

ON A DEGENERATE SECOND ORDER TRAFFIC MODEL: EXISTENCE OF DISCRETE EVOLUTIONS, DETERMINISTIC MANY-PARTICLE LIMIT AND FIRST ORDER APPROXIMATION

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ABSTRACT. We propose and analyse a new microscopic second order Follow-the-Leader type scheme to describe traffic flows. The main novelty of this model consists in multiplying the second order term by a nonlinear function of the global density, with the intent of considering the attentiveness of the drivers in dependence of the amount of congestion. Such term makes the system highly degenerate; indeed, coherently with the modellistic viewpoint, we allow for the nonlinearity to vanish as soon as consecutive vehicles are very close to each other. We first show existence of solutions to the degenerate discrete system. We then perform a rigorous discrete-to-continuum limit, as the number of vehicles grows larger and larger, by making use of suitable piece-wise constant approximations of the relevant macroscopic variables. The resulting continuum system turns out to be described by a degenerate pressure-less Euler-type equation, and we discuss how this could be considered an alternative to the groundbreaking Aw-Rascle-Zhang traffic model. Finally, we study the singular limit to first order dynamics in the spirit of a vanishing-inertia argument. This eventually validates the use of first order macroscopic models with nonlinear mobility to describe a congested traffic stream.

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1. INTRODUCTION

The study of mathematical models for traffic flows has considerably spread in the recent decades due to the twofold need of preventing vehicular congestion on road networks and, at the same time, managing the increasing demand for mobility. Moreover, these models contain several challenging and interesting mathematical issues to tackle. For a general introduction to the literature on this topic we refer to the books [29, 31, 42] and to the survey papers [32, 39].

1.1. Overview of the literature and of the existing models. Mathematical models for traffic can be classified according to several features which become relevant or not depending on the envisaged application. Among such features, we mention the level of description of the system (microscopic, mesoscopic, macroscopic) and the representation of the process (deterministic, stochastic). Deterministic microscopic models give a detailed description of the traffic flow and characterise individually each vehicle and its interaction with the other vehicles and the environment. However, they are computationally too costly when the number of vehicles is big and do not provide insights into the relevant features of the traffic flow. Deterministic macroscopic models, on the other hand, describe traffic as a flow without distinguishing the constituent vehicles. They are computationally less demanding as they involve a limited number of variables (density, velocity and flow). Moreover, they are very suited for analysing macroscopic characteristics of vehicular traffic, like shock waves and queue lengths.

Macroscopic models further split into the two main subcategories of *equilibrium* and *non-equilibrium* models, depending on the relation between the velocity and the density of vehicles. In equilibrium models (or *first-order* models) the velocity is expressed as a function of the density, while in non-equilibrium ones (or *second-order* ones) density and velocity are coupled through a partial differential equation. First-order models should be preferred when describing applications where the shock structures are irrelevant, because they do not take into account the distribution of the desired velocities across vehicles and, therefore, are not able to predict many traffic instabilities (e.g. stop & go waves, hysteresis phenomena, phantom jams, etc). Among first-order models we recall the Lighthill-Whitham-Richards (LWR) model [37, 41], which represents the starting point for the modelling of vehicular flows, and its extension to road networks as introduced in [33, 34] and further studied in [5]. The second-order model which is currently the most used is called Aw-Rascle-Zhang (ARZ) and has been introduced independently by Aw and Rascle [3] and Zhang [45]. It has been proposed to solve the inconsistencies (pointed out in [16]) of a previous second-order model by Payne [38] and Whitham [44]. We mention that macroscopic models for traffic are often treated in the fluid dynamics framework, as the traffic stream can be regarded as a fluid flow, see for instance [10, 11].

The present work is more concerned with the link between microscopic and macroscopic models. Such connection for the LWR model has been well established in a series of works [13, 20, 24, 35, 43] in the framework of many-particle limits of first-order Follow-the-Leader models. Second-order Follow-the-Leader type models, instead, have been investigated by Aw et al. [2], Greenberg [30] and Di Francesco et al. [21] in connection with the ARZ model. However, these latter results all deal with the simplified homogeneous version of the ARZ model (see Section 1.4 for more details), thus leaving the many-particle derivation of the complete nonlinear model still an open question.

We conclude this overview by mentioning related works on the extensions of the aforementioned models to road networks and to multi-scale analysis. Among multi-scale models, which couples the traffic description at different scales separated by an interface that can be either fixed or solution-dependent, we mention [36] for a second-order micro and macro coupling and [14] for second-order micro and first-order macro one. On the other hand, many-particle limits for road networks traffic flows were considered in [15] in the case of first order Follow-the-Leader schemes. Consistently, as observed in [33] and further investigated in [4, 5], the corresponding macroscopic models appear to be suitable extensions of the LWR model to networks.

1.2. A new second order model. In the present work we propose a novel second-order deterministic individual-based model to describe traffic dynamics with congestion. First, we prove an existence result for a new microscopic scheme, then we study the convergence, as the number of individuals grows larger and larger, towards a second-order macroscopic model resulting in a hydrodynamic pressureless Euler-type system. Finally, we investigate the passage to the first-order dynamics in the spirit of a vanishing-inertia asymptotics. This aims at validating the use of first-order schemes, reducing the level of complexity of the problem still being close to the original system.

Our modellistic starting point lies in the observation that the drivers' reaction-time to inputs should be directly proportional to the local amount of congestion. We postulate that the drivers' attention is higher in overcrowded regions while, in areas of lower density (and thus also of lower danger) the level of alertness may decrease.

Precisely, we consider a population of $N + 1$ (thinking) indistinguishable individuals (vehicles), located at positions $x_0(t), \dots, x_N(t)$ along the evolution, moving in one fixed direction according to the following system of second-order ordinary differential equations

$$\varepsilon\zeta(\rho_i(t))\ddot{x}_i(t) + \gamma\dot{x}_i(t) = \vartheta(\rho_i(t))F(t, x_i(t)), \quad t \in (0, T), \quad (1.1)$$

which we will show to fall in the class of Follow-the-Leader type models (that is, when the traffic dynamics is governed by the interaction between a vehicle and the vehicle immediately in front).

The right-hand side of (1.1) represents the forcing term, and is the product of two quantities: a congestion function ϑ and a drift F . The simplest, though most popular, choice for drifts in traffic flow models is $F \equiv 1$. The (possibly nonlinear) function ϑ describes the amount of congestion and is evaluated at $\rho_i(t)$, representing a local reconstruction of the global particle density at $x_i(t)$. We consider approximations ρ_i of the form

$$\rho_i(t) = \frac{1}{N(x_{i+1}(t) - x_i(t))}, \quad (1.2)$$

since, according to the one directional motion of vehicular flows on a line, the traffic density detected by the i -th particle depends on its proximity to the particle ahead. Hard congested traffic models are characterised by a threshold density above which the particles are stuck and cannot move anymore. This feature is encoded in (1.1) by requiring ϑ to be nonincreasing and compactly supported on $[0, \bar{\rho}_\vartheta]$. We emphasize that the monotonicity of ϑ attenuates the effect of the drift F as the density increases. Typical examples of ϑ appearing in traffic flows [37], pedestrian flows [22], animal swarming [1, 12], bacterial chemotaxis [6, 9, 23, 26], are given by $\vartheta(\rho) = (1 - \rho)_+^\alpha$ for $\alpha \geq 1$.

For what concerns the left-hand side of (1.1), $\gamma, \varepsilon > 0$ are fixed parameters, \dot{x}_i, \ddot{x}_i denote the velocity and the acceleration of the i -th particle respectively, while ζ is another nonnegative function of the density ρ_i describing the effect of the congestion on the drivers' response-time, namely on their alertness. Indeed, in the literature, the presence of second order terms in individual-based systems usually models the attentiveness of the particle to the surrounding environment. Smaller values of the coefficient in front of \ddot{x}_i correspond to a faster response of the particle; in the limit case where the coefficient is null, i.e. there is no second order term in the equation, the velocity of the particle instantaneously changes accordingly to the forcing term. For this reason, when modelling traffic flows, it feels natural to require that the coefficient of the second order term depends on how crowded the space is in front of the driver in the direction of the motion. The term $\varepsilon\zeta(\rho_i)\ddot{x}_i$ in (1.1) has precisely this role: in crowded regions (where ρ_i is close to or above a certain threshold $\bar{\rho}_\zeta$) we expect ζ to be very small or even null, thus the particle will almost instantaneously adapt its velocity, while in low-density areas (ρ_i close to 0) we expect $\zeta \approx 1$ as the level of attention drops and the particle can slowly react to inputs. Accordingly to these observations, we require ζ to be supported on an interval $[0, \bar{\rho}_\zeta]$ for some $\bar{\rho}_\zeta \leq \bar{\rho}_\vartheta$. We emphasize that the alertness function ζ plays a similar role to that of ϑ in accounting for the local level of congestion, however the two functions may differ from each other since the effect of the congestion on the forcing term or on the second order term may, in principle, not be related at all.

1.3. Main results. In this work we first establish the well-posedness of the system (1.1) under general assumptions on ζ, ϑ, F and the initial data. Notice that the presence of ζ in front of the leading order term \ddot{x}_i makes the equation highly degenerate since ζ can vanish when x_i is too close to the particle ahead. Such degeneracy leads to several mathematical difficulties and prevents one from applying standard arguments to prove existence. Moreover, equation (1.1) makes sense only if ρ_i can be defined, namely when the distance $x_{i+1} - x_i$ remains positive along the evolution, see (1.2). This is an expected behaviour of the system, indeed when x_i and x_{i+1} are very close, ζ and ϑ both vanish and formally $\dot{x}_i = 0$. Surprisingly, the rigorous proof of this observation is far from being trivial and requires a careful study of the equations. Another crucial feature of microscopic traffic flow models is that all particles move in the same direction. We show that the proposed scheme fulfills the expected behaviour, despite this being not evident from the expression (1.1).

We then study uniform bounds for suitable approximations of some relevant macroscopic quantities, e.g. density, first and second order moment, whose definitions depend on the particles trajectories

solving (1.1). The presence of the nonlinearities ζ, ϑ prevents from approximating the macroscopic density ρ in the continuum setting via the standard approach via empirical measures $\frac{1}{N+1} \sum_{i=0}^N \delta_{x_i}$ as done in [8] for first-order dynamics and in [7] for second-order ones.

Exploiting the one-dimensional structure of the dynamics, we instead rely on piece-wise constant approximation techniques as those introduced in [24] and further extended to more involved first-order transport mechanisms, including nonlocal interactions [19, 28, 40], external potentials [25] and diffusive operators [17, 18, 27], and to deal with the second-order homogeneous ARZ model [21]. We show that such approximations of the density and the first and second moment enjoy suitable uniform bounds, and thus also weak compactness in suitable topologies.

We then perform the many-particle limit of the system (1.1) as the number N of vehicles goes to infinity in the regime $\zeta = k\vartheta$, for some $k > 0$. In this case, we derive the following continuum degenerate pressureless Euler-type model

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, & \text{in } (0, T) \times \mathbb{R}, \\ \varepsilon k \vartheta(\rho) \left(\partial_t(\rho u) + \partial_x(\rho u^2) \right) + \gamma \rho u = \rho \vartheta(\rho) F, & \text{in } (0, T) \times \mathbb{R}, \end{cases} \quad (1.3)$$

where ρ represents the traffic density and u the average velocity. In this setting, we are able to prove that the weak limits obtained by the discrete approximations of the density ρ and first and second moment ρu and ρu^2 respectively, solve system (1.3) in a weak sense (see Definition 2.7).

To the best of our knowledge, this is the first convergence result for second-order models involving the nonlinearity ϑ . We do not consider here the many-particle limit for $\zeta \neq \vartheta$, since this more general case requires stronger compactness properties of the piece-wise interpolants which are highly nontrivial and will be treated in future works. However, the uniform bounds satisfied by the discrete approximations do not depend explicitly on the parameter ε and this allows for the degeneracy of the small parameter $\varepsilon = \varepsilon_N$ in the many-particle limit. We can thus perform a combined many-particle limit and asymptotic analysis as $\varepsilon \rightarrow 0$ to prove that the piece-wise interpolations converge in the limit as $N \rightarrow \infty$, and hence $\varepsilon_N \rightarrow 0$, to a solution of the first order macroscopic traffic model

$$\partial_t \rho + \frac{1}{\gamma} \partial_x(\rho \vartheta(\rho) F) = 0, \quad \text{in } (0, T) \times \mathbb{R},$$

in the sense of Theorem 2.10.

1.4. Comparison with the ARZ model. We conclude this introduction with a comparison between the proposed model and the ARZ model where, for simplicity, we set $F \equiv 1$ and the parameters $\varepsilon = \gamma = 1$. In this case, the ARZ model in its macroscopic nonlinear formulation reads as

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, & \text{in } (0, T) \times \mathbb{R}, \\ \partial_t(\rho w) + \partial_x(\rho w u) = A \rho(\vartheta(\rho) - u), & \text{in } (0, T) \times \mathbb{R}, \end{cases} \quad (1.4)$$

where $A \geq 0$ is a nonnegative constant, while w is usually referred to in the literature as a *Lagrangian marker* expressing a sort of average desired velocity of the vehicles flow and is related to the actual average velocity through the relation $w = u + P(\rho)$, where P is a pressure function depending on the density.

