CHARACTERIZATIONS OF SOBOLEV SPACES ON SUBLEVEL SETS IN ABSTRACT WIENER SPACES

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ABSTRACT. In this paper we consider an abstract Wiener space (X,γ,H) and an open subset $O\subseteq X$ which satisfies suitable assumptions. For every $p\in(1,+\infty)$ we define the Sobolev space $W_0^{1,p}(O,\gamma)$ as the closure of Lipschitz continuous functions which have support with positive distance from ∂O with respect to the natural Sobolev norm, and we show that under the assumptions on O the space $W_0^{1,p}(O,\gamma)$ can be characterized as the space of functions in $W^{1,p}(O,\gamma)$ which have null trace at the boundary ∂O, or, equivalently, as the space of functions defined on O whose trivial extension belongs to $W^{1,p}(X,\gamma)$.

1. INTRODUCTION

In this paper we consider an abstract Wiener space (X,γ,H), i.e., X is a real separable Banach space endowed with a centered non-degenerate Gaussian measure γ and H is the associated Cameron-Martin space, and a subset $O\subseteq X$ with $O=G^{-1}((-\infty,0))$, where G is a function which satisfies suitable assumptions (see Hypotheses 3.1).

The topic of Sobolev spaces $W^{k,p}(X,\gamma)$ in a Wiener space X is well established (see e.g. [5]), while Sobolev spaces in subsets of a Wiener space admits different definitions, and they have been treated for example in [7], [12] and [13]. Following [7] (see also [3]), we consider $W^{1,p}(O,\gamma)$ as the domain of closure ∇_H of the H-gradient operator on Lipschitz continuous functions.

In [7], the set O is the sublevel $G^{-1}((-\infty,0))$ of a function G. Under some regularity assumptions on G it is possible to define a surface measure ρ (Hausdorff-Gauss infinite dimensional measure or Feyel-de La Pradelle measure, see e.g. [11]). Moreover, in [7] the authors show the existence of a bounded operator (trace operator) Tr from $W^{1,p}(O,\gamma)$ to $L^q(\partial O,\rho)$ with $p>1$ and $q\in[1,p)$. Thanks to this operator, it is possible to introduce an integration-by-parts formula on O which generalize that on the whole space. Namely, for every $\varphi\in W^{1,p}(O,\gamma)$ and every $h\in H$, we have

$$\int_O(\partial_h\varphi-\hat h\varphi)\,d\gamma=\int_{\partial O}\mathrm{Tr}\,\varphi\frac{\langle\nabla_H G,h\rangle_H}{\|\nabla_H G\|_H}\,d\rho, \tag{1.1}$$

with $\partial_h\varphi(x)=\langle\nabla_H\varphi(x),h\rangle_H$ and $\hat h=R_\gamma^{-1}(h)$, where R_γ is the covariance operator of γ. Integration-by-parts formula as (1.1) on domains have been also proved in [3, 4] with different techniques. Strengthening the assumptions on G, the trace operator can be extended as an operator from $W^{1,p}(O,\gamma)$ onto $L^p(\partial O,\rho)$ for every $p>1$. Finally, in [7] the authors prove that the subspace of $f\in W^{1,p}(O,\gamma)$, consisting of functions with null trace on ∂O,

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coincides with the subspace of $f \in W^{1,p}(O,\gamma)$, whose elements are those functions whose trivial extension to the whole X belongs to $W^{1,p}(X,\gamma)$.

In this paper we consider $O = G^{-1}((-\infty,0))$ and we define the space $W_0^{1,p}(O,\gamma)$ as the closure, with respect to the $W^{1,p}(O,\gamma)$-norm, of Lipschitz continuous functions whose support has positive distance from O^c. Eventually we prove that, under suitable conditions on G, for every $p > 1$, $W_0^{1,p}(O,\gamma)$ is the space of functions in $W^{1,p}(O,\gamma)$ with null trace on ∂O (Theorem 4.1).

Examples of spaces O to which our results apply can be found in the Section 5: they include subgraphs of functions with some regularity, and subsets of X in the particular case in which X is the Wiener space which models the Brownian motion or in the case in which X is the Wiener space which models the pinned Brownian motion.

We stress that these examples may not comprehend many regular sets like balls, neither if X is a Hilbert space. This limitation is strictly related to our approach, and also appears in [7], in the case when the operator Tr maps $W^{1,p}(O,\gamma)$ onto $L^p(\partial O,\rho)$ when $p > 1$. To the best of our knowledge, nowadays there is no other result about the definition of a trace operator from $W^{1,p}(O,\gamma)$ onto $L^p(\partial O,\rho)$ for more general subsets O, and also the case $p = 1$ is not reached. This is one of the main gap with respect to the finite dimension, where the theory of traces for Sobolev functions is well-understood and complete. For the case $p = 1$, a possible alternative approach is to consider BV functions in open domains in Wiener spaces, which are studied and characterized in [2]. However, it is still not clear how to extend the theory of traces for BV functions in finite dimension to this setting.

Beside traces, another open question in infinite dimension is what domains O allow the construction of extension operators for Sobolev functions. Again, the case when O is an open ball is still an open problem, even when X is a Hilbert space. A negative answer is given in [6], where the authors provide an example of open convex subset O of a Hilbert space X such that, for every $p > 1$, there exists a function $f \in W^{1,p}(O,\gamma)$ which does not admit a Sobolev extension to the whole X. On the contrary, an example of extension operator can be found in [1], where the authors show that if O is an half-plane that it admits an extension operator, and explicitly provide such an extension.

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2. Notations and preliminary results

In the following, for any $k,d \in \mathbb{N}$ we denote by $C_b^k(\mathbb{R}^d)$ the set of k-times differentiable functions on $\mathbb{R}^d$ with all derivatives uniformly bounded. $C_b^\infty(\mathbb{R}^d)$ is the set of bounded smooth functions on $\mathbb{R}^d$ which belongs to $C_b^k(\mathbb{R}^d)$ for every $k \in \mathbb{N}$. $C_c^\infty(\mathbb{R}^d)$ is the set of functions in $C_b^\infty(\mathbb{R}^d)$ with compact support.

For every real-valued function f defined in a subset $A \subseteq X$, we denote by $\overline{f}$ its trivial extension on X, i.e., $\overline{f} = f$ on A and $\overline{f} = 0$ on A^c.

For every $A \subseteq X$, we denote by $\mathbb{1}_A$ the characteristic function of A, i.e., $\mathbb{1}_A(x) = 1$ if $x \in A$ and $\mathbb{1}_A(x) = 0$ if $x \notin A$.

Let K be a real separable Hilbert space. and let $\mathscr{L}(K)$ be the space of linear bounded operators on K. We denote by $\mathscr{L}_2(K)$ the subspace of $\mathscr{L}(K)$ whose elements L satisfy

$$\|L\|^2_{\mathscr{L}_2(K)} := \sum_{n=1}^{\infty} \|Le_n\|_K^2 < +\infty,$$

where $\{e_n : n \in \mathbb{N}\}$ is any orthonormal basis of K. The elements of $\mathscr{L}_2(K)$ are called Hilbert-Schmidt operators, and the norm $\|\cdot\|_{\mathscr{L}_2(K)}$ is the Hilbert-Schmidt norm. The space $(\mathscr{L}_2(K), \|\cdot\|_{\mathscr{L}_2(K)})$ is a separable Hilbert space if endowed with the inner product

$$[L,M]_{\mathscr{L}_2(K)} = \sum_{n=1}^{\infty} \langle Le_n, Me_n\rangle_K, \qquad L,M \in \mathscr{L}_2(K),$$

where $\{e_n : n \in \mathbb{N}\}$ is any orthonormal basis of K. Given a real separable Banach space X, we denote by $\mathscr{B}(X)$ the Borel subsets of X.

We denote by $\mathrm{Lip}(X)$ the set of Lipschitz continuous functions from X onto $\mathbb{R}$. For every open set $O \subseteq X$ we denote by $\mathrm{Lip}(O)$ the set of Lipschitz continuous functions on O, by $\mathrm{Lip}_b(O)$ the set of bounded Lipschitz continuous functions on O, and by $\mathrm{Lip}_c(O)$ the set of Lipschitz continuous functions on O whose support has positive distance from O^c.

We recall some definitions and properties of abstract Wiener spaces (see e.g. [5]). Let X be a separable Banach space, let X^* be its dual and let X^{**} be the dual of X^*. We will suppose that γ is a centered non-degenerate Gaussian measure on X.

We consider the embedding $j : X^* \hookrightarrow L^2(X,\gamma)$, and we define the reproducing kernel X^*_γ as the closure in $L^2(X,\gamma)$ of $j(X^*)$. It is a separable Hilbert space endowed with the L^2-norm, and we introduce the covariance operator $R_\gamma : X^*_\gamma \to X^{**}$ defined as

$$R_\gamma f(g) = \int_X f j(g) d\gamma, \qquad f \in X^*_\gamma,\ g \in X^*.$$

R_γ is injective, and its range is contained in X, by identifying X with its natural embedding in X^{**}. We define the Cameron-Martin space H as $R_\gamma(X^*_\gamma) \subseteq X$; H inherits a structure of separable Hilbert space through R_γ: we define $\langle\cdot,\cdot\rangle_H$ as the inner product in H and $\|\cdot\|_H$ as the associated norm. As a subspace of X, H is dense. If $h \in H$, we define $\hat{h} = R_\gamma^{-1}(h)$, so that $\hat{h} \in X^*_\gamma \subseteq L^2(X,\gamma)$. The triple (X,H,γ) is called abstract Wiener space.

