VARIATIONAL NONLINEAR AND NONLOCAL CURVATURE FLOWS

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Abstract. We prove that the minimizing movements scheme á la Almgren-Taylor-Wang converges towards level-set solutions to a nonlinear version of nonlocal curvature flows with time-depending forcing term, in the rather general framework of variational curvatures introduced in [16]. The nonlinearity involved is assumed to satisfy minimal assumptions, namely continuity, monotonicity, and vanishing at zero. Under additional assumptions only on the curvatures involved, we establish uniqueness for level-set solutions.

1. Introduction

This paper establishes existence via minimizing movements and uniqueness results for a nonlinear modification of variational and nonlocal curvature flows in presence of mobility and time-dependent forcing. This nonlinear and nonlocal generalization of the classical mean curvature flow (MCF in short) is defined as follows: given a continuous, non-decreasing function $\mathtt{G}:\mathbb{R}\to\mathbb{R}$ with $\mathtt{G}(0)=0$, we consider the evolution of a family of sets $t\mapsto E_t$ formally governed by the evolution law

$$V(x,t)=\psi(\nu_{E_t}(x))\mathtt{G}\Big(-\kappa(x,E_t)+\mathtt{f}(t)\Big),\qquad \text{for all } x\in\partial E_t,\ t\geq 0, \tag{1.1}$$

where ψ is an anisotropy (usually called the mobility), ν_{E_t} denotes the outer normal vector to E_t and $\mathtt{f}$ is a forcing term constant in space. In (1.1) the curvature $\kappa(\cdot,E)$ denotes a variational curvature, belonging to a class of generalized nonlocal curvatures introduced in [16].

Generalised curvatures are functions defined on pairs (x,E), where E is a set of class C^2 with compact boundary and $x\in\partial E$, that are non-decreasing with respect to inclusion of sets touching at x, continuous w.r.t. C^2-convergence of sets, and translation invariant (see conditions (A)-(C) below). We will focus on a particular instance of generalized curvatures, namely variational curvature. These curvatures arise as the first variation (in a suitable sense) of perimeter-like functionals, which are called generalized perimeters. A generalized perimeter $J:\mathscr{M}\to[0,+\infty]$ is a translation invariant functional on the class of measurable sets $\mathscr{M}$, which is insensitive to modifications on negligible sets, finite on C^2-sets with compact boundary, lower semicontinuous w.r.t. the L^1_{loc}-convergence, and satisfies a submodularity condition: $J(E\cap F)+J(E\cup F)\leq J(E)+J(F)$ for every $E,F\in\mathscr{M}$.

The evolution law (1.1) is relevant even in the specific instance where κ is the classical mean curvature, arising as first variation of the perimeter. From a numerical point of view, as suggested e.g. in [17, Remark 3.5], a truncation of the classical evolution speed $V=-\kappa$ is usually encoded in algorithms for the MCF, which corresponds to choosing $\mathtt{G}(s)=(-M)\vee s\wedge M$ in (1.1), for $M>0$ large. Another interesting choice could be $\mathtt{G}(s)=-s^-$ (so that $\mathtt{G}(-\kappa)=-\kappa^+$), which amounts to consider a purely shrinking evolution. Moreover, evolution by powers of the mean curvature have been previously studied in the smooth or convex setting [3, 18, 27, 29] and have been used to prove isoperimetric inequalities [30], or considered in the setting of image processing algorithms [2, 28]. In particular, in [2, Section 4.5] it is remarked that the evolution law (1.1) with $\mathtt{G}(s)=s^{\frac{1}{3}}$ and $\psi=|\cdot|$, $\mathtt{f}=0$ is particularly interesting as it is invariant under affine transformations (isometries and rescalings). See also [20] for interesting links between motion by powers of the mean curvature and a time-fractional Allen-Cahn equation, and [5], where flat flows solutions to the power (anisotropic) mean curvature flow are studied.

On the other hand, being able to address this study in the framework of generalized curvatures and general nonlinearity $\mathtt{G}$, allows us to prove new results for different geometric flows. This notion of generalized curvature has been introduced in [16] to deal with a wide class of local and nonlocal translation-invariant geometric flows in a unified framework. Some previous contributions can be found in [6, 7, 8, 24, 31]. As detailed in [16, Section 5], some instances of geometric flows driven by variational curvatures are the following: classical *anisotropic MCF* (driven by a suitably smooth and translation invariant anisotropy), *fractional MCF*, *capacity flows*, and flows driven by the curvature associated to the *regularized pre-Minkowski content*. See also [4] for some extensions.

Given the definition of variational curvature, and the formal gradient flow structure of the MCF, one is naturally led to consider the minimizing movements approach, in the spirit of [1, 25], as a way to prove existence for (1.1). This scheme provides a discrete-in-time approximation of the evolution law (1.1) by iteratively solving a variational problem, where the energy to minimize consists of the sum of J and a suitable dissipation term that penalizes the L^2-distance between sets. In our setting, we will modify the iterative scheme of [16] (reminiscent of [1, 25]), tailored for the present general setting, by taking into account the nonlinearity in the dissipation term.

We provide here existence via minimizing movements and uniqueness of viscosity solutions to nonlinear and possibly nonlocal curvature flows in the presence of continuous time-dependent forcing and mobility, in the form (1.1). Our first main result concerns the instance of (1.1) where κ is a variational curvature. In this case, we show in Theorem 3.5 that the minimizing movements scheme produces discrete-in-time functions that converge, as the time-step parameter tends to zero, towards a viscosity solution to (1.1). Subsequently, we establish uniqueness for the parabolic Cauchy problem associated with the level set formulation of (1.1). This result, presented in Theorem 4.9, does not require the curvature κ to be variational, though it must satisfy specific additional conditions. Remarkably, no further assumptions on $\mathtt{G}$ are needed. In particular, κ is required to be either of first-order type or to satisfy a strengthened uniform regularity condition in the second-order case (see conditions (FO) and (C') in Section 4 for details). All the relevant examples of generalized curvatures presented above satisfy these assumptions.

The proofs are inspired by the techniques developed in [16], coupled with recent insights we developed in [11] (see also [9]). In [16], the authors prove existence and uniqueness of viscosity solutions to curvature flows of the form $V = -\kappa$, with κ being a generalized curvature (the uniqueness result requires additional assumptions on κ, the same we will require in the last section). In the specific case of variational curvatures, existence can also be proved by using the minimizing movements scheme, similar to the one sketched above. The starting observation is that, under our assumptions on $\mathtt{G}$, if κ is a generalized curvature, then $-\mathtt{G}(-\kappa)$ is still a generalized curvature. Therefore, the same viscosity theory of [16] applies to evolution laws of the form

$$V = \mathtt{G}(-\kappa), \tag{1.2}$$

providing existence of viscosity solutions, convergence of the minimizing movements scheme and uniqueness under further assumptions on $\mathtt{G}$ and κ. Anyhow, when dealing with (1.1) two problems arise. Firstly, it is no longer true in general that if κ is a *variational* curvature, then so is $-\mathtt{G}(-\kappa)$. In particular, convergence of the minimizing movements scheme does not follow immediately from [16]. It is thus interesting to modify the minimizing movements scheme to account for the nonlinear term, even in the simplified version of (1.1) given by (1.2). In this regard, nontrivial difficulties arise in the case where $\mathtt{G}$ is bounded from above or below, as some tools heavily employed in the linear setting are no longer available (see e.g. the commonly used reformulation (2.15)). This issue will be circumvented by an approximation procedure. One of the main goals of this paper was indeed considering $\mathtt{G}$ with minimal regularity assumptions.

Secondly, the introduction of a time-dependent forcing term and a mobility requires some care. Indeed, the level set formulation for (1.1) with time-dependent forcing and mobility does not fall in the framework of [16]. In particular, the proof of the comparison principle needs some careful work. It is inspired by [16] with some insights coming from the classical theory of viscosity solutions (see for instance [21]).

This work is an extension and an improvement of the unpublished (and unfinished) preprint [10], where the authors show the convergence of the minimizing movements scheme towards (1.2), where κ the *classical* mean curvature and $\mathtt{G}$ is a smooth function with polynomial growth.

To conclude, it would be interesting to study the much more challenging case where the subjacent perimeter is of crystalline type. In this setting the availability of the viscosity solutions of [22, 23] and the development of distribution solutions of [13, 14, 15] may suggest the possibility of a future investigation in this direction. Another interesting instance is the non translation invariant case, and a first step could be considering the same setting of [11]. A simplified model would consist in considering a forcing term $\mathtt{f}$ that also depends on the spatial variable. In this case, we expect the convergence result to still hold, by suitably adapting the arguments from the present work and from [11]. However, the uniqueness result appears to be more delicate as the proof presented here does not carry effortlessly to this instance (see (4.19)).

The paper is structured as follows. In Section 2 we introduce some notation and the minimizing movements scheme. Then, in Sections 3 we show the convergence of the minimizing movements scheme towards viscosity solutions to (1.1). Uniqueness of viscosity solutions to (1.1), under additional assumptions on κ is the subject of Section 4.

2. The minimizing movements scheme

2.1. **Preliminaries.** We start introducing some notations. We will use both $B_r(x)$ and $B(x, r)$ to denote the Euclidean ball in $\mathbb{R}^N$ centered in x and of radius r. If the ball is centered in zero, we simply write B_r. We let $\mathscr{M}$ denote the family of the measurable sets in $\mathbb{R}^N$, and $E \in C^2$ to say that the set E is of class C^2. In the following, we will always speak about measurable sets and refer to a set as the union of all the points of density 1 of that set i.e. $E = E^{(1)}$. Moreover, if not otherwise stated, we implicitly assume that the function spaces considered are defined on $\mathbb{R}^N$, e.g $L^\infty = L^\infty(\mathbb{R}^N)$. Moreover, we often drop the measure with respect to which we are integrating, if clear from the context.

Definition 2.1. We define anisotropy a function $\psi : \mathbb{R}^N \to [0, +\infty)$ which is continuous, convex, even and positively 1-homogeneous. Moreover, there exists $c_\psi > 0$ such that $\forall p \in \mathbb{R}^N$ it holds

$$\frac{1}{c_\psi}|p| \le \psi(p) \le c_\psi |p|. \tag{2.1}$$

We recall that the polar function ψ° of an anisotropy ψ is defined by

$$\psi^\circ(v) := \sup_{\psi(\xi)\le 1} \xi \cdot v.$$

The following identities hold for smooth anisotropies: $\forall v, \xi \in \mathbb{R}^N$

$$\psi(v)\psi^\circ(\xi) \ge v \cdot \xi, \qquad \psi^\circ(\nabla\psi(v)) = v, \qquad \nabla\psi(v) \cdot v = \psi(v).$$

Definition 2.2. Given an anisotropy ψ and a set E, we define the ψ-distance from E as

$$\mathrm{dist}^{\psi}_{E}(x) = \inf_{y\in E} \psi^{\circ}(x-y),$$

and the signed ψ-distance from E as

$$\mathrm{sd}^{\psi}_{E}(x) = \mathrm{dist}^{\psi}_{E}(x) - \mathrm{dist}^{\psi}_{E^c}(x).$$

For $\delta \in \mathbb{R}$ and $E \in \mathscr{M}$, we denote

$$E_{\delta} = \{x \in \mathbb{R}^N \ : \ \mathrm{sd}^{\psi}_{E}(x) \le \delta\},$$

and use the notation $E_{-\infty} := \emptyset, E_{+\infty} := \mathbb{R}^N$.

Note that (2.1) implies that

$$\frac{1}{c_{\psi}}\mathrm{dist}_E(x) \le \mathrm{dist}^{\psi}_{E}(x) \le c_{\psi}\mathrm{dist}_E(x), \tag{2.2}$$

where dist_E denotes the Euclidean distance from the set E.

In this section we extend the previous study to nonlocal instances, in the spirit of [16]. We recall some notation. For any given $E \in C^2$, we consider[1] a function $x \mapsto \kappa(x, E)$, defined for $x \in \partial E$, and that we will call (generalized) curvature of E at x. This function must satisfy the following axioms:

(A) Monotonicity: If $E, F \in C^2$ and $x \in \partial E \cap \partial F$ with $E \subseteq F$, then $\kappa(x, E) \ge \kappa(x, F)$;
(B) translation invariance: For every $E \in C^2$, $x \in \partial E$ and $y \in \mathbb{R}^N$, it holds $\kappa(x, E) = \kappa(x + y, E + y)$;
(C) Continuity: If $E_n \to E$ in C^2 and $x_n \in \partial E_n \to x \in \partial E$, then $\kappa(x_n, E_n) \to \kappa(x, E)$.

Defining for $x \in \mathbb{R}^N$ and $\rho > 0$

$$\begin{aligned} \overline{c}(\rho) &= \max_{x\in\partial B_\rho} \max\left\{\kappa(x, B_\rho), -\kappa(x, B^c_\rho)\right\}, \\ \underline{c}(\rho) &= \min_{x\in\partial B_\rho} \min\left\{\kappa(x, B_\rho), -\kappa(x, B^c_\rho)\right\}, \end{aligned} \tag{2.3}$$

we note that by (C) these functions are continuous in ρ. We further require

(D) Curvature of balls: There exists $K > 0$ such that $\underline{c}(\rho) \ge -K > -\infty$.

In the following we will focus on the study of the geometric evolution equation

$$V(x, t) = \psi(\nu_{E_t})(x)\mathtt{G}(-\kappa(x, E_t) + \mathtt{f}(t)), \quad \text{for } x \in \partial E_t \text{ and } t > 0, \tag{2.4}$$

starting from an initial bounded set E_0 (or an unbounded set with bounded complement), where ψ is an anisotropy, $\kappa(\cdot, E_t)$ is a variational curvature in the sense above, and $\mathtt{f}$ is a bounded forcing term. Here and in the following, we fix $T > 0$ and consider the evolution for $t \in (0, T)$. The functions $\mathtt{G}, \mathtt{f}$ are required to satisfy the following conditions:

- $\mathtt{G} : \mathbb{R} \to \mathbb{R}$ is a continuous, non-decreasing function, with $\mathtt{G}(0) = 0$;
- $\mathtt{f} \in C^0_b(\mathbb{R})$;

[1] One can slightly generalize this definition by considering sets in $C^{k,\beta}$ with $k \ge 2$, $\beta \in [0, 1]$, but for simplicity we consider the C^2 case only.

We then set

$$\lim_{s\to-\infty} \mathtt{G}(s) = -a \in [-\infty, 0], \qquad \lim_{s\to+\infty} \mathtt{G}(s) = b \in [0, +\infty].$$

Consider a function $u : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$ whose superlevel sets $E_s := \{u(\cdot, t) \geq s\}$ evolve according to the nonlinear mean curvature equation (2.4). By classical computations (see for instance [21]), the function u satisfies

$$\begin{cases} \partial_t u(x,t) - \psi(\nabla u(x,t))\mathtt{G}(-\kappa(x, \{u(\cdot,t) \geq u(x,t)\}) + \mathtt{f}(t)) = 0 \\ u(\cdot, 0) = u_0. \end{cases} \tag{2.5}$$

Let us recall the notion of viscosity solutions employed in [16]. One first introduces a family of auxiliary functions.

Definition 2.3. Given a curvature κ defined as above, we consider a family $\mathcal{L}$ of functions $\ell \in C^\infty([0, +\infty))$, such that $\ell(0) = \ell'(0) = \ell''(0) = 0, \ell(\rho) > 0$ for all ρ in a neighborhood of 0, ℓ is constant in $[M, +\infty)$ for some $M > 0$ (depending on ℓ), and

$$\lim_{\rho\to 0^+} \ell'(\rho)\, \mathtt{G}(\overline{c}(\rho)) = 0,$$

where $\overline{c}$ is as in (2.3).

We refer to [21, Lemma 3.1.3] for a proof that the the family $\mathcal{L}$ is not empty. The notion of admissible test function is the following. With a slight abuse of notation, in the following we will say that a function is spatially constant outside a compact set even if the value of such constant is time-dependent.

