

Interior regularity of area minimizing currents within a $C^{2,\alpha}$ -submanifold

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Abstract

Given an area-minimizing integral m -current in Σ , we prove that the Hausdorff dimension of the interior singular set of T cannot exceed $m - 2$, provided that Σ is an embedded $(m + \bar{n})$ -submanifold of \mathbb{R}^{m+n} of class $C^{2,\alpha}$, where $\alpha > 0$. This result establishes the complete counterpart, in the arbitrary codimension setting, of the interior regularity theory for area-minimizing integral hypercurrents within a Riemannian manifold of class $C^{2,\alpha}$.

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1 Introduction

Let $m \geq 2$, $n \geq 1$, $\bar{n} \geq 0$, $\alpha > 0$, and $\Sigma \subset \mathbb{R}^{m+n}$ be a $C^{2,\alpha}$ -submanifold of dimension $m + \bar{n}$. In order to state our main result, we need to introduce the notion of regular interior points in the context of integral currents. This notion asserts that the support of T is equal to a $C^{2,\alpha}$ -submanifold of Σ around points that are far from the boundary. We refer the reader to Section 2 notation and basic definitions.

Definition. Let T be an m -dimensional integral current in Σ . We say that $p \in \text{spt}(T) \setminus \text{spt}(\partial T)$ is an *interior regular point* of T , if there exists a neighborhood $U \ni p$ and an m -dimensional $C^{2,\alpha}$ -submanifold $\mathcal{S} \subset \Sigma$ such that $\text{spt}(T) \cap U = \mathcal{S} \cap U$. The set of such points will be denoted by $\text{Reg}_i(T)$. We call $\text{Sing}_i(T) := \text{spt}(T) \setminus (\text{Reg}_i(T) \cup \text{spt}(\partial T))$ the set of *interior singular points* of T .

We now define the notion of area minimality that we will consider throughout this note.

Definition. Let $U \subset \mathbb{R}^{m+n}$ be an open set, we say that an integral m -current T is *area minimizing* in $\Sigma \cap U$ where Σ is an embedded $C^{2,\alpha}$ of \mathbb{R}^{m+n} , if

$$\|T\|(U) \leq \|T + \partial S\|(U),$$

for every integral $(m + 1)$ -current S with $\text{spt}(S) \subseteq \Sigma \cap U$.

Given the definitions above, we are ready to state the main result of this paper.

Theorem 1.1. *Let $\alpha > 0$, $m \geq 2$, $n \geq 1$, $\bar{n} \geq 0$, and let Σ be an embedded $(m + \bar{n})$ -submanifold of class $C^{2,\alpha}$ of \mathbb{R}^{m+n} . If T be an area minimizing integral m -current in Σ , then*

$$\dim_{\mathcal{H}}(\text{Sing}_i(T)) \leq m - 2.$$

This result provides a complete counterpart in the context of arbitrary codimension for the interior regularity theory concerning area-minimizing integral currents in codimension 1. Specifically, through an application of Nash's embedding theorem (for a comprehensive exposition of Nash's embedding theorem, see [3, Section 4.1]; for Nash's original proof assuming at least C^3 regularity, see [12]; and for a generalization assuming $C^{k,\beta}$ -regularity with $k + \beta > 2$, see [11]), Theorem 1.1 implies the following.

Theorem 1.2. *Let $\alpha > 0$, $m \geq 2$, $\bar{n} \geq 0$, and let $(\bar{\Sigma}, g)$ be a Riemannian $(m + \bar{n})$ -manifold of class $C^{2,\alpha}$. If T be an area minimizing integral m -current in $\bar{\Sigma}$, then*

$$\dim_{\mathcal{H}_g}(\text{Sing}_i(T)) \leq m - 2.$$

Remark 1.3. Note that the assumption on Theorem 1.1 is weaker than what we ask for in Theorem 1.2. Indeed, if we consider Σ already embedded in \mathbb{R}^{m+n} and of class $C^{2,\alpha}$ as in Theorem 1.1, then its Riemannian structure (i.e., the metric tensor) is of class $C^{1,\alpha}$ which is not covered by Theorem 1.2.

Nash's embedding theorem remains a wide open problem for C^2 Riemannian manifolds. Specifically, the challenge lies in ensuring that the embedded Riemannian structure inherits C^2 regularity. While it is known that one can achieve $C^{1,\alpha}$ -regularity for the embedded Riemannian structure, the question of attaining C^2 regularity remains unanswered. With that being said, one would need to approach Theorem 1.2 intrinsically in order to relax the assumption from $C^{2,\alpha}$ to C^2 . Although

the authors expect such a result to be true, rerunning all the machinery presented in [6, 5, 7, 8] intrinsically seems quite challenging. Indeed, in the case that the density is 1, an intrinsic proof of Allard's ε -regularity theorem in C^2 Riemannian manifolds will be provided in [10].

It is well-known that the dimensional bound in Theorem 1.1 is optimal. Indeed, consider the 2-dimensional integral current T in \mathbb{R}^4 induced by $\{(z, w) \in \mathbb{C}^2 : z^2 = w^3\}$ (which, being a holomorphic subvariety of \mathbb{C}^2 , is area-minimizing by a famous theorem due to Federer), this current T clearly has 0 as a singular point, demonstrating the optimality of the dimensional bound in Theorem 1.1.

Theorem 1.1 was first proved by Almgren in his celebrated work [1], considering Σ an already embedded submanifold of some higher dimensional Euclidean space and of class C^5 . While Almgren's approach was innovative and groundbreaking, it is also intricate and lengthy. In their series of papers [4, 6, 5, 7, 8], De Lellis and Spadaro introduced new techniques to tackle the regularity theory developed by Almgren. They provided a much simpler and shorter proof of Theorem 1.1, in which they also achieved an improvement by weakening the assumption on the regularity of Σ from C^5 to $C^{3,\alpha}$. In this work, we further improve upon their results by extending them to the case where Σ is merely $C^{2,\alpha}$.

Outline of the proof

We will follow the approach introduced in [4, 6, 5, 7, 8] to prove Theorem 1.1. The techniques presented in [4, 5] do not depend on the $C^{3,\alpha}$ -regularity of Σ . In [5], the authors do require Σ to be at least C^2 , which is then applicable to the setting considered in the present work.

In contrast, in [6, 7, 8], the authors utilize the full strength of the $C^{3,\alpha}$ -regularity assumption. Specifically, in [7], they need to construct the so-called center manifold $\mathcal{M} \subset \Sigma$, which enjoys $C^{3,\alpha}$ -regularity due to $\Sigma \in C^{3,\alpha}$, to subsequently leverage this regularity in controlling certain error terms arising in [8] after some applications of [6]. The control over these error terms, with very specific decay exponents, is delicate and crucial for proceeding with proving the so-called almost-monotonicity of the frequency function and the blow-up argument in [8]. Therefore, the question of whether the $C^{3,\alpha}$ -regularity assumption could be eliminated has remained open.

In our approach, we weaken the $C^{3,\alpha}$ -regularity assumption by constructing what we term the external center manifold \mathcal{M}^* , potentially lying outside Σ (contrary to \mathcal{M} which is always contained in Σ), which culminates in several complexities when approaching Theorem 1.1. Nonetheless, our method of proof can still establish that \mathcal{M}^* is $C^{3,\alpha}$ (Theorem 3.7) using only the elliptic system induced by the $C^{2,\alpha}$ -regularity of Σ and the stationarity of T in Σ .

Another delicate issue in this work is the construction of the so-called \mathcal{M} -normal approximations N , as introduced in [7], which serve as an average-free approximation of T . These approximations, termed \mathcal{M}^* -normal approximations and denoted by N^* , are defined on \mathcal{M}^* in our setting. This enables us to utilize all the results in [6], as they solely rely on the $C^{3,\alpha}$ -regularity of the domain of N^* . However, it is crucial that N^* ranges within Σ , as we aim to utilize the stationarity of T in Σ to compare it with the first variation of the multi-valued graph of N^* , thereby obtaining precise error terms that lead to the almost-monotonicity of the frequency function. In Section 4, we provide a construction to ensure that the image of N^* is trapped within Σ .

The geometric constructions and the proof of the almost-monotonicity of the frequency function in [8] heavily rely also on the fact that $\mathcal{M} \subset \Sigma$, which is not necessarily true for \mathcal{M}^* . Hence, we provide a less geometric approach (compared to [8]) in Section 5 to carry out a blow-up argument that allow us to demonstrate Theorem 1.1.

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2 Preliminaries and Whitney decomposition

We will use the following notations: $\mathbf{B}(p, r) := \{x \in \mathbb{R}^{m+n} : |x - p| < r\}$, $\pi_0 := \mathbb{R}^m \times \{0\} \subset \mathbb{R}^{m+n}$. For any m -plane π and $x \in \pi$, we define $B_r(x, \pi) := \mathbf{B}(x, r) \cap \pi$ and $\mathbf{C}(x, r, \pi) = B_r(x, \pi) \times \pi^\perp$, when $\pi = \pi_0$ we simply omit it from the notation. We also fix \mathcal{H}^s for the s -dimensional Hausdorff measure in \mathbb{R}^{m+n} , $\omega_m := \mathcal{H}^m(\mathbf{B}_1(0))$, and $\dim_{\mathcal{H}}(S)$ to be the Hausdorff dimension of a set S . Let $\mathbf{E}(T, \mathbf{B}(p, r), \pi)$ to be the spherical excess of T in $\mathbf{B}(p, r)$ with respect to π as in [5]. For basics on Q -valued functions, we refer the reader to [4, 6]. We refer the reader to [9] for classical theory of currents. We will call any element T of $\mathbf{I}_m(U)$ an integral m -current in $U \subset \mathbb{R}^{m+n}$ and $\|T\|$ denotes its total variation measure.

Assumption 1. Let T be an m -dimensional integral current of \mathbb{R}^{m+n} with support in $\Sigma \cap \mathbf{B}_{6\sqrt{m}}$, where Σ is given by the graph of $\Psi_p : T_p \Sigma \cap \mathbf{B}_{7\sqrt{m}} \rightarrow T_p \Sigma^\perp$. We assume that

$$\Sigma \text{ is a } C^{2,\alpha}\text{-submanifold of } \mathbf{B}_{7\sqrt{m}}, \quad (2.1)$$

$$T \text{ is an area minimizing current in } \Sigma \cap \mathbf{B}_{6\sqrt{m}}, \quad (2.2)$$

$$\partial T \llcorner \mathbf{B}_{6\sqrt{m}} = 0, \quad (2.3)$$

$$\Theta^m(T, 0) = Q \in \mathbb{N} \setminus \{0\}, \quad (2.4)$$

$$\frac{\|T\|(\mathbf{B}_{6\sqrt{mr}})}{r^m} - Q\omega_m (6\sqrt{m})^m \leq \varepsilon_{cm}^2, \quad \forall r \leq 1, \quad (2.5)$$

$$\mathbf{m}_0 = \max\{\mathbf{E}, c(\Sigma)\} \leq \varepsilon_{cm}^2 \ll 1, \quad (2.6)$$

where ε_{cm} is a small positive number to be specified later, $\mathbf{E} := \mathbf{E}(T, \mathbf{B}_{6\sqrt{m}}) = \mathbf{E}(T, \mathbf{B}_{6\sqrt{m}}, \pi_0)$ and $c(\Sigma) := \sup_{p \in \Sigma \cap \mathbf{B}_{6\sqrt{m}}} \|D\Psi_p\|_{C^{1,\alpha}}$.

We will say that a constant is a *dimensional constant* when it depends only on m and n , and *geometric constant* when it depends on m, n and Q . We always use the notations C_0 and c_0 for large and small geometric constants, respectively. We recall that from [6, Lemma 1.6], we have as a consequence of Assumption 1, that

$$(\mathbf{p}\#T) \llcorner \mathbf{B}_{11\sqrt{m}/2} = Q \llbracket \mathbf{B}_{6\sqrt{m}} \rrbracket, \quad (2.7)$$

and the height bound estimate that will be helpful to derive more refined bounds. Precisely, there exists positive geometric constants $C_0, c_0 > 0$ such that

$$\mathbf{h} \left(T \llcorner \mathbf{B}_{\frac{23\sqrt{m}}{4}}, \mathbf{C}_{5\sqrt{m}} \right) \leq C_0 \mathbf{m}_0^{\frac{1}{2m}}, \text{ whenever } \varepsilon_{cm} \leq c_0. \quad (2.8)$$

2.1 Whitney decomposition, behaviour of the various parameters, and stopping conditions

We begin by setting up the Whitney decomposition that will allow us to construct the external center manifold. To define this Whitney decomposition, we define the so-called stopping conditions in which we use precise rates of decay for the excess and height of the current T to determine whether or not a m -cube should belong to such Whitney decomposition.

We will fix some notations on m -cubes and Whitney decomposition that will be used in this work. For each $j \in \mathbb{N}$, we denote by \mathcal{C}^j the family of closed dyadic cubes L of $\pi_0 := \mathbb{R}^m \times \{0\} \cong \mathbb{R}^m$ (such identification will be used without further mention) of the form

$$[a_1, a_1 + 2\ell] \times \cdots \times [a_m, a_m + 2\ell] \subset \pi_0,$$

where $\ell(L) := \ell$ and $2^{-j} = \ell(L)$ is half of the side-length of the cube, $a_i \in 2^{1-j}\mathbb{Z}$, and we also set

$$-4 \leq a_i \leq a_i + 2\ell(L) \leq 4. \quad (2.9)$$

We fix the notation for the center x_L of the cube L , i.e. $x_L := (a_1 + \ell, \dots, a_m + \ell)$. Next we set $\mathcal{C} := \bigcup_{j \in \mathbb{N}} \mathcal{C}^j$.

Definition 2.1. If $H, L \in \mathcal{C}$, we say that:

- (a) H is a *descendant* of L and L is an *ancestor* of H , if $H \subset L$;
- (b) H is a *child* of L and L is the *parent* of H , if $H \subset L$ and $\ell(H) = \frac{1}{2}\ell(L)$;
- (c) H and L are *neighbors* if $\frac{1}{2}\ell(L) \leq \ell(H) \leq 2\ell(L)$ and $H \cap L \neq \emptyset$;
- (d) $\{H_i\}_{i=N_0}^{i_0}$ is called *the genealogical tree of H* , if $H_{i_0} := H$, $H_{N_0} \in \mathcal{C}^{N_0}$, and each H_i is the parent of H_{i+1} .

Let us recall the classical well known notion of Whitney decomposition in which we decompose $[-4, 4]^m$ satisfies nice interactions among the cubes.

Definition 2.2. A *Whitney decomposition* of $[-4, 4]^m \subset \pi_0$ consists of a closed set $\Gamma \subset [-4, 4]^m$ and a family $\mathcal{W} \subset \mathcal{C}$ satisfying the following properties:

- (w1) $\Gamma \cup \bigcup_{L \in \mathcal{W}} L = [-4, 4]^m$, and Γ does not intersect any element of \mathcal{W} ;
- (w2) the interiors of any pair of distinct cubes $H, L \in \mathcal{W}$ are disjoint, i.e., $\mathring{H} \cap \mathring{L} = \emptyset$;
- (w3) if $H, L \in \mathcal{W}$ have nonempty intersection, then $\frac{1}{2}\ell(H) \leq \ell(L) \leq 2\ell(H)$.

Note that, if $H, L \in \mathcal{W}$ have nonempty intersection and (Γ, \mathcal{W}) is a Whitney decomposition, then H and L are neighbors. We now show the behaviour of the various parameters that we need to prove the existence of the external center manifold. This hierarchy of the parameters ensures that we can indeed choose such parameters in a way that all the assumptions of the statements of Section 3 are satisfied. Henceforth, we denote by $\varepsilon_{la} > 0$ and $\gamma_{la} > 0$ the geometric constants ε_1 and γ_1 given by the strong Lipschitz approximation, [5, Thm 2.4]. Note that the author in [5] only assume C^2 regularity of Σ .

Assumption 2 (Hierarchy of the parameters for the external center manifold). γ_h and γ_e are two fixed exponents satisfying:

$$\gamma_h = 4\gamma_e = \min \left\{ \frac{1}{2m}, \frac{\gamma_{ta}}{100} \right\}.$$

M_0 is positive real number and $N_0 = N_0(M_0)$ a natural number under the following assumptions:

$$M_0 \geq C_0(m, n, \bar{n}, Q) \geq 4, \text{ and } \sqrt{m}M_0 2^{7-N_0} \leq 1.$$

C_e and C_h are positive real numbers for which, throughout the paper, we always assume:

- $C_e = C_e(\gamma_e, M_0, N_0) \geq 6^m M_0^{-m} 2^{(N_0+6)(m+2-2\gamma_e)}$,
- $C_h = C_h(\gamma_h, M_0, N_0, C_e(\gamma_e, M_0, N_0)) \geq C_e^{1/2} C_0 \left(M_0 + 2^{N_0(1+\gamma_h)} \right)$.

Finally, $\varepsilon_{cm} = \varepsilon_{cm}(\gamma_e, \gamma_h, M_0, N_0, C_e, C_h) > 0$ will be a small parameter.

This hierarchy is the very same that appears in [7, Assump 1.8 and 1.9]. From now on, we give several properties of our Whitney decomposition given by the stopping cubes conditions. First of all, thanks to property (2.7), given a cube $H \in \mathcal{C}$, there exists $y_L \in \mathbb{R}^n$ such that $p_L := (x_L, y_L) \in \text{spt}(T)$. We also define

$$r_L := M_0 \sqrt{m} \ell(L).$$

We will usually call *generation of the cube* L the number $-\log_2(\ell(L)) \in \mathbb{N}$ and *satellite balls* the balls $\mathbf{B}(p_L, 2^6 r_L)$. We use an abuse of notation on the balls that we work with, for instance $B_r(p_L, \pi_H)$ is not precise since $p_L \notin \pi_H$ can totally occur, however, we understand it as $B_r(\mathbf{p}_{\pi_H}(p_L), \pi_H)$. This abuse of notation will make the text more fluid and clear, furthermore it does not affect the mathematics in it, since up to translations everything work properly.

