LOCAL REGULARITY OF VERY WEAK s-HARMONIC FUNCTIONS VIA FRACTIONAL DIFFERENCE QUOTIENTS

ALESSANDRO CARBOTTI, SIMONE CITO, DOMENICO ANGELO LA MANNA, AND DIEGO PALLARA

ABSTRACT. The aim of this paper is to give a new proof that any very weak s-harmonic function u in the unit ball B is smooth. As a first step, we improve the local summability properties of u. Then, we exploit a suitable version of the difference quotient method tailored to get rid of the singularity of the integral kernel and gain Sobolev regularity and local linear estimates of the H^s_{loc} norm of u. Finally, by applying more standard methods, such as elliptic regularity and Schauder estimates, we reach real analyticity of u. Up to the authors' knowledge, the difference quotient techniques are new.

1. INTRODUCTION

This paper comes from our attempt to generalize the by now classical difference quotient method due to L. Nirenberg to nonlocal operators. It has been introduced in [31] and is now presented in all the textbooks dealing with regularity properties of solutions of elliptic equations. After the introduction of weak, or even distributional, solutions of partial differential equations, the problem of their regularity has been tackled by various techniques. Probably the first result in this direction has been the proof of regularity of weakly harmonic functions, obtained in the fifties by Hermann Weyl in [39] and by Renato Caccioppoli, see [29, page 122]. Subsequently, much more general operators have been considered and one of the most fruitful and flexible techniques has proved to be that of difference quotients, which - as it is - appears to be strictly depending on the *local* character of differential operators.

On the other hand, the notion of distributional solution is well established also for equations coming from *nonlocal* operators and the question on the regularity of such solutions is in turn quite natural. One of the first examples of nonlocal operator, and probably the simplest one, is the fractional power of the laplacian and solutions of the equation $(-\Delta)^s u = 0$, $s \in (0,1)$, are called s-harmonic. The class of s-harmonic functions has been broadly studied in the last years. Though they have many features relating them with harmonic functions, see for instance [17,19], s-harmonic functions exhibit a different behaviour in other aspects, due to the nonlocal nature of the fractional Laplacian. Among these facts we mention the local density of s-harmonic functions among smooth functions ([18]), a purely nonlocal phenomenon that has some interesting consequences, such as the failure of the classical Harnack inequality and a quantitative version of an inverse mean value formula in the fractional case. See [11,24] for more precise statements, [14] for more general density results and [1,12] for other applications.

There are several equivalent ways of defining $(-\Delta)^s$, see [27], and the first proof of local regularity of s-harmonic distributions has been obtained via pseudodifferential techniques by R. T. Seeley [33]. See [34] and [35] for more general operators. The Dirichlet problem for the fractional Laplacian

$$\begin{cases} (-\Delta)^s w = f & \text{in} \quad \Omega \\ w = 0 & \text{in} \quad \Omega^c \end{cases} \tag{1.1}$$

has been studied in [28,32] providing basic summability estimates according to the summability of the source f and boundary regularity, respectively. Improved Sobolev regularity for weak solutions of equations more general than (1.1) is proved in [15]. We notice that when Ω is the whole of $\mathbb{R}^N$ more than $H^s(\mathbb{R}^N)$ regularity is available for weak solution of $(-\Delta)^s w = f$, see [26].

It seemed to us that a natural question about the regularity of very weak solutions of nonlocal equations, see (2.4) for the precise definition in the case of the fractional laplacian, is if it is possible to extend the difference quotient method to such solutions. As a first check, we have considered the operator $(-\Delta)^s$, where the difficulty of getting rid of the singularity of the kernel in its definition, see (2.2) below, already appears.

Very weak solutions of (1.1) have been treated in [3], where the authors observe that such solutions, when Ω is a bounded smooth domain and $f \in L^1(\Omega)$, are actually pointwise solutions, that is they are given in terms of the Green operator applied to the source f. See [10], where explicit representation formulae when Ω is a ball are given. We also mention that in the fractional setting regularity estimates for weak solutions of (1.1) are also available. In [21] the author characterises the maximal domain of the fractional Laplacian in $L^p(\Omega)$ as the fractional Sobolev space $W^{2s,p}_{\text{loc}}(\Omega)$ or the Besov space $B^{2s}_{p,2,\text{loc}}(\Omega)$ according to the values of s and p. See also [5,6]. Though not explicitly written, these results are sharp, but our arguments do not rely on the estimates proved there.

We consider very weak solutions of the following problem

$$\begin{cases} (-\Delta)^s w = 0 & \text{in} \quad B \\ w = g & \text{in} \quad B^c, \end{cases} \tag{1.2}$$

where B is the unit ball in $\mathbb{R}^N$ and the outer datum g belongs to the weighted Lebesgue space $L^1_s(\mathbb{R}^N)$ that is defined in (2.1). Notice that if the external datum is assumed to be bounded, one can also exploit the explicit representation of the solution in terms of the fractional Poisson kernel, see [7,10,20].

Our main result is the following

Main Theorem. *Let u be a very weak solution of (1.2). Then*

(1) *(Sobolev regularity) $u \in H^s_{\text{loc}}(B)$ and the estimate*

$$\|u\|_{H^s(B')} \le c(B') \|u\|_{L^1_s(\mathbb{R}^N)}$$

holds for every $B' \Subset B$.

(2) *(Classical regularity)* u *is real analytic in* B *and the estimate*

$$\|D^\iota u\|_{L^\infty(B_{r_0})} \leq c^{|\iota|} \iota! C(R, r_0, N, s) \left(\|u\|_{L^\infty(B_R)} + \|u\|_{L^1_s(\mathbb{R}^N)} \right)$$

holds for any $\iota \in \mathbb{N}_0^N$ *and* $0 < r_0 < R < 1$.

We prove our Main Theorem in several steps. In Theorem 3.1 we prove that the solution u, which is by definition in $L^1(B)$, is actually in $L^2_{\mathrm{loc}}(B)$. This local improvement of summability is done by suitable localisation methods joint with regularity results for the fractional Poisson equation with homogeneous external condition proved in [2, 32]. The second step is done in Theorem 4.1, where we prove that the $L^2_{\mathrm{loc}}(B)$ solution u belongs to the fractional Sobolev space $H^{2s}_{\mathrm{loc}}(B)$, see Section 2 for the relevant definitions. The main point in the proof of this result consists in showing the $H^s_{\mathrm{loc}}(B)$ regularity, as the final step from $H^s_{\mathrm{loc}}(B)$ to $H^{2s}_{\mathrm{loc}}(B)$ follows from L^2 estimates on the operator I_s which is the *carré du champ* of the fractional Laplacian that arises using the relevant fractional Leibniz rule. This kind of estimates, that we also prove for every $p \geq 1$, are different with respect to the one proved in [25]. We exploit a suitable variant of the classical Nirenberg difference quotient method: we introduce a cut-off function that vanishes near the origin and allows us to get rid of the singularity of the kernel and to obtain the fractional Sobolev regularity $H^s_{\mathrm{loc}}(B)$. In the previous literature difference quotient methods have been used in [8, 13] in a different fashion to improve regularity of solutions to more general nonlocal equations. We point out once more that the core of the paper are the linear estimates and the new techniques introduced to prove Claim 1 in the Main Theorem.

In the third step we prove that for a cut-off function η the function $\eta^2 u$ solves the equation $(-\Delta)^s w = f$ in the whole space, with $f \in L^2(\mathbb{R}^N)$, and as a consequence u belongs to $H^{2s}_{\mathrm{loc}}(B)$.

The fourth step relies on fractional De Giorgi estimates proved in [9] that allow us to gain local boundedness of u in B_1 and also local Hölder regularity. The usefulness of those estimates with respect to the previous literature (see e.g. [22, 23]) leans on the fact that the Hölder continuity exponent is quantified. Namely the authors prove that $u \in C^{0,\gamma}_{\mathrm{loc}}(B_1)$ for every $\gamma \in (0, \min\{2s, 1\})$.

In the fifth step we use again that $\eta^2 u$ solves the equation $(-\Delta)^s w = f$ in $\mathbb{R}^N$ but with $f \in C^{0,\gamma}(\mathbb{R}^N)$ and as a consequence u belongs to $C^{\gamma+2s}_{\mathrm{loc}}(B)$ and is s-harmonic in the classical sense in B. To do this we also exploit suitable Hölder continuity properties of the operator I_s.

In the last step we use the explicit Poisson representation formula to give a pointwise expression for u in a small ball. By well known estimates on the derivatives of the Poisson kernel we conclude our proof by proving the real analyticity of u.

Finally, we notice that all our results are stated and proved using the unit ball as reference domain in order to avoid technical issues and to focus on the core of our strategy though the same results also hold true for every bounded and sufficiently smooth domain.

Data Availability Statement Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

2. Notations and Preliminary results

In the whole paper we always assume that $N \geq 2$.

The space $L_s^1(\mathbb{R}^N)$ is the weighted Lebesgue space defined as

$$L_s^1(\mathbb{R}^N) := \left\{ u \in \mathcal{M}(\mathbb{R}^N); \quad \|u\|_{L_s^1(\mathbb{R}^N)} < \infty \right\}, \tag{2.1}$$

where $\mathcal{M}(\mathbb{R}^N)$ denotes the space of Lebesgue measurable functions on $\mathbb{R}^N$ and

$$\|u\|_{L_s^1(\mathbb{R}^N)} := \int_{\mathbb{R}^N} \frac{|u(x)|}{1 + |x|^{N+2s}} dx.$$

It is very easy to check that $L^p(\mathbb{R}^N) \subsetneq L_s^1(\mathbb{R}^N) \subsetneq L_{\text{loc}}^1(\mathbb{R}^N)$ for every $p \geq 1$.

The space $L_s^1(\mathbb{R}^N)$ is a natural setting for very weak s-harmonic functions. Indeed, it encodes local integrability and a growth condition at infinity. This is equivalent to requiring that the nonlocal tail of u

$$\text{Tail}(u; x_0, R) := R^{2s} \int_{B_R^c(x_0)} \frac{|u(x)|}{|x - x_0|^{N+2s}} dx$$

is finite for every $x_0 \in \mathbb{R}^N$ and $R > 0$.

For $u \in C_{\text{loc}}^{2s+\gamma}(B) \cap L_s^1(\mathbb{R}^N)$, $\gamma \in (0, 1)$, the s-Laplacian $(-\Delta)^s u$ is pointwise defined for every $x \in B$ and the following representation formula holds

$$(-\Delta)^s u(x) = C_{N,s} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy = \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{N+2s}} dy, \tag{2.2}$$

where $C_{N,s} := \frac{s2^{2s}\Gamma\left(\frac{N}{2}+s\right)}{\pi^{N/2}\Gamma(1-s)}$ and Γ denotes the Euler Gamma function. This choice of the normalisation constant makes the fractional Laplacian a Fourier multiplier with symbol $|\cdot|^{2s}$ whenever for $u \in L^1(\mathbb{R}^N)$ the Fourier transform $\mathcal{F}$ is defined as $\mathcal{F}u(\xi) = \int_{\mathbb{R}^N} u(x)e^{-2\pi i x \cdot \xi} dx$.

Notice that if u only belongs to $L_s^1(\mathbb{R}^N)$, formula (2.2) still holds true by taking the integrals in the Cauchy principal value sense.

For $s \in (0, 1)$, $1 \leq p < \infty$ and $\Omega \subseteq \mathbb{R}^N$ we define the fractional Sobolev space $W^{s,p}(\Omega)$ as

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) : \quad [u]_{W^{s,p}(\Omega)} < \infty \right\},$$

where

$$[u]_{W^{s,p}(\Omega)} := \left(\int_\Omega dx \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dy \right)^{1/p},$$

endowed with the norm $\|\cdot\|_{W^{s,p}(\Omega)} := \left(\|\cdot\|_{L^p(\Omega)}^p + [\cdot]_{W^{s,p}(\Omega)}^p \right)^{1/p}$.