We fix an orthonormal basis $\{h_i : i \in \mathbb{N}\}$ of H such that $h_i \in R_\gamma(j(X^*))$ for every $i \in \mathbb{N}$. We have that $\{\hat{h}_i : i \in \mathbb{N}\}$ is an orthonormal basis of $X^*_\gamma \subseteq L^2(X,\gamma)$, and for every $f \in L^2(X,\gamma)$ we get

$$\sum_{i=1}^{+\infty} |\langle f, \hat{h}_i\rangle_{L^2(X,\gamma)}|^2 \leq \|f\|^2_{L^2(X,\gamma)}. \tag{2.1}$$

For every $n \in \mathbb{N}$ we define the projection $\pi_n : X \to \mathrm{span}\{h_1,\ldots,h_n\} \subseteq H$ as

$$\pi_n(x) = \sum_{i=1}^{n} \hat{h}_i(x)h_i, \qquad x \in X.$$

We denote by $L^p(X,\gamma;H)$ the space of (equivalence classes of) Bochner integrable functions $f : X \to H$ such that

$$\|f\|_{L^p(X,\gamma;H)} := \left(\int_X \|f\|_H^p \, d\gamma\right)^{1/p} < \infty.$$

$L^p(X,\gamma;H)$ is a Banach space endowed with the norm $\|\cdot\|_{L^p(X,\gamma,H)}$ (see e.g. [9]).

Let $n \in \mathbb{N}$ and let F be a n-dimensional subspace of $R_\gamma(j(X^*)) \subseteq H$. If $\{e_i, \dots, e_n\}$ is an orthonormal basis of F, we define the projection π_F of X on F as the bounded linear function

$$\pi_F(x) = \sum_{i=1}^{n} \hat{e}_i(x) e_i$$

for every $x \in X$. π_F is uniquely defined, independently from the choice of the basis.

We denote by γ_F the image measure $\gamma \circ \pi_F^{-1}$ on F, i.e.,

$$\gamma_F(A) = \gamma(\pi_F^{-1}(A))$$

for every A Borel set in F. It follows that γ_F is a non-degenerate centered Gaussian measure, and there exist a Banach space $X_{F^\perp}$ and a non-degenerate centered Gaussian measure $\gamma_{F^\perp}$ such that we have an isometry between $F \times X_{F^\perp}$ and X, and $\gamma = \gamma_F \otimes \gamma_F^\perp$. This is said factorization of γ with respect to F.

We define the space of bounded infinitely many times differentiable cylindrical functions $\mathscr{F}C_b^\infty(X)$ as the set of functions $f : X \to \mathbb{R}$ such that

$$f(x) = g(l_1(x), \dots, l_n(x)), \qquad x \in X,$$

where $l_1, \dots, l_n \in X^*$ are bounded linear functions on X and $g \in C_b^\infty(\mathbb{R}^n)$ for some $n \in \mathbb{N}$. We recall that $\mathscr{F}C_b^\infty(X)$ is dense in $L^p(X, \gamma)$ for every $p \in [1, +\infty)$. $\mathscr{F}C_b^\infty(X; H)$ denotes the set of functions $f : X \to H$ with finite dimensional range such that, for every $l \in H$, we have $x \mapsto \langle l, f(x) \rangle_H \in \mathscr{F}C_b^\infty(X)$. In particular, $\mathscr{F}C_b^\infty(X; H)$ is spanned by functions ϕh with $\phi \in \mathscr{F}C_b^\infty(X)$ and $h \in H$. It is easy to prove that $\mathscr{F}C_b^\infty(X; H)$ is dense in $L^p(X, \gamma; H)$.

For every smooth function $f : X \to \mathbb{R}$, every $h \in H$ and every $x \in X$, we define the partial derivative $\partial_h f(x)$ of f at x along h as

$$\partial_h f(x) := \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon h) - f(x)}{\varepsilon} \tag{2.2}$$

and the partial logarithmic derivative $\partial_h^* f(x)$ of f at x along h as

$$\partial_h^* f(x) := \partial_h f(x) - f(x)\hat{h}(x).$$

We say that f is H-differentiable in $x \in X$ if there exists $\nabla_H f(x) \in H$ such that

$$\partial_h f = \langle \nabla_H f, h \rangle_H, \qquad h \in H.$$

If $f \in \mathscr{F}C_b^\infty(X)$ then it is everywhere H-differentiable, $\nabla_H f = R_\gamma D f$, where Df is the Fréchet derivative of f, and $\nabla_H f \in L^\infty(X, \gamma; H)$. Further, the operator ∇_H is well defined for any Lipschitz continuous function f and $\nabla_H f \in L^\infty(X, \gamma; H)$ (see e.g. [5, Theo. 5.11.2]).

For every $p \in [1, +\infty)$, $\nabla_H : \mathscr{F}C_b^\infty(X) \to L^p(X, \gamma; H)$ is a closable operator in $L^p(X, \gamma)$. We still denote its closure as ∇_H and we define the Sobolev space $W^{1,p}(X, \gamma)$ as the domain of this closure (see [5, Sec. 5.2]). Moreover, if $f \in W^{1,p}(X, \gamma)$, then $\nabla_H f \in L^p(X, \gamma; H)$ and for every $h \in H \setminus \{0\}$ we set $\partial_h f = \langle \nabla_H f, h \rangle_H$.

Let $f : X \to H$. We say that f is H-differentiable at $x \in X$ if there exists a Hilbert-Schmidt operator $D_H f(x)$ on H such that

$$D_H f(x) h = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon h) - f(x)}{\varepsilon}, \qquad x \in X,\ h \in H.$$

For every $p \in [1, +\infty)$, $D_H : \mathscr{F}C_b^\infty(X; H) \to L^p(X, \gamma; \mathscr{L}_2(H))$ is a closable operator on $L^p(X, \gamma, H)$. We still denote by D_H its closure and we define $W^{1,p}(X, \gamma; H)$ as the domain of this closure (see [5, Sec. 5.2]).

Let $f : X \to \mathbb{R}$ be such that ∇_H is defined at each point $x \in X$. We say that f is twice H-differentiable at $x \in X$ if $\nabla_H f$ is H-differentiable at x. We set $D^2_H f(x) := D_H(\nabla_H f)(x)$, and we recall that the operator $D^2_H f(x) : H \times H \to \mathbb{R}$ is a Hilbert-Schmidt operator on H. $D^2_H f(x)$ is said H-second derivative of f at x.

The operator $(\nabla_H, D^2_H) : \mathscr{F}C^\infty_b(X) \to L^p(X,\gamma;H) \times L^p(X,\gamma;\mathscr{L}_2(H))$ is a closable operator on $L^p(X,\gamma)$ for every $p \in [1,+\infty)$. We denote by $W^{2,p}(X,\gamma)$ the domain of the closure of the operator (∇_H, D^2_H) (see [5, Sec. 5.2]).

We recall the concept of H-divergence (see [5, Sec. 5.8]). For every $f \in \mathscr{F}C^\infty_b(X;H)$ we define the H-divergence div_γ with respect to γ as

$$\mathrm{div}_\gamma f = \sum_{i=1}^\infty (\partial_{h_i} f_i - \hat{h}_i f_i) = \sum_{i=1}^\infty \partial^*_{h_i} f_i, \tag{2.3}$$

where $\{h_i : i \in \mathbb{N}\}$ is an orthonormal basis of H and $f_i = \langle f, h_i \rangle_H$ for every $i \in \mathbb{N}$. The definition of div_γ does not depend on the choice of the basis of H. Further, if $f : X \to H$ is everywhere H-differentiable with $D_H f$ uniformly bounded, then $\mathrm{div}_\gamma f$ is defined everywhere (through formula (2.3)).

Let $f \in W^{1,2}(X;H)$. For every $n \in \mathbb{N}$ we define $f_n(x) = \pi_n \circ f(x)$ for every $x \in X$. It follows that the divergence $\mathrm{div}_\gamma f_n$ is defined γ-a.e. in X, it belongs to $L^2(X,\gamma)$ and it converges in $L^2(X,\gamma)$ to a function $g \in L^2(X,\gamma)$ which we denote by $\mathrm{div}_\gamma f$ (see [5, Theo. 5.8.3]). Moreover, the operator div_γ is the adjoint of $-\nabla_H$ in $L^2(X,\gamma)$ in the sense that, if $f \in W^{1,2}(X;H)$ then

$$\int_X \langle f, \nabla_H g \rangle_H \, d\gamma = -\int_X \mathrm{div}_\gamma f g \, d\gamma$$

for every $g \in W^{1,2}(X,\gamma)$.

2.1. **The Hausdorff-Gauss spherical measure.** In the above setting, by following [11], it is possible to define a Borel measure $\mathscr{S}^{\infty-1}$ on X which replace the $(d-1)$-Hausdorff measure in $\mathbb{R}^d$ in abstract Wiener spaces (hence, it can be seen as an area measure for $(\infty-1)$-hypersurfaces). The measure $\mathscr{S}^{\infty-1}$ is called the Hausdorff-Gauss measure or Feyel-de la Pradelle measure and we denote it by ρ. Let us briefly show the construction of ρ.

