1-\gamma}}),$$

where K_1 depends only on R, and therefore, since by (2.11) and (2.12), we have

$$\|u\|_{C^{0,1-\gamma}} \leq K_2\|g\|_{C^{0,1-\gamma}},$$

we obtain

$$\beta^{\frac{1}{2}-\gamma}\|\tilde{v}_1 - \tilde{h}_1\|_{C^{0,\gamma}} \leq K_0K_1K_2\|g\|_{C^{0,1-\gamma}},$$

where K_2 depends only on R. Combining the above inequality with the well known Schauder estimate, we finally obtain

$$\|\nabla^2\tilde{v}_1\|_\infty \leq K_3\|\tilde{f}\mathrm{div}(\tilde{A}\nabla\tilde{v}_1)\|_{C^{0,\gamma}} = K_3\beta^{\frac{1}{2}+\gamma}\beta^{\frac{1}{2}-\gamma}\|\tilde{v}_1 - \tilde{h}_1\|_{C^{0,\gamma}} \leq K_3K_0K_1K_2\beta^{\frac{1}{2}+\gamma}\|g\|_{C^{0,1-\gamma}}, \tag{3.14}$$

where K_3 depends only on $C^{1,\gamma}$-norm of A and f and therefore only on R and $\overline{\Lambda}$.

Estimate for $\tilde{v}_2$. Arguing exactly as in the previous point, we obtain

$$\beta^{\frac{1}{2}-\gamma}\|\tilde{v}_2 - \tilde{h}_2\|_{C^{0,\gamma}} \leq K_0K_1\beta^{-1}\|\nabla u\|_{C^{0,1-\gamma}}. \tag{3.15}$$

By the Sobolev Embedding Theorem and by estimate (3.9) (with $p = \frac{n}{\gamma}$) we have, for $\beta \geq \beta_0$ and for every $x \in \Omega\setminus\overline{\Omega'}$,

$$[\nabla u]_{C^{0,1-\gamma}\left((\Omega\setminus\overline{\Omega'})\cap B\left(x,\beta^{-\frac{1}{2}}\right)\right)} \leq Q_0\|\nabla^2 u\|_{L^{\frac{n}{\gamma}}(\Omega\setminus\overline{\Omega'}\cap B(x,\beta^{-\frac{1}{2}}))} \leq Q_0Q_1\beta^{1-\frac{\gamma}{2}}\|g\|_{W^{1,\infty}}, \tag{3.16}$$

and

$$\|\nabla u\|_\infty \leq Q_1\|g\|_{W^{1,\infty}}, \tag{3.17}$$

where Q_0 is the constant of Sobolev Embedding and depends only on γ while Q_1 depends only on R. If $|x-y| \geq \beta^{-\frac{1}{2}}$, then, by (3.17), we infer

$$\frac{|\nabla u(x) - \nabla u(y)|}{|x-y|^{1-\gamma}} \leq \beta^{\frac{1-\gamma}{2}}2\|\nabla u\|_\infty \leq 2Q_1\beta^{\frac{1-\gamma}{2}}\|g\|_{W^{1,\infty}}. \tag{3.18}$$

Combining (3.16), (3.17), and (3.18), we get

$$\|\nabla u\|_{C^{0,1-\gamma}} \le Q_1(Q_0+1)\beta^{1-\frac{\gamma}{2}}\|g\|_{W^{1,\infty}},$$

and substitution in (3.15), together with Shauder Estimate, yields

$$\|\nabla^2\tilde v_2\|_\infty \le K_3\|\tilde f \mathrm{div}(\tilde A\nabla\tilde v_2)\|_{C^{0,\gamma}} = K_3\beta^{\frac12+\gamma}\beta^{\frac12-\gamma}\|\tilde v_2-\tilde h_2\|_{C^{0,\gamma}} \le K_3K_0K_1Q_1(Q_0+1)\beta^{\frac{1+\gamma}{2}}\|g\|_{W^{1,\infty}}. \tag{3.19}$$

Estimate for $\tilde v_3$. First we note that, by (2.11) and (2.12),

$$\|\tilde v_3\|_{C^{0,\gamma}} \le K_4\|\tilde h_3\|_{C^{0,\gamma}},$$

with K_4 depending only on R; so we can estimate

$$\begin{aligned}\|\tilde v_3-\tilde h_3\|_{C^{0,\gamma}} &\le \|\tilde v_3\|_{C^{0,\gamma}}+\|\tilde h_3\|_{C^{0,\gamma}}\\ &\le (K_4+1)\|\tilde h_3\|_{C^{0,\gamma}} \le \beta^{-1}(K_4+1)M\|u\|_{C^{0,\gamma}} \le \beta^{-1}(K_4+1)K_4M\|g\|_{W^{1,\infty}}.\end{aligned}$$

By Shauder Estimate we finally obtain,

$$\|\nabla^2\tilde v_3\|_\infty \le K_3(K_4+1)K_4\|g\|_{W^{1,\infty}}. \tag{3.20}$$

By (3.11) and again (3.9) we have

$$\begin{aligned}\|\nabla^2 u\|_{L^\infty(2^{-1}C^\eta)} &\le C\left(\|\nabla^2\tilde v\|_{L^\infty(\mathbb{R}^n_+)}+\|\tilde v\|_{W^{1,\infty}(\mathbb{R}^n_+)}\right)\\ &\le CC'\left(\|\nabla^2\tilde v_1\|_{L^\infty(\mathbb{R}^n_+)}+\|\nabla^2\tilde v_2\|_{L^\infty(\mathbb{R}^n_+)}+\|\nabla^2\tilde v_3\|_{L^\infty(\mathbb{R}^n_+)}+\|g\|_{W^{1,\infty}(\mathbb{R}^n_+)}\right),\end{aligned}$$

where C and C' depend only on R. Using (3.14), (3.19), and (3.20), we finally deduce for $\beta \ge \beta_0 \vee 1$

$$\|\nabla^2 u\|_{L^\infty(2^{-1}C^\eta)} \le CC'C''\left(\beta^{\frac12+\gamma}\|g\|_{W^{1,\infty}(\mathbb{R}^n_+)}+\|g\|_{W^{1,\infty}(\mathbb{R}^n_+)}\right) \le 2CC'C''\beta^{\frac12+\gamma}\|g\|_{W^{1,\infty}(\mathbb{R}^n_+)},$$

where C'' depends only on γ, R, and $\overline{\Lambda}$. Repeating all the above argument for every $p\in\partial\Omega'$ we get *ii)*.

The proof of statement *i)* can be done in a similar way: by localizing, straightening the boundary, and using Theorem 2.7. □

3.2 Estimates in domains with angles

In the following $\Omega\subset\mathbb{R}^2$ will denote a curvilinear polygon which means that $\partial\Omega$ is given by the union of a finite number of simple connected curves $\tau_1,\dots,\tau_k$ of class C^3 (up to their endpoints) meeting at corners with different angles $\alpha_j\in(0,\pi)$ $(j=1,\dots,k)$. Finally we will denote by $\mathcal{S}$ the set of the vertices, i.e. the set of the singular points of $\partial\Omega$.

Proposition 3.3 *Let* Ω *be as above. Then there exists* $\beta_0>0$ *and* $K>0$ *such that for every* $\beta>\beta_0$ *and for every* $g\in L^\infty(\Omega)$, *the solution* u *of*

$$\begin{cases}\Delta u=\beta(u-g) & \text{in } \Omega\\ \partial_\nu u=0 & \text{on } \partial\Omega,\end{cases} \tag{3.21}$$

satisfies

$$\|u\|_\infty+\beta^{-\frac12}\|\nabla u\|_\infty \le K\|g\|_\infty. \tag{3.22}$$

PROOF. The estimate is proved in [12] for the corresponding Dirichlet problem in a polygon, but one easily sees that the same proof actually works also in our case: indeed the change of boundary conditions does not affect the argument, and the main tool, which is a Calderon-Zygmund type inequality, proved in [13], is actually available also for curvilinear polygon, as shown, for example, in [19]. □

The following proposition is proved in [19].

Proposition 3.4 *Let Ω be as above. Then there exists $K>0$ such that for every $\beta>0$ $\beta>0$ and for every $g\in W^{1,\infty}(\Omega)$, the function u solution of (3.21), satisfies:*

$$\beta^{\frac{1}{2}}\|u-g\|_\infty \le K\|\nabla g\|_\infty. \tag{3.23}$$

Proposition 3.5 *Let Ω be as above. Then there exists a positive constant K such that for every $\beta\ge 1$ and for every $g\in W^{1,\infty}(\Omega)$, the solution u of (3.21) satisfies:*

$$\|\nabla u\|_\infty \le K\|g\|_{W^{1,\infty}}\beta^{\frac{1}{4}}. \tag{3.24}$$

PROOF. Fix $\beta\ge 1$; by Proposition 3.3 there exists $\lambda_0>0$ independent of β such that, setting $g_\lambda=\frac{\Delta u-\lambda u}{\lambda}$, for $\lambda\ge\lambda_0$ we have

$$\|\nabla u\|_\infty \le K\sqrt{\lambda}\|g_\lambda\|_\infty \le K\sqrt{\lambda}\left(\frac{\|\Delta u\|_\infty}{\lambda}+\|u\|_\infty\right) = K\left(\frac{\|\Delta u\|_\infty}{\sqrt{\lambda}}+\sqrt{\lambda}\|u\|_\infty\right). \tag{3.25}$$

Now set $\lambda_{\min}:=\frac{\|\Delta u\|_\infty}{\|u\|_\infty}$ and suppose that $\|\Delta u\|_\infty\ge\lambda_0\|g\|_\infty$. It follows that $\lambda_{\min}\ge\lambda_0$ (recall that $\|u\|_\infty\le\|g\|_\infty$): therefore, taking $\lambda=\lambda_{\min}$ in (3.25), we obtain

$$\|\nabla u\|_\infty \le 2K\|\Delta u\|_\infty^{\frac{1}{2}}\|u\|_\infty^{\frac{1}{2}}$$

and therefore, by Proposition 3.4,

$$\|\nabla u\|_\infty \le 2K\|g\|_\infty^{\frac{1}{2}}\left(K'\beta^{\frac{1}{2}}\|\nabla g\|_\infty\right)^{\frac{1}{2}} \le K''\|g\|_{W^{1,\infty}}\beta^{\frac{1}{4}},$$

where K'' is independent of β.

If $\|\Delta u\|_\infty<\lambda_0\|g\|_\infty$, then we simply use the Calderon-Zygmund type estimate proved in [19] (it is crucial here the hypothesis that all the angles are less than π) to get the existence of a constant $C>0$, depending only on Ω, such that

$$\|u\|_{W^{2,p}} \le C\|g\|_\infty \le C\|g\|_\infty\beta^{\frac{1}{4}}.$$

We conclude by applying the Sobolev Embedding Theorem. □

Proposition 3.6 *Let Ω and $\mathcal{S}$ be as above and let Γ be a simple connected curve in Ω joining two points x_1 and x_2 belonging to $\partial\Omega\setminus\mathcal{S}$. Suppose in addition that Γ is of class C^3 up to x_1 and x_2 (actually it would be enough to take Γ of class C^3 in two neighbourhoods U_1 and U_2 of x_1 and x_2 respectively, and of class $C^{2,\alpha}$, for some $\alpha>0$, outside those neighbourhood). Let us call Ω_1 and Ω_2 the two connected components of $\Omega\setminus\Gamma$. Finally set $\overline{d}:=\operatorname{dist}(x_1,\mathcal{S})\wedge\operatorname{dist}(x_2,\mathcal{S})$. Then for every $\delta<\overline{d}$ and $\gamma\in\left(0,\frac{1}{2}\right)$, there exist two positive constants β_0 and K depending on δ, γ, and Γ, such that, for every $\beta\ge\beta_0$ and for every $g\in W^{1,\infty}(\Omega_i)$ $(i=1,2)$, the solution u_i of*

$$\begin{cases}\Delta u_i=\beta(u_i-g) & \text{in } \Omega_i\\ \partial_\nu u_i=0 & \text{on } \partial\Omega_i,\end{cases} \tag{3.26}$$

satisfies

$$\|\nabla u_i\|_{L^\infty((\Gamma)_\delta\cap\Omega_i)}+\beta^{-\left(\frac{1}{2}+\gamma\right)}\|\nabla^2 u_i\|_{L^\infty((\Gamma)_\delta\cap\Omega_i)} \le K\|g\|_{W^{1,\infty}}. \tag{3.27}$$

PROOF. The estimate can be performed by a localization procedure as for Theorem 3.2 and in fact we have only to look at what happens in a neighbourhood of x_1 and x_2. We will look only at x_1 considered as a point of $\partial\Omega_1$, the other cases being analogous.

First of all, as in [19], we can find a neighbourhood $U = B(x_1, r) \cap \Omega_1$ of x_1, for a suitable $r \leq \delta$, and a diffeomorphism which transforms U into a right angle, more precisely we can construct a one-to-one map $\Phi = (\Phi_1, \Phi_2) : U \cap \Omega_1 \to \Phi(U \cap \Omega_1)$ of class $C^{1,1}$ such that $\nabla\Phi(0,0) = I$ and $\Phi(U) = \{w = (w_1, w_2) \in \mathbb{R}^2 : w_1 > 0, w_2 > 0\} \cap V$, where V is a neighbourhood of the origin; we can endow Φ with the further property that if v is a function defined in U with normal derivative vanishing on $\partial\Omega \cap \overline{U}$, then $v \circ \Phi^{-1}$ has normal derivative vanishing on $\Phi(\partial\Omega \cap \overline{U})$ and vice-versa. It follows, in particular, that $\Phi_2(x)$ has the following properties:

- $\Phi_2(x) = 0$ for every $x \in \Gamma \cap U$;
- $\partial_\nu \Phi_2 = 0$ on $\partial\Omega \cap \overline{U}$.