When $p = \infty$ any $f \in W^{s,\infty}(\Omega)$ has a representative $\tilde{f} \in C^s(\overline{\Omega})$.

As usual, when $p = 2$ we use the notation $H^s(\Omega)$ to indicate the Hilbert space $W^{s,2}(\Omega)$. See [16] for a gentle introduction to the fractional Sobolev spaces.

Let us also define higher order fractional Sobolev spaces, confining to the non-integer case: for $\sigma \in (1, \infty)$, $\sigma = k + s$, $k \in \mathbb{N}$, $s \in (0, 1)$ and $1 \leq p < \infty$, the fractional Sobolev space $W^{\sigma, p}(\Omega)$ is defined as follows:

$$W^{\sigma, p}(\Omega) = \left\{ u \in W^{k, p}(\Omega) : \quad D^\alpha u \in W^{s, p}(\Omega), \quad \forall \, \alpha \in \mathbb{N}_0^N, \ |\alpha| \leq k \right\}.$$

Set $\mathcal{Q}_A := (A \times A) \cup (A \times A^c) \cup (A^c \times A)$ for every open set A. We define

$$\mathbb{H}^s(B) = \left\{ u \in L^2(B) : \quad [u]_{\mathbb{H}^s(B)} < \infty \right\}, \tag{2.3}$$

where

$$[u]_{\mathbb{H}^s(B)} := \left(\iint_{\mathcal{Q}_B} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{1/2}.$$

We say that $u \in \mathbb{H}^s(B)$ is a weak solution for (1.2) if for every $\varphi \in \mathbb{H}_0^s(B) = H_0^s(B) = \overline{C_c^\infty(B)}^{\|\cdot\|_{H^s(B)}}$ it holds that

$$\begin{cases} \displaystyle\int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} dy = 0 \\ u = g \quad \text{in} \quad B^c. \end{cases}$$

Notice that for $u \in \mathbb{H}^s(B)$ and $\varphi \in \mathbb{H}_0^s(B)$ the definition is well posed. Indeed, let $A \Subset B$, $A \supset \operatorname{supp} \varphi$

$$\left| \int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} dy \right| \leq \iint_{\mathcal{Q}_A} \frac{|u(x) - u(y)||\varphi(x) - \varphi(y)|}{|x - y|^{N+2s}} dx dy$$

$$\leq [u]_{\mathbb{H}^s(A)} [\varphi]_{\mathbb{H}^s(A)}.$$

We notice that if $g \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $u \in \mathbb{H}^s(B) \cap L^\infty(\mathbb{R}^N)$ is a weak solution of (1.2) then u is also a solution in the viscosity sense for (1.2), as proved in [36, Theorem 1] also in the inhomogeneous case for continuous sources.

For $s \in (0, 1)$ we also introduce the space

$$L_s^\infty(\mathbb{R}^N) := \left\{ u \in L^\infty(\mathbb{R}^N); \quad \sup_{x \in \mathbb{R}^N} (1 + |x|^{N+2s}) |u(x)| < \infty \right\},$$

equipped with the norm

$$\|u\|_{L_s^\infty(\mathbb{R}^N)} := \sup_{x \in \mathbb{R}^N} (1 + |x|^{N+2s}) |u(x)|.$$

We say that $u \in L_s^1(\mathbb{R}^N)$ is a very weak solution of (1.2) if, for every φ compactly supported in B such that $(-\Delta)^s \varphi \in L_s^\infty(\mathbb{R}^N)$, it holds

$$\begin{cases} \displaystyle\int_{\mathbb{R}^N} u(-\Delta)^s \varphi \, dx = 0 \\ u = g \quad \text{in} \quad B^c. \end{cases} \tag{2.4}$$

Notice that the chosen class of test functions is not empty. Indeed, let $\varphi \in C_c^{2s+\gamma}(B)$ for some $\gamma > 0$. We have

$$\left| \int_{\mathbb{R}^N} u(-\Delta)^s \varphi \, dx \right| \leq \int_{B_2} |u(-\Delta)^s \varphi| dx + \int_{B_2^c} |u(-\Delta)^s \varphi| dx$$

$$\leq \|u\|_{L^1(B_2)}\|\varphi\|_{C^{2s+\gamma}(B)} + \int_{B_2^c}\frac{|u(x)|}{1+|x|^{N+2s}}dx\int_B\frac{1+|x|^{N+2s}}{|x-y|^{N+2s}}|\varphi(y)|dy$$

$$\leq \|u\|_{L^1(B_2)}\|\varphi\|_{C^{2s+\gamma}(B)} + \int_{B_2^c}\frac{|u(x)|}{1+|x|^{N+2s}}dx\int_B\left(1+\frac{1+|y|}{|x-y|}\right)^{N+2s}|\varphi(y)|dy$$

$$\leq \|u\|_{L^1(B_2)}\|\varphi\|_{C^{2s+\gamma}(B)} + 3^{N+2s}\|u\|_{L_s^1(\mathbb{R}^N)}\|\varphi\|_{L^1(B)}$$

$$\leq C_{N,s}\|\varphi\|_{C^{2s+\gamma}(B)}\|u\|_{L_s^1(\mathbb{R}^N)},$$

and then φ is a test function.

We notice that a weak solution is a very weak solution. Indeed, using the symmetry of the double integral in x and y

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dxdy = \int_{\mathbb{R}^N}u(x)dx\int_{\mathbb{R}^N}\frac{\varphi(x)-\varphi(y)}{|x-y|^{N+2s}}dy$$

for every $\varphi \in C_c^\infty(B)$.

Now, we recall some useful results. From now on, for $r \geq 1$ we denote with $r' := \frac{r}{r-1}$ the Hölder conjugate of r, and for $\tau > 0$ such that $r\tau < N$ we denote with $r_\tau^* := \frac{Nr}{N-r\tau}$ the Sobolev conjugate of r with respect to τ. First, we state the Sobolev embedding theorem.

Proposition 2.1. [4, Theorem 7.63] *Let $\Omega \subseteq \mathbb{R}^N$ an open and smooth set and let $k, h \geq 0$, $p \geq 1$. If $u \in W^{k,p}(\Omega)$, $k > h$ and $N > (k-h)p$ then the following continuity estimate holds*

$$\|u\|_{W^{h,q}(\Omega)} \leq C\,\|u\|_{W^{k,p}(\Omega)}$$

- *for every $1 \leq q \leq \frac{Np}{N-(k-h)p}$ if Ω has finite measure*
- *for every $p \leq q \leq \frac{Np}{N-(k-h)p}$ if Ω has infinite measure.*

We notice that if $k = h \in \mathbb{N}_0$ and if Ω has finite measure then the statement of Proposition 2.1 still holds true, but if $k = h \in (0, \infty) \setminus \mathbb{N}$ then the embedding may fail in general even if Ω is a ball, see [30].

The following results give regularity properties of the weak solutions of (1.1) under suitable assumptions on f.

Theorem 2.2. [2, Corollary 1.7] *Let $N \geq 2$, $\Omega \subset \mathbb{R}^N$ a bounded C^2 domain, $s \in (0,1)$ and let u be the unique solution of*

$$\begin{cases}(-\Delta)^s w = f & in \quad \Omega \\ w = 0 & in \quad \Omega^c\end{cases}$$

with $f \in L^m(\Omega)$.

   (i) *If $1 \leq m < \frac{N}{s}$, then for all $1 < p < m_s^*$ there exists $C > 0$ such that*

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} \leq C\,\|f\|_{L^m(\Omega)}.$$

   (ii) *If $m > \frac{N}{s}$, then for all $1 < p < \infty$ there exists $C > 0$ such that*

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} \leq C\,\|f\|_{L^m(\Omega)}.$$

Theorem 2.3. [32, Proposition 1.4] *Let $s \in (0,1)$, $N > 2s$, $\Omega \subset \mathbb{R}^N$ a bounded $C^{1,1}$ domain, $f \in C(\overline{\Omega})$ and let u be the weak solution of*

$$\begin{cases} (-\Delta)^s w = f & in \quad \Omega \\ w = 0 & in \quad \Omega^c. \end{cases}$$

(i) *For each $1 \le r < \left(\frac{N}{2s}\right)'$ there exists $C = C(n, r, s, |\Omega|) > 0$ such that*

$$\|u\|_{L^r(\Omega)} \le C \|f\|_{L^1(\Omega)}.$$

(ii) *Let $1 < p < \frac{N}{2s}$ and $p_{2s}^* = \frac{Np}{N-2sp}$. Then there exists $C = C(n,s,p) > 0$ such that for any $1 \le q \le p_{2s}^*$*

$$\|u\|_{L^q(\Omega)} \le C \|f\|_{L^p(\Omega)}.$$

(iii) *Let $\frac{N}{2s} < p < \infty$. Then there exists $C = C(n,s,p,\Omega) > 0$ such that*

$$\|u\|_{C^\beta(\mathbb{R}^N)} \le C \|f\|_{L^p(\Omega)},$$

where $\beta := \min\left\{s, 2s - \frac{N}{p}\right\}$.

3. Improvement of summability

Now we are ready to state and prove the following

Theorem 3.1. *Let $u \in L_s^1(\mathbb{R}^N)$ a very weak solution of (1.2). Then $u \in L_{\mathrm{loc}}^2(B)$.*

Proof. To ease the presentation, we divide the proof in three steps.

Step one: first summability improvement. In this first step we prove

$$u \in L_{\mathrm{loc}}^r(B) \quad \text{for all} \quad r < \frac{N}{N-s}. \tag{3.1}$$

Let $p > \frac{N}{s}$ and $\psi \in C^\infty(B) \cap C(\overline{B})$. Now, let v be the unique solution of the Dirichlet problem

$$\begin{cases} (-\Delta)^s w = \psi & in \quad B_{1-\delta} \\ w = 0 & in \quad B_{1-\delta}^c. \end{cases} \tag{3.2}$$

for some $\delta > 0$ sufficiently small to be conveniently chosen in the sequel. Now let $\eta \in C_c^\infty(B)$ be such that $\eta = 1$ in $B_{1-4\delta}$, $\eta = 0$ in $B_{1-2\delta}^c$ and $|\nabla \eta| \le \frac{1}{\delta}$. Notice that by Theorem 2.3 we have that $v \in C^s(\mathbb{R}^N)$, and this easily implies that $(-\Delta)^s(\eta^2 v) \in L_s^\infty(\mathbb{R}^N)$ and $\operatorname{supp} \eta^2 v \subset B_{1-2\delta}$. Indeed, for $|x| > 2$ it holds

$$|(-\Delta)^s(\eta^2 v)(x)| \le \left| \int_B \frac{\eta^2(y)v(y)}{|x-y|^{N+2s}} dy \right| \le \frac{3^{N+2s} \|v\|_{L^1(B)}}{1+|x|^{N+2s}},$$

while the boundedness of $(-\Delta)^s(\eta^2 v)$ in B_2 is an immediate consequence of Theorem 2.3.

Therefore, we can use $\varphi = \eta^2 v$ as a test function in the definition of very weak solution (2.4). Then

$$0 = \int_{\mathbb{R}^N} \eta^2 u \psi dx + 2 \int_{\mathbb{R}^N} uv\eta(-\Delta)^s \eta dx - \int_{\mathbb{R}^N} u\eta I_s(\eta, v) dx - \int_{\mathbb{R}^N} u I_s(\eta, \eta v) dx \tag{3.3}$$

where for any f_1, f_2 measurable we have set

$$I_s(f_1, f_2)(x) := C_{N,s} \int_{\mathbb{R}^N} \frac{(f_1(x) - f_1(y))(f_2(x) - f_2(y))}{|x - y|^{N+2s}} dy, \tag{3.4}$$

and the constant $C_{N,s}$ is that one in the definition of $(-\Delta)^s$.