Let $F \subseteq R_\gamma(j(X^*))$ be an m-dimensional subspace of H. We identify F with $\mathbb{R}^m$ by choosing an orthonormal basis of F in H, and by identifying it with the canonical basis of $\mathbb{R}^m$. For every $m \in \mathbb{N}$, $\mathscr{S}^{m-1}$ denotes the spherical $(m-1)$-dimensional Hausdorff measure on space F, and for every $y \in X_{F^\perp}$ and every $B \in \mathscr{B}(X)$, we denote by B_y the section

$$B_y = \{z \in F : \; y + z \in B\}$$

and the function

$$G_m(y) = (2\pi)^{-\frac{m}{2}} e^{-\frac{\|y\|_H^2}{2}}.$$

The spherical $(\infty-1)$-dimensional Hausdorff-Gauss measure in X with respect to F is

$$\mathscr{S}_F^{\infty-1}(B) = \int_{X_{F^\perp}} \left(\int_{B_y} G_m(z) \, d\mathscr{S}^{m-1}(z) \right) d\gamma_F^\perp(y), \qquad B \subseteq X.$$

$\mathscr{S}_F^{\infty-1}$ is a σ-additive Borel measure on X, and for every Borel set $B \in \mathscr{B}(X)$, the map $y \mapsto \int_{B_y} G_m \, d\mathscr{S}^{m-1}$ is $\gamma^\perp$-measurable in $F^\perp$. Since the measures $\mathscr{S}_F^{\infty-1}$ are monotone with respect to F, we set

$$\rho(B) = \sup_{F \le R_\gamma(X^*)} \mathscr{S}_F^{\infty-1}(B)$$

for every $B \in \mathscr{B}(X)$, where the supremum is meant as a supremum in a direct set. It turns out that ρ is a Borel measure.

2.2. Definition of the Sobolev spaces $W^{1,p}(O,\gamma)$ and $W_0^{1,p}(O,\gamma)$.

Let $O \subseteq X$ be an open set. We denote by $\mathscr{F}C_b^\infty(O)$ the set of the restrictions to O of elements of $\mathscr{F}C_b^\infty(X)$. The next Lemma is proved, for instance, in [3, Lem. 2.1].

Lemma 2.1. *For every $p \in [1,+\infty)$, the operator $\nabla_H : \mathscr{F}C_b^\infty(O) \to L^p(O,\gamma,H)$ is closable in $L^p(O,\gamma)$. The same is true if we use Lip(O) instead of $\mathscr{F}C_b^\infty(O)$, and the domains of the closures coincide. We still denote by ∇_H the closure of ∇_H.*

Proof. The proof is the same of [3, Lemma 2.1] for both the space functions. The closures coincide because $\mathscr{F}C_b^\infty(O) \subseteq \mathrm{Lip}(O)$, and every Lipschitz continuous function can be extended to X by the McShane extension, and then approximated in L^p by $\mathscr{F}C_b^\infty(X)$ functions. □

Actually, the proof in [3] uses spaces $\mathscr{F}C_b^1(O)$ and $\mathrm{Lip}_b(O)$, respectively, but the arguments are the same. From Lemma 2.1 we introduce the following spaces.

Definition 2.1. We denote by $W^{1,p}(O,\gamma)$ the domain of the closure of ∇_H in $L^p(O,\gamma)$. If endowed with the norm

$$\|f\|_{W^{1,p}(O,\gamma)} := \left(\|f\|^p_{L^p(O,\gamma)} + \|\nabla_H f\|^p_{L^p(O,\gamma;H)} \right)^{1/p}, \quad f \in W^{1,p}(O,\gamma),$$

the space $W^{1,p}(O,\gamma)$ is a Banach space. If $p = 2$ then $W^{1,2}(O,\gamma)$ is a Hilbert space with inner product

$$\langle f,g\rangle_{W^{1,2}(O,\gamma)} = \langle f,g\rangle_{L^2(O,\gamma)} + \langle \nabla_H f, \nabla_H g\rangle_{L^2(O,\gamma;H)}, \quad f,g \in W^{1,2}(O,\gamma).$$

We now define the Sobolev spaces $W_0^{1,p}(O,\gamma)$.

Definition 2.2. For every $p \in [1,+\infty)$, we denote by $W_0^{1,p}(O,\gamma)$ the closure of $\mathrm{Lip}_c(O)$ in $W^{1,p}(O,\gamma)$.

We want to prove that $W_0^{1,p}(O)$ actually coincides with the closure of different subspaces of $W^{1,p}(O,\gamma)$. To this aim, we introduce the following spaces of functions.

Definition 2.3. The space $\mathrm{Lip}_{c,H}(O)$ is the space of functions $f : O \to \mathbb{R}$, with support contained in an open set A with positive distance from O^c, such that there exists a positive constant ℓ such that for γ-a.e. $x \in O$ we have

$$|f(x+h) - f(x)| \le \ell \|h\|_H, \quad \forall h \in H. \tag{2.4}$$

Definition 2.4. With $\mathscr{H}^1(X)$ we denote the set of all continuous functions f (not necessarily bounded) which are H-differentiable on X and such that $\nabla_H f : X \to H$ is bounded and continuous.

$\mathscr{H}_0^1(O)$ is the subset of $\mathscr{H}^1(X)$ of functions f whose support has positive distance from O^c.

The following result shows that $\mathscr{H}_0^1(O)$ is not empty. To prove this fact, we introduce the Ornstein-Uhlenbeck semigroup $(T_t)_{t\geq 0}$ on X, characterized by the Mehler formula

$$T_t f(x) = \int_X f\left(e^{-t}x + \sqrt{1-e^{-2t}}y\right) d\gamma(y), \qquad f \in C_b(X),\ x \in X,\ t \geq 0, \tag{2.5}$$

which extends to a bounded strongly continuous semigroup on $L^p(X,\gamma)$ for every $p \in [1,+\infty)$, which we again denote by $(T_t)_{t\geq 0}$. We recall that for every $f \in C_b(X)$, we have $T_t f$ is H-differentiable and

$$\langle \nabla_H T_t f(x), h\rangle_H = \frac{e^{-t}}{\sqrt{1-e^{-2t}}} \int_X f\left(e^{-t}x + \sqrt{1-e^{-2t}}y\right) \widehat{h}(y) d\gamma(y), \qquad x \in X,\ t \geq 0.$$

Lemma 2.2. *$\mathscr{H}_0^1(O)$ is not empty.*

Proof. Let us fix a bounded closed set $B \subseteq X$ and $\varepsilon > 0$. From [14, Lemma 2.5] and its proof, we infer that there exists a Lipschitz continuous function f with Lipschitz constant $2\varepsilon^{-1}$, $t > 0$ and a smooth function $\Phi \in C_b^\infty(\mathbb{R})$ with $0 \leq \Phi \leq 1$, $\Phi = 0$ on $(-\infty, 1/3)$ and $\Phi = 1$ on $(2/3, +\infty)$, such that the function $F_{B,\varepsilon} = \Phi(T_t f)$ equals 1 on B and $F_{B,\varepsilon} = 0$ on $X \setminus \{x \in X : d(x,B) < \varepsilon\}$.

Hence, for every bounded open set $A \subseteq O$ with positive distance d from O^c, with the choice $B = \overline{A}$ and $\varepsilon < d$, the function $F = F_{B,\varepsilon}$ has $\nabla_H F$ everywhere defined and bounded by [5, Theorem 5.11.2]. F belongs to $\mathscr{H}_0^1(O)$, providing that we prove that $\nabla_H F$ is continuous. To this aim, for every $x, y \in X$ we have

$$\begin{aligned}
&\|\nabla_H F(x) - \nabla_H F(y)\|_H^2 \\
\leq &\frac{e^{-2t}\|\Phi\|^2_{C_b^1(\mathbb{R})}}{1-e^{-2t}} \sum_{n=1}^\infty \left(\int_X \left(f(e^{-t}x + \sqrt{1-e^{-2t}}z) - f(e^{-t}y + \sqrt{1-e^{-2t}}z)\right) \hat{h}_n(z) d\gamma(z)\right)^2 \\
\leq &\frac{e^{-2t}\|\Phi\|^2_{C_b^1(\mathbb{R})}}{1-e^{-2t}} \|f(e^{-t}x + \sqrt{1-e^{-2t}}\cdot) - f(e^{-t}y + \sqrt{1-e^{-2t}}\cdot)\|^2_{L^2(X,\gamma)} \\
\leq &\frac{4e^{-4t}\|\Phi\|^2_{C_b^1(\mathbb{R})}}{\varepsilon^2(1-e^{-2t})} \|x-y\|_X^2,
\end{aligned}$$

where the second inequality is a consequence of (2.1), and this gives the continuity of $\nabla_H F$. Furthermore, F is Lipschitz continuous due to the Lipschitz continuity of f, the definition of T_t and the smoothness of Φ. ☐

From the definition of $\mathscr{H}_0^1(O)$, it follows that $\mathscr{H}_0^1(O) \subseteq \mathrm{Lip}_{c,H}(O)$.

Furthermore, for every $f \in \mathrm{Lip}_{c,H}(O)$, its trivial extension $\overline{f}$ belongs to $W^{1,p}(X,\gamma)$ for every $p \in (1,+\infty)$, and so $\mathrm{Lip}_{c,H}(O) \subseteq W^{1,p}(O,\gamma)$ for every $p \in (1,+\infty)$ (see [5, Theorem 5.11.2]).

Clearly, also $\mathrm{Lip}_c(O) \subseteq \mathrm{Lip}_{c,H}(O)$, and so

$$W_0^{1,p}(O,\gamma) \subseteq \overline{\mathrm{Lip}_{c,H}(O)}^{W^{1,p}(O,\gamma)}, \qquad \overline{\mathscr{H}_0^1(O)}^{W^{1,p}(O,\gamma)} \subseteq \overline{\mathrm{Lip}_{c,H}(O)}^{W^{1,p}(O,\gamma)}.$$

We prove that the above inclusions are indeed equalities.

Lemma 2.3. *We have*

$$W_0^{1,p}(O,\gamma) = \overline{\mathscr{H}_0^1(O)}^{W^{1,p}(O,\gamma)} = \overline{\mathrm{Lip}_{c,H}(O)}^{W^{1,p}(O,\gamma)},$$

The closure of $\mathscr{H}_0^1(O)$ in $W^{1,p}(O,\gamma)$ coincides with $W_0^{1,p}(O,\gamma)$ for every $p \in (1,+\infty)$.

