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Stable periodic configurations in nonlocal sharp interface models

Abstract. This paper collects results obtained by the author together with Chen Chao-Nien, Choi Yung Sze, Nicola Fusco, Vesa Julin and Massimiliano Morini (in various groupings) in the last years; it is intended to be an introduction to the "geometric" perspective on some physical problems. Equilibrium models based on energy competition between volume and surface terms, in connection with nonlocal effects, got special attention in recent investigations, as their critical points exhibit various patterns with high degree of symmetry. There is interest in both finding the possible equilibrium shapes, and (which is the object of the present works) proving that they actually are (local) isolated minimizers. Particularly the latter has been thoroughly investigated for lamellar configurations in a model with long-range interaction governed by a screened Coulomb kernel. A section with open problems concludes the paper.

Keywords. Lamella; stability; sharp interface model; nonlocal geometric variational problem.

Mathematics Subject Classification: 49J20, 49K20, 49Q10, 92C15, 35K57.

1 - Introduction

In the last decades, many researchers devoted tremendous efforts in studying mathematical mechanisms responsible for pattern formation in nature; the present bibliography will therefore be necessarily hugely incomplete. Only some references, regardless of priority, will be given here; surely most contributors to the theory will find their names here or in the bibliographies of quoted papers. Also, no attempt will be made to rigorously explain the underlying physical or chemical phenomena. Fields in which pattern formation occurs include, for example, ferroelectric and ferromagnetic films, diblock copolymers and degenerate ferromagnetic semiconductors, [8, 10, 19, 21, 22, 23, 26, 27, 28, 29, 32]. The cases reported in this paper concern equilibrium models based on a free energy functional with long range interaction whose typical form is [7, 13, 14, 15, 24, 30, 31]

$$\begin{aligned}\mathcal{J}_\epsilon(u) &= MM_\epsilon(u) + NL(u) \\ &= \int_\Omega \left(\frac{\epsilon}{2}|\nabla u|^2 + \epsilon^{-1}F(u)\right) dx \, + \gamma \int_\Omega \int_\Omega \psi(u(x))G(x,\xi)\psi(u(\xi))d\xi dx \,, \qquad (1.1)\end{aligned}$$

where u is a scalar function, F is a double-well potential, G is a positive kernel, ψ is a given smooth function, ϵ is a small positive parameter and $\Omega \subset \mathbb{R}^N$ is a given bounded domain. A well-known example of G is the Green's function associated with a uniformly elliptic operator. These energies are made of a (local) Modica-Mortola term MM_ϵ governing short-range behaviour [24] and a nonlocal term NL which takes into account the long-range effects; the first part favours the presence of coexisting phases induced by the two wells, with a layer of rapid change in between representing the interface; when ϵ is small, the resulting structure of nearly sharp transition interfaces defines the pattern. This turns (1.1) into a competition between short and long-range interactions; who is winning depends on the precise tuning of the control parameters. The short-range MM_ϵ term leads to congregation, favoring large domains of pure phases with boundary shape that minimizes its surface area; at the same time the long-range effect NL is repulsive in nature biasing towards small domains. Further players in the game may be volume constraints, or boundary effects.

We will focus on two cases, the Ohta-Kawasaki model for diblock copolymers and the (simplified) FitzHugh-Nagumo model of reaction-diffusion. For diblock copolymers, the observed mesoscopic domains are highly regular periodic structures that include spheres, cylindrical tubes, lamellae and double-gyroids [10]. On the mathematical side, it was proposed [29] to study the critical points of a functional like (1.1) with G being the Green's function for the Laplace operator subject to the homogeneous Neumann boundary conditions or periodic boundary conditions, see also [5, 6, 16, 18, 20, 25, 28]:

$$(1.2) \qquad (OK_\epsilon) \quad \int_\Omega \left(\frac{\epsilon}{2}|\nabla u|^2 + \frac{(u^2-1)^2}{4\epsilon})\right) dx + \gamma \int_\Omega \int_\Omega (u(x)-m)G(x,\xi)(u(\xi)-m)d\xi dx$$