It is easy to see that we can choose a positive convex function f such that

$$f(0) = 0,\ f'(0) = 0, \text{ and } \Delta(f \circ \Phi_2) \geq 0 \text{ on } U' := B(x_1, r') \cap \Omega_1,$$

with $r' \leq r$, if needed. Thus we see that $f \circ \Phi_2$ is a subsolution of

$$\begin{cases} \Delta u = 0 & \text{in } U' \\ u = 0 & \text{on } \Gamma \cap \overline{U'} \\ \partial_\nu u = 0 & \text{on } \partial\Omega \cap \overline{U'} \\ u = f \circ \Phi_2 & \text{on } \partial U' \setminus (\partial\Omega \cup \Gamma) \end{cases}$$

and therefore $f \circ \Phi_2 \leq u$ in U'. By Theorem 5.1.3.1 of [13] (actually it is stated only for polygons, but it can be extended to curvilinear polygons, by the continuity method used, for example, in [19]) and the Sobolev Embedding Theorem, u is in $C^2(\overline{U''})$, where $U'' = B(x_1, r'') \cap \Omega$, with $r'' < r'$. Therefore, since $\nabla(f \circ \Phi_2)(x_1) \neq 0$, and so $\nabla u(x_1) \neq 0$, we can say that the map $\Psi := (v, u)$, where v is the harmonic anticonjugate of u, is conformal in a neighbourhood $U''' := B(x_1, r''') \cap \Omega_1$, with $r''' \leq r''$, it belongs to $C^2(\overline{U'''})$ and $\Psi(U''') = \{w = (w_1, w_2) \in \mathbb{R}^2 : w_1 > 0, w_2 > 0\} \cap V$, where V is a neighbourhood of the origin. Now take a cut-off function θ of class C^3 such that $\theta \equiv 1$ on $B(x_1, r'''/2) \cap \Omega_1$, $\theta(x) = 0$ for $|x| \geq (2/3)r'''$, and $\partial_\nu \theta = 0$ on $\partial\Omega \cup \Gamma \cap \overline{U'''}$; note that $v_1 := (\theta u_1) \circ \Psi^{-1}$ solves

$$\begin{cases} A(w)\Delta v_1 = \beta(v_1 - h) & \text{in } \Psi(U''') \\ \partial_\nu v_1 = 0 & \text{on } \{w_1 = 0\} \cup \{w_2 = 0\} \cap \overline{\Psi(U''')}, \end{cases}$$

where $h := [\theta g + \beta^{-1}(\Delta\theta u + 2\nabla u \nabla\theta)] \circ \Psi^{-1}$ and $A := |\nabla u|^2 \circ \Psi^{-1}$.

Moreover we have that $\partial_\nu A = 0$ on $\{w_1 = 0\} \cap \overline{\Psi(U''')}$, indeed, in view of the conformality of Ψ, this is equivalent to say that $\partial_\nu |\nabla u|^2 = 0$ on $\partial\Omega \cap \overline{U'''}$, which is true by the following computation

$$\partial_\nu |\nabla u|^2 = \partial_\nu (\partial_\tau u)^2 = 2\partial_\tau u \partial^2_{\nu\tau} u = 0,$$

where we used the fact that $u \in C^2(\overline{U'''})$ and $\partial_\nu u \equiv 0$ on $\partial\Omega \cap U'''$. As a consequence, the function

$$\tilde{A} := \begin{cases} A(w_1, w_2) & \text{if } w_1 > 0 \text{ and } (w_1, w_2) \in \Psi(U''') \\ A(-w_1, w_2) & \text{if } w_1 < 0 \text{ and } (-w_1, w_2) \in \Psi(U''') \end{cases}$$

turns out to be of class C^1 up to the boundary; in particular it can be extended to a function, still denoted by $\tilde{A}$, belonging to $C^1(\overline{\mathbb{R}^2_+}) \cap W^{1,\infty}(\mathbb{R}^2_+)$. Now it is easy to check that, denoting by $\tilde{v}_1$ and $\tilde{h}$ the extensions by reflection of v_1 and h respectively,

$$\begin{cases} \tilde{A}(w)\Delta \tilde{v}_1 = \beta(\tilde{v}_1 - \tilde{h}) & \text{in } \mathbb{R}^2_+ \\ \partial_\nu \tilde{v}_1 = 0 & \text{on } \{w_2 = 0\}; \end{cases}$$

at this point we are in a position to apply the regularity theorems stated in Subsection 3.1, obtaining the desired estimate for $\tilde{v}_1$. To complete the proof we can now proceed exactly as we did for Theorem 3.2. □

4 The calibration

4.1 The regular case

Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset of class $C^{1,1}$ and let $\Omega_1 \subset \Omega$ be an open set belonging to $\mathcal{U}_R(\Omega)$ (see (2.2)). We set $\Omega_2 := \Omega \setminus \overline{\Omega_1}$, $\Gamma := \partial\Omega_1$, and, for every $x \in \Gamma$, we denote the unit outer normal to $\partial\Omega_1$ at x by $\nu(x)$.

Lemma 4.1 *There exist two positive constants c and β_0, depending only on R, such that, for every $\beta \geq \beta_0$, we can find two functions $z_{1,\beta} : \Omega_1 \to \mathbb{R}$ and $z_{2,\beta} : \Omega_2 \to \mathbb{R}$ of class $W^{2,\infty}$ with the following properties:*

i) $\frac{1}{2} \leq z_{i,\beta} \leq 1$ *in Ω_i, for $i = 1, 2$ and $z_{2,\beta} \equiv \frac{1}{2}$ in a neighbourhood of $\partial\Omega$;*

ii) $\Delta z_{i,\beta} \leq c\beta z_{i,\beta}$ *in Ω_i, for $= 1, 2$;*

iii) $z_{1,\beta}(x) = z_{2,\beta}(x) = 1$ *and $\partial_\nu z_{1,\beta}(x) = -\partial_\nu z_{2,\beta}(x) \geq \sqrt{\beta}$ for every $x \in \Gamma$;*

iv) $\|\nabla z_{i,\beta}\|_\infty \leq c\sqrt{\beta}$ *and* $\|\nabla^2 z_{i,\beta}\|_\infty \leq c\beta$.

Proof. Let us denote the signed distance function from Ω_1 by d and let π the projection on Γ which, by Lemma 2.5, is well defined in $(\Gamma)_{\frac{R}{2}}$; we begin by constructing $z_{2,\beta}$. Let $w_\beta : [0, +\infty) \to (0, +\infty)$ be the solution of the following problem

$$\begin{cases} w_\beta'' = 16\beta w_\beta, \\ w_\beta(0) = 1/2, \\ w_\beta'(R/2) = 0, \end{cases}$$

which can be explicitly computed and it is given by

$$w_\beta(t) = \frac{1}{2}\frac{\mathrm{e}^{-4\sqrt{\beta}\frac{R}{2}}}{\mathrm{e}^{4\sqrt{\beta}\frac{R}{2}} + \mathrm{e}^{-4\sqrt{\beta}\frac{R}{2}}}\mathrm{e}^{4\sqrt{\beta}t} + \frac{1}{2}\frac{\mathrm{e}^{4\sqrt{\beta}\frac{R}{2}}}{\mathrm{e}^{4\sqrt{\beta}\frac{R}{2}} + \mathrm{e}^{-4\sqrt{\beta}\frac{R}{2}}}\mathrm{e}^{-4\sqrt{\beta}t}, \tag{4.1}$$

and let $\theta : [0, +\infty) \to [0, 1]$ be a C^∞ function such that

$$\theta \equiv 1 \quad \text{in } [0, R/4] \qquad \theta \equiv 0 \quad \text{in } [R/2, +\infty) \qquad \text{and} \qquad \|\theta\|_{C^2} \leq c_0, \tag{4.2}$$

with c_0 depending only on R. We are now ready to define $z_{2,\beta} : \Omega_2 \to \mathbb{R}$ as

$$z_{2,\beta}(x) := \begin{cases} \theta(d(x))((w_\beta(d(x)) + 1/2) + (1 - \theta(d(x)))1/2 & \text{if } 0 < d(x) \leq R/2, \\ 1/2 & \text{otherwise in } \Omega_2. \end{cases}$$

First of all note that, as it is a convex combination of two functions with range contained in $[1/2, 1]$, $z_{2,\beta}$ itself has range in $[1/2, 1]$. Using the expression in (4.1) it is easy to see that there exist $\beta_0 > 1$ and $c_1 > 1$ depending only on R such that

$$w_\beta'(0) \leq -\sqrt{\beta} \qquad |w_\beta'| \leq c_1\sqrt{\beta} \quad \text{in } [0, R/2] \qquad \text{and} \qquad |w_\beta''| \leq c_1\beta \quad \text{in } [0, R/2], \tag{4.3}$$

for every $\beta \geq \beta_0$. From the first inequality we obtain immediately *iii)* for $z_{2,\beta}$. Moreover, by (4.2) and (4.3), we can estimate

$$\begin{aligned} |\nabla z_{2,\beta}| &= |\theta(d)w_\beta'(d)\nabla d + \theta'\nabla d\, w_\beta(d)| \\ &\leq |w_\beta'| + |\theta'| \leq c_1\sqrt{\beta} + c_0 \leq c\sqrt{\beta}, \end{aligned}$$

with c depending only on R. Finally, using again (4.2),(4.3), and Lemma 2.5, we have

$$\begin{aligned}|\nabla^2 z_{2,\beta}| &\leq |w'_\beta||\nabla^2 d| + |w'_\beta||\theta'| + |w''_\beta| + |\theta''| + |\theta'||\nabla^2 d| + |\theta'||w'_\beta| \\ &\leq c_1 c_2\sqrt{\beta} + c_0 c_1\sqrt{\beta} + c_1\beta + c_0 + c_0 c_2 + c_0 c_1\sqrt{\beta},\end{aligned}$$

where all the constants depend only on R so that we can state the existence of $c > 0$, still depending only on R, such that

$$|\nabla^2 z_{2,\beta}| \leq c\beta \qquad \forall \beta \geq \beta_0.$$

To conclude, we define $z_{1,\beta} : \Omega_1 \to \mathbb{R}$ as follows:

$$z_{1,\beta}(x) := \begin{cases}\theta(-d(x))((w_\beta(-d(x)) + 1/2) + (1 - \theta(-d(x)))1/2 & \text{if } 0 > d(x) \geq -R/2, \\ 1/2 & \text{otherwise in } \Omega_1.\end{cases}$$ □

Theorem 4.2 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of class $C^{1,1}$ and let $\Omega_1 \subset \Omega$ be an open set of class $C^{2,\alpha}$ for some $\alpha \in (0,1)$ and compactly contained in Ω. Let $R > 0$ such that $\Omega_1 \in \mathcal{U}_R(\Omega)$ (see (2.2) and Remark 2.2)and set $\Gamma := \partial\Omega_1$. Then for every function g belonging $W^{1,\infty}(\Omega \setminus \Gamma)$, discontinuous along Γ (i.e., $S_g = \Gamma$) and such that $g^+(x) - g^-(x) > S > 0$ for every $x \in \Gamma$, there exists $\beta_0 > 0$ depending on R, S, $\Lambda^\alpha(\Gamma)$ (see (3.1)), and $\|g\|_{W^{1,\infty}}$, such that for $\beta \geq \beta_0$ the solution u_β of*

$$\begin{cases}\Delta u_\beta = \beta(u_\beta - g) & \text{in } \Omega \setminus \Gamma, \\ \partial_\nu u_\beta = 0 & \text{on } \partial\Omega \cup \Gamma,\end{cases} \tag{4.4}$$

is discontinuous along Γ ($S_{u_\beta} = \Gamma$) and it is the unique absolute minimizer of $F_{\beta,g}$ over $SBV(\Omega)$.

PROOF. In the sequel we will denote the signed distance from Ω_1 by d and the projection on Γ by π: by Lemma 2.5, the two functions are well defined in $(\Gamma)_{R/2}$. Moreover, in that neighbourhood, d and π are at least of class $W^{2,\infty}$ and $W^{1,\infty}$ respectively.

As announced in the Introduction, the proof will be performed by constructing a calibration ϕ; adopting the notation introduced there, the vector field ϕ will be written as

$$\phi(x,z) = (\phi^x(x,z), \phi^z(x,z)),$$

where $\phi^x(x,z)$ is a n-dimensional "horizontal" component, while ϕ^z is the (one dimensional) "vertical" component.

- *Preparation.*

Without loss of generality we can suppose that g^+ coincides with the trace on Γ of g from Ω_1, while g^- is trace from Ω_2. First of all let us choose β', depending only on R, S, and $\|g\|_{W^{1,\infty}}$ and G depending on R, such that, for $\beta \geq \beta'$,

$$\|u_\beta - g\|_\infty \leq \frac{S}{16} \quad \text{and} \quad \sqrt{\beta}\|u_\beta - g\|_\infty \leq G\|g\|_{W^{1,\infty}} \quad i = 1,2: \tag{4.5}$$

this is possible by virtue of Theorem 3.2.