In the ARZ model, the congestion effects play a role exactly in the latter relation. Indeed, by the monotonicity of P , higher values of ρ (i.e. crowded regions) correspond to higher values of the pressure, and so to higher mismatch between the actual and the desired velocities u and w . Our model, on the other hand, directly deals with the first and second moment related to the actual average velocity u and encompasses congestion effects via the alertness function ζ . Indeed, in the same setting $F \equiv 1$ and $\varepsilon = \gamma = 1$, the system (1.3) (here we are not requiring $\zeta = k\vartheta$) can be written in the following form

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, & \text{in } (0, T) \times \mathbb{R}, \\ \zeta(\rho) \left(\partial_t(\rho u) + \partial_x(\rho u^2) \right) = \rho(\vartheta(\rho) - u), & \text{in } (0, T) \times \mathbb{R}. \end{cases}$$

We recall that many-particle limit results for the ARZ model (1.4) are nowadays available only in the homogeneous case $A = 0$ and, therefore, deal with microscopic models of the form

$$\begin{cases} \dot{x}_i(t) = w_i(t) - P(\rho_i(t)), \\ \dot{w}_i(t) = A(\vartheta(\rho_i(t)) - \dot{x}_i(t)) = 0, \end{cases} \quad t \in (0, T), \quad (1.5)$$

which is actually a first order system since from the second equation w_i turns out to be constant along the evolution. Thus, it is possible to apply many first-order tools [21] to deal with the homogeneous problem. To the best of our knowledge, in the present work we consider for the first time existence and many-particle limit results for a second-order traffic model involving the nonlinearity ϑ . Indeed, system (1.1) is truly of second-order nature as it can also be seen by the following reformulation (still when $F \equiv 1$ and $\varepsilon = \gamma = 1$)

$$\begin{cases} \dot{x}_i(t) = w_i(t), \\ \zeta(\rho_i(t))\dot{w}_i(t) = \vartheta(\rho_i(t)) - \dot{x}_i(t), \end{cases} \quad t \in (0, T). \quad (1.6)$$

As a common feature between the microscopic models (1.5) (also in the non-homogeneous case $A > 0$) and (1.6), we mention that in both cases the term $\vartheta(\rho_i)$ may represent an *equilibrium velocity* for the i -th particle.

Plan of the paper. In Section 2 we detail the degenerate microscopic traffic model we propose and analyse in the paper. We list all the required assumptions and we state our main results regarding existence of solutions, the limit passage to a continuum second order traffic model as the number of particles increases, and the asymptotic analysis as $\varepsilon \rightarrow 0$ leading to a first order model with nonlinear mobility. We refer respectively to Theorems 2.3, 2.8 and 2.10. Section 3 collects useful results concerning the behaviour of solutions to degenerate second order differential equations which will be employed in the work. The last three sections are devoted to the proofs of the main results: we prove the existence theorem for the discrete system in Section 4, the many-particle limit in Section 5, and finally the first-order approximation in Section 6.

Notations. The maximum (resp. minimum) of two extended real numbers $\alpha, \beta \in \mathbb{R} \cup \{\pm\infty\}$ is denoted by $\alpha \vee \beta$ (resp. $\alpha \wedge \beta$). The interior of a set A is denoted by $\overset{\circ}{A}$ or $\text{int}(A)$.

For functions $f(t)$ depending on one scalar variable representing time, we denote by $\dot{f}(t)$ its derivative. If $f = f(t, x)$ instead, we write $\partial_t f$, $\partial_x f$ for the partial derivatives with respect to time t and space x , respectively.

We adopt standard notations for Lebesgue and Sobolev spaces or spaces of continuous, Lipschitz continuous, or continuously differentiable functions. A superscript $+$ is added when referring to subspaces of nonnegative functions. By $\mathcal{M}([0, T] \times \mathbb{R})$ we denote the set of Radon measures on $[0, T] \times \mathbb{R}$, and we write $\mathcal{M}([0, T] \times \mathbb{R})^+$ for its subset of positive measures.

2. SETTING AND MAIN RESULTS

2.1. Degenerate microscopic traffic model. The microscopic traffic flow scheme we consider in this paper models the evolution of $N + 1$ ordered vehicles (or thinking particles), whose position at time $t \in [0, T]$, for some time horizon $T > 0$, is represented by the function $x_i(t)$, for $i = 0, \dots, N$. In order to describe the system, we introduce several quantities. The distance between two consecutive vehicles is denoted by $d_i(t) := x_{i+1}(t) - x_i(t)$, for $i = 0, \dots, N - 1$, and we set

$$\rho_i(t) := \frac{1}{Nd_i(t)}, \quad \text{for } i = 0, \dots, N - 1, \quad (2.1)$$

modelling a reconstruction of the macroscopic particle density at $x_i(t)$.

We also consider a nonnegative time-dependent external drift $F \in C^0([0, T] \times \mathbb{R})$, while the alertness function $\zeta \in C^0([0, \infty))$ and the congestion function $\vartheta \in C^0([0, \infty))$ are assumed to be nonnegative and to satisfy

$$\zeta(r) = 0 \quad \text{if and only if} \quad r \geq \bar{\rho}_\zeta, \quad \text{and} \quad \vartheta(r) = 0 \quad \text{if and only if} \quad r \geq \bar{\rho}_\vartheta,$$

for certain values $\bar{\rho}_\vartheta \geq \bar{\rho}_\zeta > 0$ modelling the hard congestion threshold and the instantaneous response threshold, respectively. If $\rho_i(t) \in [\bar{\rho}_\zeta, \bar{\rho}_\vartheta]$, we say that the particle $x_i(t)$ is saturated at time t , otherwise we say it is unsaturated. In the particular case $\rho_i(t) = \bar{\rho}_\vartheta$ we instead say that the particle is congested.

Given an initial configuration $x_0^0 < x_1^0 < \dots < x_N^0$ such that $\frac{1}{N(x_{i+1}^0 - x_i^0)} =: \rho_i^0 \leq \bar{\rho}_\vartheta$ for $i = 0, \dots, N-1$, we define the set of saturated indexes at the initial time as

$$\Sigma^0 = \left\{ i \in \{0, \dots, N-1\} : \rho_i^0 \in [\bar{\rho}_\zeta, \bar{\rho}_\vartheta] \right\}.$$

We allow the particle in front to move freely, namely we assume that $x_N \in C^0([0, T])$ is a given nondecreasing function which fulfils $x_N(0) = x_N^0$.

The traffic scheme we want to study consists in the following degenerate second order system of nonlinear ordinary differential equations, set in the time interval $[0, T]$:

$$\begin{cases} \varepsilon \zeta(\rho_i(t)) \ddot{x}_i(t) + \gamma \dot{x}_i(t) = \vartheta(\rho_i(t)) F(t, x_i(t)), & \text{for } i = 0, \dots, N-1, \\ x_i(0) = x_i^0, & \text{for } i = 0, \dots, N-1, \\ \dot{x}_i(0) = v_i^0, & \text{for } i \notin \Sigma^0, \end{cases} \quad (2.2)$$

where $\rho_i(t)$ has been introduced in (2.1), $\varepsilon, \gamma > 0$ are two positive parameters and $v_i^0 \geq 0$, for $i \notin \Sigma^0$, are nonnegative initial velocities. Note that prescribing initial velocities is meaningful only for unsaturated initial indexes, since for them the system is really of second order. If instead $\rho_i^0 \in [\bar{\rho}_\zeta, \bar{\rho}_\vartheta]$, i.e. $i \in \Sigma^0$, then ζ vanishes at $t = 0$ and the system collapses to a first order one, so there is no need to enforce an initial velocity (which actually, as we will see, may make the system ill-posed).

The rigorous definition of solution to (2.2) that we adopt in this paper is the following one.

Definition 2.1. Under the previous assumptions, we say that $x = (x_0, \dots, x_{N-1}) \in C^{0,1}([0, T]; \mathbb{R}^N)$ is a solution to the microscopic traffic model if each component x_i , for $i = 0, \dots, N-1$, solves (2.2) in the sense that:

- (1) $x_i(0) = x_i^0$;
- (2) the map $t \mapsto x_i(t)$ is nondecreasing in $[0, T]$;
- (3) for all $t \in [0, T]$ there hold $x_i(t) < x_{i+1}(t)$ and $\rho_i(t) \leq \bar{\rho}_\vartheta$;
- (4) the function x_i is of class C^2 in the (relatively) open set

$$\{t \in [0, T] : \rho_i(t) < \bar{\rho}_\zeta\}, \quad (2.3)$$

and therein it solves the second order equation

$$\varepsilon \zeta(\rho_i(t)) \ddot{x}_i(t) + \gamma \dot{x}_i(t) = \vartheta(\rho_i(t)) F(t, x_i(t)); \quad (2.4)$$

- (5) the function x_i is of class C^1 in the (relatively) open set

$$\{t \in [0, T] : \rho_i(t) \in (\bar{\rho}_\zeta, \bar{\rho}_\vartheta)\} \cup \text{int}\{t \in [0, T] : \rho_i(t) = \bar{\rho}_\zeta\} \cup \text{int}\{t \in [0, T] : \rho_i(t) = \bar{\rho}_\vartheta\}, \quad (2.5)$$

and therein it solves the first order equation

$$\gamma \dot{x}_i(t) = \vartheta(\rho_i(t)) F(t, x_i(t)); \quad (2.6)$$

- (6) $\dot{x}_i(0) = v_i^0$ for $i \notin \Sigma^0$, namely for the unsaturated indexes, which satisfy $\rho_i^0 < \bar{\rho}_\zeta$.

Remark 2.2. Conditions (4) and (5), together with the attainment of the initial data (1) and (6), completely specify the behaviour of the solution in the whole interval $[0, T]$. Indeed, in the remaining set $\partial\{t \in [0, T] : \rho_i(t) = \bar{\rho}_\zeta\} \cup \partial\{t \in [0, T] : \rho_i(t) = \bar{\rho}_\vartheta\}$, where a differential equation can not be imposed since the set does not contain any interval, by definition (2.1) of ρ_i one has

$$\text{either } x_i(t) = x_{i+1}(t) - \frac{1}{N\bar{\rho}_\zeta}, \quad \text{or } x_i(t) = x_{i+1}(t) - \frac{1}{N\bar{\rho}_\vartheta}.$$

Hence, the position of x_i is prescribed by the position of x_{i+1} .

We state now the first result of the paper, ensuring existence of solutions to the microscopic traffic scheme. We will prove it in Section 4.

Theorem 2.3. *Under the previous assumptions, there exists a solution x to the microscopic traffic model (2.2) in the sense of Definition 2.1 satisfying the following bound on the velocity for $i = 0, \dots, N - 1$:*

$$\dot{x}_i(t) \leq \tilde{v}_i^0 \vee \left(\frac{\bar{\vartheta}}{\gamma} \max_{[0, T] \times [x_0^0, x_N(T)]} F \right), \quad \text{for almost every } t \in (0, T), \quad (2.7)$$

where we set

$$\tilde{v}_i^0 := \begin{cases} v_i^0, & \text{if } i \notin \Sigma^0, \\ 0, & \text{if } i \in \Sigma^0, \end{cases} \quad (2.8)$$

and $\bar{\vartheta} := \max_{[0, \bar{\rho}_\vartheta]} \vartheta$.

If in addition x_N is strictly increasing and F is positive, then x_i is strictly increasing for all $i = 0, \dots, N - 1$.

By slightly strenghtening the assumptions, we also show that every solution of the microscopic traffic model never reaches the hard congestion threshold $\bar{\rho}_\vartheta$. This property will be crucial for the many-particle limit and the first-order approximation we will perform later.

Proposition 2.4. *In addition to the previous assumptions, suppose that the external drift F is positive, that the alertness function ζ is differentiable in $[0, \bar{\rho}_\zeta]$ with bounded derivative, and that $x_N \in C^1([0, T])$ fulfils $\dot{x}_N(t) > 0$ for all $t \in (0, T]$.*

Then, any solution x in the sense of Definition 2.1 which furtherly fulfils

$$\text{if } \bar{\rho}_\zeta < \bar{\rho}_\vartheta, \text{ then the set } \partial\{t \in [0, T] : \rho_i(t) = \bar{\rho}_\zeta\} \text{ is finite for all } i = 0, \dots, N - 1, \quad (2.9)$$

satisfies:

- x belongs to $C^1([0, T]; \mathbb{R}^N)$;
- $\rho_i(t) < \bar{\rho}_\vartheta$ and $\dot{x}_i(t) > 0$ for all $t \in (0, T]$ and $i = 0, \dots, N - 1$;
- $\dot{x}_i(0) = \frac{1}{\gamma} \vartheta(\rho_i^0) F(0, x_i^0)$ for $i \in \Sigma^0$.

Moreover, if $\bar{\rho}_\zeta = \bar{\rho}_\vartheta$ one also has $x \in C^2((0, T]; \mathbb{R}^N)$.

Remark 2.5. The request (2.9) is a conditional assumption we directly put on the solution x . It forbids wild oscillations of ρ_i around the value $\bar{\rho}_\zeta$ along the evolution. Although we strongly believe that every solution of the microscopic traffic model should satisfy (2.9) (at least far from $t = 0$), we are not able to show its validity. However, in the case $\bar{\rho}_\zeta = \bar{\rho}_\vartheta$, which will be the case on which we focus in the sequel when dealing with the many-particle limit and the first-order approximation, that conditional assumption is not needed and the result is completely applicable.

Remark 2.6. In case $\bar{\rho}_\zeta = \bar{\rho}_\vartheta$, the above proposition states that a solution x of the microscopic traffic model is composed by particles moving with positive velocity ($\dot{x}_i(t) > 0$) and which are never congested ($\rho_i(t) < \bar{\rho}_\vartheta$). In particular, they solve equation (2.4) for all times $t \in (0, T)$.

2.2. Discrete-to-continuum limit. The first problem we aim to analyse concerns the asymptotic behaviour of the microscopic system (2.2) as the number of individuals N grows bigger and bigger. It is well-known [2, 7], and natural, that such limiting behaviour can be captured by means of a system of partial differential equations of Euler-type. This latter continuum point of view is usually called macroscopic approach to traffic models [29, 42].