with prescribed mass constraint $\int_\Omega u \, dx = m$ and small ϵ, so that u expecially favours taking values -1 or 1. It is clear that studying minimizers of OK_ϵ is a difficult task, but Γ-convergence comes to our aid, in that [24] as $\epsilon \to 0$ the L^1 norm Γ-limit of the functional (1.2) goes to ($\gamma' = 3/16$ is a fixed multiplicative constant)

$$\frac{\gamma'}{2}|Du|(\Omega) + \gamma \int_\Omega \left(|\nabla v_u|^2\right)\, dx, \tag{1.3}$$

where u is a BV function from Ω to $\{-1, 1\}$ with prescribed integral m, its total variation measure is $|Du|$ and

$$v_u(x) = \int_\Omega G(x, \xi)(u(\xi) - m) d\xi\,,$$

the inverse Laplacian of $u - m$ with zero average. The core business of Γ-convergence (devised for equi-coercive functionals) is that if u is a strict local minimizer of (1.3) then [24] it is the L^1-limit of a sequence $\{u_\epsilon\}$ of local minimizers of (1.2); thus instead of solving the original ϵ-problem we are allowed to deal with the possibly simpler problem (1.3), knowing that if it has, say, a bubble as a local minimizer then for ϵ small (1.2) will have local minimizers which resemble a bubble.

A further step towards an analytic-geometric isoperimetric problem is to abandon the function setting and switch to sets: indeed if $E = \{x : u(x) = 1\}$ so that $u(x) = u_E(x) = \chi_E - \chi_{\Omega\setminus E}$, the above problem (we drop the harmless γ' from now on, simply thinking it is incorporated into the other constant) may be rephrased as

$$(OK_{\text{geom}}) \qquad P_\Omega(E) + \gamma \int_\Omega |\nabla v_E|^2\, dx$$

where P is the perimeter [33] and

$$-\Delta v_E = u_E - m\,, \qquad \int_\Omega v_E = 0\,.$$

When Ω is a very large domain, one expects that the effect of boundary conditions diminishes in its interior and the minimizer may settle down into a natural minimal energy periodic configuration. It is known that in one space dimension, minimizers of (1.2) and (1.3) are periodic [13, 31], see also [7] for an investigation in higher dimension. On these grounds, to separate boundary effects from energy-induced pattern formation we replace the generic Ω by a periodic torus $\mathbb{T}$, i.e., a box $[0, T]^N$ with periodic boundary conditions; we now collect results on the Ohta-Kawasaki model, then we will present analogies and differences with the FitzHugh-Nagumo case.

2 - Ohta-Kawasaki, stationary points and stability

With the previous notation, in particular with $u_E = \chi_E - \chi_{\mathbb{T}\setminus E}$, the Ohta-Kawasaki energy we consider is thus

$$J_{OK}(E) = P_{\mathbb{T}}(E) + \gamma \int_{\mathbb{T}} |\nabla v_E|^2 \, dx \ , \tag{2.1}$$

with

$$|E| - |\mathbb{T} \setminus E| = m \ , \qquad -\Delta v_E = u_{\mathcal{E}} - m \ , \qquad v_E \text{ is } \mathbb{T}\text{-periodic and } \int_{\mathbb{T}} v_E = 0 \ .$$

It is quite easy to show [17] that the Euler-Lagrange equation satisfied by local minimizers of class C^2 is

$$\mathcal{H}_{\partial E}(x) + 4\gamma v_E(x) = \lambda \tag{2.2}$$

where $\mathcal{H}$ is the curvature (sum of the principal curvatures, or in a geometric-measure-theoretic vocabulary the tangential divergence of the unit outward normal ν), the number 4 is due to the fact that u_E jumps 2 units across the boundary of E and λ is a Lagrange multiplier due to the volume constraint. The equation may be derived by deforming E through the time flow associated with a (regular) vector field X on $\mathbb{T}$ into a time-indexed family E_t, and taking the time derivative of $J_{OK}(E_t)$ at $t = 0$. We call stationary points all regular solutions of (2.2). Several authors found instances of sets satisfying (2.2), among them balls, cylinders, lamellae and gyroids, but proving they actually are minimizers is a task of a different magnitude. Analogous to the positive second derivative criterion in $\mathbb{R}$, one may compute the second derivative of J_{OK}, and call “stable” those stationary points at which the second derivative is positive (in some sense). The difficulties one faces are many: first, actually computing the second derivative requires some effort, see [17]; second, in the periodic setting all translates of E share its same energy, so the concept of “positive” second variation has to be made precise by the use of equivalence classes: one has to replace the usual L^1 distance of sets, $d(E,F) = |E \triangle F|$, with

