Intermediate Domains for Scalar Conservation Laws

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Abstract

For a scalar conservation law with strictly convex flux, by Oleinik's estimates the total variation of a solution with initial data $\overline{u} \in \mathbf{L}^\infty(\mathbb{R})$ decays like t^{-1}. This paper introduces a class of intermediate domains $\mathcal{P}_\alpha$, $0 < \alpha < 1$, such that for $\overline{u} \in \mathcal{P}_\alpha$ a faster decay rate is achieved: Tot.Var.$\{u(t,\cdot)\} \sim t^{\alpha-1}$. A key ingredient of the analysis is a "Fourier-type" decomposition of $\overline{u}$ into components which oscillate more and more rapidly. The results aim at extending the theory of fractional domains for analytic semigroups to an entirely nonlinear setting.

Key words: Scalar conservation law, total variation decay, intermediate domain.

1 Introduction

Consider a scalar conservation law

$$u_t + f(u)_x \;=\; 0, \tag{1.1}$$

with strictly convex flux. By a classical result [9, 19], there exists a contractive semigroup $S : \mathbf{L}^1(\mathbb{R}) \times \mathbb{R}_+ \mapsto \mathbf{L}^1(\mathbb{R})$ such that, for every initial datum

$$u(0,\cdot) \;=\; \overline{u} \in \mathbf{L}^1(\mathbb{R}), \tag{1.2}$$

the trajectory $t \mapsto u(t) = S_t\overline{u}$ is the unique entropy weak solution of the Cauchy problem.

It is well known that, even for smooth initial data, the solution can develop shocks in finite time. Taking an abstract point of view, consider the operator $Au \doteq \frac{\partial}{\partial x} f(u)$ which generates the semigroup. Then there exists data $\overline{u} \in Dom(A)$ in the domain of the generator, such that

$S_\tau \overline{u} \notin Dom(A)$ for some $\tau > 0$. In other words, the domain of the generator is not positively invariant.

To address this issue, the paper [10] introduced a definition of "generalized domain" $\mathcal{D}$ for the operator. This consists of all initial data $\overline{u}$ for which the map $t \mapsto S_t\overline{u}$ is globally Lipschitz continuous. Notice that for the conservation law (1.1) one has

$$\mathbf{L}^1 \cap BV \ \subseteq\ \mathcal{D}.$$

Using the fact that the semigroup is contractive, it is easy to show that this generalized domain is positively invariant. Indeed, the quantity

$$\limsup_{\varepsilon\to 0+} \frac{\big\|S_{t+\varepsilon}\overline{u} - S_t\overline{u}\big\|_{\mathbf{L}^1}}{\varepsilon}$$

is a non-increasing function of t.

Our present aim is to study intermediate domains

$$\mathcal{D}_\alpha \ \subset\ \mathbf{L}^1(\mathbb{R}), \qquad\qquad 0<\alpha<1, \tag{1.3}$$

related to the decay properties of the corresponding trajectories of (1.1). As in [1] we define

$$\mathcal{D}_\alpha \ \doteq\ \Big\{\overline{u}\in \mathbf{L}^1(\mathbb{R})\,;\quad \sup_{0<t<1} t^{-\alpha}\big\|S_t\overline{u}-\overline{u}\|_{\mathbf{L}^1} < +\infty\Big\}. \tag{1.4}$$

We also consider the domains (slightly different from the ones considered in [1])

$$\widetilde{\mathcal{D}}_\alpha \ \doteq\ \Big\{\overline{u}\in \mathbf{L}^1(\mathbb{R})\cap\mathbf{L}^\infty(\mathbb{R})\,;\quad \sup_{0<t<1} t^{1-\alpha}\cdot \text{Tot.Var.}\big\{S_t\overline{u}\big\} < +\infty\Big\}. \tag{1.5}$$

The domains $\mathcal{D}_\alpha$ arise naturally in connection with balance laws:

$$u_t + f(u)_x \ =\ g(t,x). \tag{1.6}$$

Indeed, as shown in [1], one has

Proposition 1.1. *Let $f\in\mathcal{C}^2$ with $f''(u)\geq c>0$ for all $u\in\mathbb{R}$. Consider a compactly supported solution $u=u(t,x)$ of (1.6), and assume that the source term satisfies*

$$\big\|g(t,\cdot)\big\|_{\mathbf{L}^1} \ \leq\ C \qquad\qquad \textit{for all } t\in[0,T]. \tag{1.7}$$

Then for every $0<t\leq T$, one has $u(t,\cdot)\in\mathcal{D}_{1/2}$.

We remark, however, that some of the functions $u(t,\cdot)$ can be unbounded. In particular, they can have infinite total variation.

In the theory of linear analytic semigroups [16, 20], intermediate domains arise naturally as domains of fractional powers of sectorial operators. The faster decay of solutions is usually related to higher Sobolev regularity of the initial data. In particular, this theory applies to semilinear equations of the form

$$u_t - \Delta u \ =\ F(x,u,\nabla u), \qquad\qquad u(0,\cdot)=\overline{u}. \tag{1.8}$$

Under natural assumptions (see [16] for details), this Cauchy problem is well posed provided that the initial datum $\overline{u}$ lies in the domain of some fractional power $(-\Delta)^\alpha$ of the generator.

Our eventual goal is to develop a similar theory of intermediate domains for nonlinear semigroups generated by conservation laws. In particular, we conjecture that the Cauchy problem for a genuinely nonlinear 2×2 hyperbolic system with $\mathbf{L}^\infty$ initial data [4, 15] is well posed within an intermediate domain such as (1.4) or (1.5).

As a first step in this research program, here we focus our attention on the scalar conservation law (1.1), seeking conditions on the initial data $\overline{u} \in \mathbf{L}^1(\mathbb{R})$ that will imply $\overline{u} \in \mathcal{D}_\alpha$ or $\overline{u} \in \widetilde{\mathcal{D}}_\alpha$, respectively.

Assumptions that imply $\overline{u} \in \mathcal{D}_\alpha$ can be readily formulated in terms of fractional Sobolev regularity. On the other hand, conditions that guarantee a faster decay rate of the total variation are more subtle. Here we consider the assumption

($\mathbf{P}_\alpha$) *For every $\lambda \in \,]0,1]$, there exists an open set $V(\lambda) \subset \mathbb{R}$ such that the following holds.*

$$meas\big(V(\lambda)\big) \;\leq\; C\,\lambda^\alpha, \tag{1.9}$$

$$Tot.Var.\big\{\overline{u}\,;\; \mathbb{R} \setminus V(\lambda)\big\} \;\leq\; C\,\lambda^{\alpha-1}, \tag{1.10}$$

for some constant C independent of λ.

Roughly speaking, $\overline{u}$ can have unbounded variation, but most of its variation should be concentrated on a set with small Lebesgue measure. Our main result establishes the implication

$$\overline{u}\ \text{ satisfies } (\mathbf{P}_\alpha) \qquad \Longrightarrow \qquad \overline{u} \in \widetilde{\mathcal{D}}_\alpha \tag{1.11}$$

when $1/2 < \alpha < 1$. On the other hand, a counterexample shows that the above implication fails for $\alpha \leq 1/2$. The proof of (1.11) relies on a structural result for functions satisfying $(\mathbf{P}_\alpha)$, which is of independent interest. Indeed, Theorem 5.1 provides a nonlinear "Fourier-type" decomposition of such functions, in components which oscillate more and more rapidly.

The remainder of the paper is organized as follows. In Section 2 we describe a general class of metric interpolation spaces, for functions defined on a set $\Omega \subseteq \mathbb{R}^N$. This yields a natural way to formulate conditions such as $(\mathbf{P}_\alpha)$, in a general setting.

Section 3 contains some examples. The first one (Fig. 1) shows how to construct an initial datum $\overline{u}$ with unbounded variation, such that $\overline{u} \in \widetilde{\mathcal{D}}_\alpha$, for any given $0 < \alpha < 1$. We then consider initial data consisting of a packet of triangular waves (Fig. 3). By suitably choosing the size and distance of these triangular blocks we show that, if $0 < \alpha \leq 1/2$, then there exists an initial datum $\overline{u}$ that satisfies $(\mathbf{P}_\alpha)$ and yet $\overline{u} \notin \widetilde{\mathcal{D}}_\beta$ for any $\beta \in \,]0,1[$. As stated in Proposition 3.2, the implication (1.11) thus cannot hold for $\alpha \leq 1/2$.

Section 4 is concerned with the intermediate domain $\mathcal{D}_\alpha$. For $0 < \alpha < 1$ we prove that any one of the conditions: (i) $\overline{u}$ lies in the fractional Sobolev space $W^{\alpha,1}(\mathbb{R})$, or (ii) $\overline{u}$ satisfies $(\mathbf{P}_\alpha)$, implies $\overline{u} \in \mathcal{D}_\alpha$. These results are valid for any flux $f \in \mathcal{C}^1$, not necessarily convex.

Section 5 establishes further properties of functions which satisfy $(\mathbf{P}_\alpha)$, proving a useful decomposition result, stated in Theorem 5.1. Finally, in Section 6 we prove our main theorem, showing that for $1/2 < \alpha < 1$ the property $(\mathbf{P}_\alpha)$ implies that $\overline{u} \in \widetilde{\mathcal{D}}_\alpha$. To simplify the exposition, the proofs will first be given for Burgers' equation. In Remark 6.2 we observe that all results remain valid for a conservation law with uniformly convex flux.

For an introduction to the theory of conservation laws we refer to [6, 7, 12, 17]. Results on the decay of solutions to conservation laws in generalized BV spaces can be found in [5, 21]. In addition to genuinely nonlinear conservation laws, several other examples of nonlinear semigroups with regularizing properties are known in the literature, see in particular [2, 3, 11, 22].

2 A family of metric interpolation spaces

Consider an open set $\Omega \subseteq \mathbb{R}^N$ and let X be a Banach space contained in the set $L^0(\Omega)$ of Lebesgue measurable functions $f : \Omega \mapsto \mathbb{R}$. Let $0 < \alpha < 1$ be given. A distance function $d(\cdot,\cdot) : L^0(\Omega) \times L^0(\Omega) \to [0, +\infty]$ can be defined as follows. For any $\lambda \in]0, 1]$, we begin by setting

$$d^\lambda(f,g) \doteq d^\lambda(f-g,0), \tag{2.1}$$

$$\begin{aligned} d^\lambda(f,0) \doteq \inf\Big\{ & C \geq 0\,;\ \text{ there exists } \widetilde{f} \in X \text{ such that} \\ & \|\widetilde{f}\|_X \leq C\,\lambda^{\alpha-1}, \qquad \text{meas}\big\{x \in \Omega\,;\ f(x) \neq \widetilde{f}(x)\big\} \leq C\,\lambda^\alpha \Big\}. \end{aligned} \tag{2.2}$$

Finally, we define

$$d(f,g) \doteq \sup_{0<\lambda\leq 1} d^\lambda(f,g). \tag{2.3}$$

By possibly identifying couples of functions f, g, such that $d(f,g) = 0$, we claim that (2.3) yields a distance on the set of Lebesgue measurable functions $f \in L^0(\Omega)$ for which $d(f,0) < \infty$. This is usually a strictly larger set than X.

Lemma 2.1. *Let $0 < \alpha < 1$ be given, and let $d(\cdot,\cdot)$ be as in (2.1)–(2.3). Then the following properties hold.*

(i) $d(f,g) = d(g,f) \geq 0$.

(ii) If $f \in X$, then $d(f,0) \leq \|f\|_X$.

(iii) $d(f,h) \leq d(f,g) + d(g,h)$.

Proof. 1. Part (i) is trivial. If $f \in X$, choosing $\widetilde{f} = f$ in (2.2), we see that $d^\lambda(f,0) \leq \|f\|_X$ for every $\lambda > 0$. This yields (ii).

2. We now check that each $d^\lambda(\cdot,\cdot)$, $0 < \lambda \leq 1$, satisfies the triangle inequality. Toward this goal, let f_1, f_2 be measurable functions such that

$$d^\lambda(f_i, 0) = C_i, \qquad i = 1, 2.$$

We then need to show that

$$d^\lambda(f_1, f_2) \doteq d^\lambda(f_1 - f_2, 0) \leq C_1 + C_2\,. \tag{2.4}$$

Given $\varepsilon > 0$, by assumption there exist functions $\widetilde{f}_1, \widetilde{f}_2 \in X$ such that

$$\|\widetilde{f}_i\|_X \;\le\; C_i\lambda^{\alpha-1} + \varepsilon, \qquad \text{meas}\Big\{x \in \Omega\,;\; \widetilde{f}_i(x) \neq f_i(x)\Big\} \;\le\; C_i\lambda^{\alpha} + \varepsilon, \qquad i = 1,2. \tag{2.5}$$

Then the function $\widetilde{f}_1 - \widetilde{f}_2$ satisfies

$$\|\widetilde{f}_1 - \widetilde{f}_2\|_X \;\le\; (C_1 + C_2)\lambda^{\alpha-1} + 2\varepsilon,$$

$$\text{meas}\Big\{x \in \Omega\,;\; \widetilde{f}_1(x) - \widetilde{f}_2(x) \neq f_1(x) - f_2(x)\Big\} \;\le\; (C_1 + C_2)\lambda^{\alpha} + 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this proves (2.4).