As a second step, it is convenient to extend the restriction of u_β to Ω_i $(i = 1,2)$ to a $C^{1,1}$ function $u_{i,\beta}$ defined in the whole Ω, in such a way that

$$u_{i,\beta}(x) = u_\beta(x) \text{ for } x \in \Omega_i, \quad \|u_{i,\beta}\|_{W^{2,\infty}} \leq c\|u_\beta\|_{W^{2,\infty}}, \quad \text{and} \quad u_{1,\beta} - u_{2,\beta} \geq \frac{3}{4}S \quad \text{for every } x \in \Omega, \tag{4.6}$$

where c is a positive constant depending only on R: this operation can be performed in many ways, for example, to construct $u_{2,\beta}$ we can extend the resctriction of u_β to Ω_2 in a neighbourhood of Γ by a standard localization procedure and then we can make a convex combination through a cut-off function with $u_\beta - (3/4)S$ (recall that by definition of S and by (4.5), we have $u_\beta^+ - u_\beta^- > (3/4)S$ on Γ); it is clear that all can be done in such a way that the constant c depends only on the "$C^{1,1}$-norm" of Γ and therefore only on R. We require also that

$$\partial_\nu u_{1,\beta} = 0 \qquad \text{on } \partial\Omega .$$

By (3.9) and (4.6), we can state the existence of two positive constants K and β'' depending only on R such that

$$\|\nabla u_{i,\beta}\|_\infty \le K\|g\|_{W^{1,\infty}} \qquad i = 1, 2, \tag{4.7}$$

for every $\beta \ge \beta''$.

Let $\beta''' > 0$ satisfying

$$\frac{1}{6}\sqrt{\beta'''} = \max\left\{4(K\|g\|_{W^{1,\infty}})^2, 64/S^2, \beta', \beta'', \beta_0\right\} + 1, \tag{4.8}$$

where β_0 is the constant appearing in Lemma 4.1. Let $z_{1,\beta'''}$ and $z_{2,\beta'''}$ be the two functions constructed in Lemma 4.1 with $\lambda = \beta'''$ and define v_1, v_2 as follows

$$v_1(x) = \begin{cases} z_{1,\beta'''}(x) & \text{if } x \in \overline{\Omega_1} \\ 2 - z_{2,\beta'''}(x) & \text{if } x \in \Omega_2 \end{cases}$$

and

$$v_2(x) = \begin{cases} z_{2,\beta'''}(x) & \text{if } x \in \overline{\Omega_2} \\ 2 - z_{1,\beta'''}(x) & \text{if } x \in \Omega_1 . \end{cases}$$

From the properties of $z_{i,\beta}$ $(i = 1, 2)$, as stated in Lemma 4.1, it follows immediately that $v_i \in W^{2,\infty}(\Omega)$ and

$$\|\nabla v_i\|_\infty \le K_1\sqrt{\beta'''}, \qquad \|\nabla^2 v_i\|_\infty \le K_1\beta''' \quad i = 1, 2 \tag{4.9}$$

where K_1 is a positive constant depending only on R. Note that $\nabla v_1(x) = -\nabla v_2(x)$ for every $x \in \Omega$. We remark also that, for $x \in \Gamma$, by construction,

$$\frac{\nabla v_1(x)}{|\nabla v_1(x)|} = -\frac{\nabla v_2(x)}{|\nabla v_2(x)|} = \nu(x), \tag{4.10}$$

where $\nu(x)$ denotes the unit normal vector at x to Γ (outer with respect to Ω_1). We set

$$\tilde{h}_(x) = \frac{1}{\sqrt{2}}|\nabla v_1|^{-\frac{1}{2}} = \frac{1}{\sqrt{2}}|\nabla v_2|^{-\frac{1}{2}} \tag{4.11}$$

for every $x \in \Gamma$. Moreover, using (4.9) and *iii)* of Lemma 4.1, we can find a positive constant $D \le R/2$, depending only on R, S, and $\|g\|_{W^{1,\infty}}$, such that

$$|\nabla v_i(x)| \ge \frac{1}{2}, \quad \tilde{h}^2(\pi(x))\frac{|\nabla v_i(x)|}{v_i(x)} < 1 - \frac{25}{32}\frac{1}{\sqrt{3}} \qquad \text{if } |d(x)| \le D,\, i = 1, 2. \tag{4.12}$$

Applying *iii)* of Lemma 4.1, we get

$$|\nabla v_i(x)|^{\frac{1}{2}} \ge \sqrt[4]{\beta'''} \ge \max\{8/S, 1\} \qquad i = 1, 2, \tag{4.13}$$

where the last inequality follows directly from (4.8).

Moreover, combining Lemma 4.1, (4.7), and (4.8), we deduce

$$4|\nabla u_{i,\beta}(x)|^2-\frac{1}{6}|\nabla v_i(x)|\le 4(K\|g\|_{W^{1,\infty}})^2-\frac{1}{6}\sqrt{\beta'''}\le -1 \tag{4.14}$$

and analogously

$$\frac{1}{\sqrt{2}}|\nabla v_i(x)|^{-\frac{1}{2}}\|\nabla u_{i,\beta}\|_\infty<\frac{1}{4\sqrt{3}}\qquad i=1,2, \tag{4.15}$$

for every $x\in\Gamma$ and for every $\beta\ge\beta'''$.

Let $\varepsilon\in(0,1)$ be such that

$$6\varepsilon\|\nabla u_{i,\beta}\|_\infty+4\varepsilon^2\|\nabla v_i\|_\infty\le\frac{1}{4}\quad\text{for } i=1,2 \text{ and } \beta\ge\beta'''; \tag{4.16}$$

by (4.7) and (4.9) (and the definition of β''') we see that ε can be chosen depending only R, S and $\|g\|_{W^{1,\infty}}$. By (4.11), it follows, for every $x\in\Gamma$,

$$4(\tilde h)^2\|\nabla v_i\|_\infty\ge 4(\tilde h)^2(-1)^{i+1}\partial_\nu v_i=4\cdot\frac{1}{2}>\frac{1}{4},$$

therefore, by (4.16),

$$\varepsilon<\tilde h(x)\quad\forall x\in\Gamma. \tag{4.17}$$

Let γ be a fixed constant belonging to $(0,\frac{1}{2}\wedge\alpha)$: by applying *ii)* of Theorem 3.2, we can find two positive constants β^{IV} and K_2 depending only on R and $\Lambda^\alpha(\Gamma)$ (and γ) such that

$$\|\nabla^2u_\beta\|_{L^\infty\left((\Gamma)_{\frac{R}{2}}\right)}\le K_2\beta^{\frac{1}{2}+\gamma}\|g\|_{W^{1,\infty}}, \tag{4.18}$$

for every $\beta\ge\beta^{\text{IV}}$.

We can define, for $\beta>0$,

$$h_\beta(x)=\begin{cases}\left(\tilde h(\pi(x))-\beta^{\frac{1}{2}+\gamma_1}|d(x)|\right)\vee\varepsilon & \text{if } |d(x)|\le D\\ \varepsilon & \text{if } |d(x)|>D,\end{cases}$$

where γ_1 is a fixed constant belonging to $(\gamma,\frac{1}{2})$. It is easy to see that there exists $\beta^{\text{V}}>0$ depending on D (and therefore only on R, S, and $\|g\|_{W^{1,\infty}}$) such that h_β is continuous (in fact Lipschitz) for $\beta>\beta^{\text{V}}$.

Using (4.13), (4.11), (4.9), and Lemma 2.5, we have

$$\|\nabla h_\beta\|_\infty\le C'\left(\frac{1}{\sqrt{2}}\left(\frac{S}{8}+1\right)^4\|\nabla^2v_i\|_\infty\|\nabla\pi\|_\infty+\beta^{\frac{1}{2}+\gamma_1}\|\nabla d\|_\infty\right)\le K_3\beta^{\frac{1}{2}+\gamma_1}, \tag{4.19}$$

where K_3 is a positive constant depending on R, S, and $\|g\|_{W^{1,\infty}}$.

Finally we set

$$\beta_1=\max\{\beta'',\beta''',\beta^{\text{IV}},\beta^{\text{V}},1\} \tag{4.20}$$

and

$$\mu_i(x)=\frac{\Delta v_i(x)}{v_i(x)}; \tag{4.21}$$

notice that by (4.9) we get

$$\mu_i(x)\le\frac{K_1\beta'''}{v_i(x)}\le 2K_1\beta_1\qquad\text{for every } x\in\Omega\ . \tag{4.22}$$

- *Definition of the calibration.*

From now on we will assume $\beta \geq \beta_1$. Let us consider the following sets

$$A_i := \{(x,z) \in \Omega \times \mathbb{R} :\ u_{i,\beta}(x) - h_\beta(x) \leq z \leq u_{i,\beta}(x) + h_\beta(x)\}, \quad i = 1,2. \tag{4.23}$$

Since, by (4.6), $u_{1,\beta}(x) - u_{2,\beta}(x) \geq \frac{3}{4}S$ everywhere, noting that $h_\beta \leq S/8$ everywhere (by (4.13) and (4.11)), we see that

$$\operatorname{dist}(A_1, A_2) \geq \frac{S}{2} \qquad \text{for } \beta \geq \beta_1 .$$

The crucial point is in constructing the vector field around the graph of u_β, i.e. in $A_i \cap (\Omega_i \times \mathbb{R})$: here we have to provide a divergence free vector field satisfying condition (d) of Section 1 and such that

$$\begin{aligned} &\phi^x(x,z) \cdot \nu_{u_\beta} \geq 0 && \text{for } x \in \Gamma \text{ and } u_{2,\beta} < z < u_{1,\beta}, \\ &\phi^x(x,z) \cdot \nu_{u_\beta} \leq 0 && \text{for } x \in \Gamma,\ z < u_{2,\beta} \text{ or } z > u_{1,\beta}. \end{aligned}$$

These properties are crucial in order to obtain (e) and (f) simultaneously.

The remaining work is a matter of finding a suitable extension which preserves all the properties of calibrations.

We start by giving the global definition of the horizontal component ϕ^x:

$$\phi^x(x,z) := \begin{cases} 2\nabla u_{i,\beta} - 2\frac{u_{i,\beta}-z}{v_i}\nabla v_i - \frac{16}{h_\beta}\left((-1)^i(z-u_{i,\beta}) - \frac{h_\beta}{2}\right)^+ \nabla u_{i,\beta} & \text{if } (x,z) \in A_i,\ i = 1,2, \\ 0 & \text{otherwise in } \Omega \times \mathbb{R}. \end{cases} \tag{4.24}$$

Concerning ϕ^z, we begin by defining it in $A_i \cap \left(\overline{\Omega}_i \times \mathbb{R}\right)$:

$$\phi^z_i(x,z) := \left|\nabla u_\beta - \frac{u_\beta - z}{v^i}\nabla v^i\right|^2 - \beta(z-g)^2 + (\beta - \mu_i)(u_\beta - z)^2 + \Psi_i(x,z) \quad \forall (x,z) \in A_i \cap \left(\overline{\Omega}_i \times \mathbb{R}\right), \tag{4.25}$$

where μ_i is the function defined in (4.21) and

$$\Psi_i(x,z) := \int_{u_{i,\beta}}^{z} \operatorname{div}_x \left[\frac{16}{h_\beta}\left((-1)^i(z-u_{i,\beta}) - \frac{h_\beta}{2}\right)^+ \nabla u_{i,\beta}\right] dt.$$

Let us clarify that in the formulas above $(\cdot)^+$ stands for $(\cdot) \vee 0$.

For $x \in \Omega_i$ and $-h_\beta < (-1)^i(z - u_\beta) < \frac{h_\beta}{2}$, the field ϕ reduces to

$$\left(2\nabla u_\beta - 2\frac{u_\beta - z}{v_i}\nabla v_i,\ \left|\nabla u_\beta - \frac{u_\beta - z}{v_i}\nabla v_i\right|^2 - \beta(z-g)^2 + (\beta-\mu_i)(u_\beta - z)^2\right) \tag{4.26}$$

and so, by some easy computation and using the definition of u_β and μ_i, we have

$$\begin{aligned} \operatorname{div}\phi(x,z) &= 2\left(\Delta u_\beta - \frac{u_\beta - z}{v_i}\Delta v_i\right) - 2\beta(z-g) - 2(\beta-\mu_i)(u_\beta - z) \\ &= 2\beta(u_\beta - g) - 2\mu_i(u_\beta - z) - 2\beta(z-g) - 2(\beta - \mu_i)(u_\beta - z) = 0. \end{aligned}$$

For $x \in \Omega_i$ and $\frac{h_\beta}{2} < (-1)^i(z-u_\beta) < h_\beta$, ϕ is the sum of the field in (4.26) and

$$\left(-\frac{16}{h_\beta}\left((-1)^i(z-u_\beta) - \frac{h_\beta}{2}\right)^+ \nabla u_\beta,\ \Psi_i(x,z)\right),$$

which is clearly divergence free by the definition of Ψ_i. Eventually we have,

$$\mathrm{div}\phi = 0 \quad \text{in } (\Omega_i \times \mathbb{R}) \cap A_i. \tag{4.27}$$

It is time now to extend the definition of ϕ^z. Before writing the explicit expression, we remark that conditions (a) and (b) of Section 1 imply that such extension is essentially unique. More precisely, if $(U_j)_{j=1,\dots,10}$ is the family of all connected components of $(\Omega \times \mathbb{R}) \setminus (\partial A_1 \cup \partial A_2 \cup (\gamma \times \mathbb{R}))$, it easy to see that ϕ^z is uniquely determined on $(\Omega \setminus \Gamma) \times \mathbb{R} = \cup_{j=1}^{10} U_j$ by (4.24), (4.25), and the two following necessary conditions:

- $\partial_z \phi^z = -\mathrm{div}_x \phi^x$ in U_j for $j = 1, \dots, 10$ (which ensures condition (a) of Section 1),
- $\phi^+ \cdot \nu_{\partial U_j} = \phi^- \cdot \nu_{\partial U_j}$ on ∂U_j for every $j = 1, \dots, 10$, where ϕ^+ and ϕ^- denote the traces of ϕ on the two sides of ∂U_j.