$$d_{\text{trasl}}(E,F) = \min_x |E \triangle (x + F)|$$

so that a strict local minimizer E of J_{OK} is an admissible set (i.e. $\int u_E = m$) such that for some $\delta > 0$

$$J_{OK}(E) < J_{OK}(F) \qquad \forall F : 0 < d_{\text{trasl}}(E,F) < \delta \ ,$$

always keeping the volume constraint $|E| = |F|$. This gives meaning to “strict” or “isolated” but does not solve the problem of the meaning of “positive”.

Assuming E is sufficiently smooth, by the results of [17] one sees that the second variation computed along the flow associated with X only requires the component of X parallel to the normal to ∂E: thus one may associate with the second variation of J_{OK} the quadratic form $J''_{OK}(E)$ defined on all functions $\phi \in H^1(\partial E)$ with $\int \phi = 0$ [the latter condition is due to the volume constraint] by

$$
\begin{aligned}
J''_{OK}(E)[\phi] &= \int_{\partial E} \big(|D_\tau \phi|^2 - |B_{\partial E}|^2 \phi^2\big) d\mathcal{H}^{N-1} \\
&\quad +8\gamma \int_{\partial E}\int_{\partial E} G(x,y)\phi(x)\phi(y)\, d\mathcal{H}^{N-1}(x) d\mathcal{H}^{N-1}(y) \\
&\quad +4\gamma \int_{\partial E} \partial_\nu v_E \, \phi^2 \, d\mathcal{H}^{N-1}
\end{aligned}
\tag{2.3}
$$

where $B_{\partial E}$ is the second fundamental form. The translation invariance condition $J_{OK}(E + t\eta) = J_{OK}(E)$ for all $\eta \in \mathbb{R}^N$ and all t, differentiated twice with respect to t, gives $J''_{OK}(E)[\eta \cdot \nu] = 0$, thus one has to get rid of the (finite dimensional) subspace spanned by the components of the normal field ν. Setting

$$
\mathcal{T}^\perp(\partial E) = \{\phi \in H^1(\partial E) : \int_{\partial E} \phi \, d\mathcal{H}^{N-1} = 0 \, , \quad \int_{\partial E} \phi \nu_i \, d\mathcal{H}^{N-1} = 0 \ \forall i\}
\tag{2.4}
$$

one may finally say that $J''_{OK}(E)$ is positive whenever $J''_{OK}(E)[\phi] > 0$ for all $\phi \in \mathcal{T}^\perp(\partial E) \setminus \{0\}$.

Even before Choksi and Sternberg [17], where the second variation is computed at any generic critical point of J_{OK}, it was known for special nice sets E represented by bubbles, cylinders or lamellae that $J''_{OK}(E)$ was positive, see the many papers by Ren and Wei quoted in [1, 4]. Here comes the third and the hardest difficulty: this intuition is a good omen, but does not yet prove that critical sets where the J''_{OK} is positive are indeed isolated local minimizers; in $\mathbb{R}$, one proves this for a function f essentially by integrating f'' from the critical point x_0 to nearby points x. For the Ohta-Kawasaki energy this was solved in [4] where the second variation is computed at all sets (and not only critical ones) and this result is used to deduce the following minimality criterion with quantitative estimate [4, Theorem 1.1].

Theorem 2.1. *Let $E \subset \mathbb{T}$ be a regular critical set of J_{OK} such that $J''_{OK}(E)[\phi] > 0$ for all $\phi \in \mathcal{T}^\perp(\partial E) \setminus \{0\}$. Then there exist $\delta, C > 0$ such that*

$$
J_{OK}(F) \geq J_{OK}(E) + C\big[d_{\mathrm{trasl}}(E,F)\big]^2
$$

for all $F \subset \mathbb{T}$ with $|F| = |E|$ and $d_{\mathrm{trasl}}(E,F) < \delta$.