3. In turn, the triangle inequality (iii) follows from

$$\begin{aligned} d(f-g,0) \;&=\; \sup_{\lambda\in]0,1]} d^\lambda(f-g,0) \;\le\; \sup_{\lambda\in]0,1]} \Big(d^\lambda(f,0) + d^\lambda(g,0)\Big) \\ &\le\; \sup_{\lambda\in]0,1]} d^\lambda(f,0) + \sup_{\lambda\in]0,1]} d^\lambda(g,0) \;=\; d(f,0) + d(g,0). \end{aligned}$$

□

Remark 2.2. One should keep in mind that, in general, the balls $\{g \in L^0(\Omega)\,;\;\; d(g,f) \le r\}$ are not convex. Moreover, the function $f \mapsto d(f,0)$ is not a norm.

In connection with $(\mathbf{P}_\alpha)$, for $0 < \alpha < 1$ we consider the distances $d^\lambda(f,g)$ as in (2.1), where now

$$\begin{aligned} d^\lambda(f,0) \;\doteq\; \inf\Big\{&C \ge 0\,;\;\; \text{there exists } \widetilde{f} \in \mathbf{L}^1(\mathbb{R}) \text{ such that} \\ &\text{Tot.Var.}\{\widetilde{f}\} \le C\,\lambda^{\alpha-1}, \quad \text{meas}\{x \in \mathbb{R}\,;\; f(x) \neq \widetilde{f}(x)\} \le C\,\lambda^{\alpha}\Big\}. \end{aligned} \tag{2.6}$$

Finally, given $\overline{u} \in \mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^\infty(\mathbb{R})$, we define

$$\|\overline{u}\|_{\mathcal{P}_\alpha} \;\doteq\; \sup_{0<\lambda\le 1} d^\lambda(\overline{u},0) \tag{2.7}$$

and write $\overline{u} \in \mathcal{P}_\alpha$ if $\|\overline{u}\|_{\mathcal{P}_\alpha} < +\infty$. Notice that this holds provided that $\overline{u}$ satisfies the condition $(\mathbf{P}_\alpha)$. Throughout the following, we shall use $\|\overline{u}\|_{\mathcal{P}_\alpha}$ as a convenient notation. However, as already pointed out in Remark 2.2, one should be aware that $\|\cdot\|_{\mathcal{P}_\alpha}$ is not a norm.

3 Examples

We present here some examples, to motivate the results proved in later sections. We consider Burgers' equation

$$u_t + \left(\frac{u^2}{2}\right)_x \;=\; 0. \tag{3.1}$$

Throughout the following, we use the semigroup notation $t \mapsto S_t\overline{u}$ to denote the solution of (3.1) with initial data (1.2).

Example 3.1. Fix $\beta > 0$ and consider the decreasing sequence of points $x_n = n^{-\beta}$, $n \geq 1$. As shown in Fig. 1, define the piecewise affine function

$$\overline{u}(x) \;=\; \begin{cases} 0 & \text{if} \quad x \notin [0,1]\,, \\ \dfrac{x - x_{n+1}}{x_n - x_{n+1}} & \text{if} \quad x_n < x < x_{n-1}\,. \end{cases} \tag{3.2}$$

We claim that this initial data lies in some of the subdomains $\widetilde{\mathcal{D}}_\alpha$, depending on the exponent β. Indeed, fix a time $t \in]0,1]$. Consider the position $x_k(t)$ of the shock which is initially located at x_k. By Oleinik's inequality, the total variation of the solution $u(t,\cdot)$ can be estimated by

$$\text{Tot.Var.}\big\{u(t,\cdot)\,;\;[0, x_k(t)]\big\} \;\leq\; 2\frac{x_k(t)}{t}\,. \tag{3.3}$$

On the other hand, for $x > x_k(t)$, still by Oleinik's estimates we have

$$\text{Tot.Var.}\big\{u(t,\cdot)\,;\;[x_k(t), x_1(t)]\big\} \;\leq\; 2\frac{x_1(t) - x_k(t)}{(x_{k-1} - x_k) + t}\,. \tag{3.4}$$

Observing that

$$x_k(t) \;\leq\; x_k + t, \qquad\qquad x_1(t) - x_k(t) \;\leq\; 1 + t,$$

from (3.3)-(3.4) we deduce

$$\text{Tot.Var.}\big\{u(t,\cdot)\big\} \;\leq\; 2\frac{x_k + t}{t} + 2\frac{1+t}{(x_{k-1} - x_k) + t} \;\leq\; 4 + 2\frac{x_k}{t} + \frac{2}{(x_{k-1} - x_k) + t}\,. \tag{3.5}$$

Since we are assuming $x_k = k^{-\beta}$, the previous estimate yields

$$\text{Tot.Var.}\big\{u(t,\cdot)\big\} \;\leq\; 4 + \frac{2k^{-\beta}}{t} + \frac{2}{\beta k^{-\beta-1} + t}\,.$$

Here $k \geq 1$ is arbitrary. Choosing $k \approx t^{-\gamma}$, we obtain

$$\text{Tot.Var.}\big\{u(t,\cdot)\big\} \;\leq\; \mathcal{O}(1)\cdot\left(\frac{t^{\beta\gamma}}{t} + \frac{1}{t^{\gamma(\beta+1)} + t}\right) \;=\; \mathcal{O}(1)\cdot\Big(t^{\beta\gamma - 1} + t^{-\gamma(\beta+1)}\Big).$$

Here and throughout the sequel, the Landau symbol $\mathcal{O}(1)$ denotes a uniformly bounded quantity. The two terms on the right hand side have similar magnitude if $\gamma = (1+2\beta)^{-1}$. With this choice, we obtain

$$\text{Tot.Var.}\big\{u(t,\cdot)\big\} \;\leq\; \mathcal{O}(1)\cdot t^{-\frac{\beta+1}{2\beta+1}},$$

hence

$$\overline{u} \in \widetilde{\mathcal{D}}_\alpha, \qquad \text{with} \qquad \alpha \;=\; 1 - \frac{\beta+1}{2\beta+1} \;=\; \frac{\beta}{2\beta+1}.$$

In the next examples we consider initial data consisting of one or more triangular blocks. As shown in Fig. 2, left, the most elementary case is

$$w(0,x) \;=\; \overline{w}(x) \;=\; \begin{cases} h - \frac{2h}{\ell}\cdot|x - \ell/2| & \text{if} \;\; x \in [0,\ell], \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

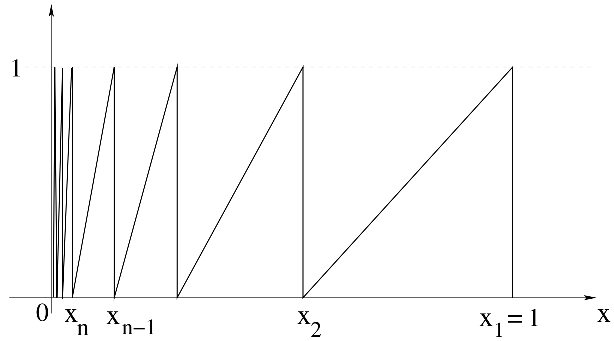

Figure 1: The initial data considered in Example 3.1.

At time $t = \ell/(2h)$, a shock is created in the solution at the point $\ell > 0$. Characteristics originating from points $0 < x < \ell/2$ start impinging on the shock, and the solution has a right triangle shape:

$$w(t,x) = \begin{cases} \frac{2hx}{2ht+\ell} & \text{if } x \in [0, L(t)], \\ 0 & \text{otherwise} \end{cases} \tag{3.7}$$

Conservation of mass implies that the shock at time $t \geq \ell/(2h)$ is located at

$$L(t) = \sqrt{\frac{\ell}{2}\,(2\,h\,t + \ell)}. \tag{3.8}$$

Always for $t \geq \ell/(2h)$, we thus have

$$\text{Tot.Var.}\{S_t \overline{w}\} = 2\,p(t), \qquad p(t) \doteq h\sqrt{\frac{2\ell}{2ht+\ell}}. \tag{3.9}$$

We notice for later use that the (decreasing) function $p(t)$ satisfies the lower bound

$$p(t) \geq \sqrt{\frac{h\,\ell}{2t}} \qquad \text{for all} \quad t \geq \ell/2h. \tag{3.10}$$

and that the (increasing) function $L(t)$ satisfies the upper bound

$$L(t) \leq \sqrt{2\,\ell\,h\,t} \qquad \text{for all} \quad t \geq \ell/2h. \tag{3.11}$$

We also consider initial data containing packets of triangular blocks, shifted by different amounts so they do not overlap with each other. See Fig. 2, right.

In the following proposition we denote by $\mathcal{C}^{0,\sigma}(\mathbb{R})$ the space of Holder functions $\overline{u} : \mathbb{R} \to \mathbb{R}$ with exponent $0 < \sigma < 1$, equipped with the norm

$$\|\overline{u}\|_{\mathcal{C}^{0,\sigma}} \doteq \|\overline{u}\|_{\mathcal{C}^0} + |\overline{u}|_{\mathcal{C}^{0,\sigma}}$$

$$|\overline{u}|_{\mathcal{C}^{0,\sigma}} \doteq \sup_{x<y} \frac{|\overline{u}(y) - \overline{u}(x)|}{|y-x|^\sigma}.$$

Proposition 3.2. *There exists a compactly supported function $\overline{u} : \mathbb{R} \to \mathbb{R}$ with the following properties:*

1. $\overline{u} \in \mathcal{P}_\alpha$ *for every* $0 < \alpha \leq 1/2$;

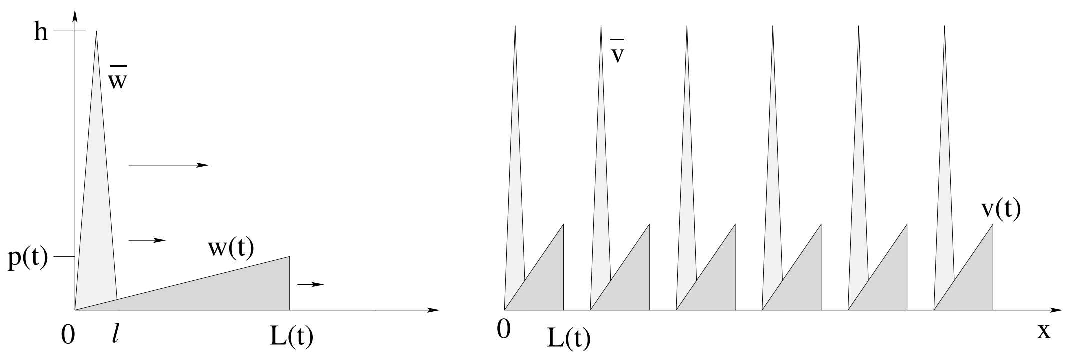

Figure 2: Left: the elementary solution to Burgers' equation considered at (3.6)-(3.7). Right: the superposition of several shifted copies of the same solution.

2. $\overline{u} \in \mathcal{C}^{0,\sigma}(\mathbb{R})$ *for every* $0 < \sigma < 1$*;*

3. $\overline{u} \notin \widetilde{\mathcal{D}}_\beta$ *for any* $0 < \beta < 1$*. Namely:*

$$\limsup_{t \to 0+} t^{1-\beta} \cdot Tot.Var.\{S_t \overline{u}\} = +\infty \qquad \text{for all } 0 < \beta < 1. \tag{3.12}$$

Remark 3.3. The function $\overline{u}$ constructed in Proposition 3.2 does not belong to any $\mathcal{P}_\alpha$ if $1/2 < \alpha < 1$. This suggests that $\alpha = 1/2$ is a critical exponent for the decay of solution with initial data in $\mathcal{P}_\alpha$. In fact, this surprising behavior will be later confirmed by Theorem 6.1.

Remark 3.4. By part 2. of Proposition 3.2 and by the embedding $\mathcal{C}^{0,\sigma}(\mathbb{R}) \hookrightarrow W^{s,p}_{loc}(\mathbb{R})$ for every $0 < s < \sigma < 1$, $p \geq 1$, we obtain that

$$W^{s,p}(\mathbb{R}) \not\subset \widetilde{\mathcal{D}}_\beta \qquad \text{for every } 0 < s < 1, \ \ 1 \leq p < \infty \text{ and } 0 < \beta < 1.$$

On the other hand, the inclusion $W^{\alpha,1}(\mathbb{R}) \subset \mathcal{D}_\alpha$ does hold for every $0 < \alpha < 1$, as proved in Proposition 4.2.