Definition 2.3 (Stopping cubes conditions). For $L \in \mathcal{C}$, we define the families of cubes $\mathcal{S} \subset \mathcal{C}$ and $\mathcal{W} = \mathcal{W}_e \cup \mathcal{W}_h \cup \mathcal{W}_n \subset \mathcal{C}$ with the convention that

$$\mathcal{S}^j = \mathcal{S} \cap \mathcal{C}^j, \quad \mathcal{W}^j = \mathcal{W} \cap \mathcal{C}^j, \quad \text{and} \quad \mathcal{W}_{\square}^j = \mathcal{W}_{\square} \cap \mathcal{C}^j, \quad \text{for } \square = h, n, e.$$

We set $\mathcal{W}^i = \mathcal{S}^i = \emptyset$ for $i < N_0$. We define the next generations, i.e., $j \geq N_0$, inductively: if no ancestor of $L \in \mathcal{C}^j$ is in \mathcal{W} , then

$$\begin{aligned} L \in \mathcal{W}_e^j & \text{ if } \mathbf{E} \left(T, \mathbf{B}(p_L, 2^6 r_L) \right) > C_e \mathbf{m}_0 \ell(L)^{2-2\gamma_e}; & \text{(Stops for the excess)} \\ L \in \mathcal{W}_h^j & \text{ if } L \notin \mathcal{W}_e^j \text{ and } \mathbf{h} \left(T, \mathbf{B}(p_L, 2^6 r_L) \right) > C_h \mathbf{m}_0^{\frac{1}{2m}} \ell(L)^{1+\gamma_h}; & \text{(Stops for the height)} \\ L \in \mathcal{W}_n^j & \text{ if } L \notin \mathcal{W}_e^j \cup \mathcal{W}_h^j \text{ but it intersects an element of } \mathcal{W}^{j-1}; & \text{(Stops by neighboring)} \\ L \in \mathcal{S}^j & \text{ if } L \notin \mathcal{W}_e^j \cup \mathcal{W}_h^j \cup \mathcal{W}_n^j. & \text{(Subdividing cubes)} \end{aligned}$$

Lastly, we define what we call the *contact set* as follows

$$\mathbf{\Gamma} := [-4, 4]^m \setminus \bigcup_{L \in \mathcal{W}} L = \bigcap_{j \geq N_0} \bigcup_{L \in \mathcal{S}^j} L.$$

Observe that, if $j > N_0$ and $L \in \mathcal{S}^j \cup \mathcal{W}^j$, then necessarily its parent belongs to \mathcal{S}^{j-1} . Otherwise, its parent would not be subdivided, i.e., it would be a stopping cube and then the children must not exist.

2.2 Fine properties of the Whitney decomposition

All results disclosed in this subsection are not proved in this paper, one can look in [7, Subsections 4.2 and 4.3] for the proofs. We warn the reader that to find out the relation among the constants depending on the parameters M_0, N_0 , it is enough to carefully keep track of all choices of constants in [7, Subsections 4.2 and 4.3]. We show that the set of cubes \mathscr{W} defined by the stopping conditions and the π_0 -contact set Γ (as in Definition 2.3) is a Whitney decomposition in the sense of Definition 2.2.

Proposition 2.4 (Whitney decomposition). *Under Assumptions 1 and assuming $M_0\sqrt{m}2^{7-N_0} \leq 1$, then (Γ, \mathscr{W}) is a Whitney decomposition of $[-4, 4]^m \subset \pi_0$. If we additionally have that*

$$\begin{aligned} \varepsilon_{cm} &\leq \left(\frac{1}{C_0} \left(\sqrt{m} - \frac{1}{2} \right) \right)^m \leq c_0, \\ \gamma_h + \gamma_e &\leq 1, \\ C_e &= C_e(\gamma_e, M_0, N_0) \geq 6^m M_0^{-m} 2^{(N_0+6)(m+2-2\gamma_e)}, \\ C_h &= C_h(\gamma_h, M_0, N_0, C_e(\gamma_e)) \geq C_e^{\frac{1}{2}} C_0 \left(M_0 + 2^{N_0(1+\gamma_h)} \right), \end{aligned} \quad (2.10)$$

then we obtain that

$$\mathscr{W}^j = \emptyset \text{ for } j \leq N_0 + 6. \quad (2.11)$$

Remark 2.5. Equation (2.11) is clearly stating that all the cubes belonging to the first 6 generations are always subdivided into other cubes. \square

Remark 2.6. We attract the reader's attention to the following: C_h depends on γ_e through C_e , it means $\frac{C_h}{C_e^{1/2}}$ only depends on m, n, γ_h, M_0, N_0 . \square

A piece of essential information about our Whitney decomposition is the behavior we can extract when comparing quantities among different cubes. More specifically, we disclose estimates on how much the optimal planes $\pi_H, H \in \mathscr{W} \cup \mathscr{S}$, deviate from each other and from the basis π_0 , and we give upper bound for the height function \mathbf{h} of the current T in cylinders with center p_L and reference plane π_H for possibly different cubes H and L . Last but not least, we state that the portion of the support of the current $\text{spt}(T)$ inside those cylinders is contained in satellite balls.

Proposition 2.7 (Comparisons among cubes and optimal planes). *Under Assumption 1. Assume that the conclusion of Proposition 2.4 holds and*

$$\begin{aligned} \max \left\{ \frac{64}{7}, \frac{3}{36}, \frac{3}{64}, \frac{1}{36} \right\} &\leq M_0 \leq \frac{2^{-7+N_0}}{\sqrt{m}}, \quad \text{and} \quad \gamma_h + \gamma_e \leq 1, \\ \varepsilon_{cm} = \varepsilon_{cm}(M_0, N_0, C_h) &\leq c_0 \min \left\{ \left(M_0 2^{-N_0} \right)^m, \left(2^{6-N_0} M_0 \right)^{m^2}, 2^{-mN_0}, \right. \\ &\quad \left. 2^{\frac{m^2(6-N_0)}{2} - mN_0} M_0^{\frac{m^2}{2} + m}, \max^{-1} \left\{ C_h^m, 2^{\frac{m^2(N_0-6)}{2}} M_0^{\frac{m^2}{2}} \right\} \right\}, \end{aligned} \quad (2.12)$$

then we get that

$$\mathbf{B}(p_H, 2^6 r_H) \subset \mathbf{B}(p_L, 2^6 r_L) \subset \mathbf{B}_{5\sqrt{m}} \text{ for all } H, L \in \mathscr{W} \cup \mathscr{S} \text{ with } H \subset L. \quad (2.13)$$

Assume further that

$$C_e = C_e(M_0, N_0) \geq \left(2^{-6+N_0} 6 M_0^{-1} \right)^{\frac{m}{2}}, \quad \text{and} \quad C_h = C_h(M_0, N_0) \geq C_0 \max \left\{ 2^{N_0}, \left(2^{N_0-6} M_0^{-1} \right)^{\frac{m}{2}} \right\}.$$

If $H, L \in \mathscr{W} \cup \mathscr{S}$, and either (a) $H \subset L$, or (b) $H \cap L \neq \emptyset$ and $2^{-1}\ell(L) \leq \ell(H) \leq \ell(L)$, then

- (i) $|\hat{\pi}_H - \pi_H| \leq C_e \mathbf{m}_0^{\frac{1}{2}} \ell(L)^{1-\gamma_e}$;
- (ii) $|\pi_H - \pi_L| \leq C_e \mathbf{m}_0^{\frac{1}{2}} \ell(L)^{1-\gamma_e}$;
- (iii) $|\pi_H - \pi_0| \leq C_e \mathbf{m}_0^{\frac{1}{2}}$;
- (iv) $\mathbf{h}(T, \mathbf{C}(p_H, 36r_H, \pi_0)) \leq C_h \mathbf{m}_0^{\frac{1}{2m}} \ell(H)$ and $\text{spt}(T) \cap \mathbf{C}(p_H, 36r_H, \pi_0) \subset \mathbf{B}(p_H, 2^6 r_H)$;
- (v) $\mathbf{h}(T, \mathbf{C}(p_L, 36r_L, \pi)) \leq C_h \mathbf{m}_0^{\frac{1}{2m}} \ell(L)^{1+\gamma_h}$ and $\text{spt}(T) \cap \mathbf{C}(p_L, 36r_L, \pi) \subset \mathbf{B}(p_L, 2^6 r_L)$ for $\pi = \pi_H, \hat{\pi}_H$.

We exhibit upper bounds for the excess and height of stopping cubes. We recall that the stopping conditions give only lower bounds to the excess and height.

Corollary 2.8 (Upper bounds for stopping cubes). *Provided the conclusions of Proposition 2.7 hold. For every $L \in \mathscr{W}$, we have that*

$$C_e \mathbf{m}_0^{\frac{1}{2}} \ell(L)^{2-2\gamma_e} < \mathbf{E}\left(T, \mathbf{B}(p_L, 2^6 r_L)\right) \leq 2^{m+2-2\gamma_e} C_e \mathbf{m}_0^{\frac{1}{2}} \ell(L)^{2-2\gamma_e}, \quad (2.14)$$

$$C_h \mathbf{m}_0^{\frac{1}{2m}} \ell(L)^{1+\gamma_h} < \mathbf{h}\left(T, \mathbf{B}(p_L, 2^6 r_L)\right) \leq 2^{1+\gamma_h} (C_h + M_0 \sqrt{m} C_e) \mathbf{m}_0^{\frac{1}{2m}} \ell(L)^{1+\gamma_h}. \quad (2.15)$$

The Constancy Lemma [9, 4.1.17] is used to prove that the property (2.7) also holds for tilted cylinders.

Lemma 2.9 (Full projection property on tilted planes). *Under Assumptions 1, assume the conclusions of Proposition 2.7 hold. We additionally assume that*

$$\varepsilon_{cm} = \varepsilon_{cm}(M_0, N_0, C_e) \leq \frac{C_0}{(C_e)^m}. \quad (2.16)$$

Let $H, L \in \mathscr{W} \cup \mathscr{S}$ such that either (a) $H \subset L$, or (b) $H \cap L \neq \emptyset$ and $2^{-1} \ell(L) \leq \ell(H) \leq \ell(L)$. Then, for $\pi = \pi_H, \hat{\pi}_H$, we have that

$$\mathbf{p}_{\pi\#}(T \llcorner \mathbf{C}(p_L, 32r_L, \pi)) = Q \llbracket \mathbf{B}_{32r_L}(p_L, \pi) \rrbracket.$$

3 Existence of the external center manifold

We start fixing a convolution kernel $\varrho \in C^\infty(\mathbf{B}_1)$ which is radial and satisfies $\int \varrho = 1$ and $\int |x|^2 \varrho(x) dx = 0$. For $t > 0$, we define ϱ_t as $\varrho_t(x) := t^{-m} \varrho(x/t)$. We will also always assume that $\vartheta \in C_c^\infty([-\frac{17}{16}, \frac{17}{16}], [0, 1])$.

3.1 Lipschitz approximations on tilted cylinders and interpolating functions

Definition 3.1 (π -approximations). Let $L \in \mathscr{S} \cup \mathscr{W}$ and π an arbitrary m -dimensional plane. If $T \llcorner \mathbf{C}(p_L, 32r_L, \pi)$ is under the assumptions of [5, Thm 2.4] in the cylinder $\mathbf{C}(p_L, 32r_L, \pi)$, then the Q -valued map $f_L : \mathbf{B}_{8r_L}(p_L, \pi) \rightarrow \mathcal{A}_Q(\pi^\perp)$ given by [5, Thm 2.4] is called π -approximation of T in $\mathbf{C}(p_L, 8r_L, \pi)$.

Definition 3.2 (Smoothed average). The single valued map $\hat{h}_L : B_{7r_L}(p_L, \pi) \rightarrow \pi^\perp$ given by $\hat{h}_L := (\boldsymbol{\eta} \circ f_L) * \varrho_{\ell(L)}$ is called *smoothed average of the π -approximation*.

Proposition 3.3 (Existence of interpolating functions). *Assume the conclusions of Proposition 2.7 holds true. Then, we have*

- (i) *For $\pi \in \{\pi_H, \hat{\pi}_H\}$, we have that $(\mathbf{p}_\pi)_\# T \lrcorner \mathbf{C}(p_L, 32r_L, \pi) = Q \llbracket B_{32r_L}(p_L, \pi) \rrbracket$ and T satisfies the assumptions of [5, Theorem 2.4].*

Furthermore, let f_{HL} be the π_H -approximation of T in $\mathbf{C}(p_L, 8r_L, \pi_H)$ and $\hat{h}_{HL} := (\boldsymbol{\eta} \circ f_{HL}) * \varrho_{\ell(L)}$ its smoothed average. Set $\varkappa := \pi_H^\perp \cap T_{p_H} \Sigma$, $\bar{h}_{HL} := \mathbf{p}_{T_{p_H} \Sigma}(\hat{h}_{HL})$, and $h_{HL} := (\bar{h}_{HL}, \Psi_{p_H} \circ \bar{h}_{HL})$. We then have

- (ii) *there is a smooth function $g_{HL}^* : B_{4r_L}(p_L, \pi_0) \rightarrow \pi_0^\perp$ such that $\mathbf{G}_{g_{HL}^*} = \mathbf{G}_{\hat{h}_{HL}} \lrcorner \mathbf{C}(p_L, 4r_L, \pi_0)$,*
 (iii) *there is a smooth function $g_{HL} : B_{4r_L}(p_L, \pi_0) \rightarrow \pi_0^\perp$ such that $\mathbf{G}_{g_{HL}} = \mathbf{G}_{h_{HL}} \lrcorner \mathbf{C}(p_L, 4r_L, \pi_0)$.*

Remark 3.4. Note that \bar{h}_{HL} 's definition includes a composition with a function depending on Σ , however the function $\mathbf{p}_{T_{p_H} \Sigma}$ is fixed for each H , i.e., it is a fixed linear function. Hence, \bar{h}_{HL} does not depend on the $C^{2,\alpha}$ -regularity of Σ .

Definition 3.5 (Glued interpolation). Set $\vartheta_L(x) := \vartheta(\frac{x-x_L}{\ell(L)})$ and define, for each $x \in (-4, 4)^m$, the *glued interpolating functions at step j* to be

$$\hat{\varphi}_j := \frac{\sum_{L \in \mathcal{P}^j} \vartheta_L g_L}{\sum_{L \in \mathcal{P}^j} \vartheta_L} \text{ and } \varphi_j^* := \frac{\sum_{L \in \mathcal{P}^j} \vartheta_L g_L^*}{\sum_{L \in \mathcal{P}^j} \vartheta_L}.$$

3.2 Construction of the external center manifold \mathcal{M}^*

The main goal of this subsection is to show (Theorem 3.7) the existence of the external center manifold \mathcal{M}^* which is a $C^{3,\kappa}$, $\kappa > 0$, submanifold of \mathbb{R}^{m+n} . The external center manifold does not necessarily lie within Σ . Furthermore, we can construct (Theorem 3.6) a center manifold \mathcal{M} similarly to the classical approach (as in [7]) that lies inside Σ , it will be, however, only of class $C^{2,\alpha}$.

Theorem 3.6 (Existence of the center manifold \mathcal{M}). *Under Assumption 1 and Assumption 2, there is a positive constant $C = C(\gamma_e, \gamma_h, M_0, N_0, C_e, C_h)$ such that, for any $j \geq N_0$, the glued interpolating functions φ_j satisfy*

- (A) $\|\varphi_j\|_{C^0} \leq C \mathbf{m}_0^{1/2m}$;
 (B) $\|D^2 \varphi_j\|_{C^{0,\alpha}} \leq C \mathbf{m}_0^{1/2}$;
 (C) *if $H \in \mathcal{W}^i$ and L is the cube concentric to H with $\ell(L) = 9\ell(H)/8$, then $\varphi_j = \varphi_k$ on L for any $j, k \geq i + 2$;*
 (D) φ_j convergence in C^2 to a map $\boldsymbol{\varphi}$ and $\mathcal{M} := \text{graph}(\boldsymbol{\varphi}|_{(-4,4)^m})$ is a $C^{2,\alpha}$ submanifold of Σ .

Theorem 3.7 (Existence of the external center manifold \mathcal{M}^*). *Under Assumption 1 and Assumption 2, there is a positive constant $C = C(\gamma_e, \gamma_h, M_0, N_0, C_e, C_h)$ such that, for any $j \geq N_0$, the glued interpolating functions φ_j^* satisfy*

- (A*) $\|\varphi_j^*\|_{C^0} \leq C\mathbf{m}_0^{1/2m}$;
- (B*) $\|D^3\varphi_j^*\|_{C^{0,\kappa}} \leq C\mathbf{m}_0^{1/2}$;
- (C*) if $H \in \mathcal{W}^i$ and L is the cube concentric to H with $\ell(L) = 9\ell(H)/8$, then $\phi_j^* = \phi_k^*$ on L for any $j, k \geq i + 2$;
- (D*) φ_j^* convergence in C^3 to a map φ^* and $\mathcal{M}^* := \text{graph}(\varphi^*|_{(-4,4)^m})$ is a $C^{3,\kappa}$ submanifold of \mathbb{R}^{m+n} .

In order to prove the existence of \mathcal{M} and \mathcal{M}^* , we implement an argument similar to [7]. We start showing how to derive both existences from the constructions estimates that we state below.