The only freedom is in the choice of ϕ^z on ∂U_j according to the condition

$$\phi \cdot \nu_{\partial U_j} = \phi^+ \cdot \nu_{\partial U_j} = \phi^- \cdot \nu_{\partial U_j}.$$

We are now ready to give the complete the definition of ϕ^z; for $(x,z) \in (\Omega_1 \times \mathbb{R}) \setminus A_1$ we define $\phi^z(x,z)$ as follows:

$$\begin{cases} \phi^x(x, u_\beta + h_\beta) \cdot (-\nabla u_\beta - \nabla h_\beta) + \phi^z(x, u_\beta + h_\beta) & \text{if } z > u_\beta + h_\beta, \\ \phi^x(x, u_\beta - h_\beta) \cdot (-\nabla u_\beta + \nabla h_\beta) + \phi^z(x, u_\beta - h_\beta) & \text{if } u_\beta - h_\beta > z \geq u_{2,\beta} + h_\beta, \\ \chi_1(x,z) + \phi^z(x, u_{2,\beta} + h_\beta) + \phi^x(x, u_{2,\beta} + h_\beta) \cdot (\nabla u_{2,\beta} + \nabla h_\beta) & \text{if } u_{2,\beta} + h_\beta > z \geq u_{2,\beta} - h_\beta, \\ \phi^x(x, u_{2,\beta} - h_\beta) \cdot (-\nabla u_{2,\beta} + \nabla h_\beta) + \phi^z(x, u_{2,\beta} - h_\beta) & \text{if } u_{2,\beta} - h_\beta > z, \end{cases} \tag{4.28}$$

where

$$\chi_1(x,z) = \int_z^{u_{2,\beta}+h_\beta} \mathrm{div}_x \phi^x(x,t)\, dt.$$

We remark that in first and in the second line we used the definition of ϕ^z already given in (4.25), in the third line we used the definition of $\phi^z(x, u_{2,\beta} + h_\beta)$ given in the second one, and finally in the last line we exploited the definition $\phi^z(x, u_{2,\beta} - h_\beta)$ given in the previous one.

Analogously, for $(x,z) \in (\Omega_2 \times \mathbb{R}) \setminus A_2$ we define $\phi^z(x,z)$ as follows:

$$\begin{cases} \phi^x(x, u_\beta - h_\beta) \cdot (-\nabla u_\beta + \nabla h_\beta) + \phi^z(x, u_\beta - h_\beta) & \text{if } z < u_\beta - h_\beta, \\ \phi^x(x, u_\beta + h_\beta) \cdot (-\nabla u_\beta - \nabla h_\beta) + \phi^z(x, u_\beta + h_\beta) & \text{if } u_\beta + h_\beta < z \leq u_{1,\beta} - h_\beta, \\ \chi_2(x,z) + \phi^z(x, u_{1,\beta} - h_\beta) + \phi^x(x, u_{1,\beta} - h_\beta) \cdot (\nabla u_{1,\beta} - \nabla h_\beta) & \text{if } u_{1,\beta} - h_\beta < z \leq u_{1,\beta} + h_\beta, \\ \phi^x(x, u_{1,\beta} + h_\beta) \cdot (-\nabla u_{1,\beta} - \nabla h_\beta) + \phi^z(x, u_{1,\beta} + h_\beta) & \text{if } u_{1,\beta} + h_\beta < z, \end{cases} \tag{4.29}$$

where

$$\chi_2(x,z) = \int_z^{u_{1,\beta}-h_\beta} \mathrm{div}_x \phi^x(x,t)\, dt.$$

Finally we set

$$\phi^z(x,z) = 0 \qquad \text{on } (\Gamma \cap \mathbb{R}) \setminus (A_1 \cup A_2);$$

this concludes the definition of ϕ which, by construction (and recalling (4.27)), satisfies conditions (a) and (b) of Section 1.

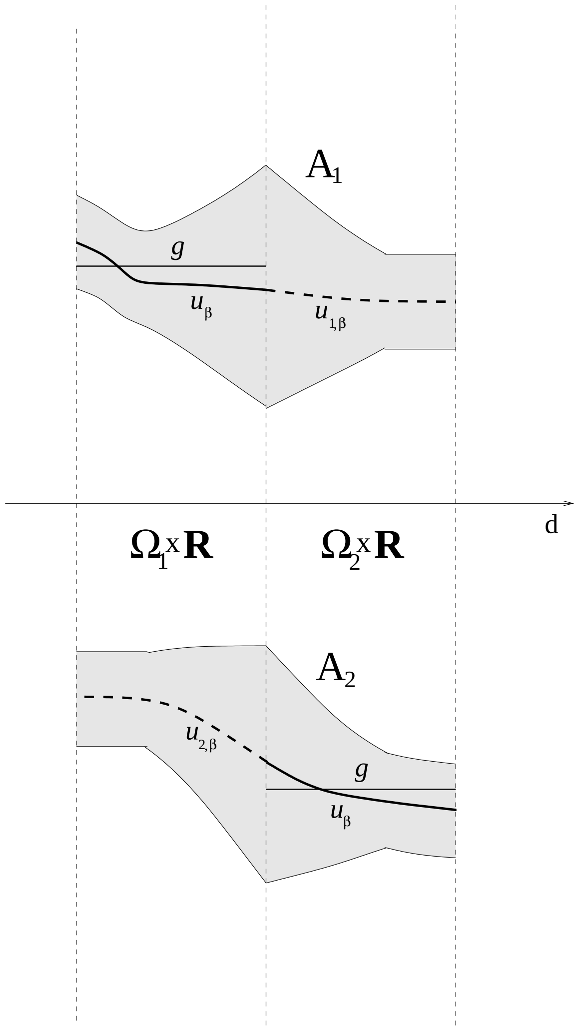

Figure 1: A cross section of the sets A_1 and A_2: the vector field ϕ is purely vertical out of the shaded regions.

- $\phi^z + \beta(z-g)^2 > |\phi^x|^2/4$ *for almost every* $(x,z) \in \Omega \times \mathbb{R}$ *with* $z \neq u(x)$.

We first prove the condition above in $A_i \cap (\Omega_i \times \mathbb{R})$, and then in the remaining. For $x \in \Omega_i$ and $-h_\beta \leq (-1)^i(z-u_\beta) \leq \frac{h_\beta}{2}$, by (4.26), we have that

$$\phi^z + \beta(z-g)^2 = \frac{|\phi^x|^2}{4} + (\beta - \mu_i)(u_\beta - z)^2 > \frac{|\phi^x|^2}{4},$$

so condition (c) of Secton 1 is trivially satisfied, with strict inequality.

For $x \in \Omega_i$ and $\frac{h_\beta}{2} < (-1)^i(z-u_\beta) \leq h_\beta$, using the definition of ϕ we see that (c) is equivalent to

$$(1) := (\beta-\mu_i)(u_\beta - z)^2 + \Psi_i(x,z) > \frac{(16)^2}{4(h_\beta)^2}|\nabla u_\beta|^2([\cdots])^2 - \frac{16}{h_\beta}[\cdots]\left(\nabla u_\beta - \frac{u_\beta - z}{v_i}\nabla v_i\right)\nabla u_\beta := (2), \tag{4.30}$$

where we wrote $[\cdots]$ instead of $\left[((-1)^i(z-u_\beta)-\frac{h_\beta}{2})^+\right]$; by (4.17), (4.5), (4.7), and (4.19), we have

$$\begin{aligned}\Psi_i(x,z) &\geq \int_{u_\beta}^{z}\left(-\frac{16}{h_\beta}[\cdots]|\Delta u_\beta|-|\nabla u_\beta|\cdot\left|\nabla\left(\frac{16}{h_\beta}[\cdots]\right)\right|\right)\,dt\\ &\geq -\frac{16}{\varepsilon}S^2|\beta(u_\beta-g)|-S|\nabla u_\beta|\left(\frac{16}{\varepsilon}(|\nabla u_\beta|+|\nabla h_\beta|)+\frac{16S}{\varepsilon^2}|\nabla h_\beta|\right)\\ &\geq -\frac{16}{\varepsilon}S^2\sqrt{\beta}G\|g\|_{W^{1,\infty}}-SK\|g\|_{W^{1,\infty}}\left(\frac{16}{\varepsilon}(K\|g\|_{W^{1,\infty}}+K_3\beta^{\frac{1}{2}+\gamma_1})+\frac{16S}{\varepsilon^2}K_3\beta^{\frac{1}{2}+\gamma_1}\right),\end{aligned}$$

therefore, recalling that the all the constants appearing in the last expression depend only on R, S, and $\|g\|_{W^{1,\infty}}$, there exists a positive constant C depending on the same quantities such that

$$\Psi_i(x,z)\geq -C\beta^{\frac{1}{2}+\gamma_1}; \tag{4.31}$$

recalling that $|u_\beta-z|\geq\frac{h_\beta}{2}\geq\frac{\varepsilon}{2}$ we finally obtain

$$(1)\geq(\beta-\mu_i)\frac{\varepsilon^2}{4}-C\beta^{\frac{1}{2}+\gamma_1}\qquad\text{for } \beta \text{ large enough .} \tag{4.32}$$

Analogously exploiting (4.16), (4.17), (4.7), and (4.9), we discover that

$$(2)\leq C_1, \tag{4.33}$$

where C_1 depends on R, S, and $\|g\|_{W^{1,\infty}}$; combining (4.32), (4.33), and recalling (4.22), we finally obtain that there exists $b_0>\beta_1$ depending only on R, S, and $\|g\|_{W^{1,\infty}}$ such that (4.30) holds true for $\beta\geq b_0$.

Before proceeding let us observe that arguing as for estimate (4.33), we easily obtain

$$|\phi_i^x(x,z)|\leq C_2(\|\nabla u_{i,\beta}\|_\infty+\|\nabla v_i\|_\infty)\leq C_3\qquad\text{for every } (x,z)\in A_i, \tag{4.34}$$

where C_3 depends only on R, S, and $\|g\|_{W^{1,\infty}}$. For $(x,z)\in(\Omega_i\times\mathbb{R})\cap A_j$ $(i\neq j)$, by the definition of ϕ^x and, by (4.17), we have

$$|\mathrm{div}_x\phi^x|\leq C_4\left(\|\nabla^2u_{j,\beta}\|_\infty+\|\nabla^2v_j\|_\infty+\|\nabla u_{j,\beta}\|_\infty^2+\|\nabla v_j\|_\infty^2+\|\nabla u_{j,\beta}\|_\infty\|\nabla h_\beta\|_\infty\right), \tag{4.35}$$

where C_4 depend only on R, and S; by using (4.6), (4.7), (4.9), (4.18), (4.19), and recalling that $\gamma_1>\gamma$, we deduce, from (4.35), that

$$|\chi_j|\leq SC_4\left(C_5\|g\|_{W^{1,\infty}}\beta^{\frac{1}{2}+\gamma}+C_5+K_1\beta_1+(K\|g\|_{W^{1,\infty}})^2+K_1^2\beta_1+K\|g\|_{W^{1,\infty}}K_3\beta^{\frac{1}{2}+\gamma_1}\right)\leq C_6\beta^{\frac{1}{2}+\gamma_1}, \tag{4.36}$$

where C_6 depends only on R, S, $\Lambda^\alpha(\Gamma)$, and $\|g\|_{W^{1,\infty}}$.