The proof of Proposition 3.2 is based on the following lemma.

Lemma 3.5. *For every fixed* $t \in (0, 1)$*, there exists a function* $\widehat{u} : \mathbb{R} \to \mathbb{R}$ *satisfying*

$$\begin{cases} \widehat{u}(x) \in [0,1] & \text{if } x \in [0,1], \\ \widehat{u}(x) = 0 & \text{if } x \notin [0,1], \end{cases} \tag{3.13}$$

$$\|\widehat{u}\|_{\mathcal{P}_\alpha} \leq C_0\,, \qquad \text{for all } 0 < \alpha \leq 1/2, \qquad\qquad Tot.Var.\{S_t \widehat{u}\} \geq \frac{1}{C_0\, t}\,. \tag{3.14}$$

$$\|\widehat{u}\|_{\mathcal{C}^{0,\sigma}} \leq C_\sigma \doteq \left(e^{2^{\left(\frac{1}{1-\sigma}\right)}}\right)^{(1-\sigma)e^{-1}} \qquad \text{for all } \sigma \in]0,1[. \tag{3.15}$$

Here C_0 *is a constant independent of* t*.*

We postpone the proof of the lemma, and begin by showing that it implies the previous proposition.

Proof of Proposition 3.2. Let $(t_j)_{j\geq 1}$ be a sequence decreasing to 0 sufficiently fast (to be specified later). Let

$$x_0 = 0, \quad x_j \doteq \sum_{k=0}^{j-1} 2\cdot 2^{-k}, \qquad I_j \doteq [x_j, x_{j+1}], \qquad j = 0,1,2\ldots$$

Let $\widehat{u}_j$ be a function satisfying the properties (3.13)-(3.14) in Lemma 3.5 for $t = t_j$. Consider the rescaled functions

$$\overline{u}_j(x) \;\doteq\; 2^{-j}\widehat{u}_j\big(2^j(x-x_j)\big), \qquad j = 0,1,2\ldots$$

By Lemma 3.5, the corresponding rescaled solution of Burgers' equation satisfies

$$\text{Tot.Var.}\{S_{t_j}\overline{u}_j\} \;=\; 2^{-j}\,\text{Tot.Var.}\{S_{t_j}\widehat{u}_j\} \;\geq\; \frac{1}{C_0\,2^j\,t_j}\,.$$

Define the initial data

$$\overline{u} \;=\; \sum_{j=0}^{\infty}\overline{u}_j. \tag{3.16}$$

Notice that

$$\text{supp}\,\overline{u}_j(t,\cdot) \;\subseteq\; [x_j, x_j + 2^{-j}(1+t)] \;\subset\; I_j \qquad\qquad \text{for} \;\; t\in[0,1].$$

In particular for every $i \neq j$ and $t < 1$ the supports of $S_t\overline{u}_i$ and $S_t\overline{u}_j$ remain disjoint. Choosing $t_j < 2^{-j}$ we thus obtain

$$\big(S_{t_j}\overline{u}\big)(x) \;=\; \big(S_{t_j}\overline{u}_j\big)(x) \qquad \text{for all } x\in I_j\,.$$

Since every $\widehat{u}_j$ satisfies (3.14), for $0 < \alpha \leq 1/2$, by the triangle inequality it now follows

$$\|\overline{u}\|_{\mathcal{P}_\alpha} \;\leq\; \sum_{j=0}^{\infty}\|\overline{u}_j\|_{\mathcal{P}_\alpha} \;\leq\; C_0\sum_{j=0}^{\infty}2^{-j} \;<\; +\infty.$$

This implies that $\overline{u}$ satisfies assumption 1. of Proposition 3.2. By (3.15) of Lemma 3.5, we infer that

$$\|\overline{u}_j\|_{\mathcal{C}^{0,\sigma}} \;\leq\; 2^{-(1-\sigma)j}\,\|\widehat{u}_j\|_{\mathcal{C}^{0,\sigma}} \;\leq\; C_\sigma\,2^{-(1-\sigma)j}.$$

Therefore since the supports of $\overline{u}_j$ are all disjoint, the function $\overline{u}$ belongs to all the Hölder spaces $\mathcal{C}^{0,\sigma}$ for $0 < \sigma < 1$. Finally, choosing for example $t_j = \exp(-2^j)$, for any $0 < \beta < 1$ we obtain

$$\begin{aligned}\limsup_{t\to 0+} t^\beta\,\text{Tot.Var.}\{S_t\overline{u}\} \;&\geq\; \lim_{j\to+\infty} t_j^\beta\,\text{Tot.Var.}\{S_{t_j}\overline{u}\} \;\geq\; \lim_{j\to+\infty} t_j^\beta\,\text{Tot.Var.}\{S_{t_j}\overline{u}_j\}\\ &\geq\; \frac{1}{C_0\,2^j\,t_j^{1-\beta}} \;\to\; +\infty,\end{aligned}$$

completing the proof of 3. of Proposition 3.2. $\square$

Proof of Lemma 3.5.

1. Let $t > 0$ be a fixed positive time. Given a sequence of positive numbers $\{\ell_k\}_k$ satisfying $\ell_k \leq 2^{-k}$ (to be chosen later), and an integer $k \geq 1$, we construct a packet of triangular waves by setting

$$v_k \;=\; \sum_{j=1}^{N_k} w_k^j\,, \tag{3.17}$$

where $w_k^j = \overline{w}_k(x - j\,L_k)$ are translations by $j\,L_k > 0$ of elementary triangular blocks as in (3.6), with width ℓ_k and height

$$h_k \doteq 2^k \ell_k \leq 1\,. \tag{3.18}$$

The distance L_k between the supports of two blocks is chosen large enough so that the supports of the corresponding solutions remain disjoint up to the given time t. By (3.11), it is sufficient to separate these elementary blocks by a distance $L_k \doteq \sqrt{2h_k\,\ell_k\,t}$, see Figure 3. With this choice, the support of v_k is contained inside an interval I_k with length

$$\mathrm{meas}(I_k) \;\leq\; N_k L_k \;=\; \sqrt{2t}\,2^{k/2} N_k \ell_k\,. \tag{3.19}$$

Moreover, since the elementary blocks do not interact with each other up to time t, assuming $\ell_k/(2h_k) \leq t$, by (3.10), one has

$$\text{Tot.Var.}\{S_t v_k\} \;=\; 2N_k p_k(t) \;\geq\; \sqrt{2}N_k \cdot \sqrt{\frac{h_k\ell_k}{t}} \;=\; \frac{\sqrt{2}\cdot 2^{k/2}\,\ell_k}{\sqrt{t}}\,N_k\,. \tag{3.20}$$

We now choose

$$\ell_k \doteq 2^{-1}\cdot 2^{-k/2}\cdot k^{-k}, \qquad\qquad N_k \doteq 2^{-1}\cdot 2^{-k/2}\cdot \ell_k^{-1} = k^k \tag{3.21}$$

and notice that, by (3.18), this implies

$$\text{Tot.Var.}\{v_k\} \;=\; 2N_k h_k \;=\; 2^{k/2}.$$

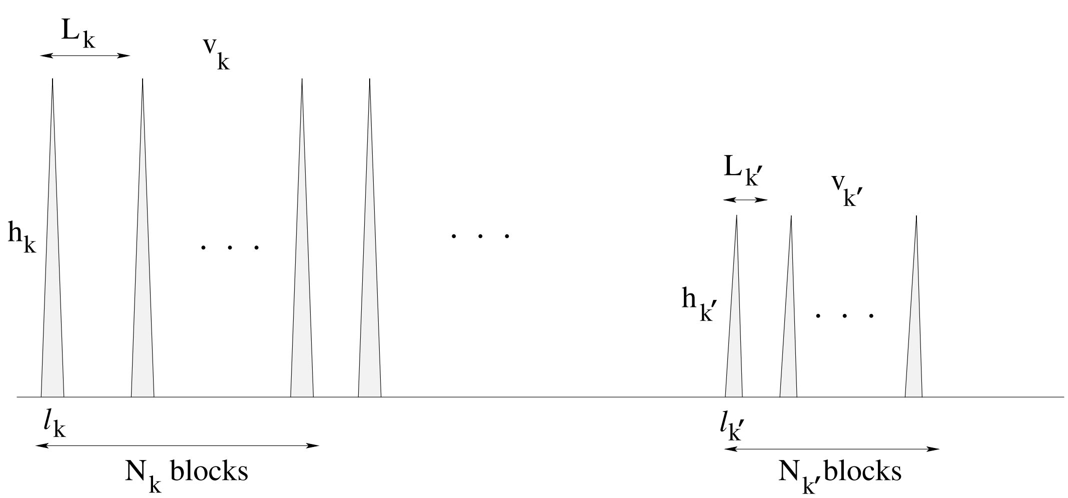

Figure 3: A family of triangular wave packets.

2. We put next to each other all the wave packets v_k, for $k_1 \le k \le k_2$, where $k_1, k_2 \in \mathbb{N}$ will be chosen later (see Figure 3), and define

$$\widehat{u} \doteq \sum_{k=k_1}^{k_2} v_k\,. \tag{3.22}$$

Recalling (2.6)-(2.7), we now show that, for $\alpha = \frac{1}{2}$, the above construction yields a uniform bound

$$\|\widehat{u}\|_{\mathcal{P}_\alpha} \le C, \tag{3.23}$$

with a constant C independent of k_1, k_2. As a consequence, the same bound holds for $\alpha \in]0, \frac{1}{2}[$.

To prove (3.23), let $\lambda \in]0,1]$ and choose $\bar{k}$ such that $\lambda \in]2^{-\bar{k}}, 2^{-\bar{k}+1}]$. Set $V(\lambda)$ in (1.9)-(1.10) to be the support of $\sum_{k=\bar{k}}^{k_2} v_k$. By (3.21) the following estimates hold:

$$\mathrm{meas}(V(\lambda)) \le \sum_{k=\bar{k}}^{k_2} N_k \ell_k = \frac{1}{2}\sum_{k=\bar{k}}^{k_2} 2^{-k/2} < \frac{1}{2}\frac{\sqrt{2}}{\sqrt{2}-1}\cdot 2^{-\bar{k}/2} < C_0\,\lambda^{1/2},$$

$$\mathrm{Tot.Var.}\left\{\sum_{k=k_1}^{\bar{k}-1} v_k;\, \mathbb{R}\right\} \le \sum_{k=k_1}^{\bar{k}-1} N_k 2^k \ell_k < \frac{1}{2}\frac{2^{\bar{k}/2}}{\sqrt{2}-1} < C_0\,\lambda^{1/2}.$$

This proves (3.23). A more general result will be obtained in Proposition 5.3, to which we refer for additional details.

3. In view of (3.19), (3.21), the support of $\widehat{u}$ is contained in an interval I whose length is

$$\mathrm{meas}(I) \le \sum_{k_1}^{k_2} \mathrm{meas}(I_k) = \sum_{k_1}^{k_2} \sqrt{2t}2^{k/2} N_k \ell_k = \sqrt{\frac{t}{2}}(k_2 - k_1 + 1). \tag{3.24}$$

We now compute the total variation of the corresponding solution $S_t\widehat{u}$ at a given time $t > 0$. If $t \ge 2^{-1}\cdot 2^{-k_1}$, i.e., if $k_1 \ge \log_2(1/2t)$, then at time t all the elementary solutions appearing in the blocks v_k, $k \ge k_1$, have a right triangle shape and we can use the estimate (3.20). Since these blocks do not interact with each other, we have

$$\mathrm{Tot.Var.}\{S_t\widehat{u}\} = \sum_{k_1}^{k_2} \mathrm{Tot.Var.}\{S_t v_k\} \ge \sqrt{\frac{2}{t}}\sum_{k_1}^{k_2} 2^{k/2} N_k \ell_k = \sqrt{\frac{1}{2t}}(k_2 - k_1 + 1). \tag{3.25}$$

Using the notation $\lceil a \rceil$ to denote the smallest integer $\ge a$, we now choose $k_1 = \lceil \log_2(1/t) \rceil$ and $k_2 = k_1 - 2 + \lceil \sqrt{\frac{2}{t}} \rceil$. By (3.24) we deduce that the support of $\widehat{u}$ is contained in the interval I whose length is

$$\mathrm{meas}(I) \le \sqrt{\frac{t}{2}}(k_2 - k_1 + 1) = \sqrt{\frac{t}{2}}\left(\left\lceil \sqrt{\frac{2}{t}} \right\rceil - 1\right) \le 1.$$

By (3.25), $S_t\widehat{u}$ has total variation

$$\mathrm{Tot.Var.}\{S_t\widehat{u}\} \ge \sqrt{\frac{1}{2t}}(k_2 - k_1 + 1) = \sqrt{\frac{1}{2t}}\left(\left\lceil \sqrt{\frac{2}{t}} \right\rceil - 1\right) \ge \sqrt{\frac{1}{2t}}\left\lceil \frac{\sqrt{2}-1}{\sqrt{t}} \right\rceil \ge \frac{1}{c}\cdot\frac{1}{t}$$

where $c > 0$ is an absolute constant.