Proposition 3.8 (Construction estimates). *Under Assumption 1 and Assumption 2, there is a positive constant $C = C(\gamma_e, \gamma_h, M_0, N_0, C_e, C_h)$ such that, setting $\kappa = \min\{\alpha, \gamma_a\}/2$, the following hold:*

- (i) $\|g_H\|_{C^0(B)} + \|g_H^*\|_{C^0(B)} \leq C\mathbf{m}_0^{1/2m}$ and $\|g_H\|_{C^{2,\alpha}(B)} + \|g_H^*\|_{C^{3,\kappa}(B)} \leq C\mathbf{m}_0^{1/2}$ for $B := B_{4r_H}(x_H, \pi_0)$;
- (ii) if $H \cap L \neq \emptyset$, then $\|g_H - g_L\|_{C^l(B_{r_H}(x_H, \pi_0))} \leq C\mathbf{m}_0^{1/2}\ell(L)^{3+\min\{\alpha, \gamma_a\}-h}$ for any $l \in \{0, 1, 2\}$ and $\|g_H^* - g_L^*\|_{C^h(B_{r_H}(x_H, \pi_0))} \leq C\mathbf{m}_0^{1/2}\ell(L)^{3+2\kappa-l}$ for every $h \in \{0, 1, 2, 3\}$;
- (iii) $|D^2g_H(x_H) - D^2g_L(x_L)| \leq C\mathbf{m}_0^{1/2}|x_H - x_L|^{1+\min\{\alpha, \gamma_a\}}$ and we also have $|D^3g_H^*(x_H) - D^3g_L^*(x_L)| \leq C\mathbf{m}_0^{1/2}|x_H - x_L|^\kappa$;
- (iv) $\|g_H - y_H\|_{C^0} + \|g_H^* - y_H\|_{C^0} \leq C\mathbf{m}_0^{1/2m}\ell(H)$ and $|\pi_H - T_{(x, g_H(x))}\mathbf{G}_{g_H}| + |\pi_H - T_{(x, g_H^*(x))}\mathbf{G}_{g_H^*}| \leq C\mathbf{m}_0^{1/2}\ell(H)^{1-\gamma_e}$ for all $x \in H$;
- (v) if L' is a cube concentric to $L \in \mathcal{W}^j$ with $\ell(L') = 9\ell(L)/8$, we have $\|\hat{\varphi}_i - g_L\|_{L^1(L')} + \|\varphi_i^* - g_L^*\|_{L^1(L')} \leq C\mathbf{m}_0^{1/2}\ell(L)^{m+3+\min\{\alpha, \gamma_a\}}$ for any $i \geq j$.

Proof of Theorem 3.6 and Theorem 3.7. Define $\chi_H := \vartheta_H / (\sum_{L \in \mathcal{P}^j} \vartheta_L)$ for each $H \in \mathcal{P}^j$ and $\varphi_j(x) := (\bar{\varphi}_j(x), \Psi(x, \bar{\varphi}_j(x)))$. Note that

$$\sum_{H \in \mathcal{P}^j} \chi_H = \sum_{H \in \mathcal{P}^j(L)} \chi_H = 1 \text{ on } [-4, 4]^m, \quad (3.1)$$

$$\|D^l \chi_H\|_{C^0} \leq C_0 \ell(H)^{-l}, \quad \forall l \in \mathbb{N}, \quad (3.2)$$

$$\|D^l \chi_H\|_{C^{0,\theta}} \leq \frac{C(\theta)}{\ell(H)^{l+\theta}}. \quad (3.3)$$

By definition of $\hat{\varphi}_j$ and φ_j^* , we have

$$\|\hat{\varphi}_j\|_{C^0} + \|\varphi_j^*\|_{C^0} \leq \sum_{L \in \mathcal{P}^j(H)} \|g_L\|_{C^0} + \|g_L^*\|_{C^0} \stackrel{\text{(ii) of Prop. 3.8}}{\leq} C\mathbf{m}_0^{1/2m},$$

where the summation sign disappeared by reasons of the cardinality of $\mathcal{P}^j(H)$ is bounded by a dimensional constant C_0 . This and the definition of φ_j and φ_j^* conclude the proof of (A) and (A*). From (3.1), whenever $x \in H$, we achieve that

$$\hat{\varphi}_j(x) = g_H(x) + \sum_{L \in \mathcal{P}^j(H)} (g_L - g_H)(x)\chi_L(x) \text{ and } \varphi_j^*(x) = g_H^*(x) + \sum_{L \in \mathcal{P}^j(H)} (g_L^* - g_H^*)(x)\chi_L(x). \quad (3.4)$$

Differentiating the first equation in (3.4), we infer that

$$\begin{aligned}
\|D^2\hat{\varphi}_j\|_{C^{0,\alpha}} &\leq \|D^2g_H\|_{C^{0,\alpha}} + C_0 \sum_{l=0}^2 \sum_{L \in \mathcal{P}^j(H)} \left(\|D^l(g_H - g_L)\|_{C^{0,\alpha}} \|D^{2-l}\chi_L\|_{C^0} \right. \\
&\quad \left. + \|D^l(g_H - g_L)\|_{C^0} \|D^{2-l}\chi_L\|_{C^{0,\alpha}} \right) \\
&\stackrel{(3.2),(3.3)}{\leq} \|D^2g_H\|_{C^{0,\alpha}} + C(\alpha) \sum_{l=0}^2 \sum_{L \in \mathcal{P}^j(H)} \ell(L)^{l-2} \left(\|D^l(g_H - g_L)\|_{C^{0,\alpha}} \right. \\
&\quad \left. + \|D^l(g_H - g_L)\|_{C^0} \ell(L)^{-\alpha} \right) \\
&\stackrel{(i) \text{ of Prop. 3.8}}{\leq} C\mathbf{m}_0^{1/2} + C(\alpha) \sum_{l=0}^2 \sum_{L \in \mathcal{P}^j(H)} \ell(L)^{l-2} \left(\|D^l(g_H - g_L)\|_{C^{0,\alpha}} \right. \\
&\quad \left. + \|D^l(g_H - g_L)\|_{C^0} \ell(L)^{-\alpha} \right).
\end{aligned}$$

Notice that for each $l \in \{0, 1, 2\}$ and $L \in \mathcal{P}^j(H)$, it holds

$$\ell(L)^{l-2} \|D^l(g_H - g_L)\|_{C^{0,\alpha}} \stackrel{(ii) \text{ of Prop. 3.8}}{\leq} C\mathbf{m}_0^{1/2} \ell(L)^{1-\alpha+\min\{\alpha, \gamma_{\ell a}\}},$$

whereas we also obtain

$$\|D^l(g_H - g_L)\|_{C^0} \ell(L)^{l-2-\alpha} \stackrel{(ii) \text{ of Prop. 3.8}}{\leq} C\mathbf{m}_0^{1/2} \ell(L)^{1-\alpha+\min\{\alpha, \gamma_{\ell a}\}},$$

The last three inequalities lead to

$$\|D^2\hat{\varphi}_j\|_{C^{0,\alpha}(H)} \leq C\mathbf{m}_0^{1/2} \ell(L)^{1-\alpha+\min\{\alpha, \gamma_{\ell a}\}}, \quad (3.5)$$

where we use that the cardinality of $\mathcal{P}^j(H)$ is bounded by a dimensional constant. Note that the estimate on the Holder norm in (3.5) is only on the cube H , we need to extend it uniformly to the whole cube $(-4, 4)^m$ to prove (B). To that end, we fix $x, y \in (-4, 4)^m$ and $H, L \in \mathcal{P}^j$ such that $x \in H$ and $y \in L$.

If $H \cap L \neq \emptyset$, the proof is trivial by (3.5) and the triangle inequality. If $H \cap L = \emptyset$, we prove the last inequality as follows. Without loss of generality, assume that $\ell(L) \geq \ell(H)$, then by simple geometric considerations, we have

$$\max\{|x - x_H|, |y - x_L|\} \leq \sqrt{m}\ell(L) \leq 2\sqrt{m}|x - y|.$$

Using that $\hat{\varphi}_j \equiv g_Y$ on a neighborhood of x_Y for any cube $Y \in \mathcal{P}^j$, we thus obtain

$$\begin{aligned}
|D^2\hat{\varphi}_j(x) - D^2\hat{\varphi}_j(y)| &\leq |D^2\hat{\varphi}_j(x) - D^2\hat{\varphi}_j(x_H)| + |D^2g_H(x_H) - D^2g_L(x_L)| + |D^2\hat{\varphi}_j(x_L) - D^2\hat{\varphi}_j(y)| \\
&\stackrel{(iii) \text{ of Prop. 3.8,(3.5)}}{\leq} C\mathbf{m}_0^{1/2} |x_H - x_L|^{1-\alpha+\min\{\alpha, \gamma_{\ell a}\}}.
\end{aligned}$$

This inequality and an easy application of the chain rule on φ_j finish the proof of (B). Notice that the proof of (B*) goes in the very same lines of the four previous displayed equations with the use of the bound on the *third* derivatives of g_H^* and g_L^* provided by Proposition 3.8 in items (i), (ii), and (iii).

Fix $H \in \mathcal{W}^i$ and $j \geq i + 2$, by construction, we have $\mathcal{P}^j(H) = \mathcal{P}^{i+2}(H) \subset \mathcal{W}$. If L is the cube concentric to H with $\ell(L) = 9\ell(H)/8$, recalling the definition of ϑ_Y , we easily ensure $\text{spt}(\vartheta_Y) \cap L = \emptyset$

for all cubes $Y \notin \mathcal{P}^j(H)$. Hence, (C) and (C*) are verified. We finally start the proof of (D). Differentiating the first equality in (3.4) and putting (3.2) into account, for $l \in \{0, 1, 2\}$, we obtain

$$\begin{aligned} \|D^l \hat{\varphi}_j\|_{C^0} &\leq \|D^l g_H\|_{C^0} + \sum_{k=0}^l \sum_{L \in \mathcal{P}^j(H)} \|D^k (g_H - g_L)\|_{C^0} \ell(L)^{k-l} \\ &\stackrel{\text{(i) of Prop. 3.8}}{\leq} C \mathbf{m}_0^{1/2} + \sum_{k=0}^l \sum_{L \in \mathcal{P}^j(H)} \|D^k (g_H - g_L)\|_{C^0} \ell(L)^{k-l} \\ &\stackrel{\text{(iii) of Prop. 3.8}}{\leq} C \mathbf{m}_0^{1/2}. \end{aligned} \quad (3.6)$$

Pick a point $x \in (-4, 4)^m$ such that $x \in H \cap L$ with $L \in \mathcal{P}^j$ and $H \in \mathcal{P}^{j+1}$. Notice that

$$\hat{\varphi}_j(x_H) = g_H(x_H) \text{ and } \hat{\varphi}_{j+1}(x_L) = g_L(x_L), \quad (3.7)$$

thus the following holds

$$\begin{aligned} |\hat{\varphi}_j(x) - \hat{\varphi}_{j+1}(x)| &\leq |\hat{\varphi}_j(x) - \hat{\varphi}_j(x_H)| + |g_H(x_H) - g_L(x_L)| + |\hat{\varphi}_{j+1}(x_L) - \hat{\varphi}_{j+1}(x)| \\ &\leq C_0 \left(\|D \hat{\varphi}_j\|_{C^0} + \|D \hat{\varphi}_{j+1}\|_{C^0(L)} \right) \ell(L) + |g_H(x_H) - g_L(x_L)| \\ &\quad + |g(x_L) - y_L| + |p_H - p_L| \\ &\stackrel{(3.6), (*)}{\leq} C \mathbf{m}_0^{1/2} \ell(L) = C 2^{-j}, \end{aligned}$$

in (*) we use the same computation as in [13, Eq. 4.46]. Notice that the same bound holds to φ_j due to the Lipschitz continuity of Ψ . So, passing the last inequality to the limit, we obtain that $(\varphi_j)_{j \geq N_0}$ uniformly converges to a map φ . It is straightforward to derive the C^2 convergence, indeed it is concluded as follows

$$\begin{aligned} |D^2 \hat{\varphi}_j(x) - D^2 \hat{\varphi}_{j+1}(x)| &\stackrel{(3.7)}{\leq} \frac{|D^h \hat{\varphi}_j(x) - D^2 \hat{\varphi}_j(x_H)|}{|x - x_H|^\alpha} |x - x_H|^\alpha + |D^2 g_H(x_H) - D^2 g_L(x_L)| \\ &\quad + \frac{|D^2 \hat{\varphi}_{j+1}(x_L) - D^2 \hat{\varphi}_{j+1}(x)|}{|x - x_H|^\alpha} |x - x_H|^\alpha \\ &\stackrel{\text{(iii) of Prop. 3.8, (3.5)}}{\leq} C \ell(L)^{1+\min\{\alpha, \gamma_{\mathbf{u}_a}\}} = C 2^{-j(1+\min\{\alpha, \gamma_{\mathbf{u}_a}\})}. \end{aligned}$$

The chain rule and the definition of φ_j assures that this bound is also true for φ_j . Again, using the same argument and the bounds on the third derivatives given in Proposition 3.8, we obtain

$$|D^3 \varphi_j^*(x) - D^3 \varphi_{j+1}^*(x)| \leq C 2^{-2j\kappa}.$$

At last, the Holder regularity of the external center manifold $\mathcal{M}^* := \text{graph}(\varphi^*|_{(-4,4)^m})$ and of the center manifold $\mathcal{M} := \text{graph}(\varphi|_{(-4,4)^m})$ are a consequence of the convergences above and (B*) and (B), respectively. \square

3.3 Proof of the construction estimates for \mathcal{M}^* through the elliptic system

Definition 3.9 (Tangential parts). Fix $H \in \mathcal{P}^j$ and let \varkappa be the orthogonal complement of π_H in $T_{p_H} \Sigma$. Given $p \in \mathbb{R}^{m+n}$, any set $\Omega \subset \pi_H$, and any function $\chi : p + \Omega \rightarrow \pi_H^\perp$, the map $\mathbf{p}_\varkappa \circ \chi$ will be called the *tangential part* of χ and denoted by $\bar{\chi}$.

As it is well-known, from the minimality of T , one can derive an elliptic systems that leads to great estimates on the π_L -approximations that are gathered in Proposition 3.10. These estimates in turn lead (not trivially, see Lemmas 3.11 and 3.13) to the bounds on the derivatives given in the construction estimates (Proposition 3.8).

We mention that the following proof differs from the proof presented in [7], since the authors rely on the C^{3,ε_0} -regularity of Σ to get the bounds on the error terms that will appear naturally when investigating the first variation of the current $\mathbf{G}_{f_{HL}}$ (see (3.18)).

Proposition 3.10 (Elliptic system). *Assume Assumption 1 and Assumption 2, we denote $B := B_{8r_L}(p_L, \pi_H)$ and $\sigma := \min\{\alpha, \gamma_{la}\}$. There is a positive constant $C = C(\gamma_e, \gamma_h, M_0, N_0, C_e, C_h)$ such that*

$$\left| \int_B D(\boldsymbol{\eta} \circ \bar{f}_{HL}) : D\zeta + (\mathbf{p}_{\pi_H}(x - p_H)^t) \cdot \mathbf{L} \cdot \zeta \right| \leq C \mathbf{m}_0 r_H^{1+\sigma} \left(r_H \|\zeta\|_{C^1(B)} + \|\zeta\|_{C^0(B)} \right), \quad (3.8)$$

for any test function ζ . Moreover

$$\|\bar{h}_{HL} - \boldsymbol{\eta} \circ \bar{f}_{HL}\|_{L^1(B_{7r_L}(x_L, \pi_H))} \leq C \mathbf{m}_0 r_L^{m+3+\sigma}. \quad (3.9)$$

Proof. We will denote geometric constants by C_0 , whereas C denotes constants depending upon the parameters $\gamma_h, \gamma_e, M_0, N_0, C_e$ and C_h . In order to simplify the notation, we fix a system of coordinates $(x, y, z) \in \pi_H \times \varkappa \times (T_{p_H} \Sigma)^\perp$ so that $p_H = (0, 0, 0)$. Although the domains of the various maps are subsets Ω of $p_L + \pi_H$, from now on we will consider them as functions of x ; i.e., we shift their domains to $\mathbf{p}_{\pi_H}(\Omega)$. We also use Ψ_H for the map Ψ_{p_H} which graph gives Σ as in Assumption 1. Recall that $\Psi_H(0, 0) = 0$, $D\Psi_H(0, 0) = 0$ and $\|\Psi_H\|_{C^{2,\alpha}} \leq \mathbf{m}_0^{1/2}$.

Given a test function ζ and any point $q = (x, y, z) \in \Sigma$, we consider the vector field $\chi(q) = (0, \zeta(x), D_y \Psi(x, y) \cdot \zeta(x))$. The vector field χ is tangent to Σ , and therefore $\delta T(\chi) = 0$. Thus, we easily conclude that

$$|\delta \mathbf{G}_{f_{HL}}(\chi)| = |\delta \mathbf{G}_{f_{HL}}(\chi) - \delta T(\chi)| \leq C_0 \int_{\mathcal{C}_{(p_L, 8r_L, \pi_H)}} |D\chi| d\|\mathbf{G}_{f_{HL}} - T\|. \quad (3.10)$$

Since we have $\|D\Psi_H\| \leq \mathbf{m}_0^{1/2}$, choosing ε_2 sufficiently small, we achieve

$$|\chi| \leq 2|\zeta| \text{ and } |D\chi| \leq 2|D\zeta| + 2|\zeta|. \quad (3.11)$$

We denote $\mathbf{E}_{HL} := \mathbf{E}(T, \mathbf{C}(p_L, 32r_L, \pi_H))$ and recall the estimates in [5, Theorem 1.4], where $K_{HL} \subset B_{8r_L}(p_L, \pi_H)$ is the bad set,

$$|Df_{HL}| \leq C_0 \mathbf{E}_{HL}^{\gamma_{la}} + C_0 r \mathbf{A} \leq C \mathbf{m}_0^{\gamma_{la}} r_H^{\gamma_{la}}, \quad (3.12)$$

$$|f_{HL}| \leq C_0 \mathbf{h}(T, \mathbf{C}(p_L, 32r_L, \pi_H)) + C_0 \left(\mathbf{E}_{HL}^{1/2} + r_H \mathbf{A} \right) r_H \leq C \mathbf{m}_0^{\frac{1}{2}} r_H^{1+\gamma_h}, \quad (3.13)$$

$$\int_{B_{8r_L}(p_L, \pi_H)} |Df_{HL}|^2 \leq C_0 \mathbf{E}_{HL} r_H^m \leq C \mathbf{m}_0 r_H^{m+2-2\gamma_e}, \quad (3.14)$$

$$|B_{8r_L}(p_L, \pi_H) \setminus K_{HL}| \leq C_0 \mathbf{E}_{HL}^{\gamma_{la}} \left(\mathbf{E}_{HL} + r_H^2 \mathbf{A}^2 \right) r_H^m \leq C \mathbf{m}_0^{1+\gamma_{la}} r_H^{m+(2-2\gamma_e)(1+\gamma_{la})}, \quad (3.15)$$

$$\begin{aligned} \left| \|T\|(\mathbf{C}(p_L, 8r_L, \pi_H)) - |B_{8r_L}(p_L, \pi_H)| - \frac{1}{2} \int_{B_{8r_L}(p_L, \pi_H)} |Df_{HL}|^2 \right| \\ \leq C_0 \mathbf{E}_{HL}^{\gamma_{la}} \left(\mathbf{E}_{HL} + r_H^2 \mathbf{A}^2 \right) r_H^m \leq C \mathbf{m}_0^{1+\gamma_{la}} r_H^{m+(2-2\gamma_e)(1+\gamma_{la})}. \end{aligned} \quad (3.16)$$