Using the definition (4.28) for $(x,z)\in(\Omega_1\times\mathbb{R})\cap A_2$, we have

$$\phi^z(x,z)\geq\chi_1-2\|\phi^x\|(\|\nabla u_{2,\beta}\|_\infty+\|\nabla h_\beta\|_\infty)+\phi^z(x,u_\beta-h_\beta), \tag{4.37}$$

where, by (4.25),

$$\phi^z(x,u_\beta-h_\beta)\geq-\beta(u_\beta-h_\beta-g)^2+\Psi_1(x,u_\beta-h_\beta). \tag{4.38}$$

Therefore, for $(x,z) \in (\Omega_1 \times \mathbb{R}) \cap A_2$, combining (4.37) and (4.38), and using (4.7), (4.19), (4.31), (4.34), and (4.36), we obtain

$$\begin{aligned}
\phi^z(x,z) + \beta(z-g)^2 - \frac{|\phi^x|^2}{4} &\geq \beta\left[(z-g)^2 - (u_\beta - h_\beta - g)^2\right] - |\chi_i| - 2\|\phi^x\|_\infty(\|\nabla h_\beta\|_\infty \\
&\quad +\|\nabla u_{2,\beta}\|_\infty) + \Psi_1(x, u_\beta - h_\beta) - \frac{\|\phi^x\|_\infty^2}{4} \\
&\geq \beta[(7/16)^2 S^2 - (3/16)^2 S^2] - C_5 \beta^{\frac{1}{2}+\gamma_1} - C_3(K_3\beta^{\frac{1}{2}+\gamma_1} + K\|g\|_{W^{1,\infty}}) \\
&\quad -C\beta^{\frac{1}{2}+\gamma_1} - \frac{(C_3)^2}{4},
\end{aligned}$$

where we used also the fact that that $|z-g| \geq |z-u_\beta| - |u_\beta - g| \geq S/2 - S/16 = (7/16)S$ and, analogously, that $|u_\beta - h_\beta - g| \leq S/16 + S/8 = (3/16)S$ (see (4.5)); as $\frac{1}{2} + \gamma_1 < 1$, there exists $b_1 > 0$ depending only on R, S, $\Lambda^\alpha(\Gamma)$, and $\|g\|_{W^{1,\infty}}$ such that

$$\phi^z(x,z) + \beta(z-g)^2 - \frac{|\phi^x|^2}{4} > 0,$$

for $\beta \geq b_1$ and for $(x,z) \in (\Omega_1 \times \mathbb{R}) \cap A_2$. Analogously we can prove the existence of a constant $b_2 > 0$ depending on the same quantities such that

$$\phi^z(x,z) + \beta(z-g)^2 - \frac{|\phi^x|^2}{4} > 0,$$

for $\beta \geq b_2$ and for $(x,z) \in (\Omega_2 \times \mathbb{R}) \cap A_1$. Arguing exactly in the same way (in fact exploiting the same estimates), one can finally check that there exists $b_3 > 0$ depending on R, S, $\Lambda^\alpha(\Gamma)$, and $\|g\|_{W^{1,\infty}}$, such that

$$\phi^z + \beta(z-g)^2 - \frac{|\phi^x|^2}{4} > 0$$

holds true for $(x,z) \in (\Omega \times \mathbb{R}) \setminus (A_1 \cup A_2)$ and $\beta \geq b_3$. If we call $\beta_2 := \max\{b_0, b_1, b_2, b_3\}$ we have that for $\beta \geq \beta_2$ condition (c) of Section 1 is satisfied for almost every (x,z) in $\Omega \times \mathbb{R}$ with strict inequality if $z \neq u(x)$.

- $\phi(x, u_\beta) = (2\nabla u_\beta, |\nabla u_\beta|^2 - \beta(u_\beta - g)^2)$ *everywhere in* $\Omega \setminus \Gamma$.

Condition (d) of Section 1 is trivially satisfied, as one can see directly from the definition of ϕ.

- $\displaystyle\int_{u_{2,\beta}(x)}^{u_{1,\beta}(x)} \phi^x(x,t)\,dt = \nu_{u_\beta} = -\nu$ $\mathcal{H}^{n-1}$*-a.e. on* Γ.

By direct computation, for $x \in \Gamma$, we have

$$\int_{u_{2,\beta}}^{u_{1,\beta}} \phi^x(x,z)\,dz = (h_\beta)^2 \frac{\nabla v_2}{v_2} - (h_\beta)^2 \frac{\nabla v_1}{v_1}. \tag{4.39}$$

Using (4.10), (4.11), and the fact that $v_i \equiv 1$ on Γ, we obtain

$$\int_{u_{2,\beta}}^{u_{1,\beta}} \phi^x(x,z)\,dz = +\frac{1}{2}\frac{\nabla v_2}{|\nabla v_2|} - \frac{1}{2}\frac{\nabla v_1}{|\nabla v_1|} = -\nu,$$

so that condition (e) of Section 1 is satisfied.

- $\displaystyle\left|\int_{t_1}^{t^2} \phi^x(x,z)\,dz\right| \leq 1$ *for every* t_1, $t_2 \in \mathbb{R}$ *and for every* $x \in \Omega$.

It is convenient to introduce the following notation: for every $x \in \Omega$ and for every $s,\ t \in \mathbb{R}$, we set

$$I(x,[s,t]) := \int_s^t \phi^x(x,z)\,dz,$$

where, with a slight abuse of notation, $[s,t]$ stands for the interval $[s\wedge t, s\vee t]$ positively oriented if $s \le t$, negatively oriented otherwise. We define

$$d_\beta(\pi(x)) := \frac{\tilde{h}(\pi(x)) - \varepsilon}{\beta^{\frac{1}{2}+\gamma_1}}.$$

If $|d(x)| > d_\beta(\pi(x))$, recalling that, by definition, $h_\beta(x) = \varepsilon$ we have

$$\begin{aligned}
|I(x,t_1,t_2)| &\le \int_{u_{1,\beta}-\varepsilon}^{u_{1,\beta}+\varepsilon} \Big(2\|\nabla u_{1,\beta}\|_\infty + \tfrac{16}{\varepsilon}\|\nabla u_{1,\beta}\|_\infty \left(u_{1,\beta} - \tfrac{\varepsilon}{2} - z\right)^+ + 4|u_{1,\beta} - z|\|\nabla v_1\|_\infty\Big)\,dz \\
&\quad + \int_{u_{2,\beta}-\varepsilon}^{u_{2,\beta}+\varepsilon} \Big(2\|\nabla u_{2,\beta}\|_\infty + \tfrac{16}{\varepsilon}\|\nabla u_{2,\beta}\|_\infty \left(-u_{2,\beta} - \tfrac{\varepsilon}{2} + z\right)^+ + 4|u_{2,\beta} - z|\|\nabla v_2\|_\infty\Big)\,dz \\
&\le 6\varepsilon\|\nabla u_{1,\beta}\|_\infty + 4\varepsilon^2\|\nabla v_1\|_\infty + 6\varepsilon\|\nabla u_{2,\beta}\|_\infty + 4\varepsilon^2\|\nabla v_2\|_\infty \le \frac{1}{2},
\end{aligned} \tag{4.40}$$

where the last inequality is due to (4.16), therefore condition (f) is satisfied.

Let us consider now the case of a point x where $|d(x)| \le d_\beta(\pi(x))$. For $x \in \Omega_i \cup \Gamma$ we set

$$n(x) := -\frac{\nabla v_1}{|\nabla v_1|} = \frac{\nabla v_2}{|\nabla v_2|};$$

note that $n(x) = \nu_{u_\beta}(x)$ for every $x \in \Gamma$. Given any vector valued function $\xi : \Omega \to \mathbb{R}^n$, we call $\xi^\perp$ and $\xi^\parallel$ the vector valued functions such that $\xi^\perp(x)$ and $\xi^\parallel(x)$ are equal to the projections of $\xi(x)$ on the orthogonal space and on the space generated by $n(x)$, respectively. We denote the open unit sphere of $\mathbb{R}^n$ centred at the origin by B and by the open ball of $\mathbb{R}^n$ centred at the point $-rn(x)$ with radius r, by $A(x,r)$. Finally, for $x \in \Omega$ and $t \in \mathbb{R}$ we introduce the following vector

$$b_i(x,t) := (-1)^i(2(t - u_{i,\beta}) - j_i(x,t))\,(\nabla u_{i,\beta})^\parallel,$$

where j_i is defined by

$$j_i(x,t) := \frac{16}{h_\beta}\int_{u_{i,\beta}}^t \Big((-1)^i(u_{i,\beta} - z) - \tfrac{h_\beta}{2}\Big)^+\,dz.$$

CLAIM 1. There exists a positive constant $c_0 > 0$, depending on R, S, $\Lambda^\alpha(\Gamma)$, and $\|g\|_{W^{1,\infty}}$, with the property that for every $x \in \Omega$ such that $|d(x)| \le d_\beta(\pi(x))$, for every $t \in \mathbb{R}$ such that $|t - u_{i,\beta}(x)| \le h_\beta(x)$, and for $\beta \ge c_0$, we have

$$(-1)^{i+1}I(x,[u_{i,\beta},t]) + b_i(x,t) \in A(x,1/3). \tag{4.41}$$

A straightforward computation gives

$$\begin{aligned}
(-1)^{i+1}I(x,[u_{i,\beta}(x),t]) + b_i(x,t) &= 2(-1)^{i+1}\nabla u_{i,\beta}(t - u_{i,\beta}) + (-1)^i j_i(x,t)\nabla u_{i,\beta} + (-1)^{i+1}\frac{\nabla v_i}{v_i}(t - u_{i,\beta})^2 \\
&\quad + 2(-1)^i(t - u_{i,\beta})\,(\nabla u_{i,\beta})^\parallel + (-1)^{i+1}j_i(x,t)\,(\nabla u_{i,\beta})^\parallel = \\
&= (-1)^{i+1}(2(t - u_{i,\beta}) - j_i(x,t))(\nabla u_{i,\beta})^\perp - \frac{|\nabla v_i|}{v_i}(t - u_{i,\beta})^2 n(x)
\end{aligned}$$

and so the claim is equivalent to prove that

$$(2-j_i(x,t)(t-u_{i,\beta})^{-1})^2[(\nabla u_{i,\beta})^\perp]^2(t-u_{i,\beta})^2+\frac{|\nabla v_i|^2}{v_i^2}(t-u_{i,\beta})^4-\frac{2}{3}\frac{|\nabla v_i|}{v_i}(t-u_{i,\beta})^2<0;$$

as $0\le 2-j_i(x,t)(t-u_{i,\beta})^{-1}\le 2$ everywhere, it is sufficient to prove that

$$(*) := 4\left|(\nabla u_{i,\beta})^\perp\right|^2+h_\beta^2\frac{|\nabla v_i|^2}{v_i^2}-\frac{2}{3}\frac{|\nabla v_i|}{v_i}<0. \tag{4.42}$$

Since, by (4.11), $h_\beta^2\frac{|\nabla v_i|}{v_i}=\frac{1}{2}$ for $x\in\Gamma$, we can estimate

$$(*) = 4[(\nabla u_{i,\beta})^\perp]^2-\frac{1}{6}\frac{|\nabla v_i|}{v_i}<0 \qquad \text{on } \Gamma, \tag{4.43}$$

where the last inequality follows from (4.14). In the following we denote by $\partial_{|d|}$ the differential operator

$$\partial_{|d|}f(x)=\nabla f(x)\cdot\nabla|d(x)|,$$

defined for $x\in(\Gamma)_{\frac{R}{2}}\setminus\Gamma$; noting that, by the estimates (4.9), we have

$$\left|\partial_{|d|}\frac{|\nabla v_i|^2}{v_i^2}\right|\le C \qquad \left|\partial_{|d|}\frac{|\nabla v_i|}{v_i}\right|\le C,$$

where C depends only on R, S, and $\|g\|_{W^{1,\infty}}$, and using (4.12), (4.16), (4.18), (4.6), and (4.19), one sees that

$$\begin{aligned}\partial_{|d|}((*)) &= 8(\nabla u_{i,\beta})^\perp\cdot\partial_{|d|}(\nabla u_{i,\beta})^\perp+2\frac{|\nabla v_i|^2}{v_i^2}h_\beta\partial_{|d|}h_\beta+h_\beta^2\partial_{|d|}\frac{|\nabla v_i|^2}{v_i^2}-\frac{2}{3}\partial_{|d|}\frac{|\nabla v_i|}{v_i}\\ &\le 8cKK_2\beta^{\frac{1}{2}+\gamma}\|g\|^2_{W^{1,\infty}}+C_1-\frac{\varepsilon}{2}K_3\beta^{\frac{1}{2}+\gamma_1}+S^2C+C;\end{aligned}$$

as $\gamma_1>\gamma$ and since all the constants appearing in the last inequality depend only on R, S, $\Lambda^\alpha(\Gamma)$, and $\|g\|_{W^{1,\infty}}$, it is clear that there exists $c_0>0$ depending on the same quantities such that $\partial_{|d|}((*))<0$, for $x\notin\Gamma$ such that $|d(x)|\le d_\beta(\pi(x))$ and for $\beta\ge c_0$. Therefore, taking into account (4.43), (4.42) follows immediately: Claim 1 is proved.

CLAIM 2. There exists a positive constant c_1, depending only on R, S, $\Lambda^\alpha(\Gamma)$, and $\|g\|_{W^{1,\infty}}$, such that for every $x\in\Omega$, t_1, $t_2\in\mathbb{R}$, with $|d(x)|\le d_\beta(\pi(x))$, $|t_1-u_{1,\beta}|\le h_\beta$, $|t_2-u_{2,\beta}|\le h_\beta$, and for every $\beta\ge c_1$, we have

$$I(x,[u_{2,\beta},u_{1,\beta}])-b_1(x,t_1)-b_2(x,t_2)=b(x,t_1,t_2)n(x), \tag{4.44}$$

with $b(x,t_1,t_2)<1$.