Finally, by (3.18), (3.21), the function $\widehat{u}$ satisfies

$$\sup_{x<y} \frac{|\widehat{u}(y) - \widehat{u}(x)|}{|y-x|^{\sigma}} \leq \sup_{k} \left(k^{-(1-\sigma)k} \cdot 2^k\right) \leq \left(e^{2^{\frac{1}{1-\sigma}}}\right)^{(1-\sigma)e^{-1}},$$

since the function $k \mapsto k^{-(1-\sigma)k} \cdot 2^k$ attains its maximum in $\mathbb{R}$ at $k = 2^{\frac{1}{1-\sigma}} e^{-1}$. □

4 The intermediate domains $\mathcal{D}_\alpha$

We consider entropy solutions to the scalar conservation law (1.1) and let $S : \mathbf{L}^1(\mathbb{R}) \times \mathbb{R}_+ \to \mathbf{L}^1(\mathbb{R})$ be the corresponding semigroup of entropy weak solutions. Notice that in this section the convexity assumption of the flux f is not necessary.

The goal of this section is to study the subdomains $\mathcal{D}_\alpha \subset \mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^\infty(\mathbb{R})$ defined by

$$\mathcal{D}_\alpha \doteq \left\{ \overline{u} \in \mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^\infty(\mathbb{R}) \ ; \quad \sup_{0<t\leq 1} t^{-\alpha} \|S_t\overline{u} - \overline{u}\|_{\mathbf{L}^1} < +\infty \right\}. \tag{4.1}$$

Since the semigroup S_t is nonlinear, the domains $\mathcal{D}_\alpha$ are not vector spaces. However, we can ask if they contain some classical linear spaces, such as fractional Sobolev spaces.

Let $1 \leq p < +\infty$ and $0 < \alpha \leq 1$ be given, together with an open set $\Omega \subset \mathbb{R}$. The fractional Sobolev space $W^{\alpha,p}(\Omega)$ is defined by (see for example [13])

$$W^{\alpha,p}(\Omega) \ \doteq \ \left\{ u \in \mathbf{L}^p(\Omega) \ ; \quad \frac{|u(x) - u(y)|}{|x-y|^{\frac{1}{p}+\alpha}} \in \mathbf{L}^p(\Omega \times \Omega) \right\}, \tag{4.2}$$

equipped with the norm

$$\|u\|_{W^{\alpha,p}} \ \doteq \ \|u\|_{\mathbf{L}^p} + \left(\int_{\Omega\times\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{1+\alpha p}} \, dxdy \right)^{\frac{1}{p}}. \tag{4.3}$$

As it is well known, functions in Sobolev spaces can be approximated by smooth functions by taking mollifications. Let $\eta : \mathbb{R} \mapsto [0,1]$ be a symmetric, $\mathcal{C}^\infty$ mollifier with compact support, so that

$$\begin{cases} \eta(s) = \eta(-s) \ \in \ [0,1] & \text{if} \quad s \in [-1,1], \\ \eta(s) \ = \ 0 & \text{if} \quad |s| \geq 1, \\ \big|\eta'(s)\big| \leq 2 & \text{for all } s \in \mathbb{R}, \end{cases} \qquad \int \eta(s)\, ds = 1. \tag{4.4}$$

Here and in the sequel, the prime $'$ denotes a derivative. For $h > 0$, define the rescaled kernels by setting

$$\eta_h(s) \ = \ \frac{1}{h}\eta\left(\frac{s}{h}\right). \tag{4.5}$$

For $u \in \mathbf{L}^1_{loc}$, consider the convolution $u_h = u \star \eta_h$. The rate of convergence of these mollifications depends on the regularity properties of the function u.

Lemma 4.1. *Assume $u \in W^{\alpha,1}(\mathbb{R})$ for some $0 < \alpha \leq 1$. Then, for every $h > 0$, the convolution $u_h = u \star \eta_h$ satisfies*

$$\|u - u_h\|_{\mathbf{L}^1} \leq \|u\|_{W^{\alpha,1}} \cdot h^\alpha, \qquad \|u_h'\|_{\mathbf{L}^1} \leq C\|u\|_{W^{\alpha,1}} \cdot \frac{1}{h^{1-\alpha}}, \tag{4.6}$$

for some constant C independent of u.

Proof. A direct computation yields

$$\begin{aligned}
\int |u(x) - u_h(x)|\, dx &\leq \iint |u(x) - u(y)|\eta_h(y-x)\, dxdy \\
&= \int \frac{1}{h}\eta(s/h) \int |u(x+s) - u(x)|\, dxds \\
&\leq \int_{-h}^{h} \left(\frac{1}{|s|} \int \big|u(x+s) - u(x)\big|\, dx \right) ds \\
&\leq h^\alpha \iint \frac{|u(x+s) - u(x)|}{|s|^{1+\alpha}}\, dxds \\
&= \|u\|_{W^{\alpha,1}} \cdot h^\alpha.
\end{aligned}$$

Moreover, the total variation of u_h is bounded by

$$\begin{aligned}
\int_{\mathbb{R}} |u_h'(x)|\, dx &= \int \left| \int u(y)\eta_h'(y-x)dy \right| dx \\
&= \int \Big| \int u(x+s)\eta_h'(s)ds \Big| dx \\
&= \int \frac{1}{h^2} \Big| \int u(x+s)\eta'(s/h)ds \Big| dx \\
&= \int \frac{1}{h^2} \Big| \int_0^h \eta'(s/h)(u(x+s) - u(x-s))ds \Big| dx \\
&\leq C \int \frac{1}{h^{1-\alpha}} \int_0^h \frac{|u(x+s) - u(x-s)|}{s^{1+\alpha}} dsdx \\
&\leq C\, \|u\|_{W^{\alpha,1}} \frac{1}{h^{1-\alpha}},
\end{aligned}$$

Notice that the constant C depends only on the mollifying kernel η. $\square$

Proposition 4.2. *Let (1.1) be any conservation law with continuously differentiable flux. For every $\alpha \in]0,1]$ we have the inclusion $\mathbf{L}^\infty(\mathbb{R}) \cap W^{\alpha,1}(\mathbb{R}) \subseteq \mathcal{D}_\alpha$.*

Proof. Let $\overline{u} \in \mathbf{L}^\infty(\mathbb{R}) \cap W^{\alpha,1}(\mathbb{R})$ and consider the mollifications $u_h \doteq \eta_h \star \overline{u}$. By Lemma 4.1 it follows

$$\|S_h u_h - u_h\|_{\mathbf{L}^1} \leq \|f'(u_h)\|_{\mathbf{L}^\infty}\, h \cdot \text{Tot.Var.}\{u_h\} \leq C\, \|f'\|_{\mathbf{L}^\infty} \|\overline{u}\|_{W^{\alpha,1}} h^\alpha,$$

where the $\mathbf{L}^\infty$ norm of f' is taken on the interval $[-\|\overline{u}\|_{\mathbf{L}^\infty}, \|\overline{u}\|_{\mathbf{L}^\infty}]$. Therefore

$$\begin{aligned}
\|S_h\overline{u} - \overline{u}\|_{\mathbf{L}^1} &\leq \|S_h\overline{u} - S_h u_h\|_{\mathbf{L}^1} + \|S_h u_h - u_h\|_{\mathbf{L}^1} + \|u_h - \overline{u}\|_{\mathbf{L}^1} \\
&\leq (2 + C\, \|f'\|_{\mathbf{L}^\infty})\, \|\overline{u}\|_{W^{\alpha,1}} h^\alpha.
\end{aligned}$$

$\square$

The second result in this section is formulated in terms of the property $(\mathbf{P}_\alpha)$.

Proposition 4.3. *Let (1.1) be a conservation law with continuously differentiable flux. For any $0 < \alpha < 1$, if $\overline{u} \in \mathbf{L}^\infty(\mathbb{R})$ satisfies $(\mathbf{P}_\alpha)$, then $\overline{u} \in \mathcal{D}_\alpha$.*

Proof. Let $\overline{u}$ satisfy $(\mathbf{P}_\alpha)$. Given $t \in]0,1]$, set $\lambda = t$ and let $V(\lambda) \subset \mathbb{R}$ be an open set satisfying (1.9)-(1.10). Observing that this open set $V(t)$ is a countable union of disjoint open intervals

$$V(t) \;=\; \bigcup_{k\geq 1}]a_j, b_k[\,,$$

we define a new function $\overline{v}$ by replacing $\overline{u}$ with an affine function on each interval $[a_j, b_j]$. Namely,

$$\overline{v}(x) \;=\; \begin{cases} \overline{u}(x) & \text{if} \quad x \notin \cup_k [a_k, b_k], \\ \dfrac{(b_j - x)\overline{u}(a_j) + (x - a_j)\overline{u}(b_j)}{b_j - a_j} & \text{if} \quad x \in [a_j, b_j]\,. \end{cases}$$

This implies

$$\text{Tot.Var.}\{\overline{v}\} \;\leq\; \text{Tot.Var.}\bigl\{\overline{u}\,;\; \mathbb{R}\setminus V(t)\bigr\} \;\leq\; C\,t^{\alpha-1},$$
$$\|\overline{v} - \overline{u}\|_{\mathbf{L}^1} \;\leq\; 2\|\overline{u}\|_{\mathbf{L}^\infty} \cdot \text{meas}(V(t)) \;\leq\; 2\|\overline{u}\|_{\mathbf{L}^\infty} \cdot Ct^\alpha.$$

We thus obtain

$$\begin{aligned} \|S_t\overline{u} - \overline{u}\|_{\mathbf{L}^1} \;&\leq\; \|S_t\overline{u} - S_t\overline{v}\|_{\mathbf{L}^1} + \|S_t\overline{v} - \overline{v}\|_{\mathbf{L}^1} + \|\overline{u} - \overline{v}\|_{\mathbf{L}^1} \\ &\leq\; \|\overline{v} - \overline{u}\|_{\mathbf{L}^1} + t\cdot\|f'\|_{\mathbf{L}^\infty} \cdot \text{Tot.Var.}\{\overline{v}\} + \|\overline{v} - \overline{u}\|_{\mathbf{L}^1} \\ &\leq\; 4\,\|\overline{u}\|_{\mathbf{L}^\infty}\, C\,t^\alpha + \|f'\|_{\mathbf{L}^\infty}\, Ct^\alpha, \end{aligned}$$

where the $\mathbf{L}^\infty$ norm of f' is taken on the interval $[-\|\overline{u}\|_{\mathbf{L}^\infty}, \|\overline{u}\|_{\mathbf{L}^\infty}]$. Since the same constant C is valid for all $t \in]0,1]$, this proves that $\overline{u} \in \mathcal{D}_\alpha$. □

5 A decomposition property for functions $\overline{u} \in \mathcal{P}_\alpha$

In this section we study properties of functions that lie in the metric space $\mathcal{P}_\alpha$ introduced at (2.7). These are functions that satisfy the property $(\mathbf{P}_\alpha)$ at (1.9)-(1.10). Our main result provides a decomposition of a function $\overline{u} \in \mathcal{P}_\alpha$, as the sum of countably many components with different degrees of regularity.