Concerning (3.13) observe that the statement of [5, Theorem 1.4] indeed bounds $\text{osc}(f_{HL})$. Moreover, in our case we have $p_H = (0, 0, 0) \in \text{spt}(T)$ and $\text{spt}(T) \cap \mathbf{G}_{f_{HL}} \neq \emptyset$. Thus we conclude $|f_{HL}| \leq C_0 \text{osc}(f_{HL}) + C_0 \mathbf{h}(T, \mathbf{C}(p_L, 32r_L, \pi_H))$. Taking selections $f_{HL} = \sum_i \llbracket f_{i,HL} \rrbracket$ and $\bar{f}_{HL} = \sum_i \llbracket \bar{f}_{i,HL} \rrbracket$, we have that $\text{spt}(\mathbf{G}_{f_{HL}}) \subset \Sigma$ implies

$$f_{HL} = \sum_{i=1}^Q \llbracket (\bar{f}_{i,HL}, \Psi_H(\bar{f}_{i,HL})) \rrbracket. \quad (3.17)$$

We have from [6, Theorem 4.1] (straightforwardly estimating Err with (3.12) and (3.14)) the following inequality for the first variation

$$\begin{aligned} \delta \mathbf{G}_{f_{HL}}(\chi) &= \text{Err} + \int_{\mathbf{B}_{8r_L}(p_L, \pi_H)} \sum_{i=1}^Q (S_1 + S_2 + S_3) : (S_4 + S_5) + D_x \zeta : D_x \bar{f}_{i,HL}, \\ \text{Err} &\leq C \|\zeta\|_{C^1} \mathbf{m}_0^{1+\gamma a} r_H^{m+2-2\gamma e+\gamma a}, \end{aligned} \quad (3.18)$$

where we are using the following notations

$$\begin{aligned} S_1 &:= D_{xy} \Psi_H(x, \bar{f}_{i,HL}(x)) \cdot \zeta, & S_2 &:= (D_{yy} \Psi_H(x, \bar{f}_{i,HL}(x)) \cdot D_x \bar{f}_{i,HL}) \cdot \zeta \\ S_3 &:= D_y \Psi_H(x, \bar{f}_{i,HL}(x)) \cdot D_x \zeta, & S_4 &:= D_x \Psi_H(x, \bar{f}_{i,HL}(x)), & S_5 &:= D_y \Psi_H(x, \bar{f}_{i,HL}(x)) \cdot D_x \bar{f}_{i,HL}. \end{aligned}$$

Our goal now is to use (3.18) and (3.10) to prove the proposition. To that end, we will estimate all related terms. Let us start with a Taylor expansion for $\Psi_H \in C^{2,\alpha}$ using [2, Proposition 2.1] and the aforementioned considerations for Ψ_H to derive the following bound

$$\begin{aligned} |D\Psi_H(x, y) - D_x D\Psi_H(0, 0) \cdot x - D_y D\Psi_H(0, 0) \cdot y| &\leq C_0 \|\Psi_H\|_{C^{2,\alpha}} (|x|^2 + |y|^2)^{1/2+\alpha/2}, \\ |D^2\Psi_H(x, y) - D^2\Psi_H(0, 0)| &\leq C_0 \|\Psi_H\|_{C^{2,\alpha}} (|x| + |y|)^\alpha. \end{aligned}$$

By (3.13) we have $\|(x, \bar{f}_{i,HL}(x))\| \leq C \|(x, f_{i,HL})\| \leq C|x|$. Joining the last fact with $\|\Psi_H\|_{C^{2,\alpha}} \leq \mathbf{m}_0^{1/2}$ and the last displayed inequalities, we infer the following

$$|D\Psi_H(x, \bar{f}_{i,HL}(x)) - D_x D\Psi_H(0, 0) \cdot x| = O\left(\mathbf{m}_0^{1/2} r_H^{1+\alpha}\right) + O\left(\mathbf{m}_0^{1/2} r_H^{1+\alpha}\right), \quad (3.19)$$

$$|D\Psi_H(x, \bar{f}_{i,HL}(x))| \leq \mathbf{m}_0^{1/2} r_H, \quad (3.20)$$

$$|D^2\Psi_H(x, \bar{f}_{i,HL}(x)) - D^2\Psi_H(0, 0)| = O\left(\mathbf{m}_0^{1/2} r_H^\alpha\right), \quad (3.21)$$

$$|D^2\Psi_H(x, \bar{f}_{i,HL}(x))| \leq \mathbf{m}_0^{1/2}. \quad (3.22)$$

We now start to estimate the several quantities appearing in (3.18). Henceforth we omit the domain of integration $\mathbf{B}_{8r_L}(p_L, \pi_H)$ for the sake of simplicity. We begin as follows

$$\begin{aligned} \int \sum_{i=1}^Q S_1 : S_4 &\stackrel{(3.22),(3.19)}{=} \int \sum_{i=1}^Q (D_{xy} \Psi_H(x, \bar{f}_{i,HL}(x)) \cdot \zeta) : (D_x D\Psi_H(0, 0) \cdot x) \\ &\quad + O\left(\mathbf{m}_0 r_H^{1+\min\{\alpha, \gamma a\}} \int |\zeta|\right) \\ &\stackrel{(3.21),(3.20)}{=} \int \sum_{i=1}^Q (D_{xy} \Psi_H(0, 0) \cdot \zeta) : (D_x D\Psi_H(0, 0) \cdot x) \\ &\quad + O\left(\mathbf{m}_0 r_H^{1+\min\{\alpha, \gamma a\}} \int |\zeta|\right) \\ &=: O\left(\mathbf{m}_0 r_H^{1+\min\{\alpha, \gamma a\}} \int |\zeta|\right) + \int x^t \cdot \mathbf{L}_{1,4} \cdot \zeta, \end{aligned} \quad (3.23)$$

where $\mathbf{L}_{1,4} = \mathbf{L}_{1,4}(D^2\Psi_H(0,0))$ is defined in the last inequality and we bring the reader's attention to the fact that it does not depend on L . It is easy to see that $\mathbf{L}_{1,4}$ is a quadratic form of $D^2\Psi_H(0,0)$. We also have

$$\int \sum_{i=1}^Q S_1 : S_5 \stackrel{(3.22),(3.20),(3.12)}{=} O\left(\mathbf{m}_0^{1+\gamma_{l_a}} r_H^{1+\gamma_{l_a}} \int |\zeta|\right). \quad (3.24)$$

Now that we handled the terms involving S_1 , let us turn the attention to S_2 . We have

$$\int \sum_{i=1}^Q S_2 : (S_4 + S_5) \stackrel{(3.22),(3.20),(3.12)}{=} O\left(\mathbf{m}_0^{1+\gamma_{l_a}} r_H^{1+\gamma_{l_a}} \int |\zeta|\right). \quad (3.25)$$

It remains to take care of S_3 , we proceed as follows

$$\int \sum_{i=1}^Q S_3 : S_5 \stackrel{(3.22),(3.20),(3.12)}{=} O\left(\mathbf{m}_0^{1+\gamma_{l_a}} r_H^{2+\gamma_{l_a}} \int |D\zeta|\right). \quad (3.26)$$

Additionally, we obtain

$$\begin{aligned} \int \sum_{i=1}^Q S_3 : S_4 &\stackrel{(3.19),(3.20)}{=} \int \sum_{i=1}^Q ((D_{yx}\Psi_H(0,0) \cdot x) \cdot D_x\zeta) : D_x\Psi_H(x, \bar{f}_{i,HL}(x)) \\ &\quad + O\left(\mathbf{m}_0 r_H^{2+\min\{\alpha, \gamma_{l_a}\}} \int |D\zeta|\right) \\ &\stackrel{(3.19),(3.22)}{=} \int \sum_{i=1}^Q ((D_{yx}\Psi_H(0,0) \cdot x) \cdot D_x\zeta) : (D_{xx}\Psi_H(0,0) \cdot x) \\ &\quad + O\left(\mathbf{m}_0 r_H^{2+\min\{\alpha, \gamma_{l_a}\}} \int |D\zeta|\right) \\ &\stackrel{(*)}{=} O\left(\mathbf{m}_0 r_H^{2+\min\{\alpha, \gamma_{l_a}\}} \int |D\zeta|\right) + \int x^t \cdot \mathbf{L}_{3,4} \cdot \zeta, \end{aligned} \quad (3.27)$$

where in $(*)$ we integrate by parts and define $\mathbf{L}_{3,4} = \mathbf{L}_{3,4}(D^2\Psi_H(0,0))$ in accordance with the equality. As before, we bring the reader's attention to the fact that $\mathbf{L}_{3,4}$ does not depend on L and is a quadratic form of $D^2\Psi_H(0,0)$. Denoting $\mathbf{L} = \mathbf{L}(D^2\Psi_H(0,0)) := \mathbf{L}_{1,4} + \mathbf{L}_{3,4}$ and putting together (3.18), (3.23), (3.24), (3.25), (3.26), and (3.27), we obtain that

$$\begin{aligned} \delta\mathbf{G}_{f_{HL}}(\chi) &= \int (D_x\boldsymbol{\eta} \circ \bar{f}_{HL} : D_x\zeta + x^t \cdot \mathbf{L} \cdot \zeta) + O\left(\|\zeta\|_{C^1} \mathbf{m}_0^{1+\gamma_{l_a}} r_H^{m+2-2\gamma_e+\gamma_{l_a}}\right) \\ &\quad + O\left(\mathbf{m}_0 r_H^{1+\min\{\alpha, \gamma_{l_a}\}} \|\zeta\|_{L^1}\right) + O\left(\mathbf{m}_0 r_H^{2+\alpha} \|D\zeta\|_{L^1}\right). \end{aligned}$$

Such equation inserted into (3.10) generates

$$\begin{aligned} \left| \int D_x\boldsymbol{\eta} \circ \bar{f}_{HL} : D_x\zeta + x^t \cdot \mathbf{L} \cdot \zeta \right| &\leq O\left(\|\zeta\|_{C^1} \mathbf{m}_0^{1+\gamma_{l_a}} r_H^{m+2-2\gamma_e+\gamma_{l_a}}\right) + O\left(\mathbf{m}_0 r_H^{2+\alpha} \|D\zeta\|_{L^1}\right) \\ &\quad + O\left(\mathbf{m}_0 r_H^{1+\min\{\alpha, \gamma_{l_a}\}} \|\zeta\|_{L^1}\right) + C_0 \int |D\zeta| d\vartheta, \end{aligned} \quad (3.28)$$

where we are using for any Borel set E

$$\vartheta(E) := \mathcal{H}^m(E \setminus K_{HL}) + \|T\|((E \setminus K_{HL}) \times \mathbb{R}^n) \text{ and } \|T - \mathbf{G}_{f_{HL}}\|(E \times \mathbb{R}^n) \leq C_0\vartheta(E).$$

The same argument provided in [7, Eq. 5.20] ensures that

$$\vartheta(\mathbf{B}_{8r_L}(p_L, \pi_H)) \leq C \mathbf{m}_0 r_H^{m+2-2\gamma_e+\gamma_{l_a}}.$$

By (3.28) and the last inequality, we derive that

$$\begin{aligned} \left| \int D_x \boldsymbol{\eta} \circ \bar{f}_{HL} : D_x \zeta + x^t \cdot \mathbf{L} \cdot \zeta \right| &\leq O\left(\|\zeta\|_{C^1} \mathbf{m}_0 r_H^{m+2-2\gamma_e+\gamma_{l_a}}\right) + O\left(\mathbf{m}_0 r_H^{2+\alpha} \|D\zeta\|_{L^1}\right) \\ &+ O\left(\mathbf{m}_0 r_H^{1+\min\{\alpha, \gamma_{l_a}\}} \|\zeta\|_{L^1}\right). \end{aligned} \quad (3.29)$$

Thanks to the choice of the exponents, we have $\gamma_{l_a} - 2\gamma_e = 49\gamma_{l_a}/50 < 0$, hence (3.8) follows. The proof of the moreover part goes along the very same line of the proof of [7, Eq. 5.2] using (3.8), we omit it here. \square

For the sake of clarity and completeness, we will state the following two lemmas that will be used in the proof of 3.8.

Lemma 3.11 (From the elliptic system to C^j estimates). *Assume Assumption 1 and Assumption 2 and set $B' := \mathbf{B}_{5r_H}(p_H, \pi_H)$ and $B := \mathbf{B}_{4r_H}(p_H, \pi_H)$. Then, we obtain that*

$$\|\bar{h}_{HL} - \bar{h}_H\|_{C^j(B')} + \|g_{HL}^* - g_H^*\|_{C^j(B)} \leq C \mathbf{m}_0^{1/2} \ell(L)^{3+\min\{\alpha, \gamma_{l_a}\}-j}, \quad j \in \{0, 1, 2, 3\}, \quad (3.30)$$

$$\|\bar{h}_{HL} - \bar{h}_H\|_{C^{3,\theta}(B')} + \|g_{HL}^* - g_H^*\|_{C^{3,\theta}(B)} \leq C \mathbf{m}_0^{1/2} \ell(L)^{\min\{\alpha, \gamma_{l_a}\}-\theta}, \quad \forall \theta \in (0, 1), \quad (3.31)$$

$$\|h_{HL} - h_H\|_{C^j(B')} + \|g_{HL} - g_H\|_{C^j(B)} \leq C \mathbf{m}_0^{1/2} \ell(L)^{3+\min\{\alpha, \gamma_{l_a}\}-j}, \quad j \in \{0, 1, 2\}, \quad (3.32)$$

$$\|h_{HL} - h_H\|_{C^{2,\alpha}(B')} + \|g_{HL} - g_H\|_{C^{2,\alpha}(B)} \leq C \mathbf{m}_0^{1/2} \ell(L)^{1-\alpha+\min\{\alpha, \gamma_{l_a}\}}. \quad (3.33)$$

Consequently, items (i) and (iv) of Proposition 3.8 hold.

Remark 3.12. Notice that \bar{h}_{HL} and g_{HL}^* are proven to be in fact of class $C^{3,\min\{\alpha, \gamma_{l_a}\}}$. On the other hand, since h_{HL} is a composition of Ψ_H and \bar{h}_{HL} , the regularity of h_{HL} (and consequently of g_{HL}) do not always exceed $C^{2,\alpha}$ which is Σ 's regularity.

Proof. The proof of [7, Lemma 5.3] works in the same lines using Proposition 3.10 instead of [7, Proposition 5.2]. \square

Lemma 3.13 (Tilted L^1 estimates). *Assume Assumption 1 and Assumption 2. We have that*

$$\|h_{HJ} - \hat{h}_{LM}\|_{L^1(\mathbf{B}_{2r_J}(p_J, \pi_H))} \leq C \mathbf{m}_0 \ell(J)^{m+3+\min\{\alpha, \gamma_{l_a}\}}. \quad (3.34)$$

Proof. The very same proof given in [7, Lemma 5.5]. \square

We are now able to derive the construction estimates for the functions g_{HL} and g_{HL}^* .

Proof of Proposition 3.8. Since $\kappa = \min\{\alpha, \gamma_{l_a}\}/2$, recall that (i) and (iv) are proven in Lemma 3.11. Let us show how to prove (ii). Take $H, L \in \mathcal{P}^j$ with nonempty intersection. We show that the inequality

$$\|h_H - \hat{h}_L\|_{L^1(\mathbf{B}_{2r_H}(p_H, \pi_H))} \leq C \mathbf{m}_0^{1/2} \ell(L)^{m+3+\min\{\alpha, \gamma_{l_a}\}} \quad (3.35)$$

holds true. If $\ell(L) = \ell(H)$, it is a direct consequence of Lemma 3.13. If $\ell(L) = 2\ell(H)$, take J to be the father of H . It is clear that $J \cap L \neq \emptyset$, therefore we can apply Lemma 3.13 to obtain

$$\|h_{HJ} - \hat{h}_L\|_{L^1(\mathbf{B}_{2r_H}(p_H, \pi_H))} \leq C \mathbf{m}_0 \ell(L)^{m+3+\min\{\alpha, \gamma_{l_a}\}}.$$

On the other hand, by (3.32), we have

$$\|h_H - h_{HJ}\|_{L^1(B_{2r_H}(p_H, \pi_H))} \leq C \mathbf{m}_0^{1/2} \ell(L)^{m+3+\min\{\alpha, \gamma_{\ell a}\}}.$$

The last two inequalities prove the validity of (3.35) in the latter case. Recalling that by construction, we have $\mathbf{G}_{g_X} \sqcup \mathbf{C}(x_H, r_H, \pi_0) = \mathbf{G}_{\hat{h}_X} \sqcup \mathbf{C}(x_H, r_H, \pi_0)$ (same with g^* and \hat{h}) for $X \in \{L, H\}$. Thus, as a consequence of [7, Lemma B.1] we deduce

$$\|g_H - g_L\|_{L^1(B_{r_H}(x_H, \pi_0))} + \|g_H^* - g_L^*\|_{L^1(B_{r_H}(x_H, \pi_0))} \leq C \mathbf{m}_0^{1/2} \ell(L)^{m+3+\min\{\alpha, \gamma_{\ell a}\}}. \quad (3.36)$$

The fact that $\|g_H - g_L\|_{C^{2,\alpha}(B_{r_H}(x_H, \pi_0))} \leq C \mathbf{m}_0^{1/2} \ell(L)^{1-\alpha+\min\{\alpha, \gamma_{\ell a}\}}$ (see (3.33)) and the last inequality together with

$$\|D^j(g_H - g_L)\|_{C^0} \leq C r_L^{-m-j} \|g_H - g_L\|_{L^1} + C r_L^{2+\alpha-j} \|D^2(g_H - g_L)\|_{C^{0,\alpha}}$$

imply (ii), for g_H^* and g_L^* the proof is exactly the same using (3.31) and

$$\|D^j(g_H^* - g_L^*)\|_{C^0} \leq C r_L^{-m-j} \|g_H^* - g_L^*\|_{L^1} + C r_L^{3+\kappa-j} \|D^3(g_H^* - g_L^*)\|_{C^{0,\kappa}}.$$