Then

$$\left| \int_{\mathbb{R}^N} \eta^2 u \psi dx \right| \leq \int_{\mathbb{R}^N} |u I_s(\eta, \eta v)| dx + 2 \int_{\mathbb{R}^N} |u v \eta (-\Delta)^s \eta| dx + \int_{\mathbb{R}^N} |u \eta I_s(\eta, v)| dx \tag{3.5}$$

$$=: A_1 + A_2 + A_3.$$

We start by estimating the term A_1. Then

$$\int_{\mathbb{R}^N} |u I_s(\eta, \eta v)| dx \leq \|u\|_{L_s^1(\mathbb{R}^N)} \|I_s(\eta, \eta v)\|_{L_s^\infty(\mathbb{R}^N)} \leq C(\delta) \|u\|_{L_s^1(\mathbb{R}^N)} \|\eta v\|_{C^s(\mathbb{R}^N)} \tag{3.6}$$

$$\leq C(\delta) \|u\|_{L_s^1(\mathbb{R}^N)} \|v\|_{C^s(\mathbb{R}^N)} \leq C(\delta) \|u\|_{L_s^1(\mathbb{R}^N)} \|\psi\|_{L^p(B)},$$

where the last inequality exploits item (iii) in Theorem 2.3, which holds true since $p > \frac{N}{s}$. For the second inequality we notice that for any $x \in \mathbb{R}^N$ and $y \in B_{1-2\delta}$ we have

$$(1 + |x|^{N+2s})|\eta(x) - \eta(y)| \leq C(\delta)\chi_{B_{1-\delta}}(x) \|\nabla \eta\|_{L^\infty(\mathbb{R}^N)} |x - y|$$

$$+ 2^{N+2s-1}\chi_{B_{1-\delta}^c}(x)(1 + |x - y|^{N+2s} + |y|^{N+2s})$$

$$\leq C(\delta)\chi_{B_{1-\delta}}(x) \|\nabla \eta\|_{L^\infty(\mathbb{R}^N)} |x - y| \tag{3.7}$$

$$+ C(N, s, \delta)\chi_{B_{1-\delta}^c}(x)|x - y|^{N+2s}$$

while for any $x \in \mathbb{R}^N$ and $y \in B_{1-2\delta}^c$

$$(1 + |x|^{N+2s})|\eta(x) - \eta(y)| \leq C(\delta)\chi_{B_{1-\delta}}(x).$$

Therefore, for any $x \in \mathbb{R}^N$

$$(1 + |x|^{N+2s}) |I_s(\eta, \eta v)(x)| \leq (1 + |x|^{N+2s}) \int_{B_{1-2\delta}} \frac{|\eta(x) - \eta(y)||\eta(x)v(x) - \eta(y)v(y)|}{|x - y|^{N+2s}} dy \tag{3.8}$$

$$+ (1 + |x|^{N+2s}) \int_{B_{1-2\delta}^c} \frac{|\eta(x) - \eta(y)||\eta(x)v(x) - \eta(y)v(y)|}{|x - y|^{N+2s}} dy$$

$$\leq (1 + |x|^{N+2s}) \int_{B_{1-2\delta}} \frac{|\eta(x) - \eta(y)||\eta(x)v(x) - \eta(y)v(y)|}{|x - y|^{N+2s}} dy$$

$$+ (1 + |x|^{N+2s})|v(x)| \int_{B_{1-2\delta}^c} \frac{(\eta(x) - \eta(y))^2}{|x - y|^{N+2s}} dy$$

$$\leq \chi_{B_{1-\delta}}(x) \|\eta v\|_{C^s(\mathbb{R}^N)} \int_{B_{1-2\delta}} \frac{dy}{|x - y|^{N+s-1}}$$

$$+ \chi_{B_{1-2\delta}}(x)C'(\delta)|v(x)| \left(\int_{B_\delta^c(x)} \frac{dy}{|x - y|^{N+2s}} + \int_{B_{2-3\delta}(x)} \frac{dy}{|x - y|^{N+2s-2}} \right)$$

$$+ C(N, s, \delta) \|\eta v\|_{L^\infty(\mathbb{R}^N)} \chi_{B_{1-\delta}^c}(x)$$

$$\leq C(N, s, \delta) \|v\|_{C^s(\mathbb{R}^N)} \leq C(N, s, \delta) \|\psi\|_{L^p(B)},$$

where in the third inequality we have used (3.7), the equality

$$|\eta(x) - \eta(y)||\eta(x)v(x) - \eta(y)v(y)| = |v(x)||\eta(x) - \eta(y)|^2$$

that holds for any $x \in B_{1-\delta}$ and $y \in B_{1-2\delta}^c$ and also that the integral term in the fourth line of (3.8) is nonzero if and only if $x \in B_{1-2\delta}$ and it can be split in the sum of two integrals over $B_{1-\delta}^c$ and $B_{1-\delta} \setminus B_{1-2\delta}$.

For A_2 we have

$$\int_{\mathbb{R}^N} |uv\eta(-\Delta)^s\eta|dx \leq C \|u\|_{L^1(B_{1-2\delta})} \|v\|_{L^\infty(B_{1-2\delta})} \leq C \|u\|_{L^1(B_{1-2\delta})} \|\psi\|_{L^p(B)}, \qquad (3.9)$$

where we used again Theorem 2.3.

To estimate A_3 we notice that

$$\int_{\mathbb{R}^N} |u\eta I_s(\eta, v)|dx = \int_{B_{1-2\delta}} |u\eta I_s(\eta, v)|dx \leq \|u\|_{L^1(B_{1-2\delta})} \|I_s(\eta, v)\|_{L^\infty(B_{1-2\delta})}$$

and for almost any $x \in B_{1-2\delta}$, if we split A_3 into the sum of the integrals over $B_{1-\delta}$ and $B_{1-\delta}^c$ we have

$$|I_s(\eta, v)(x)| \leq C \|v\|_{C^s(\mathbb{R}^N)} \left(\int_{B_{2-3\delta}(x)} \frac{dy}{|x-y|^{N+s-1}} + \int_{B_\delta^c(x)} \frac{dy}{|x-y|^{N+2s}} \right) \leq C \|\psi\|_{L^p(B)},$$

where we have used again Theorem 2.3(iii), that holds true for $p > \frac{N}{s}$ with $\beta = s$. Now we have

$$\left| \int_{\mathbb{R}^N} u\eta^2\psi dx \right| \leq C \|u\|_{L_s^1(\mathbb{R}^N)} \left(\|v\|_{W^{s,p}(\mathbb{R}^N)} + \|v\|_{C^s(\mathbb{R}^N)} \right) \leq C \|u\|_{L_s^1(\mathbb{R}^N)} \|\psi\|_{L^p(B)} \qquad (3.10)$$

for all $\psi \in C^\infty(B) \cap C(\overline{B})$, with $p > \frac{N}{s}$. By the density of $C^\infty(B) \cap C(\overline{B})$ in $L^p(B)$ we have that (3.10) holds for all $\psi \in L^p(B)$, which implies that $\eta^2 u \in L^{p'}(B)$, hence $\eta^2 u \in L^r(B)$ for all $r < \frac{N}{N-s}$. The arbitrariness of δ gives the claim.

Step two: higher summability. Our next goal is to show

$$u \in L_{\text{loc}}^r(B) \quad \text{for} \quad r \in \left(\frac{N}{N-s}, \frac{N}{N-2s} \right) \qquad (3.11)$$

In order to improve the summability of the solution u we still use a duality argument, but in a bit different way. Let $\psi \in C^\infty(B) \cap C(\overline{B})$, take $m \in \left(\frac{N}{(1+\alpha)s}, \frac{N}{s} \right)$ for some $\alpha \in (0,1)$ and let v, η as before, where now δ is the double of the previous one. Since we know that $u \in L_{\text{loc}}^r(B)$ for all $r < \frac{N}{N-s}$, let $p' \in (\frac{N}{N-\alpha s}, \frac{N}{N-s})$. It is very easy to check that the function $v\eta^2$ is admissible as test function in definition (2.4). We estimate again the three terms appearing in (3.5), but this time we can use the higher summability of u proved in Step one to estimate

$$
\begin{aligned}
A_2 &\leq C \|u\|_{L^{p'}(B_{1-2\delta})} \|v\|_{L^p(B_{1-2\delta})} \leq C \|u\|_{L^{p'}(B_{1-2\delta})} \|v\|_{W^{s,q}(B_{1-2\delta})} \\
&\leq C \|u\|_{L^{p'}(B_{1-2\delta})} \|\psi\|_{L^m(B)},
\end{aligned}
\qquad (3.12)
$$

where the second inequality exploits Sobolev embedding Theorem for $q \geq \frac{Np}{N+sp} \geq \frac{N}{2s}$ and the third inequality exploits Theorem 2.2 and it holds for all $q \in (1, m_s^*)$ (recall that $m_s^* = \frac{Nm}{N-ms}$) whenever $1 \leq m < \frac{N}{s}$.

Now we estimate again the term A_1 in (3.5)

$$
\begin{aligned}
A_1 \leq & \int_{B_{1-\delta}} |u(x)| dx \int_{B_{1-2\delta}^c} \frac{|\eta(x)-\eta(y)||\eta(x)v(x)-\eta(y)v(y)|}{|x-y|^{N+2s}} dy \\
& + \int_{B_{1-\delta}} |u(x)| dx \int_{B_{1-2\delta}} \frac{|\eta(x)-\eta(y)||\eta(x)v(x)-\eta(y)v(y)|}{|x-y|^{N+2s}} dy \\
& + \int_{B_{1-\delta}^c} |u(x)| dx \int_{\mathbb{R}^N} \frac{|\eta(x)-\eta(y)||\eta(x)v(x)-\eta(y)v(y)|}{|x-y|^{N+2s}} dy \\
=: & B_1 + B_2 + B_3.
\end{aligned}
\tag{3.13}
$$

To bound B_1 we first observe that since $\eta(y)=0$ for $y \in B_{1-2\delta}^c$ we have

$$
\begin{aligned}
B_1 = & \int_{B_{1-2\delta}} |u(x)||v(x)|\eta^2(x) dx \int_{B_{1-2\delta}^c} \frac{dy}{|x-y|^{N+2s}}, \\
= & \int_{B_{1-2\delta}} |u(x)||v(x)| dx \int_{B_{1-2\delta}^c} \frac{(\eta(x)-\eta(y))^2}{|x-y|^{N+2s}} dy \\
\leq & \int_{B_{1-2\delta}} |u(x)||v(x)| dx \int_{B_{1-\delta}^c} \frac{dy}{|x-y|^{N+2s}} \\
& + \int_{B_{1-2\delta}} |u(x)||v(x)| dx \int_{B_{1-\delta} \setminus B_{1-2\delta}} \frac{(\eta(x)-\eta(y))^2}{|x-y|^{N+2s}} dy \\
\leq & C(\delta) \left(\|u\|_{L^{p'}(B_{1-2\delta})} \|v\|_{L^p(B_{1-2\delta})} + \int_{B_{1-2\delta}} |u(x)||v(x)| dx \int_{B_{1-\delta}} \frac{1}{|x-y|^{N+2s-2}} dy \right) \\
\leq & C(\delta) \|u\|_{L^{p'}(B_{1-2\delta})} \|\psi\|_{L^m(B)}
\end{aligned}
\tag{3.14}
$$