Definition 2.4. Let $\hat{z} = (\hat{x}, \hat{t}) \in \mathbb{R}^N \times (0, T)$ and let $A \subseteq (0, T)$ be any open interval containing $\hat{t}$. We say that $\eta \in C^0(\mathbb{R}^N \times \overline{A})$ is admissible at the point $\hat{z}$ if it is of class C^2 in a neighborhood of $\hat{z}$, if it is constant out of a compact set, and, in case $\nabla\eta(\hat{z}) = 0$, the following holds: there exists $\ell \in \mathcal{L}$ and $\omega \in C^\infty([0, +\infty))$ with $\omega'(0) = 0, \omega(\rho) > 0$ for $\rho > 0$ such that

$$|\eta(x,t) - \eta(\hat{z}) - \eta_t(\hat{z})(t - \overline{t})| \leq \ell(|x - \hat{x}|) + \omega(|t - \hat{t}|)$$

for all (x, t) in $\mathbb{R}^N \times A$.

Then, the notion of viscosity solutions employed in [16] is the following.

Definition 2.5. An upper semicontinuous function $u : \mathbb{R}^N \times [0, T] \to \mathbb{R}$, constant outside a compact set, is a viscosity subsolution of the Cauchy problem (2.5) if $u(\cdot, 0) \leq u_0$ and, for all $z := (x, t) \in \mathbb{R}^N \times [0, T]$ and all C^∞-test functions η such that η is admissible at z and $u - \eta$ has a maximum at z, the following holds:

i) If $\nabla\eta(z) = 0$, then

$$\eta_t(z) \leq 0; \tag{2.6}$$

ii) If $\nabla\eta(z) \neq 0$, then

$$\partial_t\eta(z) + \psi(\nabla\eta(x,t))\mathtt{G}(-\kappa(x, \{\eta(\cdot,t) \geq \eta(z)\}) + \mathtt{f}(t)) \leq 0. \tag{2.7}$$

A lower semicontinuous function $u : \mathbb{R}^N \times [0, T] \to \mathbb{R}$, constant outside a compact set, is a viscosity supersolution of the Cauchy problem (2.5) if $u(\cdot, 0) \geq u_0$ and, for all $z := (x, t) \in \mathbb{R}^N \times [0, T]$ and all C^∞-test functions η such that η is admissible at z and $u - \eta$ has a minimum at z, the following holds:

i) If $\nabla\eta(z) = 0$, then $\eta_t(z) \geq 0$,

ii) If $\nabla\eta(z) \neq 0$, then $\partial_t\eta(z) + \psi(\nabla\eta(x,t))\mathtt{G}(-\kappa(x,\{\eta(\cdot,t) \geq \eta(x,t)\}) + \mathtt{f}(t)) \geq 0$.

Finally, a function u is a viscosity solution for the Cauchy problem (2.5) if it is both a subsolution and a supersolution of (2.5).

Remark. By classical arguments, one could assume that the maximum of $u - \eta$ is strict in the definition of subsolution above (an analogous remark holds for supersolutions).

In the rest of the section we will consider a particular instance of generalized curvatures, namely the variational curvatures introduced in [16]. We start by recalling the notion of generalized perimeters.

Definition 2.6. We will say that a functional $J : \mathscr{M} \to [0, +\infty]$ is a generalized perimeter if it satisfies the following properties: for every E, E' measurable sets and $x \in \mathbb{R}^N$

(i) $J(E) < +\infty$ for every bounded C^2-set E;
(ii) $J(\emptyset) = J(\mathbb{R}^N) = 0$;
(iii) $J(E) = J(E')$ if $|E \triangle E'| = 0$;
(iv) J is lower semicontinuous in L^1_{loc};
(v) J is submodular, that is

$$J(E \cap E') + J(E \cup E') \leq J(E) + J(E'); \tag{2.8}$$

(vi) J is translation invariant: for every $E \in C^2$ and $x \in \mathbb{R}^N$ it holds $J(x + E) = J(E)$.

A generalized perimeter J can be extended to a functional on $L^1_{loc}(\mathbb{R}^N)$ enforcing a generalized co-area formula:

$$J(u) = \int_{-\infty}^{+\infty} J(\{u \geq s\})\,\mathrm{d}s \quad \text{for every } u \in L^1_{loc}(\mathbb{R}^N). \tag{2.9}$$

It turns out that the functional above is a convex *lsc* functional on $L^1_{loc}(\mathbb{R}^N)$ see [12].

Definition 2.7. Given a bounded C^2-set E and $x \in \partial E$, we define

$$\kappa^+(x, E) = \inf\left\{\liminf_{\varepsilon\to 0} \frac{J(E \cup W_\varepsilon) - J(E)}{|W_\varepsilon \setminus E|} : \overline{W_\varepsilon} \xrightarrow{\mathcal{H}} \{x\}, |W_\varepsilon \setminus E| > 0\right\}, \tag{2.10}$$

and

$$\kappa^-(x, E) = \inf\left\{\liminf_{\varepsilon\to 0} \frac{J(E) - J(E \setminus W_\varepsilon)}{|W_\varepsilon \cap E|} : \overline{W_\varepsilon} \xrightarrow{\mathcal{H}} \{x\}, |W_\varepsilon \cap E| > 0\right\},$$

where $\xrightarrow{\mathcal{H}}$ denotes Hausdorff convergence. We say that $\kappa(x, E)$ is the curvature of E at x if $\kappa^+(x, E) = \kappa^-(x, E) =: \kappa(x, E)$.

In the rest of the section we will assume that κ exists for all sets of class C^2, and furthermore that it satisfies assumption (C) and (D). Assumptions (A) and (B) follow from the assumptions on J, furthermore one can prove that the weak notion of curvature of Definition 2.7 coincides with the more standard one based on the first variation of the functional J, whenever the latter exists (see [16, Section 4] for details).

2.2. **The minimizing movements scheme.** We set g as a selection of the set-valued inverse of $\mathtt{G}$, that is $g(x) \in \mathtt{G}^{-1}(x)$ for every $x \in (-a, b)$ and extend it setting $g = -\infty$ for every $x \leq -a$, $g = +\infty$ for every $x \geq b$. Here, we extended $\mathtt{G}$ to $[-\infty, +\infty]$ setting $\mathtt{G}(\pm\infty) = \lim_{x \to \pm\infty} \mathtt{G}(x)$. We assume also that $g(0) = 0$. Note that these definitions imply $\mathtt{G} \circ g = id$ in $[-a, b]$. Moreover, g is strictly increasing. In the following we will denote for $k \in \mathbb{N}, h > 0$

$$f(kh) = \fint_{kh}^{(k+1)h} \mathtt{f}(s)\,\mathrm{d}s.$$

Given a bounded set $E \in \mathscr{M}$ and $h > 0, t \in (0, +\infty)$ we define a functional on the measurable sets as

$$\mathscr{F}^E_{h,t}(F) = J(F) + \int_{E \triangle F} \left| g\left(\frac{\mathrm{sd}^\psi_E}{h}\right) \right| - f([t/h]h)|F|, \tag{2.11}$$

where $[\cdot]$ denotes the integer part. Before proving existence for the functional 2.11 we recall the following existence result for a related problem, see [16, Proposition 6.1].

Lemma 2.8. *Assume that η is a measurable function satisfying $(-\eta) \vee 0 = \eta^- \in L^1(\mathbb{R}^N)$. Then, the problem*

$$\min \left\{ J(F) + \int_F \eta(x)\,\mathrm{d}x \right\} \tag{2.12}$$

admits a minimal and a maximal solution (with respect to inclusion). Moreover, if $\eta_1 \leq \eta_2$ then the minimal (resp. maximal) solution to (2.12) with η_1 replacing η contains the minimal (resp. maximal) solution to (2.12) with η_2 replacing η.

We then prove existence of minimizers to $\mathscr{F}^E_{h,t}$. The proof of the boundedness of minimizers has been taken from [26].

Lemma 2.9. *Let $E \in \mathscr{M}$ be a bounded set and $h > 0, t \in [0, +\infty)$. Then, there exist minimizers of $\mathscr{F}^E_{h,t}$ and, denoting E' one such minimizer, it has the following properties: it is a bounded set such that (up to negligible sets)*

$$E_{-ah} \subseteq E' \subseteq E_{bh}.$$

Moreover, there exist a maximal and a minimal minimizer (with respect to inclusion) of $\mathscr{F}^E_{h,t}$.

Proof. We fix $h > 0$ and $t \in (0, T)$, and $c = f([t/h]h)$. Let $n \in \mathbb{N}$ and denote $g_n := g(\frac{\mathrm{sd}^\psi_E}{h}) \vee -n$ and $\tilde{g} := g(\frac{\mathrm{sd}^\psi_E}{h})$. We note that $g_n^- \in L^1_{loc}$, thus Lemma 2.8 implies that the functional

$$J(F) + \int_F (g_n - c)$$

admits a minimal minimizer E_n. Since $\int_E g_n$ is finite, one can check that E_n minimizes also

$$J(F) + \int_{E \triangle F} |g_n| - c|F|. \tag{2.13}$$

Note that $E_n \subseteq E_{n+1}$ by Lemma 2.8, therefore $E_n \to E' = \bigcup_{n \in \mathbb{N}} E_n$ in L^1_{loc}. Since $|\tilde{g}|$ is coercive, there exists $R > 0$ such that $|\tilde{g}| \geq 2\|f\|_{L^\infty(\mathbb{R})}+1$ in B_R^c and $E \subseteq B_R$. Testing (2.13) with $\emptyset$, we deduce

$$0 \geq J(E_n) + \int_{E_n} (g_n - c) \geq (\|f\|_{L^\infty(\mathbb{R})} + 1)|E_n \setminus B_R|,$$

that implies $E_n \subseteq B_R$ for every $n \in \mathbb{N}$. By semicontinuity and Fatou's lemma we get

$$\mathscr{F}^E_{h,t}(E') \le \lim_{n\to\infty} J(E_n) + \int_{E_n \triangle E} |g_n| - c|E_n|.$$

Since $|g_n| \le |\tilde{g}|$, we conclude that E' is a minimizer of $\mathscr{F}^E_{h,t}$. By classical arguments, one can check that if E_1', E_2' are minimizers of $\mathscr{F}^E_{h,t}$, then so are $E_1' \cap E_2', E_1' \cup E_2'$, implying the existence of a minimal and a maximal solution (see e.g. [16, Proposition 6.1]).

Let now $\tilde{E}$ denote a minimizer of $\mathscr{F}^E_{h,t}$. Since $\tilde{E}$ has finite energy, it is straightforward to check that $|\tilde{E}| < +\infty$ and $\mathrm{sd}^\psi_E \in [-ah, bh]$ a.e. on $\tilde{E} \triangle E$. If $b < +\infty$ this clearly implies that $\tilde{E}$ is bounded; if $b = +\infty$ we use a different argument. We first prove some preliminary results. □

The first one is a comparison principle, in the spirit of [25].

Lemma 2.10 (Weak comparison principle)**.** *Fix $h > 0, t \in (0, +\infty)$ and assume that F_1, F_2 are bounded sets with $F_1 \subset\subset F_2$. Then, for any two minimizers E_i of $\mathscr{F}^{F_i}_{h,t}$ for $i = 1, 2$, we have $E_1 \subseteq E_2$. If, instead, $F_1 \subseteq F_2$, then we have that the minimal (respectively, maximal) minimizer of $\mathscr{F}^{F_1}_{h,t}$ is contained in the minimal (respectively, maximal) minimizer of $\mathscr{F}^{F_2}_{h,t}$.*

Proof. Firstly, we assume $F_1 \subset\subset F_2$, Testing the minimality of E_1, E_2 with their intersection and union, respectively, we obtain

$$J(E_1) + \int_{(E_1\setminus E_2)\setminus F_1} g\left(\frac{\mathrm{sd}^\psi_{F_1}}{h}\right) + \int_{(E_1\setminus E_2)\cap F_1} g\left(\frac{\mathrm{sd}^\psi_{F_1}}{h}\right) \le J(E_1 \cap E_2) + f([t/h]h)|E_1 \setminus E_2|$$

$$J(E_2) \le J(E_1 \cup E_2) + \int_{(E_1\setminus E_2)\setminus F_2} g\left(\frac{\mathrm{sd}^\psi_{F_2}}{h}\right) + \int_{(E_1\setminus E_2)\cap F_2} g\left(\frac{\mathrm{sd}^\psi_{F_2}}{h}\right) - f([t/h]h)|E_1 \setminus E_2|.$$

Summing the two inequalities above and using the submodularity of J we get

$$\begin{aligned}(2.14)\qquad \int_{(E_1\setminus E_2)\setminus F_1} g\left(\frac{\mathrm{sd}^\psi_{F_1}}{h}\right) &+ \int_{(E_1\setminus E_2)\cap F_1} g\left(\frac{\mathrm{sd}^\psi_{F_1}}{h}\right) \\ &\le \int_{(E_1\setminus E_2)\cap F_2} g\left(\frac{\mathrm{sd}^\psi_{F_2}}{h}\right) + \int_{(E_1\setminus E_2)\setminus F_2} g\left(\frac{\mathrm{sd}^\psi_{F_2}}{h}\right).\end{aligned}$$

Assume by contradiction that $|E_1 \setminus E_2| > 0$. Since $\mathrm{sd}^\psi_{F_2} < \mathrm{sd}^\psi_{F_1}$ and by the strict monotonicity of g, we estimate the *rhs* of (2.14) by

$$\int_{(E_1\setminus E_2)\setminus F_2} g\left(\frac{\mathrm{sd}^\psi_{F_2}}{h}\right) + \int_{(E_1\setminus E_2)\cap F_2} g\left(\frac{\mathrm{sd}^\psi_{F_2}}{h}\right) < \int_{(E_1\setminus E_2)\setminus F_2} g\left(\frac{\mathrm{sd}^\psi_{F_1}}{h}\right) + \int_{(E_1\setminus E_2)\cap F_1} g\left(\frac{\mathrm{sd}^\psi_{F_1}}{h}\right)$$

and plug it in (2.14) to reach the desired contradiction. The other cases follow analogously, reasoning by approximation if $F_1 \subseteq F_2$. □

Lemma 2.11. *Let $c \in \mathbb{R}$. Consider a bounded set $E \in \mathscr{M}$ and non-decreasing functions $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ such that $g_1 < g_2$ in $\mathbb{R} \setminus \{0\}$ and $g_1(0) = g_2(0) = 0$. Then, if E_i solves*

$$\min_F \left\{ J(F) + \int_{E\triangle F} \left| g_i(\mathrm{sd}^\psi_E(x)) \right| \,\mathrm{d}x + c|F| \right\}$$

for $i = 1, 2$, we have that $E_2 \subseteq E_1$. If $g_1 \le g_2$ instead, an analogous statement holds for the maximal and minimal solutions.

Proof. Denote $\tilde g_i = g_i \circ \mathrm{sd}_E^\psi$ for $i=1,2$ and assume by contradiction that $|E_2\setminus E_1|>0$. Reasoning as in Lemma 2.10, one gets

$$\int_{E_1\triangle E}|\tilde g_1| + \int_{E_2\triangle E}|\tilde g_2| \le \int_{(E_1\cup E_2)\triangle E}|\tilde g_1| + \int_{(E_1\cap E_2)\triangle E}|\tilde g_2|.$$

Simplifying[2] the above expression and recalling that $\tilde g_i\ge 0$ on E^c, $\tilde g_i \le 0$ on E, we reach

$$0\le \int_{(E_2\setminus E_1)\setminus E}(\tilde g_1-\tilde g_2) + \int_{(E_2\setminus E_1)\cap E}(\tilde g_1-\tilde g_2) = \int_{E_2\setminus E_1}(\tilde g_1-\tilde g_2),$$

which implies the contradiction. The case $g_1\le g_2$ follows by approximation. $\square$

We can then conclude the proof of the boundedness of minimizers to $\mathscr{F}_{h,t}^E$.