We now prove (v) since it is an easy consequence of (3.35) instead of (3.33). Indeed, if $L \in \mathscr{W}^j$ and $i \geq j$, consider the subset $\mathscr{P}^i(L) := \{X \in \mathscr{P}^i : X \cap L \neq \emptyset\}$. Note that the cardinality of $\mathscr{P}^i(L)$ is bounded by a dimensional constant. If L' is a cube concentric to L with $\ell(L') = 9\ell(L)/8$, by definition of $\hat{\varphi}_j$ and the last considerations, we have

$$\begin{aligned} \|\hat{\varphi}_i - g_L\|_{L^1(L')} + \|\varphi_i^* - g_L^*\|_{L^1(L')} &\leq C \sum_{X \in \mathscr{P}^i(L)} \left(\|g_H - g_L\|_{L^1(B_{r_X}(x_X, \pi_0))} + \|g_H^* - g_L^*\|_{L^1(B_{r_X}(x_X, \pi_0))} \right) \\ &\stackrel{(3.36)}{\leq} C \mathbf{m}_0^{1/2} \ell(L)^{m+3+\min\{\alpha, \gamma_{\ell a}\}}, \end{aligned}$$

which is exactly (v). Aiming at proving (iii), we take J to be any ancestor of H and $H_0 = H, \dots, H_{i_0} = J$ a chain of cubes such that H_i is the father of H_{i-1} . Then, we have

$$|D^2 g_H(x_H) - D^2 g_J(x_J)| \leq \sum_{i=0}^{i_0} |D^2 g_{H_i}(x_{H_i}) - D^2 g_{H_{i+1}}(x_{H_{i+1}})| \stackrel{(3.33)}{\leq} C \mathbf{m}_0^{1/2} \ell(J)^{1+\min\{\alpha, \gamma_{\ell a}\}}. \quad (3.37)$$

If we now take any random par of cubes $H, L \in \mathscr{P}^j$, we choose J_H and J_L to be the first ancestors of H and L , respectively, such that $J_H \cap J_L \neq \emptyset$. We then have

$$\begin{aligned} |D^2 g_H(x_H) - D^2 g_L(x_L)| &\leq |D^2 g_H(x_H) - D^2 g_{J_H}(x_{J_H})| + |D^2 g_{J_H}(x_{J_H}) - D^2 g_{J_L}(x_{J_L})| \\ &\quad + |D^2 g_L(x_L) - D^2 g_{J_L}(x_{J_L})| \\ &\stackrel{(3.37), (3.33)}{\leq} C \mathbf{m}_0^{1/2} \max\{\ell(J_H), \ell(J_L)\}^{1+\min\{\alpha, \gamma_{\ell a}\}}. \end{aligned}$$

By construction, one can check that $|x_H - x_L| \geq c_0 \max\{\ell(J_H), \ell(J_L)\}$, the last inequality and this consideration finish the proof of (iii). Again, it is a straightforward adaptation to prove the statement in (iii) for g_H^* and g_L^* . \square

4 Normal approximation on the external center manifold

Having now the existence of the external center manifold, Theorem 3.7, we can construct the \mathcal{M}^* -normal approximation (Theorem 4.7) which is a Q -valued function defined on \mathcal{M}^* whose graph induces a current that approximates T in cylinders defined by Whitney regions $(\mathbf{p}^{-1}(\mathcal{L}))$ for \mathcal{L} being a Whitney region). Let us make this notion precise below.

Definition 4.1 (Whitney regions). Let \mathcal{M}^* be the external center manifold relative to π_0 and (Γ, \mathscr{W}) the Whitney decomposition associated to it. Defining $\Phi^*(x) := (x, \varphi^*(x))$, we call $\Phi^*(\Gamma)$ the *contact set*. Moreover, to each $L \in \mathscr{W}$, we set the Whitney region \mathcal{L} on \mathcal{M}^* to be the following set $\mathcal{L} := \Phi^*(J \cap [-\frac{7}{2}, \frac{7}{2}]^m)$ where J is the cube concentric to L with side-length equal to $17\ell(L)/16$.

Definition 4.2. Let \mathcal{M}^* be an external manifold as in Theorem 3.7, we define

- $\mathbf{U}^* := \{x \in \mathbb{R}^{m+n} : \exists! y = \mathbf{p}^*(x) \in \mathcal{M}^*, |x - y| < 1, \text{ and } (x - y) \perp \mathcal{M}^*\}$,
- $\mathbf{p}^* : \mathbf{U}^* \rightarrow \mathcal{M}^*$ is the map defined by the previous bullet,
- $\partial_l \mathbf{U}^* := (\mathbf{p}^*)^{-1}(\partial \mathcal{M}^*)$ is the lateral boundary of \mathcal{M}^* .

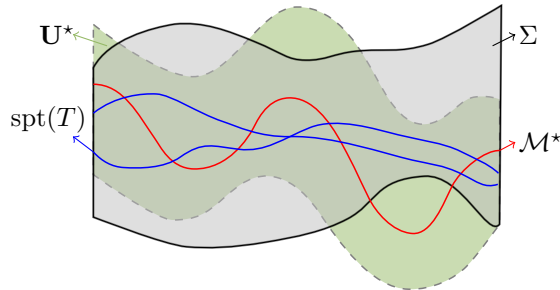
Assumption 3. Under Assumptions 1 and 2, we further possibly decrease $\varepsilon_{cm} > 0$ in order to have that $\mathbf{p}^* \in C^{1,\alpha}(\mathbf{U}^*)$ and $(\mathbf{p}^*)^{-1}(y) = y + B_1(0, T_p \mathcal{M}^*)$ for every $y \in \mathcal{M}^*$.

We are now in position to prove the following corollary of the constructions made so far.

Corollary 4.3. Under Assumption 1, Assumption 2, and Assumption 3, we have

- $\text{spt}(\partial(T \llcorner \mathbf{U}^*)) \subset \partial_l \mathbf{U}^*$, $\text{spt}(T \llcorner [-\frac{7}{2}, \frac{7}{2}] \times \mathbb{R}^n) \subset \mathbf{U}^*$, and $\mathbf{p}^*(T \llcorner \mathbf{U}^*) = Q \llbracket \mathcal{M}^* \rrbracket$;
- $\text{spt}(\langle T, \mathbf{p}^*, \Phi^*(q) \rangle) \subset \{y : |\Phi^*(q) - y| \leq C \mathbf{m}_0^{1/2m} \ell(L)^{1+\gamma_h}\}$ for every $q \in L \in \mathscr{W}$, where $C = C(\gamma_e, \gamma_h, M_0, N_0, C_e, C_h)$;
- $\langle T, \mathbf{p}^*, p \rangle = Q \llbracket p \rrbracket$ for every $p \in \Phi^*(\Gamma)$.

Remark 4.4. Notice that, the set \mathbf{U}^* is not necessarily contained in Σ which is not an issue when paralleled to the approach in [7, Corollary 2.2]. Indeed, one can see that the set \mathbf{U} in [7, Corollary 2.2] is not guaranteed to be within Σ .



Proof. The proof of the first and third affirmations in item (i), items (ii), and (iii), are the same as presented in [7, Corollary 2.2]. Let us prove $\text{spt}(T \llcorner [-\frac{7}{2}, \frac{7}{2}] \times \mathbb{R}^n) \subset \mathbf{U}^*$. Pick a point $p \in \text{spt}(T \llcorner [-\frac{7}{2}, \frac{7}{2}] \times \mathbb{R}^n)$ and denote by p' the first m coordinates of $\mathbf{p}_{\pi_0}(p)$, then $p_0 := (p', \varphi^*(p')) \in \mathcal{M}^*$.

If $p' \in \mathbf{\Gamma}$, it follows from item (iii) that $p_0 \in \mathbf{\Phi}^*(\mathbf{\Gamma}) \cap \text{spt}(T)$ and thus, by (2.8), $|p - p_0| < 1$. If $p' \in H$ where $H \in \mathcal{W}^j$ for some j , we can proceed as follows:

$$|p - p_0| \leq |p - p_H| + |p_H - p_0| \stackrel{\text{Prop. 2.7 (iv)}}{\leq} 2^6 r_H + |p_H - p_0| = 2^6 M_0 \sqrt{m} \ell(L) + |p_H - p_0|.$$

Therefore, it remains to bound $|p_H - p_0|$. We show this bound below:

$$\begin{aligned} |p_H - p_0| &\leq |p' - x_H| + |\varphi_j^*(p') - y_H| + |\varphi_j^*(p') - \varphi^*(p')| \\ &\stackrel{p' \in H}{\leq} \sqrt{m} \ell(H) + |\varphi_j^*(p') - y_H| + |\varphi_j^*(p') - \varphi^*(p')| \\ &\stackrel{\text{Thm. 3.7 (A}^*)}{\leq} \sqrt{m} \ell(H) + C \mathbf{m}_0^{1/2m} + |y_H| + |\varphi_j^*(p') - \varphi^*(p')| \\ &\stackrel{\text{Prop. 2.7 (iv)}}{\leq} \sqrt{m} \ell(H) + C \mathbf{m}_0^{1/2m} + C_h \mathbf{m}_0^{1/2m} \ell(H) + |\varphi_j^*(p') - \varphi^*(p')| \\ &\stackrel{p_0 \in \mathcal{M}^*, \text{Thm. 3.7 (A}^*, \text{D}^*)}{\leq} C \mathbf{m}_0^{1/2m} < 1. \end{aligned}$$

We proved that $|p - p_0| < 1$ always holds true. Therefore, by Assumption 3, we get that $p \in \mathbf{U}^*$. \square

A notion of Lipschitz approximation for nonlinear domains is given below with respect to \mathcal{M}^* . Opportunely, we highlight the fact that the \mathcal{M}^* -normal approximations are required to take values in Σ , see item (ii) in Definition 4.5. Even though F and N will be defined on the external center manifold (which possibly lies outside Σ , in contrast with [7]), their values are trapped in Σ which will be important for computing inner and outer variations in what follows.

Definition 4.5 (\mathcal{M}^* -normal approximation). An \mathcal{M}^* -normal approximation of T is given by a pair (\mathcal{K}, F) such that

- (i) $F : \mathcal{M}^* \rightarrow \mathcal{A}_Q(\mathbf{U}^*)$ is Lipschitz w.r.t. the geodesic distance in \mathcal{M}^* and $F(p) = \sum_i \llbracket p + N_i(p) \rrbracket$ where $N : \mathcal{M}^* \rightarrow \mathcal{A}_Q(\mathbb{R}^{m+n})$ is called *the normal part of F* ,
- (ii) $N_i(p) \perp T_p \mathcal{M}^*$ and $p + N_i(p) \in \Sigma$ for any $p \in \mathcal{M}^*$ and i ,
- (iii) $\mathcal{K} \subset \mathcal{M}^*$ is a closed set that contains $\mathbf{\Phi}^*(\mathbf{\Gamma} \cap [-\frac{7}{2}, \frac{7}{2}]^m)$ and $\mathbf{T}_F \mathbb{L}(\mathbf{p}^*)^{-1}(\mathcal{K}) = T \mathbb{L}(\mathbf{p}^*)^{-1}(\mathcal{K})$.

Remark 4.6. We point out that N_i is not necessarily tangent to Σ . In fact, this is often the case, it can be explicitly seen in the proof of Theorem 4.7 when we construct the map Ξ .

We now state the existence and fine properties of an \mathcal{M}^* -normal approximation. The strategy of the proofs of Theorem 4.7 and Corollary 4.8 follow [7, Thm 2.4 and Cor 2.5] and the final bounds are indeed the same. Nevertheless, given the difficulties of the $C^{2,\alpha}$ setting, some estimates have to be carefully carried out and the definition of the map N is also subtler. In fact, the authors in [7] rely on the fact that $\mathcal{M} \subset \Sigma$ to construct a trivialization of class $C^{2,\alpha}$ (given that in their setting $\mathcal{M}, \Sigma \in C^{3,\kappa}$) of the normal bundle of \mathcal{M} . A crucial fact in their analysis is that the normal bundle of \mathcal{M} is a subspace of the tangent bundle of Σ which is not guaranteed for the external center manifold.

Theorem 4.7 (Local estimates for the \mathcal{M}^* -normal approximation). *Assume Assumptions 1, 2, and 3, and let $\gamma_{na} := \min\{\gamma_{ta}/4, \alpha\}$. If $\varepsilon_{cm} = \varepsilon_{cm}(\gamma_e, \gamma_h, M_0, N_0, C_e, C_h) > 0$ is sufficiently small, then there exists a constant $C = C(\gamma_e, \gamma_h, M_0, N_0, C_e, C_h) > 0$ and an \mathcal{M}^* -normal approximation*

(\mathcal{K}^*, F^*) such that, for every Whitney region \mathcal{L} associated to a cube $L \in \mathscr{W}$, the following estimates hold true:

$$\text{Lip}(N^*|_{\mathcal{L}}) \leq C\mathbf{m}_0^{\gamma_{na}} \ell(L)^{\gamma_{na}} \text{ and } \|N^*|_{\mathcal{L}}\|_{C^0} \leq C\mathbf{m}_0^{1/2m} \ell(L)^{1+\gamma_h}, \quad (4.1)$$

$$\mathcal{H}^m(\mathcal{L} \setminus \mathcal{K}^*) + \|\mathbf{T}_{F^*} - T\| \left((\mathbf{p}^*)^{-1}(\mathcal{L}) \right) \leq C\mathbf{m}_0^{1+\gamma_h} \ell(L)^{m+2+\gamma_h}, \quad (4.2)$$

$$\int_{\mathcal{L}} |DN^*|^2 \leq C\mathbf{m}_0 \ell(L)^{m+2-2\gamma_e}. \quad (4.3)$$

Moreover, for any $a > 0$ and any Borel $\mathcal{V} \subset \mathcal{L}$, we have

$$\int_{\mathcal{V}} |\boldsymbol{\eta} \circ N^*| \leq C\mathbf{m}_0 \left(\ell(L)^{m+3+\gamma_e/3} + a\ell(L)^{2+\gamma_{na}/2} |\mathcal{V}| \right) + \frac{C}{a} \int_{\mathcal{V}} \mathcal{G}(N^*, Q[\boldsymbol{\eta} \circ N^*])^{2+\gamma_{na}}. \quad (4.4)$$

Corollary 4.8 (Global estimates for the \mathcal{M}^* -normal approximation). *Under the Assumptions of Theorem 4.7, N^* is the map from Theorem 4.7, and denote $\mathcal{M}_0^* := \Phi^*([-7/2, 7/2]^m)$. Then there exists a constant $C = C(\gamma_e, \gamma_h, M_0, N_0, C_e, C_h) > 0$ such that*

$$\text{Lip}(N^*|_{\mathcal{M}_0^*}) \leq C\mathbf{m}_0^{\gamma_{na}} \text{ and } \|N^*|_{\mathcal{M}_0^*}\|_{C^0} \leq C\mathbf{m}_0^{1/2m} \quad (4.5)$$

$$\mathcal{H}^m(\mathcal{M}_0^* \setminus \mathcal{K}^*) + \|\mathbf{T}_{F^*} - T\| \left((\mathbf{p}^*)^{-1}(\mathcal{M}_0^*) \right) \leq C\mathbf{m}_0^{1+\gamma_h} \quad (4.6)$$

$$\int_{\mathcal{M}_0^*} |DN^*|^2 \leq C\mathbf{m}_0. \quad (4.7)$$

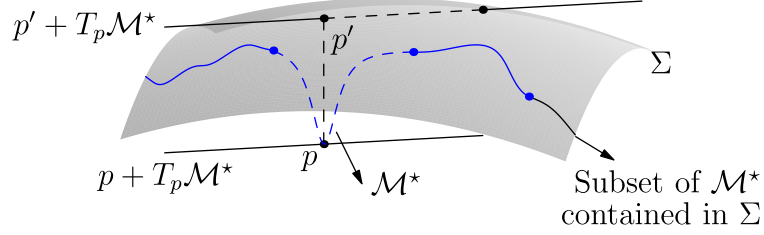
Proof of Theorem 4.7. Running the same argument as in [8, Subsection 6.2] with $\mathcal{M}^* \in C^{3,\kappa}$ (Theorem 3.7) in place of the center manifold \mathcal{M} which in our setting is only $C^{2,\alpha}$ (Theorem 3.6), we can apply [6, Theorem 5.1] for \mathcal{M}^* to get, for each $L \in \mathscr{W}^J$, Q -valued maps N_L and F_L satisfying the following properties:

- $N_L : \mathcal{L}' \subset \mathcal{M}^* \rightarrow \mathcal{A}_Q(\mathbb{R}^{m+n})$ and $F_L : \mathcal{L}' \subset \mathcal{M}^* \rightarrow \mathcal{A}_Q(\mathbf{U}^*)$, where $\mathcal{L}' := \Phi^*(J)$ with J being the cube concentric to L with side-length $\ell(J) = 9\ell(L)/8$,
- $F_L(p) = \sum_i \llbracket p + (N_L)_i(p) \rrbracket$ and $(N_L)_i(p) \perp T_p \mathcal{M}^*$ for every $p \in \mathcal{L}'$,
- and $\mathbf{G}_{f_L} \llcorner ((\mathbf{p}^*)^{-1}(\mathcal{L}')) = \mathbf{T}_{F_L} \llcorner ((\mathbf{p}^*)^{-1}(\mathcal{L}'))$.

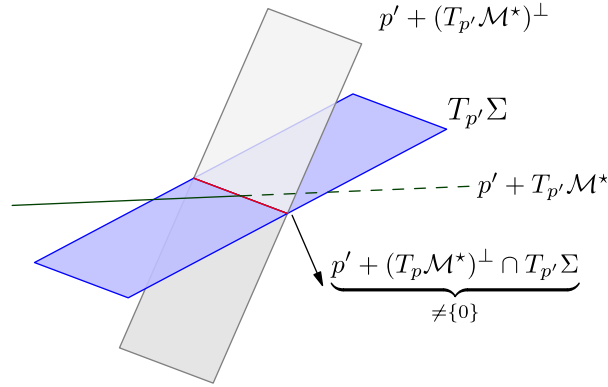
The functions N_L and F_L do not satisfy all the properties required in Definition 4.5. For this reason, we have to extend their domains of definition and modify them in order to take values within Σ and be orthogonal to \mathcal{M}^* . Indeed, when we run the argument on [7, page 537], we obtain, using the same notation of [7], functions \hat{F} and \hat{N} defined on the whole *external* center manifold \mathcal{M}^* satisfying that $\hat{N}_i(p) \perp T_p \mathcal{M}^*$ holds for every $p \in \mathcal{M}^*$. Moreover, these functions also verify items (i) and (iii) of Definition 4.5.