Analogously for B_3 we use that $\eta(x)=0$ for $x \in B_{1-\delta}^c$ and that $(1+|x|^{N+2s}) \leq C_\delta |x-y|^{N+2s}$ for any $x \in B_{1-\delta}^c$ and $y \in B_{1-2\delta}$ to find

$$
\begin{aligned}
B_3 \leq & \int_{B_{1-\delta}^c} \frac{|u(x)|}{1+|x|^{N+2s}} dx \int_{B_{1-2\delta}} (1+|x|^{N+2s}) \frac{|v(y)|\eta^2(y)}{|x-y|^{N+2s}} dy \\
& \leq C_\delta \|u\|_{L_s^1(\mathbb{R}^N)} \|v\|_{L^1(B_{1-2\delta})} \leq C_\delta \|u\|_{L_s^1(\mathbb{R}^N)} \|\psi\|_{L^m(B)}
\end{aligned}
\tag{3.15}
$$

and again the Sobolev embedding Theorem holds true for any $q \geq \frac{N}{2s}$. To bound B_2 we use the Hölder inequality in the following way:

$$
\begin{aligned}
B_2 \leq & \int_{B_{1-\delta}} |u(x)| dx \int_{B_{1-2\delta}} \frac{|\eta(x)-\eta(y)||\eta(x)v(x)-\eta(y)v(y)|}{|x-y|^{N+2s}} dy \\
\leq & \|u\|_{L^{p'}(B_{1-\delta})} \left(\int_{B_{1-\delta}} \left(\int_{B_{1-\delta}} \frac{|\eta(x)v(x)-\eta(y)v(y)|}{|x-y|^{N+2s-1}} dy \right)^p dx \right)^{1/p} \\
\leq & \|u\|_{L^{p'}(B_{1-\delta})} \left(\int_{B_{1-\delta}} \int_{B_{1-\delta}} \frac{|\eta(x)v(x)-\eta(y)v(y)|^p}{|x-y|^{N+sp}} dy \cdot \left(\int_{B_{1-\delta}} \frac{dy}{|x-y|^{N-(1-s)p'}} \right)^{p-1} dx \right)^{1/p} \\
\leq & C\|u\|_{L^{p'}(B_{1-\delta})} \|\eta v\|_{W^{s,p}(\mathbb{R}^N)} \leq C\|u\|_{L^{p'}(B_{1-\delta})} \|\psi\|_{L^m(B)},
\end{aligned}
\tag{3.16}
$$

provided that $p < \frac{Nm}{N-sm}$. Since $p' > \frac{N}{N-\alpha s}$ we get $p < \frac{N}{\alpha s}$, hence $\|v\|_{W^{s,p}(B)} \leq \|\psi\|_{L^m(B)}$ if $m > \frac{N}{(1+\alpha)s}$. Inequalities (3.14),(3.15),(3.16) give

$$|A_2| \leq C(\delta)\|\psi\|_{L^m(B)}\|u\|_{L^{p'}(B_{1-2\delta})}. \tag{3.17}$$

To estimate A_3 we proceed as in the previous estimate:

$$A_3 = \int_{B_{1-2\delta}} |u\eta I_s(\eta, v)|dx \leq \int_{B_{1-2\delta}} |\eta(x)u(x)| \int_{\mathbb{R}^N} \frac{|\eta(x)-\eta(y)||v(x)-v(y)|}{|x-y|^{N+2s}} dy\, dx.$$

We split the integral on $\mathbb{R}^N$ in $B_{1-\delta}$ and $B_{1-\delta}^c$ and we use again the Hölder inequality and Theorem 2.2 to infer

$$\int_{B_{1-2\delta}} |\eta(x)u(x)| \int_{B_{1-\delta}} \frac{|\eta(x)-\eta(y)||v(x)-v(y)|}{|x-y|^{N+2s}} dy\, dx$$

$$\leq \left(\int_B |\eta u|^{p'} dx\right)^{1/p'} \left(\int_{B_{1-\delta}} \left(\int_{B_{1-\delta}} \frac{|v(x)-v(y)|}{|x-y|^{N+2s-1}} dy\right)^p dx\right)^{1/p} \tag{3.18}$$

$$\leq C\|\eta u\|_{L^{p'}(B)} \left(\int_{B_{1-\delta}} dx \int_{B_{1-\delta}} \frac{|v(x)-v(y)|^p}{|x-y|^{N+sp}} dy\right)^{1/p}$$

$$\leq C\|\eta u\|_{L^{p'}(B)} \|v\|_{W^{s,p}(\mathbb{R}^N)} \leq C\|\eta u\|_{L^{p'}(B)} \|\psi\|_{L^m(B)}.$$

Concerning the second integral, since $v(y) = \eta(y) = 0$ for $y \in B_{1-\delta}^c$, we have

$$\int_B |\eta(x)u(x)|dx \int_{B_{1-\delta}^c} \frac{|\eta(x)-\eta(y)||v(x)-v(y)|}{|x-y|^{N+2s}} dy$$

$$= \int_{B_{1-2\delta}} |\eta(x)u(x)v(x)|dx \int_{B_{1-\delta}^c} \frac{dy}{|x-y|^{N+2s}} \tag{3.19}$$

$$\leq C(\delta)\|\eta u\|_{L^{p'}(B)}\|v\|_{L^p(B)} \leq C(\delta)\|\eta u\|_{L^{p'}(B)}\|\psi\|_{L^m(B)}$$

provided that $m > \frac{N}{(1+\alpha)s}$. Inequalities (3.18) and (3.19) give $|A_3| \leq C(\delta)\|\eta u\|_{L^{p'}(B)}\|\psi\|_{L^m(B)}$. Hence, using the latter and inequalities (3.13), (3.17) we deduce

$$\left| \int_B u\eta^2 \psi\, dx \right| \leq C(\delta)\|\eta u\|_{L^{p'}(B)}\|\psi\|_{L^m(B)}$$

for all $\psi \in C^\infty(B) \cap L^\infty(B)$. By density the inequality holds for $\psi \in L^m(B)$ and hence $\eta^2 u \in L^{m'}(B)$ with $m' \in \left(\frac{N}{N-s}, \frac{N}{N-(1+\alpha)s}\right)$. Since this is true for any $\alpha \in (0,1)$ we get our claim.

Step three. We finally show that $u \in L^p_{\text{loc}}(B)$ for $p < \frac{N}{s}$.
We first prove recursively that

$$u \in L^r_{\text{loc}}(B) \quad \text{for} \quad r < \frac{N}{N-ks}$$

for all $k \in \mathbb{N}$ such that $k < \frac{N}{s}$. We notice that we already proved the claim for $k = 1, 2$. Hence let us assume that $u \in L^{p'}_{\text{loc}}(B)$ for $p' \in \left(\frac{N}{N-(k-\alpha)s}, \frac{N}{N-ks}\right)$ for some $\alpha \in (0,1)$. Fix $\delta > 0$ to be chosen again as the double of the one selected in the previous step, let $\psi \in C^\infty(B) \cap L^\infty(B)$, take $m \in \left(\frac{N}{(k+1-\alpha)s}, \frac{N}{ks}\right)$, and let v be the unique solution of (1.1). For a cut-off function η supported in $B_{1-2\delta}$ we use $\eta^2 v$ as a test function in (2.4) to find again

$$\left| \int_B \eta^2 u\psi dx \right| \leq A_1 + A_2 + A_3,$$

with A_1, A_2, A_3 defined as in (3.5). As before, to estimate A_1 we split it in three terms:

$$A_1 \le \int_{B_{1-\delta}} |u(x)| dx \int_{B_{1-2\delta}^c} \frac{|\eta(x) - \eta(y)||\eta(x)v(x) - \eta(y)v(y)|}{|x-y|^{N+2s}} dy \qquad (3.20)$$

$$+ \int_{B_{1-\delta}} |u(x)| dx \int_{B_{1-2\delta}} \frac{|\eta(x) - \eta(y)||\eta(x)v(x) - \eta(y)v(y)|}{|x-y|^{N+2s}} dy$$

$$+ \int_{B_{1-\delta}^c} |u(x)| dx \int_{\mathbb{R}^N} \frac{|\eta(x) - \eta(y)||\eta(x)v(x) - \eta(y)v(y)|}{|x-y|^{N+2s}} dy$$

$$=: B_1 + B_2 + B_3$$

The same argument used to bound B_1 and B_3 in Step two provides

$$|B_1| \le C(\delta)\|u\|_{L^{p'}(B_{1-2\delta})}\|v\|_{W^{s,p}(\mathbb{R}^N)}$$

and

$$|B_3| \le C(\delta)\|u\|_{L^1_s(\mathbb{R}^N)}\|v\|_{W^{s,p}(\mathbb{R}^N)}$$

where at this stage $p \in \left(\frac{N}{ks}, \frac{N}{(k-\alpha)s}\right)$. For B_2 we use Hölder inequality to have

$$|B_2| \le \int_{B_{1-\delta}} |u(x)| dx \int_{B_{1-2\delta}} \frac{|\eta(x) - \eta(y)||\eta(x)v(x) - \eta(y)v(y)|}{|x-y|^{N+2s}} dy$$

$$\le \|u\|_{L^{p'}(B_{1-\delta})}\left(\int_{B_{1-\delta}}\left(\int_{B_{1-\delta}} \frac{|\eta(x)v(x) - \eta(y)v(y)|}{|x-y|^{N+2s-1}} dy\right)^p dx\right)^{1/p}$$

$$\le \|u\|_{L^{p'}(B_{1-\delta})}\left(\int_{B_{1-\delta}}\int_{B_{1-\delta}} \frac{|\eta(x)v(x) - \eta(y)v(y)|^p}{|x-y|^{N+sp}} dy \cdot \left(\int_{B_{1-\delta}} \frac{dy}{|x-y|^{N+(s-1)p'}}\right)^{p-1} dx\right)^{1/p}$$

$$\le C\|u\|_{L^{p'}(B_{1-\delta})}\|\eta v\|_{W^{s,p}(\mathbb{R}^N)}$$

Since $p > 1$ we have

$$|A_2| \le C(\delta)\|u\|_{L^{p'}(B_{1-\delta})}\|v\|_{W^{s,p}(\mathbb{R}^N)}$$

and using Theorem 2.2 we get

$$\|v\|_{W^{s,p}(\mathbb{R}^N)} \le C\|\psi\|_{L^m(B)}$$

whenever

$$p < \frac{Nm}{N-sm} \qquad \text{i.e.,} \qquad m > \frac{Np}{N+sp}.$$

Since $p < \frac{N}{(k-\alpha)s}$ we get

$$|A_2| \le C(\delta)\|u\|_{L^{p'}(B_{1-\delta})}\|\psi\|_{L^m(B)}$$

if $m > \frac{N}{(k+1-\alpha)s}$. The estimate for A_3 follows from the same argument and gives

$$|A_3| \le C(\delta)\|u\|_{L^{p'}(B_{1-\delta})}\|\psi\|_{L^m(B)}.$$

Thus we arrive at

$$\left|\int_B \eta^2 u\psi \, dx\right| \le C(\delta)\|u\|_{L^{p'}(B_{1-\delta})}\|\psi\|_{L^m(B)}$$

for any $m > \frac{N}{(k+1-\alpha)s}$. Using again a duality argument and since the latter is true for all $\alpha > 0$ we get

$$u \in L^{m'}_{\text{loc}}(B)$$

for any $m' < \frac{N}{N-(k+1)s}$.

Hence we now run this argument k_0 times, where $k_0 := \max\{d \in \mathbb{N}; \, d \leq \overline{k}\}$ and $\overline{k} := \frac{N}{s} - 1$ to find

$$u \in L^r_{\text{loc}}(B)$$

for all $r < \frac{N}{N-k_0s} \leq \frac{N}{s}$. Since in particular $2 < \frac{2}{s} \leq \frac{N}{s}$ by our assumptions in Section 2 the proof is complete. □

Remark 3.2. In each step k of the proof of Theorem 3.1 we choose $\delta_k > 0$ such that $\delta_k < \frac{1}{4}$ and $\delta_k = 2\delta_{k-1} < 1$, for $k \in \{1, \ldots, k_0\}$ and these conditions imply that in Step one we have to fix $\delta_1 := \delta < \frac{1}{2^{k_0+2}}$.