Proof. Let us fix $p \in (1,+\infty)$. To prove the statement, we show that $\overline{\mathscr{H}_0^1(O)}^{W^{1,p}(O,\gamma)} \subseteq W_0^{1,p}(O,\gamma)$ and that $\overline{\mathrm{Lip}_{c,H}(O)}^{W^{1,p}(O,\gamma)} \subseteq \overline{\mathscr{H}_0^1(O)}^{W^{1,p}(O,\gamma)}$. Without loss of generality, we assume that the support of the considered functions is bounded.

Let $g \in \mathscr{H}_0^1(O)$. Its trivial extension $\overline{g}$ belongs to $W^{1,p}(X,\gamma)$ and so there exists a sequence $(g_n)_{n\in\mathbb{N}} \subseteq \mathscr{F}\mathscr{C}_b^\infty(X)$ which converges to $\overline{g}$ in $W^{1,p}(X,\gamma)$ as $n \to +\infty$. Let $A \subseteq O$ be a bounded open set with positive distance from O^c such that $\mathrm{supp}(g) \subseteq A$, and let F be the function defined in Lemma 2.2. F is Lipschitz continuous: indeed, for every $x,y \in X$ we have

$$\begin{aligned} |F(x)-F(y)| \leq & \|\Phi'\|_\infty |T_t f(x) - T_t f(y)| \\ \leq & \|\Phi'\|_\infty \int_X \left| f(e^{-t}x+\sqrt{1-e^{2t}}z) - f(e^{-t}y+\sqrt{1-e^{2t}}z)\right| d\gamma(z) \\ \leq & \frac{2}{\varepsilon} e^{-t}\|\Phi'\|_\infty \|x-y\|_X, \end{aligned}$$

where we have used the fact that f is a $\frac{2}{\varepsilon}$-Lipschitz continuous function. Then, the sequence $(Fg_n)_{n\in\mathbb{N}} \subseteq \mathrm{Lip}_c(O)$ and it converges to g in $W^{1,p}(O,\gamma)$.

This implies that $\overline{\mathscr{H}_0^1(O)}^{W^{1,p}(O,\gamma)} \subseteq W_0^{1,p}(O,\gamma)$.

Let $g \in \mathrm{Lip}_{c,H}(O)$. Its trivial extension $\overline{g}$ belongs to $W^{1,p}(X,\gamma)$. Hence, there exists a sequence $(g_n)_{n\in\mathbb{N}} \subseteq \mathscr{F}\mathscr{C}_b^\infty(X)$ such that $g_n \to \overline{g}$ in $W^{1,p}(X,\gamma)$ as $n \to +\infty$. Let $A \subseteq O$ be a bounded open set with positive distance from O^c such that $\mathrm{supp}(g) \subseteq A$, and let F be the function defined in Lemma 2.2. The sequence $(Fg_n)_{n\in\mathbb{N}}$ converges to g in $W^{1,p}(O,\gamma)$, with $F \in \mathscr{H}_0^1(O)$ and $g_n \in \mathscr{F}\mathscr{C}_b^\infty(X)$, which give $Fg_n \in \mathscr{H}_0^1(O)$ for every $n \in \mathbb{N}$.

This gives $\overline{\mathrm{Lip}_{c,H}(O)}^{W^{1,p}(O)} \subseteq \overline{\mathscr{H}_0^1(O)}^{W^{1,p}(O)}$. □

By the operator theory, there exists a unique unbounded operator L_O, with dense domain in $W_0^{1,2}(O,\gamma)$, such that, for every $f \in D(L_O)$ and $g \in W_0^{1,2}(O,\gamma)$, we get

$$\int_O L_O f \cdot g \, d\gamma = -\int_O \langle \nabla_H f, \nabla_H g\rangle_H \, d\gamma.$$

Definition 2.5. The operator $L_O : D(L_O) \to L^2(O,\gamma)$ is called Ornstein-Uhlenbeck operator on O with homogeneous Dirichlet boundary conditions.

When $O = X$, we denote L_X by L, and it is the infinitesimal generator of the Ornstein-Uhlenbeck semigroup $(T_t)_{t\geq 0}$.

3. Traces in regular sets

In the following, for every $\delta > 0$ we denote by I_δ the real interval $(-\delta,\delta) \subseteq \mathbb{R}$.

Inspired by [7, Hypothesis 3.1], we state our assumptions on O.

Hypotheses 3.1. *Let $G : X \to \mathbb{R}$ and $\delta > 0$ satisfy:*

(i) *G is a continuous function which, for some positive constant c, satisfies $|G(x+h) - G(x)| \leq c\|h\|_H$ for every $x \in X$ and $h \in H$;*
(ii) *$G \in W^{2,p}(X,\gamma)$ for some $p > 1$ and $\mathrm{esssup}_X \|D_H^2 G\|_{\mathscr{L}_2(H)} < +\infty$;*
(iii) *$\|\nabla_H G\|_H^{-1} \in L^\infty(X)$;*
(iv) *$LG \in L^\infty(G^{-1}(I_\delta))$.*

Hereafter, we set $O := G^{-1}((-\infty,0))$ and we assume that O and ∂O are not the empty set.

Remark 3.1. *Let us comment the above assumptions.*

i) *O is an open set and $\partial O = G^{-1}(\{0\})$. Hence, $\gamma(O) > 0$ since every open set has positive measure by an immediate consequence of* [5, Prop. 2.4.10]. *Further, Hypotheses 3.1(i) implies that G belongs to $W^{1,p}(X,\gamma)$ for every $p \in (1,+\infty)$ and $\|\nabla_H G\|_H \le c$ for γ-a.e. in X (see e.g.* [5, Theorem 5.11.2]*).*

ii) *From Hypothesis 3.1(ii) we infer that $G \in W^{2,q}(X,\gamma)$ for all $q > 1$, and so* [7, Hypothesis 3.1] *is fulfilled.*

iii) *By the points (ii) and (iii) of the Hypotheses 3.1 it follows that*

$$\frac{\nabla_H G}{\|\nabla_H G\|_H} \in W^{1,2}(X;H).$$

By adding the point (iv) we have also that $\mathrm{div}_\gamma\left(\frac{\nabla_H G}{\|\nabla_H G\|_H}\right) \in L^\infty(G^{-1}(I_\delta))$*, since*

$$\mathrm{div}_\gamma\left(\frac{\nabla_H G}{\|\nabla_H G\|_H}\right) = \frac{LG}{\|\nabla_H G\|_H} - \frac{\left\langle D^2_H G(\nabla_H G), \nabla_H G\right\rangle_H}{\|\nabla_H G\|^3_H}$$

iv) *The Hypothesis (iii) is very restrictive, for example it is not satisfied by $\|\cdot\|^2_X$ when X is a Hilbert space, which would allow to consider balls.*

v) *For $-\delta < \varepsilon < \delta$ the assumption remains true if we replace G with $G+\varepsilon$ or with $-G+\varepsilon$, with the value δ replaced by $\delta' = \delta - |\varepsilon|$.*

vi) *From Hypotheses 3.1 it follows that LG, and so* $\mathrm{div}_\gamma(\nabla_H G/\|\nabla_H G\|_H)$*, belongs to $L^p(X,\gamma)$ for every $p \in (1,+\infty)$.*

In the sequel, we will need the Sobolev regularity of the modulus of elements of $W^{1,p}(X)$, which is proved in the following lemma.

Lemma 3.1. *Let $u \in W^{1,p}(X,\gamma)$ with $p > 1$. Then, for every $q \in (1,p)$, the function $|u|^q$ belongs to $W^{1,p/q}(X,\gamma)$, and $\nabla_H |u|^q = q\,\mathrm{sgn}(u)|u|^{q-1}\nabla_H u$.*

Proof. The classical method consists in introducing the function $\eta_n(\xi) := \left(\xi^2 + \frac{1}{n}\right)^{q/2}$ and approximating $|u|^q$ by means of the sequence $(\eta_n \circ u_n) \subseteq \mathscr{F}C^\infty_b(X)$ in $W^{1,p/q}(O,\gamma)$, where $(u_n) \subseteq \mathscr{F}C^\infty_b(X)$ is a sequence which converges to u in $W^{1,p}(O,\gamma)$. However, we provide a different proof.

At first, we notice that, for every $q \ge 1$ and every $a,b \in \mathbb{R}$, we have

$$|\mathrm{sgn}(a)\,|a|^q - \mathrm{sgn}(b)\,|b|^q| \le q\,|a-b|\left||a|^{q-1} + |b|^{q-1}\right|. \tag{3.1}$$

Let $(u_n)_{n\in\mathbb{N}} \subseteq \mathscr{F}C^\infty_b(X)$ be a sequence which converges to u in $W^{1,p}(X,\gamma)$. Without loss of generality, we may suppose that $(u_n)_{n\in\mathbb{N}}$ pointwise converges to u. We split the proof into two steps.

Let us notice that, since $u_n \to u$ in $L^p(X,\gamma)$ as n goes to ∞, then $|u_n|^q$ converges to $|u|^q$ in $L^{p/q}(X,\gamma)$ as $n \to +\infty$.