Proving Theorem 2.1 does not simply reduce to computing J'' and integrating it along a flow E_t: this treatment only proves the quantitative inequality above [4, Theorem 3.9] for a set F whose boundary is the graph over ∂E of a regular function, that is its boundary may be written as $\{\sigma + \nu(\sigma)\psi(\sigma) : \sigma \in \partial E\}$; moreover, due to the volume constraint one should produce a volume preserving flow, just going straight along ν is not enough, [4, Theorem 3.7]. Getting rid of possible translations is a tough issue, reducing to sets whose boundary is in a tubular neighbourhood of ∂E is another one, and the celebrated result by Almgren [9], stating that any sequence E_h of ω-minimizing sets of the area functional with equibounded perimeters which converges in L^1 to some regular set E is made (for h large) of graphs over ∂E, allows us to deduce the general result from the one on graphs (tackling, among others, problems arising from the unknown Lagrange multipliers).

With some hard work, see [3], it is possible to show an even deeper stability result: if E is such a stable critical set, it is an attractor, in the sense that starting from any sufficiently close set F and letting it evolve along the gradient flow associated with J_{OK}, the evolution will converge to (a translate of) E.

3 - FitzHugh-Nagumo, stationary points and stability

We start directly from the geometric version of the problem, which is

$$J_{FHN}(E) = P_{\mathbb{T}}(E) - \alpha|E| + \frac{\sigma}{2}\int_E \mathcal{N}_E \, dx \ ,$$

with no volume constraint on E; $\alpha, \sigma > 0$ and $\mathcal{N}_E$ is the solution of the modified Helmholtz equation:

$$-\Delta\mathcal{N}_E + \mathcal{N}_E = \chi_E \text{ in } \mathbb{T}, \qquad \mathcal{N}_E \text{ is periodic in } \mathbb{T} \ .$$

At the same time $\mathcal{N}_E$ is the unique $\mathbb{T}$-periodic minimizer of

$$v \mapsto \int_{\mathbb{T}} \Big(\frac{|Dv|^2}{2} + \frac{v^2}{2} - v\chi_E \Big) \, dx$$

and it takes values between 0 and 1. By its definition

$$\int_E \mathcal{N}_E \, dx = \int_{\mathbb{T}} \mathcal{N}_E(x)\chi_E(x) \, dx = \int_{\mathbb{T}} \big(|\mathcal{N}_E|^2 + |\nabla\mathcal{N}_E|^2 \big) \, dx \ ,$$

so we may rewrite J_{FHN} in a way similar to J_{OK}, compare (2.1):

$$J_{FHN}(E) = P_{\mathbb{T}}(E) - \alpha|E| + \frac{\sigma}{2}\int_{\mathbb{T}} \big(|\mathcal{N}_E|^2 + |\nabla\mathcal{N}_E|^2 \big) \, dx \ .$$

Local minimizers will be those sets E such that $J_{FHN}(E) \leq J_{FHN}(F)$ for all sets F such that $d_{\text{trasl}}(E,F) < \delta$ for some $\delta > 0$. A classical stationary set of J has a C^2 interface that satisfies the Euler-Lagrange equation

$$\mathcal{H}(\partial E) - \alpha + \sigma \mathcal{N}_E = 0 \text{ on } \partial E \ , \tag{3.1}$$

see for example [11, 12]. A first difference between OK and FHN is that proving some easy set satisfies (2.2) requires combining curvature and the solution of the Poisson equation, which were done for lamellae, spheres, cylinders, but checking (3.1) involves the Helmholtz equation, which limits the easy case to that of lamellar solutions; the studies of spherical bubbles in [11, 12] are for infinite domain $\mathbb{R}^N$, rather than in periodic setting. The ratio between parameters α and σ plays an important role, and it is useful to introduce the "emptiness parameter"

$$c := 1 - \frac{2\alpha}{\sigma} \ ;$$

indeed, if $c > 0$ the empty set $E_\emptyset = \emptyset$ is more energetically favourable than the full set $E_\mathbb{T} = \mathbb{T}$, and the opposite is true when $c < 0$. In fact the empty set is the unique global minimizer when $c \geq 1$, while the full set is the unique global minimizer when $c \leq -1$. One may go a step further: naming c_N the isoperimetric constant in the torus in $\mathbb{R}^N$