First of all observe that for every $x\in\Omega$ $I(x,u_{2,\beta},u_{1,\beta})$ is a vector parallel to $n(x)$, by (4.39); it is also clear that

$$\begin{aligned}|I(x,[u_{2,\beta},u_{1,\beta}])-b_1(x,t)-b_2(x,t)| &\le |I(x,[u_{2,\beta},u_{1,\beta}])|+\left|(\nabla u_{1,\beta})^{\parallel}\right||(2h_\beta+j_1(x,h_\beta))\\ &\quad +\left|(\nabla u_{2,\beta})^{\parallel}\right|(2h_\beta+j_2(x,h_\beta))\\ &\le |I(x,[u_{2,\beta},u_{1,\beta}])|+4h_\beta\left(\left|(\nabla u_{1,\beta})^{\parallel}\right|+\left|(\nabla u_{2,\beta})^{\parallel}\right|\right)\\ &=: m_\beta(x);\end{aligned}$$

therefore it is sufficient to prove that $m_\beta(x)<1$ for $|d(x)|\le d_\beta(\pi(x))$, if β is large enough. Since $m_\beta(x)=|I(x,u_{2,\beta},u_{1,\beta})|=1$ for every $x\in\Gamma$, it will be enough to show that $\partial_{|d|}m_\beta(x)<0$ for x

such that $|d(x)| \leq d_\beta(\pi(x))$. We don't enter all the details, indeed arguing as above, that is using (4.7), (4.18), (4.9), and (4.19), one easily sees that the derivative of h_β which is negative and of the same order as $\beta^{\frac{1}{2}+\gamma_1}$, dominates the other terms and so there exists a positive constant $c_1 > 0$ depending on R, S, $\Lambda^\alpha(\Gamma)$, and $\|g\|_{W^{1,\infty}}$, such that $\partial_{|d|} m_\beta(x) < 0$ for $\beta \geq c_1$: Claim 2 is proved.

We set $\beta_3 = \max\{c_0, c_1\}$ and we are going to prove that condition (f) of Section 1 is satisfied for $\beta \geq \beta_3$. We will check the condition only in $\Omega_1 \times \mathbb{R}$: for $\Omega_2 \times \mathbb{R}$ the argument would be analogous. Let $x \in \Omega_1$ and $t_2 < t_1$ two real numbers such that $|t_2 - u_{2,\beta}(x)| \leq h_\beta(x)$ and $|t_1 - u_\beta(x)| \leq h_\beta(x)$; first of all it is easy to see, by explicit computation, that

$$I(x, [t_2, t_1]) \cdot n(x) \geq 0; \tag{4.45}$$

recalling that, by Claim 1,

$$I(x, [u_\beta, t_1]) + b_1(x, t_1) \in A(x, 1/3) \qquad \text{and} \qquad I(x, [t_2, u_{2,\beta}]) + b_2(x, t) \in A(x, 1/3),$$

we have

$$\begin{aligned} I(x, [t_2, t_1]) \;\; &= \;\; I(x, [u_{2,\beta}, u_\beta]) - b_1(x, t_1) - b_2(x, t_2) + I(x, [t_2, u_{2,\beta}]) + b_2(x, t_2) \\ &\quad + I(x, [u_\beta, t_1]) + b_1(x, t_1) \in I(x, [u_{2,\beta}, u_\beta]) - b_1(x, t_1) - b_2(x, t_2) + 2A(x, 1/3), \end{aligned}$$

therefore, taking into account (4.45),

$$I(x, [t_2, t_1]) \in (I(x, [u_{2,\beta}, u_\beta]) - b_1(x, t_1) - b_2(x, t_2) + A(x, 2/3)) \cap H^+, \tag{4.46}$$

where H^+ is the half-space $\{\xi \in \mathbb{R}^n : \xi \cdot n(x) \geq 0\}$. By elementary geometry it is easy to see that $(bn(x) + A(x, r)) \cap H^+ \subset B$ for $b < 1$ and for $r \in (0, 1)$, and hence, invoking Claim 2, we get

$$\begin{aligned} I(x, [t_1, t_2]) \;\; &\in \;\; (I(x, [u_{2,\beta}, u_\beta]) - b_1(x, t_1) - b_2(x, t_2) + A(x, 2/3)) \cap H^+ \\ &= (b(x, t_1, t_2)n(x) + A(x, 2/3)) \cap H^+ \subset B. \end{aligned} \tag{4.47}$$

If (x, t_1) and (x, t_2) belong to A_i it is easy to see, by explicitly computing the integral, that

$$|I(x, [t_1, t_2])| \leq h_\beta^2(x) \frac{|\nabla v_i|}{v_i} + \frac{25}{8} h_\beta |\nabla u_{i,\beta}| < 1 - \frac{25}{32} \frac{1}{\sqrt{3}} + \frac{25}{32} \frac{1}{\sqrt{3}} = 1, \tag{4.48}$$

where the last inequality follows from (4.12), (4.15), and (4.11) (we recall that for β large enough $d_\beta(\pi(x)) \leq D$, for every x, being D the constant introduced in (4.12)).

We now consider the general case. Let $x \in \Omega_1$, t_1, $t_2 \in \mathbb{R}$ with $t_1 < t_2$; since ϕ^x vanishes out of the regions A_1 and A_2, we have

$$I(x, [t_1, t_2]) = I(x, [t_1, t_2] \cap [u_{2,\beta} - h_\beta, u_{2,\beta} + h_\beta]) + I(x, [t_1, t_2] \cap [u_\beta - h_\beta, u_\beta + h_\beta]);$$

by (4.48), each integral in the expression above has modulus less than 1, so that if one of the two is vanishing condition (f) is verified. If both are non-vanishing, then

$$[t_1, t_2] \cap [u_{2,\beta} - h_\beta, u_\beta + h_\beta] = [s_1, s_2],$$

with $|s_1 - u_{2,\beta}| \leq h_\beta$ and $|s_2 - u_\beta| \leq h_\beta$, so that, again taking into account the fact that ϕ^x vanishes out of the regions A_1 and A_2,

$$|I(x, [t_1, t_2])| = |I(x, [t_1, t_2] \cap [u_{2,\beta} - h_\beta, u_\beta + h_\beta])| = |I(x, [s_1, s_2])| < 1,$$

where the last inequality follows from (4.47): condition (f) of Section 1 is proved.

Since, by construction, ϕ has vanishing normal component on $\partial\Omega \times \mathbb{R}$, if we set $\overline{\beta} := \max\{\beta_1, \beta_2, \beta_3\}$ we have that conditions of Section 1 are all satisfied for $\beta \geq \overline{\beta}$: the theorem is proved. □

A similar result holds true also if Γ is made up of several connected components, as the following theorem states: we omit the proof, since it is essentially the same as the previous one.

Theorem 4.3 *Let Ω as above and let $\Omega_1, \dots, \Omega_k$ a family of open disjoint subsets belonging of class $C^{2,\alpha}$ and let $R > 0$ such that $\Omega_i \in \mathcal{U}_R(\Omega)$ for $i = 1, \dots, k$ and $\operatorname{dist}(\Omega_i, \Omega_j) \geq R$ for every $i \neq j$. Set $\Gamma := \partial\Omega_1 \cup \dots \cup \partial\Omega_k$. Then for every function g belonging $W^{1,\infty}(\Omega \setminus \Gamma)$, discontinuous along Γ (i.e. $S_g = \Gamma$) and such that $g^+(x) - g^-(x) > S > 0$ for every $x \in \Gamma$, there exists $\beta_0 > 0$ depending on R, S, $\Lambda^\alpha(\Gamma)$ (see (3.1)), and $\|g\|_{W^{1,\infty}}$, such that for $\beta \geq \beta_0$ the solution u_β of (4.4) is discontinuous along Γ ($S_{u_\beta} = \Gamma$) and it is the unique absolute minimizer of $F_{\beta,g}$ over $SBV(\Omega)$.*

Remark 4.4 We remark that refining a little the construction, it is possible to improve the result of Theorem 4.2 as follows:
there exist $\delta^* > 0$ and $\beta_0 > 0$ such that, for every $\beta \geq \beta_0$ and for every $g \in W^{1,\infty}(\Omega \setminus \Gamma)$, with $\|g\|_{W^{1,\infty}} \leq \beta^{\delta^*}$ and such that $\inf_\Gamma(g^+ - g^-) > S$, the solution u_β of (4.4) is the unique absolute minimizer of $F_{\beta,g}$ over $SBV(\Omega)$.
The main difficulty comes from the fact that instead of (4.7) we have the weaker estimate

$$\|\nabla u_\beta\|_\infty \leq K\beta^{\delta^*}.$$

Such a difficulty can be overcome replacing, in the construction above, v_1 and v_2 by $v_{1,\beta}$ and $v_{2,\beta}$ defined as

$$v_{1,\beta}(x) = \begin{cases} z_{1,c\beta^{4\delta^*}}(x) & \text{if } x \in \overline{\Omega_1} \\ 2 - z_{2,c\beta^{4\delta^*}}(x) & \text{if } x \in \Omega_2 \end{cases}$$

and

$$v_{2,\beta}(x) = \begin{cases} z_{2,c\beta^{4\delta^*}}(x) & \text{if } x \in \overline{\Omega_2} \\ 2 - z_{1,c\beta^{4\delta^*}}(x) & \text{if } x \in \Omega_1, \end{cases}$$

where $z_{1,c\beta^{4\delta^*}}$ and $z_{2,c\beta^{4\delta^*}}$ are the two functions constructed in Lemma 4.1 with $\lambda = c\beta^{4\delta^*}$. One can check that if δ^* is sufficiently small and c sufficiently large, all the conditions of Section 1 are still satisfied for β large enough.

4.2 The two-dimensional case

As stated in the Introduction, in dimension two we are able to treat the case of Ω with piecewise smooth boundary (curvilinear polygon) and of Γ touching (orthogonally) $\partial\Omega$.

Lemma 4.5 *Let Ω, $\mathcal{S}$, and Γ be as in Proposition 3.6 and denote by Ω_1, Ω_2 the two connected components of $\Omega \setminus \Gamma$. Then for every $\delta > 0$ there exist two positive constants c and β_0 depending on Γ and δ (and Ω of course) such that, for $\beta \geq \beta_0$, we can find two functions $z_{1,\beta} : \Omega_1 \to \mathbb{R}$ and $z_{2,\beta} : \Omega_2 \to \mathbb{R}$ of class $W^{2,\infty}$ with the following properties:*

i) $\frac{1}{2} \leq z_{i,\beta} \leq 1$ *in* Ω_i, *for* $i = 1, 2$ *and* $z_{i,\beta} \equiv \frac{1}{2}$ *in* $\Omega \setminus (\Gamma)_\delta$;

ii) $\Delta z_{i,\beta} \leq c\beta z_{i,\beta}$ *in* Ω_i, *for* $= 1, 2$;

iii) $z_{1,\beta}(x) = z_{2,\beta}(x) = 1$ *and* $\partial_\nu z_{1,\beta}(x) = -\partial_\nu z_{2,\beta}(x) \geq \sqrt{\beta}$ *for every* $x \in \Gamma$;

iv) $\|\nabla z_{i,\beta}\|_\infty \leq c\sqrt{\beta}$ *and* $\|\nabla^2 z_{i,\beta}\|_\infty \leq c\beta$.

Proof. Let us denote by x_1 and x_2 the two intersection points of Γ with $\partial\Omega$. If we are able to find a function $\tilde{d}$ belonging to $W^{2,\infty}((\Gamma)_{\delta'} \cap \Omega)$ (for a suitable $\delta' < \operatorname{dist}(\mathcal{S}, \Gamma)$) such that $\tilde{d}$ is vanishing on Γ, positive in $\Omega_2 \cap (\Gamma)_{\delta'}$, negative in $\Omega_1 \cap (\Gamma)_{\delta'}$, satisfying $\partial_\nu \tilde{d} = 0$ on $\partial\Omega \cap \overline{\Gamma^{\delta'}}$ and $\partial_\nu \tilde{d} \neq 0$ on Γ, we are done: indeed we can proceed exactly as in Lemma 4.1 using $\tilde{d}$ in place of d. We briefly describe a possible construction: as in Proposition 3.6 we can find a neighbourhood U_i of x_i ($i = 1, 2$) and a $C^{1,1}$ function ψ_i vanishing on $\Gamma \cap U_i$, positive in $\Omega_2 \cap U_i$, negative in $\Omega_1 \cap U_i$ and such that $\partial_\nu \psi_i = 0$ on $\partial\Omega \cap U_i$ and $\partial_\nu \psi_i \neq 0$ in $\Gamma \cap U_i$. Now we can define $\tilde{d} := \theta_1\psi_1 + \theta d + \theta_2\psi_2$, where θ_1, θ_2, and θ_3

are suitable positive cut-off functions such that $\theta_1+\theta_2+\theta_3\equiv 1$, while d is the usual signed distance function from Γ, positive in Ω_2 and negative in Ω_1 (it is well defined in $\Gamma^{\delta'}$ if δ' is small enough). $\square$

Theorem 4.6 *Let Ω, Ω_1, Ω_2, and Γ as in the previous Lemma and let g be a function in $W^{1,\infty}(\Omega\setminus\Gamma)$, discontinuous along Γ (i.e. $S_g=\Gamma$) and such that $g^+(x)-g^-(x)>S>0$ for every $x\in\Gamma$. Then there exists $\beta_0>0$ depending on Γ, S, and $\|g\|_{W^{1,\infty}}$, such that for $\beta\geq\beta_0$ the solution u_β of (4.4) is discontinuous along Γ ($S_{u_\beta}=\Gamma$) and it is the unique absolute minimizer of $F_{\beta,g}$ over $SBV(\Omega)$.*

PROOF. As above, let us denote by $\mathcal{S}$ the set of the singular points of $\partial\Omega$. If Ω is regular (i.e. $\mathcal{S}=\emptyset$) we can recycle exactly the same construction of Theorem 4.2. If $\mathcal{S}\neq\emptyset$, an additional difficulty is due to the fact that we are not able to prove that $\|\nabla u_\beta\|_{L^\infty(\Omega)}\leq C$ with C independent of β. Since we can perform such an estimate only in a neighbourhood of Γ which does not intersect $\mathcal{S}$, the idea will be to keep the construction of Theorem 4.2 in that neighbourhood and to suitably modify it near the singular points in order to exploit estimate (3.24).