Step 1. We want to show that $\mathrm{sgn}(u_n)|u_n|^{q-1}$ converges to $\mathrm{sgn}(u)|u|^{q-1}$ in $L^{p/(q-1)}(X,\gamma)$. Let us suppose $q \ge 2$, hence $q-1 \ge 1$, and from (3.1) we infer that

$$\begin{aligned}
&\int_X \left|\mathrm{sgn}(u_n)\,|u_n|^{q-1} - \mathrm{sgn}(u)\,|u|^{q-1}\right|^{p/(q-1)} d\gamma \\
\le&(q-1)^{p/(q-1)} \int_X |u_n - u|^{p/(q-1)} \left||u_n|^{q-2} + |u|^{q-2}\right|^{p/(q-1)} d\gamma \\
\le&(q-1)^{p/(q-1)} \left(\int_X |u_n - u|^p\, d\gamma\right)^{\frac{1}{q-1}}
\end{aligned}$$

$$\times\left(\int_X\left||u_n|^{q-2}+|u|^{q-2}\right|^{p/(q-2)}d\gamma\right)^{(q-2)/(q-1)}$$
$$=:(q-1)^{p/(q-1)}A_nB_n.$$

A_n converges to 0 as $n\to+\infty$, and

$$B_n^{(q-1)/p}\le\left\||u_n|^{q-2}+|u|^{q-2}\right\|_{L^{p/(q-2)}(X,\gamma)}\le\left\||u_n|^{q-2}\right\|_{L^{p/(q-2)}(X,\gamma)}+\left\||u|^{q-2}\right\|_{L^{p/(q-2)}(X,\gamma)}$$
$$\le\|u_n\|^{q-2}_{L^p(X,\gamma)}+\|u\|^{q-2}_{L^{p(q-2)}(X,\gamma)},$$

which is uniformly bounded with respect to $n\in\mathbb{N}$.

Instead, if $q\in[1,2)$,

$$\left\|\operatorname{sgn}(u_n)\,|u_n|^{q-1}-\operatorname{sgn}(u)\,|u|^{q-1}\right\|_{L^{p/(q-1)}(X,\gamma)}$$
$$\le\left\|\operatorname{sgn}(u_n)\,|u_n|^{q-1}-\operatorname{sgn}(u_n)\,|u|^{q-1}\right\|_{L^{p/(q-1)}(X,\gamma)}$$
$$+\left\|\operatorname{sgn}(u_n)\,|u|^{q-1}-\operatorname{sgn}(u)\,|u|^{q-1}\right\|_{L^{p/(q-1)}(X,\gamma)}$$
$$\le\left\||u_n|^{q-1}-|u|^{q-1}\right\|_{L^{p/(q-1)}(X,\gamma)}$$
$$+\left(\int_X\left((\operatorname{sgn}(u_n)-\operatorname{sgn}(u))\,|u|^{q-1}\right)^{p/(q-1)}d\gamma\right)^{(q-1)/p}=:P_n+Q_n.$$

Since $t\mapsto t^{q-1}$ is concave, we get

$$P_n\le\left\|||u_n|-|u||^{q-1}\right\|_{L^{p/(q-1)}(X,\gamma)}=\|u_n-u\|^{q-1}_{L^p(X,\gamma)}$$

which converges to 0 as $n\to+\infty$, while, if we set $U:=\{x\in X|u(x)\neq0\}$, then

$$Q_n=\int_U(\operatorname{sgn}(u_n)-\operatorname{sgn}(u))\,|u|^p\,d\gamma,$$

and it converges to 0 by the dominated convergence, since $(u_n)_{n\in\mathbb{N}}$ pointwise converges to u.

Step 2. By approximation, it is possible to show that $\nabla_H|u_n|^q=q\operatorname{sgn}(u_n)|u_n|^{q-1}\nabla_Hu_n$ for every $n\in\mathbb{N}$. In this last step we prove that $\nabla_H|u_n|^q$converges to $q\operatorname{sgn}(u)|u|^{q-1}\nabla_Hu$ in $L^{p/q}(X,\gamma,H)$ as $n\to+\infty$. We have

$$\left\|q\operatorname{sgn}(u_n)|u_n|^{q-1}\nabla_Hu_n-q\operatorname{sgn}(u)|u|^{q-1}\nabla_Hu\right\|_{L^{p/q}(X,\gamma)}$$
$$\le\left\|q\operatorname{sgn}(u_n)|u_n|^{q-1}\nabla_Hu_n-q\operatorname{sgn}(u)|u|^{q-1}\nabla_Hu_n\right\|_{L^{p/q}(X,\gamma,H)}$$
$$+\left\|q\operatorname{sgn}(u)|u|^{q-1}\nabla_Hu_n-q\operatorname{sgn}(u)|u|^{q-1}\nabla_Hu\right\|_{L^{p/q}(X,\gamma,H)}=:R_n+S_n.$$

As far as R_n is concerned, by applying the Hölder inequality with q and $q/(q-1)$ we get

$$R_n\le q\left\|\operatorname{sgn}(u_n)|u_n|^{q-1}-\operatorname{sgn}(u_n)|u_n|^{q-1}\right\|^{(q-1)/q}_{L^{p/(q-1)}(X,\gamma)}\cdot\|\nabla_Hu_n\|^{\frac{1}{q}}_{L^p(X,H,\gamma)}.$$

The first factor converges to 0 as $n\to+\infty$ from Step 1 and the second one is uniformly bounded with respect to $n\in\mathbb{N}$, while

$$S_n\le q\left\||u|^{q-1}\right\|^{(q-1)/q}_{L^{p/(q-1)}(X,\gamma)}\cdot\|\nabla_Hu_n-\nabla_Hu\|^{\frac{1}{q}}_{L^p(X,H,\gamma)}$$
$$=\left\|q^{1/(q-1)}|u|\right\|^{1/q}_{L^p(X,\gamma)}\cdot\|\nabla_Hu_n-\nabla_Hu\|^{\frac{1}{q}}_{L^p(X,H,\gamma)},$$

and the last term converges to 0 as $n \to +\infty$, where again we have used the Hölder inequality with q and $q/(q-1)$.

The proof is concluded.

□

From [7], under Hypotheses 3.1 (i)-(iii), for every $t \in (-\delta, \delta)$ and every $q < p$ it is defined the *trace operator*

$$\mathrm{Tr}_t : W^{1,p}(G^{-1}((-\infty,t)),\gamma) \to L^q(G^{-1}(\{t\}),\rho).$$

If $f \in W^{1,p}(G^{-1}((-\infty,t)),\gamma)$ is the restriction of a continuous function on X, then

$$\mathrm{Tr}_t f = f_{|G^{-1}(\{t\})}.$$

The following three results are proved in [7].

Lemma 3.2. [7, Cor. 3.2] *Assume Hypotheses 3.1 (i)-(iii), let $\delta_0 > 0$ and $O_{\delta_0} := G^{-1}(I_{\delta_0})$. Then, for every $f \in \mathrm{Lip}(X) \subseteq L^1(O_{\delta_0},\gamma)$, the function*

$$q_f(\xi) := \int_{G^{-1}(\{\xi\})} \frac{f}{\|\nabla_H G\|_H}\, d\rho, \quad -\delta_0 < \xi < \delta_0,$$

belongs to $L^1(I_{\delta_0},\mathscr{L}^1)$ (with $\mathscr{L}^1$ being the 1-dimensional Lebesgue measure). Moreover, q_f is a density of the measure $f\gamma \circ G^{-1}$ with respect to $\mathscr{L}^1$, and

$$\|q_f\|_{L^1(-\delta_0,\delta_0)} \le \|f\|_{L^1(O_{\delta_0},\gamma)}.$$

Lemma 3.3. [7, Prop. 4.10] *Under Hypotheses 3.1, for every $p > 1$, every $t \in I_\delta$ and every $f \in W^{1,p}(G^{-1}((-\infty,t)),\gamma)$, $\mathrm{Tr}_t f \equiv 0$ if and only if the trivial extension $\overline{f}$ of f out of $G^{-1}((-\infty,t))$ belongs $W^{1,p}(X,\gamma)$.*

Lemma 3.4. [7, Prop. 4.1] *Under Hypotheses 3.1, for every $p > 1$, every $q \in [1,p)$ and every $t \in I_\delta$, if $f \in W^{1,p}(X,\gamma)$ then*

$$\begin{aligned}
&\int_{G^{-1}(\{t\})} |\mathrm{Tr}_t f|^q d\rho \\
=& q\int_{G^{-1}((-\infty,t))} |f|^{q-2} f \frac{\langle \nabla_H f, \nabla_H G\rangle_H}{\|\nabla_H G\|_H} d\gamma + \int_{G^{-1}((-\infty,t))} \mathrm{div}_\gamma\left(\frac{\nabla_H G}{\|\nabla_H G\|_H}\right)|f|^q\, d\gamma \\
=& q\int_{G^{-1}((t,+\infty))} |f|^{q-2} f \frac{\langle \nabla_H f, \nabla_H G\rangle_H}{\|\nabla_H G\|_H} d\gamma + \int_{G^{-1}((t,+\infty))} \mathrm{div}_\gamma\left(\frac{\nabla_H G}{\|\nabla_H G\|_H}\right)|f|^q\, d\gamma. \quad (3.2)
\end{aligned}$$

Remark 3.2. *An easy consequence of Lemma 3.2 is that $\gamma(G^{-1}(\{t\})) = 0$ for every $t \in I_\delta$. Further, if we take $f = 1$ in (3.2), we infer that*

$$\begin{aligned}
\rho(G^{-1}(\{t\})) &= \int_{G^{-1}(\{t\})} d\rho \\
&\le \left\|\mathrm{div}_\gamma\left(\frac{\nabla_H G}{\|\nabla_H G\|_H}\right)\right\|_{L^1(O,\gamma)} + \left\|\mathrm{div}_\gamma\left(\frac{\nabla_H G}{\|\nabla_H G\|_H}\right)\right\|_{L^\infty(G^{-1}(I_\delta))} < +\infty. \quad (3.3)
\end{aligned}$$

4. Equivalent definitions of $W^{1,p}_0(O,\gamma)$

We set $A_t := G^{-1}((t,\delta))$, and we prove the following two intermediate results.