Remark 3.3. We notice that from estimates (3.14), (3.15), (3.16) and (3.18), we deduce that for any $w \in L^1_s(\mathbb{R}^N) \cap W^{s,p}_{\text{loc}}(B)$ for some $1 < p < \infty$, by definition $\eta w \in W^{s,p}(\mathbb{R}^N)$ for any $\eta \in C^\infty_c(B)$ cut-off function and, if $\operatorname{supp}\eta = B_{1-2\delta} \Subset B_{1-\delta} \Subset B$ there exists C, that depends only on δ, s, N, p but independent of w, such that

$$\|I_s(\eta, \eta w)\|_{L^p(\mathbb{R}^N)} \leq C \|\eta w\|_{W^{s,p}(\mathbb{R}^N)}. \tag{3.21}$$

Moreover, for any $x \in \mathbb{R}^N$ we have

$$
\begin{aligned}
|\eta(x) I_s(\eta, w)(x)| &\leq \eta(x) \int_{B_{1-\delta}} \frac{|\eta(x) - \eta(y)||w(x) - w(y)|}{|x-y|^{N+2s}} dy \\
&\quad + \eta(x) \int_{B^c_{1-\delta}} \frac{|\eta(x) - \eta(y)||w(x) - w(y)|}{|x-y|^{N+2s}} dy \\
&\leq C(\delta)\eta(x) \int_{B_{1-\delta}} \frac{|w(x) - w(y)|}{|x-y|^{N+2s-1}} dy \\
&\quad + 2\eta(x) \int_{B^c_{1-\delta}} \frac{|w(x) - w(y)|}{|x-y|^{N+2s}} dy \\
&\leq C(\delta)\eta(x) \left(\int_{B_{1-\delta}} \frac{|w(x) - w(y)|^p}{|x-y|^{N+sp}} dy \right)^{1/p} \left(\int_{B_{1-\delta}} \frac{dy}{|x-y|^{N-p'(1-s)}} \right)^{1/p'} \\
&\quad + 2\eta(x) \left(|w(x)| \int_{B^c_{1-\delta}(x)} \frac{dy}{|x-y|^{N+2s}} + \int_{\mathbb{R}^N} \frac{|w(y)|}{1+|y|^{N+2s}} dy \right).
\end{aligned}
\tag{3.22}
$$

From (3.22) we deduce that there exists a positive constant $C = C(N, s, \delta, p)$ such that

$$\|\eta I_s(\eta, w)\|_{L^p(\mathbb{R}^N)} \leq C(N, s, \delta, p) \left(\|w\|_{W^{s,p}(B_{1-\delta})} + \|w\|_{L^1_s(\mathbb{R}^N)} \right). \tag{3.23}$$

The estimates in the cases $p = 1$ and $p = \infty$ also hold true with analogous computations. In the case $p = \infty$ we recall that for any Ω open and smooth set $W^{s,\infty}(\Omega) = C^{0,s}(\overline{\Omega})$, see e.g. [16, Pag. 59].

4. Fractional Sobolev regularity

In this section we prove the local H^s regularity of very weak solutions, which we know to be in L^2_{loc}. In the classical case, Sobolev regularity is usually obtained via the Nirenberg difference quotients method, but in the nonlocal case this method does not work directly because of the

presence of a divergent kernel. Therefore, we have devised a different approach, which consists in using another cut-off function (the η_τ below) that eliminates the singularity and makes the relevant integrals convergent.

Theorem 4.1. *Let u be a very weak solution of* (1.2) *in* $L^2_{\text{loc}}(B)$. *Then* $u \in H^{2s}_{\text{loc}}(B)$.

Proof. Let u be given as in the statement. Let $\tau \in (0, 1/2)$ and let $\eta_\tau : [0, +\infty) \to [0, 1]$ be the cut-off function defined as

$$\eta_\tau(t) := \begin{cases} 0 & \text{if } 0 \leq t \leq \tau/2, \\ \frac{2}{\tau}t - 1 & \text{if } \tau/2 \leq t \leq \tau, \\ 1 & \text{if } \tau \leq t, \end{cases}$$

For $\delta \in \left(0, \frac{1}{4}\right)$, let us consider another cut-off function $\eta : \mathbb{R}^N \to [0, 1]$ such that

$$\eta = 1 \text{ in } B_{1-4\delta}, \quad \eta = 0 \text{ in } B^c_{1-2\delta}, \quad |\nabla \eta| \leq \frac{1}{\delta}.$$

For any $x \in \mathbb{R}^N$ let us define the function

$$D^s_{\eta_\tau, \eta} u(x) := \int_{\mathbb{R}^N} \eta_\tau(|x - y|) \frac{\eta(x)u(x) - \eta(y)u(y)}{|x - y|^{N+2s}} \, dy. \tag{4.1}$$

In order to prove the required regularity, as a test function in (2.4) we choose

$$\varphi(x) := \eta(x)v(x),$$

where v is the solution of the problem

$$\begin{cases} (-\Delta)^s w = D^s_{\eta_\tau, \eta} u & \text{in } B_{1-\delta} \\ w = 0 & \text{in } B^c_{1-\delta}. \end{cases} \tag{4.2}$$

We notice that φ is an admissible test function since $(-\Delta)^s \varphi \in L^\infty_s(\mathbb{R}^N) \subset L^2(\mathbb{R}^N)$. Since η is supported in $B_{1-2\delta}$, we have

$$0 = \int_{\mathbb{R}^N} u(-\Delta)^s \varphi \, dx = \int_{\mathbb{R}^N} u(-\Delta)^s(\eta v) \, dx$$

$$= \int_{B_{1-\delta}} uv(-\Delta)^s \eta \, dx + \int_{B_{1-2\delta}} u\eta(-\Delta)^s v \, dx - \int_{\mathbb{R}^N} uI_s(\eta, v) dx,$$

where I_s is defined in (3.4). It follows

$$\left| \int_{B_{1-2\delta}} u\eta D^s_{\eta_\tau, \eta} u \, dx \right| \leq \left| \int_{B_{1-\delta}} uv(-\Delta)^s \eta \, dx \right| + \left| \int_{\mathbb{R}^N} uI_s(\eta, v) dx \right| =: |C_1| + |C_2|. \tag{4.3}$$

First of all, rewrite the left hand side of (4.3) as

$$\int_{\mathbb{R}^N} u\eta D^s_{\eta_\tau, \eta} u \, dx = \int_{\mathbb{R}^N} u(x)\eta(x) dx \int_{\mathbb{R}^N} \eta_\tau(|x - y|) \frac{\eta(x)u(x) - \eta(y)u(y)}{|x - y|^{N+2s}} \, dy \tag{4.4}$$

$$= \frac{1}{2} \int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} \eta_\tau(|x - y|) \frac{(\eta(x)u(x) - \eta(y)u(y))^2}{|x - y|^{N+2s}} \, dy =: \frac{1}{2} G^s_{\eta_\tau, \eta}(u).$$

Let us estimate the term C_2. We write $C_2 = C_3 + C_4$, where

$$C_3 := \int_{B_{1-\delta}} u(x) dx \int_{\mathbb{R}^N} \frac{(v(x) - v(y))(\eta(x) - \eta(y))}{|x - y|^{N+2s}} \, dy$$

and

$$C_4 := \int_{B_{1-\delta}^c} u(x)dx \int_{\mathbb{R}^N} \frac{(v(x) - v(y))(\eta(x) - \eta(y))}{|x-y|^{N+2s}} \, dy.$$

We have

$$
\begin{aligned}
|C_3| &= \left| \int_{B_{1-\delta}} u(x)dx \int_{\mathbb{R}^N} \frac{(v(x) - v(y))(\eta(x) - \eta(y))}{|x-y|^{N+2s}} \, dy \right| \\
&\leq \|u\|_{L^2(B_{1-\delta})} \left(\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{(v(x) - v(y))(\eta(x) - \eta(y))}{|x-y|^{N+2s}} \, dy \right)^2 dx \right)^{1/2} \\
&\leq \|u\|_{L^2(B_{1-\delta})} \left(\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{(v(x) - v(y))^2}{|x-y|^{N+2s}} \, dy \right) \left(\int_{\mathbb{R}^N} \frac{(\eta(x) - \eta(y))^2}{|x-y|^{N+2s}} \, dy \right) dx \right)^{1/2} \\
&\leq C(\eta) \|u\|_{L^2(B_{1-\delta})} \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(v(x) - v(y))^2}{|x-y|^{N+2s}} \, dy\, dx \right)^{1/2} = C(\eta) \|u\|_{L^2(B_{1-\delta})} [v]_{H^s(\mathbb{R}^N)}.
\end{aligned}
\tag{4.5}
$$

Now, in order to estimate the right hand side in (4.5), we use that v is a weak solution of (4.2); by testing against v itself we obtain

$$
\begin{aligned}
[v]_{H^s(\mathbb{R}^N)}^2 &= \int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} \frac{(v(x) - v(y))^2}{|x-y|^{N+2s}} \, dy = 2 \int_{\mathbb{R}^N} v(-\Delta)^s v \, dx = 2 \int_{\mathbb{R}^N} v D_{\eta_\tau, \eta}^s u \, dx \\
&= 2 \int_{\mathbb{R}^N} v(x)dx \int_{\mathbb{R}^N} \frac{(\eta(x)u(x) - \eta(y)u(y))}{|x-y|^{N+2s}} \eta_\tau(|x-y|) \, dy \\
&= \int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} \left(\frac{(v(x) - v(y))}{|x-y|^{\frac{N+2s}{2}}} \sqrt{\eta_\tau(|x-y|)} \right) \left(\frac{(\eta(x)u(x) - \eta(y)u(y))}{|x-y|^{\frac{N+2s}{2}}} \sqrt{\eta_\tau(|x-y|)} \right) dy \\
&\leq \left(\int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} \frac{(v(x) - v(y))^2}{|x-y|^{N+2s}} \eta_\tau(|x-y|) \, dy \right)^{1/2} \cdot \\
&\quad \left(\int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} \frac{(\eta(x)u(x) - \eta(y)u(y))^2}{|x-y|^{N+2s}} \eta_\tau(|x-y|) dy \right)^{1/2} \\
&\leq [v]_{H^s(\mathbb{R}^N)} \sqrt{G_{\eta_\tau, \eta}^s(u)},
\end{aligned}
\tag{4.6}
$$

where in the first estimate we applied the Hölder inequality with exponent 2 and in the second one we took into account that $\|\eta_\tau\|_{L^\infty((0,\infty))} = 1$. Summarising,

$$[v]_{H^s(\mathbb{R}^N)} \leq \sqrt{G_{\eta_\tau, \eta}^s(u)} \tag{4.7}$$

and thus by (4.5) we get

$$|C_3| \leq C(\eta) \|u\|_{L^2(B_{1-\delta})} \sqrt{G_{\eta_\tau, \eta}^s(u)}. \tag{4.8}$$

Let us estimate C_4. Since $\eta(x) = \eta(y) = 0$ for $x \in B_{1-\delta}^c$ and $y \in B_{1-2\delta}^c$ we have

$$
\begin{aligned}
|C_4| &= \left| \int_{B_{1-\delta}^c} u(x)dx \int_{\mathbb{R}^N} \frac{(v(x) - v(y))(\eta(x) - \eta(y))}{|x-y|^{N+2s}} \, dy \right| \\
&= \left| \int_{B_{1-\delta}^c} u(x)dx \int_{B_{1-2\delta}} \frac{(v(x) - v(y))(\eta(x) - \eta(y))}{|x-y|^{N+2s}} \, dy \right|,
\end{aligned}
$$

and then

$$
\begin{aligned}
|C_4| &= \left| \int_{B_{1-\delta}^c} \frac{u(x)}{1+|x|^{N+2s}} dx \int_{B_{1-2\delta}} \left(\frac{1+|x|^{N+2s}}{|x-y|^{N+2s}} \right) v(y)\eta(y)\, dy \right| \\
&\leq \int_{B_{1-\delta}^c} \frac{|u(x)|}{1+|x|^{N+2s}} dx \int_{B_{1-2\delta}} \left(1 + \frac{1+|y|}{|x-y|} \right)^{N+2s} |v(y)|\, dy \\
&\leq C(\delta)\|u\|_{L_s^1(\mathbb{R}^N)}[v]_{H^s(\mathbb{R}^N)} \\
&\leq C(\delta)\|u\|_{L_s^1(\mathbb{R}^N)}\sqrt{G_{\eta_\tau,\eta}^s(u)}.
\end{aligned}
\tag{4.9}
$$

In the third inequality we exploited (4.7), the second one follows from the Hölder inequality and the fractional Sobolev inequality, see [16, Theorem 6.5]; taking into account that v has compact support in $B_{1-\delta}$ we have:

$$
\|v\|_{L^1(B_{1-2\delta})} \leq |B_{1-2\delta}|^{\frac{2_s^*-1}{2_s^*}} \|v\|_{L^{2_s^*}(\mathbb{R}^N)} \leq C(\delta, N, s)[v]_{H^s(\mathbb{R}^N)}.
$$

Now, let us estimate the term C_1:

$$
\begin{aligned}
|C_1| &= \left| \int_{B_{1-\delta}} uv(-\Delta)^s\eta\, dx \right| \leq \|u\|_{L^{(2_s^*)'}(B_{1-\delta})}\|v\|_{L^{2_s^*}(\mathbb{R}^N)}\|(-\Delta)^s\eta\|_{L^\infty(\mathbb{R}^N)} \\
&\leq C(\delta)[v]_{H^s(\mathbb{R}^N)} \leq C(\delta)\sqrt{G_{\eta_\tau,\eta}^s(u)},
\end{aligned}
\tag{4.10}
$$

where $C(\delta) > 0$ can be explicitly computed and in the second and third inequalities we exploited again the fractional Sobolev inequality and (4.7), respectively.