End of proof of Lemma 2.9. We prove that any minimizer $\tilde E$ of $\mathscr{F}_{h,t}^E$ is bounded. Recall that $|\tilde E|<+\infty$. We assume by contradiction the existence of points $\{x_n\}_{n\in\mathbb{N}}\subseteq\mathbb{R}^N$ of density one for $\tilde E$, with $|x_n|\to+\infty$ as $n\to+\infty$. For fixed $M>0$, since $|g|$ is coercive there exists $R>0$ such that $|\tilde g|\ge M$ in B_R^c. We can assume that $E\subseteq B_R$, and, up to extracting an unrelabelled subsequence, that $|x_n-x_m|>2R$ for $n\ne m$ and $|x_n|>3R$ for all $n\in\mathbb{N}$. We note that

$$M\chi_{B_{2R}^c} < |\tilde g|(\cdot+\tau) \qquad \text{for all } |\tau|\le R.$$

Let us denote by E_M a minimizer of

$$J(F)+\int_{F\triangle E} M\chi_{B_{2R}^c} = J(F) + M|F\setminus B_{2R}|.$$

By translation invariance $\tilde E+\tau$ minimizes (2.11) with $|\tilde g|(\cdot+\tau)$ substituting $|\tilde g|$, thus by comparison

$$\tilde E+\tau\subseteq E_M \qquad \text{for all } |\tau|\le R.$$

In particular, the disjoint balls $B_R(x_n)$ are all contained (up to negligible sets) in E_M. This implies

$$J(E_M)+M|E_M\setminus B_{2R}| \ge M\Big|\bigcup_{n\in\mathbb{N}} B_R(x_n)\Big| = +\infty,$$

a contradiction. $\square$

If $\int_E g(\mathrm{sd}_E^\psi/h)<+\infty$, minimizers of $\mathscr{F}_{h,t}^E$ minimize also the functional

$$F\mapsto J(F)+\int_F g\left(\tfrac{1}{h}\mathrm{sd}_E^\psi\right) - f([t/h]h)|F|, \tag{2.15}$$

as can be see adding the (constant term) $\int_E g(\mathrm{sd}_E^\psi/h)$ to the functional $\mathscr{F}_{h,t}^E$. In the present setting, since $\int_E g(\mathrm{sd}_E^\psi/h)$ may be infinite in the case $a<+\infty$, we can not draw this conclusion straightforwardly. We can nonetheless recover the minimal and the maximal solution to (2.16) by means of a sequence of minimizers of a functional similar to (2.15), essentially as in the proof of Lemma 2.9.

[2]Noting that

$$\begin{aligned}
E_1\triangle E &= ((E_1\setminus E_2)\setminus E)\cup((E_1\cap E_2)\setminus E)\cup((E\setminus E_1)\setminus E_2)\cup((E\cap E_2)\setminus E_1)\\
(E_1\cup E_2)\triangle E &= (E_2\setminus E_1\setminus E)\cup((E_1\cap E_2)\setminus E)\cup((E_1\setminus E_2)\setminus E)\cup((E\setminus E_1)\setminus E_2)\\
(E_1\cap E_2)\triangle E &= ((E_2\cap E_1)\setminus E)\cup((E\setminus E_1)\setminus E_2)\cup((E\cap E_1)\setminus E_2)\cup((E\cap E_2)\setminus E_1).
\end{aligned}$$

For a given bounded set $E \in \mathscr{M}$ and $t \in (0,+\infty)$, we denote

$$T^-_{h,t}E = \min \operatorname{argmin} \mathscr{F}^E_{h,t}, \qquad T^+_{h,t}E = \max \operatorname{argmin} \mathscr{F}^E_{h,t}, \tag{2.16}$$

where the minimum and maximum above are made with respect to inclusion. We will often denote $T_{h,t} := T^-_{h,t}$. From the previous results, we deduce this corollary.

Corollary 2.12. *Assume $a < +\infty$. Let $E \in \mathscr{M}$ be a bounded set and $t \in (0,+\infty), h > 0$. Then, there exists a sequence of uniformly bounded sets $(E_n)_{n\in\mathbb{N}}$ such that $E_n \nearrow T^-_{h,t}E$ and for any $n \in \mathbb{N}$, E_n is a minimizer of*

$$F \mapsto J(F) + \int_F g\left(\frac{\mathrm{sd}^\psi_E}{h}\right) \vee (-n) - f([t/h]h)|F| =: \mathscr{F}^{E,n}_{h,t}(F). \tag{2.17}$$

Analogously, there exists a sequence of uniformly bounded sets $(E_n)_{n\in\mathbb{N}}$ such that $E_n \searrow T^+_{h,t}E$ in L^1 and for any $n \in \mathbb{N}$, E_n is a solution to

$$\min\left\{J(F) + \int_{B_R\setminus F} g\left(\frac{\mathrm{sd}^\psi_E}{h}\right) \wedge n - f([t/h]h)|F| : F \subseteq B_R\right\}, \tag{2.18}$$

where $T^\pm_{h,t}E \subseteq B_R$.

Proof. We prove the statement for $T^-_{h,t}E$, the other case being analogous. We set $c = f([t/h]h)$, $g_n := g(\mathrm{sd}^\psi_E/h) \vee (-n)$, and $E' = T^-_{h,t}E$. Arguing as in the proof of Lemma 2.9, one builds a sequence of sets $(E_n)_{n\in\mathbb{N}}$, each being the minimal minimizer of $\mathscr{F}^{E,n}_{h,t}$, $E_n \subseteq B_R$ for all $n \in \mathbb{N}$ and $E_n \nearrow \bigcup_{n\in\mathbb{N}} E_n =: \tilde{E}$. Note that $E' \supseteq E_n$ as $g \le g_n$, therefore $\tilde{E} \subseteq E'$ and also $\chi_{E_n \triangle E'} = |\chi_{E_n} - \chi_{E'}| \to \chi_{\tilde{E}\triangle E'}$ a.e. as $n \to \infty$. By lower semicontinuity of J and Fatou's lemma we get

$$\begin{aligned}\mathscr{F}^E_{h,t}(\tilde{E}) = J(\tilde{E}) - c|\tilde{E}| + \int_{\tilde{E}\triangle E'} |g(\mathrm{sd}^\psi_E/h)| &= J(\tilde{E}) - c|\tilde{E}| + \int_{\mathbb{R}^N} \liminf_{n\to\infty} (|g_n|\chi_{E_n\triangle E}) \\ &\le \liminf_{n\to\infty}\left(J(E_n) - c|E_n| + \int_{E_n\triangle E} |g_n|\right).\end{aligned}$$

Since E_n minimizes $\mathscr{F}^{E,n}_{h,t}$ we get

$$\mathscr{F}^E_{h,t}(\tilde{E}) \le \liminf_n \left(J(E') + \int_{E'\triangle E} |g_n| - c|E'|\right) \le \mathscr{F}^E_{h,t}(E'), \tag{2.19}$$

where in the last inequality we used that $|g_n| \le |g|$. Since E' is the minimal minimizer of $\mathscr{F}^E_{h,t}$ we conclude $\tilde{E} = E'$. The functional (2.17) is obtained from (2.11) adding $\int_E g_n(\mathrm{sd}^\psi_E/h)$. Finally, the functional in (2.18) is obtained from functional (2.11) adding the (finite) term $-\int_{B_R\setminus E} g(\mathrm{sd}^\psi_E/h)\wedge n$ and restricting the family of competitors. $\square$

We now provide an estimate on the evolution speed of balls. It is interesting to note that, in the isotropic setting ($\psi = \phi = |\cdot|$) and under the assumption of strict monotonicity of $\mathtt{G}$, an explicit evolution law for the radii of evolving balls can be obtained. In our more general case we need to employ the variational proofs of [16, 11]. By Lemma 2.9, the relevant case is $b = +\infty$.

Lemma 2.13. *Assume $b = +\infty$. There exists a positive constant C such that, for every $R > 0$ and every $t \in (0,+\infty), h > 0$ it holds*

$$T^\pm_{h,t}B_R \subseteq B_{R+Ch}.$$

Proof. It is sufficient to prove the claim for $T^+_{h,t}B_R$. We fix $h > 0$ and set $E = T^+_{h,t}B_R$ and $\tilde g = g(\mathrm{sd}^\psi_{B_R}/h)$. We define
$$\bar\rho = \inf\{\rho \in (0,+\infty) : |E \setminus B_\rho| = 0\},$$
and note that $\bar\rho < +\infty$ since E is bounded. We can assume *wlog* $\bar\rho > R$. Let $\bar x \in \partial B_{\bar\rho}$ such that $|E \cap B(\bar x, \varepsilon)| > 0$ for any $\varepsilon > 0$, and let $\rho > \bar\rho$. Let $\tau = (\frac{\rho}{\bar\rho} - 1)\bar x$ and note that $B(-\tau, \rho) \supseteq B_{\bar\rho}$ and $\partial B(-\tau,\rho)$ is tangent to $\partial B_{\bar\rho}$ at $\bar x$.

We let for $\varepsilon > 0$ small $B^\varepsilon = B(-(1+\varepsilon)\tau, \rho)$ and $W^\varepsilon = E \setminus B^\varepsilon$. We note that by construction $|W^\varepsilon| > 0$ and it converges to $\bar x$ in the Hausdorff sense as $\varepsilon \to 0$.

Testing the minimality of E against $E \cap B^\varepsilon$, we find
$$J(E) - J(B^\varepsilon \cap E) \le f([t/h]h)|W_\varepsilon| + \int_{B^\varepsilon \cap E \triangle B_R} |\tilde g| - \int_{E \triangle B_R} |\tilde g|. \tag{2.20}$$
We remark that, by the choice of $\bar\rho$ and τ, taking ε small it holds $B_R \subseteq B^\varepsilon \cap E$. Therefore, (2.20) reads
$$J(E) - J(B^\varepsilon \cap E) \le f([t/h]h)|W_\varepsilon| + \int_{B^\varepsilon \cap E \setminus B_R} |\tilde g| - \int_{E \setminus B_R} |\tilde g|,$$
implying
$$J(E) - J(B^\varepsilon \cap E) \le f([t/h]h)|W_\varepsilon| - \int_{(E\setminus B^\varepsilon)\setminus B_R} |\tilde g| = f([t/h]h)|W_\varepsilon| - \int_{W^\varepsilon} |\tilde g|. \tag{2.21}$$
By submodularity (2.8), using the definition of $\underline{c}$ and assumption (D) we conclude
$$-K + o_\varepsilon(1) \le \|f\|_\infty - ⨍_{W^\varepsilon} |\tilde g| \le \|f\|_\infty - ⨍_{W^\varepsilon} g(c_\psi(|x| - R)/h).$$
Passing to the limit $\varepsilon \to 0$ we get
$$K + \|f\|_\infty \ge \liminf_{s \to c_\psi(\bar\rho - R)h} g(s),$$
from which the thesis follows applying $\mathtt{G}$ on both sides. □

Note that the previous result implies, in particular, that the discrete evolution starting from an initial bounded set remains bounded in every bounded time interval $(0, T)$.

We then provide an upper bound on the evolution speed of balls in the spirit of [16, 11]. We remark that the relevant case is $a = +\infty$ as otherwise Lemma 2.9 yields
$$T^\pm_{h,t}B_R \supseteq B_{R-ah}.$$

Lemma 2.14. *Let $R_0 > 0$ and $\sigma > 1$ be fixed. Assume $a = +\infty$. Then, there exist a positive constant c such that, if $h > 0$ is small enough, for all $R \ge R_0$ and $t \in (0,+\infty)$ it holds*
$$T^\pm_{h,t}B_R \supseteq B_{R + \frac{h}{c_\psi}\mathtt{G}(-\bar c(R/\sigma) - \|f\|_\infty)}. \tag{2.22}$$

Proof. We prove the result for $E := T^-_{h,t}B_R$. Take h small enough so that $T_{h,t}B_{\frac14 R_0} \ne \emptyset$. By translation invariance and taking h small, one can see that[3] $B_{\frac{R}{4}} \subseteq E$. We set
$$\bar\rho = \sup\{\rho \in [0,+\infty) : |B_\rho \setminus E| = 0\} \in [\tfrac{R}{4}, +\infty), \tag{2.23}$$

[3] Indeed, by translation invariance it holds
$$T_{h,t}B_{\frac{R}{4}} + B_{\frac34 R} \subseteq T_{h,t}B_R,$$
and for h small (depending on R) the set $T^\pm_{h,t}B_{R/4}$ is not empty.

and note that $\bar{\rho} < +\infty$ by the boundedness of E. Assume *wlog* $\bar{\rho} < R$. Let $\bar{x} \in \partial B_{\bar{\rho}}$ be such that $|B(\bar{x}, \varepsilon) \setminus E| > 0$ for any $\varepsilon > 0$. Set $\rho \in (0, \bar{\rho})$ and $\tau = (1 - \rho/\bar{\rho})\bar{x}$ such that $\partial B(\tau, \rho) \cap \partial B_{\bar{\rho}} = \{\bar{x}\}$. Setting $B^\varepsilon := ((1+\varepsilon)\tau, \rho)$, consider the sets

$$W^\varepsilon := B^\varepsilon \setminus E.$$

Notice that by construction, for ε small, W^ε has positive measure and it converges to $\{x\}$ as $\varepsilon \to 0$ in the Hausdorff sense. Since E minimizes (2.15) (as $a = +\infty$), we use its minimality to get

$$J(T^{\pm}_{h,t} B_R) - J(B^\varepsilon \cup T^{\pm}_{h,t} B_R) \leq f([t/h]h)|W_\varepsilon| + \int_{W^\varepsilon} g\left(\frac{\mathrm{sd}^{\psi}_{B_R}}{h}\right).$$

Dividing by $|W_\varepsilon| > 0$ the equation above reads

$$\frac{J(T^{\pm}_{h,t} B_R) - J(B^\varepsilon \cup T^{\pm}_{h,t} B_R)}{|W_\varepsilon|} \leq f([t/h]h) + \fint_{W^\varepsilon} g\left(\frac{\mathrm{sd}^{\psi}_{B_R}}{h}\right). \tag{2.24}$$

By submodularity and the definition of variational curvature (2.10) we see that

$$J(T^{\pm}_{h,t} B_R) - J(B^\varepsilon \cup T^{\pm}_{h,t} B_R) \geq J(B^\varepsilon \setminus W_\varepsilon) - J(B^\varepsilon) \geq |W_\varepsilon| \left(-\kappa(\bar{x}, B^\varepsilon) + o_\varepsilon(1)\right),$$

where $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$. We plug the estimate above in (2.24) and send $\varepsilon \to 0$ to conclude

$$-\bar{c}(\bar{\rho}) - \|f\|_\infty \leq \limsup_{s \to c_\psi(\bar{\rho} - R)/h} g(s).$$

Applying $\mathtt{G}$ to both sides of (2.24), we conclude

$$\bar{\rho} \geq R + \frac{h}{c_\psi} \mathtt{G}\left(-\bar{c}(\bar{\rho}) - \|f\|_\infty\right) \geq R + \frac{h}{c_\psi} \mathtt{G}\left(-\bar{c}(R/4) - \|f\|_\infty\right), \tag{2.25}$$

where in the last inequality we recalled that $\bar{\rho} \geq R/4$. Using again the previous analysis with the bound (2.25), we show (2.22) by taking h small enough. □

2.3. **The scheme for unbounded sets.** We now define the discrete evolution scheme for unbounded sets having compact boundary. Let us introduce the generalized perimeter

$$\tilde{J}(E) := J(E^c) \quad \text{for all } E \in \mathscr{M}.$$

Is is easily checked that $\tilde{J}$ satisfies all the assumptions of Definition 2.6, and, denoting $\tilde{\kappa}$ the corresponding curvature, that

$$\tilde{\kappa}(x, E) = -\kappa(x, E^c).$$

Therefore, one has the bounds

$$\bar{c}(\rho) = \max_{x \in \partial B_\rho} \max\left\{\tilde{\kappa}(x, B_\rho), -\tilde{\kappa}(x, B^c_\rho)\right\},$$
$$\underline{c}(\rho) = \min_{x \in \partial B_\rho} \min\left\{\tilde{\kappa}(x, B_\rho), -\tilde{\kappa}(x, B^c_\rho)\right\},$$

where the functions $\bar{c}, \underline{c}$ are defined in (2.3). For every compact set K and $h, t > 0$, we let $\tilde{T}^{\pm}_{h,t} K$ denote the maximal and the minimal minimizer of $\tilde{\mathscr{F}}^K_{h,t}$, which corresponds to (2.11) with $\tilde{g}(s) := -g(-s)$ instead of $g(s)$ and $-f$ instead of f. By changing variable $\tilde{F} := F^c$ in (2.11), we see that $(\tilde{T}^{-}_{h,t} K)^c$ is the maximal solution to

$$\min\left\{J(\tilde{F}) + \int_{\tilde{F} \triangle K^c} \left|g\left(\mathrm{sd}^{\psi}_{K^c}/h\right)\right| + f([t/h]h)|\tilde{F}^c|\right\}. \tag{2.26}$$

Therefore, for every unbounded set E with compact boundary we define[4]

$$T^{\pm}_{h,t}E := \left(\tilde{T}^{\mp}_{h,t}E^c\right)^c. \tag{2.27}$$

As in the case of compact sets, we set $T_{h,t}E := T^{-}_{h,t}E$. Since $\tilde{g}$ has the same properties of g, one easily checks that analogous results to Lemmas 2.13, 2.10 and 2.14 hold also for (2.27).