Theorem 5.1. *Let $\bar{u} : \mathbb{R} \mapsto \mathbb{R}$ be a measurable function and let $0 < \alpha < 1$ be given. Then $\overline{u} \in \mathcal{P}_\alpha$ if and only if it can be decomposed as*

$$\overline{u}(x) \;=\; \sum_{k=0}^\infty v_k(x) \qquad \textit{for a.e. } x \in \mathbb{R}, \tag{5.1}$$

where the v_k satisfy the following properties. For some constant $C = \mathcal{O}(1) \cdot \|\overline{u}\|_{\mathcal{P}_\alpha}$ one has

(i) - bounds on the support and on the total variation:

$$\text{Tot.Var.}\{v_0\} \;\leq\; C, \tag{5.2}$$

$$\text{Tot.Var.}\{v_k\} \;\leq\; C\cdot 2^{(1-\alpha)k}, \qquad \text{meas}\bigl(\{v_k \neq 0\}\bigr) \;\leq\; C\cdot 2^{-\alpha k}, \qquad \textit{for all } \; k \geq 1. \tag{5.3}$$

(ii) - one-sided Lipschitz bound:

$$v_k(x_2) - v_k(x_1) \;\leq\; 2^k \cdot (x_2 - x_1) \qquad \textit{for all } x_1 < x_2\,. \tag{5.4}$$

(iii) - a further decomposition:

For each $k \geq 1$ we can further decompose

$$v_k \;=\; \sum_{p=1}^{\infty} v_k^p$$

so that the following conditions hold: the functions v_k^p satisfy (5.4)*, their supports have disjoint interiors, and setting $\ell_k^p \doteq meas(\mathrm{supp}\, v_k^p)$ it holds*

$$\left|v_k^p(x)\right| \;\leq\; h_k^p \;\doteq\; 2^k \ell_k^p, \qquad \textit{for all } \; x \in \mathrm{supp}\, v_k^p\,,$$

and

$$\sum_{p\geq 1} \ell_k^p \;\leq\; C \cdot 2^{-\alpha k}. \tag{5.5}$$

The proof of Theorem 5.1 will be achieved in three steps. We first show in Lemma 5.3 that the existence of a decomposition as in (5.1) which satisfies property **(i)** is equivalent to the statement that $\overline{u} \in \mathcal{P}_\alpha$. Next, in Lemma 5.4 we show that this decomposition can be refined so to satisfy also property **(ii)**, still with some constant C of the same order of $\|\overline{u}\|_{\mathcal{P}_\alpha}$. Finally, Lemma 5.5 shows that **(iii)** is an easy consequence of **(i)** and **(ii)**

Remark 5.2. One can estimate the $\mathbf{L}^p$ norm of $\overline{u} \in \mathcal{P}_\alpha$, for $1 \leq p < +\infty$, by

$$\|\overline{u}\|_{\mathbf{L}^p} \;\leq\; \sum_{k=0}^{+\infty} \|v_k\|_{\mathbf{L}^p} \;\leq\; \sum_{k=0}^{+\infty} \|v_k\|_{\mathbf{L}^\infty} \mathrm{meas}(\{v_k \neq 0\})^{\frac{1}{p}} \leq\; \frac{C}{2} \cdot C^{1/p} \cdot \sum_{k=0}^{+\infty} 2^{k(1-\alpha-\alpha p^{-1})},$$

where $C = \mathcal{O}(1) \cdot \|\overline{u}\|_{\mathcal{P}_\alpha}$ is the same constant as in Theorem 5.1. If $p < \frac{\alpha}{1-\alpha}$, we thus have the embedding $\mathcal{P}_\alpha \hookrightarrow \mathbf{L}^p_{loc}$. Indeed, for every compact set $K \subset \mathbb{R}$, there holds

$$\|\overline{u}\|_{\mathbf{L}^p(K)} \;\leq\; c(K) \cdot \frac{1}{1 - 2^{(1-\alpha-\alpha p^{-1})}} \cdot \|\overline{u}\|_{\mathcal{P}_\alpha}^{1+\frac{1}{p}}\,.$$

In particular $\mathcal{P}_\alpha \hookrightarrow \mathbf{L}^1_{loc}$ if $\alpha > 1/2$. Notice that this is consistent with the scaling property

$$\|u\|_{\mathcal{P}_\alpha} \;=\; \|u_\mu\|_{\mathcal{P}_\alpha} \qquad \text{with} \quad u_\mu(x) \doteq \mu^{\frac{1-\alpha}{\alpha}} u(\mu x) \qquad \text{for } \mu \geq 1.$$

More generally, if $p < \frac{\alpha}{1-\alpha}$, the immersion $\mathcal{P}_\alpha \hookrightarrow \mathbf{L}^p_{loc}$ is compact, namely a bounded sequence $\{\overline{u}_n\}_{n\in\mathbb{N}} \subset \mathcal{P}_\alpha$ admits a convergent subsequence in $\mathbf{L}^p_{loc}$.

Indeed, if $\overline{u}_n$ is a sequence of functions with $\|\overline{u}_n\|_{\mathcal{P}_\alpha}$ uniformly bounded in n, and if $\{v_k^n\}_{k\in\mathbb{Z}}$ are the functions appearing in the decomposition of $\overline{u}_n$, by a diagonal argument using Helly's compactness theorem one extracts a subsequence $\{n_i\}_{i\in\mathbb{N}}$ such that $v_k^{n_i}$ converges in $\mathbf{L}^p(K)$, for every k and every compact set $K \subset \mathbb{R}$.

Lemma 5.3. *Let $\bar{u} : \mathbb{R} \mapsto \mathbb{R}$ be a measurable function. Then $\overline{u} \in \mathcal{P}_\alpha$ if and only if it can be decomposed as (5.1), where the v_k satisfy (5.2) and (5.3). The smallest constant C for which (5.2) and (5.3) hold for every $k \geq 1$ is of the same order of $\|\overline{u}\|_{\mathcal{P}_\alpha}$.*

Proof. 1. Assume that $\overline{u}$ admits a decomposition as in (5.1)-(5.3). We show that

$$\|\overline{u}\|_{\mathcal{P}_\alpha} \;=\; \mathcal{O}(1)\cdot C.$$

Consider the case where $\lambda \in]0,1]$ is of the form $\lambda = 2^{-q}$ for some integer $q \geq 1$. Then we set

$$\widetilde{v} \doteq \sum_{k=0}^{q} v_k$$

and estimate

$$\text{Tot.Var.}\{\widetilde{v}\} \;\leq\; \sum_{k=0}^{q} \text{Tot.Var.}\{v_k\} \;\leq\; C\cdot\sum_{k=0}^{q} 2^{(1-\alpha)k} \;=\; \mathcal{O}(1)\cdot C\cdot 2^{(1-\alpha)q}.$$

Moreover

$$\text{meas}\big(\{\overline{u}\neq\widetilde{v}\}\big) \;\leq\; C\cdot\sum_{k=q+1}^{+\infty} 2^{-\alpha k} \;=\; \mathcal{O}(1)\cdot C\cdot 2^{-\alpha q}.$$

This proves

$$\sup_{q\geq 1} d^{(2^{-q})}(\overline{u},0) \;=\; \mathcal{O}(1)\cdot C.$$

A simple argument now shows that the estimate holds when the supremum is taken over all $0<\lambda\leq 1$.

2. Assume now that $\overline{u}\in\mathcal{P}_\alpha$. By the definition of $\|\cdot\|_{\mathcal{P}_\alpha}$ at (2.6)-(2.7), choosing $\lambda = 2^{-k}$, for every $k\geq 0$ we obtain a function u_k such that

$$\text{Tot.Var.}\{u_k\} \;\leq\; \|\overline{u}\|_{\mathcal{P}_\alpha}\cdot 2^{(1-\alpha)k}, \qquad \text{meas}\Big(\{\overline{u}\neq u_k\}\Big) \;\leq\; \|\overline{u}\|_{\mathcal{P}_\alpha}\cdot 2^{-\alpha k}. \tag{5.6}$$

One can choose the functions u_k so that they also satisfy

$$\{\overline{u}\neq u_k\} \;\subseteq\; \{\overline{u}\neq u_{k-1}\} \qquad \text{for all } \; k\geq 1.$$

We define the functions $\{v_k\}_{k\geq 1}$ by setting

$$v_0 \doteq u_0, \qquad v_k \doteq u_k - u_{k-1} \qquad \text{for all } \; k\geq 1. \tag{5.7}$$

By the second inequality in (5.6) it follows

$$\lim_{k\to+\infty} u_k(x) = \overline{u}(x), \qquad \text{pointwise for a.e. } x\in\mathbb{R}.$$

Using a telescopic sum one obtains

$$\overline{u}(x) \;=\; \lim_{N\to+\infty} u_N(x) \;=\; \lim_{N\to+\infty}\sum_{k=0}^{N} v_k(x) \qquad \text{pointwise for a.e. } x\in\mathbb{R}.$$

We conclude by observing that

$$\text{Tot.Var.}\{v_k\} \;\leq\; \text{Tot.Var.}\{u_k\} + \text{Tot.Var.}\{u_{k-1}\} \;\leq\; 2\cdot\|\overline{u}\|_{\mathcal{P}_\alpha}\cdot 2^{(1-\alpha)k}$$

and

$$\text{meas}\big(\{u_k\neq u_{k-1}\}\big) \;\leq\; \text{meas}\big(\{\overline{u}\neq u_{k-1}\}\big) \;\leq\; 2^\alpha\cdot\|\overline{u}\|_{\mathcal{P}_\alpha}\cdot 2^{-\alpha k}.$$

□

We now show that one can choose the decomposition in Lemma 5.3 in such a way that all functions v_k are one-sided Lipschitz.

Lemma 5.4. *Consider any function $\overline{u} \in \mathcal{P}_\alpha$. Then, it is possible to choose the functions v_k in (5.1)–(5.3) in such a way that the additional one-sided Lipschitz bound (5.4) holds.*

Proof. 1. In the following, given $p > 0$ and a function f, we denote by $\mathcal{E}_p(f)$ its lower one-sided p-Lipschitz envelope:

$$\mathcal{E}_p f(x) \doteq \sup\Big\{u(x)\,;\ u:\mathbb{R}\to\mathbb{R},\ u(y)\le f(y)\,,\ u(y')-u(y)\le p(y'-y)\ \text{ for all }\ y,y'\in\mathbb{R},\ y<y'\Big\}.$$

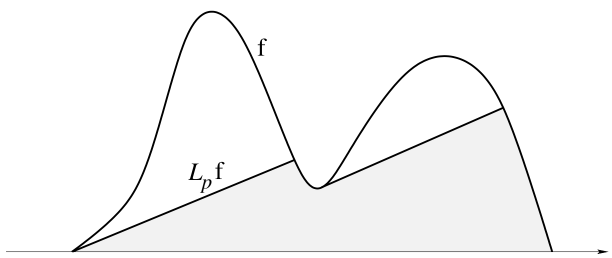

Figure 4: The lower one-sided p-Lipschitz envelope $\mathcal{E}_p f$. Here the straight lines have slope p.

The function $\mathcal{E}_p f$ is the largest one-sided p-Lipschitz function whose graph lies below the graph of f. Denoting by Tot.Var.$^+\{g\}$ the positive variation of a function g

$$\text{Tot.Var.}^+\{g\} \doteq \sup \sum_{x_0\le\cdots\le x_n} \big[g(x_i)-g(x_{i-1})\big]^+, \qquad \big[g(x_i)-g(x_{i-1})\big]^+ = \max\big\{0,\, g(x_i)-g(x_{i-1})\big\},$$

the following relations hold:

$$\begin{cases} \text{Tot.Var.}^+\{\mathcal{E}_p f\} \ \le\ \text{Tot.Var.}^+\{f\}, \\ \text{meas}\Big\{x\in\mathbb{R}\ ;\ \ \mathcal{E}_p f(x) < f(x)\Big\} \ \le\ \frac{1}{p}\cdot \text{Tot.Var.}^+\{f\}, \\ \text{Tot.Var.}^+\{f-\mathcal{E}_p f\} \ \le\ \text{Tot.Var.}^+\{f\}, \end{cases} \tag{5.8}$$

Notice that the second inequality is a consequence of Riesz' sunrise lemma (see for example [18], p.319), while the other two inequalities are straightforward.

2. Let $\{p_k^i\}_{i,k\in\mathbb{N}}$ be positive numbers (to be chosen later) such that

$$\lim_{i\to+\infty} p_k^i = +\infty \qquad \text{for all }\ k\ge 0, \tag{5.9}$$

and consider the decomposition (5.1) constructed in Lemma 5.3. As an intermediate step, we claim that for every $k \ge 0$, the function v_k can be further decomposed as

$$v_k(x) \ =\ \sum_{i=0}^{+\infty} v_k^i(x) \qquad \text{for a.e. } x\in\mathbb{R}, \tag{5.10}$$

where each v_k^i is one-sided p_k^i-Lipschitz and satisfies

$$\text{meas}\big(\{v_k^i \ne 0\}\big) \ \le\ C\cdot \frac{2^{k(1-\alpha)}}{p_k^{i-1}}, \qquad \text{Tot.Var.}^+\{v_k^i\} \ \le\ C\cdot 2^{(1-\alpha)k} \quad \text{if } i+k\ne 0, \qquad \text{Tot.Var.}\{v_0^0\} \ \le\ C, \tag{5.11}$$

with $C = \mathcal{O}(1) \cdot \|\overline{u}\|_{\mathcal{P}_\alpha}$, as in Lemma 5.3.