In our setting, the many-particle limit of (2.2) formally reads as the following system composed by a transport equation coupled with a degenerate partial differential equation:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, & \text{in } (0, T) \times \mathbb{R}, \\ \varepsilon \zeta(\rho) \left(\partial_t(\rho u) + \partial_x(\rho u^2) \right) + \gamma \rho u = \rho \vartheta(\rho) F, & \text{in } (0, T) \times \mathbb{R}, \\ \rho(0) = \rho^0, \quad (\rho u)(0) = e_1^0. \end{cases} \quad (2.10)$$

Above, $\rho = \rho(t, x)$ represents the density of vehicles, while $u = u(t, x)$ is their average velocity. The product terms ρu and ρu^2 instead are called first and second moments, respectively. Note that the initial conditions (ρ^0, e_1^0) are given in terms of the density and of the first moment.

We restrict our attention to the situation

$$\zeta \equiv \vartheta, \quad (2.11)$$

so that in particular there holds

$$\bar{\rho}_\zeta = \bar{\rho}_\vartheta =: \bar{\rho}. \quad (2.12)$$

Since the congestion function ϑ may vanish, we are again led to introduce a weak notion of solution for the macroscopic traffic model. The definition below is motivated by the fact that, whenever $\vartheta(\rho) > 0$, the second equation in (2.10) can be written as

$$\varepsilon \left(\partial_t(\rho u) + \partial_x(\rho u^2) \right) + \gamma \frac{\rho u}{\vartheta(\rho)} = \rho F.$$

Definition 2.7. In case (2.11), given an initial pair $(\rho^0, e_1^0) \in (L^\infty(\mathbb{R}^+))^2$ with compact support, we say that a quadruple $(\rho, e_1, e_2, \lambda)$ is a *measure solution* to the macroscopic traffic model (2.10) if:

- (i) $\rho, e_1, e_2 \in L^\infty((0, T) \times \mathbb{R})^+$ and $\lambda \in \mathcal{M}([0, T] \times \mathbb{R})^+$;
- (ii) $\|\rho\|_{L^\infty((0, T) \times \mathbb{R})} \leq \bar{\rho}$ and the (essential) supports of ρ, e_1, e_2, λ are compact;
- (iii) for all $\varphi \in C^1([0, T] \times \mathbb{R})$ with $\varphi(T) \equiv 0$ there holds

$$\begin{cases} \int_0^T \int_{\mathbb{R}} (\rho \partial_t \varphi + e_1 \partial_x \varphi) dx dt = - \int_{\mathbb{R}} \rho^0 \varphi(0) dx, \\ \varepsilon \int_0^T \int_{\mathbb{R}} (e_1 \partial_t \varphi + e_2 \partial_x \varphi) dx dt - \gamma \int_{[0, T] \times \mathbb{R}} \varphi d\lambda = -\varepsilon \int_{\mathbb{R}} e_1^0 \varphi(0) dx - \int_0^T \int_{\mathbb{R}} \rho F \varphi dx dt. \end{cases} \quad (2.13)$$

The presence of the nonlinearity ϑ in the model prevents us from performing the many-particle limit via standard approximations by empirical measures [7, 8]. We instead adopt a technique introduced in [24] (see also [18, 40]), which exploits piece-wise constant approximations. In view of Definition 2.7, the quantities that we aim to approximate and pass to the limit are the density and the first and second moments. For technical reasons, it is convenient to choose a mixed piece-wise constant and piece-wise linear approximation for the second moment, instead of a piece-wise constant one. Let us also stress (only in this and in the next section) the dependence on N of the involved discrete quantities, by adding a supplementary exponent (N) .

Let $x^{(N)}$ be a solution of the microscopic traffic model provided by Theorem 2.3, under the additional assumptions of Proposition 2.4. In particular, by (2.12), the solution $x^{(N)} \in C^1([0, T]; \mathbb{R}^N) \cap C^2((0, T]; \mathbb{R}^N)$ satisfies $\rho_i^{(N)}(t) < \bar{\rho}$ for all $i = 0, \dots, N-1$. For $(t, x) \in [0, T] \times \mathbb{R}$, we then set

$$\begin{aligned} \rho^N(t, x) &:= \sum_{i=0}^{N-1} \rho_i^{(N)}(t) \mathbb{1}_{[x_i^{(N)}(t), x_{i+1}^{(N)}(t))}(x), \\ e_1^N(t, x) &:= \sum_{i=0}^{N-1} \rho_i^{(N)}(t) \dot{x}_i^{(N)}(t) \mathbb{1}_{[x_i^{(N)}(t), x_{i+1}^{(N)}(t))}(x), \\ e_2^N(t, x) &:= \sum_{i=0}^{N-1} \rho_i^{(N)}(t) \dot{x}_i^{(N)}(t) \left(\dot{x}_i^{(N)}(t) + \frac{\dot{x}_{i+1}^{(N)}(t) - \dot{x}_i^{(N)}(t)}{x_{i+1}^{(N)}(t) - x_i^{(N)}(t)} (x - x_i^{(N)}(t)) \right) \mathbb{1}_{[x_i^{(N)}(t), x_{i+1}^{(N)}(t))}(x). \end{aligned} \quad (2.14)$$

In addition to the assumptions of section 2.1 and Proposition 2.4, we also require that there are universal constants $s, S, C > 0$ such that

$$s \leq x_0^{0(N)} \leq x_N^{(N)}(T) \leq S, \quad \sup_{i \notin \Sigma^{0(N)}} v_i^{0(N)} \leq C, \quad \sup_{t \in [0, T]} \dot{x}_N^{(N)}(t) \leq C. \quad (2.15)$$

We are now in position to state the second result of the paper, regarding the convergence, as N diverges to infinity, of the microscopic system (2.2) to the macroscopic one (2.10) when $\zeta = \vartheta$. The proof of the theorem below will be given in Section 5.

Theorem 2.8. *In addition to the hypotheses of Theorem 2.3 and Proposition 2.4, let us assume (2.11), (2.15), and that the map $x \mapsto F(t, x)$ is Lipschitz in \mathbb{R} uniformly with respect to $t \in [0, T]$. Then, the following convergences as $N \rightarrow \infty$ hold, up to a nonrelabelled subsequence, for the quantities introduced in (2.14):*

$$x_0^{(N)} \rightarrow \underline{x}, \quad x_N^{(N)} \rightarrow \bar{x}, \quad \text{uniformly in } [0, T]; \quad (2.16a)$$

$$\rho^N(0) \overset{*}{\rightharpoonup} \rho^0, \quad e_1^N(0) \overset{*}{\rightharpoonup} e_1^0, \quad \text{weakly}^* \text{ in } L^\infty(\mathbb{R}); \quad (2.16b)$$

$$\rho^N \overset{*}{\rightharpoonup} \rho, \quad e_k^N \overset{*}{\rightharpoonup} e_k, \quad \text{weakly}^* \text{ in } L^\infty((0, T) \times \mathbb{R}), \text{ for } k = 1, 2; \quad (2.16c)$$

$$\frac{e_1^N}{\vartheta(\rho^N)} dt dx \overset{*}{\rightharpoonup} \lambda, \quad \text{weakly}^* \text{ in } \mathcal{M}([0, T] \times \mathbb{R}). \quad (2.16d)$$

Moreover, ρ^0 and e_1^0 are nonnegative and have compact support, $\|\rho^0\|_{L^\infty(\mathbb{R})} \leq \bar{\rho}$ and the limit quadruple $(\rho, e_1, e_2, \lambda)$ is a measure solution to the macroscopic traffic model (2.10) with initial data (ρ^0, e_1^0) in the sense of Definition 2.7.

Finally, the (essential) supports of the four limit objects are contained in the set

$$\bigcup_{t \in [0, T]} \{t\} \times [\underline{x}(t), \bar{x}(t)].$$

Remark 2.9. We point out that in (2.16d) dividing by $\vartheta(\rho^N)$ is allowed since Proposition 2.4 and Remark 2.6 ensure that $\vartheta(\rho_i(t)) > 0$ for all $t \in (0, T]$ and $i = 0, \dots, N - 1$.

2.3. Asymptotics to first-order dynamics. A further problem we want to discuss is the joint many-particle limit and vanishing-inertia type analysis for the discrete model (2.2). More precisely, we now allow the small parameter ε in front of the second-order derivative to depend on N , i.e. $\varepsilon = \varepsilon_N$, and we let $\varepsilon_N \rightarrow 0$ as $N \rightarrow \infty$. In this way we aim at recovering the first order traffic model with mobility analysed in [24, 25, 28, 40], thus rigorously justifying it as an approximation of a more precise second order model. Indeed, at least formally, by letting the parameter $\varepsilon \rightarrow 0$ in (2.10), one obtains the following system

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, & \text{in } (0, T) \times \mathbb{R}, \\ \gamma \rho u = \rho \vartheta(\rho) F, & \text{in } (0, T) \times \mathbb{R}, \\ \rho(0) = \rho^0, \end{cases} \quad (2.17)$$

which can be equivalently written as

$$\begin{cases} \partial_t \rho + \frac{1}{\gamma} \partial_x(\rho \vartheta(\rho) F) = 0, & \text{in } (0, T) \times \mathbb{R}, \\ \rho(0) = \rho^0, \end{cases}$$

which is exactly the problem considered in [24] when $F = 1$ and $\gamma = 1$.

We will show that, still in the particular case (2.11), the joint limit $\varepsilon_N \rightarrow 0$, $N \rightarrow \infty$ of the microscopic system (2.2) provides a weak solution of (2.17), in a sense specified below. The notion of solution we recover in the limit is motivated by the formal observation that, whenever $\vartheta(\rho) > 0$, the second equation in (2.17) reads as

$$\frac{1}{\gamma} \rho F = \frac{e_1}{\vartheta(\rho)},$$

where $e_1 = \rho u$ represents the first moment.

More precisely, in Section 6 we will prove the following result.

Theorem 2.10. *Consider the same assumptions as in Theorem 2.8 and let $\varepsilon = \varepsilon_N$ in the particle system (2.2) be such that $\varepsilon_N \rightarrow 0$ as $N \rightarrow \infty$. Then, up to a nonrelabelled subsequence, the following*

convergences as $N \rightarrow \infty$ hold for the quantities introduced in (2.14):

$$x_0^{(N)} \rightarrow \underline{x}, \quad x_N^{(N)} \rightarrow \bar{x}, \quad \text{uniformly in } [0, T]; \quad (2.18a)$$

$$\rho^N(0) \overset{*}{\rightharpoonup} \rho^0, \quad \text{weakly* in } L^\infty(\mathbb{R}); \quad (2.18b)$$

$$\rho^N \overset{*}{\rightharpoonup} \rho, \quad e_1^N \overset{*}{\rightharpoonup} e_1, \quad \text{weakly* in } L^\infty((0, T) \times \mathbb{R}); \quad (2.18c)$$

$$\frac{e_1^N}{\vartheta(\rho^N)} dt dx \overset{*}{\rightharpoonup} \frac{1}{\gamma} \rho F dt dx, \quad \text{weakly* in } \mathcal{M}([0, T] \times \mathbb{R}). \quad (2.18d)$$

Moreover, $\|\rho^0\|_{L^\infty(\mathbb{R})} \leq \bar{\rho}$ and ρ^0 is nonnegative and has compact support; also, ρ and e_1 are nonnegative, $\|\rho\|_{L^\infty((0, T) \times \mathbb{R})} \leq \bar{\rho}$, and their supports are contained in the set

$$\bigcup_{t \in [0, T]} \{t\} \times [\underline{x}(t), \bar{x}(t)].$$

Finally, the pair (ρ, e_1) solves the transport equation with initial datum ρ^0 in the sense that for all $\varphi \in C^1([0, T] \times \mathbb{R})$ with $\varphi(T) \equiv 0$ one has

$$\int_0^T \int_{\mathbb{R}} (\rho \partial_t \varphi + e_1 \partial_x \varphi) dx dt = - \int_{\mathbb{R}} \rho^0 \varphi(0) dx. \quad (2.19)$$

3. PRELIMINARY TOOLS

In this section we collect some useful results regarding the behaviour of solutions to the second order differential equation

$$\beta(t) \ddot{y}(t) + \dot{y}(t) = \alpha(t) F(t, y(t)), \quad \text{for } t \in (a, b), \quad (3.1)$$

where $F \in C^0([a, b] \times \mathbb{R})$, $\alpha, \beta \in C^0([a, b])$ and β is positive in (a, b) . The link between (3.1) and (2.4) is evident.

The first lemma provides a representation formula for the derivative \dot{y} which will be used often throughout the paper.

Lemma 3.1. *Let $y \in C^2(a, b)$ be a solution to (3.1). Then, for all $a < s \leq t < b$, the following representation formula holds true:*

$$\dot{y}(t) = \dot{y}(s) e^{-\int_s^t \frac{1}{\beta(r)} dr} + \int_s^t \frac{\alpha(\tau) F(\tau, y(\tau))}{\beta(\tau)} e^{-\int_\tau^t \frac{1}{\beta(r)} dr} d\tau. \quad (3.2)$$

In particular, if in addition α and F are nonnegative, and if $y \in C^1([a, b])$ fulfils $\dot{y}(a) \geq 0$, then for all $t \in [a, b)$ one has

$$0 \leq \dot{y}(t) \leq \dot{y}(a) \vee \left(\max_{\tau \in [a, t]} \alpha(\tau) F(\tau, y(\tau)) \right). \quad (3.3)$$

Proof. In order to obtain (3.2) it is enough to divide equation (3.1) by $\beta(\tau)$ and to multiply it by $e^{\int_s^\tau \frac{1}{\beta(r)} dr}$, deducing

$$\frac{d}{d\tau} \left(\dot{y}(\tau) e^{\int_s^\tau \frac{1}{\beta(r)} dr} \right) = \frac{\alpha(\tau) F(\tau, y(\tau))}{\beta(\tau)} e^{\int_s^\tau \frac{1}{\beta(r)} dr}.$$

Then one concludes by integrating between s and t .