Lemma 4.1. *Let Hypotheses 3.1 be satisfied. Then, for every $q\in[1,+\infty)$, there exists $C>0$ such that for every $t\in(-\delta,0)$ and $f\in W^{1,q}(X,\gamma)$ we have*

$$\|\operatorname{Tr}_t f\|^q_{L^q(G^{-1}(\{t\}),\rho)} \le C(\|f\|^{q-1}_{L^q(A_t,\gamma)}\|\nabla_H f\|_{L^q(A_t,\gamma,H)} + \|f\|^q_{L^q(A_t,\gamma)}). \tag{4.1}$$

Proof. By density, it is enough to consider Lipschitz continuous functions f.

Arguing as in the proof of [7, Prop. 4.1], we introduce a function $\theta\in C^\infty_b(\mathbb{R})$ such that $\theta=1$ in $(-\infty,0]$, $\theta=0$ in $[\delta,+\infty)$ and $\theta(x)\in[0,1]$ for every $x\in\mathbb{R}$. We define the function $\psi := f\cdot(\theta\circ G)$. The function ψ belongs to $W^{1,s}(X,\gamma)$ for every $s\in[1,+\infty)$ (because f is Lipschitz continuous and $G\in\mathscr{H}^1(X)$) with $\nabla_H\psi=(\theta\circ G)\nabla_H f + f(\theta'\circ G)\nabla_H G$. From its definition, $\psi=0$ on $G^{-1}((\delta,+\infty))$, $|\psi|\le|f|$ on X, and

$$\|\psi\|_{L^s(G^{-1}((t,+\infty)),\gamma)} = \|\psi\|_{L^s(A_t,\gamma)}, \qquad \|\nabla_H\psi\|_{L^s(G^{-1}((t,+\infty));\gamma,H)} = \|\nabla_H\psi\|_{L^s(A_t,\gamma,H)}.$$

Finally, $\psi_{|G^{-1}(\{t\})}\equiv f$. Hence, we can apply Lemma 3.4, which gives

$$\begin{aligned}
\int_{G^{-1}(\{t\})}|\operatorname{Tr}_t f|^q d\rho &= \int_{G^{-1}(\{t\})}|\operatorname{Tr}_t \psi|^q d\rho\\
&=q\int_{A_t}|\psi|^{q-2}\psi\frac{\langle\nabla_H\psi,\nabla_H G\rangle_H}{\|\nabla_H G\|_H}\,d\gamma + \int_{A_t}\operatorname{div}_\gamma\left(\frac{\nabla_H G}{\|\nabla_H G\|_H}\right)|\psi|^q\,d\gamma\\
&\le q\int_{A_t}|\psi|^{q-1}\|\nabla_H\psi\|_H\,d\gamma + \int_{A_t}\operatorname{div}_\gamma\left(\frac{\nabla_H G}{\|\nabla_H G\|_H}\right)|\psi|^q\,d\gamma.
\end{aligned}$$

Recalling that $\operatorname{div}_\gamma(\nabla_H G/\|\nabla_H G\|_H)\in L^\infty(A_t,\gamma)$ (see Remark 3.1(*iii*)), we infer that

$$\begin{aligned}
&\int_{G^{-1}(\{t\})}|\operatorname{Tr}_t f|^q d\rho\\
\le& q\|\psi\|^{q-1}_{L^q(A_t,\gamma)}\|\nabla_H\psi\|_{L^q(A_t,\gamma,H)} + \left\|\operatorname{div}_\gamma\left(\frac{\nabla_H G}{\|\nabla_H G\|_H}\right)\right\|_{L^\infty(A_t,\gamma)}\|\psi\|^q_{L^q(A_t,\gamma)}\\
\le& C_1\|f\|^{q-1}_{L^q(A_t,\gamma)}(\|\nabla_H f\|_{L^q(A_t,\gamma,H)} + \|f\|_{L^q(A_t,\gamma)}) + C_2\|f\|^q_{L^q(A_t,\gamma)}\\
\le&(C_1+C_2)(\|f\|^q_{L^q(A_t,\gamma)} + \|\nabla_H f\|_{L^q(A_t,\gamma;H)}),
\end{aligned}$$

where $C_1 = q(1+\|\theta'\|_\infty\|\nabla_H G\|_{L^\infty(X;H)})$ and $C_2=\left\|\operatorname{div}_\gamma\left(\frac{\nabla_H G}{\|\nabla_H G\|_H}\right)\right\|_{L^\infty(A_t,\gamma)}$. The proof is now complete. □

Lemma 4.2. *Let Hypotheses 3.1 be satisfied. Then, for every $p>1$ there exists $C_0>0$ and $\delta_0\in(0,\delta)$ such that, for every $f\in W^{1,p}(O,\gamma)$ with $\operatorname{Tr}_0 f\equiv 0$ on $G^{-1}(\{0\})$ and every $t\in(-\delta_0,0)$, we have*

$$\|f\|_{L^p(G^{-1}((t,0)),\gamma)} \le 2C_1|t|\|\nabla_H f\|_{L^p(G^{-1}((t,0)),\gamma;H)}.$$

Proof. By density, it is enough to consider Lipschitz continuous functions f.

Let us assume that $\|f\|_{L^p(G^{-1}((t,0),\gamma)}\neq 0$. From Lemma 3.3, the trivial extension $\overline{f}$ of f belongs to $W^{1,p}(X,\gamma)$. From Lemma 3.1 it follows that $|\overline{f}|^q\in L^{p/q}(X,\gamma)$ for every $q\in(1,p)$, and by applying Lemma 4.1 to the function $|\overline{f}|^q$, for every $s\in(-\delta,0)$ we get

$$\begin{aligned}
\|\operatorname{Tr}_s|f|^q\|^{p/q}_{L^{p/q}(G^{-1}(\{s\}),\rho)} =&\|\operatorname{Tr}_s|\overline{f}|^q\|^{p/q}_{L^{p/q}(G^{-1}(\{s\}),\rho)}\\
&\le C(\|f\|^{p-q}_{L^p(G^{-1}((s,0)),\gamma)}\||f|^{q-1}\nabla_H f\|_{L^{p/q}(G^{-1}((s,0)),\gamma;H)}
\end{aligned}$$

$$
\begin{aligned}
&+\|f\|^q_{L^p(G^{-1}((s,0)),\gamma)})\\
\le &C(\|f\|^{p-q}_{L^p(G^{-1}((s,0)),\gamma)}\|f\|^{q-1}_{L^p(G^{-1}((s,0),\gamma)}\|\nabla_H f\|_{L^p(G^{-1}((s,0)),\gamma;H)}\\
&+\|f\|^q_{L^p(G^{-1}((s,0)),\gamma)})\\
\le &C(\|f\|^{p-1}_{L^p(G^{-1}((s,0)),\gamma)}\|\nabla_H f\|_{L^p(G^{-1}((s,0)),\gamma;H)}+\|f\|^p_{L^p(G^{-1}((s,0)),\gamma)}),
\end{aligned}
$$

where $C=\frac{p}{q}(1+\|\theta'\|_\infty\|\nabla_H G\|_{L^\infty(X;H)})+\left\|\mathrm{div}_\gamma\left(\frac{\nabla_H G}{\|\nabla_H G\|_H}\right)\right\|_{L^\infty(A_t,\gamma)}$ and we have used the fact that $\bar f\equiv 0$ in $G^{-1}((0,+\infty))$ (hence the norm of f in $G^{-1}((s,+\infty))$ coincides with that in $G^{-1}((s,0))$). In particular, it follows that

$$
\|\,\mathrm{Tr}_s|f|^q\|^{p/q}_{L^{p/q}(G^{-1}(\{s\}),\rho)}\le \widetilde C(\|f\|^{p-1}_{L^p(G^{-1}((s,0)),\gamma)}\|\nabla_H f\|_{L^p(G^{-1}((s,0)),\gamma;H)}+\|f\|^p_{L^p(G^{-1}((s,0)),\gamma)}), \tag{4.2}
$$

where $\widetilde C:=p(1+\|\theta'\|_\infty\|\nabla_H G\|_{L^\infty(X;H)})+\left\|\mathrm{div}_\gamma\left(\frac{\nabla_H G}{\|\nabla_H G\|_H}\right)\right\|_{L^\infty(I_\delta,\gamma)}$, for every $q\in(1,p)$ and every $s\in(-\delta,0)$.

Let $q\in(1,p)$. By applying the coarea formula in Gaussian setting and Lemma 3.2, for every $t\in(-\delta,0)$ we have

$$
\begin{aligned}
\int_{G^{-1}((t,0))}|f|^q d\gamma=&\int_t^0\int_{G^{-1}(\{\xi\})}\frac{\mathrm{Tr}_\xi|f|^q}{\|\nabla_H G\|_H}d\rho d\xi\\
\le&|t|\left\|\frac{1}{\|\nabla_H G\|_H}\right\|_{L^\infty(G^{-1}((-\delta,0))}\sup_{s\in(t,0)}\|\mathrm{Tr}_s|f|^q\|_{L^1(G^{-1}(\{s\}),\rho)}\\
\le&C'|t|\sup_{s\in(t,0)}\|\mathrm{Tr}_s|f|^q\|^{q/p}_{L^{p/q}(G^{-1}(\{s\}),\rho)},
\end{aligned} \tag{4.3}
$$

where

$$
C':=\left\|\frac{1}{\|\nabla_H G\|_H}\right\|_{L^\infty(G^{-1}((-\delta,0))}(1\wedge\sup_{s\in(t,0)}\rho(G^{-1}(\{s\}))^{(p-1)/p}),
$$

and we have used the fact that $\sup_{s\in(-\delta,0)}\rho(G^{-1}(\{s\}))<+\infty$ (see estimate (3.3)). By replacing estimate (4.2) in (4.3), it follows that

$$
\begin{aligned}
\|f\|^q_{L^q(G^{-1}((t,0),\gamma)}\le &C_1|t|\sup_{s\in(t,0)}(\|f\|^{p-1}_{L^p(G^{-1}((s,0)),\gamma)}\|\nabla_H f\|_{L^p(G^{-1}((s,0)),\gamma;H)}+\|f\|^p_{L^p(G^{-1}((s,0)),\gamma)})\\
\le &C_1|t|(\|f\|^{p-1}_{L^p(G^{-1}((t,0)),\gamma)}\|\nabla_H f\|_{L^p(G^{-1}((t,0)),\gamma;H)}+\|f\|^p_{L^p(G^{-1}((t,0)),\gamma)}),
\end{aligned} \tag{4.4}
$$

with $C_1:=C'\widetilde C$. Letting $q\to p$ in the left-hand side of (4.4) we infer that

$$
\|f\|^p_{L^p(G^{-1}((t,0),\gamma)}\le C_1|t|(\|f\|^{p-1}_{L^p(G^{-1}((t,0)),\gamma)}\|\nabla_H f\|_{L^p(G^{-1}((t,0)),\gamma;H)}+\|f\|^p_{L^p(G^{-1}((t,0)),\gamma)}), \tag{4.5}
$$