By putting together (4.5), (4.9) and (4.10) in (4.3) and using equality (4.4) we have

$$
G_{\eta_\tau,\eta}^s(u) \leq C(\|u\|_{L_s^1(\mathbb{R}^N)}, \|u\|_{L^2(B_{1-\delta})}, \delta, \eta).
\tag{4.11}
$$

Now, we recall that η_τ depends on the parameter $\tau \in (0, 1/2)$ but the estimate (4.11) is uniform with respect τ because the right-hand side is independent of τ. Therefore, estimate (4.11) finally yields

$$
[\eta u]_{H^s(\mathbb{R}^N)}^2 \leq \sup_{\tau \in (0,1/2)} G_{\eta_\tau,\eta}^s(u) < \infty,
\tag{4.12}
$$

where the first inequality holds true in view of Fatou's Lemma. Thus $\eta u \in H^s(\mathbb{R}^N)$ and then $u \in H_{\text{loc}}^s(B)$.

In order to complete the proof, let us show that $\eta^2 u$ is a compactly supported weak solution of $(-\Delta)^s w = f$, with $f \in L^2(\mathbb{R}^N)$. This implies that $\eta^2 u \in H^{2s}(\mathbb{R}^N)$.

For any $\varphi \in C_c^\infty(\mathbb{R}^N)$ we have

$$
\begin{aligned}
\frac{1}{2} \int_{\mathbb{R}^N} &dx \int_{\mathbb{R}^N} \frac{(\eta^2(x)u(x) - \eta^2(y)u(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}}\, dy \\
&= \int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} \frac{\eta^2(x)u(x)(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}}\, dy \\
&= C_{N,s}^{-1} \int_{\mathbb{R}^N} u(x)(-\Delta)^s(\eta^2\varphi)(x)\, dx + \int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} \frac{u(x)\varphi(y)(\eta^2(y) - \eta^2(x))}{|x-y|^{N+2s}}\, dy \\
&= \int_{\mathbb{R}^N} \varphi(x)\, dx \int_{\mathbb{R}^N} \frac{u(y)(\eta^2(x) - \eta^2(y))}{|x-y|^{N+2s}} dy,
\end{aligned}
$$

where in the last equality we used that $\eta^2\varphi \in C_c^\infty(B)$ and that u is a very weak solution of (1.2). To conclude, we show that the function f defined a.e. by

$$f(x) := C_{N,s} \int_{\mathbb{R}^N} \frac{u(y)(\eta^2(x) - \eta^2(y))}{|x-y|^{N+2s}} dy \tag{4.13}$$

belongs to $L^2(\mathbb{R}^N)$. We point out that

$$f(x) = C_{N,s} \int_{\mathbb{R}^N} \frac{u(y)(\eta(x) + \eta(y))(\eta(x) - \eta(y))}{|x-y|^{N+2s}} dy \tag{4.14}$$

$$= C_{N,s} \int_{\mathbb{R}^N} \frac{[2u(x)\eta(x) + u(y)\eta(y) - u(x)\eta(x) + \eta(x)(u(y) - u(x))]\,(\eta(x) - \eta(y))}{|x-y|^{N+2s}} dy$$

$$= 2\eta(x)u(x)(-\Delta)^s\eta(x) - I_s(\eta, \eta u)(x) - \eta(x)I_s(\eta, u)(x).$$

The last equality shows that $f \in L^2(\mathbb{R}^N)$, as $I_s(\eta, \eta u)$ and $\eta I_s(\eta, u)$ belong to $L^2(\mathbb{R}^N)$ in view of Remark 3.3 and $\eta u(-\Delta)^s\eta$, supported in $B_{1-2\delta}$, belongs to $L^2(\mathbb{R}^N)$ since $u \in L^2_{\text{loc}}(B)$.

We thus obtain that $\eta^2 u \in H^{2s}(\mathbb{R}^N)$, and so $u \in H^{2s}_{\text{loc}}(B)$. $\qquad\square$

5. Full regularity

In this section we prove that a very weak s-harmonic function u is actually a classical s-harmonic function hence locally smooth. To do this, we use the fact that u is locally bounded and by fractional De Giorgi estimates proved in [9, Theorem 1.4.] also belongs to $C^{0,\gamma}_{\text{loc}}(B)$ for every $\gamma \in (0, \min\{2s, 1\})$. Then, in Propositions 5.2 and 5.3 we prove that the operator I_s enjoys useful Hölder continuity properties that allow us to exploit that the function $\eta^2 u$ solves the equation $(-\Delta)^s w = f \in C^{0,\gamma}(\mathbb{R}^N)$, and then $u \in C^{\gamma+2s}_{\text{loc}}(B) \cap L^1_s(\mathbb{R}^N)$. Therefore u is s-harmonic in B in the classical sense and, by Theorem 5.5, is real analytic in B.

Theorem 5.1. *Let $u \in H^{2s}_{\text{loc}}(B) \cap L^1_s(\mathbb{R}^N)$ a very weak s-harmonic function in B. Then $u \in C^{0,\alpha}_{\text{loc}}(B)$ for every $\alpha = \alpha(s) \in (0, \min\{2s, 1\})$.*

Proof. Since $u \in H^{2s}_{\text{loc}}(B) \cap L^1_s(\mathbb{R}^N)$ then u is a local weak solution of (1.2) and by [9, Theorem 3.2., Remark 3.3.] one has that $u \in L^\infty_{\text{loc}}(B)$. Then the claim plainly follows by [9, Theorem 1.4.]. $\qquad\square$

Proposition 5.2. *Let $s \in (0,1)$ and $\alpha \in (s, \min\{2s, 1\})$. For any $f \in C^{0,\alpha}(\mathbb{R}^N)$ and $g \in C^{0,1}(\mathbb{R}^N)$ we have that $I_s(f,g) \in C^{0,\gamma(\alpha,s)}(\mathbb{R}^N)$ where $\gamma(\alpha,s) := \begin{cases} 2\alpha - 2s & \text{if} \quad 0 < s \le \frac{1}{2} \\ \alpha - 2s + 1 & \text{if} \quad \frac{1}{2} < s < 1 \end{cases}$*

and

$$[I_s(f,g)]_{C^{0,\gamma(\alpha,s)}(\mathbb{R}^N)} \le C[f]_{C^{0,\alpha}(\mathbb{R}^N)}[g]_{C^{0,1}(\mathbb{R}^N)}$$

Proof. Let $x, x' \in \mathbb{R}^N$, $x \neq x'$, and let $R := |x - x'|$. We estimate

$$I_s(f,g)(x) - I_s(f,g)(x') \tag{5.1}$$

$$= \int_{\mathbb{R}^N} \frac{(f(x) - f(x-y))(g(x) - g(x-y)) - (f(x') - f(x'-y))(g(x') - g(x'-y))}{|y|^{N+2s}} dy.$$

By adding ad subtracting $(f(x) - f(x - y))(g(x') - g(x' - y))$ in the numerator of the integrand in (5.1) we can equivalently write

$$I_s(f, g)(x) - I_s(f, g)(x') = J_1 + J_2 + J_3 + J_4$$

where

$$J_1 := \int_{B_R} \frac{(f(x) - f(x - y))\left[(g(x) - g(x - y)) + (g(x' - y) - g(x'))\right]}{|y|^{N+2s}} dy,$$

$$J_2 := \int_{B_R^c} \frac{(f(x) - f(x - y))\left[(g(x) - g(x')) + (g(x' - y) - g(x - y))\right]}{|y|^{N+2s}} dy,$$

$$J_3 := \int_{B_R} \frac{(g(x') - g(x' - y))\left[(f(x) - f(x - y)) + (f(x' - y) - f(x'))\right]}{|y|^{N+2s}} dy,$$

$$J_4 := \int_{B_R^c} \frac{(g(x') - g(x' - y))\left[(f(x) - f(x')) + (f(x' - y) - f(x - y))\right]}{|y|^{N+2s}} dy.$$

Now if $s \leq \frac{1}{2}$ we have

$$|J_1| \leq C[f]_{C^\alpha(\mathbb{R}^N)}[g]_{C^{0,1}(\mathbb{R}^N)} \int_{B_R} \frac{dy}{|y|^{N+2s-\alpha-1}} = C[f]_{C^\alpha(\mathbb{R}^N)}[g]_{C^{0,1}(\mathbb{R}^N)}|x - x'|^{\alpha-2s+1},$$

and

$$|J_2| \leq C[f]_{C^\alpha(\mathbb{R}^N)}[g]_{C^{0,1}(\mathbb{R}^N)}|x - x'| \int_{B_R^c} \frac{dy}{|y|^{N+2s-\alpha}} = C[f]_{C^\alpha(\mathbb{R}^N)}[g]_{C^{0,1}(\mathbb{R}^N)}|x - x'|^{\alpha-2s+1}.$$

The estimate of J_3 is analogous to the one of J_1 while for J_4 we have

$$|J_4| \leq C[f]_{C^\alpha(\mathbb{R}^N)}[g]_{C^\alpha(\mathbb{R}^N)}|x - x'|^\alpha \int_{B_R^c} \frac{dy}{|y|^{N+2s-\alpha}} = C[f]_{C^\alpha(\mathbb{R}^N)}[g]_{C^\alpha(\mathbb{R}^N)}|x - x'|^{2\alpha-2s}.$$

Since $2\alpha - 2s < \alpha - 2s + 1$ we get the thesis.

If $s > \frac{1}{2}$ the estimates of J_1, J_2, J_3 are analogous to the previous case, while for J_4 we can write

$$|J_4| \leq C[f]_{C^\alpha(\mathbb{R}^N)}[g]_{C^{0,1}(\mathbb{R}^N)}|x - x'|^\alpha \int_{B_R^c} \frac{dy}{|y|^{N+2s-1}} = C[f]_{C^\alpha(\mathbb{R}^N)}[g]_{C^{0,1}(\mathbb{R}^N)}|x - x'|^{\alpha-2s+1}.$$

Hence $I_s(f, g) \in C^{\alpha-2s+1}(\mathbb{R}^N)$. □

Proposition 5.3. *Let $s \in (0, 1)$, $\alpha \in (s, \min\{2s, 1\})$ and $\beta = \beta(\alpha) := \frac{\alpha}{\alpha+1} < \frac{\alpha}{2s}$. If $f \in C^{0,\alpha}_{loc}(B) \cap L^1_s(\mathbb{R}^N)$ and $\eta \in C^\infty_c(B)$ then $\eta I_s(\eta, f) \in C^{0,\gamma}(\mathbb{R}^N)$ where $\gamma := (\alpha - 2s + 1)\beta$ and*

$$\|\eta I_s(\eta, f)\|_{C^{0,\gamma}(\mathbb{R}^N)} \leq C\left(\|f\|_{C^{0,\alpha}(B_{r'})} + \|f\|_{L^1_s(\mathbb{R}^N)}\right)$$

where $r' \in (0, 1)$ is such that $B_{r'} = \operatorname{supp}\eta$.