Under the additional assumptions, the lower bound $\dot{y}(t) \geq 0$ is then immediate. The upper bound in (3.3) instead can be proved as follows: we first estimate by using (3.2)

$$\begin{aligned} \dot{y}(t) &\leq \dot{y}(s) e^{-\int_s^t \frac{1}{\beta(r)} dr} + \max_{\tau \in [s, t]} \alpha(\tau) F(\tau, y(\tau)) \int_s^t \frac{d}{d\tau} e^{-\int_\tau^t \frac{1}{\beta(r)} dr} d\tau \\ &\leq \dot{y}(s) e^{-\int_s^t \frac{1}{\beta(r)} dr} + \max_{\tau \in [a, t]} \alpha(\tau) F(\tau, y(\tau)) \left(1 - e^{-\int_s^t \frac{1}{\beta(r)} dr} \right) \\ &\leq \dot{y}(s) \vee \left(\max_{\tau \in [a, t]} \alpha(\tau) F(\tau, y(\tau)) \right), \end{aligned}$$

and then we send $s \searrow a$. \square

Next lemma concerns the behaviour of the derivative \dot{y} near the boundary points a and b whenever β vanishes at some of them. As the Reader may notice comparing equation (3.1) with (3.4a) and (3.4b), the result is not surprising. In fact, it would be trivial if one had some a priori control on the second derivative \ddot{y} . The power of the lemma, which exploits formula (3.2), consists indeed in the validity of the result without assuming any bound on the second derivative.

Lemma 3.2. *Let $y \in C^2(a, b) \cap C^0([a, b])$ be a solution to (3.1), and suppose in addition that α and F are nonnegative, while β is differentiable in (a, b) with bounded derivative. If $\beta(b) = 0$, then there holds*

$$\lim_{t \rightarrow b^-} \dot{y}(t) = \alpha(b)F(b, y(b)). \quad (3.4a)$$

If $\beta(a) = 0$ and if \dot{y} is bounded in a right neighborhood of a , then there also holds

$$\lim_{t \rightarrow a^+} \dot{y}(t) = \alpha(a)F(a, y(a)). \quad (3.4b)$$

Proof. The proof will distinguish the cases when β^{-1} is not summable near b (near a) and when β^{-1} is summable near b (near a).

Case 1. β^{-1} *not summable near b (near a).*

We start showing (3.4a). We fix $\varepsilon > 0$, and by continuity let (b_ε, b) be such that

$$|\alpha(t)F(t, y(t)) - \alpha(b)F(b, y(b))| \leq \varepsilon, \quad \text{for } t \in (b_\varepsilon, b).$$

Then for all $t \in (b_\varepsilon, b)$, by exploiting (3.2), we can estimate

$$\begin{aligned} & |\dot{y}(t) - \alpha(b)F(b, y(b))| \\ &= \left| \left(\dot{y}(b_\varepsilon) - \alpha(b)F(b, y(b)) \right) e^{-\int_{b_\varepsilon}^t \frac{1}{\beta(r)} dr} + \int_{b_\varepsilon}^t \left(\alpha(\tau)F(\tau, y(\tau)) - \alpha(b)F(b, y(b)) \right) \frac{d}{d\tau} e^{-\int_\tau^t \frac{1}{\beta(r)} dr} d\tau \right| \\ &\leq \left| \left(\dot{y}(b_\varepsilon) - \alpha(b)F(b, y(b)) \right) e^{-\int_{b_\varepsilon}^t \frac{1}{\beta(r)} dr} + \varepsilon \left(1 - e^{-\int_{b_\varepsilon}^t \frac{1}{\beta(r)} dr} \right) \right|. \end{aligned}$$

Sending $t \rightarrow b^-$, since β^{-1} is not summable (and positive) near b , we deduce

$$\limsup_{t \rightarrow b^-} |\dot{y}(t) - \alpha(b)F(b, y(b))| \leq \varepsilon,$$

whence (3.4a) follows by the arbitrariness of ε .

In order to show (3.4b) when β^{-1} is not summable near a , we begin by sending $s \rightarrow a^+$ in (3.2), obtaining

$$\dot{y}(t) = \int_a^t \alpha(\tau)F(\tau, y(\tau)) \frac{d}{d\tau} e^{-\int_\tau^t \frac{1}{\beta(r)} dr} d\tau, \quad \text{for all } t \in (a, b). \quad (3.5)$$

We now fix $\varepsilon > 0$, and by continuity let (a, a_ε) be such that

$$|\alpha(t)F(t, y(t)) - \alpha(a)F(a, y(a))| \leq \varepsilon, \quad \text{for } t \in (a, a_\varepsilon).$$

By using (3.5) and the fact that $e^{-\int_a^t \frac{1}{\beta(r)} dr} = 0$, for all $t \in (a, a_\varepsilon)$ we now infer

$$|\dot{y}(t) - \alpha(a)F(a, y(a))| = \left| \int_a^t \left(\alpha(\tau)F(\tau, y(\tau)) - \alpha(a)F(a, y(a)) \right) \frac{d}{d\tau} e^{-\int_\tau^t \frac{1}{\beta(r)} dr} d\tau \right| \leq \varepsilon,$$

and so also (3.4b) is proved.

Case 2. β^{-1} *summable near b (near a).*

By (3.2) we observe that, fixing $s \in (a, b)$, for all $t \in [s, b)$ we have

$$\dot{y}(t) = e^{-\int_s^t \frac{1}{\beta(r)} dr} \left(\dot{y}(s) + \int_s^t \frac{\alpha(\tau)F(\tau, y(\tau))}{\beta(\tau)} e^{\int_s^\tau \frac{1}{\beta(r)} dr} d\tau \right).$$

Since β^{-1} is summable near b , the above identity together with the boundedness of α and F yield the existence of the finite limit

$$v_b := \lim_{t \rightarrow b^-} \dot{y}(t). \quad (3.6)$$

Analogously, if β^{-1} is summable near a , the identity

$$\dot{y}(s) = e^{\int_s^t \frac{1}{\beta(r)} dr} \left(\dot{y}(t) - \int_s^t \frac{\alpha(\tau)F(\tau, y(\tau))}{\beta(\tau)} e^{-\int_\tau^t \frac{1}{\beta(r)} dr} d\tau \right), \quad \text{for } s \in (a, t],$$

provides the existence of the finite limit

$$v_a := \lim_{s \rightarrow a^+} \dot{y}(s). \quad (3.7)$$

Now, from equation (3.1) we deduce that for all $a < s \leq t < b$ we have

$$\beta(t)\dot{y}(t) - \beta(s)\dot{y}(s) = \int_s^t \left(\dot{\beta}(\tau)\dot{y}(\tau) + \beta(\tau)\ddot{y}(\tau) \right) d\tau = \int_s^t \left(\dot{\beta}(\tau)\dot{y}(\tau) + \alpha(\tau)F(\tau, y(\tau)) - \dot{y}(\tau) \right) d\tau.$$

Letting $t \rightarrow b^-$ (and relabelling s as t), or $s \rightarrow a^+$, since $\beta(b) = 0$ or $\beta(a) = 0$, for all $t \in (a, b)$ we obtain

$$\begin{aligned} \dot{y}(t) &= \frac{1}{\beta(t)} \int_b^t \left(\dot{\beta}(\tau)\dot{y}(\tau) + \alpha(\tau)F(\tau, y(\tau)) - \dot{y}(\tau) \right) d\tau \\ &= \frac{1}{\beta(t)} \int_a^t \left(\dot{\beta}(\tau)\dot{y}(\tau) + \alpha(\tau)F(\tau, y(\tau)) - \dot{y}(\tau) \right) d\tau. \end{aligned}$$

We denote the integrals in the first and second line above by $N_b(t)$ and $N_a(t)$, respectively, and we focus on N_b , being the other case analogous (using (3.7) instead of (3.6)). Since both N_b and β are continuous in $[t, b]$, differentiable in (t, b) for all $t \in (a, b)$, and since $N_b(b) = \beta(b) = 0$, by Cauchy Theorem there exists $\xi_t \in (t, b)$ such that

$$N_b(t)\dot{\beta}(\xi_t) = \dot{N}_b(\xi_t)\beta(t). \quad (3.8)$$

Fix now an arbitrary sequence $t_k \rightarrow b^-$ (hence $\xi_{t_k} \rightarrow b^-$). Then there are two possibilities.

1. $\dot{\beta}(\xi_{t_k}) \neq 0$ *definitively*.

In this case, by combining (3.6) and (3.8), we have

$$\begin{aligned} v_b &= \lim_{k \rightarrow \infty} \dot{y}(t_k) = \lim_{k \rightarrow \infty} \frac{N_b(t_k)}{\beta(t_k)} = \lim_{k \rightarrow \infty} \frac{\dot{N}_b(\xi_{t_k})}{\dot{\beta}(\xi_{t_k})} = \lim_{k \rightarrow \infty} \left(\dot{y}(\xi_{t_k}) + \frac{\alpha(\xi_{t_k})F(\xi_{t_k}, y(\xi_{t_k})) - \dot{y}(\xi_{t_k})}{\dot{\beta}(\xi_{t_k})} \right) \\ &= v_b + \lim_{k \rightarrow \infty} \frac{\alpha(\xi_{t_k})F(\xi_{t_k}, y(\xi_{t_k})) - \dot{y}(\xi_{t_k})}{\dot{\beta}(\xi_{t_k})}, \end{aligned}$$

whence

$$\lim_{k \rightarrow \infty} \frac{\alpha(\xi_{t_k})F(\xi_{t_k}, y(\xi_{t_k})) - \dot{y}(\xi_{t_k})}{\dot{\beta}(\xi_{t_k})} = 0. \quad (3.9)$$

Since $\dot{\beta}$ is bounded by assumption, the numerator above must vanish in order for (3.9) to be true. Hence we obtain $v_b = \alpha(b)F(b, y(b))$ and (3.4a) is proved.

2. *There exists a subsequence for which $\dot{\beta}(\xi_{t_{k_j}}) = 0$.*

In this second case, since β is positive in (a, b) by assumption, by using (3.8) we infer

$$0 = \dot{N}_b(\xi_{t_{k_j}}) = \alpha(\xi_{t_{k_j}})F(\xi_{t_{k_j}}, y(\xi_{t_{k_j}})) - \dot{y}(\xi_{t_{k_j}}),$$

whence we deduce $v_b = \alpha(b)F(b, y(b))$ by letting $j \rightarrow \infty$ and recalling (3.6). So (3.4a) is proved also in this case, and we conclude.

The validity of (3.4b) follows by the very same argument. \square

We conclude this section by showing how a function of the form $\tau \mapsto \frac{1}{\beta(\tau)} e^{-\int_\tau^t \frac{1}{\beta(r)} dr}$, appearing in (3.2), is an approximation of a Dirac delta centered at t , as soon as β is small enough.

Lemma 3.3. *Let $\{f_n\}$ and $\{g_n\}$ be sequences of continuous functions in $[a, b]$. Assume that f_n uniformly converges to f in $[a, b]$ as $n \rightarrow \infty$, and that g_n is positive and vanishes uniformly in $[a, b]$ as $n \rightarrow \infty$. Then for all $t \in (a, b]$ there holds*

$$\lim_{n \rightarrow \infty} \int_a^t \frac{f_n(\tau)}{g_n(\tau)} e^{-\int_\tau^t \frac{1}{g_n(r)} dr} d\tau = f(t).$$

Proof. We begin by observing that for all $t \in (a, b]$ the following facts are true:

- (a) $\int_a^t \frac{1}{g_n(\tau)} e^{-\int_\tau^t \frac{1}{g_n(r)} dr} d\tau = 1 - e^{-\int_a^t \frac{1}{g_n(r)} dr}$;
- (b) $\lim_{n \rightarrow \infty} e^{-\int_a^t \frac{1}{g_n(r)} dr} = 0$;
- (c) $\lim_{n \rightarrow \infty} \int_a^t e^{-\int_\tau^t \frac{1}{g_n(r)} dr} d\tau = 0$.

Identity (a) is simply the Fundamental Theorem of Calculus, while the limits in (b) and (c) can be proved arguing as follows. Since g_n vanishes uniformly, for any $\varepsilon > 0$ we definitively have $\|g_n\|_{C^0([a, b])} \leq \varepsilon$, whence

$$e^{-\int_a^t \frac{1}{g_n(r)} dr} \leq e^{-\frac{t-a}{\varepsilon}}, \quad \text{and} \quad \int_a^t e^{-\int_\tau^t \frac{1}{g_n(r)} dr} d\tau \leq \int_a^t e^{-\frac{t-\tau}{\varepsilon}} d\tau = \varepsilon \left(1 - e^{-\frac{t-a}{\varepsilon}}\right).$$

Sending first $n \rightarrow \infty$ and then $\varepsilon \searrow 0$ we obtain (b) and (c).

We now fix $t \in (a, b]$, and by means of (a) we estimate

$$\begin{aligned} & \left| \int_a^t \frac{f_n(\tau)}{g_n(\tau)} e^{-\int_\tau^t \frac{1}{g_n(r)} dr} d\tau - f(t) \right| \\ & \leq \int_a^t |f_n(\tau) - f(\tau)| \frac{1}{g_n(\tau)} e^{-\int_\tau^t \frac{1}{g_n(r)} dr} d\tau + \left| \int_a^t (f(\tau) - f(t)) \frac{1}{g_n(\tau)} e^{-\int_\tau^t \frac{1}{g_n(r)} dr} d\tau \right| + |f(t)| e^{-\int_a^t \frac{1}{g_n(r)} dr} \\ & \leq \|f_n - f\|_{C^0([a, b])} + |f(t)| e^{-\int_a^t \frac{1}{g_n(r)} dr} + \left| \int_a^t (f(\tau) - f(t)) \frac{1}{g_n(\tau)} e^{-\int_\tau^t \frac{1}{g_n(r)} dr} d\tau \right|. \end{aligned} \quad (3.10)$$

The first two terms in the last line above vanish as $n \rightarrow \infty$ by assumption and by (b), respectively; so we conclude if we prove that also the third term vanishes.