We set $\delta_0:=\delta\wedge\frac12 C_1^{-1}$. Then, for every $t\in(-\delta_0,0)$ we get

$$
\|f\|^p_{L^p(G^{-1}((t,0),\gamma)}\le C_1|t|\|f\|^{p-1}_{L^p(G^{-1}((t,0)),\gamma)}\|\nabla_H f\|_{L^p(G^{-1}((t,0)),\gamma;H)}+\frac12\|f\|^p_{L^p(G^{-1}((t,0)),\gamma)},
$$

which gives

$$
\|f\|^p_{L^p(G^{-1}((t,0),\gamma)}\le 2C_1|t|\|f\|^{p-1}_{L^p(G^{-1}((t,0)),\gamma)}\|\nabla_H f\|_{L^p(G^{-1}((t,0)),\gamma;H)}. \tag{4.6}
$$

By dividing both the sides of (4.6) by $\|f\|^{p-1}_{L^p(G^{-1}((t,0),\gamma)}$ (which we assumed different from 0), we infer that

$$\|f\|_{L^p(G^{-1}((t,0),\gamma)} \leq 2C_1|t|\|\nabla_H f\|_{L^p(G^{-1}((t,0)),\gamma;H)},$$

and the thesis is proved.

□

The next result (based on [7, Prop. 4.10]) is the infinite-dimensional version of a well-known theorem (see e.g. [10, Thm. 5.5.2]). We recall that the space $W^{1,p}_0(O,\gamma)$ was defined in Definition 2.2, and characterized as the closure of $\mathscr{H}^1_0(O)$ in $W^{1,p}(O,\gamma)$ in Lemma 2.3.

Theorem 4.1. *Let $O = G^{-1}((-\infty,0))$ with G satisfying Hypotheses 3.1, and let $f \in W^{1,p}(O,\gamma)$ for some $p \in (1,+\infty)$. then, the following are equivalent:*

i) $f \in W^{1,p}_0(O,\gamma)$;
ii) $\mathrm{Tr}_0 f \equiv 0$;
iii) the trivial extension $\overline{f}$ of f belongs to $W^{1,p}(X,\gamma)$.

Proof. The points ii) and iii) are equivalent by Lemma 3.3.

By definition, if $f \in W^{1,p}_0(O,\gamma)$, then it is the limit of a sequence $(f_n) \subseteq \mathrm{Lip}_c(O)$; clearly f_n can be extended as 0 out of O, and the sequence of trivial extensions $(\overline{f}_n)$ converges to the trivial extension $\overline{f} \in W^{1,p}(X,\gamma)$ of f as $n \to +\infty$. This gives $i) \Rightarrow iii)$.

Now We prove $iii) \Rightarrow i)$. Let us set

$$O_m := G^{-1}\left(\left(-\frac{2}{m},\frac{2}{m}\right)\right), \qquad m \in \mathbb{N}.$$

(O_m) is a sequence of open decreasing sets such that

$$\bigcap_{m\in\mathbb{N}} O_m = G^{-1}(0). \tag{4.7}$$

Let $\eta \in C^\infty(\mathbb{R})$ be such that $0 \leq \eta \leq 1$, $\eta = 1$ in $(-\infty,-1]$, $\eta = 0$ in $[-1/2,+\infty)$, and $\eta' \leq 0$ everywhere. For every $m \in \mathbb{N}$, we define χ_m as

$$\chi_m(x) = \begin{cases} \eta(mG(x)+1), & \text{if } x \in O, \\ 0, & \text{if } x \notin O. \end{cases}$$

It follows that $\chi_m \in \mathrm{Lip}_{c,H}(O)$ for every $m \in \mathbb{N}$, that $\chi_m = 1$ on $G^{-1}\left((-\infty,-\frac{2}{m}]\right)$ and $\chi_m = 0$ on $G^{-1}\left((-\frac{1}{m},+\infty)\right)$. Further,

$$\nabla_H \chi_m(x) = (m\eta'(mG(x)+1))\nabla_H G(x), \qquad \gamma\text{-a.e. } x \in O, \tag{4.8}$$

$\nabla_H \chi_{m|X\setminus O_m} = 0$ and

$$\|\nabla_H \chi_m\|_{L^\infty(X;H)} \leq Cm, \tag{4.9}$$

for some positive constant C independent of m. Let $f \in W^{1,p}(O,\gamma)$ be such that $\mathrm{Tr}_0 f = 0$. We have

$$\int_O |f|^p \|\nabla_H \chi_m\|^p_H \, d\gamma \leq C^p m^p \int_{O_m} |f|^p \, d\gamma,$$

and from Lemma 4.2 it follows that

$$\int_O |f|^p \|\nabla_H \chi_m\|^p_H \, d\gamma \leq C_0 \|f\|^p_{W^{1,p}(O_m,\gamma)}, \tag{4.10}$$

for some positive constant C_0 independent of m and φ.

Now we prove that $f \in W_0^{1,p}(O,\gamma)$, by finding a sequence $(f_m) \subseteq \mathrm{Lip}_{c,H}(O)$ which converges to f in $W^{1,p}(O,\gamma)$ as $m \to +\infty$. We know that there exists a sequence $(g_n) \subseteq \mathscr{F}C_b^\infty(X)$ such that $g_n \to \overline{f}$ in $W^{1,p}(X,\gamma)$ as $n \to +\infty$. We fix $n \in \mathbb{N}$. Then,

$$\int_O |g_n\chi_m - f|^p \, d\gamma \le 2^{p-1}\left(\int_O |g_n - f|^p \, d\gamma + \int_O |f|^p|\chi_m - 1|^p d\gamma\right),$$

and the right-hand side converges to 0 as $m \to \infty$, because $\chi_m \to \mathbb{1}_O$ in $L^r(X,\gamma)$ as $m \to +\infty$ for every $r \in (1,+\infty)$ by the dominated convergence theorem, and $g_n \to f$ in $L^p(O,\gamma)$ as $n \to +\infty$. Further, for every $n,m \in \mathbb{N}$ we have

$$\begin{aligned}
&\int_O \|\nabla_H(g_n\chi_m) - \nabla_H f\|_H^p \, d\gamma \\
=&\int_O \|g_n\nabla_H\chi_m + \chi_m\nabla_H g_n - \nabla_H f - f\nabla_H\chi_m + f\nabla_H\chi_m - \chi_m\nabla_H f + \chi_m\nabla_H f\|_H^p \, d\gamma \\
\le&4^{p-1}\bigg(\int_O (g_n - f)^p \|\nabla_H\chi_m\|_H^p \, d\gamma + \int_O \chi_m^p \|\nabla_H g_n - \nabla_H f\|_H^p \, d\gamma \\
&+\int_O (\chi_m - 1)^p \|\nabla_H f\|_H^p \, d\gamma + \int_O f^p \|\nabla_H\chi_m\|_H^p \, d\gamma\bigg) \\
\le&4^{p-1}\bigg(m^p \|g_n - f\|_{L^p(O_m,\gamma)}^p + \|\nabla_H g_n - \nabla_H f\|_{L^p(O,\gamma,H)}^p + \int_{O_m} \|\nabla_H f\|_H^p d\gamma \\
&+\int_O f^p \|\nabla_H\chi_m\|_H^p d\gamma\bigg) \\
\le&4^{p-1}(m^p \|g_n - f\|_{L^p(O_m,\gamma)}^p + \|\nabla_H g_n - \nabla_H f\|_{L^p(O,\gamma,H)}^p + (1+C_0)\|f\|_{W^{1,p}(O_m,\gamma)}^p),
\end{aligned}$$

where in the last inequality we exploit (4.10). By recalling that $g_n \to f$ in $W^{1,p}(X,\gamma)$ for $n \to +\infty$, it follows that, for every fixed $m \in \mathbb{N}$, there exists $n_m \ge m$ such that $\|g_{n_m} - f\|_{W^{1,p}(X,\gamma)}^p \le m^{-p-1}$, which gives

$$\int_O \|\nabla_H(g_{n_m}\chi_m) - \nabla_H f\|_H^p \, d\gamma \le 4^{p-1}(m^{-1} + (1+C_0)\|f\|_{W^{1,p}(O_m,\gamma)}^p). \tag{4.11}$$

We recall that $\gamma(G^{-1}(\{0\})) = 0$ by Remark 3.2. Hence, from (4.7) the last addend in the right-hand side of (4.11) converges to 0 as $m \to \infty$. We now define $f_m = g_{n_m}\chi_m$ for every $m \in \mathbb{N}$. Then, (f_m) converges to f in $W^{1,p}(O,\gamma)$ as $m \to \infty$, and it is not hard to show that $f_m \in \mathrm{Lip}_{c,H}(O)$ for every $m \in \mathbb{N}$. □

5. Examples

5.1. **Region below graphics.** As above, we consider a basis $\{h_i\}_{i\in\mathbb{N}}$ in $R_\gamma(j(X^*))$. In the following, we will denote $\pi_{\hat{h}_1}$ with π_1. We will define a function G such that $O = G^{-1}((-\infty,0))$ is the region below the graph of a smooth function.