The next and last step is to modify \hat{F} in order to ensure item (ii) of Definition 4.5, i.e., that $p + N_i(p) \in \Sigma$ and $N_i(p) \perp T_p \mathcal{M}^*$ for every $p \in \mathcal{M}^*$. Such step has to be done cautiously, since the authors in [7] use the fact that $\mathcal{M} \subset \Sigma$, thus we perform the complete proof.

Take $p \in \mathcal{M}^*$ and p' to be the nearest point to p in Σ . Due to the $C^{2,\alpha}$ -regularity of Σ , the correspondence $p \mapsto p'$ is $C^{1,\alpha}$. Since \mathcal{M}^* is $C^{3,\kappa}$ and Σ is $C^{2,\alpha}$, arguing as in [6, App. A], there is a global $C^{1,\alpha}$ trivialization of $\cup_{p \in \mathcal{M}^*} (T_p \mathcal{M}^*)^\perp \cap T_p \Sigma$.



Notice that $(T_p\mathcal{M}^*)^\perp \cap T_{p'}\Sigma$ always has dimension bigger or equal than one. In fact, the dimensions of $(T_p\mathcal{M}^*)^\perp$ and $T_{p'}\Sigma$ are, respectively, equal to n and $m + \bar{n}$, thus the dimension of their sum as vector spaces exceeds $m + n$.



Denote $\varkappa_p := (T_p\mathcal{M}^*)^\perp \cap T_{p'}\Sigma$. Having a $C^{1,\alpha}$ -trivialization of $\cup_{p \in \mathcal{M}^*} \varkappa_p$, we can get a map

$$\Xi : \cup_{p \in \mathcal{M}^*} \varkappa_p \rightarrow \mathbb{R}^{m+n} \text{ such that } \begin{cases} \bullet & p' + \Xi(p, v) \text{ is the only point in } \Sigma \\ & \text{that is orthogonal to } T_p\mathcal{M}^*, \\ \bullet & \mathbf{p}_{\varkappa_p}(\Xi(p, v)) = v. \end{cases}$$

Finally, we can define the \mathcal{M}^* -normal approximation (which does *not* necessarily satisfy $N_i(p) \in T_{p'}\Sigma$) as follows:

$$N : \mathcal{M}^* \rightarrow \mathcal{A}_Q(\mathbb{R}^{m+n}) \text{ defined by } \sum_{i=1}^Q \left[\Xi(p, \mathbf{p}_{\varkappa_p}(\hat{N}_i(p))) \right].$$

Since $p + \hat{N}_i(p)$ belongs to the support of T and then to Σ , it is clear that $N(p) = \hat{N}(p)$ for any point in the good set \mathcal{K}^* . To simplify the notation, denote $\Omega(p, v) := \Xi(p, \mathbf{p}_{\varkappa_p}(v))$. As mentioned before, Ω is a $C^{1,\alpha}$ map. Therefore, we obtain

$$|\Omega(p, v) - \Omega(p, u)| \leq C_0|v - u|. \quad (4.8)$$

Moreover, since $\Omega(p, 0) = 0$ for every p , we have $D_p\Omega(p, 0) = 0$, thus $|D_p\Omega(p, v)| \leq C_0|v|^\alpha$. Hence, we obtain that

$$|\Omega(p, v) - \Omega(q, v)| \leq C_0|v|^\alpha|p - q|. \quad (4.9)$$

Fix two points $p, q \in \mathcal{L}$ and assume that $\mathcal{G}(\hat{N}(p), \hat{N}(q))^2 = \sum_i |\hat{N}_i(p) - \hat{N}_i(q)|^2$. We now have

$$\begin{aligned} \mathcal{G}(N(p), N(q))^2 &\leq 2 \sum_i |\Omega(p, \hat{N}_i(p)) - \Omega(p, \hat{N}_i(q))|^2 + 2 \sum_i |\Omega(p, \hat{N}_i(q)) - \Omega(q, \hat{N}_i(q))|^2 \\ &\stackrel{(4.8), (4.9)}{\leq} C_0 \mathcal{G}(\hat{N}(p), \hat{N}(q))^2 + C \sum_i |\hat{N}_i(q)|^{2\alpha} |p - q|^2 \\ &\leq C^2 \mathbf{m}_0^{2\gamma_{na}} \ell(L)^{2\gamma_{na}} |p - q|^2 + C^2 \mathbf{m}_0^{1/m} \ell(L)^{2\alpha(1+\gamma_h)} |p - q|^2. \end{aligned}$$

Thanks to the fact that $\gamma_{na} \leq \alpha$, the last inequality concludes the proof of (4.1). The proof of the remaining estimates goes along the same lines as [7, Thm 2.4]. \square

5 Frequency function and blow-up argument

In this section, we show how to prove Theorem 1.1 using the approach in [1] and [8]. To this end, we set some notations and definitions. We start by setting the rescaling functions by $\iota_{p,r}(x) := \frac{x-p}{r}$ for any $p, x \in \mathbb{R}^{m+n}$ and $r > 0$ and denoting the rescaled currents as $T_{p,r} := (\iota_{p,r})_{\#} T$.

Assumption 4 (Contradiction assumption). Assume that Theorem 1.1 is false. Precisely, assume that: there exists $m \geq 2, \bar{n} \geq 0, n \geq 1, \alpha > 0, \Sigma$ and T such that Σ is an $C^{2,\alpha}$ embedded $(m + \bar{n})$ -submanifold of \mathbb{R}^{m+n} , T is an integral m -current in Σ that minimizes area, and

$$\mathcal{H}^{m-2+\tau}(\text{Sing}(T)) > 0, \text{ for some } \tau > 0.$$

Under Assumption 4, we apply [8, Prop 1.3] to obtain the following contradiction sequence, which will allow us to derive a contradiction from Assumption 4 and therefore conclude the proof of Theorem 1.1.

Lemma 5.1 (Contradiction sequence). *Assume Assumption 4 and let $\mathbf{m}_0 < \varepsilon_{bu}$ where $\varepsilon_{bu} \in (0, \varepsilon_{cm})$. There exist two real numbers $\tau, \eta > 0$ and a sequence $r_k \downarrow 0$ such that:*

- (i) $\Theta^m(T, 0) = Q$;
- (ii) $\mathbf{E}(T_{0,r_k}, \mathbf{B}_{6\sqrt{m}})$ goes to 0 as k goes to $+\infty$;
- (iii) $\lim_{k \rightarrow +\infty} \mathcal{H}_{\infty}^{m-2+\tau}(\{p : \Theta^m(T_{0,r_k}, p) = Q\} \cap \mathbf{B}_1) > \eta$;
- (iv) $\mathcal{H}^m((\mathbf{B}_1 \cap \text{spt}(T_{0,r_k})) \setminus \{p : \Theta^m(T_{0,r_k}, p) = Q\}) > 0$ for all $k \in \mathbb{N}$;
- (v) $T_0 \Sigma = \mathbb{R}^{m+\bar{n}}, \partial T \llcorner \mathbf{B}_{6\sqrt{m}} = 0$;
- (vi) $c(\Sigma \cap \mathbf{B}_{7\sqrt{m}}) \leq \varepsilon_{bu}$;
- (vii) $\|T\|(\mathbf{B}_{6\sqrt{mr}}) \leq r^m (Q\omega_m(6\sqrt{m})^m + \varepsilon_{bu})$ for any $r \in (0, 1)$.

We bring the readers' attention to the fact that the current T of Assumption 4 satisfies, thanks to Lemma 5.1, all the requirements in Assumption 1 *except* the smallness of the excess. However, (ii) of Lemma 5.1 ensures that the excess of the rescalings of T with respect to the sequence $\{r_k\}_k$ converges to 0. This motivates us to define the *intervals of flattening* that capture all radii $r \in (0, 6\sqrt{m})$ for which the rescaled current $T_{0,r}$ is under Assumption 4. Thus, the external center manifold and the normal approximations can be constructed for each $T_{0,r}$.

Definition 5.2 (Intervals of flattening and sequence of external center manifolds \mathcal{M}_j^*). Assume Assumption 4. We define the following set of radii:

$$\mathcal{R} := \left\{ r \in (0, 1] : \mathbf{E}(T_{0,r}, \mathbf{B}_{6\sqrt{m}}) \leq \varepsilon_{bu}^2 \right\}.$$

If $\{s_k\} \subset \mathcal{R}$ and $s_k \uparrow s$, then $s \in \mathcal{R}$. Now, we cover \mathcal{R} with a family of interval $\mathcal{F} = \{(s_j, t_j]\}_j$ called *intervals of flattening*, defined as follows: $t_0 := \max\{t : t \in \mathcal{R}\}$, by induction, assume that t_j is defined, and hence also $t_0 > s_0 \geq t_1 > s_1 \geq \dots > s_{j-1} \geq t_j$. We also define the following objects:

- $T_j := T_{0,t_j} \lfloor \mathbf{B}_{6\sqrt{m}}, \Sigma_j := \iota_{0,t_j}(\Sigma) \cap \mathbf{B}_{7\sqrt{m}}$. Moreover, consider for each j an orthonormal system of coordinates so that $\pi_0 := \mathbb{R}^m \times \{0\}$ is the optimal m -plane for the excess, i.e., $\mathbf{E}(T_j, \mathbf{B}_{6\sqrt{m}}, \pi_0) = \mathbf{E}(T_j, \mathbf{B}_{6\sqrt{m}})$;
- Let \mathcal{M}_j^* be the external center manifold and N_j^* the \mathcal{M}_j^* -normal approximation constructed in Theorem 3.7 and Theorem 4.7 applied to T_j and Σ_j with respect to the m -plane π_0 . Notice that T_j and Σ_j are under Assumption 1, 2, and 3, thanks to Lemma 5.1 and the definition of the intervals of flattening.

With this, we consider the Whitney decomposition $\mathscr{W}^{(j)}$ of $[-4, 4]^m \subset \pi_0$ with respect to T_j , and we define

$$s_j := t_j \max \left(\left\{ c_s^{-1} \ell(L) : L \in \mathscr{W}^{(j)} \text{ and } c_s^{-1} \ell(L) \geq \text{dist}(0, L) \right\} \cup \{0\} \right).$$

As in [8], one can see that $s_j/t_j < 2^{-5}$ which then ensures that $(s_j, t_j]$ is a nontrivial interval. Next, if $s_j = 0$, we stop the induction. Otherwise, we let t_{j+1} be the largest element in $\mathcal{R} \cap [0, s_j]$ and repeat the procedure above.

5.1 The frequency function

Definition 5.3. Let $\phi : [0, +\infty) \rightarrow [0, 1]$ be the piecewise linear function given by

$$\phi(r) := \begin{cases} 1, & r \in \left[0, \frac{1}{2}\right] \\ 2 - 2r, & r \in \left[\frac{1}{2}, 1\right] \\ 0, & r \in [1, +\infty). \end{cases}$$

Now we fix, for each j , d_j to be the geodesic distance in \mathcal{M}_j^* between p and $\Phi_j^*(0)$. Then, we define

- $\mathbf{D}_j(r) := \int_{\mathcal{M}_j^*} \phi\left(\frac{d_j(p)}{r}\right) |DN_j^*|^2(p) d\mathcal{H}^m(p)$;
- $\mathbf{H}_j(r) := - \int_{\mathcal{M}_j^*} \phi'\left(\frac{d_j(p)}{r}\right) \frac{|N_j^*|^2(p)}{d_j(p)} d\mathcal{H}^m(p)$.

The *frequency function* is then defined as $\mathbf{I}_j(r) := \frac{r\mathbf{D}_j(r)}{\mathbf{H}_j(r)}$ whenever $\mathbf{H}_j(r) > 0$.

Remark 5.4. A main ingredient to prove Theorem 1.1 is the monotonicity of the frequency function. To prove this, the authors in [8, Thm 3.2] take the inner and outer variations of \mathbf{I}_j and prove fine estimates for them. A crucial information in this analysis is the stationarity of T , more precisely, that the first variation of T vanishes with respect to vector fields *tangent* to Σ . To use this fact, we need to make sure that our normal approximations are supported in Σ , which we ensure in Theorem 4.7. Moreover, to carry out the computations in [8], it is also essential to have that $\text{spt}(T)$ is a subset of \mathbf{U} (i.e., [8, Corollary 2.2]) which is also guaranteed in our setting with \mathbf{U}^* as we have shown in Corollary 4.3, see also Remark 4.4.

Remark 5.5. We mention that one of the main reasons the authors in [8] need the $C^{3,\alpha}$ -regularity for the center manifold is to utilize the Hölder continuity of $DH_{\mathcal{M}}$ in proving the monotonicity of the frequency function, where $H_{\mathcal{M}}$ denotes the mean curvature of \mathcal{M} . In the present setting, however, \mathcal{M} only possesses $C^{2,\alpha}$ -regularity, which precludes us from using the derivative of $H_{\mathcal{M}}$. Nevertheless, we have constructed \mathcal{M}^* so that normal approximations are defined over \mathcal{M}^* , which has $C^{3,\alpha}$ -regularity. This ensures that $DH_{\mathcal{M}^*}$ is well-defined and exhibits Hölder continuity.

Taking into consideration Remark 5.4 and Remark 5.5, we are therefore in position to apply the same machinery as in [8] to obtain the monotonicity of the frequency function, i.e., Theorem 5.6.

Theorem 5.6 (Main frequency estimate). *Provided ε_{bu} is chosen small enough, there exists a geometric constant $C_0 > 0$ such that*

$$\mathbf{I}_j(a) \leq C_0(1 + \mathbf{I}_j(b)), \text{ for every } [a, b] \subset \left[\frac{s_j}{t_j}, 3 \right] \text{ such that } \mathbf{H}_j|_{[a,b]} > 0.$$

5.2 Blow-up argument

We now show how to obtain a function at the limit that provides us the desired contradiction to conclude Theorem 1.1. The proof of Theorem 5.7 resembles that of [8, Thm 6.2], however, it has to be subtly changed when dealing with the external center manifolds \mathcal{M}_j^* and the \mathcal{M}_j^* -normal approximations N_j^* since the arguments in [8] strongly rely on the fact that $\mathcal{M} \subset \Sigma$ and the $C^{3,\alpha}$ -regularity of Σ and \mathcal{M} .

Theorem 5.7 (Final blow-up). *The maps $N_k^{*,b}$ strongly converge, up to subsequences, in $L^2(B_{3/2})$ to a function $N_\infty^{*,b} \in W^{1,2}(B_{3/2}, \mathcal{A}_Q(\{0\} \times \mathbb{R}^{\bar{n}} \times \{0\}))$ that satisfies the following:*

- (i) $N_\infty^{*,b}$ is a minimizer of the Dirichlet energy in B_t for any $t \in (5/3, 3/2)$;
- (ii) $\|N_\infty^{*,b}\|_{L^2(B_{3/2})} = 1$;
- (iii) $\boldsymbol{\eta} \circ N_\infty^{*,b} \equiv 0$.

Proof of Theorem 1.1. The approach implemented in [8, Pf of Thm 0.3] can be carried out along the very same lines to get Theorem 1.1 from the above properties of the multivalued limit function $N_\infty^{*,b}$. For this reason, we omit the proof here. \square

To prove Theorem 5.7, we need to introduce some notation as follows. Consider a current T satisfying Assumption 4. As in [8, Proposition 2.2], for each radius r_k produced by Lemma 5.1, there is an interval of flattening such that $r_k \in (s_{j(k)}, t_{j(k)})$. We denote by \bar{s}_k the radius $2t_{j(k)}/3 \leq \bar{s}_k \leq 3r_k$ given by the Reverse Sobolev inequality [8, Corollary 5.3] applied to $r = r_k$. We then put $\bar{r}_k := 2\bar{s}_k$ and rescale and translate our objects as below:

- $\bar{T}_k := (\iota_{0, \bar{r}_k})_\# T_{j(k)} = ((\iota_{0, \bar{r}_k t_{j(k)}}))T \lrcorner \mathbf{B}_{\frac{6\sqrt{m}}{\bar{r}_k}}, \bar{\Sigma}_k := \iota_{0, \bar{r}_k}(\Sigma_{j(k)})$ and $\bar{\mathcal{M}}_k^* := \iota_{0, \bar{r}_k}(\mathcal{M}_{j(k)}^*)$,
- $\bar{N}_k^* : \bar{\mathcal{M}}_k^* \rightarrow \mathbb{R}^{m+n}$ are the rescaled $\bar{\mathcal{M}}_k^*$ -normal approximations given by

$$\bar{N}_k^*(p) = \bar{r}_k^{-1} N_{j(k)}^*(\bar{r}_k p).$$

We can assume $T_0\Sigma = \mathbb{R}^{m+\bar{n}} \times \{0\}$, thus the ambient manifolds $\bar{\Sigma}_k$ converge to $\mathbb{R}^{m+\bar{n}} \times \{0\}$ locally in $C^{2,\alpha}$. Furthermore, since $\frac{1}{2} < \frac{r_k}{\bar{r}_k t_{j(k)}} < 1$, Lemma 5.1 implies that

$$\mathbf{E}(\bar{T}_k, \mathbf{B}_{1/2}) \leq C\mathbf{E}(T, \mathbf{B}_{r_k}) \rightarrow 0.$$

Without loss of generality, we can assume that \bar{T}_k locally converges to $Q \llbracket \pi_0 \rrbracket$. Moreover, from Lemma 5.1, it follows that

$$\mathcal{H}_\infty^{m+2+\tau}(D_Q(\bar{T}_k) \setminus \mathbf{B}_1) \leq C_0 r_k^{-(m+2+\tau)} \mathcal{H}_\infty^{m+2+\tau}(D_Q(T) \setminus \mathbf{B}_{r_k}) \geq \eta > 0, \quad (5.1)$$

where C_0 is a geometric constant. As in [8, Lemma 6.1], we can show that $\bar{\mathcal{M}}_k^*$ locally converge in $C^{3,\kappa/2}$ to the m -plane π_0 . We thus define the blow-up maps $N_k^{*,b} : \mathcal{B}_3 \subset \pi_0 = \mathbb{R}^m \rightarrow \mathcal{A}_Q(\mathbb{R}^{m+n})$:

$$N_k^{*,b}(x) := \frac{\bar{N}_k^*(\mathbf{e}_k(x))}{\mathbf{h}_k^*}, \quad \mathbf{h}_k^* := \left\| \bar{N}_k^* \right\|_{L^2(\mathcal{B}_{3/2})}, \quad \text{and} \quad \bar{p}_k^* := \frac{\bar{\Phi}_{j(k)}^*(0)}{\bar{r}_k}, \quad (5.2)$$

where $\mathbf{e}_k := \exp_{\bar{p}_k^*}$ denotes the exponential map of $\bar{\mathcal{M}}_k^*$ at \bar{p}_k^* . Henceforth, we assume, without loss of generality, that we have applied a suitable rotation to each \bar{T}_k so that the tangent plane $T_{\bar{p}_k^*} \bar{\mathcal{M}}_k^*$ coincides with $\mathbb{R}^m \times \{0\}$.