For the sake of clarity we set $h(\tau) := f(\tau) - f(t)$, and we consider a sequence $\{h_m\}$ of smooth functions uniformly converging to h in $[a, t]$. Denoting by $J_n(t)$ the last term in (3.10), integrating by parts we infer

$$\begin{aligned} J_n(t) & \leq \int_a^t |h(\tau) - h_m(\tau)| \frac{1}{g_n(\tau)} e^{-\int_\tau^t \frac{1}{g_n(r)} dr} d\tau + \left| \int_a^t h_m(\tau) \frac{1}{g_n(\tau)} e^{-\int_\tau^t \frac{1}{g_n(r)} dr} d\tau \right| \\ & \leq \|h - h_m\|_{C^0([a, t])} + \left| h_m(t) - h_m(a) e^{-\int_a^t \frac{1}{g_n(r)} dr} - \int_a^t \dot{h}_m(\tau) e^{-\int_\tau^t \frac{1}{g_n(r)} dr} d\tau \right| \\ & \leq \|h - h_m\|_{C^0([a, t])} + |h_m(t)| + |h_m(a)| e^{-\int_a^t \frac{1}{g_n(r)} dr} + \|\dot{h}_m\|_{C^0([a, t])} \int_a^t e^{-\int_\tau^t \frac{1}{g_n(r)} dr} d\tau. \end{aligned}$$

Letting $n \rightarrow \infty$, by using (b) and (c) we deduce

$$\limsup_{n \rightarrow \infty} J_n(t) \leq \|h - h_m\|_{C^0([a, t])} + |h_m(t)|,$$

and sending then $m \rightarrow \infty$ we conclude since $h(t) = f(t) - f(t) = 0$. \square

4. EXISTENCE OF SOLUTIONS TO THE DISCRETE SYSTEM

This section is devoted to the proofs of Theorem 2.3 and Proposition 2.4.

Proof of Theorem 2.3. The proof is entirely based on the following inductive step.

If $x_{i+1} \in C^0([0, T])$ is nondecreasing and fulfils $x_{i+1}(0) = x_{i+1}^0$, then there exists $x_i \in C^{0,1}([0, T])$ solution to (2.2) in the sense of Definition 2.1 satisfying (2.7). Assuming additionally that F is positive, if x_{i+1} is strictly increasing, then x_i is strictly increasing as well.

It is clear that the inductive step above entails the claim of the theorem. For this reason, what follows is devoted to its proof, which we split in several steps for simplicity.

Step 1. Approximated problem and compactness.

For $\delta > 0$ we consider the approximated problem

$$\begin{cases} \varepsilon(\zeta(\rho_i^\delta(t)) + \delta)\ddot{x}_i^\delta(t) + \gamma\dot{x}_i^\delta(t) = \vartheta(\rho_i^\delta(t))F(t, x_i^\delta(t)), & \text{for } t \in (0, T), \\ x_i^\delta(0) = x_i^0, & \dot{x}_i^\delta(0) = \tilde{v}_i^0, \end{cases} \quad (4.1)$$

where \tilde{v}_i^0 has been introduced in (2.8), while we define $\rho_i^\delta(t) := (Nd_i^\delta(t))^{-1}$ and $d_i^\delta(t) = x_{i+1}(t) - x_i^\delta(t)$. Noting that in (4.1) the coefficient of the second derivative is no more degenerate thanks to the addition of the parameter δ , by the Peano Theorem there exists a local solution $x_i^\delta \in C^2([0, \hat{T}_\delta])$ for some $\hat{T}_\delta > 0$, which we will show to be independent of δ .

Without loss of generality, we can assume that \hat{T}_δ is the first time at which $x_i^\delta(\hat{T}_\delta) = \frac{x_{i+1}^0 + x_i^0}{2}$; indeed, if $x_i^\delta(t) < \frac{x_{i+1}^0 + x_i^0}{2}$ for all the evolution, then x_i^δ exists in the whole $[0, T]$.

By Lemma 3.1 (with $\alpha(t) = \frac{1}{\gamma}\vartheta(\rho_i^\delta(t))$ and $\beta(t) = \frac{\varepsilon}{\gamma}(\zeta(\rho_i^\delta(t)) + \delta)$) we deduce that

$$0 \leq \dot{x}_i^\delta(t) \leq \tilde{v}_i^0 \vee \left(\frac{\bar{\vartheta}}{\gamma} \max_{[0, T] \times [x_0^0, x_N(T)]} F \right), \quad \text{for all } t \in [0, \hat{T}_\delta].$$

Since from the above estimate the velocity \dot{x}_i^δ is bounded by a constant independent of δ , and since clearly the distance $x_{i+1}^0 - x_i^0$ does not depend on δ , we deduce that \hat{T}_δ does not depend on δ as well.

Summarising, we have proved that there exists $\hat{T} > 0$ such that $x_i^\delta \in C^2([0, \hat{T}])$ satisfies

$$x_i^0 \leq x_i^\delta(t) \leq \frac{x_{i+1}^0 + x_i^0}{2}, \quad 0 \leq \dot{x}_i^\delta(t) \leq \tilde{v}_i^0 \vee \left(\frac{\bar{\vartheta}}{\gamma} \max_{[0, T] \times [x_0^0, x_N(T)]} F \right), \quad \text{for all } t \in [0, \hat{T}]. \quad (4.2)$$

Hence, by Ascoli-Arzelà Theorem, we have that, up to passing to a subsequence, $x_i^\delta \rightarrow x_i$ uniformly in $[0, \hat{T}]$ as $\delta \rightarrow 0$, for a certain nondecreasing function $x_i \in C^{0,1}([0, \hat{T}])$, which satisfies

$$0 \leq \dot{x}_i(t) \leq \tilde{v}_i^0 \vee \left(\frac{\bar{\vartheta}}{\gamma} \max_{[0, T] \times [x_0^0, x_N(T)]} F \right), \quad \text{for almost every } t \in [0, \hat{T}],$$

attains the initial condition $x_i(0) = x_i^0$, and fulfils $x_i(t) < x_{i+1}(t)$ in $[0, \hat{T}]$. Moreover, the fact that the distance between x_i^δ and x_{i+1}^δ is uniformly bounded from below and far from 0 entails a uniform bounds on the densities and, hence, the uniform convergence $\rho_i^\delta \rightarrow \rho_i$ in $[0, \hat{T}]$.

Step 2. Existence of a local solution x_i .

We now aim to show that the limit function x_i is a local solution of the microscopic traffic model. For the moment, we have seen that points (1), (2) of Definition 2.1 are fulfilled by x_i (in $[0, \hat{T}]$). We then need to show that

$$\rho_i(t) \leq \bar{\rho}_\vartheta, \quad \text{for all } t \in [0, \hat{T}]. \quad (4.3)$$

For the sake of contradiction, if the claim above is false, then there exists $\bar{t} \in (0, \hat{T}]$ such that $\rho_i(\bar{t}) > \bar{\rho}_\vartheta$ (i.e. $d_i(\bar{t}) < (N\bar{\rho}_\vartheta)^{-1}$). By continuity, there exists $a \in [0, \bar{t})$ such that $\rho_i(a) = \bar{\rho}_\vartheta$ and $\rho_i(t) > \bar{\rho}_\vartheta$ in $(a, \bar{t}]$. We consider now a sequence $\{a_k\} \subseteq (a, \bar{t})$ converging to a as $k \rightarrow \infty$. Observe that $\rho_i^\delta(t) \geq \bar{\rho}_\vartheta$ definitively in $[a_k, \bar{t}]$, thanks to the uniform convergence of ρ_i^δ to ρ_i in $[0, \hat{T}]$.

In particular, $\zeta(\rho_i^\delta(t)) = \vartheta(\rho_i^\delta(t)) = 0$ for $t \in [a_k, \bar{t}]$, and so by (3.2) we obtain

$$\dot{x}_i^\delta(t) = \dot{x}_i^\delta(a_k) e^{-\frac{\gamma}{\varepsilon} \frac{t-a_k}{\delta}}, \quad \text{for all } t \in [a_k, \bar{t}],$$

whence $\lim_{\delta \rightarrow 0} \dot{x}_i^\delta(t) = 0$ for all $t \in (a_k, \bar{t}]$.

Since x_{i+1} is nondecreasing, we have

$$d_i^\delta(\bar{t}) = x_{i+1}(\bar{t}) - x_i^\delta(\bar{t}) \geq x_{i+1}(a_k) - x_i^\delta(a_k) - \int_{a_k}^{\bar{t}} \dot{x}_i^\delta(\tau) d\tau = d_i^\delta(a_k) - \int_{a_k}^{\bar{t}} \dot{x}_i^\delta(\tau) d\tau.$$

Passing to the limit as $\delta \rightarrow 0$, by Dominated Convergence Theorem we thus deduce that $d_i(\bar{t}) \geq d_i(a_k)$ and eventually passing to the limit as $k \rightarrow \infty$ (so that $a_k \rightarrow a$), we obtain

$$d_i(\bar{t}) \geq d_i(a) = \frac{1}{N\rho_i(a)} = \frac{1}{N\bar{\rho}_\vartheta},$$

which contradicts the fact that $d_i(\bar{t}) < (N\bar{\rho}_\vartheta)^{-1}$. Therefore the claim (4.3) is proved, and so x_i satisfies point (3) of Definition 2.1, too.

Let us now focus on proving points (4) and (6). For all $k \in \mathbb{N}$, we set

$$A_i^k := \left\{ t \in [0, \widehat{T}] : \zeta(\rho_i(t)) > \frac{1}{k} \right\}, \quad \text{so that} \quad \bigcup_{k \in \mathbb{N}} A_i^k = \left\{ t \in [0, \widehat{T}] : \zeta(\rho_i(t)) > 0 \right\},$$

namely the set in (2.3). Clearly, it is enough to prove that x_i solves equation (2.4) in each connected component I_j of A_i^k , where for the sake of clarity we omit the dependence on i, k . By uniform convergence, one has $\zeta(\rho_i^\delta) \geq \frac{1}{2k}$ in I_j for δ sufficiently small, thus from the equation we deduce, recalling (4.2), that

$$\varepsilon |\ddot{x}_i^\delta(t)| \leq \frac{\bar{\theta}}{\frac{1}{2k}} \max_{[0, T] \times [x_0^0, x_N(T)]} F + \gamma \frac{\dot{x}_i^\delta(t)}{\frac{1}{2k}} \leq 4k \left[(\gamma v_i^0) \vee \left(\bar{\vartheta} \max_{[0, T] \times [x_0^0, x_N(T)]} F \right) \right], \quad \text{in } I_j.$$

By Ascoli-Arzelà Theorem, we now obtain that

$$\dot{x}_i^\delta \rightarrow \dot{x}_i, \quad \text{uniformly in } \bar{I}_j, \text{ as } \delta \rightarrow 0,$$

so that $\dot{x}_i \in C^{0,1}(\bar{I}_j)$ and condition (6) is automatically fulfilled by x_i , since for unsaturated initial positions one has $0 \in A_i^k$ for some k .

Now, for all $\varphi \in C_c^\infty(\bar{I}_j)$, recalling that $\zeta(\rho_i^\delta) \geq \frac{1}{2k}$ in I_j (for δ sufficiently small), we can test equation (4.1) with φ , divide by $\zeta(\rho_i^\delta(\tau)) + \delta$ and integrate over I_j , obtaining

$$\gamma \int_{I_j} \frac{\dot{x}_i^\delta(\tau)}{\zeta(\rho_i^\delta(\tau)) + \delta} \varphi(\tau) d\tau = \int_{I_j} \left(\varepsilon \dot{x}_i^\delta(\tau) \dot{\varphi}(\tau) + \frac{\vartheta(\rho_i^\delta(\tau))}{\zeta(\rho_i^\delta(\tau)) + \delta} F(\tau, x_i^\delta(\tau)) \varphi(\tau) \right) d\tau.$$

Letting $\delta \rightarrow 0$, we deduce (recalling that $\zeta(\rho_i) > \frac{1}{k}$ in I_j)

$$\gamma \int_{I_j} \frac{\dot{x}_i(\tau)}{\zeta(\rho_i(\tau))} \varphi(\tau) d\tau = \int_{I_j} \left(\varepsilon \dot{x}_i(\tau) \dot{\varphi}(\tau) + \frac{\vartheta(\rho_i(\tau))}{\zeta(\rho_i(\tau))} F(\tau, x_i(\tau)) \varphi(\tau) \right) d\tau,$$

whence, by the arbitrariness of φ , we infer that $x_i \in C^2(\bar{I}_j)$ and that (4) is proved.