Let Φ be a real-valued function on X such that $\partial_{h_1}(\Phi) \equiv 0$. Hence, for every x we have $\Phi(x) = \Phi(x - \pi_1(x))$. We set

$$G(x) = \hat{h}_1(x) - \Phi(x), \qquad x \in X.$$

In this case, O is just the region below the graph of Φ. We assume that Φ is continuous and also satisfies the following conditions:

(1) $\Phi \in \mathrm{Lip}_H(X)$;
(2) $\Phi \in W^{2,p}(X,\gamma)$ for some $p > 1$ and $\|D_H^2\Phi\|_{\mathscr{L}_2(H)}$ is essentially bounded;

(3) $\Phi + L\Phi \in L^\infty(X,\gamma)$.

Under these assumptions, it follows that

$$\begin{aligned}\nabla_H G(x) &= h_1 - \nabla_H \Phi(x-\pi_1(x)),\\ D^2_H G(x) &= -D^2_H \Phi(x-\pi_1(x)),\\ LG(x) &= -\hat{h}_1(x) - L\Phi(x-\pi_1(x)),\end{aligned}$$

γ-a.e. $x \in X$, and G satisfies Hypotheses 3.1. Indeed, we have

$$\begin{aligned}LG(x) =& -\hat{h}_1(x) - L\Phi(x-\pi_1(x)) + \Phi(x-\pi_1(x)) - \Phi(x-\pi_1(x))\\ =&(-\hat{h}(x) + \Phi(x-\pi_1(x))) - (\Phi(x-\pi_1(x)) + L\Phi(x-\pi_1(x)))\\ =&G(x) - (\Phi(x-\pi_1(x)) + L\Phi(x-\pi_1(x))),\end{aligned}$$

γ-a.e. $x \in X$, and both the addends are bounded on $G^{-1}(I_\delta)$. If we take for instance $\Phi = c \in \mathbb{R}$, we get that open half-planes satisfy our assumptions.

5.2. **Brownian motion and pinned Brownian motion.** For the following examples we refer to [8, Section 5].

We recall (see [5, Section 2.3]) that a Brownian motion starting from 0 can be modelled by a Wiener space (X,γ_W) where $X = L^2(0,1)$ (with Lebesgue measure), and γ_W concentrates on the set of the elements of $L^2(0,1)$ which have a continuous representative f such that $f(0) = 0$. The Cameron-Martin space H is the set of the elements of $L^2(0,1)$ which have an absolutely continuous representative f such that $f' \in L^2(0,1)$ and $f(0) = 0$. Finally, for every $f_1, f_2 \in H$ the inner product in H is defined as $\langle f_1, f_2\rangle_H = \int_0^1 f_1'(s)f_2'(s)\,ds$. In the following, for every $h \in H$, we will identify h with its absolutely continuous representative.

We define an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ of $L^2(0,1)$ as

$$e_n(s) = \sqrt{2}\sin\Big(\frac{s}{\sqrt{\lambda_n}}\Big) = \sqrt{2}\sin\Big(\frac{2n+1}{2}\pi s\Big)$$

where

$$\lambda_n = \frac{1}{\pi^2\left(n+\frac{1}{2}\right)^2}.$$

For every $n \in \mathbb{N}$ we set $h_n = \sqrt{\lambda_n}e_n$. It follows that $\{h_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of H.

We consider a function $g \in C^2(\mathbb{R})$, with bounded first and second order derivative, such that there exists $C > 0$ such that

$$|g''(\xi) - g''(\eta)| \le C|\xi-\eta|(|\xi|+|\eta|), \tag{5.1}$$

for every $\xi, \eta \in \mathbb{R}$. Further, we assume that there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that $|g'(\xi)| \ge a$ (hence $g'(\xi) \ne 0$) for every $\xi \in \mathbb{R}$ and

$$\alpha_1 g(\xi) + \beta_1 \le \xi g'(\xi) \le \alpha_2 g(\xi) + \beta_2 \tag{5.2}$$

for every $\xi \in \mathbb{R}$.

The above assumptions are satisfied, for instance, by the function $g = p/q$, where q is a positive polynomial of degree $m \in \mathbb{N}$ and p polynomial of degree $m+1$ such that $p'(\xi) \ne 0$ for every $\xi \in \mathbb{R}$.

Proposition 5.1. *Let us assume that* $g \in C^2(\mathbb{R})$ *satisfies* (5.1) *and* (5.2), *and let r belong to the range of g. We define the function*

$$G(x) := \int_0^1 g(x(s))\, ds - r$$

for every $x \in X = L^2(0,1)$. *Then, G satisfies Hypotheses 3.1.*

Proof. It is not hard to show that G is H-differentiable. For every $h \in H$ and every $x \in X$ we have

$$\langle \nabla_H G(x), h \rangle_H = \int_0^1 g'(x(s))h(s)ds$$

and

$$\|\nabla_H G(x)\|_H \leq \sqrt{\int_0^1 |g'(x(s))|^2 ds} \leq \|g'\|_\infty.$$

Moreover, for every $x, y \in X$,

$$\begin{aligned}\|\nabla_H G(x) - \nabla_H G(y)\|_H^2 \leq & \int_0^1 |g'(x(s)) - g'(y(s))|^2 ds \leq \int_0^1 \|g''\|_\infty^2 |x(s) - y(s)|^2 ds \\ \leq & \|g\|_\infty^2 \|x - y\|_X^2,\end{aligned}$$

from which it follows that $G \in \mathscr{H}^1(X)$. Further, $D_H^2 G$ is everywhere defined and, for every $h, k \in H$ and every $x \in X$ we get

$$\left\langle \left(D_H^2 G(x)\right)(h), k \right\rangle_H = \int_0^1 g''(x(s))h(s)k(s)ds.$$

Hence,

$$\|D_H^2 G(x)\|_{\mathscr{L}_2(H)} = \sum_{n=1}^\infty \|D_H^2 G(x) h_n\|_H^2 \leq \sum_{n=1}^\infty \lambda_n \|g''\|_\infty^2 < +\infty,$$

for every $x \in X$. Let us consider the function $\bar{h} \in H$ defined by $\bar{h}(s) = s$ for every $s \in [0,1]$. It follows that $\bar{h} > 0$ and $\|\bar{h}\|_H = 1$. Further, since g' has constant sign (from $|g'| \geq a$), we have

$$|\left\langle \nabla_H G(x), \bar{h} \right\rangle_H| = \int_0^1 |g'(x(s))| \bar{h}(s)\, ds \geq a \int_0^1 \bar{h}(s)\, ds = \frac{a}{2},$$

for every $x \in X$, which implies that

$$\|\nabla_H G\|_H^{-1} \leq \frac{2}{a}. \tag{5.3}$$

If we consider the sequence $\{h_k = \sqrt{\lambda_k} e_k\}_{k \in \mathbb{N}}$, then the series $\sum_{k=1}^\infty h_k^2$ uniformly converges to a function $f \in C([0,1])$. Moreover,

$$\begin{aligned}LG(x) &= \sum_{i=1}^\infty \langle D_H^2 G(x)(h_i), h_i \rangle_H - \sum_{i=1}^\infty \langle \nabla_H G(x), h_i \rangle_H \widehat{h_i}(x) \\ &= \sum_{i=1}^\infty \int_0^1 g''(x(s)) h_i^2(s) ds - \int_0^1 g'(x(s)) x(s) ds \\ &= \int_0^1 g''(x(s)) f(s) ds - \int_0^1 g'(x(s)) x(s) ds.\end{aligned}$$

The first addend in the last right-hand side of the above chain of equality is bounded because $g'' \in C_b(\mathbb{R})$ and $f \in C([0,1])$. Further, from (5.2) we infer that

$$\int_0^1 g'(x(s))x(s)ds \geq \int_0^1 (\alpha_1 g(x(s)) + \beta_1)\, ds = \alpha_1 G(x) - \alpha_1 r + \beta_1$$

and

$$\int_0^1 g'(x(s))x(s) \leq \int_0^1 (\alpha_2 g(x(s)) + \beta_2)\, ds = \alpha_2 G(x) - \alpha_2 r + \beta_2.$$

Therefore, LG is bounded in $G^{-1}((-\delta,\delta))$ for every $\delta > 0$.

Finally, since r belongs to range of g it follows that that $G^{-1}(\{0\}) \neq \varnothing$, we conclude that G fulfills Hypotheses 3.1. □

An analogous example can be provided for pinned Wiener space, which models Brownian bridge with starting point at 0 and subject to the condition that in 1 the arriving point is 0. $(X, \tilde{\gamma}_W)$ where $X = L^2(0,1)$, the Cameron-Martin space is $H = W_0^{1,2}(0,1)$. We recall that $\{e_n\}_{n\in\mathbb{N}}$ with $e_n = \sqrt{2}\sin(n\pi\cdot)$ for every $n \in \mathbb{N}$ is an orthonormal basis of X, and $\{h_n\}_{n\in\mathbb{N}}$, where $h_n = \sqrt{2}\pi^{-1}n^{-1}\sin(n\pi\cdot)$ for every $n \in \mathbb{N}$ is an orthonormal basis of H.

If we consider a function $g \in C^2(\mathbb{R})$ which enjoys (5.1) and (5.2), arguing as in the proof of Proposition 5.1 we gain the following result.

Proposition 5.2. *Given r in the range of g, we define*

$$G(x) = \int_0^1 g(x(s))\, ds - r$$

for every $x \in X = L^2(0,1)$. Then, the function G satisfies Hypotheses 3.1.

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