To prove Theorem 5.7, we will proceed as follows. We approximate, by smoothing, both $\bar{\mathcal{M}}_k^*$ and $\bar{\Sigma}_k$ with suitable C^∞ -submanifolds of the ambient space \mathbb{R}^{m+n} which we denote by $\bar{\mathcal{M}}_{k,\varepsilon}^*$ and $\bar{\Sigma}_{k,\varepsilon}$, respectively. Using these smoothed approximations, we have the existence of a well-defined projection $\mathfrak{P}_{k,\varepsilon} = \mathfrak{P}_{\bar{\Sigma}_{k,\varepsilon}, \bar{\mathcal{M}}_{k,\varepsilon}^*}$ satisfying:

$$\mathfrak{P}_{k,\varepsilon}(\bar{\mathcal{M}}_{k,\varepsilon}^*) \subseteq \bar{\Sigma}_{k,\varepsilon} \quad \text{and} \quad \mathfrak{P}_{k,\varepsilon} \text{ is of class } C^\infty.$$

From this, we can run similar constructions to those in [8, Section 7.2] to the pair $(\bar{\mathcal{M}}_{k,\varepsilon}^*, \bar{\Sigma}_{k,\varepsilon})$ where we need to factorize the normal approximation N_j^* through the projection above and let $\varepsilon \rightarrow 0$ and $k \rightarrow +\infty$ to conclude the proof. The subtleties of controlling all the error terms involved with this smoothing procedure are detailed below.

Proof of Theorem 5.7. We divide the proof into some steps.

Step 1: Proof of the existence of $N_\infty^{,b} \in W^{1,2}(\mathcal{B}_{3/2}, \mathcal{A}_Q(\{0\} \times \mathbb{R}^{\bar{n}} \times \{0\}))$, (ii), (iii), and L^2 -strong convergence.*

Without loss of generality, we translate the manifolds $\bar{\mathcal{M}}_k^*$ so that the rescaled points $\bar{p}_k^* = \bar{r}_k^{-1} \bar{\Phi}_{j(k)}^*(0)$ all coincide with the origin $0_{\mathbb{R}^{m+n}}$. Let us define

$$\bar{F}_k^* : \mathcal{B}_{3/2}^* \subset \bar{\mathcal{M}}_k^* \rightarrow \mathcal{A}_Q(\mathbb{R}^{m+n}) \quad \text{and} \quad \bar{F}_k^*(x) := \sum_i \left[x + \left(\bar{N}_k^* \right)_i(x) \right].$$

To simplify the notation, set $\mathbf{p}_k^* := \mathbf{p}_{\bar{\mathcal{M}}_k^*}$. We start by showing the existence of a suitable exponent $\gamma > 0$ such that

$$\text{Lip} \left(\bar{N}_k^* \Big|_{\mathcal{B}_{3/2}^*} \right) \leq C \mathbf{h}_k^{*\gamma na}, \quad \left\| \bar{N}_k^* \right\|_{C^0(\mathcal{B}_{3/2}^*)} \leq C \left(\mathbf{m}_{0,j(k)} \bar{r}_k \right)^{\gamma na}, \quad (5.3)$$

$$\mathbf{M} \left(\left(\mathbf{T}_{\bar{F}_k^*} - \bar{T}_k \right) \Big|_{\mathbf{p}_k^{*-1}(\mathcal{B}_{3/2}^*)} \right) \leq C \mathbf{h}_k^{*2+2\gamma na}, \quad \text{and} \quad \int_{\mathcal{B}_{3/2}^*} \left| \boldsymbol{\eta} \circ \bar{N}_k^* \right| \leq C \mathbf{h}_k^{*2}. \quad (5.4)$$

Using the fact that $3\bar{r}_k/2 \in (s_{j(k)}/t_{j(k)}, 3)$ and (4.4) with $a = \bar{r}_k$, we infer as in [8, Section 7.1] that

$$\begin{aligned} \left\| N_{j(k)}^* \right\|_{C^0(\mathcal{B}_{3\bar{r}_k/2}^*(p_{j(k)}^*))} &\leq C \mathbf{m}_{0,j(k)}^{1/2m} \bar{r}_k^{-1+\gamma_h}, \quad \text{Lip} \left(N_{j(k)}^* \Big|_{\mathcal{B}_{3\bar{r}_k/2}^*(p_{j(k)}^*)} \right) \leq C \mathbf{m}_{0,j(k)}^{\gamma_2} \max_i \ell_i^{\gamma_2}, \\ \mathbf{M} \left(\left(\mathbf{T}_{F_{j(k)}^*} - T_{j(k)} \right) \lrcorner \mathbf{p}_k^{*-1} \left(\mathcal{B}_{3\bar{r}_k/2}^*(p_{j(k)}^*) \right) \right) &\leq \sum_i \mathbf{m}_{0,j(k)}^{1+\gamma_2} \ell_i^{m+2+\gamma_2}, \\ \int_{\mathcal{B}_{3\bar{r}_k/2}^*(p_{j(k)}^*)} \left| \boldsymbol{\eta} \circ N_{j(k)}^* \right| &\leq C \mathbf{m}_{0,j(k)} \bar{r}_k \sum_i \ell_i^{2+m+\gamma_{la}/2} + \frac{C}{\bar{r}_k} \int_{\mathcal{B}_{3\bar{r}_k/2}^*(p_{j(k)}^*)} \left| N_{j(k)}^* \right|^2. \end{aligned}$$

Arguing similarly to [8, (4.12), (4.13), and (4.14)] and using the Reverse Sobolev inequality proved in [8, Corollary 5.3], we see that

$$\sum_i \mathbf{m}_{0,j(k)} \ell_i^{m+2+\frac{\gamma_2}{4}} \leq C_0 \int_{\mathcal{B}_{3\bar{r}_k/2}^*(p_{j(k)}^*)} \left(\left| DN_{j(k)}^* \right|^2 + \left| N_{j(k)}^* \right|^2 \right) \leq \frac{C_T}{\bar{r}_k^2} \int_{\mathcal{B}_{s_k}^*(p_{j(k)}^*)} \left| N_{j(k)}^* \right|^2, \quad (5.5)$$

from which (5.3) and (5.4) follow by a simple rescaling. The constant C_T on the right-hand side of (5.5) depends on T but not on k . It is a consequence of these bounds and the Sobolev embedding (cf. [4, Prop. 2.11]) that the sequence $\{N_k^{*,b}\}_k$ weakly converges in $W^{1,2}(B_{3/2}, \mathcal{A}_Q(\{0\} \times \mathbb{R}^{\bar{n}} \times \{0\}))$ (in the sense of [4, Definition 2.9]) to a Q -valued function $N_\infty^{*,b} \in W^{1,2}(B_{3/2}, \mathcal{A}_Q(\mathbb{R}^{m+n}))$. From this convergence and (5.4), we derive that

$$\int_{B_{3/2}} \left| \boldsymbol{\eta} \circ N_\infty^{*,b} \right| = \lim_{k \rightarrow +\infty} \int_{B_{3/2}} \left| \boldsymbol{\eta} \circ N_k^{*,b} \right| \leq C \lim_{k \rightarrow +\infty} \mathbf{h}_k^* = 0.$$

We now check that $N_\infty^{*,b}$ must take its values in $\mathcal{A}_Q(\{0\} \times \mathbb{R}^{\bar{n}} \times \{0\})$. Consider the tangential part of \bar{N}_k^* as in

$$\bar{N}_k^{*,T}(x) := \sum_i \left[\mathbf{p}_{T_x \bar{\Sigma}_k} \left(\bar{N}_k^* \right)_i(x) \right].$$

One can verify that $\mathcal{G}(\bar{N}_k^*, \bar{N}_k^{*,T}) = \mathcal{G}(Q[0], \bar{N}_k^{*,\perp}) \leq C_0 \left| \bar{N}_k^* \right|^2$ which leads to

$$\int_{B_{3/2}} \mathcal{G} \left(N_k^{*,b}, \mathbf{h}_k^{*-1} \bar{N}_k^{*,T} \circ \mathbf{e}_k \right)^2 \leq C_0 \mathbf{h}_k^{*-2} \int_{B_{3/2}} \left| \bar{N}_k^* \right|^4 \stackrel{(5.3)}{\leq} C \left(\mathbf{m}_{0,j(k)} \bar{r}_k \right)^{2\gamma_{ma}} \rightarrow 0 \text{ as } k \rightarrow +\infty.$$

By the $C^{2,\alpha}$ -convergence of $\bar{\Sigma}_k$ to $\mathbb{R}^{m+\bar{n}} \times \{0\}$, we conclude Step 1.

Step 2: A suitable trivialization of the normal bundle through smoothing estimates and $W^{1,2}$ -convergence.

The difficulties we face in exploiting the arguments of [8, Section 7.2] are twofold: (1) the first is that we have to deal with an external center manifold $\bar{\mathcal{M}}_k^*$ not lying inside $\bar{\Sigma}_k$, which introduce error terms that we have to account for, and (2) the fact that $\bar{\Sigma}_k$ is merely of class $C^{2,\alpha}$, which we deal with by approximating with $\bar{\Sigma}_{k,\varepsilon}$ of class C^∞ obtained from Σ first by rescaling by a factor $t_{j(k)}$, then mollifying it, and finally rescaling it again by a factor \bar{r}_k . Precisely, $\bar{\Sigma}_{k,\varepsilon}$ is obtained as follows. We introduce a parameter $\varepsilon > 0$, mollify the functions $\boldsymbol{\Psi}_{j(k)}$ to get the functions $C^\infty \boldsymbol{\Psi}_{k,\varepsilon}$, and then define

$$\Sigma_{k,\varepsilon} := \text{graph}(\boldsymbol{\Psi}_{j(k),\varepsilon}) \text{ and } \bar{\Sigma}_{k,\varepsilon} := \iota_{0,\bar{r}_k} \left(\Sigma_{j(k),\varepsilon} \right).$$

Furthermore, since \mathcal{M}_k^* is of class $C^{3,\kappa}$ the projection \mathfrak{P}_k is only $C^{2,\kappa}$, we will also need to mollify \mathcal{M}_k^* . Precisely, we define $\Phi_{k,\varepsilon} \in C^\infty$ to be the mollified function obtained from $\Phi_{j(k)}$,

$$\mathcal{M}_{k,\varepsilon}^* := \text{graph}(\Phi_{j(k),\varepsilon}), \quad \overline{\mathcal{M}}_{k,\varepsilon}^* := \iota_{0,\bar{r}_k}(\mathcal{M}_{k,\varepsilon}^*), \quad \text{and} \quad \widetilde{\mathcal{M}}_{k,\varepsilon} := \mathfrak{P}_{k,\varepsilon}(\overline{\mathcal{M}}_{k,\varepsilon}^*) \subset \overline{\Sigma}_{k,\varepsilon}.$$

By construction, $\overline{\Sigma}_{k,\varepsilon}$ is approaching $\overline{\Sigma}_k$ in the $C^{2,\beta}$ -topology for every $0 < \beta < \alpha$ and $\overline{\mathcal{M}}_{k,\varepsilon}^* \rightarrow \overline{\mathcal{M}}_k^*$ in C^{3,β_1} topology for every $0 < \beta_1 < \kappa$, when $\varepsilon \rightarrow 0^+$ for any fixed $k \in \mathbb{N}$. Observe that an application of [8, Lemma 6.1], for each $\overline{\mathcal{M}}_{k,\varepsilon}^*$, permits the construction of an orthonormal frame of $(T\overline{\mathcal{M}}_{k,\varepsilon}^*)^\perp$ which we denote by

$$\begin{aligned} \nu_1^{k,\varepsilon}(p), \dots, \nu_{\bar{n}}^{k,\varepsilon}(p), \varpi_1^{k,\varepsilon}(p), \dots, \varpi_l^{k,\varepsilon}(p), \text{ at each point } p \in \overline{\mathcal{M}}_{k,\varepsilon}^*, \\ \nu_j^{k,\varepsilon}(p) \in T_{\mathfrak{P}_{k,\varepsilon}(p)}\overline{\Sigma}_{k,\varepsilon}, \text{ and } \varpi_j^{k,\varepsilon}(p) \perp T_{\mathfrak{P}_{k,\varepsilon}(p)}\overline{\Sigma}_{k,\varepsilon}. \end{aligned}$$

Thus, we have that, for every fixed $\varepsilon > 0$, it holds

$$\nu_j^{k,\varepsilon} \rightarrow e_{m+j} \quad \text{and} \quad \varpi_j^{k,\varepsilon} \rightarrow e_{m+\bar{n}+j} \quad \text{in } C^{2,\kappa/2}(\overline{\mathcal{M}}_{k,\varepsilon}^*) \text{ as } k \rightarrow \infty,^1 \quad (5.6)$$

where $e_1, \dots, e_{m+\bar{n}+l}$ is the standard basis of $\mathbb{R}^{m+\bar{n}+l} = \mathbb{R}^{m+n}$. We can find $\delta > 0$ (independent of k and ε) such that, for $k \geq k_0(\delta) \in \mathbb{N}$ large enough and $0 < \varepsilon < \varepsilon_0(\delta, k)$ small enough, there is a map $\psi_{k,\varepsilon}^* : \overline{\mathcal{M}}_{k,\varepsilon}^* \times \mathbb{R}^{\bar{n}} \rightarrow \mathbb{R}^l$ converging to 0 in $C^{2,\kappa/2}$ (uniformly bounded in C^{2,β_1} with $\frac{\kappa}{2} < \beta_1 < \kappa \leq \alpha$) given by the following property:

$$\mathfrak{P}_{k,\varepsilon}(p) + v \in \overline{\Sigma}_{k,\varepsilon} \iff v^\perp = \psi_{k,\varepsilon}^*(p, v^T), \quad \forall v \in (T_p\overline{\mathcal{M}}_{k,\varepsilon}^*)^\perp \text{ with } |v| \leq \delta.$$

where $v^T = (\langle v, \nu_1^{k,\varepsilon} \rangle, \dots, \langle v, \nu_{\bar{n}}^{k,\varepsilon} \rangle) \in \mathbb{R}^{\bar{n}}$ and $v^\perp = (\langle v, \varpi_1^{k,\varepsilon} \rangle, \dots, \langle v, \varpi_l^{k,\varepsilon} \rangle) \in \mathbb{R}^l$. Observe $\mathfrak{P}_{k,\varepsilon}$ are well defined for every $0 < \varepsilon \leq \varepsilon_0(k, \delta)$ due to the C^2 -convergence of the mollified manifolds $\Sigma_{k,\varepsilon}$. Now, to see that $\psi_{k,\varepsilon}^* \rightarrow 0$, in $C^{2,\kappa/2}$ -topology, consider the map

$$\mathcal{J}_{k,\varepsilon} : \overline{\mathcal{M}}_k^* \times \mathbb{R}^{\bar{n}} \times \mathbb{R}^l \ni (p, z, w) \mapsto p + z^j \nu_j^{k,\varepsilon} + w^j \varpi_j^{k,\varepsilon} \in \mathbb{R}^{m+n}, \quad (5.7)$$

where we use the Einstein convention of summation over repeated indices. It is simple to show that the frame can be chosen so that $D\mathcal{J}_{k,\varepsilon}(0, 0) = \text{Id}$ and, hence, using the implicit function theorem, $\mathcal{J}_{k,\varepsilon}^{-1}(\overline{\Sigma}_{k,\varepsilon})$ can be written locally as a graph of a function $\psi_{k,\varepsilon}^*$ that meets the claimed property. By construction, we also have $\psi_{k,\varepsilon}^*(x, 0) = 0$, (which yields $D_x \psi_{k,\varepsilon}^*(x, 0) = 0$) and $|D_z \psi_{k,\varepsilon}^*(x, 0)| = 0$ for every $x \in \overline{\mathcal{M}}_{k,\varepsilon}^*$, which in turn implies

$$|D_x \psi_{k,\varepsilon}^*(x, z)| \leq C|z|^{1+\beta_1}, \quad |D_z \psi_{k,\varepsilon}^*(x, z)| \leq C|z|, \quad \text{and} \quad |\psi_{k,\varepsilon}^*(x, z)| \leq C|z|^2, \quad (5.8)$$

where $C > 0$ is a constant that can be chosen independently of ε and k . Indeed, it suffices to remember that $\overline{\Sigma}_{k,\varepsilon} \rightarrow \overline{\Sigma}_k$ in the $C^{2,\beta}$ -topology and also that $\overline{\Sigma}_k$ converges to $\mathbb{R}^{m+\bar{n}} \times \{0\}$ to deduce the existence of the constant $C > 0$ independent of ε and k such that $\|\psi_{k,\varepsilon}^*\|_{C^{2,\beta_1}} \leq C$. We define ψ_k^* analogously substituting in (5.7) $\overline{\Sigma}_{k,\varepsilon}$ by $\overline{\Sigma}_k$ and $\overline{\mathcal{M}}_{k,\varepsilon}^*$ by $\overline{\mathcal{M}}_k^*$. Notice that ψ_k^* is of class $C^{1,\kappa}$ and that the convergence of $\psi_{k,\varepsilon}^*$ to ψ_k^* is in the $C^{1,\kappa/2}$ -topology, which is enough for the approximation procedure that we will perform later.