For $k \geq 0$, we first prove the claim assuming $v_k \geq 0$. Define

$$v_k^0 \doteq \mathcal{E}_{p_k^0}(v_k).$$

By (5.8) one has

$$\begin{aligned} &\text{Tot.Var.}^+\{v_k^0\} \;\leq\; \text{Tot.Var.}^+\{v_k\} \;\leq\; C \cdot 2^{(1-\alpha)k}, &&\text{for all } k \geq 0. \\ &\text{meas}(\{v_k^0 \neq 0\}) \;\leq \text{meas}(\{v_k \neq 0\}) \;\leq\; C \cdot 2^{-\alpha k} &&\text{for all } k \geq 1. \end{aligned}$$

Setting $\psi_0 = v_k - v_k^0$ and using again (5.8) we get

$$\text{meas}(\{\psi_0 \neq 0\}) \;\leq\; C \cdot \frac{1}{p_k^0} \cdot 2^{(1-\alpha)k}, \qquad \text{Tot.Var.}^+\{\psi_0\} \;\leq\; C \cdot 2^{(1-\alpha)k}.$$

Defining $v_k^1 = \mathcal{E}_{p_k^1}\psi_0$ we obtain

$$\text{meas}(\{v_k^1 \neq 0\}) \;\leq\; C \cdot \frac{1}{p_k^0} \cdot 2^{(1-\alpha)k}, \qquad \text{Tot.Var.}^+\{v_k^1\} \;\leq\; C \cdot 2^{(1-\alpha)k}.$$

By induction, assume we are given ψ_{i-2} and $v_k^{i-1} = \mathcal{E}_{p_k^{i-1}}\psi_{i-2}$, $i \geq 2$, both with positive variation $\leq C \cdot 2^{(1-\alpha)k}$. We then define $\psi_{i-1} = \psi_{i-2} - v_k^{i-1}$ and $v_k^i = \mathcal{E}_{p_k^i}\psi_{i-1}$. This yields

$$\text{meas}(\{\psi_{i-1} \neq 0\}) \;\leq\; C \cdot \frac{1}{p_k^{i-1}} \cdot 2^{(1-\alpha)k}, \qquad \text{Tot.Var.}^+\{\psi_{i-1}\} \;\leq\; C \cdot 2^{(1-\alpha)k}$$

and hence

$$\text{meas}(\{v_k^i \neq 0\}) \;\leq\; C \cdot \frac{2^{(1-\alpha)k}}{p_k^{i-1}}, \qquad \text{Tot.Var.}^+\{v_k^i\} \;\leq\; C \cdot 2^{(1-\alpha)k}. \tag{5.12}$$

Therefore, by induction (5.12) holds for every i and every k.

Finally, we prove that (5.10) holds. Indeed, by the second inequality in (5.8) it follows

$$\text{meas}\left(\left\{v_k \neq \sum_{i=0}^{i^*} v_k^i\right\}\right) \;\leq\; \frac{1}{p_k^{i^*}} \cdot \text{Tot.Var.}\, v_k \qquad \text{for all } \; i^* \geq 1.$$

Letting $i^* \to +\infty$, this proves our claim in the case $v_k \geq 0$.

3. Next, we show how to handle the general case where $v_k = v_k^+ - v_k^-$ has a positive and a negative part. We already know how to decompose the positive part v_k^+.

We treat the negative part v_k^- in the same way, but using instead the lower one-sided Lipschitz envelope, defined by

$$\mathcal{E}_p^- f(x) \;\doteq\; \sup\Big\{u(x)\,; \;\; u(y) \leq f(y)\,, \;\; u(y') - u(y) \geq -p(y'-y) \;\text{ for all }\; y, y' \in \mathbb{R}, \;\; y < y'\Big\}.$$

This yields a decomposition

$$v_k^- = \sum_{i=0}^{\infty} w_k^i$$

where w_k^i are positive one sided Lipschitz functions which satisfy the same inequalities as in (5.11), and whose distributional derivatives satisfy $Dw_i^k \geq -p_k^i$.

Then

$$v_k(x) \;=\; v_k^+(x) - v_k^-(x) \;=\; \sum_{i=0}^{\infty} v_k^i(x) + \sum_{i=0}^{\infty}\big(-w_k^i(x)\big).$$

is the desired decomposition.

4. We now conclude the proof of the lemma, relying on (5.10)-(5.11). Defining

$$\widetilde{v}_q \;=\; \sum_{i=0}^{q} v_{q-i}^i\,, \tag{5.13}$$

we obtain

$$\overline{u} \;=\; \sum_{q=0}^{+\infty} \widetilde{v}_q\,. \tag{5.14}$$

Next, we choose

$$p_k^i \;\doteq\; \frac{6}{\pi^2}\cdot\frac{2^{k+i}}{(i+2)^2}\,.$$

By (5.11), for every $q \geq 1$ we obtain

$$\mathrm{meas}\big(\{\widetilde{v}_q \neq 0\}\big) \;\leq\; \frac{\pi^2}{6} C\cdot 2^{-\alpha q}\sum_{i=0}^{q}\frac{2^q\cdot(i+1)^2}{2^{(1-\alpha)i}\cdot 2^{q-1}} \;\leq\; \frac{\pi^2}{3}\cdot C\cdot 2^{-\alpha q}\cdot\sum_{i=0}^{q}\frac{(i+1)^2}{2^{(1-\alpha)i}} \;=\; \mathcal{O}(1)\cdot C\cdot 2^{-\alpha q}.$$

Moreover, always for $q \geq 1$, the one-sided Lipschitz constant for $\widetilde{v}_q$ is estimated by

$$D\widetilde{v}_q \;\leq\; \frac{6}{\pi^2}\sum_{i=0}^{q} p_{q-i}^i \;\leq\; 2^q\frac{6}{\pi^2}\sum_{i=0}^{q}\frac{1}{(i+1)^2} \;\leq\; 2^q.$$

The one-sided Lipschitz property and the estimate on the support of $\widetilde{v}_q$ readily imply

$$\mathrm{Tot.Var.}\{\widetilde{v}_q\} \;=\; \mathcal{O}(1)\cdot C\cdot 2^{(1-\alpha)q}.$$

If $q = 0$, by definition we have $\widetilde{v}_q = v_0^0$, so that by (5.11)

$$\mathrm{Tot.Var.}\,\{\widetilde{v}_0\} \;\leq\; C, \qquad D\widetilde{v}_0 \;\leq\; p_0^0 \;\leq\; 1.$$

The conclusion of the lemma is achieved by renaming $v_k \doteq \widetilde{v}_q$, with $q = k$. □

Lemma 5.5. *For every $k \geq 0$, the function v_k constructed in Lemma 5.4 satisfies 3 of Theorem 5.1.*

Proof. Since v_k is one-sided Lipschitz, the set $\{v_k \neq 0\}$ has at most countably many connected components, that we denote by $\{I_k^p\}_{p\geq 1}$. Consider then the restrictions $v_k^p \doteq v_k\Big|_{I_k^p}$. On every interval I_k^p, having length ℓ_k^p, since v_k is one-sided 2^k-Lipschitz, one has

$$|v_k(x)| \;\leq\; 2^k\ell_k^p \;=\; h_k^p, \qquad \text{for all } \; x\in I_k^p$$

Therefore

$$\sum_{p\geq 1}\ell_k^p \;=\; \mathrm{meas}(\{v_k \neq 0\}) \;\leq\; C2^{-\alpha k},$$

which proves (5.5). □

6 Decay rate of the Total Variation

In this section we prove that if $1/2 < \alpha < 1$, then the conjectured decay of the total variation with rate $t^{\alpha-1}$ holds.

Theorem 6.1. *Consider a bounded, compactly supported initial datum $\overline{u} \in \mathcal{P}_\alpha$, with $1/2 < \alpha < 1$. Then the solution to Burgers' equation (3.1) satisfies*

$$\limsup_{t\to 0+} \Big(t^{1-\alpha} \cdot TV\left\{S_t\overline{u}\right\}\Big) \;\leq\; C_0 \,\frac{\|\overline{u}\|_{\mathcal{P}_\alpha}}{2\alpha - 1}\,, \tag{6.1}$$

where C_0 is some absolute constant.

Proof. 1. Let $u = u(t,x)$ be a solution of Burgers' equation and let $t > 0$ be given. Denote by $J_t \subset \mathbb{R}$ the jump set of $u(t,\cdot)$. By the Lax-Oleinik formula, for every $x \in \mathbb{R}\setminus J_t$ there is a unique backward characteristic from the point (t,x) along which the solution is constant: namely

$$u(t,x) = u(s, x - u(t,x)(t-s)) \qquad \text{for all } \; s \in]0,t]. \tag{6.2}$$

If (6.2) holds we say that the couple

$$(x_0, v) \;\doteq\; \big(x - u(t,x)t\,,\; u(t,x)\big)$$

survives up to time t, and we denote by $\mathcal{Q}(t)$ the set of couples which survive up to time t. The point x_0 is the starting point of the characteristic passing through (t,x) and v is the value of u along the characteristic. Notice however, for example in the case of a centered rarefaction, that we can have $\overline{u}(x_0) \neq v$. Indeed, (6.2) does not extend to $t = 0$, in general. We will estimate Tot.Var.$\{S_t\overline{u}\}$ by means of the equality

$$\text{Tot.Var.}\big\{S_t\overline{u}\,;\;\mathbb{R}\big\} \;=\; \text{Tot.Var.}\big\{S_t\overline{u}\,;\;\mathbb{R}\setminus J_t\big\} \;=\; \sup \sum_{\substack{x_1\leq\cdots\leq x_n,\\ (x_i,v_i)\in\mathcal{Q}(t)}} |v_i - v_{i-1}|, \tag{6.3}$$

where in the last sum we assume that if $x_{i-1} = x_i$, then $v_{i-1} \leq v_i$. By the Lax formula, the constraint $(x_0, u_0) \in \mathcal{Q}(t)$ is satisfied if and only if

$$\begin{aligned} &\int_{x_0}^{y} \overline{u}(z) - \Big[u_0 - \frac{1}{t}(z - x_0)\Big]\,dz \;\geq\; 0 \qquad \text{for all } \; y \geq x_0\,,\\ &\int_{y}^{x_0} \overline{u}(z) - \Big[u_0 - \frac{1}{t}(z - x_0)\Big]\,dz \;\leq\; 0 \qquad \text{for all } \; y \leq x_0\,. \end{aligned} \tag{6.4}$$

Equivalently:

$$\int_{x_0}^{y} \Big[\overline{u}(z) - \Big(u_0 - \frac{z-x_0}{t}\Big)\Big]^+ dz \;\geq\; \int_{x_0}^{y} \Big[\overline{u}(z) - \Big(u_0 - \frac{z-x_0}{t}\Big)\Big]^- dz \qquad \text{for all } \; y \geq x_0\,, \tag{6.5}$$

$$\int_{y}^{x_0} \Big[\overline{u}(z) - \Big(u_0 - \frac{z-x_0}{t}\Big)\Big]^+ dz \;\leq\; \int_{y}^{x_0} \Big[\overline{u}(z) - \Big(u_0 - \frac{z-x_0}{t}\Big)\Big]^- dz \qquad \text{for all } \; y \leq x_0\,, \tag{6.6}$$

where we used the notation

$$[z]^+ \doteq \max\{z, 0\}, \qquad [z]^- \doteq -\min\{z, 0\}.$$

The interpretation of (6.5) is that for every $y > x_0$ the area of the hypograph of $\overline{u}$ in $[x_0, y]$ that lies above the line passing through (x_0, u_0) with slope $-1/t$ must be bigger then the area of the epigraph lying below the same line. Analogously, the interpretation of (6.6) is that for every $y < x_0$ the area of the hypograph of $\overline{u}$ in $[x_0, y]$ that lies above the line passing through (x, u_0) with slope $-1/t$ must be smaller then the area of the epigraph lying below the same line.

2. It suffices to prove the decay estimate (6.1) for all times of the form $t = 2^{-k}$, $k \geq 1$. We thus need to estimate the quantity

$$\limsup_{k \to +\infty} 2^{(\alpha-1)k} \, \text{Tot.Var.}\big\{S_{2^{-k}}\overline{u}\big\}$$

In the following we fix a time $t = 2^{-k}$ and show that

$$\text{Tot.Var.}\big\{S_{2^{-k}}\overline{u}\big\} \;\leq\; c \cdot \frac{2^{(1-\alpha)k}}{2\alpha - 1} \cdot \|\overline{u}\|_{\mathcal{P}_\alpha}\,, \tag{6.7}$$

where c is some constant depending only on $\|\overline{u}\|_{\mathbf{L}^\infty}$.

Let $\overline{u} = \sum_{q=0}^{\infty} v_q$ be a decomposition satisfying all the properties listed in Theorem 5.1. We write $\overline{u}$ as the sum of two terms:

$$\overline{u} \;=\; \sum_{q=0}^{k-1} v_q + \sum_{q=k}^{\infty} v_q \;\doteq\; \widetilde{u}_k + \widehat{u}_k. \tag{6.8}$$

We regard the function $\widetilde{u}_k$ as the *regular part of* $\overline{u}$, in the sense that it has total variation that is bounded, of size $\mathcal{O}(1) \cdot 2^{(1-\alpha)k}$. In fact, since each v_q is one-sided Lipschitz with constant 2^q and with total variation bounded by $C\,2^{(1-\alpha)q}$, the function $\widetilde{u}_k$ is one-sided Lipschitz with constant 2^k and with total variation bounded by $C2^{(1-\alpha)k}$.