It remains only to show that x_i also satisfies condition (5). Observe that it is enough to prove the statement for any open interval (a, b) contained in (2.5). Since ρ_i^δ uniformly converges to ρ_i in the whole $[0, \widehat{T}]$ as $\delta \rightarrow 0$, in particular we have

$$\zeta(\rho_i^\delta) \rightarrow \zeta(\rho_i) \equiv 0, \quad \text{uniformly in } [a, b] \text{ as } \delta \rightarrow 0.$$

We now recall that formula (3.2) yields

$$\gamma \dot{x}_i^\delta(t) = \gamma \dot{x}_i^\delta(a) e^{-\frac{\gamma}{\varepsilon} \int_a^t \frac{1}{\zeta(\rho_i^\delta(\tau)) + \delta} dr} + \frac{\gamma}{\varepsilon} \int_a^t \frac{\vartheta(\rho_i^\delta(\tau)) F(\tau, x_i^\delta(\tau))}{\zeta(\rho_i^\delta(\tau)) + \delta} e^{-\frac{\gamma}{\varepsilon} \int_\tau^t \frac{1}{\zeta(\rho_i^\delta(\tau)) + \delta} dr} d\tau, \quad \text{for } t \in (a, b).$$

So, by exploiting Lemma 3.3 (see also (b) within its proof) with $f_\delta(t) = \vartheta(\rho_i^\delta(t)) F(t, x_i^\delta(t))$ and $g_\delta(t) = \frac{\varepsilon}{\gamma} (\zeta(\rho_i^\delta(t)) + \delta)$, we deduce that

$$\lim_{\delta \rightarrow 0} \gamma \dot{x}_i^\delta(t) = \vartheta(\rho_i(t)) F(t, x_i(t)), \quad \text{for all } t \in (a, b).$$

Fixing $\varphi \in C_c^\infty(a, b)$, the above limit implies

$$-\gamma \int_a^b x_i(\tau) \dot{\varphi}(\tau) d\tau = \lim_{\delta \rightarrow 0} -\gamma \int_a^b x_i^\delta(\tau) \dot{\varphi}(\tau) d\tau = \lim_{\delta \rightarrow 0} \gamma \int_a^b \dot{x}_i^\delta(\tau) \varphi(\tau) d\tau = \int_a^b \vartheta(\rho_i(\tau)) F(\tau, x_i(\tau)) \varphi(\tau) d\tau,$$

whence we obtain that $x_i \in C^1(a, b)$ and that equation (2.6) is fulfilled in (a, b) . This yields (5) and so x_i is a local solution in the sense of Definition 2.1.

Step 3. Extension of the local solution to the whole interval $[0, T]$.

Let \bar{T} be the maximal time of existence of x_i , namely

$$\bar{T} = \sup \left\{ t \in (0, T] : \begin{array}{l} x_i \in C^{0,1}([0, t]) \text{ satisfies conditions (1)-(6) in } [0, t], \\ \text{and } \dot{x}_i(\tau) \leq \tilde{v}_i^0 \vee \left(\frac{\bar{\vartheta}}{\gamma} \max_{[0, T] \times [x_0^0, x_N(T)]} F \right) \text{ for almost every } \tau \in [0, t] \end{array} \right\}. \quad (4.4)$$

We argue for the sake of contradiction, and assume that $\bar{T} < T$. Since $x_{i+1} \in C^0([0, T])$ by inductive assumption, by conditions (2) and (3) the following limit exists by monotonicity and is finite:

$$x_i(\bar{T}) = \lim_{t \rightarrow \bar{T}^-} x_i(t).$$

Moreover, by (3), we also deduce that $\rho_i(\bar{T}) \leq \bar{\rho}_\vartheta$, so that $x_i(\bar{T}) < x_{i+1}(\bar{T})$. By the uniform bound on the derivative, we finally infer that $x_i \in C^{0,1}([0, \bar{T}])$ with $\dot{x}_i(\tau) \leq \tilde{v}_i^0 \vee \left(\frac{\bar{\vartheta}}{\gamma} \max_{[0, T] \times [x_0^0, x_N(T)]} F \right)$ almost everywhere in $[0, \bar{T}]$.

If $\rho_i(\bar{T}) \in [\bar{\rho}_\zeta, \bar{\rho}_\vartheta]$, we do not need to give a velocity for x_i at \bar{T} . On the other hand, if $\rho_i(\bar{T}) < \bar{\rho}_\zeta$, by continuity it must exist $\tilde{\rho} \in (0, \bar{\rho}_\zeta)$ and $\tilde{T} \in (0, \bar{T})$ such that $\rho_i(t) \leq \tilde{\rho} < \bar{\rho}_\zeta$ for all $t \in [\tilde{T}, \bar{T}]$. Thus, there exists $\tilde{\zeta} > 0$ such that $\zeta(\rho_i(t)) \geq \tilde{\zeta} > 0$ for all $t \in [\tilde{T}, \bar{T}]$ and so x_i solves the equation

$$\varepsilon \zeta(\rho_i(t)) \ddot{x}_i(t) + \gamma \dot{x}_i(t) = \vartheta(\rho_i(t)) F(t, x_i(t)), \quad \text{in } (\tilde{T}, \bar{T}).$$

Since $\zeta(\rho_i(t)) \geq \tilde{\zeta} > 0$ for all $t \in [\tilde{T}, \bar{T}]$, we also deduce that the solution x_i is actually defined and of class C^2 in $[\tilde{T}, \bar{T}]$, whence $\dot{x}_i(\bar{T})$ exists and is (necessarily) nonnegative and satisfies the bound in (4.4).

In both the saturated and unsaturated cases it is then possible to check that $x_i \in C^{0,1}([0, \bar{T}])$ satisfies all conditions (1)-(6) in $[0, \bar{T}]$ and thus \bar{T} actually realizes the maximum in (4.4).

We can now repeat the local existence argument of Steps 1 and 2 with \bar{T} as starting time, thus finding a solution $\check{x}_i \in C^{0,1}([\bar{T}, \check{T}])$ with initial data $x_i(\bar{T})$ and $\dot{x}_i(\bar{T})$, fulfilling conditions (1)-(6) in $[\bar{T}, \check{T}]$ for some $\check{T} \in (\bar{T}, T]$ and satisfying

$$\dot{\check{x}}_i(\tau) \leq \dot{x}_i(\bar{T}) \vee \left(\frac{\bar{\vartheta}}{\gamma} \max_{[0, T] \times [x_0^0, x_N(T)]} F \right) \leq \tilde{v}_i^0 \vee \left(\frac{\bar{\vartheta}}{\gamma} \max_{[0, T] \times [x_0^0, x_N(T)]} F \right), \quad \text{for almost every } \tau \in (\bar{T}, \check{T}).$$

By gluing x_i and \check{x}_i we finally obtain a new Lipschitz continuous function satisfying conditions (1)-(6) in the whole $[0, \check{T}]$ and with Lipschitz constant bounded by $\tilde{v}_i^0 \vee \left(\frac{\bar{\vartheta}}{\gamma} \max_{[0, T] \times [x_0^0, x_N(T)]} F \right)$. We thus

contradict the maximality of \bar{T} , and we conclude Step 3.

Step 4. If F is positive and x_{i+1} is strictly increasing, then x_i is strictly increasing.

Assume by contradiction that there exist $0 \leq a < b \leq T$ such that $x_i(a) = x_i(b)$. Since x_i is nondecreasing, this means that x_i is constant in $[a, b]$. Observe that the interval (a, b) can not intersect the open set $\{\rho_i < \bar{\rho}_\zeta\} \cup \{\bar{\rho}_\zeta < \rho_i < \bar{\rho}_\vartheta\}$, indeed constant functions do not solve equation (2.4) nor equation (2.6) due to the positivity of F . We thus infer that (a, b) is contained either in the set $\{\rho_i = \bar{\rho}_\zeta\}$ or in the set $\{\rho_i = \bar{\rho}_\vartheta\}$. In both cases, by the definition (2.1) of ρ_i , one deduces that x_{i+1} is constant in (a, b) . We thus contradict the inductive assumption, and we conclude the proof of Step 4 and of the whole theorem. \square

We now move to the proof of Proposition 2.4.

Proof of Proposition 2.4. Let x be a solution to the microscopic traffic model fulfilling in addition (2.9). As in the proof of Theorem 2.3 we argue by induction, showing the following:

If $x_{i+1} \in C^1([0, T])$ satisfies $\dot{x}_{i+1}(t) > 0$ for all $t \in (0, T]$, then x_i satisfies

- x_i belongs to $C^1([0, T])$;

- $\rho_i(t) < \bar{\rho}_\vartheta$ and $\dot{x}_i(t) > 0$ for all $t \in (0, T]$;
- $\dot{x}_i(0) = \frac{1}{\gamma} \vartheta(\rho_i^0) F(0, x_i^0)$ if $i \in \Sigma^0$;
- if $\bar{\rho}_\zeta = \bar{\rho}_\vartheta$, then x_i belongs to $C^2((0, T])$.

Step 1. $\rho_i(t) < \bar{\rho}_\vartheta$ **for all** $t \in (0, T]$.

First of all, let us prove that the set $\{t \in (0, T] : \rho_i(t) < \bar{\rho}_\vartheta\}$ is nonempty and 0 is an accumulation point for it. By contradiction, assume there exists an open interval $(0, s) \subseteq \{t \in (0, T] : \rho_i(t) = \bar{\rho}_\vartheta\}$. Thus, by condition (5) in Definition 2.1, we infer that $x_i \in C^1(0, s)$ satisfies

$$\gamma \dot{x}_i(t) = \vartheta(\rho_i(t)) F(t, x_i(t)) = \vartheta(\bar{\rho}_\vartheta) F(t, x_i(t)) = 0, \quad \text{for } t \in (0, s),$$

namely $x_i(t) \equiv x_i^0$ for all $t \in [0, s]$. By definition (2.1) of ρ_i , we hence deduce that

$$x_{i+1}(t) = x_i^0 + \frac{1}{N \bar{\rho}_\vartheta}, \quad \text{for } t \in (0, s),$$

which contradicts the inductive assumption $\dot{x}_{i+1}(t) > 0$ for all $t \in (0, T]$.

We now prove the claim of Step 1. Assume by contradiction that there exists $b \in (0, T]$ such that $\rho_i(b) = \bar{\rho}_\vartheta$. Since we proved that 0 is an accumulation point for the set $\{t \in (0, T] : \rho_i(t) < \bar{\rho}_\vartheta\}$, there also exists $a \in (0, b)$ satisfying $\rho_i(a) < \bar{\rho}_\vartheta$. Without loss of generality, we may assume that b is the first point greater than a at which ρ_i reaches the threshold $\bar{\rho}_\vartheta$, namely $\rho_i(t) < \bar{\rho}_\vartheta$ for all $t \in (a, b)$ and $\rho_i(b) = \bar{\rho}_\vartheta$. Equivalently, we may write

$$d_i(t) > \frac{1}{N \bar{\rho}_\vartheta}, \quad \text{for all } t \in (a, b), \quad \text{and} \quad d_i(b) = \frac{1}{N \bar{\rho}_\vartheta}. \quad (4.5)$$

If $\bar{\rho}_\zeta < \bar{\rho}_\vartheta$, up to possibly choosing a larger a , we can assume without loss of generality that $\bar{\rho}_\zeta < \rho_i(t) < \bar{\rho}_\vartheta$ in (a, b) , and so by condition (5) we know that $x_i \in C^1(a, b)$ satisfies

$$\gamma \dot{x}_i(t) = \vartheta(\rho_i(t)) F(t, x_i(t)), \quad \text{for } t \in (a, b).$$

In particular, from the continuity of ϑ one deduces

$$\lim_{t \rightarrow b^-} \dot{x}_i(t) = 0. \quad (4.6)$$

We now show that we reach the same conclusion also in the case $\bar{\rho}_\zeta = \bar{\rho}_\vartheta$. Indeed, by condition (4) of Definition 2.1, we know that $x_i \in C^2(a, b)$ satisfies

$$\varepsilon \zeta(\rho_i(t)) \ddot{x}_i(t) + \gamma \dot{x}_i(t) = \vartheta(\rho_i(t)) F(t, x_i(t)), \quad \text{for } t \in (a, b).$$

We aim to apply Lemma 3.2, so we need to check that the map $t \mapsto \zeta(\rho_i(t))$ is differentiable in (a, b) with bounded derivative. We may directly compute

$$\frac{d}{dt} (\zeta \circ \rho_i)(t) = \zeta'(\rho_i(t)) \dot{\rho}_i(t) = -N \zeta'(\rho_i(t)) \rho_i(t)^2 (\dot{x}_{i+1}(t) - \dot{x}_i(t)), \quad \text{for all } t \in (a, b). \quad (4.7)$$

Since ζ' is bounded by assumption, $\rho_i(t) \leq \bar{\rho}_\vartheta$, x_{i+1} is of class $C^1([0, T])$ and x_i is Lipschitz continuous in the whole $[0, T]$, we deduce that the right-hand side of (4.7) is bounded in (a, b) . We are thus in position to apply Lemma 3.2 and deduce that (4.6) holds true also in this case.

Summing up, the limit in (4.6) yields that \dot{x}_i is differentiable from the left in b and $\dot{x}_i^-(b) = 0$. In particular, also d_i is differentiable from the left in b and it holds

$$\dot{d}_i^-(b) = \dot{x}_{i+1}(b) - \dot{x}_i^-(b) = \dot{x}_{i+1}(b). \quad (4.8)$$

This leads to a contradiction. Indeed, by (4.5) it must hold $\dot{d}_i^-(b) \leq 0$, which contradicts the inductive assumption on x_{i+1} , namely $\dot{x}_{i+1}(b) > 0$, by using (4.8).

Step 1 is thus concluded.

Step 2. $x_i \in C^1([0, T])$ **satisfies** $\dot{x}_i(t) > 0$ **for all** $t \in (0, T]$.

We distinguish between the cases $\bar{\rho}_\zeta < \bar{\rho}_\vartheta$ and $\bar{\rho}_\zeta = \bar{\rho}_\vartheta$.

We begin with the former case. Fix $\bar{t} \in (0, T]$. If \bar{t} belongs to the set (2.3), then x_i is differentiable at \bar{t} and $\dot{x}_i(\bar{t}) > 0$ by the expression (3.2). If instead \bar{t} belongs to the set (2.5), again x_i is differentiable at \bar{t} and the property $\dot{x}_i(\bar{t}) > 0$ directly follows from the equation (2.6) (we recall that by Step 1 we know that $\rho_i(\bar{t}) < \bar{\rho}_\vartheta$). We just need to check the case $\bar{t} \in \partial\{t \in [0, T] : \rho_i(t) = \bar{\rho}_\zeta\}$. Since by (2.9)

that set consists of isolated points, we can find $a < \bar{t} < b$ such that the intervals (a, \bar{t}) and (\bar{t}, b) are contained in one of the open sets in (2.3) or (2.5). By using directly the first order equation (2.6), or by means of Lemma 3.2, we deduce that in any case the derivative $\dot{x}_i(\bar{t})$ exists and it holds $\gamma \dot{x}_i(\bar{t}) = \vartheta(\bar{\rho}_\zeta)F(\bar{t}, x_i(\bar{t})) > 0$. A similar argument shows that x_i is differentiable (from the right) also in 0 and that x_i is of class $C^1([0, T])$.