¹In fact, they converge in C^∞ topology.

Now given any Q -valued map $u_{k,\varepsilon} = \sum_i \llbracket u_{k,\varepsilon,i} \rrbracket : \overline{\mathcal{M}}_{k,\varepsilon}^* \rightarrow \mathcal{A}_Q(\{0\} \times \mathbb{R}^{\bar{n}} \times \{0\})$ with $\|u_{k,\varepsilon}\|_{L^\infty} \leq \delta$, we can consider the map $\mathbf{u}_{k,\varepsilon}$ from $\overline{\mathcal{M}}_{k,\varepsilon}^*$ to $\mathcal{A}_Q(\nu(\overline{\mathcal{M}}_{k,\varepsilon}^*))$, where $\nu(\overline{\mathcal{M}}_{k,\varepsilon}^*)$ is the total space of the normal bundle of $\overline{\mathcal{M}}_{k,\varepsilon}^*$, defined by

$$\mathbf{u}_{k,\varepsilon}(x) := \sum_i \left[\sum_{j=1}^{\bar{n}} (u_{k,\varepsilon,i})^j(x) \nu_j^{k,\varepsilon}(x) + \sum_{j=1}^l \psi_{k,\varepsilon}^{*,j}(x, u_{k,\varepsilon,i}(x)) \varpi_j^{k,\varepsilon}(x) \right],$$

where we set $(u_{k,\varepsilon,i})^j(x) := \langle u_{k,\varepsilon,i}(x), e_{m+j} \rangle$, $\psi_{k,\varepsilon}^{*,j}(x, u_{k,\varepsilon,i}(x)) := \langle \psi_{k,\varepsilon}^*(x, u_{k,\varepsilon,i}(x)), e_{m+\bar{n}+j} \rangle$. Then, we get that

$$\begin{aligned} D(\mathbf{u}_{k,\varepsilon})_i &= D(u_i)^j \nu_j^{k,\varepsilon} + \left[D_x \psi_{k,\varepsilon}^{*,j}(x, u_i) + D_z \psi_{k,\varepsilon}^{*,j}(x, u_i) D u_i \right] \varpi_j^{k,\varepsilon} \\ &\quad + (u_i)^j D \nu_j^{k,\varepsilon} + \psi_{k,\varepsilon}^{*,j}(x, u_i) D \varpi_j^{k,\varepsilon}, \text{ a.e. } \mathcal{H}^m \llcorner \overline{\mathcal{M}}_{k,\varepsilon}^*, \end{aligned}$$

where we used the Einstein summation convention on the index j . We define \mathbf{u}_k analogously and also derive the analogous of the last displayed inequality for \mathbf{u}_k . Taking into account that

$$\lim_{k \rightarrow +\infty} \left(\|D \nu_i^k\|_{C^0} + \|D \varpi_j^k\|_{C^0} \right) = 0 \text{ and } \lim_{\varepsilon \rightarrow 0} \left(\|D \nu_i^{k,\varepsilon} - D \nu_i^k\|_{C^0} + \|D \varpi_j^{k,\varepsilon} - D \varpi_j^k\|_{C^0} \right), \quad (5.9)$$

we obtain the existence of a double sequence $(\varepsilon_0(k, \delta))_{k \in \mathbb{N}, \delta \in (0, +\infty)}$, generating a sequence $0 < \varepsilon_{0,k} = \varepsilon_0(k, \delta_k) > 0$ such that $\varepsilon_{0,k} \rightarrow 0^+$ and $\delta_k \rightarrow 0^+$ as $k \rightarrow +\infty$. Furthermore, we have the following

$$\forall \{\varepsilon_k\}_{k \in \mathbb{N}} \text{ with } 0 < \varepsilon_k \leq \varepsilon_{0,k}, \forall k \in \mathbb{N}, \text{ we have } \lim_{k \rightarrow +\infty} \left(\|D \nu_i^{k,\varepsilon_k}\|_{C^0} + \|D \varpi_j^{k,\varepsilon_k}\|_{C^0} \right) = 0.$$

By (5.8), we readily derive that, for some constant $C > 0$ independent of k and k large enough, it holds

$$\left| \int \left(|D \mathbf{u}_{k,\varepsilon_k}|^2 - |D u|^2 \right) \right| \leq C \int |u|^{2+2\beta_1} + |u|^2 |D u|^2 + o(1) |u|^2.$$

On the other hand, since $\psi_{k,\varepsilon}^* \rightarrow \psi_k^*$ in $C^{1,\kappa/2}$ -topology as $\varepsilon \rightarrow 0$ for any fixed k large enough, we have

$$\lim_{\varepsilon \rightarrow 0} \left| \int \left(|D \mathbf{u}_{k,\varepsilon}|^2 - |D \mathbf{u}_k|^2 \right) \right| = 0.$$

We now choose $\varepsilon_{k,\delta} > 0$, depending on $\delta > 0$ and $k \in \mathbb{N}$ large enough, to get from the last two displayed inequalities that

$$\begin{aligned} \left| \int \left(|D \mathbf{u}_k|^2 - |D u|^2 \right) \right| &\leq \left| \int \left(|D \mathbf{u}_{k,\varepsilon_{k,\delta}}|^2 - |D u|^2 \right) \right| + \left| \int \left(|D \mathbf{u}_{k,\varepsilon_{k,\delta}}|^2 - |D \mathbf{u}_k|^2 \right) \right| \\ &\leq C \int |u|^{2+2\beta_1} + |u|^2 |D u|^2 + o(1) |u|^2. \end{aligned} \quad (5.10)$$

By (4.1), we can write $\overline{N}_k^*(x) = Q \llbracket \mathfrak{P}_k(x) \rrbracket \oplus \psi_k^*(x, \bar{u}_k(x))$ for some Lipschitz $\bar{u}_k := \sum_i \llbracket \bar{u}_{k,i} \rrbracket : \overline{\mathcal{M}}_k^* \rightarrow \mathcal{A}_Q(\{0\} \times \mathbb{R}^{\bar{n}} \times \{0\})$ with $\|\bar{u}_k\|_{L^\infty} = o(1)$. Setting

$$u_k^b(x) := \sum_i \llbracket \mathfrak{P}_k(\mathbf{e}_k(x)) + \bar{u}_{k,i}(\mathbf{e}_k(x)) \rrbracket,$$

we conclude, from the Reverse Sobolev inequality ([8, (5.11)]), (4.1), and (5.10), that

$$\lim_{k \rightarrow +\infty} \int_{B_{3/2}} \left(|D N_k^{*,b}|^2 - \mathbf{h}_k^{-2} |D u_k^b|^2 \right) = 0, \quad (5.11)$$

and N_∞^{*b} is the limit of $\mathbf{h}_k^{-1}u_k^b$ in $W^{1,2}(\mathcal{B}_{3/2}, \mathcal{A}_Q(\{0\} \times \mathbb{R}^{\bar{n}} \times \{0\}))$.

*Step 3: Proof of the Dir-minimizing property of N_∞^{*b} ((i) of Theorem 5.7)*

There is nothing to prove if its Dirichlet energy vanishes. Therefore, we assume existence of $c_0 > 0$ such that $0 < c_0 \mathbf{h}_k^{*2} \leq \int_{\mathcal{B}_{3/2}^*} |D\bar{N}_k^*|^2$. We will argue by contradiction, if (i) of Theorem 5.7 were to be false, we argue follow the same argument in [8, Section 7.3] to find $r \in (t, 2)$ and $v_k^{*,b}$ with the following property. If we define $\tilde{N}_{k,\varepsilon}^* = \psi_{k,\varepsilon}^* (x, v_k^b \circ \mathbf{e}_{k,\varepsilon}^{-1})$, then we have

$$\begin{aligned} \tilde{N}_{k,\varepsilon}^* &\equiv \bar{N}_{k,\varepsilon}^* \quad \text{on } \mathcal{B}_{3/2} \setminus \mathcal{B}_t, \quad \text{Lip}(\tilde{N}_{k,\varepsilon}^*) \leq C \mathbf{h}_{k,\varepsilon}^{*\gamma na}, \quad |\tilde{N}_{k,\varepsilon}^*| \leq C (\mathbf{m}_{0,k}^* \bar{T}_k)^{\gamma na} \\ \int_{\mathcal{B}_{\frac{3}{2}}} |\boldsymbol{\eta} \circ \tilde{N}_{k,\varepsilon}^*| &\leq C \mathbf{h}_{k,\varepsilon}^{*2} \quad \text{and} \quad \int_{\mathcal{B}_{3/2}} |D\tilde{N}_{k,\varepsilon}^*|^2 \leq \int_{\mathcal{B}_{3/2}} |D\bar{N}_k^*|^2 - \delta \mathbf{h}_{k,\varepsilon}^{*2}. \end{aligned}$$

We finally construct the competitor current that will violate the minimality of T . Set $\tilde{F}_{k,\varepsilon}^*(x) = \sum_i \llbracket x + \tilde{N}_{k,\varepsilon,i}^* \rrbracket$. The currents $\mathbf{T}_{\tilde{F}_{k,\varepsilon}^*}$ coincides with $\mathbf{T}_{\bar{F}_{k,\varepsilon}^*}$ in $(\mathbf{p}_{k,\varepsilon}^*)^{-1}(\mathcal{B}_{3/2} \setminus \mathcal{B}_t)$ and both of them lie in $\bar{\Sigma}_{k,\varepsilon}$. Define the function $\varphi_{k,\varepsilon}^*(p) = \text{dist}_{\bar{\mathcal{M}}_{k,\varepsilon}^*}(0, \mathbf{p}_{k,\varepsilon}^*(p))$, and, for each $s \in (t, \frac{3}{2})$, consider the slices $\langle \mathbf{T}_{\tilde{F}_{k,\varepsilon}^*} - \bar{T}_{k,\varepsilon}, \varphi_{k,\varepsilon}^*, s \rangle$, where $\bar{T}_{k,\varepsilon} := (\mathcal{T}_{k,\varepsilon})_{\#} \bar{T}_k$ and $\mathcal{T}_{k,\varepsilon} : \bar{\Sigma}_k \rightarrow \bar{\Sigma}_{k,\varepsilon}$ are the natural $C^{2,\beta}$ diffeomorphisms induced by the mollification process which by definition converge to $\mathbf{1}_{\bar{\Sigma}_k}$ in $C^{2,\beta}$ topology. By (5.3), we have

$$\int_t^{\frac{3}{2}} \mathbf{M} \left(\langle \mathbf{T}_{\tilde{F}_{k,\varepsilon}^*} - \bar{T}_{k,\varepsilon}, \varphi_{k,\varepsilon}^*, s \rangle \right) \leq C \mathbf{h}_{k,\varepsilon}^{*2+\gamma na}.$$

Thus, for each $k \in \mathbb{N}$ and $\varepsilon > 0$, we can find a radius $\sigma_{k,\varepsilon} \in (t, \frac{3}{2})$ for which

$$\mathbf{M} \left(\langle \mathbf{T}_{\tilde{F}_{k,\varepsilon}^*} - \bar{T}_{k,\varepsilon}, \varphi_{k,\varepsilon}^*, \sigma_{k,\varepsilon} \rangle \right) \leq C \mathbf{h}_{k,\varepsilon}^{*2+\gamma na}.$$

By the isoperimetric inequality (see [5, Rem. 4.3]), there is a current $S_{k,\varepsilon}$ such that

$$\partial S_{k,\varepsilon} = \langle \mathbf{T}_{\tilde{F}_{k,\varepsilon}^*} - \bar{T}_{k,\varepsilon}, \varphi_{k,\varepsilon}^*, \sigma_{k,\varepsilon} \rangle, \quad \mathbf{M}(S_{k,\varepsilon}) \leq C \mathbf{h}_{k,\varepsilon}^{*(2+\gamma)m/(m-1)} \quad \text{and} \quad \text{spt}(S_{k,\varepsilon}) \subset \bar{\Sigma}_{k,\varepsilon}.$$

Our competitor current is then given by

$$\tilde{T}_{k,\varepsilon} := \bar{T}_{k,\varepsilon} \llcorner \left((\mathbf{p}_{k,\varepsilon}^*)^{-1}(\bar{\mathcal{M}}_{k,\varepsilon}^* \setminus \mathcal{B}_{\sigma_{k,\varepsilon}}) \right) + S_{k,\varepsilon} + \mathbf{T}_{\tilde{F}_{k,\varepsilon}^*} \llcorner \left((\mathbf{p}_{k,\varepsilon}^*)^{-1}(\mathcal{B}_{\sigma_{k,\varepsilon}}) \right).$$

Note that $\tilde{T}_{k,\varepsilon}$ is supported in $\bar{\Sigma}_{k,\varepsilon}$. On the other hand, by (5.3) and the bound on $\mathbf{M}(S_{k,\varepsilon})$, we have

$$\mathbf{M}(\tilde{T}_{k,\varepsilon}) - \mathbf{M}(\bar{T}_{k,\varepsilon}) \leq \mathbf{M}(\mathbf{T}_{\bar{F}_{k,\varepsilon}^*}) - \mathbf{M}(\mathbf{T}_{\tilde{F}_{k,\varepsilon}^*}) + C (\mathbf{h}_{k,\varepsilon}^*)^{2+2\gamma na}.$$

Denote by $\bar{A}_{k,\varepsilon}^*$ and $\bar{H}_{k,\varepsilon}^*$ the second fundamental forms and mean curvatures of the manifold $\bar{\mathcal{M}}_{k,\varepsilon}^*$, respectively. Using the above inequality and the Taylor expansion of [6, Th. 3.2], for every

$0 < \varepsilon \leq \varepsilon_{0,k}$, we achieve

$$\begin{aligned}
\mathbf{M}(\tilde{T}_{k,\varepsilon}) - \mathbf{M}(\bar{T}_{k,\varepsilon}) &\leq \frac{1}{2} \int \left(|D\tilde{N}_{k,\varepsilon}^*|^2 - |D\bar{N}_{k,\varepsilon}^*|^2 \right) \\
&\quad + C \|\bar{H}_{k,\varepsilon}^*\|_{C^0} \int \left(|\boldsymbol{\eta} \circ \bar{N}_{k,\varepsilon}^*| + |\boldsymbol{\eta} \circ \tilde{N}_{k,\varepsilon}^*| \right) \\
&\quad + \|\bar{A}_{k,\varepsilon}^*\|_{C^0}^2 \int \left(|\bar{N}_{k,\varepsilon}^*|^2 + |\tilde{N}_{k,\varepsilon}^*|^2 \right) + o(\mathbf{h}_{k,\varepsilon}^{*2}) \\
&\leq -\frac{\delta}{2} \mathbf{h}_{k,\varepsilon}^{*2} + o(\mathbf{h}_{k,\varepsilon}^{*2}),
\end{aligned} \tag{5.12}$$

where in the last inequality we take into account [8, Lemma 6.1], which can be used here in this form because, by construction, our $\bar{\mathcal{M}}_{k,\varepsilon}^*$ center manifolds are all of class C^∞ and so in particular of class C^{3,β_1} having also C^2 norm uniformly bounded with respect to k and ε .

To finish the proof, we have to take the limit on ε in (5.12). We argue as follows. For every one parameter family pair of abstract pointed Riemannian manifolds $((\Sigma_\varepsilon, g_\varepsilon, p_\varepsilon), (\Sigma'_\varepsilon, g'_\varepsilon, p'_\varepsilon))_\varepsilon$ whose underlying differentiable structure is of class C^1 and with metric tensors $g_\varepsilon, g'_\varepsilon$ of class C^0 converging in topology C^0 to the pair $((\Sigma_0, g_0, p_0), (\Sigma'_0, g'_0, p'_0))$, and every family of C^1 diffeomorphisms $\mathfrak{D}_\varepsilon \in C^1(\Sigma_\varepsilon, \Sigma'_\varepsilon)$ and $\mathfrak{D}_\varepsilon^{-1} \in C^1(\Sigma'_\varepsilon, \Sigma_\varepsilon)$, it is easy to check that $(\mathfrak{D}_\varepsilon)_\#(\mathbf{I}_m(\Sigma_\varepsilon, g_\varepsilon)) = \mathbf{I}_m(\Sigma'_\varepsilon, g'_\varepsilon)$ (here $\mathbf{I}_m(\Sigma_\varepsilon, g_\varepsilon)$ denotes all integral m -currents in $(\Sigma_\varepsilon, g_\varepsilon)$). Moreover, if we assume that there exists an isometry of pointed metric spaces $\mathfrak{f} : (\Sigma_0, g_0, p_0) \rightarrow (\Sigma'_0, g'_0, p'_0)$ with $\mathfrak{f} \in C^1(\Sigma, \Sigma')$ and $\mathfrak{f}^{-1} \in C^1(\Sigma', \Sigma)$ satisfying $\mathfrak{D}_\varepsilon \rightarrow \mathfrak{f}$ in C^1 topology, $T_\varepsilon \in \mathbf{I}_m(\Sigma_\varepsilon, g_\varepsilon)$, $T_\varepsilon \rightarrow T_0 \in \mathbf{I}_m(\Sigma_0, g_0)$ in the intrinsic flat topology, then it holds

$$\mathbf{M}_{(\Sigma'_\varepsilon, g'_\varepsilon)}((\mathfrak{D}_\varepsilon)_\# T_\varepsilon) \rightarrow \mathbf{M}_{(\Sigma'_0, g'_0)}(\mathfrak{f}_\# T_0) = \mathbf{M}_{(\Sigma_0, g_0)}(T_0), \text{ when } \varepsilon \rightarrow 0^+. \tag{5.13}$$

Applying (5.13) to our setting immediately provides the following

$$\mathbf{M}(\tilde{T}_k) - \mathbf{M}(\bar{T}_k) = \lim_{\varepsilon \rightarrow 0^+} \mathbf{M}(\tilde{T}_{k,\varepsilon}) - \mathbf{M}(\bar{T}_{k,\varepsilon}),$$

and, by Dominated convergence theorem, we have that $\mathbf{h}_{k,\varepsilon}^* \rightarrow \mathbf{h}_k^*$ as $\varepsilon \rightarrow 0^+$ for each fixed k large enough. This together with (5.12) contradicts the minimizing property of \bar{T}_k for k large enough and $\varepsilon_{0,k}$ small enough. Hence, we conclude the proof. \square

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