Proof. Let $B_r \Subset B$ with $0 < r' < r$. Then $f \in C^{0,\alpha}(B_r)$. Since

$$\|\eta I_s(\eta, f)\|_{C^{0,\gamma}(\mathbb{R}^N)} \leq \|\eta\|_{C^{0,\gamma}(\mathbb{R}^N)}\|I_s(\eta, f)\|_{L^\infty(B_{r'})} + \|\eta\|_{L^\infty(\mathbb{R}^N)}\|I_s(\eta, f)\|_{C^{0,\gamma}(B_{r'})}$$

we reduce to prove that $I_s(\eta, f) \in C^{0,\gamma}(B_{r'})$. By (3.23) we already know that $I_s(\eta, f) \in L^\infty(B_{r'})$.

Now, let $x, x' \in B_{r'}$, $x \neq x'$. We set $R := |x - x'|$. We assume that $R < \left(\frac{1}{2}\right)^{\frac{1}{1-\beta}}$ otherwise the proof is done. We observe that thanks to the required upper bound on R the following inclusions of sets

$$B_{R^\beta}(x') \subset B_{\frac{3}{2}R^\beta}(x) \quad \text{and} \quad B_{R^\beta}^c(x') \subset B_{\frac{R^\beta}{2}}^c(x),$$

hold true. We write

$$|I_s(\eta, f)(x) - I_s(\eta, f)(x')| \tag{5.2}$$

$$= \left| \int_{\mathbb{R}^N} \frac{(f(x) - f(y))(\eta(x) - \eta(y))}{|x - y|^{N+2s}} dy - \int_{\mathbb{R}^N} \frac{(f(x') - f(y))(\eta(x') - \eta(y))}{|x' - y|^{N+2s}} dy \right| \tag{5.3}$$

$$\leq \int_{B_{R^\beta}(x')} \frac{|f(x) - f(y)||\eta(x) - \eta(y)|}{|x - y|^{N+2s}} dy$$

$$+ \int_{B_{R^\beta}(x')} \frac{|f(x') - f(y)||\eta(x') - \eta(y)|}{|x' - y|^{N+2s}} dy$$

$$+ \left| \int_{B_{R^\beta}^c(x')} \frac{(f(x) - f(y))(\eta(x) - \eta(y)) - \frac{|x-y|^{N+2s}}{|x'-y|^{N+2s}}(f(x') - f(y))(\eta(x') - \eta(y))}{|x - y|^{N+2s}} dy \right|$$

$$\leq \int_{B_{\frac{3}{2}R^\beta}(x)} \frac{|f(x) - f(y)||\eta(x) - \eta(y)|}{|x - y|^{N+2s}} dy$$

$$+ \int_{B_{R^\beta}(x')} \frac{|f(x') - f(y)||\eta(x') - \eta(y)|}{|x' - y|^{N+2s}} dy$$

$$+ \left| \int_{B_{R^\beta}^c(x')} \frac{(f(x) - f(y))(\eta(x) - \eta(y)) - \frac{|x-y|^{N+2s}}{|x'-y|^{N+2s}}(f(x') - f(y))(\eta(x') - \eta(y))}{|x - y|^{N+2s}} dy \right|$$

$$\leq C [f]_{C^{0,\alpha}(B_r)} R^{(\alpha - 2s + 1)\beta}$$

$$+ \left| \int_{B_{R^\beta}^c(x')} \frac{(f(x) - f(y))(\eta(x) - \eta(y)) - \frac{|x-y|^{N+2s}}{|x'-y|^{N+2s}}(f(x') - f(y))(\eta(x') - \eta(y))}{|x - y|^{N+2s}} dy \right|$$

$$=: C [f]_{C^{0,\alpha}(B_r)} R^{(\alpha - 2s + 1)\beta} + |B|.$$

By adding and subtracting $\frac{(f(x') - f(y))(\eta(x') - \eta(y))}{|x-y|^{N+2s}}$ to the integrand defining B we have

$$|B| \leq \left| \int_{B_{R^\beta}^c(x')} \frac{(f(x) - f(y))(\eta(x) - \eta(y)) - (f(x') - f(y))(\eta(x') - \eta(y))}{|x - y|^{N+2s}} dy \right|$$

$$+ \int_{B_{R^\beta}^c(x')} \left| \frac{1}{|x - y|^{N+2s}} - \frac{1}{|x' - y|^{N+2s}} \right| |f(x') - f(y)||\eta(x') - \eta(y)| dy$$

$$\leq \int_{B_{R^\beta}^c(x')} \frac{|f(x)\eta(x) - f(x')\eta(x')|}{|x - y|^{N+2s}} dy + \int_{B_{R^\beta}^c(x')} \frac{|f(y)||\eta(x) - \eta(x')|}{|x - y|^{N+2s}} dy$$

$$+ \int_{B_{R^\beta}^c(x')} \frac{|\eta(y)||f(x) - f(x')|}{|x - y|^{N+2s}} dy$$

$$+ \int_{B^c_{R^\beta}(x')} \left| \frac{1}{|x-y|^{N+2s}} - \frac{1}{|x'-y|^{N+2s}} \right| |f(x') - f(y)||\eta(x') - \eta(y)| dy$$

$$\leq \int_{B^c_{\frac{R^\beta}{2}}(x)} \frac{|f(x)\eta(x) - f(x')\eta(x')|}{|x-y|^{N+2s}} dy + \int_{B^c_{\frac{R^\beta}{2}}(x)} \frac{|f(y)||\eta(x) - \eta(x')|}{|x-y|^{N+2s}} dy$$

$$+ \int_{B^c_{\frac{R^\beta}{2}}(x)} \frac{|\eta(y)||f(x) - f(x')|}{|x-y|^{N+2s}} dy$$

$$+ \int_{B^c_{R^\beta}(x')} \left| \frac{1}{|x-y|^{N+2s}} - \frac{1}{|x'-y|^{N+2s}} \right| |f(x') - f(y)||\eta(x') - \eta(y)| dy$$

$$=: B_1 + B_2 + B_3 + B_4.$$

Now we estimate B_i for any $i \in \{1, 2, 3, 4\}$.

$$B_1 \leq C[\eta u]_{C^{0,\alpha}(\mathbb{R}^N)} R^\alpha \int_{B^c_{R^\beta/2}(x)} \frac{dy}{|x-y|^{N+2s}} \leq C[\eta u]_{C^{0,\alpha}(\mathbb{R}^N)} R^{\alpha - 2\beta s}. \tag{5.4}$$

$$B_2 \leq CR \left(\int_{B^c_{R^\beta/2}(x) \cap B_r} \frac{|f(y)|}{|x-y|^{N+2s}} dy + \int_{B^c_{R^\beta/2}(x) \cap B^c_r} \frac{|f(y)|}{|x-y|^{N+2s}} dy \right)$$

$$\leq CR \left(\|f\|_{L^\infty(B_r)} \int_{B^c_{R^\beta/2}(x)} \frac{dy}{|x-y|^{N+2s}} + C(r,r') \int_{B^c_r} \frac{|f(y)|}{1+|y|^{N+2s}} dy \right) \tag{5.5}$$

$$\leq CR \left(C_1 \|f\|_{L^\infty(B_r)} R^{-2\beta s} + C(r,r') \|f\|_{L^1_s(\mathbb{R}^N)} \right)$$

$$\leq C(r, r', N, s) \left(\|f\|_{L^\infty(B_r)} + \|f\|_{L^1_s(\mathbb{R}^N)} \right) R^{1-2\beta s}$$

$$\leq C(r, r', N, s) \left(\|f\|_{L^\infty(B_r)} + \|f\|_{L^1_s(\mathbb{R}^N)} \right) R^{\alpha - 2\beta s}.$$

$$B_3 \leq C[f]_{C^{0,\alpha}(B_{r'})} R^\alpha \int_{B^c_{R^\beta/2}(x)} \frac{dy}{|x-y|^{N+2s}} \leq C[f]_{C^{0,\alpha}(B_{r'})} R^{\alpha - 2\beta s}. \tag{5.6}$$

For B_4 we write

$$B_4 = \int_{B^c_{R^\beta}(x') \cap B_r} \left| \frac{1}{|x-y|^{N+2s}} - \frac{1}{|x'-y|^{N+2s}} \right| |f(x') - f(y)||\eta(x') - \eta(y)| dy$$

$$+ \int_{B^c_{R^\beta}(x') \cap B^c_r} \left| \frac{1}{|x-y|^{N+2s}} - \frac{1}{|x'-y|^{N+2s}} \right| |f(x') - f(y)||\eta(x') - \eta(y)| dy$$

$$=: D_1 + D_2.$$

Using the fundamental Theorem of Calculus we write

$$\left| \frac{1}{|x-y|^{N+2s}} - \frac{1}{|x'-y|^{N+2s}} \right| \leq C_{N,s}|x - x'||x' - y|^{-N-2s-1}.$$

Therefore

$$
\begin{aligned}
D_1 &\leq C\,[f]_{C^{0,\alpha}(B_{r'})}\,R\int_{B^c_{R^\beta}(x')}|x'-y|^{\alpha+1}|x'-y|^{-N-2s-1}dy \\
&= C\,[f]_{C^{0,\alpha}(B_{r'})}\,R^{1+\beta(\alpha-2s)} \\
&\leq C\,[f]_{C^{0,\alpha}(B_{r'})}\,R^{(\alpha-2s+1)\beta}.
\end{aligned}
\tag{5.7}
$$

To conclude for D_2 we have

$$
\begin{aligned}
D_2 &\leq CR\int_{B^c_{R^\beta}(x')\cap B^c_r}|f(x')-f(y)||x'-y||x'-y|^{-N-2s-1}dy \\
&\leq CR\left(\|f\|_{L^\infty(B_{r'})}\int_{B_{R^\beta}(x')}\frac{dy}{|x'-y|^{N+2s}}+\int_{B^c_r}\frac{|f(y)|}{|x'-y|^{N+2s}}dy\right) \\
&\leq CR\left(C_1\,\|f\|_{L^\infty(B_{r'})}\,R^{-2\beta s}+C(r,r')\int_{\mathbb{R}^N}\frac{|f(y)|}{1+|y|^{N+2s}}dy\right) \\
&\leq C(r,r',N,s)\left(\|f\|_{L^\infty(B_{r'})}+\|f\|_{L^1_s(\mathbb{R}^N)}\right)R^{1-2\beta s} \\
&\leq C(r,r',N,s)\left(\|f\|_{L^\infty(B_{r'})}+\|f\|_{L^1_s(\mathbb{R}^N)}\right)R^{\alpha-2\beta s}.
\end{aligned}
\tag{5.8}
$$

Putting (5.4), (5.5), (5.6), (5.7),(5.8) into (5.2) and taking into account that $(\alpha-2s+1)\beta = \alpha-2\beta s$ by the choice of β, our claim is proved. $\qquad\square$

Theorem 5.4. *Let $u\in C^{0,\gamma}_{loc}(B)\cap L^1_s(\mathbb{R}^N)$ for some $\gamma\in(0,1)$ a very weak s-harmonic function in B. Then $u\in C^{\gamma+2s}_{loc}(B)\cap L^1_s(\mathbb{R}^N)$ hence u is s-harmonic in the classical sense in B.*