Lemma 2.15. *Let $t, h > 0$. The following statements hold.*

- *Let $F_1 \subseteq F_2$ be unbounded sets with compact boundary. Then, $T^{\pm}_{h,t}F_1 \subseteq T^{\pm}_{h,t}F_2$.*
- *There exists $C > 0$ such that for every $R > 0, h > 0$ it holds $T^{\pm}_{h,t}B^c_R \supseteq B^c_{R+Ch}$.*
- *Let $R_0 > 0$ and $\sigma > 1$ be fixed. Then, if $a = +\infty$ there exist $c > 0$ such that for $h > 0$ small enough and for all $R \geq R_0$, it holds*
$$T^{\pm}_{h,t}B^c_R \subseteq B^c_{R+\frac{h}{c_\psi}\mathtt{G}(-\sigma\frac{c}{R}-\|f\|_\infty)}.$$
If instead $a < +\infty$ it holds
$$T^{\pm}_{h,t}B^c_R \subseteq B^c_{R-ah}.$$

Furthermore, Corollary 2.12 implies straightforwardly the following approximation result.

Corollary 2.16. *Set $t, h > 0$ and let $E \in \mathscr{M}$ be an unbounded set with bounded complement. Then, there exists two sequences of sets $(E_n)_{n\in\mathbb{N}}, (E'_n)_{n\in\mathbb{N}}$ with uniformly bounded complement with the following property. Each $(E_n)^c$ is a minimizer of (2.26) with $g \vee (-n)$ substituting g, and $(E'_n)^c$ is a minimizer of (2.26) with $g \wedge n$ substituting g. Moreover $E_n \nearrow T^{-}_{h,t}E$ and $E'_n \searrow T^{+}_{h,t}E$.*

We now deduce an equivalent version of (2.26), which will be used in the final proof. Let us consider E such that $E^c \subseteq B_R$ and assume $a = +\infty$. Recall that $T^{\pm}_{h,t}E \supseteq B^c_{R+Ch}$ for some $C > 0$ by Lemma 2.15. Adding to the functional in (2.26) the term $\int_{B_{R+Ch}\setminus(T_{h,t}E)^c} g(\mathrm{sd}^{\psi}_E/h)$ and restricting the family of competitors, we note that $T^{-}_{h,t}E$ is the minimal solution to

$$\min\left\{J(\tilde{F}) + \int_{\tilde{F}\cap B_{R+ch}} g\left(\mathrm{sd}^{\psi}_E/h\right) + f([t/h]h)|\tilde{F}^c| : \tilde{F}^c \subseteq B_{R+ch}\right\}. \tag{2.28}$$

The case $a < +\infty$ needs to be treated by approximation using Corollary 2.16. Lastly, we state a comparison principle between bounded and unbounded sets. Its proof follows the one of [16, Lemma 6.10], up to employing Corollary 2.16.

Lemma 2.17. *Let E_1 be a compact set and let E_2 be an open, unbounded set with compact boundary, and such that $E_1 \subseteq E_2$. Then, for every $h, t > 0$ it holds $T^{\pm}_{h,t}E_1 \subseteq T^{\pm}_{h,t}E_2$.*

3. Main result

We start by introducing the discrete approximation scheme. Given a continuous function $u_0 : \mathbb{R}^N \to \mathbb{R}$ which is constant outside a compact set, we define the transformation

$$T_{h,t}u(x) = \sup\left\{s \in \mathbb{R} \ : \ x \in T_{h,t}\{u_0 \geq s\}\right\}, \tag{3.1}$$

[4]To justify this, one can check that if a set E is moving according to (1.1), its complement moves according to
$$V(x,t) = -\psi(\nu_{E^c}(x))\mathtt{G}(\kappa(x,E^c) + \mathtt{f}) \quad \text{in the direction } \nu_{E^c},$$
from which the incremental problem follows.

where the operators $T^{\pm}_{h,t}$ have been introduced in (2.16), and we recall that $T_{h,t} := T^{-}_{h,t}$. We then set $u_h(x,t) = u_0(x)$ for $t \in [0,h)$ and define

$$u_h(x,t) := \left(T_{h,t-h} u_h(\cdot, t-h)\right)(x). \tag{3.2}$$

By lemmas 2.10 and 2.15, one can see that the operator $T_{h,t}$ maps functions into functions. The following properties of the operator $T_{h,t}$ hold.

Lemma 3.1. *Given $t, h > 0$, the operator $T_{h,t}$ defined in (3.1) satisfies the following properties:*

- *$T_{h,t}$ is monotone, meaning that $u_0 \le v_0$ implies $T_{h,t}u_0 \le T_{h,t}v_0$;*
- *$T_{h,t}$ is translation invariant, as for any $z \in \mathbb{R}^N$, setting $\tau_z u_0(x) := u_0(x-z)$, it holds $T_{h,t}(\tau_z u_0) = \tau_z(T_{h,t}u_0)$;*
- *$T_{h,t}$ commutes with constants, meaning $T_{h,t}(u+c) = (T_{h,t}u) + c$ for every $c \in \mathbb{R}$.*

Proof. The first assertion follows from Lemma 2.10 and 2.15. The second one follows easily employing the definition (3.1), recalling the fact that the functional defined in (2.11) is invariant under translations and that $\{\tau_z u_0 \ge \lambda\} = \{u_0 \ge \lambda\} + z$ for all $\lambda \in \mathbb{R}$. The last result follows analogously. $\square$

The previous properties satisfied by the operator, in turn, preserve the continuity in space of the initial function. Indeed, assume u_0 is uniformly continuous and let $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing, continuous modulus of continuity for u_0. Then, for any $s > s'$ we have

$$\{u > s\} + B_{\omega^{-1}(s-s')} \subseteq \{u > s'\},$$

thus, by translation invariance we deduce

$$T_{h,t}\{u > s\} + B_{\omega^{-1}(s-s')} \subseteq T_{h,t}\{u > s'\}.$$

This inclusion implies that the function $T_{h,t}u_0$ is uniformly continuous in space, with the same modulus of continuity ω of u_0.

The following lemma provides an estimate on the continuity in time of u_h.

Lemma 3.2. *Fix $t, h > 0$ and u_0 a uniformly continuous function. For all $\lambda \in \mathbb{R}$ it holds*

$$T_{h,t}\{u_h(\cdot,t) > \lambda\} = \{u_h(\cdot, t+h) > \lambda\}, \quad T^{+}_{h,t}\{u_h(\cdot,t) \ge \lambda\} = \{u_h(\cdot,t+h) \ge \lambda\}.$$

Proof. Given $\varepsilon > 0$, by definition it is easy to see that

$$\{T_{h,0}u_0 > \lambda + \varepsilon\} \subseteq T^{\pm}_{h,0}\{u_0 > \lambda\} \subseteq \{T_{h,0}u_0 > \lambda - \varepsilon\}.$$

Passing to the limit $\varepsilon \to 0$, we deduce

$$\{u_h(\cdot,h) > \lambda\} \subseteq T^{\pm}_{h,0}\{u_0 > \lambda\} \subseteq \{u_h(\cdot,h) \ge \lambda\}.$$

Finally, since $u_h(\cdot,h)$ is a continuous function, the equalities $\{u_h(\cdot,h) > \lambda\} = \mathrm{int}\{u_h(\cdot,h) \ge \lambda\}$ and $\{u_h(\cdot,h) \ge \lambda\} = \overline{\{u_h(\cdot,h) \ge \lambda\}}$ hold and we prove the result for $t = h$. The other cases follow by iteration. $\square$

With the previous results and reasoning exactly as in [16, Lemma 6.13], we can prove that the functions u_h are uniformly continuous in time.

Lemma 3.3. *For any $\varepsilon > 0$, there exists $\tau > 0$ and $h_0 = h_0(\varepsilon) > 0$ such that for all $|t-t'| \le \tau$ and $h \le h_0$ we have $|u_h(\cdot,t) - u_h(\cdot,t')| \le \varepsilon$.*

Thus, the family $\{u_h\}_{h>0}$ is equicontinuous and uniformly bounded as implied by Lemma 2.13. By the Ascoli-Arzelà theorem we can pass to the limit $h \to 0$ (up to subsequences) to conclude that $u_h \to u$ uniformly in any compact in time subset of $\mathbb{R}^N \times [0, +\infty)$, with u being a uniformly continuous function. Moreover, the function u is bounded and constant outside a compact set (as implied by Lemma 2.13).

Proposition 3.4. *Let $T > 0$. Up to a subsequence, the family $\{u_h\}_{h>0}$ defined in (3.2) converges uniformly on $\mathbb{R}^N \times [0, T]$ to a uniformly continuous function u, which is bounded and constant out of a compact set.*

We can thus state our main result.

Theorem 3.5. *The function u defined in Proposition 3.4 is a continuous viscosity solution to the Cauchy problem (2.5).*

We finally recall the notion of a level-set solution to the evolution equation (1.1) (see e.g. [21]).

Definition 3.6. Given an initial bounded set E_0 (or unbounded set with bounded complement) define an uniformly continuous function $u_0 : \mathbb{R}^N \to \mathbb{R}$ such that $\{u_0 > 0\} = E_0$. Then, setting u as the solution to (2.5) with initial datum u_0 given by Theorem 3.5, we define the level-set solution to the nonlinear mean curvature evolution (1.1) of E_0 as

$$E_t := \{u(\cdot, t) > 0\}.$$

3.1. **Proof of the main result.** We start by an estimate on the evolution speed. For every $r > 0$, using the notation of Lemma 2.14, we set

$$\hat{\kappa}(r) = \min\left\{-1, \tfrac{1}{c_\psi}\mathtt{G}\left(-\bar{c}(r) - \|f\|_\infty\right)\right\}$$

and, given $r_0 > 0$, we set $r(t)$ as the unique solution to

$$\begin{cases} \dot{r}(t) = \hat{\kappa}(r(t)) \\ r(0) = r_0. \end{cases} \tag{3.3}$$

Note that, in general, the solution $r(t)$ will exist in a finite time interval $[0, T^*(r_0)]$, where $T^*(r_0)$ denotes the extinction time of the solution starting from r_0 i.e. the first time t such that $r(t) = 0$.

Lemma 3.7. *Let u be the function given by Proposition 3.4 and assume that there exists $\lambda \in \mathbb{R}$ such that $B(x_0, r_0) \subseteq \{u(\cdot, t_0) > \lambda\}$. Then, if $a = +\infty$, it holds*

$$B(x_0, r(t - t_0)) \subseteq \{u(\cdot, t) > \lambda\}$$

for every $t \leq T^*(r_0) + t_0$, where $r(t)$ is the solution to (3.3) with extinction time $T^*(r_0)$. If instead $a < +\infty$ it holds

$$B(x_0, r_0 - a(t - t_0)) \subseteq \{u(\cdot, t) > \lambda\}$$

for all t such that $r_0 - a(t - t_0) \geq 0$. The same result holds for sublevels substituting superlevel sets.

Proof. The result in the case $a < +\infty$ follows directly by Lemma 2.9, so we assume $a = +\infty$. We consider *wlog* $\{u(\cdot, t_0) > \lambda\}$ bounded, as the other case is analogous. For a fixed $R_0 < r_0$, taking $h(R_0)$ small enough, we can ensure that $B(x_0, R_0) \subseteq \{u_h(\cdot, t_0) > \lambda\}$. We then fix $\sigma > 1$ and define recursively the radii R_n by

$$R_{n+1} = R_n + \tfrac{h}{c_\psi}\mathtt{G}\left(-\bar{c}(R_n/\sigma) - \|f\|_\infty\right).$$

By Lemmas 2.10, 2.14 and 3.2, we see that $B(x_0, R_{[(t-t_0)/h]+1}) \subseteq \{u(\cdot,t) > \lambda\}$ for every $t \geq t_0$ such that $R_{[(t-t_0)/h]+1} > 0$. Let then r_σ be the unique solution to the ODE

$$\begin{cases} \dot{r}_\sigma(t) = \hat{\kappa}(r_\sigma(t)/\sigma) \\ r_\sigma(0) = R_0. \end{cases} \tag{3.4}$$

Employing the monotonicity of $\hat{\kappa}$, if $r_\sigma(t) \leq R_n$, then

$$\begin{aligned} r_\sigma((n+1)h) &\leq R_n + \int_{nh}^{(n-1)h} \hat{\kappa}\left(\frac{r_\sigma(s)}{\sigma}\right) \mathrm{d}s \leq R_n + \int_{nh}^{(n-1)h} \hat{\kappa}\left(\frac{R_n}{\sigma}\right) \mathrm{d}s \\ &\leq R_n + \int_{nh}^{(n-1)h} \tfrac{1}{c_\psi}\mathtt{G}\left(-\bar{c}(R_n/\sigma) - \|f\|_\infty\right) \mathrm{d}s = R_{n+1}. \end{aligned}$$

Therefore, $B(x_0, r_\sigma(h[(t-t_0)/h] + h) \subseteq \{u_h(\cdot,t) > \lambda\}$ for $t \geq t_0$ as long as the radius is positive. We conclude sending $h \to 0$, then $R_0 \to r_0$ and $\sigma \to 1$. □

We are now in the position to prove our main result.

Proof of Theorem 3.5. Consider u as defined in (3.4): we show that u is a subsolution, as proving that it is a supersolution is analogous. Let $\eta(x,t)$ be an admissible test function in $\bar{z} := (\bar{x}, \bar{t})$ and assume that $(\bar{x}, \bar{t})$ is a strict maximum point for $u - \eta$. Assume furthermore that $u - \eta = 0$ in such point.

Case 1: We assume that $\nabla\eta(\bar{z}) \neq 0$. Firstly, in the case $a < +\infty$ we remark that if $\partial_t\eta/\psi(\nabla\eta(\hat{z})) \leq -a$, then (2.7) is trivially satisfied, thus we can assume *wlog* that

$$\frac{\partial_t\eta(\bar{z})}{\psi(\nabla\eta(\hat{z}))} > -a. \tag{3.5}$$

By classical arguments (recalled in [11]) we can assume that each function $u_{h_k} - \eta$ assumes a local supremum in $B_\rho(\bar{z})$ at a point $z_{h_k} =: (x_k, t_k)$ and that $u_{h_k}(z_{h_k}) \to u(\bar{z})$ as $k \to \infty$. Moreover, we can assume that $\nabla\eta(z_k) \neq 0$ for k large enough.