The case $\bar{\rho}_\zeta = \bar{\rho}_\vartheta$ is simpler. Indeed, by Step 1 we know that $(0, T] \subseteq \{t \in [0, T] : \rho_i(t) < \bar{\rho}_\zeta\}$, and so by Definition 2.1 we automatically have $x_i \in C^2((0, T])$, and by the expression (3.2) we may infer $\dot{x}_i(t) > 0$ for $t \in (0, T]$. If $\rho_i^0 < \bar{\rho}_\zeta$, the same argument yields $x_i \in C^2([0, T])$; otherwise we may use Lemma 3.2, deducing that $\dot{x}_i(0)$ exists, it holds $\gamma \dot{x}_i(0) = \vartheta(\bar{\rho}_\vartheta)F(0, x_i(0)) = 0$, and also x_i is of class $C^1([0, T])$.

Step 3. Conclusion of the proof.

We are left to show the last two points of the inductive step. Actually, the last one have been already proved at the end of Step 2.

So, assume that $i \in \Sigma^0$. If $\rho_i^0 > \bar{\rho}_\zeta$, then necessarily x_i solves the first order equation (2.6) in a right neighborhood of 0. This implies $\dot{x}_i(0) = \frac{1}{\gamma} \vartheta(\rho_i^0)F(0, x_i^0)$. If instead $\rho_i^0 = \bar{\rho}_\zeta$, by assumption (2.9) there exists a right neighborhood of 0 in which x_i solves either the second order equation (2.4) or the first order one (2.6). By using Lemma 3.2 in the former case, or directly by the equation in the second case, one obtains $\dot{x}_i(0) = \frac{1}{\gamma} \vartheta(\rho_i^0)F(0, x_i^0)$, and so we conclude the proof of the proposition. \square

5. MANY-PARTICLE LIMIT

The content of this section is the proof of Theorem 2.8, so we tacitly assume its hypotheses. To this aim, we first provide uniform estimates on the discrete interpolants (2.14) which will also be used in the next section to prove Theorem 2.10. We stress that we are assuming that (2.11) is in force, therefore (2.12) applies.

Proposition 5.1. *Under the assumptions of Theorem 2.8, the following uniform bounds hold:*

- $\|\rho^N\|_{L^\infty((0, T) \times \mathbb{R})} \leq \bar{\rho}$;
- $\|e_k^N\|_{L^\infty((0, T) \times \mathbb{R})} \leq C$ for $k = 1, 2$;
- $\left\| \frac{e_1^N}{\vartheta(\rho^N)} \right\|_{L^1((0, T) \times \mathbb{R})} \leq C$;

where the constant $C > 0$ depends neither on N nor on ε .

In particular, either if ε is fixed or if $\varepsilon = \varepsilon_N \rightarrow 0$, up to nonrelabelled subsequences, as $N \rightarrow \infty$ we infer

$$\rho^N \overset{*}{\rightharpoonup} \rho, \quad e_k^N \overset{*}{\rightharpoonup} e_k, \quad \text{weakly}^* \text{ in } L^\infty((0, T) \times \mathbb{R}), \text{ for } k = 1, 2; \quad (5.1a)$$

$$\frac{e_1^N}{\vartheta(\rho^N)} dt dx \overset{*}{\rightharpoonup} \lambda, \quad \text{weakly}^* \text{ in } \mathcal{M}([0, T] \times \mathbb{R}). \quad (5.1b)$$

Proof. By using condition (3) in Definition 2.1 it immediately follows that

$$\|\rho^N\|_{L^\infty((0, T) \times \mathbb{R})} \leq \bar{\rho}.$$

Moreover, the latter bound together with (2.7), (2.15) and the very definition of the interpolants (2.14), imply that

$$\|e_k^N\|_{L^\infty((0, T) \times \mathbb{R})} \leq C, \quad \text{for } k = 1, 2.$$

Finally, we note that, by using also equation (2.4) (recall that here $\zeta = \vartheta$), there holds

$$\begin{aligned} \gamma \left\| \frac{e_1^N}{\vartheta(\rho^N)} \right\|_{L^1((0, T) \times \mathbb{R})} &= \sum_{i=0}^{N-1} \int_0^T \int_{x_i(t)}^{x_{i+1}(t)} \gamma \frac{\rho_i(t) \dot{x}_i(t)}{\vartheta(\rho_i(t))} dx dt = \frac{1}{N} \sum_{i=0}^{N-1} \int_0^T \gamma \frac{\dot{x}_i(t)}{\vartheta(\rho_i(t))} dt \\ &= \frac{1}{N} \sum_{i=0}^{N-1} \int_0^T (F(t, x_i(t)) - \varepsilon \ddot{x}_i(t)) dt \leq T \max_{[0, T] \times [s, S]} F - \frac{\varepsilon}{N} \sum_{i=0}^{N-1} (\dot{x}_i(T) - \dot{x}_i(0)) \\ &\leq T \max_{[s, S]} F + \varepsilon \sup_{i \notin \Sigma^0} v_i^0 \leq C. \end{aligned}$$

So all the listed uniform bounds are proved. The weak* convergences in (5.1a) and (5.1b) now directly follow from classical weak* compactness. \square

We now compute the remainder terms which appear when plugging the previously introduced discrete interpolants in (2.10) (actually in (2.13)). We will then prove that such remainders vanish as $N \rightarrow \infty$.

To this aim, we will make use several times of the following identity

$$\dot{\rho}_i(t) = -\frac{1}{N} \frac{\dot{x}_{i+1}(t) - \dot{x}_i(t)}{(x_{i+1}(t) - x_i(t))^2} = -\rho_i(t) \frac{\dot{x}_{i+1}(t) - \dot{x}_i(t)}{x_{i+1}(t) - x_i(t)}, \quad \text{for } t \in (0, T) \text{ and } i = 0, \dots, N-1. \quad (5.2)$$

We also recall the Leibniz rule, namely for $\varphi \in C^1([0, T] \times \mathbb{R})$ there holds

$$\frac{d}{dt} \left(\int_{x_i(t)}^{x_{i+1}(t)} \varphi(t, x) dx \right) = \int_{x_i(t)}^{x_{i+1}(t)} \partial_t \varphi(t, x) dx + \dot{x}_{i+1}(t) \varphi(t, x_{i+1}(t)) - \dot{x}_i(t) \varphi(t, x_i(t)). \quad (5.3)$$

Finally, let us state the following simple summation by parts lemma, which will be used later on.

Lemma 5.2. *For any real sequences $\{a_i\}$, $\{b_i\}$, and for all $N \geq 1$ one has:*

$$\sum_{i=0}^{N-1} (a_{i+1} - a_i) b_i = a_N b_{N-1} - a_0 b_0 - \sum_{i=0}^{N-2} (b_{i+1} - b_i) a_{i+1}.$$

Proof. It is enough to compute

$$\sum_{i=0}^{N-1} (a_{i+1} - a_i) b_i = \sum_{i=0}^{N-1} a_{i+1} b_i - \sum_{i=0}^{N-1} a_i b_i = a_N b_{N-1} + \sum_{i=0}^{N-2} a_{i+1} b_i - a_0 b_0 - \sum_{i=0}^{N-2} a_{i+1} b_{i+1},$$

and we conclude. \square

We start our analysis by showing that the pair (ρ^N, e_1^N) is an almost solution of the transport equation, i.e. the first equation in (2.13).

Proposition 5.3. *For all $\varphi \in C^2([0, T] \times \mathbb{R})$ with $\varphi(T) \equiv 0$, it holds*

$$\int_0^T \int_{\mathbb{R}} \left(\rho^N \partial_t \varphi + e_1^N \partial_x \varphi \right) dx dt + \int_{\mathbb{R}} \rho^N(0) \varphi(0) dx = \frac{\mathcal{R}^N(\varphi)}{N}, \quad (5.4)$$

where the term $\mathcal{R}^N(\varphi)$ is given by

$$\mathcal{R}^N(\varphi) = \frac{1}{2} \sum_{i=0}^{N-1} \int_0^T \left(\dot{x}_{i+1}(t) - \dot{x}_i(t) \right) \left(\int_{x_i(t)}^{x_{i+1}(t)} \partial_{xx} \varphi(t, x) \left(\frac{x - x_i(t)}{x_{i+1}(t) - x_i(t)} \right)^2 dx - \partial_x \varphi(t, x_{i+1}(t)) \right) dt.$$

Proof. Using the definitions of ρ^N and e_1^N we obtain

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} \left(\rho^N \partial_t \varphi + e_1^N \partial_x \varphi \right) dx dt \\ &= \sum_{i=0}^{N-1} \int_0^T \rho_i(t) \left(\int_{x_i(t)}^{x_{i+1}(t)} \partial_t \varphi(t, x) dx + \dot{x}_i(t) \left(\varphi(t, x_{i+1}(t)) - \varphi(t, x_i(t)) \right) \right) dt = (*). \end{aligned}$$

By exploiting (5.3) and integrating by parts in time we continue the above equality

$$\begin{aligned} (*) &= - \sum_{i=0}^{N-1} \rho_i^0 \int_{x_i^0}^{x_{i+1}^0} \varphi(0, x) dx \\ &\quad - \sum_{i=0}^{N-1} \int_0^T \left(\dot{\rho}_i(t) \int_{x_i(t)}^{x_{i+1}(t)} \varphi(t, x) dx + \rho_i(t) \varphi(t, x_{i+1}(t)) \left(\dot{x}_{i+1}(t) - \dot{x}_i(t) \right) \right) dt. \end{aligned}$$

By using (5.2), we now infer

$$\begin{aligned}
(*) &= - \int_{\mathbb{R}} \rho^N(0) \varphi(0, x) dx \\
&+ \frac{1}{N} \sum_{i=0}^{N-1} \int_0^T (\dot{x}_{i+1}(t) - \dot{x}_i(t)) \frac{\int_{x_i(t)}^{x_{i+1}(t)} \varphi(t, x) dx - (x_{i+1}(t) - x_i(t)) \varphi(t, x_{i+1}(t))}{(x_{i+1}(t) - x_i(t))^2} dt.
\end{aligned}$$

We can then conclude by using Taylor expansion with integral remainder:

$$\begin{aligned}
&\frac{\int_{x_i(t)}^{x_{i+1}(t)} \varphi(t, x) dx - (x_{i+1}(t) - x_i(t)) \varphi(t, x_{i+1}(t))}{(x_{i+1}(t) - x_i(t))^2} \\
&= \frac{1}{2} \left(\int_{x_i(t)}^{x_{i+1}(t)} \partial_{xx} \varphi(t, x) \frac{(x - x_i(t))^2}{(x_{i+1}(t) - x_i(t))^2} dx - \partial_x \varphi(t, x_{i+1}(t)) \right).
\end{aligned}$$

□

Let us now show that the numerator $\mathcal{R}^N(\varphi)$ of the remainder term in (5.4) is uniformly bounded.

Proposition 5.4. *With the notations and the assumptions of Proposition 5.3, we have that*

$$|\mathcal{R}^N(\varphi)| \leq C \left(\|\partial_x \varphi\|_{L^\infty((0,T) \times (s,S))} + \|\partial_{xx} \varphi\|_{L^\infty((0,T) \times (s,S))} \right), \quad (5.5)$$

where the constant C depends neither on N nor on ε .

Proof. By the explicit expression of $\mathcal{R}^N(\varphi)$ we can estimate

$$\begin{aligned}
|\mathcal{R}^N(\varphi)| &\leq \frac{1}{2} \sum_{i=0}^{N-1} \int_0^T (\dot{x}_{i+1}(t) + \dot{x}_i(t)) \int_{x_i(t)}^{x_{i+1}(t)} |\partial_{xx} \varphi(t, x)| \left(\frac{x - x_i(t)}{x_{i+1}(t) - x_i(t)} \right)^2 dx dt \\
&+ \frac{1}{2} \left| \sum_{i=0}^{N-1} \int_0^T (\dot{x}_{i+1}(t) - \dot{x}_i(t)) \partial_x \varphi(t, x_{i+1}(t)) dt \right| =: I^N + II^N.
\end{aligned}$$

For the sake of clarity, in the following estimates we will use the shortcut $\|\cdot\|_\infty$ in place of the heavy expression $\|\cdot\|_{L^\infty((0,T) \times (s,S))}$. The first term I^N can be bounded by

$$\begin{aligned}
I^N &\leq \frac{\|\partial_{xx} \varphi\|_\infty}{2} \sum_{i=0}^{N-1} \int_0^T (\dot{x}_{i+1}(t) + \dot{x}_i(t)) (x_{i+1}(t) - x_i(t)) dt \\
&= \frac{\|\partial_{xx} \varphi\|_\infty}{2} \left(\int_0^T (\dot{x}_N(t) + \dot{x}_{N-1}(t)) (x_N(t) - x_{N-1}(t)) dt + \sum_{i=0}^{N-2} \int_0^T (\dot{x}_{i+1}(t) + \dot{x}_i(t)) (x_{i+1}(t) - x_i(t)) dt \right) \\
&\leq \frac{\|\partial_{xx} \varphi\|_\infty}{2} \left[(S-s)(x_N(T) - x_N^0 + x_{N-1}(T) - x_{N-1}^0) + 2 \sup_{i=0, \dots, N-1} \|\dot{x}_i\|_{L^\infty(0,T)} \int_0^T (x_{N-1}(t) - x_0(t)) dt \right] \\
&\leq \|\partial_{xx} \varphi\|_\infty (S-s) \left(S-s + T \sup_{i=0, \dots, N-1} \|\dot{x}_i\|_{L^\infty(0,T)} \right). \quad (5.6)
\end{aligned}$$

Concerning the second term II^N we exploit Lemma 5.2 obtaining

$$\begin{aligned}
II^N &= \frac{1}{2} \left| \int_0^T (\dot{x}_N(t) \partial_x \varphi(t, x_N(t)) - \dot{x}_0(t) \partial_x \varphi(t, x_1(t))) dt - \sum_{i=0}^{N-2} \int_0^T (\partial_x \varphi(t, x_{i+2}(t)) - \partial_x \varphi(t, x_{i+1}(t))) \dot{x}_{i+1}(t) dt \right| \\
&\leq \frac{\|\partial_x \varphi\|_\infty}{2} (x_N(T) - x_N^0 + x_0(T) - x_0^0) + \frac{\|\partial_{xx} \varphi\|_\infty}{2} \int_0^T \sum_{i=0}^{N-2} \dot{x}_{i+1}(t) (x_{i+2}(t) - x_{i+1}(t)) dt \\
&\leq \|\partial_x \varphi\|_\infty (S - s) + \frac{\|\partial_{xx} \varphi\|_\infty}{2} \sup_{i=0, \dots, N-1} \|\dot{x}_i\|_{L^\infty(0, T)} \int_0^T (x_N(t) - x_1(t)) dt \\
&\leq (S - s) \left(\|\partial_x \varphi\|_\infty + T \frac{\|\partial_{xx} \varphi\|_\infty}{2} \sup_{i=0, \dots, N-1} \|\dot{x}_i\|_{L^\infty(0, T)} \right). \tag{5.7}
\end{aligned}$$

By combining (5.6) and (5.7) we now conclude by means of (2.7) and (2.15). \square

We now focus on the second equation in (2.13).