We recall that C is the constant coming from Lemmas 5.3 and 5.4, of the same order of $\|\overline{u}\|_{\mathcal{P}_\alpha}$.

3. To simplify the exposition, we first give a proof under two additional assumptions:

(H1) the regular part $\widetilde{u}_k$ is zero, i.e.

$$\overline{u} \;=\; \sum_{q=k}^{+\infty} v_q \;=\; \widehat{u}_k. \tag{6.9}$$

(H2) All functions v_q are positive:

$$v_q(x) \;\geq\; 0 \qquad \text{for all } \; x \in \mathbb{R}.$$

This implies that all the functions v_q^p constructed in Lemma 5.5 are positive as well,

Assuming (H1) and (H2), let $x_0 \leq x_1 \leq \cdots \leq x_n$ and $v_0, \ldots, v_n$, be such that $(x_i, v_i) \in \mathcal{Q}(2^{-k})$. According to (6.3), it suffices to estimate the total variation over these points:

$$\sum_{i=1}^{n} |v_i - v_{i-1}|.$$

Since $\overline{u}$ is compactly supported, it is enough to estimate the negative variation, namely

$$\sum_{i=1}^{n} \big[v_i - v_{i-1}\big]^- . \tag{6.10}$$

The set of downward jumps

$$\mathcal{N} \ \doteq\ \Big\{ i \in \{1, \ldots, n\}\,; \quad v_i \ <\ v_{i-1} \Big\}.$$

can be partitioned as $\mathcal{N} = \mathcal{I} \cup \mathcal{J}$, where

$$\mathcal{I} \doteq \Big\{ i \in \mathcal{N}\,;\ \ ]x_{i-1}, x_i[\subseteq \{\widehat{u}_k \neq 0\} \Big\}, \qquad \mathcal{J} \doteq \mathcal{N} \setminus \mathcal{I}.$$

In the next two steps, the negative variation (6.10) will be estimated by considering the terms $i \in \mathcal{I}$ and $i \in \mathcal{J}$ separately.

4. Let $i \in \mathcal{I}$. Since the characteristics starting at x_i, x_{i+1} do not cross up to time $t = 2^{-k}$, this implies

$$v_{i-1} - v_i \ \leq\ (x_i - x_{i-1}) \cdot 2^k.$$

Summing over $\mathcal{I}$ one obtains

$$\sum_{i \in \mathcal{I}} \big[v_{i-1} - v_i\big] \ \leq\ 2^k \sum_{i \in \mathcal{I}} (x_i - x_{i-1}) \leq 2^k \cdot \mathrm{meas}\big(\{\widehat{u}_k \neq 0\}\big) \ \leq\ 2C \cdot 2^k \cdot 2^{-\alpha k} \ =\ c_1\, C \cdot 2^{(1-\alpha)k}, \tag{6.11}$$

where we used the inequalities

$$\mathrm{meas}\big(\{\widehat{u}_k \neq 0\}\big) \ \leq\ \sum_{q=k}^{\infty} \mathrm{meas}\big(\{v_q \neq 0\}\big) \ \leq\ C \sum_{q=k}^{\infty} 2^{-\alpha q} \ \leq\ \frac{2^\alpha}{2^\alpha - 1} C \cdot 2^{-\alpha k}.$$

5. Next, consider the case where $i \in \mathcal{J}$. As shown in Fig. 5, consider the triangle

$$\mathcal{T}_i \ \doteq\ \Big\{ (x, v) \in \mathbb{R}^2\,; \quad x_{i-1} < x < x_i, \quad v_i \leq v \leq v_{i-1} - 2^k (x - x_{i-1}) \Big\}.$$

with height $\delta_i \doteq v_{i-1} - v_i$ and base of length $\delta_i 2^{-k}$.

In the following, we denote by $\widehat{U}_k \subset \mathbb{R}^2$ the region below the graph of $\widehat{u}_k$:

$$\widehat{U}_k \ \doteq\ \Big\{ (x, v) \in \mathbb{R}^2\,; \quad 0 \leq v \leq \widehat{u}_k(x) \Big\} \ \subset\ \mathbb{R}^2.$$

By (6.5), the fact that $(x_{i-1}, v_{i-1}) \in \mathcal{Q}(2^{-k})$, implies that the area of the triangle $\mathcal{T}_i$ is bounded by the area of $\widehat{U}$ in the strip $[x_{i-1}, x_i] \times \mathbb{R}$:

$$\mathrm{meas}(\mathcal{T}_i) \ =\ 2^{-(k+1)} \delta_i^2 \ \leq\ \mathrm{meas}\Big(\widehat{U}_k \cap \big([x_{i-1}, x_i] \times \mathbb{R} \big) \Big).$$

This implies

$$\delta_i \ \leq\ 2^{\frac{k+1}{2}} \cdot \mathrm{meas}\Big(\widehat{U}_k \cap \big([x_{i-1}, x_i] \times \mathbb{R} \big) \Big)^{1/2}. \tag{6.12}$$

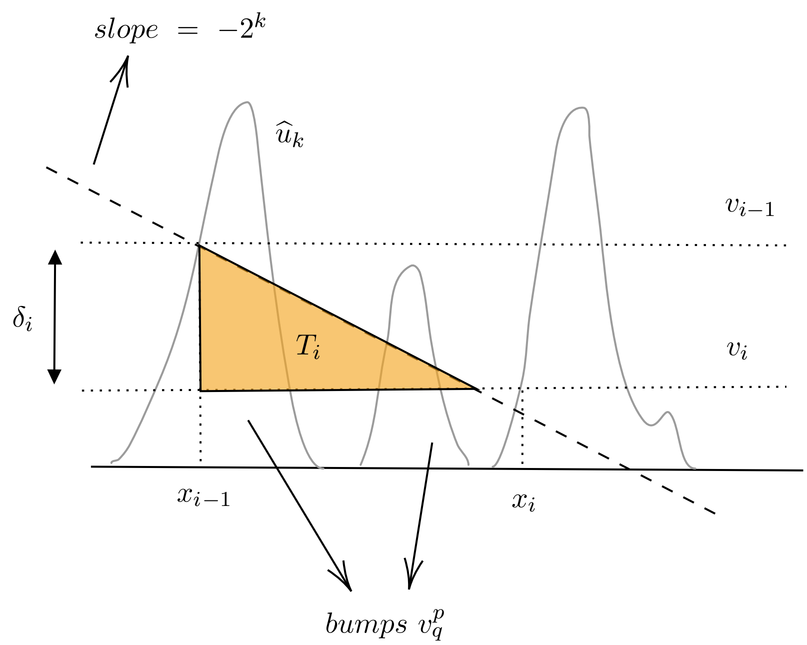

Figure 5: The configuration considered in the estimate for $i \in \mathcal{J}$, in step **5** of the proof.

For each $i \in \mathcal{J}$, we now consider the set of indices $\mathcal{Z}(i) \subset \mathbb{N} \times \mathbb{N}$ defined by

$$\mathcal{Z}(i) \doteq \Big\{ (p,q) \in \mathbb{N}^2\,; \quad q \geq k, \quad \mathrm{supp}\, v_q^p \cap]x_{i-1}, x_i[\neq \emptyset \Big\}.$$

This is the set of all functions v_q^p in the decomposition of $\widehat{u}_k$ whose support intersects the open interval $]x_{i-1}, x_i[$.

At this stage, we make an important observation:

- *For any couple (p,q) with $q \geq k$, there can be at most two indices $i \in \mathcal{J}$ such that $(p,q) \in \mathcal{Z}(i)$*

Indeed, assume that this were not the case, i.e. for some $i_1 < i_2 < i_3$ one had $(p,q) \in \mathcal{Z}(i_1) \cap \mathcal{Z}(i_2) \cap \mathcal{Z}(i_3)$. Since the set $\{v_q^p \neq 0\}$ is connected, we would have

$$]x_{i_2-1}, x_{i_2}[\ \subset\ \{v_q^p \neq 0\}\ \subseteq\ \{\widehat{u}_k \neq 0\}.$$

But this is a contradiction because $i \notin \mathcal{I}$.

As in Lemma 5.5, call $\ell_q^p \doteq \mathrm{meas}\big(\mathrm{supp}\, v_q^p\big)$. By (6.12) it follows

$$\delta_i \ \leq\ \sqrt{2} \cdot 2^{k/2} \cdot \left(\sum_{(p,q) \in \mathcal{Z}(i)} h_q^p \cdot \ell_q^p \right)^{1/2} \ \leq\ \sqrt{2}\ \cdot 2^{k/2} \cdot \sum_{(p,q) \in \mathcal{Z}(i)} \Big(h_q^p \cdot \ell_q^p \Big)^{1/2}.$$

Summing over $i \in \mathcal{J}$, and using the fact that each couple (p,q) can appear in the sum at most twice, we obtain

$$\begin{aligned}\sum_{i\in\mathcal{J}}\delta_i \leq&\ \sqrt{2}\ \cdot 2^{k/2}\cdot\sum_{i\in\mathcal{J}}\ \sum_{(p,q)\in\mathcal{Z}(i)}\Big(h_q^p\cdot\ell_q^p\Big)^{1/2}\\ \leq&\ 2^{3/2}\ \cdot 2^{k/2}\cdot\sum_{q=k}^{\infty}\sum_{p\in\mathbb{N}}\Big(h_q^p\cdot\ell_q^p\Big)^{1/2}.\end{aligned}\tag{6.13}$$

Observing that

$$\Big(\ell_q^p\cdot h_q^p\Big)^{1/2} = 2^{q/2}\ell_q^p$$

and using (5.5), from (6.13) we obtain

$$\begin{aligned}\sum_{i\in\mathcal{J}}\delta_i\ \leq&\ 2^{3/2}\cdot 2^{k/2}\cdot\sum_{q=k}^{\infty}\sum_{p\in\mathbb{N}}2^{q/2}\ell_q^p\ \leq\ 2^{3/2}\cdot C\cdot 2^{k/2}\cdot\sum_{q=k}^{\infty}2^{q/2}2^{-\alpha q}\\ \leq&\ c_2\,C\cdot\frac{1}{1-2\alpha}2^{k/2}\cdot 2^{(1/2-\alpha)k}\ =\ c_2\,C\cdot\frac{1}{2\alpha-1}\cdot 2^{(1-\alpha)k},\end{aligned}\tag{6.14}$$

where c_2 is another absolute constant. Combining (6.11) with (6.14), we obtain the desired decay rate, under the additional assumptions (H1)-(H2).

6. In the remaining steps we complete the proof of the theorem, removing the assumptions (H1)-(H2).

Recalling the decomposition (6.8), we observe that the function $\widetilde{u}_k$ is one-sided 2^k-Lipschitz, because each v_q is one-sided 2^q Lipschitz.

Let $x_0\leq x_1\leq\ldots\leq x_n$ and v_i, $i=0,\ldots,n$ be such that $(x_i,v_i)\in\mathcal{Q}(2^{-k})$. As before, it suffices to estimate the negative variation, i.e.

$$\sum_{i\in\mathcal{N}}\big[v_{i-1}-v_i\big],$$

where

$$\mathcal{N}\ \doteq\ \Big\{i\in\{1,\ldots,n\}\,;\quad v_i<v_{i-1}\Big\}.$$

We partition the above set of indices as $\mathcal{N}=\mathcal{I}\cup\mathcal{J}$, where

$$\mathcal{I}\doteq\left\{i\in\mathcal{N}\,;\quad]x_{i-1},x_i[\ \subset\ \bigcup_{q\geq k}\{v_q\neq 0\}\right\},\qquad \mathcal{J}\doteq\mathcal{N}\setminus\mathcal{I}.$$

Set

$$\delta_i\doteq v_{i-1}-v_i\qquad\text{for all}\ \ i\in\mathcal{N}.$$

We further partition $\mathcal{J}=\mathcal{J}_1\cup\mathcal{J}_2$, by setting

$$\mathcal{J}_1\doteq\Big\{i\in\mathcal{J}\ \mid\ \widetilde{u}_k(x_{i-1})\geq v_{i-1}-\frac{\delta_i}{3}\quad\text{and}\quad\widetilde{u}_k(x_i)\leq v_i+\frac{\delta_i}{3}\Big\},\qquad \mathcal{J}_2\doteq\mathcal{J}\setminus\mathcal{J}_1\,.$$

We will estimate the quantity

$$\sum_{i\in\mathcal{N}}\delta_i\ =\ \sum_{i\in\mathcal{I}}\delta_i+\sum_{i\in\mathcal{J}_1}\delta_i+\sum_{i\in\mathcal{J}_2}\delta_i$$

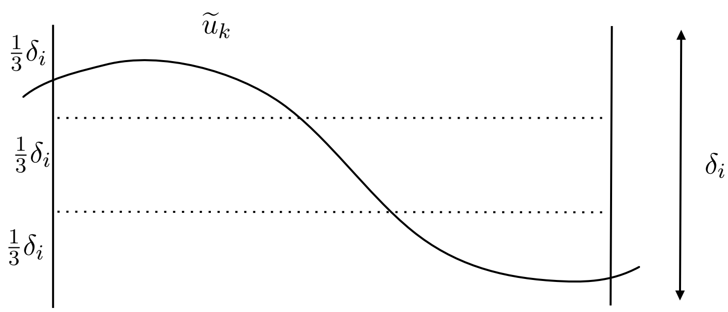

Figure 6: Illustration of the case $i \in \mathcal{J}_1$. At least one third of the variation is due to the regular part $\widetilde{u}_k$.

by providing a bound on each term on the right hand side, in the following three steps.