Step 1: We define a suitable competitor for the minimality of the level sets of u_h. By the previous remarks we have that

$$u_h(x,t) \leq \eta(x,t) + c_k \tag{3.6}$$

where $c_k := u_{h_k}(x_k,t_k) - \eta(x_k,t_k)$, with equality if $(x,t) = (x_k,t_k)$. Let $\sigma > 0$ and set

$$\eta^\sigma_{h_k}(x) := \eta(x,t_k) + c_k + \frac{\sigma}{2}|x - x_k|^2.$$

Then, for all $x \in \mathbb{R}^N$,

$$u_{h_k}(x,t_k) \leq \eta^\sigma_{h_k}(x)$$

with equality if and only if $x = x_k$. We set $l_k = u_{h_k}(x_k,t_k) = \eta^\sigma_{h_k}(x_k)$. We fix $\varepsilon > 0$, to be chosen later, and define $E^k_\varepsilon := \{u_{h_k}(\cdot,t_k) > l_k - \varepsilon\} = T_{h_k,t_k-h_k}\{u_{h_k}(\cdot,t_k - h_k) > l_k - \varepsilon\}$[5] and

$$W^k_\varepsilon := E^k_\varepsilon \setminus \left\{\eta^\sigma_{h_k}(\cdot) > l_k + \varepsilon\right\}. \tag{3.7}$$

[5]The choice of working with the open superlevel sets is motivated by our need to employ (2.17)

Assume that E^k_ε is bounded and let us define $E^k_{\varepsilon,n}$ as the sets constructed by Corollary 2.12 where $\{u_{h_k}(\cdot,t_k-h_k)\ >l_k-\varepsilon\}\,, E^k_\varepsilon$ substitute $E, T^-_{h,t}E$ respectively. We thus have that $E^k_{\varepsilon,n}\nearrow E^k_\varepsilon$ as $n\to\infty$ and that each $E^k_{\varepsilon,n}$ is the minimal minimizer of a problem in the form (2.15). We define

$$W^k_{\varepsilon,n} := E^k_{\varepsilon,n}\setminus\{\eta^\sigma_{h_k}(\cdot)>l_k+\varepsilon,\}\,. \tag{3.8}$$

It is easy to see that, along any subsequence $n(\varepsilon)\to\infty$ as $\varepsilon\to 0$, it holds $W^k_{\varepsilon,n(\varepsilon)}\to\{x\}$ as $\varepsilon\to 0$ in the Hausdorff sense. Furthermore, we check that for every $\varepsilon,k>0$ there exists $n(\varepsilon,k)$ large enough such that $|W^k_{\varepsilon,n}|>0$ for all $n\geq n(\varepsilon,k)$. Indeed, by the continuity of η^σ and since $|\nabla\eta(\bar z)|\neq 0$ there exists a positive radius r such that

$$(B(x_k,r)\cap E^k_\varepsilon)\subseteq W^k_\varepsilon.$$

Since $x_k\in E^k_\varepsilon$ and it is an open set, it holds $|W^k_\varepsilon|>0$. Recalling that $E^k_{\varepsilon,n}\to E^k_\varepsilon$ in L^1, we conclude that $|W^k_{\varepsilon,n}|>0$ for all $n=n(\varepsilon,k)$ large enough. Note also that, for every fixed k, $n(\varepsilon,k)\to\infty$ as $\varepsilon\to 0$.

By minimality of $E^k_{\varepsilon,n}$ we have

$$\begin{aligned}&J(E^k_{\varepsilon,n})+\int_{E^k_{\varepsilon,n}} g\left(\tfrac{1}{h_k}\mathrm{sd}^\psi_{\{u_{h_k}(\cdot,t_k-h_k)>l_k-\varepsilon\}}(x)\right)\vee(-n)\,\mathrm{d}x - f\left(\left[\tfrac{t}{h_k}\right]h_k\right)|W^k_{\varepsilon,n}|\\ &\leq J\left(E^k_{\varepsilon,n}\cap\{\eta^\sigma_{h_k}>l_k\}\right)+\int_{E^k_{\varepsilon,n}\cap\{\eta^\sigma_{h_k}>l_k\}} g\left(\tfrac{1}{h_k}\mathrm{sd}^\psi_{\{u_{h_k}(\cdot,t_k-h_k)>l_k-\varepsilon\}}(x)\right)\vee(-n)\,\mathrm{d}x.\end{aligned}\tag{3.9}$$

Adding to both sides $J\left(\{\eta^\sigma_{h_k}>l_k\}\cup E^k_{\varepsilon,n}\right)$ and using the submodularity of J, we obtain

$$\begin{aligned}&J(\{\eta^\sigma_{h_k}>l_k+\varepsilon\}\cup W^k_{\varepsilon,n})-J(\{\eta^\sigma_{h_k}>l_k+\varepsilon\})-f\left(\left[\tfrac{t}{h_k}\right]h_k\right)|W^k_{\varepsilon,n}|\\ &\qquad+\int_{W^k_{\varepsilon,n}} g\left(\tfrac{1}{h_k}\mathrm{sd}^\psi_{\{u_{h_k}(\cdot,t_k-h_k)>l_k-\varepsilon\}}(x)\right)\vee(-n)\,\mathrm{d}x\leq 0.\end{aligned}$$

Equation (3.6) implies $\{u_{h_k}(\cdot,t_k-h_k)>l_k-\varepsilon\}\subseteq\{\eta(\cdot,t_k-h_k)>l_k-c_k-\varepsilon\}$, therefore by monotonicity we get

$$\begin{aligned}&J(\{\eta^\sigma_{h_k}>l_k+\varepsilon\}\cup W^k_{\varepsilon,n})-J(\{\eta^\sigma_{h_k}>l_k+\varepsilon\})-f\left(\left[\tfrac{t}{h_k}\right]h_k\right)|W^k_{\varepsilon,n}|\\ &\qquad+\int_{W^k_{\varepsilon,n}} g\left(\tfrac{1}{h_k}\mathrm{sd}^\psi_{\{\eta(\cdot,t_k-h_k)>l_k-c_k-\varepsilon\}}(x)\right)\vee(-n)\,\mathrm{d}x\leq 0.\end{aligned}\tag{3.10}$$

If instead E^k_ε is an unbounded set with compact boundary, we employ (2.28) instead of (3.9) to obtain (3.10) in the computations above. See [16, 11] for details.

Step 2: We now estimate the terms appearing in (3.10). We start with the first two terms $J(\{\eta^\sigma_{h_k}>l_k+\varepsilon\}\cup W^k_{\varepsilon,n})-J(\{\eta^\sigma_{h_k}>l_k+\varepsilon\})$. By definition of variational curvature, we get

$$J(\{\eta^\sigma_{h_k}>l_k+\varepsilon\}\cup W^k_{\varepsilon,n})-J(\{\eta^\sigma_{h_k}>l_k+\varepsilon\})\geq|W^k_{\varepsilon,n}|\left(\kappa(x_k,\{\eta^\sigma_{h_k}>l_k+\varepsilon\})+o_\varepsilon(1)\right), \tag{3.11}$$

The last term in (3.10) can be treated as follows. For any $z\in W_\varepsilon$, we have

$$\eta(z,t_k)+c_k+\frac{\sigma}{2}|z-x_k|^2\leq l_k+\varepsilon. \tag{3.12}$$

Since, in turn, $\eta(z,t_k)+c_k>l_k-\varepsilon$ it follows that $\sigma|z-x_k|^2<4\varepsilon$ and thus, for ε small enough,

$$W^k_\varepsilon\subseteq B_{c\sqrt{\varepsilon}}(x_k). \tag{3.13}$$

Therefore, by Hausdorff convergence it holds that for every $\varepsilon, k > 0$ there exists $n = n(\varepsilon, k)$ large enough such that

$$W^k_{\varepsilon,n} \subseteq B_{2c\sqrt{\varepsilon}}(x_k). \tag{3.14}$$

On the other hand, by a Taylor expansion, for every $z \in W^k_{\varepsilon,n}$ we have

$$\eta(z, t_k - h_k) = \eta(z, t_k) - h_k \partial_t \eta(z, t_k) + h_k^2 \int_0^1 (1-s)\partial^2_{tt}\eta(z, t_k - sh_k)\,\mathrm{d}s. \tag{3.15}$$

Then, we consider $y \in \{\eta(\cdot, t_k - h_k)(y) = l_k - c_k - \varepsilon\}$ being a point of minimal ψ-distance from z, that is, $\psi^\circ(z-y) = |\mathrm{sd}^\psi_{\{\eta(\cdot,t_k-h_k)(y)>l_k-c_k-\varepsilon\}}(z)|$. One can prove (see [11, eq. (4.26)] for details) that

$$|z - y| = O(h_k). \tag{3.16}$$

Moreover, it holds (see [16, eq (6.26)] for details)

$$(z-y)\cdot \frac{\nabla\eta(y, t_k - h_k)}{|\nabla\eta(y, t_k - h_k)|} = \pm\psi\left(\frac{\nabla\eta(y, t_k - h_k)}{|\nabla\eta(y, t_k - h_k)|}\right)\mathrm{dist}^\psi_{\{\eta(\cdot,t_k-h_k)(y)=l_k-c_k-\varepsilon\}}(z),$$

with a "+" if $z \in \{\eta(\cdot, t_k - h_k)(y) \le l_k - c_k - \varepsilon\}$ and a "-" otherwise. We get

$$\begin{aligned}
\eta(z, t_k - h_k) &= \eta(y, t_k - h_k) + (z-y)\cdot\nabla\eta(y, t_k - h_k) \\
&\quad + \int_0^1 (1-s)\left(\nabla^2\eta(y + s(z-y), t_k - h_k)(z-y)\right)\cdot(z-y)\,\mathrm{d}s \\
&= l_k - c_k - \varepsilon - \mathrm{sd}^\psi_{\{\eta(\cdot,t_k-h_k)(y)=l_k-c_k-\varepsilon\}}(z)\psi(\nabla\eta(y, t_k - h_k)) \\
&\quad + \int_0^1 (1-s)\left(\nabla^2\eta(y + s(z-y), t_k - h_k)(z-y)\right)\cdot(z-y)\,\mathrm{d}s.
\end{aligned} \tag{3.17}$$

Note that, in view of (3.12) it holds $|\eta(z, t_k) - \eta(y, t_k)| \le c\varepsilon + ch_k = O(h_k)$, provided $\varepsilon \ll h_k$ and small enough. Thus, using also (3.14),(3.16) we deduce

$$\begin{aligned}
\frac{1}{h_k}\mathrm{sd}^\psi_{\{\eta(\cdot,t_k-h_k)>l_k-c_k-\varepsilon\}}(z) &\ge \frac{\partial_t\eta(z, t_k) - \frac{2\varepsilon}{h_k} - O(h_k) - O_{h_k}(1)}{\psi(\nabla\eta(y, t_k - h_k))} \\
&= \frac{\partial_t\eta(x_k, t_k) + O(\sqrt{\varepsilon}) - \frac{2\varepsilon}{h_k} - O(h_k) - O_{h_k}(1)}{\psi(\nabla\eta(x_k, t_k - h_k)) + O(\sqrt{\varepsilon}) + O(h_k)},
\end{aligned}$$

and we apply g to both sides to conclude

$$g\left(\tfrac{1}{h_k}\mathrm{sd}^\psi_{\{\eta(\cdot,t_k-h_k)>l_k-c_k-\varepsilon\}}(z)\right) \ge g\left(\frac{\partial_t\eta(x_k, t_k) - O_{h_k}(1)}{\psi(\nabla\eta(x_k, t_k - h_k)) + O(h_k)}\right) \tag{3.18}$$

Step 4: We conclude the proof. Combining (3.10), (3.11) and (3.18), we arrive at

$$\begin{aligned}
0 \ge |W^k_{\varepsilon,n}|\Big(\kappa(x_k, \{\eta^\sigma_{h_k} > l_k + \varepsilon\}) + o_\varepsilon(1) - f\left(\left[\tfrac{t}{h_k}\right]h_k\right) + \\
g\left(\frac{\partial_t\eta(x_k, t_k) - O_{h_k}(1)}{\psi(\nabla\eta(x_k, t_k - h_k)) + O(h_k)}\right) \vee (-n)\Big).
\end{aligned} \tag{3.19}$$

Choosing $n = n(\varepsilon, k)$, we can divide by $|W^k_{\varepsilon,n(\varepsilon,k)}| > 0$ and apply $\mathtt{G}$ to both sides to get

$$\mathtt{G}\left(-\kappa(x_k, \{\eta^\sigma_{h_k} > l_k + \varepsilon\}) + o_\varepsilon(1) + f\left(\left[\tfrac{t}{h_k}\right]h_k\right)\right) \ge$$

$$\mathtt{G}\left(g\left(\frac{\partial_t\eta(x_k,t_k)-O_{h_k}(1)}{\psi(\nabla\eta(x_k,t_k-h_k))+O(h_k)}\right)\vee(-n(\varepsilon,k))\right).$$

Let us fix $k>0$ and send $\varepsilon\to 0$ (thus also $n(\varepsilon,k)\to 0$). Thanks to the continuity of $\mathtt{G}$ and recalling also that $W^k_{\varepsilon,n(\varepsilon,k)}\to\{x\}$ as $\varepsilon\to 0$, we let $\varepsilon\to 0$ and arrive at

$$\mathtt{G}\left(-\kappa(x_k,\{\eta^\sigma_{h_k}>l_k+\varepsilon\})+f\left(\left[\tfrac{t}{h_k}\right]h_k\right)\right)\geq\frac{\partial_t\eta(x_k,t_k)-O_{h_k}(1)}{\psi(\nabla\eta(x_k,t_k))+O(h_k)},$$

which finally implies the thesis by letting simultaneously $\sigma\to 0$ and $k\to+\infty$.

Case 2: We assume $\nabla\eta(\bar x,\bar t)=0$ and prove that $\partial_t\eta(\bar x,\bar t)\leq 0$. The proof follows the line of the one in [16]. We focus on the case $a=+\infty$, the other being simpler.

Since $\nabla\eta(\bar z)=0$, there exist $\ell\in\mathcal{L}$ and $\omega\in C^\infty(\mathbb{R})$ with $\omega'(0)=0$ such that

$$|\eta(x,t)-\eta(\bar z)-\partial_t\eta(\bar z)(t-\bar t)|\leq\ell(|x-\bar x|)+\omega(|t-\bar t|)$$

thus, we can define

$$\begin{aligned}\tilde\eta(x,t)&=\partial_t\eta(\bar z)(t-\bar t)+2\ell(|x-\bar x|)+2\omega(|t-\bar t|)\\ \tilde\eta_k(x,t)&=\tilde\eta(x,t)+\frac{1}{k(\bar t-t)}.\end{aligned}$$

We remark that $u-\tilde\eta$ achieves a strict maximum in $\bar z$ and the local maxima of $u-\tilde\eta_k$ in $\mathbb{R}^N\times[0,\bar t]$ are in points $(x_k,t_k)\to\bar z$ as $k\to\infty$, with $t_n\leq\bar t$. From now on, the only difference from [16] is in the case $x_k=\bar x$ for an (unrelabeled) subsequence. We thus assume $x_k=\bar x$ for all $k>0$ and define $b_k=\bar t-t_k>0$ and the radii

$$r_k:=\ell^{-1}(a_kb_k),$$

where $a_k\to 0$ must be chosen such that the extinction time for the solution of (3.3) satisfies $T^*(r_k)\geq\bar t-t_k$, for k large enough. To show that such a choice for a_k is possible, we set

$$\beta(t)=\sup_{0\leq s\leq t}\hat\kappa(\ell^{-1}(s))\ell'(\ell^{-1}(s)),\tag{3.20}$$

where $\hat\kappa$ is as in (3.3). Note that by Definition 2.3 it holds $\beta(t)\leq\hat\kappa(t)$ for t small, β is non decreasing in t and $g(t)\to 0$ as $t\to 0$. We then have

$$\begin{aligned}\frac{T^*(r_k)}{b_k}&\geq\frac{1}{b_k}\int_{r_k/2}^{r_k}\frac{1}{\hat\kappa(s)}\,\mathrm{d}s=\frac{1}{b_k}\int_{\ell^{-1}(a_kb_k/2)}^{\ell^{-1}(a_kb_k)}\frac{1}{\hat\kappa(s)}\,\mathrm{d}s\\ &=\frac{a_k}{2}\fint_{a_kb_k/2}^{a_kb_k}\frac{1}{\hat\kappa(\ell^{-1}(r))\ell'(\ell^{-1}(r))}\,\mathrm{d}r\geq\frac{a_k}{2}\frac{1}{\beta(b_k)}=2,\end{aligned}\tag{3.21}$$

where in the last equality we chose $a_k:=4\beta(b_k)$ which tends to 0 as $k\to\infty$.