Proof. Let $\eta\in C^\infty_c(B)$. By Theorem 4.1 the function $\eta^2 u$ is a weak solution of $(-\Delta)^s w = f$ in $\mathbb{R}^N$ with $f:=2\eta u(-\Delta)^s\eta-I_s(\eta,\eta u)-\eta I_s(\eta,u)$. Moreover by applying Theorem 5.1 with $\alpha\in(s,\min\{2s,1\})$ and Propositions 5.2, 5.3 we have that $f\in C^{0,\gamma}(\mathbb{R}^N)$. Therefore by using Schauder estimates for bounded weak solutions to $(-\Delta)^s w = f$ (see [37, Proposition 2.8] or [38, Theorem 15]) it follows that $\eta^2 u\in C^{0,\gamma+2s}(\mathbb{R}^N)$ if $0<\gamma+2s<1$ or $\eta^2 u\in C^{1,\gamma+2s-1}(\mathbb{R}^N)$ if $1<\gamma+2s<2$ or $\eta^2 u\in C^{2,\gamma+2s-2}(\mathbb{R}^N)$ if $2<\gamma+2s<3$. If $\gamma+2s\in\mathbb{N}$ apply Propositions 5.2, 5.3 replacing α with α_1 such that $\alpha_1+2s\notin\mathbb{N}$. By the arbitrariness of η we get the thesis. $\quad\square$

Theorem 5.5. *Let $u\in C^{\gamma+2s}_{loc}(B)\cap L^1_s(\mathbb{R}^N)$ for some $\gamma\in(0,1)$ a very weak s-harmonic function in B. Then u is real analytic in B.*

Proof. Let $\delta\in(0,1)$ and $R=R(\delta):=1-\frac{\delta}{4}$, $r=r(\delta):=1-\frac{\delta}{2}$, $r_0=r_0(\delta):=1-\delta$. By Theorem 5.4 u is a classical solution of

$$
\begin{cases}
(-\Delta)^s w = 0 & \text{in}\quad B_r \\
w = h & \text{in}\quad B^c_r,
\end{cases}
\tag{5.9}
$$

where

$$
h := \begin{cases}
u & \text{in}\quad B \\
g & \text{in}\quad B^c.
\end{cases}
$$

Since $h\in C(\overline{B_R})\cap L^1_s(\mathbb{R}^N)$ if for $\rho>0$ we set

$$
P_\rho(x,y):=C_{N,s}\left(\frac{\rho^2-|x|^2}{|y|^2-\rho^2}\right)^s\frac{1}{|x-y|^N}
$$

the fractional Poisson kernel (see [10]) for B_ρ for any $x \in B_\rho$ and $y \in B_\rho^c$, the function

$$u_h(x) := \int_{B_r^c} P_r(x,y)h(y)dy \tag{5.10}$$

is well posed for every $x \in B_r$. Indeed

$$
\begin{aligned}
|u_h(x)| &\le \int_{B_R \setminus B_r} P_r(x,y)|h(y)|dy + \int_{B \setminus B_R} P_r(x,y)|h(y)|dy + \int_{B^c} P_r(x,y)|h(y)|dy \\
&= \int_{B_R \setminus B_r} P_r(x,y)|u(y)|dy + \int_{B \setminus B_R} P_r(x,y)|u(y)|dy + \int_{B^c} P_r(x,y)|g(y)|dy \\
&\le \|u\|_{L^\infty(B_R)} \int_{B_r^c} P_r(x,y)dy + \frac{r^{2s}}{(R^2 - r^2)^s (R-r)^N} \|u\|_{L^1(B)} + C_{r,R,N,s} \int_{\mathbb{R}^N} \frac{|u(y)|}{1+|y|^{N+2s}}dy \\
&\le C(r,R,N,s) \left(\|u\|_{L^\infty(B_R)} + \|u\|_{L^1_s(\mathbb{R}^N)} \right).
\end{aligned}
\tag{5.11}
$$

By [10, Theorem 2.10.], the function u_h is a classical solution of (5.9). By uniqueness of solutions of (5.9) (see [12, Theorem 3.3.2.]) we conclude that $u_h = u$ in B_r.

Moreover, for every $y \in B_r^c$ the function

$$B_{r_0} \ni x \mapsto \frac{(r^2 - |x|^2)^s}{|x-y|^N}$$

is smooth, and it's easy to check that

$$|(\partial_x^\iota P_r)(x,y)| \le c^{|\iota|} \iota! C(r, r_0, N, s) P_r(x,y) \tag{5.12}$$

for every $x \in B_{r_0}$, $y \in B_r^c$ and $\iota \in \mathbb{N}_0^N$. Therefore by differentiating under integral sign formula (5.10) by estimates (5.11), (5.12) we have

$$\|D^\iota u\|_{L^\infty(B_{r_0})} \le c^{|\iota|} \iota! C(R, r, r_0, N, s) \left(\|u\|_{L^\infty(B_R)} + \|u\|_{L^1_s(\mathbb{R}^N)} \right)$$

for any $\iota \in \mathbb{N}_0^N$. From the arbitrariness of $\delta \in (0,1)$ we get the thesis. $\qquad\square$

References

[1] N. Abatangelo and E. Valdinoci, *Getting acquainted with the fractional Laplacian*, Springer INdAM Series (2019).

[2] B. Abdellaoui, A. J. Fernández, T. Leonori, and A. Younes, *Global fractional Calderón-Zygmund type regularity*, Preprint (2023), available at https://arxiv.org/pdf/2107.06535.pdf.

[3] B. Abdellaoui and I. Peral, *Towards a deterministic KPZ equation with fractional diffusion: the stationary problem*, Nonlinearity **31** (2016), 1260–1298.

[4] R. A. Adams, *Sobolev spaces*, Pure and Applied Mathematics, Vol. 65, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975.

[5] U. Biccari, M. Warma, and E. Zuazua, *Local elliptic regularity for the Dirichlet fractional Laplacian*, Adv. Nonlinear Stud. **17** (2017), no. 2, 387–409.

[6] ———, *Addendum: Local elliptic regularity for the Dirichlet fractional Laplacian [MR3641649]*, Adv. Nonlinear Stud. **17** (2017), no. 4, 837–839.

[7] K. Bogdan and T. Byczkowski, *Potential theory for the α-stable Schrödinger operator on bounded Lipschitz domains*, Studia Math. **133** (1999), no. 1, 53–92.

[8] L. Brasco and E. Lindgren, *Higher Sobolev regularity for the fractional p-Laplace equation in the superquadratic case*, Adv. Math. **304** (2017), 300–354.

[9] L. Brasco, E. Lindgren, and A. Schikorra, *Higher Hölder regularity for the fractional p-Laplacian in the superquadratic case*, Adv. Math. **338** (2018), 782–846.

[10] C. Bucur, *Some observations on the Green function for the ball in the fractional Laplace framework*, Communications on Pure and Applied Analysis **15** (2016), no. 2, 657–699.

[11] C. Bucur, S. Dipierro, and E. Valdinoci, *On the mean value property of fractional harmonic functions*, Nonlinear Anal. **201** (2020), no. 25.

[12] C. Bucur and E. Valdinoci, *Nonlocal diffusion and applications*, Lecture Notes of the Unione Matematica Italiana, vol. 20, Springer; Unione Matematica Italiana, Bologna, 2016.

[13] L. Caffarelli and L. Silvestre, *Regularity theory for fully nonlinear integro-differential equations*, Comm. Pure Appl. Math. **62** (2009), no. 5, 597–638.

[14] A. Carbotti, S. Dipierro, and E. Valdinoci, *Local density of solutions to fractional equations*, De Gruyter Studies in Mathematics, De Gruyter, 2019.

[15] M. Cozzi, *Interior regularity of solutions of non-local equations in Sobolev and Nikol'skii spaces*, Annali di Matematica Pura ed Applicata **196** (2017), 555-578.

[16] E. Di Nezza, G. Palatucci, and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bulletin Des Sciences Mathematiques **136** (2011), 521-573.

[17] S. Dipierro, G. Giacomin, and E. Valdinoci, *The fractional Malmheden theorem*, Mathematics in Engineering (2022).

[18] S. Dipierro, O. Savin, and E. Valdinoci, *All functions are locally s-harmonic up to a small error*, J. Eur. Math. Soc. (JEMS) **19** (2017), no. 4, 957–966.

[19] M. M. Fall and T. Weth, *Liouville theorems for a general class of nonlocal operators*, Potential Anal. **45** (2016), no. 1, 187–200.

[20] X. Fernández-Real and X. Ros-Oton, *Integro-differential elliptic equations*, Birkhäuser, 2024.

[21] G. Grubb, *Fractional Laplacians on domains, a development of Hörmander's theory of μ-transmission pseudodifferential operators*, Adv. Math. **268** (2015), 478–528.

[22] M. Kassmann, *The theory of De Giorgi for non-local operators*, C. R. Math. Acad. Sci. Paris **345** (2007), no. 11, 621–624 (English, with English and French summaries).

[23] _____, *A priori estimates for integro-differential operators with measurable kernels*, Calc. Var. Partial Differential Equations **34** (2009), no. 1, 1–21.

[24] _____, *A new formulation of Harnack's inequality for nonlocal operators*, C. R. Math. Acad. Sci. Paris **349** (2011), no. 11-12, 637–640 (English, with English and French summaries).

[25] C. E. Kenig, G. Ponce, and L. Vega, *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle*, Comm. Pure Appl. Math. **46** (1993), no. 4, 527–620.

[26] T. Kuusi, G. Mingione, and Y. Sire, *Nonlocal self-improving properties*, Analysis & PDE **8** (2015), 57-114.

[27] M. Kwaśnicki, *Ten equivalent definitions of the fractional Laplace operator*, Fract. Calc. Appl. Anal. **20** (2017), no. 1, 7–51.

[28] T. Leonori, I. Peral, A. Primo, and F. Soria, *Basic estimates for solutions of a class of nonlocal elliptic and parabolic equations*, Discrete Contin. Dyn. Syst. **35** (2015), no. 12, 6031–6068.

[29] C. Miranda, *Partial Differential Equations of Elliptic Type*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 2, Springer, 1970. Second revised edition.

[30] P. Mironescu and W. Sickel, *A Sobolev non embedding*, Rendiconti Lincei-matematica E Applicazioni **26** (2015), 291-298.

[31] L. Nirenberg, *Remarks on strongly elliptic partial differential equations*, Comm. Pure Appl. Math. **8** (1955), 649–675.

[32] X. Ros-Oton and J. Serra, *The extremal solution for the fractional Laplacian*, Calculus of Variations and Partial Differential Equations **50** (2013), 723–750.

[33] R. T. Seeley, *Integro-differential operators on vector bundles*, Trans. Amer. Math. Soc. **117** (1965), 167–204.

[34] _____, *Refinement of the functional calculus of Calderón and Zygmund*, Indag. Math. **27** (1965), 521–531. Nederl. Akad. Wetensch. Proc. Ser. A **68**.

[35] _____ , *Complex powers of an elliptic operator*, Singular Integrals (Proc. Sympos. Pure Math., Chicago, Ill., 1966), Amer. Math. Soc., Providence, RI, 1967, pp. 288–307.

[36] R. Servadei and E. Valdinoci, *Weak and viscosity solutions of the fractional Laplace equation*, Publ. Mat. **58** (2014), no. 1, 133–154.

[37] L. Silvestre, *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Comm. Pure Appl. Math. **60** (2007), no. 1, 67–112.

[38] P. R. Stinga, *User's guide to the fractional Laplacian and the method of semigroups*, Handbook of fractional calculus with applications. Vol. 2, De Gruyter, Berlin.

[39] H. Weyl, *The method of orthogonal projection in potential theory*, Duke Mathematical Journal **7** (1940), 411-444.

Dipartimento di Matematica e Fisica "E. De Giorgi", Università del Salento, Via Per Arnesano, 73100 Lecce, Italy.

Email address: `alessandro.carbotti@unisalento.it`

Dipartimento di Matematica e Fisica "E. De Giorgi", Università del Salento, Via Per Arnesano, 73100 Lecce, Italy.

Email address: `simone.cito@unisalento.it`

Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università degli studi di Napoli "Federico II", Complesso di Monte Sant'Angelo, Via Cintia, 80126 Naples, Italy.

Email address: `domenicoangelo.lamanna@unina.it`

Dipartimento di Matematica e Fisica "E. De Giorgi", Università del Salento, and INFN, Sezione di Lecce, Via Per Arnesano, 73100 Lecce, Italy.

Email address: `diego.pallara@unisalento.it`