Denote by γ_1 and γ_2 the two curvilinear edges of Ω containing the intersection points of Γ with $\partial\Omega$ and choose $\delta>0$ so small that $(\Gamma)_\delta\cap\mathcal{S}=\emptyset$, $(\Gamma)_\delta\cap\partial\Omega=(\Gamma)_\delta\cap(\gamma_1\cup\gamma_2)$, and d and π are well defined and smooth (according to Lemma 2.3) in that neighbourhood.

Let us choose $\beta'>0$ and $G>0$ such that, for $\beta\geq\beta'$,

$$\|u_\beta-g\|_{L^\infty(\Omega)}\leq\frac{S}{16}\quad\text{and}\quad\sqrt{\beta}\|u_\beta-g\|_{L^\infty(\Omega)}\leq G\|g\|_{W^{1,\infty}(\Omega)}\quad i=1,2,: \tag{4.49}$$

this is possible by virtue of Proposition 3.4.

Again it is convenient to extend the restriction of u_β to Ω_i ($i=1,2$) to a $C^{1,1}$ function $u_{i,\beta}$ defined in the whole Ω, in such a way that

$$u_{i,\beta}(x)=u_\beta(x)\text{ in }\Omega_i,\quad \|u_{i,\beta}\|_{W^{2,\infty}(\Omega)}\leq c\|u_\beta\|_{W^{2,\infty}((\Gamma)_\delta\cap\Omega)},\quad\text{and}\quad u_{1,\beta}-u_{2,\beta}\geq\frac{3}{4}S\quad\text{everywhere,} \tag{4.50}$$

where c is a positive constant independent of β. We require also that

$$\partial_\nu u_{i,\beta}=0\qquad\text{on }\partial\Omega.$$

By (3.27) and (4.50), and by (3.24), we can state the existence of two positive constants K and β'' depending only on Γ, such that

$$\|\nabla u_{i,\beta}\|_{L^\infty(\Omega_i\cup(\Gamma)_\delta\cap\Omega)}\leq K\|g\|_{W^{1,\infty}(\Omega)}\quad\text{for}\quad i=1,2,\qquad\text{and}\qquad \|\nabla u_\beta\|_{L^\infty(\Omega)}\leq\beta^{\frac14}K\|g\|_{W^{1,\infty}(\Omega)} \tag{4.51}$$

for every $\beta\geq\beta''$ (above and in the sequel, $\hat\imath$ denotes the comlement of i, i.e., $\hat\imath$ is such that $i,\hat\imath=\{1,2\}$).

Let $\beta'''>0$ satisfying

$$\frac{1}{6}\sqrt{\beta'''}=\max\left\{4(K\|g\|_{W^{1,\infty}})^2,64/S^2,\beta',\beta'',\beta_0\right\}+1, \tag{4.52}$$

where β_0 is the constant appearing in Lemma 4.5 and $z_{1,\beta'''}$, and let $z_{2,\beta'''}$ be the two functions constructed in Lemma 4.5 with $\lambda=\beta'''$. We denote by v_1, v_2 the functions defined as follows:

$$v_1(x)=\begin{cases}z_{1,\beta'''}(x) & \text{if }x\in\overline{\Omega_1}\\ 2-z_{2,\beta'''}(x) & \text{if }x\in\Omega_2\end{cases}$$

$$v_2(x)=\begin{cases}z_{2,\beta'''}(x) & \text{if }x\in\overline{\Omega_2}\\ 2-z_{1,\beta'''}(x) & \text{if }x\in\Omega_1,\end{cases}$$

and we choose $0 < D < \delta$ in such a way that

$$|\nabla v_i(x)| \geq \frac{1}{2}, \quad \tilde{h}^2(\pi(x))\frac{|\nabla v_i|}{v_i} \leq 1 - \frac{25}{32}\frac{1}{\sqrt{3}}, \qquad \text{if } |d(x)| \leq D\,,\ i = 1,2,$$

where

$$\tilde{h}(x) = \frac{1}{\sqrt{2}}|\nabla v_1(x)|^{-\frac{1}{2}} = \frac{1}{\sqrt{2}}|\nabla v_2(x)|^{-\frac{1}{2}} \qquad \forall x \in \Gamma. \tag{4.53}$$

Then we choose $\varepsilon \in (0,1)$ in such a way that

$$12\varepsilon\|\nabla\tilde{u}_{i,\beta}\|_{L^\infty(\Omega_i\cup(\Gamma)_\delta)\cap\Omega} + 4\varepsilon^2\|\nabla v_i\|_{L^\infty(\Omega)} < \frac{1}{4} \quad \text{for } i = 1,2 \text{ and } \beta \geq \beta'''. \tag{4.54}$$

Let γ be a fixed constant belonging to $(0, \frac{1}{2} \wedge \alpha)$: by Proposition 3.6, we can find two positive constants β^{IV} and K_2 such that

$$\|\nabla^2 u_\beta\|_{L^\infty((\Gamma)_\delta\cap\Omega)} \leq K_2\beta^{\frac{1}{2}+\gamma}\|g\|_{W^{1,\infty}(\Omega)}, \tag{4.55}$$

for every $\beta \geq \beta^{\text{IV}}$.

Now we can define, for $\beta > 0$,

$$h_\beta(x) = \begin{cases} \left(\tilde{h}(\pi(x)) - \beta^{\frac{1}{2}+\gamma_1}|d(x)|\right) \vee \varepsilon & \text{if } |d(x)| \leq \frac{D}{2} \\ f_\beta(|d(x)|) & \text{if } |d(x)| > \frac{D}{2}, \end{cases}$$

where γ_1 is a fixed constant belonging to $(\gamma, \frac{1}{2})$ and $f_\beta : [\delta, +\infty) \to \mathbb{R}$ is the continuous function satisfying

$$f_\beta\left(\frac{D}{2}\right) = \varepsilon \qquad f_\beta(t) \equiv s_\beta := \left(\beta^{\frac{1}{4}}8K\|g\|_{W^{1,\infty}(\Omega)}\right)^{-1} \quad \text{for } t \geq D \qquad f_\beta \text{ is affine in } \left[\frac{D}{2}, D\right]. \tag{4.56}$$

It is easy to see that there exists $\beta^{\text{V}} > 0$ such that h_β is continuous (in fact Lipschitz) for $\beta > \beta^{\text{V}}$.

Finally we introduce a new function $\hat{u}_{i,\beta}$ which is a modification of $u_{i,\beta}$ in the region where we cannot perform a uniform control of the L^∞-norm of its gradient; such a function must satisfy, for $i = 1,2$:

$$\hat{u}_{i,\beta}(x) = u_{i,\beta}(x) \quad \text{for } x \in \Omega_i \cup (\Gamma)_{\frac{D}{2}} \cap \Omega \qquad \text{and} \qquad \hat{u}_{i,\beta}(x) = g(x) \quad \text{for } x \in \Omega_i \setminus (\Gamma)_D, \tag{4.57}$$

$$\|\nabla\hat{u}_{i,\beta}\|_{L^\infty(\Omega)} \leq c(\|\nabla u_\beta\|_{L^\infty((\Gamma)_D)} \vee \|\nabla g\|_{L^\infty(\Omega)}) \qquad \text{and} \qquad \|\hat{u}_{i,\beta} - g\|_{L^\infty(\Omega)} \leq \|u_\beta - g\|_{L^\infty(\Omega)}, \tag{4.58}$$

where $c > 0$ is independent of β: a possible construction is given by

$$\hat{u}_{i,\beta}(x) = \theta\left((-1)^i d(x)\right)\tilde{u}_{i,\beta} + \left[1 - \theta\left((-1)^i d(x)\right)\right]g(x),$$

where θ is a smooth positive function such that $\theta(t) = 1$ for $t \leq D/2$ and $\theta(t) = 0$ for $t \geq D$. Now for $\beta \geq \beta_1 := \max\{\beta'', \beta''', \beta^{\text{IV}}, \beta^{\text{V}}\}$ we consider the sets

$$A_i := \{(x,z) \in \Omega \times \mathbb{R} : \ \hat{u}_{i,\beta}(x) - h_\beta(x) \leq z \leq \hat{u}_{i,\beta}(x) + h_\beta(x)\}, \quad i = 1,2; \tag{4.59}$$

setting

$$\hat{h}_\beta(x) := \left[1 + (2/D)(|d(x)| - D/2)^+\right]h_\beta(x),$$

we can define

$$\phi^x(x,z) := \begin{cases} 2\nabla u_{i,\beta} - 2\frac{u_{i,\beta}-z}{v_i}\nabla v_i - \frac{16}{\hat{h}_\beta}\left((-1)^i(z-u_{i,\beta}) - \frac{\hat{h}_\beta}{2}\right)^+ \nabla u_{i,\beta} & \text{if } (x,z) \in A_i, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\phi^z|_{A_i\cap(\overline{\Omega}_i\times\mathbb{R})} := \left|\nabla u_\beta - \frac{u_\beta - z}{v^i}\nabla v^i\right|^2 - \beta(z-g)^2 + (\beta-\mu_i)(u_\beta - z)^2 + \Psi_i,$$

where the functions Ψ_i and μ_i are defined exactly as in the proof of Theorem 4.2. At this point, as in the proof of Theorem 4.2, the vertival component ϕ^z can be extended to the whole $\Omega\times\mathbb{R}$ in order to satisfy conditions (a) and (b) of Section 1 (we do not rewrite the explicit expression). First of all observe that in $\left((\Gamma)_{\frac{D}{2}}\cap\Omega\right)\times\mathbb{R}$ the definition of ϕ is the same as in Theorem 4.2, then, we can state the existence of a constant $\beta_0' > 0$ depending on Γ, S, and $\|g\|_{W^{1,\infty}}$ such that ϕ satisfies (a), (b), (c), (d), (e), (f), and (g) of Section 1 in $\left((\Gamma)_{\frac{D}{2}}\cap\Omega\right)\times\mathbb{R}$.

From now on we focus our attention on what happens in $\left(\Omega_i\setminus(\Gamma)_{\frac{D}{2}}\right)\times\mathbb{R}$.

Concerning (d), we have only to check that for β large enough the graph of $u_{i,\beta}$ belongs to A_i, but this follows from the fact that, by (4.56), A_i contains the s_β-neighbourhood of the graph of $\hat{u}_{i,\beta}$, where s_β is of order $\beta^{-\frac{1}{4}}$, and from the fact that, by (4.49) and (4.58), it holds

$$\|u_{i,\beta} - \hat{u}_{i,\beta}\|_\infty \le \|u_{i,\beta} - g\|_\infty + \|\hat{u}_{i,\beta} - g\|_\infty \le C\beta^{-\frac{1}{2}}.$$

Concerning condition (c), it is clearly satisfied in A_i, then it remains to check, for β large enough, the inequality $\phi^z(x,z) + \beta(z-g)^2 > 0$ holds true outside A_i. For $x \in \left(\Omega_i\setminus(\Gamma)_{\frac{D}{2}}\right)\cap\Gamma_\delta$ such an estimate can be performed using estimates (4.51), (4.55), (4.58) and arguing as in the proof of Theorem 4.2. Now let (x,z) belong to $[(\Omega_i\setminus(\Gamma)_D)\times\mathbb{R}]\setminus A_i$ and suppose also that $\hat{u}_{2,\beta}(x) + h_\beta(x) \le z \le \hat{u}_{1,\beta}(x) - h_\beta(x)$ (the other cases would be analogous); since $\phi^z(x,z) = \phi(x, \hat{u}_{i,\beta} + (-1)^i h_\beta)\cdot(-\nabla\hat{u}_{i,\beta} + (-1)^{i+1}\nabla h_\beta, 1)$ and observing that $\phi(x, \hat{u}_{i,\beta} + (-1)^i h_\beta)$ reduces to

$$\left(2\nabla u_\beta,\, |\nabla u_\beta|^2 - \beta(z-g)^2 + \beta(u_\beta - z)^2\right),$$

we obtain

$$\begin{aligned} \phi^z(x,z) + \beta(z-g)^2 &\ge -|\nabla u_\beta||\nabla\hat{u}_{i,\beta}| - 2|\nabla u_\beta||\nabla h_\beta| + \beta(u_\beta - z)^2 \\ &\ge -|\nabla u_\beta||\nabla\hat{u}_{i,\beta}| - 2|\nabla u_\beta||\nabla h_\beta| + \beta s_\beta^2; \end{aligned}$$

in the last expression the positive term βs_β^2, which behaves like $\beta^{\frac{1}{2}}$ (see the definition of s_β) dominates the negative ones, indeed these are either bounded or of the same order of $|\nabla u_\beta|$ which is less or equal to the order of $\beta^{\frac{1}{4}}$, thanks to (4.51): therefore for β large enough we get the desired inequality.