$$P_\mathbb{T}(E) \geq c_N \min\{|E|, |\mathbb{T} \setminus E|\}^{1-1/N} \qquad \forall E \subset \mathbb{T} \ ,$$

then we have a global minimality result near the extreme cases [1, Proposition 1.5]: in the case $1 > c > 0$, the empty set remains the unique global minimizer of J_{FHN} for all $\alpha \leq c_N \sqrt[N]{2}/T$ where, as we recall, T is the side of the torus. Conversely in the case $-1 < c < 0$ the full torus is still the unique global minimizer for all $\alpha \geq \sigma - c_N \sqrt[N]{2}/T$. Finding global minimizers is a rare event; from now on we focus on characterizing local minimizers with easy structures, and in particular seeing the conditions under which lamellar sets (i.e. sets E made of a lot of slabs, all parallel to one face of the torus) are stationary points and if so, do they represent stable local minimizers.

Our set E will then be a k-lamella $\mathbb{L}$, made of k layers all orthogonal to the x_1 axis (not necessarily the same size), and we denote by x' all the remaining variables so that $x = (x_1, x')$. Using the total width x_0 of the lamellae composing $\mathbb{L}$, it is readily verified that not only $\mathbb{L}$ but also $\mathcal{N}_\mathbb{L}$ has a one-dimensional structure, in that it depends on x_1 alone, and we have [1, Theorem 2.9]

Theorem 3.1. *If* $\mathbb{L}$ *is a stationary point of* J_{FHN} *then*

(i) necessarily $\alpha \leq \sigma$*, i.e.* $c \geq -1$*;*

(ii) all lamellae are equally spaced and have the same width

$$\frac{x_0}{k} = \frac{T}{2k} - \operatorname{arc\,sinh}\Big(c\ \sinh\frac{T}{2k}\Big)\ ;$$

(iii) the function $\mathcal{N}_{\mathbb{L}}$ *takes the same value on the sides of all lamellae, and its derivative takes the same value* d_0 *on the left sides, and* $-d_0$ *on right sides of all lamellae;*

(iv) stationary lamellae depend only on c*, but not on* σ *(as long as we adjust* α *accordingly).*

(v) the corresponding energy is

$$\begin{aligned} J_{FHN}(\mathbb{L}) \;&=\; kT^{N-1}\Big\{2 + c\frac{\sigma}{2}\Big[\frac{T}{2k} - \operatorname{arc\,sinh}\Big(c\sinh\frac{T}{2k}\Big)\Big] \\ &\quad -\frac{\sigma}{2\sinh\frac{T}{2k}}\left(\cosh\frac{T}{2k} - \sqrt{1+c^2\sinh^2\frac{T}{2k}}\right)\Big\}\,. \end{aligned} \tag{3.2}$$

Thus, to every value of c there corresponds a single value of x_0 for which the lamella is stationary, and x_0 decreases as the emptiness parameter increases, which gives meaning to the name "emptyness" parameter; the exact forms of the energy and of x_0 are shown only to see that they are heavy, nonlinear expressions. One may wonder if, among all stationary lamellae (one for every value of k) there is an optimal k whose energy is the lowest. The expression of $J_{FHN}(\mathbb{L})$ shows no clear signs of convexity with respect to any parameter— and indeed it is not convex. Nevertheless, it is evident that $T/2k$ appears almost everywhere, so if we call $\mathcal{F}(c,\sigma,T/2k)$ the quantity between curly braces in (3.2) we have

$$\frac{2}{T^N}J_{FHN}(\mathbb{L}) = \frac{\mathcal{F}(c,\sigma,T/2k)}{T/2k}\ ;$$

the miracle is [1, Proposition 3.3] that the function $t\mapsto\mathcal{F}(c,\sigma,t)$ is decreasing and strictly convex, but (apart from a positive multiplicative constant) $J_{FHN}(\mathbb{L})$ is the slope of the line connecting the origin and the point $\big(t,\mathcal{F}(c,\sigma,t)\big)$; in addition, the asymptote of $\mathcal{F}$ as $t\to\infty$ involves the "threshold function"

$$\Gamma(c) = |c| - 1 - |c|\log|c|$$

which appears in the following Theorem. It turns out that for certain values of the parameters the slope (i.e. our energy!) decreases continuously as t increases, while for other values of the parameters the graph of $\mathcal{F}$ has a tangent line from the origin to a certain point with abscissa $t = t_0(c,\sigma)$, thus the slope first decreases then increases. This leads to [1, Corollary 3.6]