By definition of $\tilde\eta_k$ it holds

$$\begin{aligned}B(\bar x,r_k)&\subseteq\{\tilde\eta_k(\cdot,t_k)\leq\tilde\eta_k(\bar x,t_k)+2\ell(r_k)\}\\ &\subseteq\{u(\cdot,t_k)\leq u(\bar x,t_k)+2\ell(r_k)\},\end{aligned}$$

by maximality of $u-\tilde\eta_k$ at z_k and since $u(z_k)=\tilde\eta_k(z_k)$. Since the balls $B(\cdot,r_k)$ are not vanishing, by Lemma 3.7 we have

$$\bar x\in\{u(\cdot,\bar t)\leq u(\bar x,t_k)+2\ell(r_k)\}.\tag{3.22}$$

Finally, using again the maximality of $u-\eta$ at $\bar z$, the choice of r_k and (3.22), we obtain

$$\frac{\eta(\bar z)-\eta(\bar x,t_k)}{\bar t-t_k}=\frac{\eta(\bar z)-\eta(\bar x,t_k)}{b_k}\le\frac{u(\bar z)-u(\bar x,t_k)}{b_k}\le\frac{2\ell(r_k)}{b_k}=2a_k.$$

Passing to the limit $k\to\infty$, we conclude that $\partial_t\eta(\bar z)\le 0$. □

4. Uniqueness of Viscosity Solutions

The viscosity theory developed in [16] shows uniqueness for viscosity solutions to the Cauchy problem

$$\begin{cases}\partial_t u(x,t)+|\nabla u(x,t)|\,\kappa(x,\{u(\cdot,t)\ge u(x,t)\})=0\\ u(\cdot,0)=u_0,\end{cases}$$

which corresponds to (2.5) for $\mathtt{G}=id,\psi=|\cdot|$ and $\mathtt{f}=0$, under some additional assumptions on the curvature considered. In particular, the curvature κ must either be of *first order* or satisfy a uniform continuity property (see conditions (FO) and (C') below). Given that the nonlinearity $\mathtt{G}$ is continuous, it follows that if κ satisfies the first-order condition, then $-\mathtt{G}(-\kappa)$ also satisfies it. Similarly, assuming $\mathtt{G}$ is uniformly continuous, we deduce that if κ satisfies the uniform continuity condition, so does $-\mathtt{G}(-\kappa)$. Consequently, uniqueness for continuous viscosity solutions to

$$\begin{cases}\partial_t u(x,t)-|\nabla u(x,t)|\mathtt{G}(-\kappa(x,\{u(\cdot,t)\ge u(x,t)\}))=0\\ u(\cdot,0)=u_0\end{cases}$$

can be deduced from [16, Theorem 3.5] (assuming (FO) below) and [16, Theorem 3.8] (assuming (C') below and $\mathtt{G}$ uniformly continuous). Note however that the curvature $\mathtt{G}(-\kappa)$ is in general not a variational one, thus the convergence of the minimizing movements scheme does not follow from the results of [16]. This is instead ensured by Theorem 3.5.

In this section we detail how one can generalize the results of [16] to show uniqueness of viscosity solutions to (2.5), under some additional assumptions on κ (but, quite surprisingly, not on $\mathtt{G}$). In particular, the major difficulty comes from the presence of a time-dependent term in the operator involving the curvature, which can not be decoupled straightforwardly (because of the presence of the nonlinearity $\mathtt{G}$), see (2.5).

4.1. Setup.

We start recalling notation and some results from [16]. We start introducing the notion of super/subjets.

Definition 4.1. Let $E\subseteq\mathbb{R}^N$, $x_0\in\partial E$, $p\in\mathbb{R}^N$, and $X\in Sym(N)$. We say $(p,X)\in\mathcal{J}^{2,+}_E(x_0)$, the superjet of E at x_0, if for every $\delta>0$ there exists a neighborhood U_δ of x_0 such that, for every $x\in E\cap U_\delta$ it holds

$$(x-x_0)\cdot p+\frac{1}{2}(X+\delta I)(x-x_0)\cdot(x-x_0)\ge 0. \tag{4.1}$$

Moreover, we say that (p,X) is in the subjet $\mathcal{J}^{2,-}_E(x_0)$ of E at x_0 if $(-p,-X)$ is in the superjet $\mathcal{J}^{2,+}_{\mathbb{R}^N\setminus E}(x_0)$ of $\mathbb{R}^N\setminus E$ at x_0. Finally, we say that (p,X) is in the jet $\mathcal{J}^2_E(x_0)$ of E at x_0 if $(p,X)\in\mathcal{J}^{2,+}_E(x_0)\cap\mathcal{J}^{2,-}_E(x_0)$.

Analogously, one introduces the notion of parabolic super/subjet.

Definition 4.2. Let $u : \mathbb{R}^N \times (0,T) \to \mathbb{R}$ be upper semicontinuous at (x,t). We say that $(a,p,X) \in \mathbb{R} \times \mathbb{R}^N \times Sym(N)$ is in the parabolic superjet $\mathcal{P}^{2,+}u(x,t)$ of u at (x,t), if

$$u(y,s) \le u(x,t) + a(s-t) \,+\, p\cdot(y-x) \,+\, \tfrac{1}{2}(X(y-x))\cdot(y-x) \,+\, o(|t-s| + |x-y|^2)$$

for (y,s) in a neighborhood of (x,t). If u is lower semicontinuous at (x,t) we can define the parabolic subjet $\mathcal{P}^{2,-}u(x,t)$ of u at (x,t) as $\mathcal{P}^{2,-}u(x,t) := -\mathcal{P}^{2,+}(-u)(x,t)$.

The notion of semijet induces a notion of convergence.

Definition 4.3. Let $E_n \subseteq \mathbb{R}^N$ and $x_0 \in \partial E_n$. We say that (p_n, X_n) are in the superjet $\mathcal{J}^{2,+}_{E_n}(x_0)$ uniformly, if for every positive $\delta > 0$ there exists a neighborhood U_δ of x_0 (independent of n) such that, for all $n \in N$,

$$(x-x_0)\cdot p_n + \frac{1}{2}(X_n + \delta I)(x-x_0)\cdot(x-x_0) \ge 0 \text{ for every } x \in E_n \cap U_\delta. \tag{4.2}$$

We say that (p_n, X_n, E_n) converge to (p,X,E) with uniform superjet at x_0 if $\overline{E}_n \to \overline{E}$ in the Hausdorff sense, the (p_n, X_n)'s are in the superjet $\mathcal{J}^{2,+}_{E_n}(x_0)$ uniformly and $(p_n, X_n) \to (p, X)$ as $n \to \infty$. Moreover, we say that (p_n, X_n, E_n) converge to (p,X,E) with uniform subjet at x_0 if $(-p_n, -X_n, E_n^c)$ converge to $(-p, -X, E^c)$ with uniform superjet.

One can then introduce semicontinuous extensions of κ.

Definition 4.4. For every $F \subseteq \mathbb{R}^N$ with compact boundary and $(p,X) \in \mathcal{J}^{2,+}_F(x)$, we define

$$\kappa_*(x,p,X,F) := \sup\Big\{\kappa(x,E) \,:\, E \in C^2\,, E \supseteq F\,, (p,X) \in \mathcal{J}^{2,-}_E(x)\Big\}$$

Analogously, for any $(p,X) \in \mathcal{J}^{2,-}_F(x)$ we set

$$\kappa^*(x,p,X,F) \;=\; \inf\Big\{\kappa(x,E) \,:\, E \in C^2\,, \mathring{E} \subseteq F\,, (p,X) \in \mathcal{J}^{2,+}_E(x)\Big\}\,.$$

As shown in [16, Lemma 2.8], one can prove that κ_*, κ^* are the l.s.c and the u.s.c. envelope of κ with respect to the convergence with uniform superjet and subjet. Noting that

$$(-\mathtt{G}(-\kappa))_* = -\mathtt{G}(-\kappa_*), \quad (-\mathtt{G}(-\kappa))^* = -\mathtt{G}(-\kappa^*),$$

one can also show the following equivalent characterization of viscosity solutions.

Lemma 4.5. *Let u be a viscosity subsolution of (2.5) in the sense of Definition 2.4. Then, for all (x,t) in $\mathbb{R}^N \times (0,T)$, if $(a,p,X) \in \mathcal{P}^{2,+}u(x,t)$, and $p \neq 0$, it holds*

$$a \,-\, \psi(|p|)\,\mathtt{G}(-\kappa_*\,(x,p,X,\{y : u(y,t) \ge u(x,t)\} + \mathtt{f}(t)) \;\le\; 0.$$

A similar statement holds for supersolutions, with $\mathcal{P}^{2,-}, \kappa^*$ replacing $\mathcal{P}^{2,+}, \kappa_*$.

4.2. **Proof of the Comparison Principle.** We now show how to adapt the proofs of Theorem 3.5 and Theorem 3.8 of [16] to our setting. We will assume that κ satisfies assumptions (A)-(D) and either:

(FO) For any $\Sigma \in C^2, x \in \partial\Sigma$ and $(p,X), (q,Y)$ in $\mathcal{J}^{2,+}_\Sigma(x), \mathcal{J}^{2,-}_\Sigma(x)$ respectively, then

$$\kappa_*(x,p,X,\Sigma) = \kappa^*(x,q,Y,\Sigma)$$

(C') Replace (C) by the following. For every $R>0$ there exists a modulus of continuity ω_R with the following property. For all $\Sigma \in C^2$, $x\in\partial\Sigma$, such that Σ has both an internal and external ball condition of radius R at x, and for all C^2-diffeomorphism $\Phi:\mathbb{R}^N\to\mathbb{R}^N$, with $\Phi(y)=y$ for $|y-x|\geq 1$, we have

$$|\kappa(x,\Sigma)-\kappa(\Phi(x),\Phi(E))|\leq\omega_R(\|\Phi-Id\|_{C^2}).$$

If (FO) holds, we say that the curvature κ is of first-order, since its relaxation depends only on the first-order elliptic jet. Otherwise, we say that the curvature κ is of second-order. As detailed in [16], an instance of first-order curvature is the one associated to the fractional perimeter, while the classical mean curvature is a second-order one satisfying (C').

Assuming (FO), the following comparison between κ_* and κ^* holds.

Lemma 4.6 (Lemma 3.4 in [16])**.** *Assume (FO), and let F,G be a closed and an open set respectively, with compact boundary and such that $F\subseteq G$. Let $x\in\partial F, y\in\partial G$ satisfy*

$$|x-y|=\operatorname{dist}(\partial F,\partial G).$$

Then, for all $(p,X)\in\mathcal{J}^{2,+}_F(x)$ and $(p,Y)\in\mathcal{J}^{2,-}_G(x)$ with $p=x-y$, it holds

$$\kappa_*(x,p,X,F)\geq\kappa^*(y,p,Y,G).$$

Assuming instead (C'), we recall the following results from [16].

Lemma 4.7. *Assume (C'). Then, given $R>0$, there exists a modulus of continuity ω_R with the following property. For any $F\in C^2$, $x\in\partial F$, with internal and external ball condition at x of radius R, any $(p,X)\in\mathcal{J}^{2,+}_F(x)$ with $p\neq 0$, $|X|/|p|\leq 1/R$, and any $\Phi:\mathbb{R}^N\to\mathbb{R}^N$ diffeomorphism of class C^2, it holds*

$$|\kappa_*(x,p,X,F)-\kappa_*(\Phi(x),D(\psi\circ\Phi^{-1})(\Phi(x)),D^2(\psi\circ\Phi^{-1})(\Phi(x)),\Phi(F))|\leq\omega_R(\|\Phi-Id\|_{C^2})$$

where $\psi(y)=(y-x)\cdot p+\frac{1}{2}X(y-x)\cdot(y-x)$. The same holds for κ^*.

Lemma 4.8. *Assume (C'). Let $x\in\mathbb{R}^N$, $F,G\in C^2$ with $F\subset G\cup\{x\}$ and $\partial F\cap\partial G=\{x\}$. Let $(p,X)\in\mathcal{J}^{2,+}_F(x)$, $(p,Y)\in\mathcal{J}^{2,-}_G(x)$, with $X\leq Y$. Then,*

$$\kappa_*(x,p,X,F)\geq\kappa^*(x,p,Y,G).$$

Our main result of this section is a comparison principle for sub/supersolutions.

Theorem 4.9. *Assume either (FO) or (C'). Let u,v be u.s.c and l.s.c functions on $\mathbb{R}^N\times[0,T]$, both constant outside a compact set, a subsolution and a supersolution to (2.5), respectively. If $u(\cdot,0)\leq v(\cdot,0)$, then $u\leq v$ in $\mathbb{R}^N\times[0,T]$.*

Proof assuming (FO). We assume wlog that $u(\cdot,0)<v(\cdot,0)$ and by contradiction that there exists $(\bar{x},\bar{t})\in\mathbb{R}^N\times(0,T]$ such that $u(\bar{x},\bar{t})-v(\bar{x},\bar{t})>0$. Setting $F(t):=\{u(\cdot,t)\geq u(\bar{x},\bar{t})\}$ and $G(t):=\{v(\cdot,t)\geq v(\bar{x},\bar{t})\}$, it holds $F(\bar{t})\nsubseteq G(\bar{t})$. Note that one can perturb the set F (respectively, the set G) so that it satisfies an internal ball condition (resp. an external ball condition), uniformly in time, and χ_F is still a subsolution (resp. χ_G is still a supersolution). This can be done replacing F by F_r and G by $\{\mathrm{sd}_G<-r\}=\mathrm{int}(G_{-r})$ for $r>0$ small so that $F(0)\subseteq G(0)$. Let $\ell\in\mathcal{L}$. We replace u,v by

$$\begin{aligned} u(x,t)&=\max_{\xi\in\mathbb{R}^N,\tau\in[t-T,t]}\chi_{F(t-\tau)}(x-\xi)-\lambda(\ell(\xi)+\tau^2)\\ v(x,t)&=\min_{\xi\in\mathbb{R}^N,\tau\in[t-T,t]}\chi_{G(t-\tau)}(x-\xi)+\lambda(\ell(\xi)+\tau^2),\end{aligned}\tag{4.3}$$

where λ is a positive parameter, big enough so that $u(\cdot,0)\le v(\cdot,0)$. The function u (respectively, the function v) is equal to one on F (resp. on G), zero outside a compact set, and each superlevel set satisfies an internal ball condition (resp. external ball condition), uniformly in time. Furthermore, for λ large enough (so that the max in (4.7) is not reached at $\tau=t$), u is a subsolution while v is a supersolution in $\mathbb{R}^N\times[2/\sqrt{\lambda},T]$. We refer to [16] for the proof of these facts.