About condition (f) we first observe that if t_1, $t_2 \in \mathbb{R}$ and $x \in \left((\Gamma)_D\setminus(\Gamma)_{\frac{D}{2}}\right)\cap\Omega$ then we obtain

$$\left|\int_{t_1}^{t_2}\phi^x(x,z)\,dz\right| \le$$

$$
\begin{aligned}
&\le \sum_{i=1}^{2}\left[\int_{\hat u_{i,\beta}-h_\beta}^{\hat u_{i,\beta}+h_\beta}\left(2\|\nabla u_{i,\beta}\|_\infty + \tfrac{16}{\hat h_\beta}\|\nabla u_{i,\beta}\|_\infty\left((-1)^i(z-u_{i,\beta})-\tfrac{\hat h_\beta}{2}\right)^+ + 4|u_{i,\beta}-z|\|\nabla v_i\|_\infty\right)dz\right]\\
&\le \sum_{i=1}^{2}\left[4\|\nabla u_{i,\beta}\|_\infty\varepsilon + 4\varepsilon^2\|\nabla v_i\|_\infty + \tfrac{16}{\hat h_\beta}\|\nabla u_{i,\beta}\|_\infty\left[(u_{i,\beta}-\hat u_{i,\beta})^2+\tfrac{\hat h_\beta^2}{4}\right]\right]\\
&\le \sum_{i=1}^{2}\left[4\|\nabla u_{i,\beta}\|_\infty\varepsilon + 4\varepsilon^2\|\nabla v_i\|_\infty + \frac{16}{s_\beta}\|\nabla u_{i,\beta}\|_\infty(u_{i,\beta}-\hat u_{i,\beta})^2 + 8\|\nabla u_{i,\beta}\|_\infty\varepsilon\right]\\
&\le \sum_{i=1}^{2}\left[12\|\nabla u_{i,\beta}\|_\infty\varepsilon + 4\varepsilon^2\|\nabla v_i\|_\infty + C\beta^{-\frac{3}{4}}\right]\\
&\qquad\Big[\text{the fact that } \tfrac{16}{s_\beta}\|\nabla u_{i,\beta}\|_\infty(u_{i,\beta}-\hat u_{i,\beta})^2 \le C\beta^{-\frac{3}{4}} \text{ follows from estimates}\\
&\qquad\ \ \text{(4.51), (4.49), (4.58), and the definition of } s_\beta\Big]\\
&\le \frac{1}{2},
\end{aligned}
$$

if β is large enough, thanks to (4.54).

If $x \in \Omega_i \setminus (\Gamma)_D$ then we can estimate

$$\left|\int_{t_1}^{t_2}\phi^x(x,z)\,dz\right| \le 2s_\beta\|\nabla u_{1,\beta}\|_\infty + 2s_\beta\|\nabla u_{2,\beta}\|_\infty \le \frac{1}{2},$$

by (4.51) and the definition of s_β. Also condition (f) is proved; since, by construction, ϕ has vanishing normal component along $\partial\Omega\times\mathbb{R}$, the theorem is completely proved. $\square$

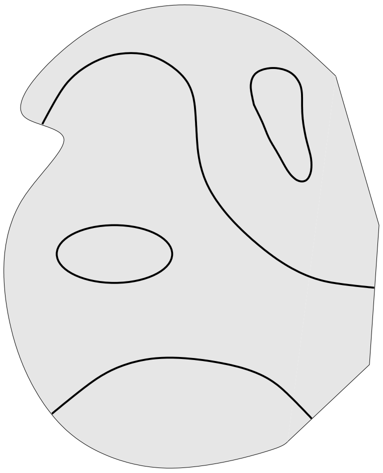

Figure 2: An admissible discontinuity set Γ.

Now we can state a theorem which is the analogous of Theorem 4.3.

Theorem 4.7 *Let Ω as in Proposition 3.6 and $\Gamma = \gamma_1 \cup \cdots \cup \gamma_k$ where for every $j = 1,\ldots,k$ γ_j is either a simple, connected, and closed curve of class $C^{2,\alpha}$ contained in Ω or a connected curve with the same regularity outside a neighbourhood of its endpoints (where it is supposed to be of class C^3), which meets orthogonally $\partial\Omega$ in two regular points (see Fig. 2); suppose in addition that $\gamma_i \cap \gamma_j = \emptyset$ if $i \neq j$.*

Then for every $g \in W^{1,\infty}(\Omega \setminus \Gamma)$ discontinuous along Γ and such that $g^+(x) - g^-(x) > S > 0$ for every $x \in \Gamma$, there exists $\beta_0 > 0$ depending on Γ, S, and $\|g\|_{W^{1,\infty}}$, such that for $\beta \geq \beta_0$ the solution u_β of (4.4) is discontinuous along Γ ($S_{u_\beta} = \Gamma$) and it is the unique minimizer of $F_{\beta,g}$ over $SBV(\Omega)$.

5 Gradient flow for the Mumford-Shah functional

In this section we are going to apply the previous results to the study of the gradient flow of the Mumford-Shah functional by the method of minimizing movements (see Section 2) with an initial datum u_0 which is regular outside a regular discontinuity set Γ: we will show that, for an initial interval of time, the discontinuity set does not move while the function evolves according to the heat equation. Our main result is stated in the following theorem:

Theorem 5.1 *Let Ω and Γ be either as in Theorem 4.3 or as in Theorem 4.7. Suppose that u_0 is a function belonging to $W^{2,\infty}(\Omega \setminus \Gamma)$, discontinuous along Γ, and such that $u_0^+(x) - u_0^-(x) > S > 0$ for every $x \in \Gamma$ and $\partial_\nu u_0 = 0$ on $\partial\Omega \cup \Gamma$. Then there exists $T > 0$ such that the minimizing movement for the Mumford-Shah functional is unique in $[0, T]$ and it is given by the function $u(x,t)$ satisfying*

$$S_{u(\cdot,t)} = \Gamma \qquad \forall t \in [0, T],$$

and

$$\begin{cases} \partial_t u = \Delta u & \text{in } (\Omega \setminus \Gamma) \times [0,T], \\ \partial_\nu u = 0 & \text{on } \partial(\Omega \setminus \Gamma) \times [0,T], \\ u(x,0) = u_0(x) & \text{in } \Omega \setminus \Gamma. \end{cases}$$

Proof. For fixed $\delta > 0$, let $v_\delta(t)$ be the affine interpolation of the discrete function

$$\begin{array}{rcl} v_\delta : \delta\mathbb{N} & \to & H^1(\Omega \setminus \Gamma) \\ v_\delta(\delta i) & \mapsto & v_{\delta,i}, \end{array}$$

where $v_{\delta,i}$ is inductively defined as follows:

$$\begin{cases} v_{\delta,0} = u_0, \\ \\ v_{\delta,i} \text{ is the unique solution of} \\ \qquad \min\limits_{z \in H^1(\Omega\setminus\Gamma)} \displaystyle\int_{\Omega\setminus\Gamma} |\nabla z|^2\, dx + \frac{1}{\delta}\int_{\Omega\setminus\Gamma} |z - v_{\delta,i-1}|^2\, dx. \end{cases} \tag{5.1}$$

Claim 1. For every $T > 0$, we have that

$$v_\delta \to v \qquad \text{in } L^\infty([0,T]; L^\infty(\Omega \setminus \Gamma)) \text{ as } \delta \to 0,$$

where v is the solution of

$$\begin{cases} \partial_t v = \Delta v & \text{in } (\Omega \setminus \Gamma) \times [0,T], \\ \partial_\nu v = 0 & \text{on } \partial(\Omega \setminus \Gamma) \times [0,T], \\ v(x,0) = u_0(x) & \text{in } \Omega \setminus \Gamma. \end{cases} \tag{5.2}$$

We will show that the functions $(v_\delta)_{\delta>0}$ are equibounded in $C^{0,1}([0,T]; L^\infty(\Omega \setminus \Gamma))$: since it is well known that, for every $T > 0$, $v_\delta \to v$ in $L^\infty([0,T]; L^2(\Omega \setminus \Gamma))$ as $\delta \to 0$ (see for example [3]), the a priori estimate in the $C^{0,1}$-norm (via Ascoli-Arzelà Theorem) will give the thesis of Claim 1. First of all we will show that

$$\|\Delta v_{i,\delta}\|_\infty \leq \|\Delta u_0\|_\infty \qquad \forall \delta > 0, \forall i \in \mathbb{N}. \tag{5.3}$$

We first prove it for $v_{\delta,1}$: if $\varepsilon \geq \|\Delta u_0\|_\infty/\beta$, then $v_1 := u_0 + \varepsilon$ and $v_2 := u_0 - \varepsilon$ satisfy:

$$\begin{cases} \Delta v_1 \leq \beta(v_1 - u_0) & \text{in } \Omega \setminus \Gamma \\ \partial_\nu v_1 = 0 & \text{on } \partial(\Omega \setminus \Gamma), \end{cases} \quad \begin{cases} \Delta v_2 \geq \beta(v_2 - u_0) & \text{in } \Omega \setminus \Gamma \\ \partial_\nu v_2 = 0 & \text{on } \partial(\Omega \setminus \Gamma), \end{cases}$$

that is v_1 and v_2 are a supersolution and a subsolution respectively of the problem solved by $v_{1,\delta}$. This implies that

$$\|v_{1,\delta} - u_0\|_\infty \leq \frac{\|\Delta u_0\|_\infty}{\beta}$$

which is equivalent to

$$\|\Delta v_{1,\delta}\|_\infty \leq \|\Delta u_0\|_\infty.$$

By the same argument we can prove that

$$\|\Delta v_{i,\delta}\|_\infty \leq \|\Delta v_{i-1,\delta}\|_\infty \qquad \forall i \geq i-1$$

and so (5.3) follows by induction on i.

By a standard truncation argument, one can prove also that

$$\|v_{\delta,i}\|_\infty \leq \|u_0\|_\infty \qquad \forall \delta > 0,\ \forall i \in \mathbb{N}. \tag{5.4}$$

Then for s, $t > 0$, using Claim 1, we can estimate

$$\|v_\delta(t) - v_\delta(s)\|_\infty \leq \int_s^t \|(v_\delta)'(\xi)\|_\infty \, d\xi \leq \int_s^t \sup_i \|\Delta v_{\delta,i}\|_\infty \, d\xi \leq \|\Delta u_0\|_\infty |t-s|;$$

this, together with (5.4) concludes the proof of Claim 1.

As a consequence of (5.3), by the well-known Calderon-Zygmund estimates, we get the existence of a constant C such that

$$\|\nabla v_{i,\delta}\|_\infty \leq C\|\Delta v_{i,\delta}\|_\infty \leq C\|\Delta u_0\|_\infty \qquad \forall \delta > 0, \forall i \in \mathbb{N}. \tag{5.5}$$

It is well known (see, for example, [14]) that

$$v(t) \to u_0 \qquad \text{in } L^\infty(\Omega \setminus \Gamma) \text{ as } t \to 0^+;$$

therefore, by our assumption on u_0, for every $0 < c < S$ we can find $T_c > 0$ such that

$$\inf_{x \in \Gamma} |v^+(x,t) - v^-(x,t)| > c \qquad \forall t \in [0, T_c], \tag{5.6}$$

and therefore, by Claim 1, we can choose $\delta_0 > 0$ such that

$$\inf_{x \in \Gamma} |v_\delta^+(t,x) - v_\delta^-(t,x)| > \frac{c}{2} \qquad \forall t \in [0, T_c],\ \forall \delta \leq \delta_0. \tag{5.7}$$

We recall now that, by Theorems 4.3 and 4.7, there exists $\overline{\beta}$ such that, for every function $g \in W^{2,\infty}(\Omega \setminus \Gamma)$ satisfying

$$\|\nabla g\|_\infty \leq C\|\Delta u_0\|_\infty \qquad \inf_{x \in \Gamma} |g^+(x) - g^-(x)| > \frac{c}{2}, \tag{5.8}$$

where C is the constant appearing in (5.5), and for every $\beta \geq \overline{\beta}$, the function $u_{\beta,g}$ solution of (4.4), minimizes the functional $F_{\beta,g}$ over $SBV(\Omega)$.

CLAIM 2. For every $\delta \leq \delta_0 \wedge (\overline{\beta})^{-1}$ the δ-approximate evolution $u_\delta(t)$ (see the end of Section 2 for the definition) coincides in the interval $[0, T_c]$ with the function $v_\delta(t)$.

Clearly it is enough to show that

$$v_{\delta,i} = u_{\delta,i} \qquad \text{for } i = 0, \ldots, \left[\frac{T_c}{\delta}\right],$$

and this can be done by induction on i: indeed for $i = 0$ the identity is trivial, and suppose it true for $i-1$ (for $i \le \left[\frac{T_c}{\delta}\right]$); this means in particular (by (5.5) and by (5.7)) that $g = u_{\delta,i-1}$ satisfies (5.8) and so, being $\frac{1}{\delta} > \overline{\beta}$, we have

$$u_{\delta,i} = u_{\frac{1}{\delta},u_{\delta,i-1}} = v_{\delta,i}.$$

Claim 2 is proved and the thesis of the theorem is now evident. □

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