Theorem 3.2. *Given a torus with side T and $-1 \leq c \leq 1$*

(i) if $\Gamma(c) \geq -4/\sigma$ the minimal energy among stationary k-lamellae is attained for $k = 1$, but either the empty set or the full torus (the trivial states) will have even less energy;

(ii) if $\Gamma(c) < -4/\sigma$ the minimal energy configuration among all k-lamellae will divide the torus in k bands with mesh (i.e. one lamella plus one interspace) size close to $T_0 = 2t_0(c, \sigma)$, and precisely with $k = T/T_0$ if this is an integer, else with k integer just above or below T/T_0.

One may play with the size T: fix k first, and choose $T = kT_0$ in order to get a minimal k-lamella: it may be shown [1, discussion just below Corollary 3.6] that for T and k large enough, the minimal k-lamella has less energy than both trivial states. Having thus identified the lamella which has minimal energy among all fellow lamellae, we turn to stability. The situation is similar to the Ohta-Kawasaki case, with some help given by the absence of the volume constraint, and some problem given by the Helmholtz operator replacing the Laplacian, but in the end one replaces (2.3),(2.4) with

$$\begin{aligned} J''_{FHN}(E)[\phi] &= \int_{\partial E} \left(|D_\tau \phi|^2 - |B_{\partial E}|^2 \phi^2\right) d\mathcal{H}^{N-1} \\ &\quad +\sigma \int_{\partial E}\int_{\partial E} G(x,y)\phi(x)\phi(y)\, d\mathcal{H}^{N-1}(x) d\mathcal{H}^{N-1}(y) \\ &\quad +\sigma \int_{\partial E} \partial_\nu \mathcal{N}_E\, \phi^2\, d\mathcal{H}^{N-1}\,, \end{aligned} \tag{3.3}$$

$$\mathcal{T}^\perp(\partial E) = \left\{ \phi \in H^1(\partial E) : \int_{\partial E} \phi \nu_i \, d\mathcal{H}^{N-1} = 0 \ \forall i \right\}$$

and one deduces the analogous of Theorem 2.1, see [2, Theorem 3.5]

Theorem 3.3. *Let $E \subset \mathbb{T}$ be a regular critical set of J such that*

$$J''_{FHN}(E)[\eta] > 0 \qquad \text{for all } \eta \in \mathcal{T}^\perp(\partial E) \setminus \{0\}\,.$$

Then there exist $\delta, C > 0$ such that

$$J_{FHN}(F) \geq J_{FHN}(E) + C\Big(d_{\mathrm{trasl}}(E,F)\Big)^2$$

for all $F \subset \mathbb{T}$ with $d_{\mathrm{trasl}}(E,F) < \delta$.

Observe that stability of lamellae depends on both parameters c and σ. The task to be completed now is to prove that, for some or all k-lamellae, the second variation is positive, so the above theorem concludes that they are isolated local minimizers. We know something about stationary lamellae, namely Theorem 3.1, and this structure leads to a simplification of (3.3), at the price of additional notations. Calling ℓ_i with $i = 1, \dots, 2k$ the x_1-coordinate of the $2k$ sides of the k lamellae (in increasing order; in particular all "left" sides have odd index), the admissible functions ϕ on which one has to check $J''_{FHN}(\mathbb{L})[\phi] > 0$ are defined only on these sides, so we may label ϕ_i the restrictions. Following an idea in [25] we decompose each $\phi_i = \mu_i + \zeta_i$ where μ_i is the average on this face of the lamella, and consequently ζ_i is periodic with zero average. Then ϕ belongs to $\mathcal{T}^\perp$ reduces to imposing that the sum of averages on left sides equals the sum on right sides, i.e., introducing for ease the vector $M = (-1, +1, -1, +1, \cdots, +1) \in \mathbb{R}^{2k}$,