Let $\alpha,\beta,\varepsilon>0$ and set

$$\Phi(x,t,y,s):=u(x,t)-v(y,s)-\alpha\ell(|x-y|)-\beta|t-s|^2-\varepsilon(t+s).$$

Noticing that Φ is u.s.c., let $z^\beta=(x^\beta,t^\beta,y^\beta,s^\beta)$ be a maximum point of ϕ. Note that choosing ε small (depending on $\bar t$), we can assume that the maximum is strictly positive and that t^β,s^β are strictly positive. Moreover, for λ large enough and $\beta\ge\lambda$, one can ensure that $t^\beta,s^\beta>2/\sqrt{\lambda}$.
Case 1: $x^\beta=y^\beta$ along a sequence $\beta_n\to+\infty$. In this case, defining

$$\begin{aligned}\varphi(x,t)&=v(y^\beta,s^\beta)+\varepsilon(t+s^\beta)+\alpha\ell(|x-y^\beta|)+\beta|t-s^\beta|^2\\ \psi(y,s)&=u(x^\beta,t^\beta)-\varepsilon(t^\beta+s)-\alpha\ell(|x^\beta-y|)-\beta|t^\beta-s|^2,\end{aligned}\tag{4.4}$$

since u,v are a sub- and supersolution respectively, we have

$$0\ge\varphi_t(x^\beta,t^\beta)=2\beta(t^\beta-s^\beta)+\varepsilon,\quad 0\le\psi_t(y^\beta,s^\beta)=2\beta(t^\beta-s^\beta)-\varepsilon,$$

which yields a contradiction.
Case 2: $x^\beta\ne y^\beta$ for all β sufficiently large. Note that

$$\Big(2\beta(t^\beta-s^\beta)+\varepsilon,\alpha f'(|p^\beta|)\frac{p^\beta}{|p^\beta|},X\Big)\in\mathcal{P}^{2,+}u(x^\beta,t^\beta),$$
$$\Big(2\beta(t^\beta-s^\beta)-\varepsilon,\alpha f'(|p^\beta|)\frac{p^\beta}{|p^\beta|},-X\Big)\in\mathcal{P}^{2,-}v(y^\beta,s^\beta),$$

where $p^\beta:=x^\beta-y^\beta$ and $X:=\nabla^2\varphi(x^\beta,t^\beta)$, with φ defined in (4.4). Thus, by Lemma 4.5, we have

$$\begin{aligned}2\beta(t^\beta-s^\beta)+\varepsilon-\psi(|p^\beta|)\,\mathtt{G}\Big(-\kappa_*(x^\beta,\alpha f'(|p^\beta|)\frac{p^\beta}{|p^\beta|},X,\{u(\cdot,t^\beta)\ge u(x^\beta,t^\beta)\})+\mathtt{f}(t^\beta)\Big)\le0,\\ 2\beta(t^\beta-s^\beta)-\varepsilon-\psi(|p^\beta|)\,\mathtt{G}\Big(-\kappa^*(y^\beta,\alpha f'(|p^\beta|)\frac{p^\beta}{|p^\beta|},-X,\{v(\cdot,t^\beta)\ge v(x^\beta,t^\beta)\})+\mathtt{f}(s^\beta)\Big)\le0.\end{aligned}\tag{4.5}$$

Let us denote $\hat p^\beta:=\alpha f'(|p^\beta|)\frac{p^\beta}{|p^\beta|}$, $F^\beta:=\{u(\cdot,t^\beta)\ge u(x^\beta,t^\beta)\}$, $G^\beta:=\{v(\cdot,t^\beta)\ge v(x^\beta,t^\beta)\}$. We then remark that

$$\{u(\cdot,t^\beta)\ge u(x^\beta,t^\beta)\}+B(0,|y^\beta-x^\beta|)\subseteq\{v(\cdot,s^\beta)>v(y^\beta,s^\beta)\}.$$

Indeed, if $x\in\{u(\cdot,t^\beta)\ge u(x^\beta,t^\beta)\}$ and $|y-x|<|y^\beta-x^\beta|$, since z^β is a maximum point for Φ, it holds

$$v(y^\beta,s^\beta)-v(y,s^\beta)\ \le\ u(x^\beta,t^\beta)-u(x,t)+\alpha\ell(|x-y|)-\alpha\ell(|x^\beta-y^\beta|)\ <\ 0$$

so that $y\in\{v(\cdot,s^\beta)>v(y^\beta,s^\beta)\}$. Thus, we can apply Lemma 4.6 to infer from (4.5) that

$$2\varepsilon\le-\psi(|p^\beta|)\left(\mathtt{G}(-\kappa^\beta+\mathtt{f}(s^\beta))-\mathtt{G}(-\kappa^\beta+\mathtt{f}(t^\beta))\right),\tag{4.6}$$

where we set $\kappa^\beta:=\kappa^*(y^\beta,\hat p^\beta,-X,G^\beta)$. Since all the superlevel sets of u,v satisfy a uniform internal, external (respectively) ball condition, and thanks to Lemma 4.6, the term κ^β is bounded as $\beta \to +\infty$, and so we can assume $(|\kappa^\beta| + \|\mathtt{f}\|_\infty) \le M$. Since $\mathtt{G}$ is uniformly continuous in $[-M, M]$, (4.6) implies

$$2\varepsilon = O(|\mathtt{f}(s^\beta) - \mathtt{f}(t^\beta)|)$$

as $\beta \to +\infty$, a contradiction.

□

Proof assuming (C'). We assume wlog that $u(\cdot, 0) < v(\cdot, 0)$ and argue by contradiction. Assume that there exists $a \in \mathbb{R}$ and $t \in (0, T]$ such that $F(t) := \{u(\cdot, t) \ge a\} \nsubseteq G(t) := \{v(\cdot, t) > a\}$. As sketched in the previous case, can assume that F satisfies an internal ball condition while G satisfies an external ball condition, uniformly in time, and χ_F, χ_G are still a sub and supersolution. For fixed $\ell \in \mathcal{L}$ and $\lambda > 0$, we can replace u, v by

$$\begin{aligned} u(x,t) &= \max_{\xi \in \mathbb{R}^N, \tau \in [t-T, t]} \chi_{F(t-\tau)}(x - \xi) - \lambda\left(\ell(\xi) + \tau^2\right) \\ v(y,s) &= \min_{\xi \in \mathbb{R}^N, \tau \in [s-T, s]} \chi_{G(s-\tau)}(y - \xi) + \lambda\left(\ell(\xi) + \tau^2\right). \end{aligned} \tag{4.7}$$

Note that it holds $u(\cdot, 0) \le v(\cdot, 0)$ for λ big enough. The function u (respectively, the function v) is equal to one on F (resp. on G), zero outside a compact set, and each superlevel set satisfies an internal ball condition (resp. external ball condition), uniformly in time. Furthermore, for λ large enough (so that the max in (4.7) is not reached at $\tau = t$), u is a subsolution while v is a supersolution in $\mathbb{R}^N \times [2/\sqrt{\lambda}, T]$. In the following, we omit the dependence on λ, as it will be a fixed parameter. For $\alpha, \beta, \varepsilon > 0$ and $\mathbb{N} \ni \beta \ge \lambda$, we define

$$\Phi(x,t,y,s) := u(x,t) - v(y,s) - \varepsilon(t+s) - \alpha\ell(|x-y|) - \beta|t-s|^2,$$

which is semiconvex. For $\varepsilon > 0$ small and α, β large enough the function Φ admits a positive maximum at some $(x^\beta, t^\beta, y^\beta, s^\beta) \in \mathbb{R}^N \times [0,T] \times \mathbb{R}^N \times [0,T]$ with t^β, $s^\beta > 0$. Note also that

$$|t^\beta - s^\beta| \to 0 \quad \text{ as } \beta \to +\infty.$$

Since u, v are constant outside a compact set, and by translation invariance it is not difficult to see that x^β, y^β admit cluster points x_0, y_0 as $\beta \to +\infty$ (see for instance [26, page 14]). We thus assume wlog that $(x^\beta, y^\beta) \to (x_0, y_0)$ as $\beta \to +\infty$. If $x^\beta = y^\beta$ infinitely often, one can conclude considering φ, ψ defined in (4.4) (see the previous proof and [16]). Thus, we assume $x^\beta \neq y^\beta$ for all β. One can also assume that $\ell(|x^\beta - y^\beta|) < 1$ (taking λ large) and check that

$$u(x^\beta, t^\beta) < 1. \tag{4.8}$$

Indeed, $Du(x^\beta, t^\beta) = D\ell(|x^\beta - y^\beta|) \neq 0$, while $u(x,t) = 1$ if and only if $x \in F(t)$, but on $F(t)$ it holds $Du = 0$.

Step 1: In this step we provide estimates for the final argument. The constructions are essentially the same introduced in [16], which we recall for the reader's convenience. We fix β and omit the dependence on it of the approximating parameters.

Let $q : [0, +\infty] \to [0, 1]$ be a smooth, nondecreasing, function with $q(r) = r^4$ for $r < 1/2$ and $q(r) = 1$ for $r > 3/2$. For $\rho > 0$ we define

$$\Phi_\rho(x,t,y,s) := \Phi(x,t,y,s) - \rho[q(|x - x^\beta|) + q(|y - y^\beta|) + q(|t - t^\beta|) + q(|s - s^\beta|)],$$

so that $(x^\beta, t^\beta, y^\beta, s^\beta)$ is a strict maximum of Φ_ρ. Let $\eta : \mathbb{R}^N \to \mathbb{R}$ be a smooth cut-off function, with compact support and equal to one in a neighborhood U of the origin. For every $\Delta := (\zeta_u, \tau_u, \zeta_v, \tau_v) \in$

$\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$, the function

$$\Phi_\rho(x,t,y,s) - \Big(\eta(x-x^\beta)\,(\zeta_u,\tau_u)\cdot(x,t) + \eta(y-y^\beta)\,(\zeta_v,\tau_v)\cdot(y,s)\Big)$$

is maximized at some $(x_\Delta, t_\Delta, y_\Delta, s_\Delta)$ converging to $(x^\beta, t^\beta, y^\beta, s^\beta)$ as $|\Delta| \to 0$. Therefore, by Jensen's Lemma [19, Lemma A.3] we may assume that for every $\delta > 0$ sufficiently small there exists $\Delta_{\rho,\delta} := (\zeta_u^{\rho,\delta}, h_u^{\rho,\delta}, \zeta_v^{\rho,\delta}, h_v^{\rho,\delta})$, with $|\Delta_{\rho,\delta}| \le \delta$, such that the function

$$\Phi_{\rho,\delta}(x,t,y,s) := \Phi_\rho(x,t,y,s) - \Big(\eta(x-x^\beta)(\xi_u^{\rho,\delta}, h_u^{\rho,\delta})\cdot(x,t) + \eta(y-y^\beta)(\xi_v^{\rho,\delta}, h_v^{\rho,\delta})\cdot(y,s)\Big)$$

attains a maximum at some $z_{\rho,\delta} := (x_{\rho,\delta}, t_{\rho,\delta}, y_{\rho,\delta}, s_{\rho,\delta})$ where $\Phi_{\delta,\rho}$ is twice differentiable and such that $x_{\rho,\delta} - x^\beta,\ y_{\rho,\delta} - y^\beta \in U$ and $t_{\rho,\delta},\ s_{\rho,\delta} > 0$. Moreover,

$$z_{\rho,\delta} \to (x^\beta, t^\beta, y^\beta, s^\beta) \qquad \text{as } \delta \to 0. \tag{4.9}$$

Notice that since Φ_ρ is twice differentiable at $z_{\rho,\delta}$ it follows that also u, v are twice differentiable at $(x_{\rho,\delta}, t_{\rho,\delta})$ and $(y_{\rho,\delta}, s_{\rho,\delta})$, respectively.

Let $\tau_u^{\rho,\delta} \in \mathbb{R}$ (resp. $\tau_v^{\rho,\delta} \in \mathbb{R}$) be the maximizing (resp. minimizing) τ in (4.7) corresponding to the point $(x_{\rho,\delta}, t_{\rho,\delta})$ (resp. $(y_{\rho,\delta}, s_{\rho,\delta})$). Setting

$$\begin{aligned}
\tilde u(x,t) &:= \max_{\xi\in\mathbb{R}^N} \Big\{\chi_{F(t-\tau_u^{\rho,\delta})}(x-\xi) - \lambda\ell(|\xi|)\Big\} - \lambda(\tau_u^{\rho,\delta})^2\\
\tilde v(y,s) &:= \min_{\xi\in\mathbb{R}^N} \Big\{\chi_{G(s-\tau_v^{\rho,\delta})}(y-\xi) + \lambda\ell(|\xi|)\Big\} + \lambda(\tau_v^{\rho,\delta})^2,
\end{aligned}$$

we note that

$$\begin{aligned}
u &\ge \tilde u, \qquad u(x_{\rho,\delta}, t_{\rho,\delta}) = \tilde u(x_{\rho,\delta}, t_{\rho,\delta}),\\
v &\le \tilde v, \qquad v(y_{\rho,\delta}, s_{\rho,\delta}) = \tilde v(y_{\rho,\delta}, s_{\rho,\delta}).
\end{aligned} \tag{4.10}$$

Set now

$$\begin{aligned}
\hat u(x,t) &:= \tilde u(x,t) - \rho\left(q(|x-x_{\rho,\delta}|) + q(|x-x^\beta|) + q(|t-t^\beta|)\right) - \eta(x-x^\beta)(\xi_u^{\rho,\delta}, h_u^{\rho,\delta})\cdot(x,t),\\
\hat v(y,s) &:= \tilde v(y,s) + \rho\left(q(|y-y_{\rho,\delta}|) + q(|y-y^\beta|) + q(|s-s^\beta|)\right) + \eta(y-y^\beta)(\xi_v^{\rho,\delta}, h_v^{\rho,\delta})\cdot(y,s).
\end{aligned}$$

Then, the function

$$\hat u(x,t) - \hat v(y,s) - \varepsilon(t+s) - \alpha\ell(|x-y|) - \beta|t-s|^2$$

has a maximum at $z_{\rho,\delta}$, which is strict with respect to the spatial variables. Thus

$$\hat F_{\rho,\delta}(t) := \{\hat u(\cdot,t) \ge \hat u(x_{\rho,\delta}, t_{\rho,\delta})\}, \qquad \hat G_{\rho,\delta}(s) := \{\hat v(\cdot,s) > \hat v(y_{\rho,\delta}, s_{\rho,\delta})\}.$$

satisfy $\hat F_{\rho,\delta}(t_{\rho,\delta}) \subseteq \hat G_{\rho,\delta}(s_{\rho,\delta})$ and moreover $x_{\rho,\delta} \in \hat F_{\rho,\delta}(t_{\rho,\delta})$ and $y_{\rho,\delta} \in \hat G_{\rho,\delta}(t_{\rho,\delta})$ are the only points realizing the distance between $\hat F_{\rho,\delta}(t_{\rho,\delta})$ and $\hat G_{\rho,\delta}(t_{\rho,\delta})$. In particular, $\hat F_{\rho,\delta}(t_{\rho,\delta})$ (respectively, $\hat G_{\rho,\delta}(t_{\rho,\delta})$) satisfies an external ball condition (resp. internal ball condition) of radius $|x^\beta - y^\beta| > 0$. We observe that at the maximum point,

$$\Big||D\hat u(x_{\rho,\delta}, t_{\rho,\delta})| - \alpha\ell'(|x^\beta - y^\beta|)\Big| = \omega(\rho,\delta)$$

where $\omega \to 0$ as its arguments tend to 0, thus since $\ell'(|x^\beta - y^\beta|) \ne 0$, the term $|D\hat u(x_{\rho,\delta}, t_{\rho,\delta})|$ is bounded below for ρ, δ small. In addition, the function $\hat u$ is semiconvex, hence $\hat F_{\rho,\delta}(t_{\rho,\delta})$ has an interior ball condition at $x_{\rho,\delta}$ with a radius depending on λ only, thus independent on ρ, δ, if small enough, and β. Analogously, $\hat G_{\rho,\delta}(s_{\rho,\delta})$ has an exterior ball condition at $y_{\rho,\delta}$ with a radius depending on λ only.