7. (Estimate of the sum over $\mathcal{I}$). With exactly the same argument as in Step 4., we obtain the estimate

$$\sum_{i\in\mathcal{I}} \big[v_{i-1} - v_i\big] \ \le c_1 \cdot C \cdot 2^{(1-\alpha)k} \tag{6.15}$$

where c_1 is an absolute constant.

8. (Estimate of the sum over $\mathcal{J}_1$). In this case (see Figure 6), by definition of $\mathcal{J}_1$ the variation of $\widetilde{u}_k$ on $[x_{i-1}, x_i]$ is at least one third of δ_i. Therefore the jump δ_i is controlled by the variation of $\widetilde{u}_k$, which is the regular part. More precisely, from the definition of $\mathcal{J}_1$ it follows

$$\delta_i \ \doteq \ v_{i-1} - v_i \ \le \ \left(\widetilde{u}_k(x_{i-1}) + \frac{\delta_i}{3}\right) - \left(\widetilde{u}_k(x_i) - \frac{\delta_i}{3}\right) \ \le \ \widetilde{u}_k(x_{i-1}) - \widetilde{u}_k(x_i) + \frac{2\delta_i}{3}\,,$$

and therefore

$$\delta_i \ \le \ 3 \cdot \big(\widetilde{u}_k(x_{i-1}) - \widetilde{u}_k(x_i)\big).$$

Summing over $i \in \mathcal{J}_1$ we obtain

$$\sum_{i\in\mathcal{J}_1} \delta_i \ \le \ 3 \sum_{i\in\mathcal{J}_1} \big(\widetilde{u}_k(x_{i-1}) - \widetilde{u}_k(x_i)\big) \ \le \ 3\,\text{Tot.Var.}\{\widetilde{u}_k\} \ \le \ c_2 \cdot C \cdot 2^{(1-\alpha)k}. \tag{6.16}$$

9. (Estimate of the sum over $\mathcal{J}_2$). The idea here is that we reduced to a situation where we can proceed as in the simplified case of the previous section, up to a modification of the definition of the triangles $\mathcal{T}_i$ that takes into account the presence of $\widetilde{u}_k$, which is one-sided 2^k-Lipschitz. Since $i \in \mathcal{J}_2$, at least one of the following inequalities is true:

$$\widetilde{u}_k(x_{i-1}) < v_{i-1} - \frac{\delta_i}{3} \qquad \text{or} \qquad \widetilde{u}_k(x_i) > v_i + \frac{\delta_i}{3}. \tag{6.17}$$

The proof splits in two cases depending on which one is true. To fix ideas, we assume that the first inequality holds. The second case is entirely similar. It can be handled by the same argument, in connection with the reversed initial datum: $\overline{v}(x) \doteq -\overline{u}(-x)$.

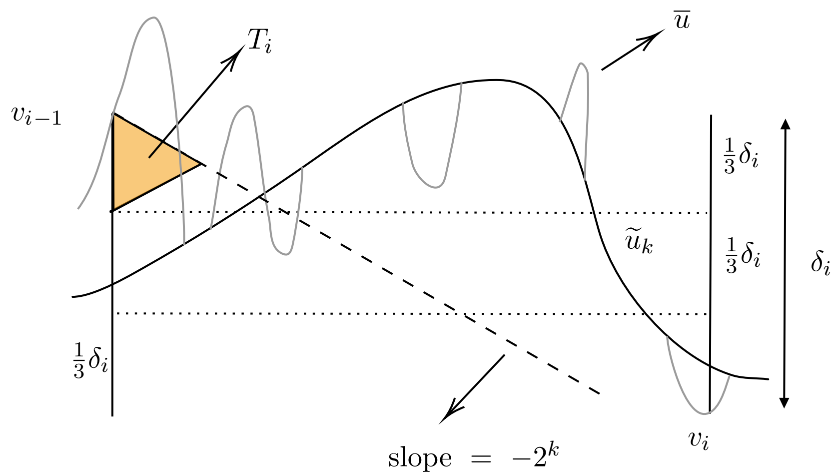

Figure 7: The estimate for $i \in \mathcal{J}_2$. From the fact that the value v_{i-1} survives up to time 2^{-k}, we deduce that the area of the yellow triangle can be controlled by the L^1 norm of all the v_q^p in the interval (x_{i-1}, x_i).

Assuming that the first inequality in (6.17) holds, define the triangle:

$$\mathcal{T}_i \doteq \Big\{(x,v) \in \mathbb{R}^2 \ \mid \ x \in (x_{i-1}, x_i), \quad v_{i-1} - \frac{\delta_i}{3} + (x - x_{i-1}) \cdot 2^k \leq v \leq v_{i-1} - (x - x_{i-1})2^k\Big\},$$

as shown in Fig. 7. We let $U \subset \mathbb{R}^2$ be the hypograph of $\overline{u}$:

$$U \doteq \Big\{(x,v) \in \mathbb{R}^2 ; \quad v \leq \overline{u}(x)\Big\} \subset \mathbb{R}^2,$$

and let $\widetilde{U}_k$ be the hypograph of $\widetilde{u}_k$:

$$\widetilde{U}_k \doteq \Big\{(x,v) \in \mathbb{R}^2 ; \quad v \leq \widetilde{u}_k(x)\Big\} \subset \mathbb{R}^2.$$

The fact that the couple (x_{i-1}, v_{i-1}) survives up to time $t = 2^{-k}$ already implies that

$$\text{meas}(\mathcal{T}_i \setminus \widetilde{U}_k) \leq \text{meas}\Big\{(x,v) \in (U \setminus \widetilde{U}) ; \quad x \in [x_{i-1}, x_i]\Big\} \doteq A_i . \tag{6.18}$$

Actually, we claim that $\widetilde{U}_k \cap \mathcal{T}_i = \emptyset$. In fact, $\widetilde{u}_k$ is one-sided 2^k-Lipschitz and satisfies $\widetilde{u}_k(x_{i-1}) \leq v_{i-1} - \frac{\delta_i}{3}$. This implies

$$\widetilde{u}_k(x) \leq v_i - \frac{\delta_i}{3} + (x - x_{i-1}) \cdot 2^k \qquad \text{for all } \ x \in]x_{i-1}, x_i[\, .$$

By definition of the triangle $\mathcal{T}_i$, this means that the hypograph of $\widetilde{u}_k$ lies entirely below the lower side of $\mathcal{T}_i$. Hence $\widetilde{U}_k \cap \mathcal{T}_i = \emptyset$. From (6.18) it thus follows

$$\text{meas}(\mathcal{T}_i) \leq A_i . \tag{6.19}$$

On the other hand, the area of the triangle $\mathcal{T}_i$ is

$$\text{meas}(\mathcal{T}_i) = \frac{\delta_i^2}{64} 2^{-k}. \tag{6.20}$$

Combining (6.19) with (6.20) we obtain

$$\delta_i \;\le\; 8\cdot 2^{k/2}\cdot A_i^{1/2}. \tag{6.21}$$

The area A_i on the right hand side is bounded above by the sum of the areas of the blocks v_q^p whose support intersects the interval $]x_i, x_{i+1}[$. More precisely, for each $i\in\mathcal{J}_2$, define the set of indices

$$\mathcal{Z}(i) \doteq \Big\{(p,q)\in\mathbb{N}^2 \;\mid\; q\ge k, \quad \mathrm{supp}\, v_q^p \cap]x_{i-1}, x_i[\neq \emptyset\Big\}.$$

With the same argument used in step **5.** we obtain that, for every couple (p,q) with $q\ge k$, there can be at most two indices $i\in\mathcal{J}_2$ such that $(p,q)\in\mathcal{Z}(i)$.

By (6.21) we now obtain

$$\delta_i \;\le\; 8\cdot 2^{k/2}\cdot\Bigg(\sum_{(p,q)\in\mathcal{Z}(i)} h_q^p\cdot\ell_q^p\Bigg)^{1/2} \;\le\; 8\cdot 2^{k/2}\cdot\sum_{(p,q)\in\mathcal{Z}(i)}\Big(h_q^p\cdot\ell_q^p\Big)^{1/2}.$$

Summing over i, and using the fact that each (p,q) appears in the sum at most twice, we obtain

$$\sum_{i\in\mathcal{J}_2}\delta_i \;\le\; 8\cdot 2^{k/2}\cdot\sum_{i\in\mathcal{J}_2}\sum_{(p,q)\in\mathcal{Z}(i)}\Big(h_q^p\cdot\ell_q^p\Big)^{1/2} \;\le\; 16\cdot 2^{k/2}\cdot\sum_{q=k}^{\infty}\sum_{p\in\mathbb{N}}\Big(h_q^p\cdot\ell_q^p\Big)^{1/2}. \tag{6.22}$$

Observing that $\Big(\ell_q^p\cdot h_q^p\Big)^{1/2} = 2^{q/2}\ell_q^p$ and using (5.5), we finally obtain

$$\begin{aligned}\sum_{i\in\mathcal{J}_2}\delta_i \;&\le\; 16\cdot 2^{k/2}\cdot\sum_{q=k}^{\infty}\sum_{p\in\mathbb{N}} 2^{q/2}\ell_q^p \;\le\; 16\cdot 2^{k/2}\cdot C\cdot\sum_{q=k}^{\infty} 2^{q/2}2^{-\alpha q}\\ &\le\; c_3\cdot\frac{1}{2\alpha-1}\cdot\|\overline{u}\|_{\mathcal{P}_\alpha}2^{k/2}\cdot 2^{(1/2-\alpha)k} \;=\; c_3\cdot\frac{1}{2\alpha-1}\cdot\|\overline{u}\|_{\mathcal{P}_\alpha}\cdot 2^{(1-\alpha)k},\end{aligned} \tag{6.23}$$

where c_3 is an absolute constant. Combining the three estimates (6.15), (6.16) and (6.23), the proof is completed. □

Remark 6.2. If Burgers' equation is replaced by a general scalar conservation law with a $\mathcal{C}^2$, uniformly convex flux f, so that $f''\ge c>0$, from the Hopf-Lax formula we obtain that $(x_0,u_0)\in\mathcal{Q}(t)$ if and only if

$$\int_{x_0}^{y}\bigg[\overline{u}(z)-(f^*)'\Big(f'(u_0)-\frac{z-x_0}{t}\Big)\bigg]^+ dz \;\ge\; \int_{x_0}^{y}\bigg[\overline{u}(z)-(f^*)'\Big(f'(u_0)-\frac{z-x_0}{t}\Big)\bigg]^- dz \tag{6.24}$$

for all $y\ge x_0$, and

$$\int_{y}^{x_0}\bigg[\overline{u}(z)-(f^*)'\Big(f'(u_0)-\frac{z-x_0}{t}\Big)\bigg]^+ dz \;\le\; \int_{y}^{x_0}\bigg[\overline{u}(z)-(f^*)'\Big(f'(u_0)-\frac{z-x_0}{t}\Big)\bigg]^- dz \tag{6.25}$$

for all $y\le x_0$. Here

$$f^*(u) \;\doteq\; \sup_{v\in\mathbb{R}}\,\big\{uv-f(v)\big\}$$

denotes the Legendre transform of f. By a well known property of the Legendre transform (see e.g. [8, 14]) we have

$$(f^*)'\big(f'(u_0)\big) \;=\; u_0\,.$$

By uniform convexity it thus follows

$$u_0 - \frac{1}{ct}(z - x_0) \ \leq \ (f^*)'\Big(f'(u_0) - \frac{z - x_0}{t}\Big) \qquad \text{for all } \ z > x_0\,,$$

$$(f^*)'\Big(f'(u_0) - \frac{z - x_0}{t}\Big) \ \leq \ u_0 - \frac{1}{ct}(z - x_0) \qquad \text{for all } \ z < x_0.$$

Using the inequalities above, the above proof remains valid up to minor modifications. Therefore, Theorem 5.1 remains valid for any uniformly convex flux f.

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