$$\phi \in \mathcal{T}^\perp \iff (\mu_1, \dots, \mu_{2k}) \cdot M = 0 \ .$$

As a consequence J''_{FHN} comes from two distinct contributions: that of the averages and that of terms containing ζ, with no interaction between the two. Denoting by $\mathcal{G}$ the Green's function of the Helmholtz operator on the periodic **segment** $[0, T]$, which is an easily computable one-dimensional function, and by $|a-b|_T$ the closest distance between two numbers a, b in $[0, T]$ if one identifies this interval as a circumference with length T, the contribution of all averages to J''_{FHN} is then

$$\sum_{i,j=1}^{2k} \mu_i \mu_j \mathcal{G}(|\ell_j - \ell_i|_T) \mu_i \mu_j - d_0 \sum_{i=1}^{2k} \mu_i^2$$

where d_0 is the slope of $\mathcal{N}_E$ on the (left) sides of the lamellae, which appeared in Theorem 3.1, (iii). It turns out that the eigenvalues of the matrix appearing in the first sum are all real and can be computed (another miracle, coming from a result by Tee [34] on block-circulant matrices); all but one are strictly greater than d_0, and the eigenvector corresponding to the smallest eigenvalue d_0 is just M, so the contribution of the average part μ to J''_{FHN} is strictly positive whenever $\phi \in \mathcal{T}^\perp$.

There remains to check the contribution of the zero-average part ζ. In general, it is easy to show that for a certain range of the parameters this is non-negative, which proves stability of lamellae, but the result is not sharp (nor satisfying); so we specialize to the 2D setting, where the torus is a square and the sides of lamellae are segments: this reduction also inspires an open problem stated in the last section. We decompose each ζ_i in Fourier series resulting in a sum of the contributions from each disturbance mode $m_1, m_2, \ldots$ where m_h corresponds to sines or cosines of $(2\pi h/T)x'$. A simple calculation shows that interaction occurs only within the same mode; within the confine of the same mode, disturbances from distinct lamella faces can influence one another. Hence to prove J'' is positive one must show that the contribution of any single mode is positive. The final study compares stability for a k-lamella for various k, stability for a k-lamella for various values of the emptiness parameter c, effect of the size T on stability, and the loss of stability due to the action of different modes. The last is the cleanest result [2, Theorem 5.11]: if the contribution to J'' of mode m_i is non-negative, so is the contribution of mode m_{i+1}; thus loss of stability can only occur due to the first disturbance mode. As for dependence on c we have [2, Theorem 5.13] that likeliness of stability increases with $|c|$, in the sense that if a stationary k-lamella with a certain value c_0 of the emptiness parameter is stable, so is the stationary k-lamella for all c with $|c| > |c_0|$: thus the most delicate case for stability is $c = 0$. Finally, regarding dependence on k, there is only a partial result [2, Theorem 5.15]: in the case $c = 0$, if the stationary k-lamella is stable so is the stationary $(k+1)$-lamella; thus the worst case for stability is the 1-lamella. The last result in this summary concerns the effect of T (see [2, Corollary 5.20]), for which we may find a number $T_0(k)$ such that in the case $c = 0$ the k-lamella is stable for $T < T_0(k)$ and unstable for $T > T_0(k)$: we may naïvely explain this by observing that periodic functions on a torus with side 10 consist of not only 1-periodic functions, but many others; disturbing stability becomes easier.

4 - Work in progress, and main open problems concerning FHN

Lamellar stationary points may lose their stability when the parameters are varied: a work in progress addresses the possible bifurcation phenomena at the point where stability starts to fail.

The last results only cover the case $c = 0$; there are at least computational complexity reasons which inhibited us from analyzing the general case, which is left open.

A majority of the stability results is done only in the 2D case, which leaves open the (physically interesting) case of 3D.

In [2, Corollary 5.7] the existence of non-lamellar sets with lower energy than all lamellae and both trivial states has been shown in suitable parameter regimes, no qualitative information is available about its structure. This will be of interest.

For the bravest, the asymptotic stability result reported for the Ohta-Kawasaki model at the end of Section 2 is missing for the FitzHugh-Nagumo case.

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