Set

$$\breve{\Phi}_{\rho,\delta}(x,t,y,s) := \Phi_{\rho,\delta}(x,t,y,s) + \alpha\ell(|x-y|) + \beta|t-s|^2$$

and

$$(\breve{a}_{\rho,\delta}, \breve{p}_{\rho,\delta}, \breve{X}_{\rho,\delta}) := (\partial_t \breve{\Phi}_{\rho,\delta}(z_{\rho,\delta}), D_x\breve{\Phi}_{\rho,\delta}(z_{\rho,\delta}), D_x^2\breve{\Phi}_{\rho,\delta}(z_{\rho,\delta})),$$
$$(\breve{b}_{\rho,\delta}, \breve{q}_{\rho,\delta}, \breve{Y}_{\rho,\delta}) := (\partial_s \breve{\Phi}_{\rho,\delta}(z_{\rho,\delta}), D_y\breve{\Phi}_{\rho,\delta}(z_{\rho,\delta}), D_y^2\breve{\Phi}_{\rho,\delta}(z_{\rho,\delta})).$$

Then, recalling (4.10), we observe that the superjet $(\breve{a}_{\rho,\delta}, \breve{p}_{\rho,\delta}, \breve{X}_{\rho,\delta})$ of

$$u(x,t) - \rho[q(|x-x^\beta|) + q(|t-t^\beta|)] - \eta(x-x^\beta)(\xi_u^{\rho,\delta}, h_u^{\rho,\delta})\cdot(x,t)$$

at $(x_{\rho,\delta}, t_{\rho,\delta})$ is also a superjet for $\hat{u}(x,t)$ at the same point. Since $\hat{u}(x,t) \geq \hat{u}(x_{\rho,\delta}, t_{\rho,\delta})\chi_{\hat{F}_{\rho,\delta}(t)}(x)$ and $x_{\rho,\delta} \in \hat{F}_{\rho,\delta}(t_{\rho,\delta})$, we have

$$(\breve{a}_{\rho,\delta}, \breve{p}_{\rho,\delta}, \breve{X}_{\rho,\delta}) \in \mathcal{P}^{2,+}_{\hat{u}(x_{\rho,\delta},t_{\rho,\delta})\chi_{\hat{F}_{\rho,\delta}}}(x_{\rho,\delta}, t_{\rho,\delta}), \tag{4.11}$$

$$(\breve{b}_{\rho,\delta}, \breve{q}_{\rho,\delta}, \breve{Y}_{\rho,\delta}) \in \mathcal{P}^{2,-}_{\hat{v}(y_{\rho,\delta},s_{\rho,\delta})\chi_{\hat{G}_{\rho,\delta}}}(y_{\rho,\delta}, s_{\rho,\delta}). \tag{4.12}$$

Since $z_{\rho,\delta}$ is a maximum of $\Phi_{\rho,\delta}$,

$$\breve{a}_{\rho,\delta} - \breve{b}_{\rho,\delta} = 2\varepsilon, \qquad \breve{p}_{\rho,\delta} = \breve{q}_{\rho,\delta}, \quad \breve{X}_{\rho,\delta} \leq \breve{Y}_{\rho,\delta}. \tag{4.13}$$

By construction, $\breve{\Phi}_{\rho,\delta}$ is also semiconvex, so that $\breve{X}_{\rho,\delta} \geq -cI$, $\breve{Y}_{\rho,\delta} \leq cI$ for a constant c that does not depend on ρ, δ.

We then let

$$c_{\rho,\delta}(x,t) = \tilde{u}(x,t) + (\hat{u}(x_{\rho,\delta}, t_{\rho,\delta}) - \hat{u}(x,t))$$

and note that $c_{\rho,\delta} \to u(x^\beta, t^\beta)$ uniformly as $\rho, \delta \to 0$. Thus, thanks to (4.8) we can assume $c_{\rho,\delta} < 1$. Note also that $c_{\rho,\delta}$ is smooth and constant away from a neighborhood of (x^β, t^β), and $c_{\rho,\delta}(x_{\rho,\delta}, t_{\rho,\delta}) = u(x_{\rho,\delta}, t_{\rho,\delta})$.

Since $\hat{F}_{\rho,\delta}(t) = \{\tilde{u}(\cdot,t) \geq c_{\rho,\delta}(\cdot,t)\}$, by definition of $\tilde{u}$ one can check that

$$\hat{F}_{\rho,\delta}(t) = \left\{x \in \mathbb{R}^N : x \in \xi + F(t - \tau_u^{\rho,\delta}) \text{ for some } \xi \in \mathbb{R}^N \text{ with } |\xi| \leq \ell^{-1}\left(\frac{1-c_{\rho,\delta}}{\lambda}\right)\right\}. \tag{4.14}$$

For ρ, δ small enough it holds $x_{\rho,\delta} \notin F(t_{\rho,\delta} - \tau_u^{\rho,\delta})$ (from (4.7)), and so we introduce $w_{\rho,\delta}$ such that $x_{\rho,\delta} + w_{\rho,\delta}$ is the projection of $x_{\rho,\delta}$ on $F(t_{\rho,\delta} - \tau_u^{\rho,\delta})$. In particular, $\xi = -w_{\rho,\delta}$ reaches the max in (4.7) for $x = x_{\rho,\delta}$. Also, $|w_{\rho,\delta}| = \ell^{-1}((1 - c_{\rho,\delta}(x_{\rho,\delta}, t_{\rho,\delta}))/\lambda)$. We define

$$\Psi_{\rho,\delta}(x) = x - \ell^{-1}\left(\frac{1 - c_{\rho,\delta}(x, t_{\rho,\delta})}{\lambda}\right)\frac{w_{\rho,\delta}}{|w_{\rho,\delta}|} + w_{\rho,\delta},$$

which is a C^2 diffeomorphism, since $c_{\rho,\delta}$ is bounded away from 1. Moreover, $\Psi_{\rho,\delta}$ is a constant small translation out of a neighborhood of x^β, converges in C^2 to the identity as $\rho, \delta \to 0$, and $\Psi_{\rho,\delta}(x_{\rho,\delta}) = x_{\rho,\delta}$. From this, define the set

$$\breve{F}_{\rho,\delta}(t) := \Psi_{\rho,\delta}(F(t - \tau_u^{\rho,\delta}) - w_{\rho,\delta}).$$

By construction, $\breve{F}_{\rho,\delta}(t_{\rho,\delta}) \subseteq \hat{F}_{\rho,\delta}(t_{\rho,\delta})$ and $x_{\rho,\delta} \in \partial\breve{F}_{\rho,\delta}(t_{\rho,\delta}) \cap \partial\hat{F}_{\rho,\delta}(t_{\rho,\delta})$. Since $\hat{F}_{\rho,\delta}(t_{\rho,\delta})$ satisfies a uniform external ball condition in $x_{\rho,\delta}$, so does $\breve{F}_{\rho,\delta}(t_{\rho,\delta})$ (with possibly a different radius). Since F satisfies an internal ball condition uniformly in time and $\Psi_{\rho,\delta}$ converges C^2 to the identity as $\rho, \delta \to 0$, we can assume additionally that $\breve{F}_{\rho,\delta}(t_{\rho,\delta})$ satisfies a uniform internal ball condition for ρ, δ small enough, with radius depending on λ.

Finally, defining

$$(p_{\rho,\delta}, X_{\rho,\delta}) := \Big(D_x(\breve{\Phi}_{\rho,\delta}(\cdot, t_{\rho,\delta}, y_{\rho,\delta}, s_{\rho,\delta}) \circ \Psi_{\rho,\delta})(x_\delta),$$
$$D_x^2(\breve{\Phi}_{\rho,\delta}(\cdot, t_{\rho,\delta}, y_{\rho,\delta}, s_{\rho,\delta}) \circ \Psi_{\rho,\delta})(x_\delta)\Big),$$

one can check that by construction (see (4.11)) it holds

$$(\breve{a}_{\rho,\delta}, p_{\rho,\delta}, X_{\rho,\delta}) \in \mathcal{P}^{2,+}_{\hat{u}(x_{\rho,\delta},t_{\rho,\delta})\chi_{F(t-\tau_u^{\rho,\delta})}}(x_{\rho,\delta} + w_{\rho,\delta}, t_{\rho,\delta})$$

Since $\hat{u}(x_{\rho,\delta}, t_{\rho,\delta})\chi_{F(t-\tau_u^{\rho,\delta})}$ is a subsolution, we have

$$\breve{a}_{\rho,\delta} + \psi(|p_{\rho,\delta}|)\mathtt{G}\left(\kappa_*(x_{\rho,\delta} + w_{\rho,\delta}, p_{\rho,\delta}, X_{\rho,\delta}, F(t_{\rho,\delta} - \tau_u^{\rho,\delta})) + \mathtt{f}(t_{\rho,\delta})\right) \le 0. \tag{4.15}$$

Note that

$$p_{\rho,\delta} \to Du(x^\beta, t^\beta) \neq 0\,,$$

as ρ, $\delta \to 0$, and thus $|p_{\rho,\delta}|$ is bounded away from zero for ρ and δ sufficiently small. Since also $\breve{X}_{\rho,\delta}$ and hence $X_{\rho,\delta}$ is bounded, the curvature terms $\kappa_*(x_{\rho,\delta}, \breve{p}_{\rho,\delta}, \breve{X}_{\rho,\delta}, \breve{F}_{\rho,\delta}(t_{\rho,\delta}))$ are uniformly bounded from above and below as $\rho, \delta \to 0$. Thus, $\mathtt{G}$ is uniformly continuous and by Lemma 4.7 we deduce from (4.15) that

$$\breve{a}_{\rho,\delta} + \psi(|\breve{p}_{\rho,\delta}|)\mathtt{G}\left(\kappa_*(x_{\rho,\delta}, \breve{p}_{\rho,\delta}, \breve{X}_{\rho,\delta}, \breve{F}_{\rho,\delta}(t_{\rho,\delta})) + \mathtt{f}(t_{\rho,\delta})\right) \le \omega(\rho,\delta), \tag{4.16}$$

where ω is a modulus of continuity that depends on β, and $\omega(\rho,\delta) \to 0$ as ρ, $\delta \to 0$. Analogously, from (4.12), (4.13) and since $\hat{v}(y_{\rho,\delta}, s_{\rho,\delta})\chi_{G(t-\tau_v^{\rho,\delta})}$ is a supersolution, we also have

$$\breve{a}_{\rho,\delta} - 2\varepsilon + \psi(|\breve{p}_{\rho,\delta}|)\mathtt{G}\left(\kappa^*(y_{\rho,\delta}, \breve{p}_{\rho,\delta}, \breve{Y}_{\rho,\delta}, \breve{G}_{\rho,\delta}(s_{\rho,\delta})) + \mathtt{f}(s_{\rho,\delta})\right) \ge \omega(\rho,\delta) \tag{4.17}$$

for a suitable set $\breve{G}_{\rho,\delta}(s_{\rho,\delta}))$ such that

$$\hat{F}(t_{\rho,\delta}) + (y_{\rho,\delta} - x_{\rho,\delta}) \subseteq \breve{G}_{\rho,\delta}(s_{\rho,\delta})) \text{ and } \partial(\breve{F}_{\rho,\delta}(t_{\rho,\delta}) + (y_{\rho,\delta} - x_{\rho,\delta})) \cap \partial\breve{G}_{\rho,\delta}(s_{\rho,\delta})) = \{y_{\rho,\delta}\}.$$

By the equation above, (4.13) and Lemma 4.8 we get

$$\kappa_*(x_{\rho,\delta}, \breve{p}_{\rho,\delta}, \breve{X}_{\rho,\delta}, \breve{F}_{\rho,\delta}(t_{\rho,\delta})) \ge \kappa^*(y_{\rho,\delta}, \breve{p}_{\rho,\delta}, \breve{Y}_{\rho,\delta}, \breve{G}_{\rho,\delta}(s_{\rho,\delta})),$$

and thus by (4.16) and (4.17) we arrive at

$$\begin{aligned} -2\varepsilon + \psi(|\breve{p}_{\rho,\delta}|)\Big[&\mathtt{G}\left(\kappa_*(x_{\rho,\delta}, \breve{p}_{\rho,\delta}, \breve{X}_{\rho,\delta}, \breve{F}_{\rho,\delta}(t_{\rho,\delta})) + \mathtt{f}(s_{\rho,\delta})\right) \\ &-\mathtt{G}\left(\kappa_*(x_{\rho,\delta}, \breve{p}_{\rho,\delta}, \breve{X}_{\rho,\delta}, \breve{F}_{\rho,\delta}(t_{\rho,\delta})) + \mathtt{f}(t_{\rho,\delta})\right)\Big] \ge 2\omega(\rho,\delta). \end{aligned} \tag{4.18}$$

Step 2: We now pass to the limit $\rho, \delta \to 0$ then $\beta \to +\infty$. Recalling that $(x^\beta, y^\beta) \to (x_0, y_0)$ as $\beta \to +\infty$, we distinguish two cases.
Case 1: Assume that $x_0 \neq y_0$. In this case, $|x^\beta - y^\beta|$ is uniformly bounded from below, and thus the term $\kappa_*(x_{\rho,\delta}, \breve{p}_{\rho,\delta}, \breve{X}_{\rho,\delta}, \breve{F}_{\rho,\delta}(t_{\rho,\delta}))$ is uniformly bounded in ρ, δ, β. Therefore, the continuity of $\mathtt{G}$ is uniform as ρ, δ, β vary, and (4.18) implies

$$\tilde{\omega}\left(|\mathtt{f}(s^\beta) - \mathtt{f}(t^\beta)|\right) + \omega(\rho,\delta) \ge \varepsilon, \tag{4.19}$$

for a modulus of continuity $\tilde{\omega}$ such that $\tilde{\omega}(r) \to 0$ as $r \to 0$. We pass to the limit $\rho, \delta \to 0$ then $\beta \to +\infty$ to arrive at a contradiction.
Case 2: It holds $x_0 = y_0$. In this case, we recall that

$$p_{\rho,\delta} \to Du(x^\beta, t^\beta) = \alpha\ell'(|x^\beta - y^\beta|), \quad \text{ as } \rho, \delta \to 0.$$

Recall that the set $\check{F}_{\rho,\delta}(t_{\rho,\delta})$ satisfies a uniform internal and external ball condition of radius $|x^\beta - y^\beta|$. In particular

$$|\kappa_*(x_{\rho,\delta}, \check{p}_{\rho,\delta}, \check{X}_{\rho,\delta}, \check{F}_{\rho,\delta}(t_{\rho,\delta}))| \leq \overline{c}(|x^\beta - y^\beta|).$$

Therefore, equation (4.18) implies

$$\psi(|p_{\rho,\delta}| + \omega(\rho,\delta))\ \mathtt{G}\left(\overline{c}(|x^\beta - y^\beta| + \|\mathtt{f}\|_\infty)\right) + 2\omega(\rho,\delta) \geq 2\varepsilon.$$

Sending $\rho, \delta \to 0$ we get

$$c_\psi |\ell'(|x^\beta - y^\beta|)|\, \mathtt{G}\left(\overline{c}(|x^\beta - y^\beta|)\right) \frac{\mathtt{G}\left(\overline{c}(|x^\beta - y^\beta| + \|\mathtt{f}\|_\infty)\right)}{\mathtt{G}\left(\overline{c}(|x^\beta - y^\beta|)\right)} \geq 2\varepsilon.$$

recalling the properties of ℓ (see Definition 2.3), we arrive at a contradiction sending $\beta \to +\infty$. $\square$

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