Proposition 5.5. *For all $\varphi \in C^1([0, T] \times \mathbb{R})$ with $\varphi(T) \equiv 0$, it holds*

$$\int_0^T \int_{\mathbb{R}} \left[\varepsilon (e_1^N \partial_t \varphi + e_2^N \partial_x \varphi) + \rho^N F \varphi - \gamma \frac{e_1^N}{\vartheta(\rho^N)} \varphi \right] dx dt + \varepsilon \int_{\mathbb{R}} e_1^N(0) \varphi(0) dx = \frac{\mathcal{S}^N(\varphi)}{N}, \tag{5.8}$$

where the term $\mathcal{S}^N(\varphi)$ is given by

$$\mathcal{S}^N(\varphi) = N \sum_{i=0}^{N-1} \int_0^T \rho_i(t) \int_{x_i(t)}^{x_{i+1}(t)} \left(F(t, x) - F(t, x_i(t)) \right) \varphi(t, x) dx dt.$$

Proof. By using the definitions of e_1^N and e_2^N we compute

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \left[\varepsilon (e_1^N \partial_t \varphi + e_2^N \partial_x \varphi) + \rho^N F \varphi - \gamma \frac{e_1^N}{\vartheta(\rho^N)} \varphi \right] dx dt \\
&= \sum_{i=0}^{N-1} \int_0^T \left[\varepsilon \rho_i(t) \dot{x}_i(t) \int_{x_i(t)}^{x_{i+1}(t)} \partial_t \varphi(t, x) dx \right. \\
&\quad + \varepsilon \rho_i(t) \dot{x}_i(t) \int_{x_i(t)}^{x_{i+1}(t)} \left(\dot{x}_i(t) + \frac{\dot{x}_{i+1}(t) - \dot{x}_i(t)}{x_{i+1}(t) - x_i(t)} (x - x_i(t)) \right) \partial_x \varphi(t, x) dx \\
&\quad \left. + \rho_i(t) \int_{x_i(t)}^{x_{i+1}(t)} F(t, x) \varphi(t, x) dx - \gamma \rho_i(t) \frac{\dot{x}_i(t)}{\vartheta(\rho_i(t))} \int_{x_i(t)}^{x_{i+1}(t)} \varphi(t, x) dx \right] dt = (*).
\end{aligned}$$

Exploiting (5.2), (5.3), and integrating by parts in time the first term of the right-hand side and in space the second one, recalling also that $\varphi(T) \equiv 0$, we can continue the above chain of equalities obtaining

$$\begin{aligned}
(*) &= -\varepsilon \sum_{i=0}^{N-1} \rho_i^0 \dot{x}_i(0) \int_{x_i^0}^{x_{i+1}^0} \varphi(0, x) dx \\
&\quad - \varepsilon \sum_{i=0}^{N-1} \int_0^T \left[\left(\dot{\rho}_i(t) \dot{x}_i(t) + \rho_i(t) \ddot{x}_i(t) \right) \int_{x_i(t)}^{x_{i+1}(t)} \varphi(t, x) dx \right. \\
&\quad \left. + \rho_i(t) \dot{x}_i(t) \left(\dot{x}_{i+1}(t) \varphi(t, x_{i+1}(t)) - \dot{x}_i(t) \varphi(t, x_i(t)) \right) \right] dt \\
&\quad + \varepsilon \sum_{i=0}^{N-1} \int_0^T \rho_i(t) \dot{x}_i(t) \left(\dot{x}_{i+1}(t) \varphi(t, x_{i+1}(t)) - \dot{x}_i(t) \varphi(t, x_i(t)) - \frac{\dot{x}_{i+1}(t) - \dot{x}_i(t)}{x_{i+1}(t) - x_i(t)} \int_{x_i(t)}^{x_{i+1}(t)} \varphi(t, x) dx \right) dt \\
&\quad + \sum_{i=0}^{N-1} \int_0^T \left[\rho_i(t) \int_{x_i(t)}^{x_{i+1}(t)} F(t, x) \varphi(t, x) dx - \gamma \rho_i(t) \frac{\dot{x}_i(t)}{\vartheta(\rho_i(t))} \int_{x_i(t)}^{x_{i+1}(t)} \varphi(t, x) dx \right] dt
\end{aligned}$$

$$\begin{aligned}
&= -\varepsilon \int_{\mathbb{R}} e_1^N(0, x) \varphi(0, x) dx - \sum_{i=0}^{N-1} \int_0^T \rho_i(t) \left(\varepsilon \ddot{x}_i(t) + \gamma \frac{\dot{x}_i(t)}{\vartheta(\rho_i(t))} \right) \int_{x_i(t)}^{x_{i+1}(t)} \varphi(t, x) dx \\
&\quad + \sum_{i=0}^{N-1} \int_0^T \rho_i(t) \int_{x_i(t)}^{x_{i+1}(t)} F(t, x) \varphi(t, x) dx dt.
\end{aligned}$$

By using equation (2.4), with $\zeta = \vartheta$, and recalling that we may divide by $\vartheta(\rho_i(t))$ since it is positive in $(0, T]$ by Proposition 2.4 and Remark 2.6 we finally infer

$$(*) = -\varepsilon \int_{\mathbb{R}} e_1^N(0, x) \varphi(0, x) dx + \sum_{i=0}^{N-1} \int_0^T \rho_i(t) \int_{x_i(t)}^{x_{i+1}(t)} \left(F(t, x) - F(t, x_i(t)) \right) \varphi(t, x) dx dt,$$

and we conclude. \square

We finally prove that also the term $\mathcal{S}^N(\varphi)$ is uniformly bounded.

Proposition 5.6. *With the notations and the assumptions of Proposition 5.5, we have that*

$$|\mathcal{S}^N(\varphi)| \leq L_F T (S - s) \|\varphi\|_{L^\infty((0, T) \times (s, S))}, \quad (5.9)$$

where L_F denotes the supremum in $[0, T]$ of the Lipschitz constants of the map $x \mapsto F(t, x)$.

Proof. It is enough to observe that

$$\begin{aligned}
|\mathcal{S}^N(\varphi)| &\leq \sum_{i=0}^{N-1} \int_0^T \int_{x_i(t)}^{x_{i+1}(t)} \frac{|F(t, x) - F(t, x_i(t))|}{x_{i+1}(t) - x_i(t)} |\varphi(t, x)| dx dt \\
&\leq L_F \|\varphi\|_{L^\infty((0, T) \times (s, S))} \sum_{i=0}^{N-1} \int_0^T (x_{i+1}(t) - x_i(t)) dt \\
&= L_F \|\varphi\|_{L^\infty((0, T) \times (s, S))} \int_0^T (x_N(t) - x_0(t)) dt \leq L_F \|\varphi\|_{L^\infty((0, T) \times \mathbb{R})} T (S - s).
\end{aligned}$$

\square

We are now in position to prove Theorem 2.8.

Proof of Theorem 2.8. The uniform bounds on the initial data and on x_N (2.15) together with the uniform estimate in (2.7) directly yield (2.16a), by means of Ascoli-Arzelà Theorem. The convergences in (2.16c) and (2.16d) have already been proved in Proposition 5.1, and similarly one can show (2.16b). The statements on the supports then follow by weak lower semicontinuity, since by definition the supports of ρ^N, e_1^N and e_2^N are contained in

$$\bigcup_{t \in [0, T]} \{t\} \times [x_0(t), x_N(t)].$$

We are just let to prove that the limit quadruple $(\rho, e_1, e_2, \lambda)$ solves (2.10) in the sense of Definition 2.7. In particular, we only need to check condition (iii), since conditions (i) and (ii) are automatically fulfilled because the interpolants (2.14) are nonnegative functions (with uniformly bounded supports). To do so, we let $N \rightarrow \infty$ in (5.4) and (5.8). By exploiting Propositions 5.4 and 5.6, together with the known weak convergence of the involved quantities, we deduce that the limit functions solve system (2.13) for all $\varphi \in C^2([0, T] \times \mathbb{R})$ with $\varphi(T) \equiv 0$. A simple density argument now shows that the equation is satisfied also for $\varphi \in C^1([0, T] \times \mathbb{R})$ with $\varphi(T) \equiv 0$, and so we conclude. \square

6. FIRST ORDER APPROXIMATION

In this last section we prove Theorem 2.10. We recall that here the parameter $\varepsilon = \varepsilon_N$ vanishes as $N \rightarrow \infty$. The strategy of the proof follows the lines of previous section, with the only difference that in the current situation we will show in addition that all terms multiplied by ε_N actually go to zero in the many-particle limit.

Proof of Theorem 2.10. Thanks to the assumption (2.15) and by the uniform estimate (2.7) we deduce that x_0 and x_N are bounded in $W^{1,\infty}(0, T)$, hence by Ascoli-Arzelà Theorem we infer (2.18a). In Proposition 5.1 we already proved the validity of (2.18c) and (2.18d), and similarly also (2.18b) can be deduced. Moreover, the statements on the sign and on the supports can easily be checked by arguing as in the proof of Theorem 2.8.

Proposition 5.1 also ensures that the sequence $\frac{e_1^N}{\vartheta(\rho^N)} dt dx$ weakly* converges in the sense of measures to some (positive) Radon measure λ on $[0, T] \times \mathbb{R}$. In order to conclude, we thus need to show that $\lambda = \frac{1}{\gamma} \rho F dt dx$ and prove that ρ and e_1 satisfy (2.19).

To this aim, we first fix $\varphi \in C^2([0, T] \times \mathbb{R})$ with $\varphi(T) \equiv 0$ and we recall that Proposition 5.3 established that

$$\int_0^T \int_{\mathbb{R}} \left(\rho^N \partial_t \varphi + e_1^N \partial_x \varphi \right) dx dt + \int_{\mathbb{R}} \rho^N(0) \varphi(0) dx = \frac{\mathcal{R}^N(\varphi)}{N}.$$

Now, by (5.5) we deduce that

$$\frac{|R^N(\varphi)|}{N} \leq C \frac{\|\partial_x \varphi\|_{L^\infty((0, T) \times (s, S))} + \|\partial_{xx} \varphi\|_{L^\infty((0, T) \times (s, S))}}{N} \longrightarrow 0, \quad \text{as } N \rightarrow \infty,$$

thus (2.19) is proved (by density one deduces its validity for $\varphi \in C^1([0, T] \times \mathbb{R})$).

On the other hand, arguing similarly as in Propositions 5.5, for any $\varphi \in C^1([0, T] \times \mathbb{R})$ (without the request on the final time condition) we obtain

$$\begin{aligned} \varepsilon_N \int_0^T \int_{\mathbb{R}} \left(e_1^N \partial_t \varphi + e_2^N \partial_x \varphi \right) dx dt + \int_0^T \int_{\mathbb{R}} \left(\rho^N F - \gamma \frac{e_1^N}{\vartheta(\rho^N)} \right) \varphi dx dt \\ + \varepsilon_N \int_{\mathbb{R}} \left(e_1^N(0) \varphi(0) - e_1^N(T) \varphi(T) \right) dx = \frac{\mathcal{S}^N(\varphi)}{N}, \end{aligned} \quad (6.1)$$

where the remainder $\mathcal{S}^N(\varphi)/N$ vanishes as $N \rightarrow \infty$ due to (5.9). By using (2.7), the uniform bounds of Proposition 5.1, and recalling (2.15), it is then easy to see that all the term multiplied by ε_N in (6.1) are bounded by a universal constant. Since $\varepsilon_N \rightarrow 0$, from the weak*-type convergences $\rho^N \xrightarrow{*} \rho$ and $\frac{e_1^N}{\vartheta(\rho^N)} dt dx \xrightarrow{*} \lambda$, we deduce that for every $\varphi \in C^1([0, T] \times \mathbb{R})$ the following identity holds true:

$$\int_0^T \int_{\mathbb{R}} \rho F \varphi dx dt = \gamma \int_{[0, T] \times \mathbb{R}} \varphi d\lambda.$$

Since, by density, the above equality is still true for all $\varphi \in C^0([0, T] \times \mathbb{R})$ we finally infer that $\lambda = \frac{1}{\gamma} \rho F dt dx$ and we conclude the proof. \square

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